A Solution Manual For

Own collection of miscellaneous problems

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Contents

CHAPTER 1

Lookup tables for all problems in current book

1.1 section 1.0

Table 1.1: Lookup table for all problems in current section

ID	problem	ODE
8389	1	$y' = \frac{\cos(y)\sec(x)}{x}$
8390	2	$y' = x(\cos(y) + y)$
8391	3	$y' = \frac{\sec(x)(\sin(y)+y)}{x}$
8392	4	$y' = \left(5 + \frac{\sec(x)}{x}\right)(\sin(y) + y)$
8393	5	$y' = y + 1$
8394	6	$y' = 1 + x$
8395	7	$y' = x$
8396	8	$y' = y$
8397	9	$y' = 0$
8398	10	$y' = 1 + \frac{\sec(x)}{x}$
8399	11	$y' = x + \frac{\sec(x)y}{x}$
8400	12	$y' = \frac{2y}{x}$
8401	13	$y' = \frac{2y}{x}$
8402	14	$y' = \frac{\ln(1+y^2)}{\ln(x^2+1)}$
8403	15	$y' = \frac{1}{x}$
8404	16	$y' = \frac{-yx-1}{4x^3y-2x^2}$
8405	17	$\frac{y'^2}{4} - xy' + y = 0$
8406	18	$y' = \sqrt{\frac{y+1}{y^2}}$
8407	19	$y' = \sqrt{1 - x^2 - y^2}$
8408	20	$y' + \frac{y}{3} = \frac{(1-2x)y^4}{3}$
8409	21	$y' = \sqrt{y} + x$
8410	23	$x^2y' + y^2 = xyy'$

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Table 1.1 Lookup table
Continued from previous page

ID	problem	ODE
8411	24	$y = xy' + x^2y'^2$
8412	25	$(x + y)\, y' = 0$
8413	26	$xy' = 0$
8414	27	$\frac{y'}{x+y} = 0$
8415	28	$\frac{y'}{x} = 0$
8416	29	$y' = 0$
8417	30	$y = xy'^2 + y'^2$
8418	31	$y' = \frac{5x^2-yx+y^2}{x^2}$
8419	32	$2t + 3x + (x + 2)\, x' = 0$
8420	33	$y' = \frac{1}{1-y}$
8421	34	$p' = ap - bp^2$
8422	35	$y^2 + \frac{2}{x} + 2xyy' = 0$
8423	36	$xf' - f = \frac{f'^2\left(1-f'^{\lambda}\right)^2}{\lambda^2}$
8424	37	$xy' - 2y + by^2 = c\,x^4$
8425	38	$xy' - y + y^2 = x^{2/3}$
8426	39	$u' + u^2 = \frac{1}{x^{4/5}}$
8427	40	$yy' - y = x$
8428	41	$y'' + 2y' + y = 0$
8429	41	$5y'' + 2y' + 4y = 0$
8430	42	$y'' + y' + 4y = 1$
8431	43	$y'' + y' + 4y = \sin(x)$
8432	44	$y = xy'^2$
8433	45	$yy' = 1 - xy'^3$

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Table 1.1 Lookup table
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ID	problem	ODE
8434	46	$f' = \frac{1}{f}$
8435	47	$ty'' + 4y' = t^2$
8436	48	$(t^2 + 9)\, y'' + 2y't = 0$
8437	49	$t^2y'' - 3y't + 5y = 0$
8438	50	$ty'' + y' = 0$
8439	51	$t^2y'' - 2y' = 0$
8440	52	$y'' + \frac{(t^2-1)y'}{t} + \frac{t^2 y}{\left(1+e^{\frac{t^2}{2}}\right)^2} = 0$
8441	53	$ty'' - y' + 4t^3y = 0$
8442	54	$y'' = 0$
8443	55	$y'' = 1$
8444	56	$y'' = f(t)$
8445	57	$y'' = k$
8446	58	$y' = -4\sin(x - y) - 4$
8447	59	$y' + \sin(x - y) = 0$
8448	60	$y'' = 4\sin(x) - 4$
8449	61	$yy'' = 0$
8450	62	$yy'' = 1$
8451	63	$yy'' = x$
8452	64	$y^2y'' = x$
8453	65	$y^2y'' = 0$
8454	66	$3yy'' = \sin(x)$
8455	67	$3yy'' + y = 5$
8456	68	$ayy'' + by = c$

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Table 1.1 Lookup table
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ID	problem	ODE
8457	69	$ay^2y'' + by^2 = c$
8458	70	$ayy'' + by = 0$
8459	71	$[x'(t) = 9x(t)+4y(t)\,, y'(t) = -6x(t)-y(t)\,, z'(t) = 6x(t)+4y(t)+3z(t)]$
8460	72	$[x'(t) = x(t) - 3y(t)\,, y'(t) = 3x(t) + 7y(t)]$
8461	73	$[x'(t) = x(t) - 2y(t)\,, y'(t) = 2x(t) + 5y(t)]$
8462	74	$[x'(t) = 7x(t) + y(t)\,, y'(t) = -4x(t) + 3y(t)]$
8463	75	$[x'(t) = x(t) + y(t)\,, y'(t) = y(t)\,, z'(t) = z(t)]$
8464	76	$[x'(t) = 2x(t)+y(t)-z(t)\,, y'(t) = -x(t)+2z(t)\,, z'(t) = -x(t)-2y(t)+ 4z(t)]$
8465	77	$x' = 4Ak\left(\frac{x}{A}\right)^{3/4} - 3kx$
8466	78	$\frac{y'y}{1+\frac{\sqrt{1+y'^2}}{2}} = -x$
8467	78	$\frac{y'y}{1+\frac{\sqrt{1+y'^2}}{2}} = -x$
8468	79	$y' = \frac{y\left(1+\frac{a^2x}{\sqrt{a^2(x^2+1)}}\right)}{\sqrt{a^2(x^2+1)}}$
8469	80	$y' = x^2 + y^2$
8470	81	$y' = 2\sqrt{y}$
8471	82	$z'' + 3z' + 2z = 24\,\mathrm{e}^{-3t} - 24\,\mathrm{e}^{-4t}$
8472	83	$y' = \sqrt{1-y^2}$
8473	84	$y' = -1 + x^2 + y^2$
8474	85	$y' = 2y\left(x\sqrt{y} - 1\right)$
8475	86	$y'' = \frac{1}{y} - \frac{xy'}{y^2}$
8476	87	$y'' + y' + y = 0$
8477	88	$y'' + y' + y = 0$

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Table 1.1 Lookup table
Continued from previous page

ID	problem	ODE
8478	88	$y'' + y' + y = 0$
8479	89	$y'' - yy' = 2x$
8480	90	$y' - y^2 - x - x^2 = 0$

1.2 section 2.0

Table 1.2: Lookup table for all problems in current section

ID	problem	ODE
8481	1	$y'' - xy' - yx - x = 0$
8482	2	$y'' - xy' - yx - 2x = 0$
8483	3	$y'' - xy' - yx - 3x = 0$
8484	4	$y'' - xy' - yx - x^2 - x = 0$
8485	5	$y'' - xy' - yx - x^3 + 2 = 0$
8486	6	$y'' - xy' - yx - x^4 - 6 = 0$
8487	7	$y'' - xy' - yx - x^5 + 24 = 0$
8488	8	$y'' - xy' - yx - x = 0$
8489	9	$y'' - xy' - yx - x^2 = 0$
8490	10	$y'' - xy' - yx - x^3 = 0$
8491	11	$y'' - axy' - bxy - cx = 0$
8492	12	$y'' - axy' - bxy - c\,x^2 = 0$
8493	13	$y'' - axy' - bxy - c\,x^3 = 0$
8494	14	$y'' - y' - yx - x = 0$
8495	15	$y'' - y' - yx - x^2 = 0$
8496	16	$y'' - y' - yx - x^2 - 1 = 0$

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Table 1.2 Lookup table

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ID	problem	ODE
8497	16	$y'' - y' - yx - x^2 - 1 = 0$
8498	17	$y'' - 2y' - yx - x^2 - 2 = 0$
8499	18	$y'' - 4y' - yx - x^2 - 4 = 0$
8500	19	$y'' - y' - yx - x^3 + 1 = 0$
8501	20	$y'' - 2y' - yx - x^3 - x^2 = 0$
8502	21	$y'' - y' - yx - x^3 + 2 = 0$
8503	22	$y'' - 2y' - yx - x^3 + 2 = 0$
8504	23	$y'' - 4y' - yx - x^3 + 2 = 0$
8505	24	$y'' - 6y' - yx - x^3 + 2 = 0$
8506	25	$y'' - 8y' - yx - x^3 + 2 = 0$
8507	26	$y'' - y' - yx - x^4 + 3 = 0$
8508	27	$y'' - y' - yx - x^3 = 0$
8509	28	$y'' - yx - x^3 + 2 = 0$
8510	29	$y'' - yx - x^6 + 64 = 0$
8511	30	$y'' - yx - x = 0$
8512	31	$y'' - yx - x^2 = 0$
8513	32	$y'' - yx - x^3 = 0$
8514	33	$y'' - yx - x^6 - x^3 + 42 = 0$
8515	34	$y'' - x^2y - x^2 = 0$
8516	35	$y'' - x^2y - x^3 = 0$
8517	36	$y'' - x^2y - x^4 = 0$
8518	37	$y'' - x^2y - x^4 + 2 = 0$
8519	38	$y'' - 2x^2y - x^4 + 1 = 0$
8520	39	$y'' - x^3y - x^3 = 0$

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Table 1.2 Lookup table
Continued from previous page

ID	problem	ODE
8521	40	$y'' - x^3 y - x^4 = 0$
8522	41	$y'' - x^2 y' - x^2 y - x^2 = 0$
8523	42	$y'' - x^3 y' - x^3 y - x^3 = 0$
8524	43	$y'' - x y' - y x - x = 0$
8525	44	$y'' - x^2 y' - y x - x^2 = 0$
8526	45	$y'' - x^2 y' - x^2 y - x^3 - x^2 = 0$
8527	46	$y'' - x^2 y' - x^3 y - x^4 - x^2 = 0$
8528	47	$y'' - \frac{y'}{x} - y x - x^2 - \frac{1}{x} = 0$
8529	48	$y'' - \frac{y'}{x} - x^2 y - x^3 - \frac{1}{x} = 0$
8530	49	$y'' - \frac{y'}{x} - x^3 y - x^4 - \frac{1}{x} = 0$
8531	50	$y'' - x^3 y' - y x - x^3 - x^2 = 0$
8532	51	$y'' - x^3 y' - x^2 y - x^3 = 0$
8533	52	$y'' - x^3 y' - x^3 y - x^4 - x^3 = 0$
8534	50	$y''' - x^3 y' - x^2 y - x^3 = 0$

1.3 section 3.0

Table 1.3: Lookup table for all problems in current section

ID	problem	ODE
8535	1	$y'' + c y' + k y = 0$
8536	2	$w' = -\frac{1}{2} - \frac{\sqrt{1-12w}}{2}$
8537	3	$y'' + y = \sin(x)$
8538	4	$y'' + y = \sin(x)$
8539	5	$y'' + y = \sin(x)$

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Table 1.3 Lookup table
Continued from previous page

ID	problem	ODE
8540	6	$y'' + y = \sin(x)$
8541	7	$y'' + y = \sin(x)$
8542	8	$y'' + y = \sin(x)$
8543	9	$y'' + y = \sin(x)$
8544	10	$y'' + y = \sin(x)$
8545	11	$y'' + y' + y = \sin(x)$
8546	12	$y'' + y' + y = \sin(x)$
8547	13	$y'' + y' + y = \sin(x)$
8548	14	$y''' + y' + y = x$
8549	15	$x^4 y'' + x^3 y' - 4x^2 y = 1$
8550	16	$x^4 y'' + x^3 y' - 4x^2 y = x$
8551	17	$x^2 y'' + x y' - 4y = x$
8552	18	$x^4 y''' + x^3 y'' + x^2 y' + yx = 0$
8553	19	$x^4 y''' + x^3 y'' + x^2 y' + yx = x$
8554	20	$5x^5 y'''' + 4x^4 y''' + x^2 y' + yx = 0$
8555	21	$(x^2+1)\, y'' + 1 + y'^2 = 0$
8556	22	$(x^2+1)\, y'' + 1 + y'^2 = x$
8557	23	$(x^2+1)\, y'' + 1 + x y'^2 = 1$
8558	24	$(x^2+1)\, y'' + y y'^2 = 0$
8559	25	$(x^2+1)\, y'' + y'^2 = 0$
8560	26	$y'' + \sin(y)\, y'^2 = 0$
8561	27	$(x^2+1)\, y'' + y'^3 = 0$
8562	28	$y' = \mathrm{e}^{-\frac{y}{x}}$
8563	29	$y' = 2x^2 \sin\left(\frac{y}{x}\right)^2 + \frac{y}{x}$

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Table 1.3 Lookup table
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ID	problem	ODE
8564	30	$4x^2y'' + y = 8\sqrt{x}\,(\ln(x) + 1)$
8565	31	$vv' = \frac{2v^2}{r^3} + \frac{\lambda r}{3}$

1.4 section 4.0

Table 1.4: Lookup table for all problems in current section

ID	problem	ODE
8566	1	$2x^2y'' - xy' + (-x^2+1)\,y = 0$
8567	2	$2x^2y'' - xy' + (-x^2+1)\,y = 1$
8568	3	$2x^2y'' - xy' + (-x^2+1)\,y = 1 + x$
8569	4	$2x^2y'' - xy' + (-x^2+1)\,y = x$
8570	5	$2x^2y'' - xy' + (-x^2+1)\,y = x^2 + x + 1$
8571	6	$2x^2y'' - xy' + (-x^2+1)\,y = x^2$
8572	7	$2x^2y'' - xy' + (-x^2+1)\,y = x^2 + 1$
8573	8	$2x^2y'' - xy' + (-x^2+1)\,y = x^4$
8574	9	$2x^2y'' - xy' + (-x^2+1)\,y = \sin(x)$
8575	10	$2x^2y'' - xy' + (-x^2+1)\,y = 1 + \sin(x)$
8576	11	$2x^2y'' - xy' + (-x^2+1)\,y = x\sin(x)$
8577	12	$2x^2y'' - xy' + (-x^2+1)\,y = \cos(x) + \sin(x)$
8578	13	$x^2y'' + (\cos(x) - 1)\,y' + y\,e^x = 0$
8579	14	$(x-2)\,y'' + \frac{y'}{x} + (1+x)\,y = 0$
8580	15	$(x-2)\,y'' + \frac{y'}{x} + (1+x)\,y = 0$
8581	16	$(1+x)(3x-1)\,y'' + \cos(x)\,y' - 3yx = 0$
8582	17	$xy'' + 2y' + yx = 0$

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Table 1.4 Lookup table

Continued from previous page

ID	problem	ODE
8583	18	$2x^2y'' + 3xy' - yx = x^2 + 2x$
8584	19	$2x^2y'' - xy' + (-x^2 + 1)\,y = 1$
8585	20	$2x^2y'' + 2xy' - yx = 1$
8586	21	$y'' + (x - 6)\,y = 0$
8587	22	$x^2y'' + (3x^2 + 2x)\,y' - 2y = 0$
8588	23	$2x^2y'' - xy' + (-x^2 + 1)\,y = x^2 + \cos(x)$
8589	24	$2x^2y'' - xy' + (-x^2 + 1)\,y = \cos(x)$
8590	24	$2x^2y'' - xy' + (-x^2 + 1)\,y = x^3 + \cos(x)$
8591	24	$2x^2y'' - xy' + (-x^2 + 1)\,y = x^3\cos(x)$
8592	24	$2x^2y'' - xy' + (-x^2 + 1)\,y = x^3\cos(x) + \sin(x)^2$
8593	24	$2x^2y'' - xy' + (-x^2 + 1)\,y = \ln(x)$
8594	25	$2x^2(x^2 + x + 1)\,y'' + x(11x^2 + 11x + 9)\,y' + (7x^2 + 10x + 6)\,y = 0$
8595	26	$x^2(x + 3)\,y'' + 5x(1 + x)\,y' - (1 - 4x)\,y = 0$
8596	27	$x^2(-x^2 + 2)\,y'' - x(4x^2 + 3)\,y' + (-2x^2 + 2)\,y = 0$
8597	28	$y'^2 + y^2 = \sec(x)^4$
8598	29	$(y - 2xy')^2 = y'^3$
8599	31	$x^2y'' + y = 0$
8600	32	$xy'' + y' - y = 0$
8601	33	$4xy'' + 2y' + y = 0$
8602	34	$xy'' + y' - y = 0$
8603	35	$xy'' + (1 + x)\,y' + 2y = 0$
8604	36	$x(x - 1)\,y'' + 3xy' + y = 0$
8605	37	$x^2(x^2 - 2x + 1)\,y'' - x(x + 3)\,y' + (4 + x)\,y = 0$
8606	38	$2x^2(x + 2)\,y'' + 5x^2y' + (1 + x)\,y = 0$

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Table 1.4 Lookup table

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ID	problem	ODE
8607	39	$2x^2y'' + xy' + (x-5)\,y = 0$
8608	40	$2x^2y'' + 2xy' - yx = \sin(x)$
8609	41	$2x^2y'' + 2xy' - yx = x\sin(x)$
8610	42	$2x^2y'' + 2xy' - yx = \sin(x)\cos(x)$
8611	43	$2x^2y'' + 2xy' - yx = x^3 + x\sin(x)$
8612	44	$\cos(x)\,y'' + 2xy' - yx = 0$
8613	45	$x^2y'' + 4xy' + (x^2+2)\,y = 0$
8614	46	$x^2y'' + xy' - yx = 0$
8615	47	$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$
8616	48	$(x^2 - x)\,y'' - xy' + y = 0$
8617	49	$x^2y'' + (x^2+6x)\,y' + yx = 0$
8618	50	$x^2y'' - xy' + (x^2-8)\,y = 0$
8619	51	$x^2y'' - 9xy' + 25y = 0$
8620	52	$x^2y'' - xy' - \left(x^2 + \frac{5}{4}\right) y = 0$
8621	53	$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$
8622	54	$xy'' + (2-x)\,y' - y = 0$
8623	55	$2x^2y'' + 3xy' - y = 0$
8624	56	$2x^2y'' + 5xy' + 4y = 0$
8625	57	$x^2y'' + 3xy' + 4x^4y = 0$
8626	58	$x^2y'' - yx = 0$
8627	59	$(-x^2+1)\,y'' + y' + y = x\,\mathrm{e}^x$
8628	60	$y' = y(1-y^2)$
8629	61	$\frac{xy''}{1-x} + y = \frac{1}{1-x}$
8630	62	$\frac{xy''}{1-x} + yx = 0$

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Table 1.4 Lookup table
Continued from previous page

ID	problem	ODE
8631	63	$\frac{xy''}{1-x}+y=\cos(x)$
8632	64	$\frac{xy''}{-x^2+1}+y=0$
8633	65	$y''=(x^2+3)\,y$
8634	66	$y''+(x-1)\,y=0$
8635	67	$[x'(t)=x(t)+2y(t)+2t+1, y'(t)=5x(t)+y(t)+3t-1]$
8636	68	$y''+20y'+500y=100000\cos(100x)$
8637	69	$y''\sin(2x)^2+y'\sin(4x)-4y=0$

1.5 section 5.0

Table 1.5: Lookup table for all problems in current section

ID	problem	ODE
8638	1	$y''=Ay^{2/3}$
8639	2	$y''+2xy'+(x^2+1)\,y=0$
8640	3	$y''+2\cot(x)\,y'-y=0$
8641	4	$x^2y''+xy'+\left(x^2-\frac{1}{4}\right)y=0$
8642	5	$4x^2y''+(-8x^2+4x)\,y'+(4x^2-4x-1)\,y=4\sqrt{x}\,e^x$
8643	6	$xy''-(2x+2)\,y'+(x+2)\,y=6x^3e^x$
8644	7	$y'+y=\frac{1}{x}$
8645	8	$y'+y=\frac{1}{x^2}$
8646	9	$xy'+y=0$
8647	10	$y'=\frac{1}{x}$
8648	11	$y''=\frac{1}{x}$
8649	12	$y''+y'=\frac{1}{x}$

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Table 1.5 Lookup table
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ID	problem	ODE
8650	13	$y'' + y = \frac{1}{x}$
8651	14	$y'' + y' + y = \frac{1}{x}$
8652	15	$h^2 + \frac{2ah}{\sqrt{1+h'^2}} = b^2$
8653	16	$y'' + 2y' - 24y = 16 - (x+2)\,\mathrm{e}^{4x}$
8654	17	$y'' + 3y' - 4y = 6\,\mathrm{e}^{2t-2}$
8655	18	$y'' + y = \mathrm{e}^{a\cos(x)}$
8656	19	$y' = \frac{y}{2y\ln(y)+y-x}$
8657	20	$xy'' - (2x+1)\,y' + (1+x)\,y = 0$
8658	21	$x^2y' + \mathrm{e}^{-y} = 0$
8659	22	$y'' + \mathrm{e}^{y} = 0$
8660	23	$y' = \frac{yx+3x-2y+6}{yx-3x-2y+6}$

CHAPTER 2

Book Solved Problems

2.1 section 1.0

2.1.1 problem 1

Internal problem ID [8389]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 1
Date solved : `Tuesday, December 17, 2024 at 12:25:26 PM`
CAS classification : `[_separable]`

Solve

$$y' = \frac{\cos(y)\sec(x)}{x}$$

Solved as first order separable ode

Time used: 0.296 (sec)

The ode $y' = \frac{\cos(y)\sec(x)}{x}$ is separable as it can be written as

$$\begin{aligned} y' &= \frac{\cos(y)\sec(x)}{x} \\ &= f(x)g(y) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{\sec(x)}{x} \\ g(y) &= \cos(y) \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(y)}\, dy &= \int f(x)\, dx \\ \int \frac{1}{\cos(y)}\, dy &= \int \frac{\sec(x)}{x}\, dx \\ \ln(\sec(y) + \tan(y)) &= \int \frac{\sec(x)}{x} dx + 2c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(y)$ is zero, since we had to divide by this above. Solving $g(y) = 0$ or $\cos(y) = 0$ for y gives

$$y = \frac{\pi}{2}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\ln(\sec(y) + \tan(y)) = \int \frac{\sec(x)}{x} dx + 2c_1$$
$$y = \frac{\pi}{2}$$

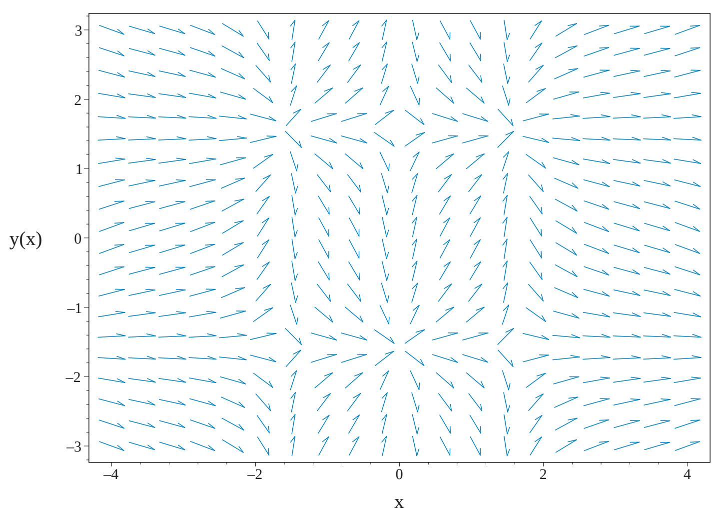


Figure 2.1: Slope field plot
$y' = \frac{\cos(y)\sec(x)}{x}$

Summary of solutions found

$$\ln(\sec(y) + \tan(y)) = \int \frac{\sec(x)}{x} dx + 2c_1$$
$$y = \frac{\pi}{2}$$

Solved as first order Exact ode

Time used: 0.176 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(\frac{\cos(y)\sec(x)}{x}\right)\mathrm{d}x$$

$$\left(-\frac{\cos(y)\sec(x)}{x}\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -\frac{\cos(y)\sec(x)}{x} \\
N(x, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\cos(y)\sec(x)}{x}\right) \\
&= \frac{\sin(y)\sec(x)}{x}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(\frac{\sin(y)\sec(x)}{x}\right) - (0)\right) \\
&= \frac{\sin(y)\sec(x)}{x}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= -x\cos(x)\sec(y)\left((0) - \left(\frac{\sin(y)\sec(x)}{x}\right)\right) \\
&= \tan(y)
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B\,\mathrm{d}y} \\ &= e^{\int \tan(y)\,\mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(\cos(y))} \\ &= \sec(y)\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \sec(y)\left(-\frac{\cos(y)\sec(x)}{x}\right) \\ &= -\frac{\sec(x)}{x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \sec(y)(1) \\ &= \sec(y)\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-\frac{\sec(x)}{x}\right) + (\sec(y))\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -\frac{\sec(x)}{x} \mathrm{d}x \\
\phi &= \int -\frac{\sec(x)}{x} dx + f(y) \qquad (3)
\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sec(y)$. Therefore equation (4) becomes

$$\sec(y) = 0 + f'(y) \qquad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \sec(y)$$

Integrating the above w.r.t y gives

$$\begin{aligned}
\int f'(y) \mathrm{d}y &= \int (\sec(y)) \mathrm{d}y \\
f(y) &= \ln(\sec(y) + \tan(y)) + c_1
\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \int -\frac{\sec(x)}{x} dx + \ln(\sec(y) + \tan(y)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \int -\frac{\sec(x)}{x} dx + \ln(\sec(y) + \tan(y))$$

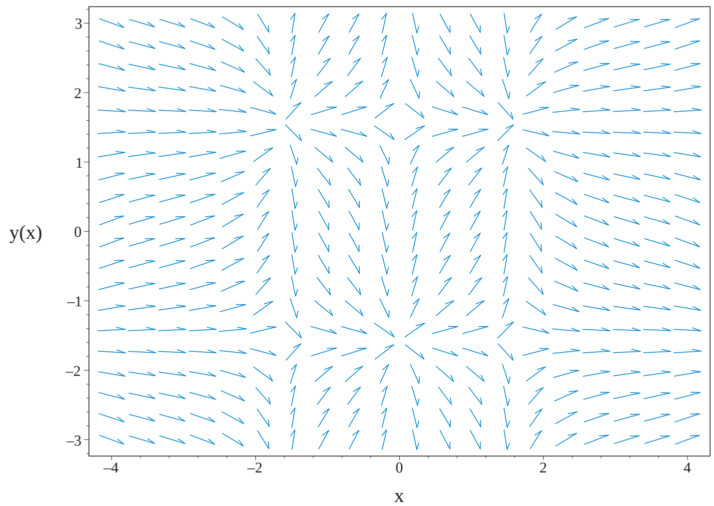


Figure 2.2: Slope field plot
$y' = \frac{\cos(y)\sec(x)}{x}$

Summary of solutions found

$$\int -\frac{\sec(x)}{x}dx + \ln(\sec(y) + \tan(y)) = c_1$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) = \frac{\cos(y(x))\sec(x)}{x}$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = \frac{\cos(y(x))\sec(x)}{x}$
- Separate variables
  $\frac{\frac{d}{dx}y(x)}{\cos(y(x))} = \frac{\sec(x)}{x}$
- Integrate both sides with respect to x
  $\int \frac{\frac{d}{dx}y(x)}{\cos(y(x))}dx = \int \frac{\sec(x)}{x}dx + C1$
- Evaluate integral
  $\ln(\sec(y(x)) + \tan(y(x))) = \int \frac{\sec(x)}{x}dx + C1$
- Solve for $y(x)$

$$y(x) = \arctan\left(\frac{\left(e^{\int \frac{\sec(x)}{x}dx+C1}\right)^2-1}{\left(e^{\int \frac{\sec(x)}{x}dx+C1}\right)^2+1}, \frac{2\,e^{\int \frac{\sec(x)}{x}dx+C1}}{\left(e^{\int \frac{\sec(x)}{x}dx+C1}\right)^2+1}\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

Maple dsolve solution

Solving time : `0.161` (sec)
Leaf size : 73

```
dsolve(diff(y(x),x) = cos(y(x))*sec(x)/x,
       y(x),singsol=all)
```

$$y = \arctan\left(\frac{e^{2\left(\int \frac{\sec(x)}{x}dx\right)}c_1^2-1}{e^{2\left(\int \frac{\sec(x)}{x}dx\right)}c_1^2+1}, \frac{2\,e^{\int \frac{\sec(x)}{x}dx}c_1}{e^{2\left(\int \frac{\sec(x)}{x}dx\right)}c_1^2+1}\right)$$

Mathematica DSolve solution

Solving time : `4.585` (sec)
Leaf size : 49

```
DSolve[{D[y[x],x]== Cos[y[x]]*Sec[x]/x,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$\begin{aligned}
y(x) &\to 2\arctan\left(\tanh\left(\frac{1}{2}\left(\int_1^x \frac{\sec(K[1])}{K[1]}dK[1]+c_1\right)\right)\right)\\
y(x) &\to -\frac{\pi}{2}\\
y(x) &\to \frac{\pi}{2}
\end{aligned}$$

2.1.2 problem 2

Internal problem ID [8390]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 2
Date solved : `Tuesday, December 17, 2024 at 12:25:29 PM`
CAS classification : `[_separable]`

Solve

$$y' = x(\cos(y) + y)$$

Solved as first order separable ode

Time used: 0.219 (sec)

The ode $y' = x(\cos(y) + y)$ is separable as it can be written as

$$\begin{aligned} y' &= x(\cos(y) + y) \\ &= f(x)g(y) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= x \\ g(y) &= \cos(y) + y \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(y)}\, dy &= \int f(x)\, dx \\ \int \frac{1}{\cos(y) + y}\, dy &= \int x\, dx \\ \int^{y} \frac{1}{\cos(\tau) + \tau} d\tau &= \frac{x^2}{2} + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(y)$ is zero, since we had to divide by this above. Solving $g(y)=0$ or $\cos(y)+y=0$ for y gives

$$y = \text{RootOf}\left(\cos\left(\_Z\right) + \_Z\right)$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

The solution $\text{RootOf}\left(\cos\left(\_Z\right) + \_Z\right)$ will not be used

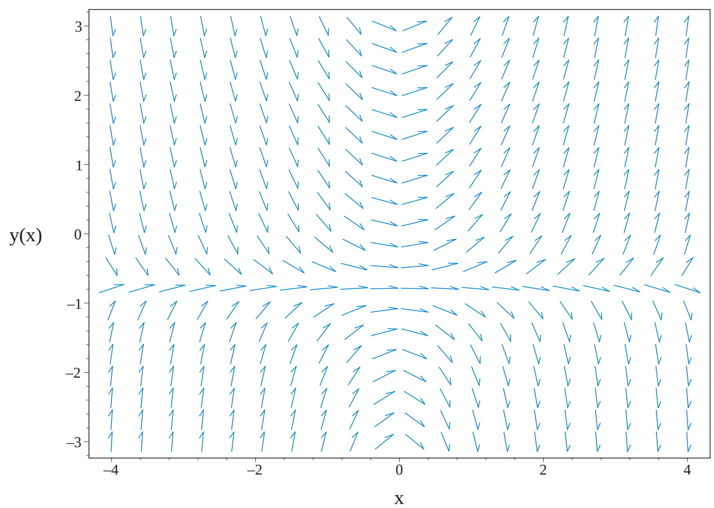


Figure 2.3: Slope field plot
$y' = x(\cos(y) + y)$

Summary of solutions found

$$\int^{y} \frac{1}{\cos(\tau)+\tau} d\tau = \frac{x^2}{2} + c_1$$

Solved as first order Exact ode

Time used: 0.175 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= (x(\cos(y) + y))\,\mathrm{d}x \\ (-x(\cos(y) + y))\,\mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x(\cos(y) + y) \\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x(\cos(y) + y)) \\ &= x(\sin(y) - 1)\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((-x(-\sin(y) + 1)) - (0)) \\
&= x(\sin(y) - 1)
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= -\frac{1}{x(\cos(y) + y)}((0) - (-x(-\sin(y) + 1))) \\
&= \frac{\sin(y) - 1}{\cos(y) + y}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int \frac{\sin(y)-1}{\cos(y)+y} \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\ln(\cos(y)+y)} \\
&= \frac{1}{\cos(y) + y}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{\cos(y) + y}(-x(\cos(y) + y)) \\
&= -x
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{\cos(y) + y}(1) \\
&= \frac{1}{\cos(y) + y}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(-x) + \left(\frac{1}{\cos(y) + y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -x\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\cos(y)+y}$. Therefore equation (4) becomes

$$
\frac{1}{\cos(y) + y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{\cos(y) + y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{\cos(y) + y}\right) \mathrm{d}y$$

$$f(y) = \int^y \frac{1}{\cos(\tau) + \tau} d\tau + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \int^y \frac{1}{\cos(\tau) + \tau} d\tau + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \int^y \frac{1}{\cos(\tau) + \tau} d\tau$$

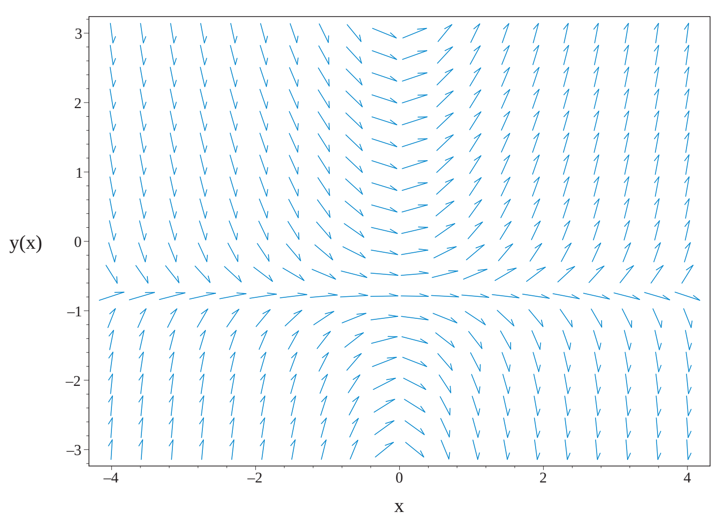


Figure 2.4: Slope field plot
$y' = x(\cos(y) + y)$

Summary of solutions found

$$-\frac{x^2}{2}+\int^{y}\frac{1}{\cos(\tau)+\tau}d\tau = c_1$$

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) = x(\cos(y(x)) + y(x))$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = x(\cos(y(x)) + y(x))$

- Separate variables

  $\frac{\frac{d}{dx}y(x)}{\cos(y(x))+y(x)} = x$

- Integrate both sides with respect to x

  $\int \frac{\frac{d}{dx}y(x)}{\cos(y(x))+y(x)}dx = \int xdx + C1$

- Cannot compute integral

  $\int \frac{\frac{d}{dx}y(x)}{\cos(y(x))+y(x)}dx = \frac{x^2}{2} + C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 23

```
dsolve(diff(y(x),x) = x*(cos(y(x))+y(x)),
       y(x),singsol=all)
```

$$\frac{x^2}{2} - \left(\int^{y} \frac{1}{\cos(\_a) + \_a} d\_a\right) + c_1 = 0$$

Mathematica DSolve solution

Solving time : 0.625 (sec)
Leaf size : 33

```
DSolve[{D[y[x],x] == x*(Cos[y[x]]+y[x]),{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \text{InverseFunction}\left[\int_1^{\#1} \frac{1}{\cos(K[1]) + K[1]} dK[1]\&\right]\left[\frac{x^2}{2} + c_1\right]$$

2.1.3 problem 3

Internal problem ID [8391]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 3
Date solved : `Tuesday, December 17, 2024 at 12:25:31 PM`
CAS classification : `[_separable]`

Solve

$$y' = \frac{\sec(x)(\sin(y) + y)}{x}$$

Solved as first order separable ode

Time used: 0.231 (sec)

The ode $y' = \frac{\sec(x)(\sin(y)+y)}{x}$ is separable as it can be written as

$$\begin{aligned} y' &= \frac{\sec(x)(\sin(y) + y)}{x} \\ &= f(x)g(y) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{\sec(x)}{x} \\ g(y) &= \sin(y) + y \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(y)}\, dy &= \int f(x)\, dx \\ \int \frac{1}{\sin(y) + y}\, dy &= \int \frac{\sec(x)}{x}\, dx \\ \int^{y} \frac{1}{\sin(\tau) + \tau} d\tau &= \int \frac{\sec(x)}{x} dx + 2c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(y)$ is zero, since we had to divide by this above. Solving $g(y) = 0$ or $\sin(y) + y = 0$ for y gives

$$y = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\int^{y} \frac{1}{\sin(\tau) + \tau} d\tau = \int \frac{\sec(x)}{x} dx + 2c_1$$
$$y = 0$$

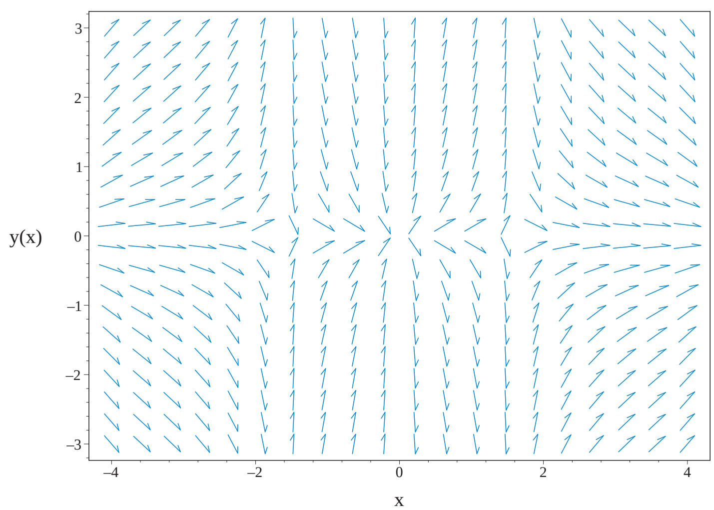


Figure 2.5: Slope field plot
$y' = \frac{\sec(x)(\sin(y)+y)}{x}$

Summary of solutions found

$$\int^{y} \frac{1}{\sin(\tau) + \tau} d\tau = \int \frac{\sec(x)}{x} dx + 2c_1$$
$$y = 0$$

Solved as first order Exact ode

Time used: 0.183 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\mathrm{d}y=\left(\frac{\sec(x)\,(\sin(y)+y)}{x}\right)\mathrm{d}x$$

$$\left(-\frac{\sec(x)\,(\sin(y)+y)}{x}\right)\mathrm{d}x+\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{\sec(x)(\sin(y)+y)}{x} \\
N(x,y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\sec(x)(\sin(y)+y)}{x}\right) \\
&= -\frac{\sec(x)(\cos(y)+1)}{x}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(-\frac{\sec(x)(\cos(y)+1)}{x}\right) - (0)\right) \\
&= -\frac{\sec(x)(\cos(y)+1)}{x}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= -\frac{x\cos(x)}{\sin(y)+y}\left((0) - \left(-\frac{\sec(x)(\cos(y)+1)}{x}\right)\right) \\
&= \frac{-\cos(y)-1}{\sin(y)+y}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int \frac{-\cos(y)-1}{\sin(y)+y} \, \mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(\sin(y)+y)} \\ &= \frac{1}{\sin(y)+y}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{\sin(y)+y}\left(-\frac{\sec(x)(\sin(y)+y)}{x}\right) \\ &= -\frac{\sec(x)}{x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{\sin(y)+y}(1) \\ &= \frac{1}{\sin(y)+y}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-\frac{\sec(x)}{x}\right) + \left(\frac{1}{\sin(y)+y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{\sec(x)}{x}\,\mathrm{d}x$$
$$\phi = \int -\frac{\sec(x)}{x}dx + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\sin(y)+y}$. Therefore equation (4) becomes

$$\frac{1}{\sin(y)+y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{\sin(y)+y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{\sin(y)+y}\right)\mathrm{d}y$$
$$f(y) = \int^{y} \frac{1}{\sin(\tau)+\tau}d\tau + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \int -\frac{\sec(x)}{x}dx + \int^{y} \frac{1}{\sin(\tau)+\tau}d\tau + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \int -\frac{\sec(x)}{x} dx + \int^{y} \frac{1}{\sin(\tau) + \tau} d\tau$$

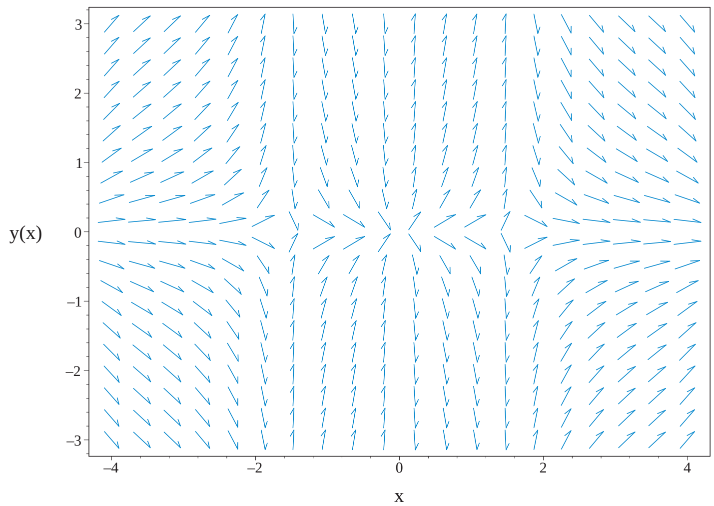


Figure 2.6: Slope field plot
$y' = \frac{\sec(x)(\sin(y)+y)}{x}$

Summary of solutions found

$$\int -\frac{\sec(x)}{x} dx + \int^{y} \frac{1}{\sin(\tau) + \tau} d\tau = c_1$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) = \frac{\sec(x)(\sin(y(x))+y(x))}{x}$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = \frac{\sec(x)(\sin(y(x))+y(x))}{x}$
- Separate variables
  $\frac{\frac{d}{dx}y(x)}{\sin(y(x))+y(x)} = \frac{\sec(x)}{x}$

- Integrate both sides with respect to x

  $\int \frac{\frac{d}{dx}y(x)}{\sin(y(x))+y(x)}dx = \int \frac{\sec(x)}{x}dx + C1$

- Cannot compute integral

  $\int \frac{\frac{d}{dx}y(x)}{\sin(y(x))+y(x)}dx = \int \frac{\sec(x)}{x}dx + C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 26

```
dsolve(diff(y(x),x) = sec(x)*(sin(y(x))+y(x))/x,
       y(x),singsol=all)
```

$$\int \frac{\sec(x)}{x}dx - \left(\int^{y} \frac{1}{\sin(\_a)+\_a}d\_a\right) + c_1 = 0$$

Mathematica DSolve solution

Solving time : 1.169 (sec)
Leaf size : 41

```
DSolve[{D[y[x],x]== Sec[x]*(Sin[y[x]]+y[x])/x,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \text{InverseFunction}\left[\int_1^{\#1} \frac{1}{K[1]+\sin(K[1])}dK[1]\&\right]\left[\int_1^x \frac{\sec(K[2])}{K[2]}dK[2]+c_1\right]$$

2.1.4 problem 4

Internal problem ID [8392]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 4
Date solved : `Tuesday, December 17, 2024 at 12:25:34 PM`
CAS classification : `[_separable]`

Solve

$$y' = \left(5+\frac{\sec\left(x\right)}{x}\right)\left(\sin\left(y\right)+y\right)$$

Solved as first order separable ode

Time used: 0.346 (sec)

The ode $y' = \frac{(5x+\sec(x))(\sin(y)+y)}{x}$ is separable as it can be written as

$$\begin{aligned} y' &= \frac{\left(5x+\sec\left(x\right)\right)\left(\sin\left(y\right)+y\right)}{x} \\ &= f(x)g(y) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{5x+\sec\left(x\right)}{x} \\ g(y) &= \sin\left(y\right)+y \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(y)}\,dy &= \int f(x)\,dx \\ \int \frac{1}{\sin\left(y\right)+y}\,dy &= \int \frac{5x+\sec\left(x\right)}{x}\,dx \\ \int^{y} \frac{1}{\sin\left(\tau\right)+\tau}d\tau &= \int \frac{5x+\sec\left(x\right)}{x}dx+2c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(y)$ is zero, since we had to divide by this above. Solving $g(y) = 0$ or $\sin(y) + y = 0$ for y gives

$$y = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\int^{y} \frac{1}{\sin(\tau) + \tau} d\tau = \int \frac{5x + \sec(x)}{x} dx + 2c_1$$
$$y = 0$$

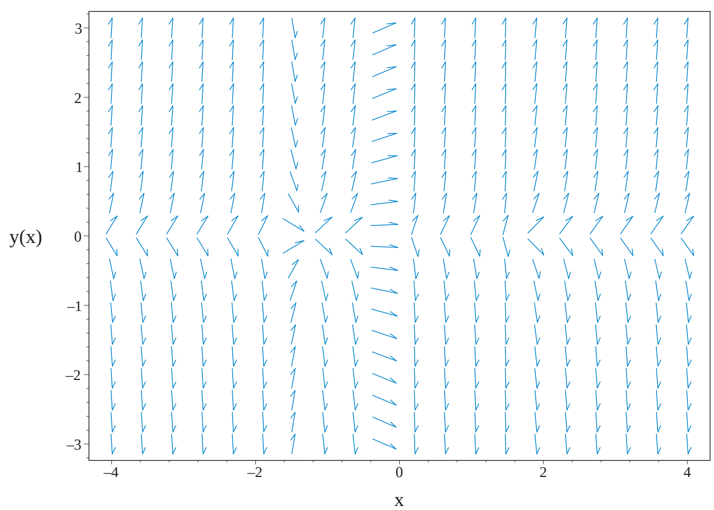


Figure 2.7: Slope field plot
$y' = \left(5 + \frac{\sec(x)}{x}\right)(\sin(y) + y)$

Summary of solutions found

$$\int^{y} \frac{1}{\sin(\tau) + \tau} d\tau = \int \frac{5x + \sec(x)}{x} dx + 2c_1$$
$$y = 0$$

Solved as first order Exact ode

Time used: 0.221 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\mathrm{d}y=\left(\left(5+\frac{\sec(x)}{x}\right)(\sin(y)+y)\right)\mathrm{d}x$$

$$\left(-\left(5+\frac{\sec(x)}{x}\right)(\sin(y)+y)\right)\mathrm{d}x+\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\left(5+\frac{\sec(x)}{x}\right)(\sin(y)+y)\\N(x,y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\left(5+\frac{\sec(x)}{x}\right)(\sin(y)+y)\right)\\&= -\frac{(5x+\sec(x))(\cos(y)+1)}{x}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\&= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\&= 1\left(\left(-\left(5+\frac{\sec(x)}{x}\right)(\cos(y)+1)\right)-(0)\right)\\&= -\frac{(5x+\sec(x))(\cos(y)+1)}{x}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)\\&= -\frac{x}{(\sin(y)+y)(5x+\sec(x))}\left((0)-\left(-\left(5+\frac{\sec(x)}{x}\right)(\cos(y)+1)\right)\right)\\&= \frac{-\cos(y)-1}{\sin(y)+y}\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \,\mathrm{d}y} \\ &= e^{\int \frac{-\cos(y)-1}{\sin(y)+y} \,\mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(\sin(y)+y)} \\ &= \frac{1}{\sin(y)+y}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{\sin(y)+y}\left(-\left(5+\frac{\sec(x)}{x}\right)(\sin(y)+y)\right) \\ &= \frac{-5x-\sec(x)}{x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{\sin(y)+y}(1) \\ &= \frac{1}{\sin(y)+y}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{-5x-\sec(x)}{x}\right)+\left(\frac{1}{\sin(y)+y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{-5x - \sec(x)}{x}\,\mathrm{d}x$$

$$\phi = \int \frac{-5x - \sec(x)}{x} dx + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\sin(y)+y}$. Therefore equation (4) becomes

$$\frac{1}{\sin(y) + y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{\sin(y) + y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{\sin(y) + y}\right)\mathrm{d}y$$

$$f(y) = \int^{y} \frac{1}{\sin(\tau) + \tau} d\tau + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \int \frac{-5x - \sec(x)}{x} dx + \int^{y} \frac{1}{\sin(\tau) + \tau} d\tau + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \int \frac{-5x - \sec(x)}{x} dx + \int^y \frac{1}{\sin(\tau) + \tau} d\tau$$

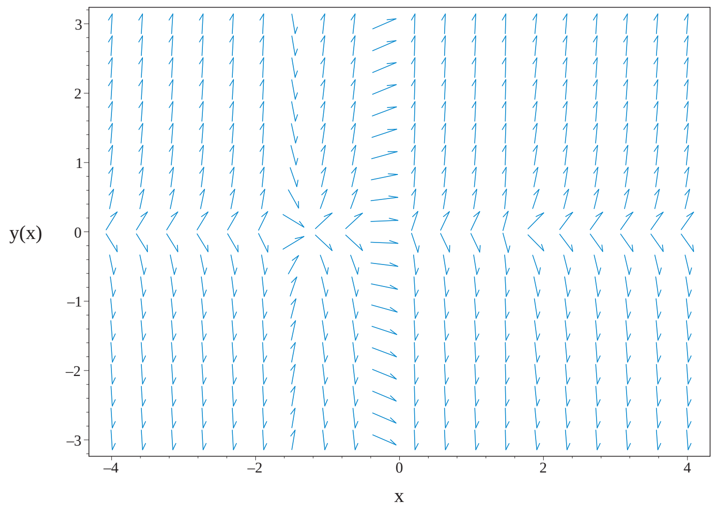


Figure 2.8: Slope field plot
$y' = \left(5 + \frac{\sec(x)}{x}\right)(\sin(y) + y)$

Summary of solutions found

$$\int \frac{-5x - \sec(x)}{x} dx + \int^y \frac{1}{\sin(\tau) + \tau} d\tau = c_1$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) = \left(5 + \frac{\sec(x)}{x}\right)(\sin(y(x)) + y(x))$

- Highest derivative means the order of the ODE is 1
$\frac{d}{dx}y(x)$
- Solve for the highest derivative
$\frac{d}{dx}y(x) = \left(5 + \frac{\sec(x)}{x}\right)(\sin(y(x)) + y(x))$
- Separate variables

$$\frac{\frac{d}{dx}y(x)}{\sin(y(x))+y(x)} = 5 + \frac{\sec(x)}{x}$$

- Integrate both sides with respect to x

$$\int \frac{\frac{d}{dx}y(x)}{\sin(y(x))+y(x)}dx = \int \left(5 + \frac{\sec(x)}{x}\right) dx + C1$$

- Cannot compute integral

$$\int \frac{\frac{d}{dx}y(x)}{\sin(y(x))+y(x)}dx = 5x + \int \frac{2\,\mathrm{e}^{\mathrm{I}x}}{\left((\mathrm{e}^{\mathrm{I}x})^2+1\right)x}dx + C1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

Maple dsolve solution

Solving time : `0.008` (sec)
Leaf size : 30

```
dsolve(diff(y(x),x) = (5+sec(x)/x)*(sin(y(x))+y(x)),
       y(x),singsol=all)
```

$$\int \frac{5x + \sec(x)}{x}dx - \left(\int^{y} \frac{1}{\sin(\_a) + \_a}d\_a\right) + c_1 = 0$$

Mathematica DSolve solution

Solving time : `23.795` (sec)
Leaf size : 168

```
DSolve[{D[y[x],x] == (5+Sec[x]/x)*(Sin[y[x]]+y[x]),{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$\text{Solve}\Bigg[\int_1^x \Bigg(-\frac{2\sec(K[1])}{K[1]}$$
$$-\frac{5(-\sec(K[1])\sin(K[1]-y(x))+\sec(K[1])\sin(K[1]+y(x))+2y(x))}{\sin(y(x))+y(x)}\Bigg)\,dK[1]$$
$$+\int_1^{y(x)}\Bigg(\frac{2}{K[2]+\sin(K[2])}$$
$$-\int_1^x\Bigg(\frac{5(\cos(K[2])+1)(2K[2]-\sec(K[1])\sin(K[1]-K[2])+\sec(K[1])\sin(K[1]+K[2]))}{(K[2]+\sin(K[2]))^2}-\frac{5(\cos(K[1]}{}$$

2.1.5 problem 5

Internal problem ID [8393]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 5
Date solved : `Tuesday, December 17, 2024 at 12:25:47 PM`
CAS classification : `[_quadrature]`

Solve

$$y' = y + 1$$

Solved as first order autonomous ode

Time used: 0.114 (sec)

Integrating gives

$$\int \frac{1}{y+1} dy = dx$$
$$\ln (y+1) = x + c_1$$

Applying the exponential to both sides gives

$$e^{\ln(y+1)} = e^{x+c_1}$$
$$y + 1 = c_1 \,\mathrm{e}^{x}$$

Singular solutions are found by solving

$$y + 1 = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = -1$$

The following diagram is the phase line diagram. It classifies each of the above equilibrium points as stable or not stable or semi-stable.

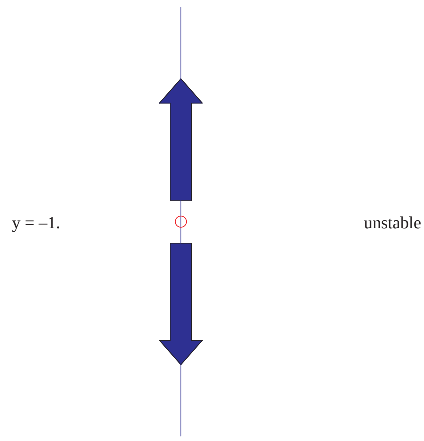


Figure 2.9: Phase line diagram

Solving for y gives

$$y = -1$$
$$y = c_1 \,\mathrm{e}^{x} - 1$$

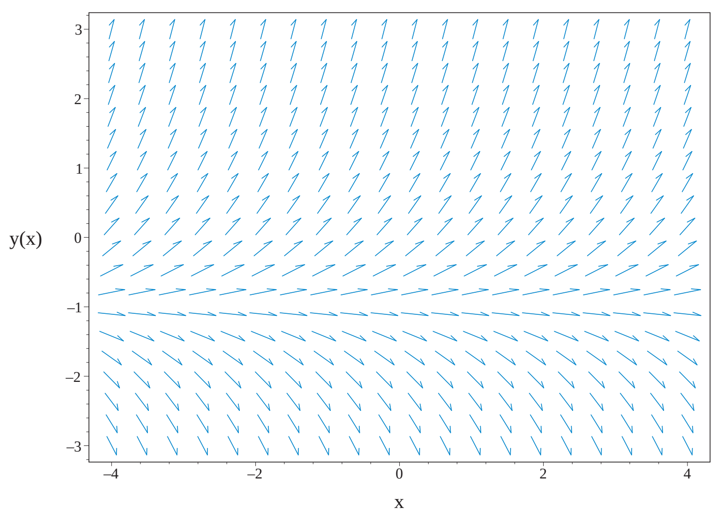


Figure 2.10: Slope field plot
$y' = y + 1$

Summary of solutions found

$$y = -1$$
$$y = c_1 \,\mathrm{e}^{x} - 1$$

Solved as first order Exact ode

Time used: 0.124 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y&=(y+1)\,\mathrm{d}x\\ (-1-y)\,\mathrm{d}x+\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-1-y\\ N(x,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-1 - y)\\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1((-1) - (0))\\ &= -1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x}\\ &= e^{\int -1 \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-x}\\ &= \mathrm{e}^{-x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{-x}(-1 - y)\\ &= -(y + 1)\,\mathrm{e}^{-x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= e^{-x}(1)\\ &= e^{-x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{dy}{dx} &= 0\\ \left(-(y+1)\,e^{-x}\right) + \left(e^{-x}\right)\frac{dy}{dx} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial y}\,dy &= \int \overline{N}\,dy\\ \int \frac{\partial \phi}{\partial y}\,dy &= \int e^{-x}\,dy\\ \phi &= e^{-x}y + f(x) \qquad (3)\end{aligned}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = -e^{-x}y + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = -(y+1)\,e^{-x}$. Therefore equation (4) becomes

$$-(y+1)\,e^{-x} = -e^{-x}y + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -e^{-x}$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int \left(-\mathrm{e}^{-x}\right)\mathrm{d}x$$
$$f(x) = \mathrm{e}^{-x} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{-x}y + \mathrm{e}^{-x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{-x}y + \mathrm{e}^{-x}$$

Solving for y gives

$$y = -\left(\mathrm{e}^{-x} - c_1\right)\mathrm{e}^{x}$$

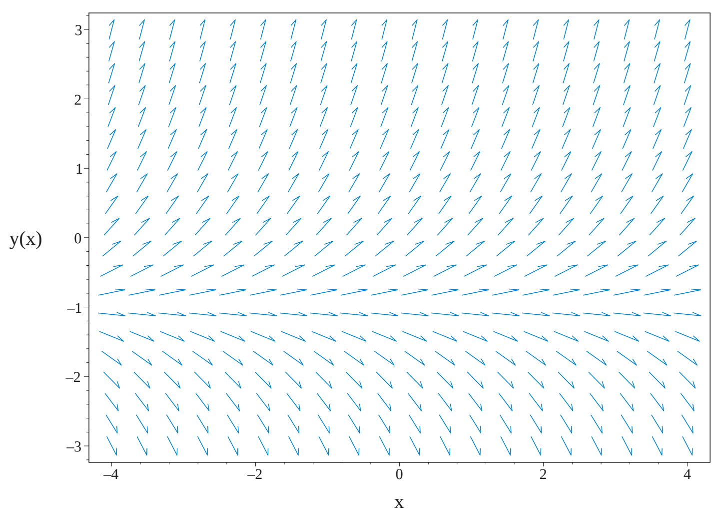


Figure 2.11: Slope field plot
$y' = y + 1$

Summary of solutions found

$$y = -\left(\mathrm{e}^{-x} - c_1\right)\mathrm{e}^{x}$$

Solved using Lie symmetry for first order ode

Time used: 0.473 (sec)

Writing the ode as

$$
\begin{aligned}
y' &= y + 1 \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + (y+1)(b_3 - a_2) - (y+1)^2 a_3 - x b_2 - y b_3 - b_1 = 0 \tag{5E}$$

Putting the above in normal form gives

$$-y^2 a_3 - x b_2 - y a_2 - 2 y a_3 - a_2 - a_3 - b_1 + b_2 + b_3 = 0$$

Setting the numerator to zero gives

$$-y^2 a_3 - x b_2 - y a_2 - 2 y a_3 - a_2 - a_3 - b_1 + b_2 + b_3 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_3v_2^2 - a_2v_2 - 2a_3v_2 - b_2v_1 - a_2 - a_3 - b_1 + b_2 + b_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-b_2v_1 - a_3v_2^2 + (-a_2 - 2a_3)\, v_2 - a_2 - a_3 - b_1 + b_2 + b_3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -a_3 &= 0 \\ -b_2 &= 0 \\ -a_2 - 2a_3 &= 0 \\ -a_2 - a_3 - b_1 + b_2 + b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= a_1 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= b_3 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 0 \\ \eta &= y + 1 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y+1} dy \end{aligned}$$

Which results in

$$S = \ln(y+1)$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = y+1$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 0 \\ S_y &= \frac{1}{y+1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x,y in terms of R,S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int 1\, dR$$
$$S(R) = R + c_2$$

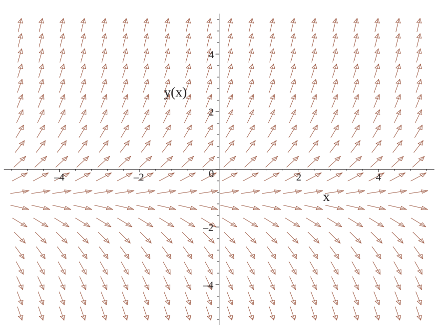
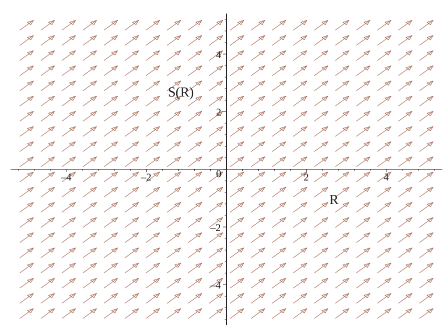
To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\ln(y+1) = x + c_2$$

Which gives

$$y = \mathrm{e}^{x+c_2} - 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = y + 1$ 	$R = x$ $S = \ln(y+1)$	$\frac{dS}{dR} = 1$ 

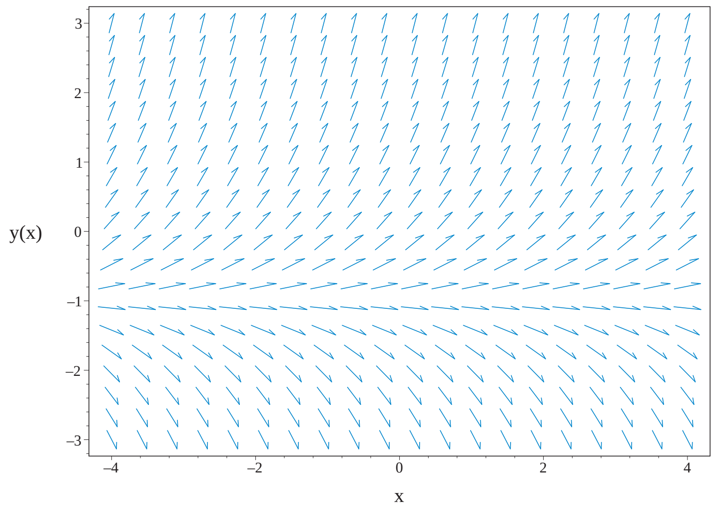


Figure 2.12: Slope field plot
$y' = y + 1$

Summary of solutions found

$$y = e^{x+c_2} - 1$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) = 1 + y(x)$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = 1 + y(x)$
- Separate variables
  $\frac{\frac{d}{dx}y(x)}{1+y(x)} = 1$
- Integrate both sides with respect to x
  $\int \frac{\frac{d}{dx}y(x)}{1+y(x)}dx = \int 1dx + C1$
- Evaluate integral
  $\ln(1 + y(x)) = x + C1$
- Solve for $y(x)$
  $y(x) = e^{x+C1} - 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 10

```
dsolve(diff(y(x),x) = y(x)+1,
       y(x),singsol=all)
```

$$y = -1 + e^{x} c_1$$

Mathematica DSolve solution

Solving time : 0.025 (sec)
Leaf size : 18

```
DSolve[{D[y[x],x] == y[x]+1,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -1 + c_1 e^x$$
$$y(x) \to -1$$

2.1.6 problem 6

Internal problem ID [8394]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 6
Date solved : `Tuesday, December 17, 2024 at 12:25:48 PM`
CAS classification : `[_quadrature]`

Solve

$$y' = 1 + x$$

Solved as first order quadrature ode

Time used: 0.040 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 1 + x\,dx$$
$$y = x + \frac{1}{2}x^2 + c_1$$

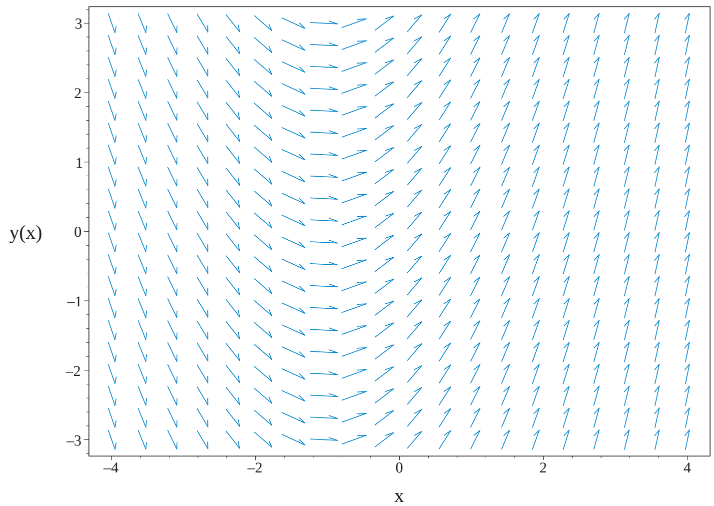


Figure 2.13: Slope field plot
$y' = 1 + x$

Summary of solutions found

$$y = x + \frac{1}{2}x^2 + c_1$$

Solved as first order Exact ode

Time used: 0.057 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= (1+x)\,\mathrm{d}x\\ (-1-x)\,\mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -1-x\\ N(x,y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-1-x)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -1 - x\,\mathrm{d}x \\
\phi &= -x - \frac{1}{2}x^2 + f(y) \qquad (3)
\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 1$. Therefore equation (4) becomes

$$1 = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 1$$

Integrating the above w.r.t y gives

$$\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int (1)\,\mathrm{d}y \\
f(y) &= y + c_1
\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x - \frac{1}{2}x^2 + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -x - \frac{1}{2}x^2 + y$$

Solving for y gives

$$y = x + \frac{1}{2}x^2 + c_1$$

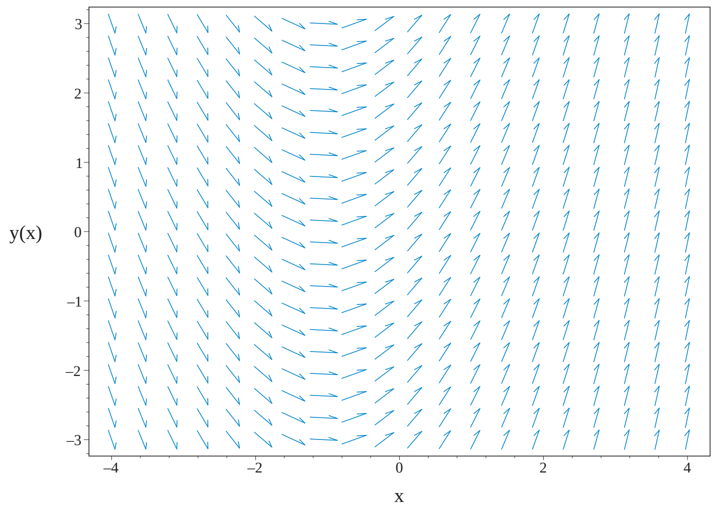


Figure 2.14: Slope field plot
$y' = 1 + x$

Summary of solutions found

$$y = x + \frac{1}{2}x^2 + c_1$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) = x + 1$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Integrate both sides with respect to x
  $\int \left(\frac{d}{dx}y(x)\right) dx = \int (x + 1)\, dx + C1$
- Evaluate integral
  $y(x) = \frac{1}{2}x^2 + x + C1$
- Solve for $y(x)$
  $y(x) = \frac{1}{2}x^2 + x + C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 12

```
dsolve(diff(y(x),x) = x+1,
       y(x),singsol=all)
```

$$y = \frac{1}{2}x^2 + x + c_1$$

Mathematica DSolve solution

Solving time : 0.003 (sec)
Leaf size : 16

```
DSolve[{D[y[x],x]== 1+x,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x^2}{2} + x + c_1$$

2.1.7 problem 7

Internal problem ID [8395]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 7
Date solved : `Tuesday, December 17, 2024 at 12:25:49 PM`
CAS classification : `[_quadrature]`

Solve

$$y' = x$$

Solved as first order quadrature ode

Time used: 0.037 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int x\,dx$$
$$y = \frac{x^2}{2} + c_1$$

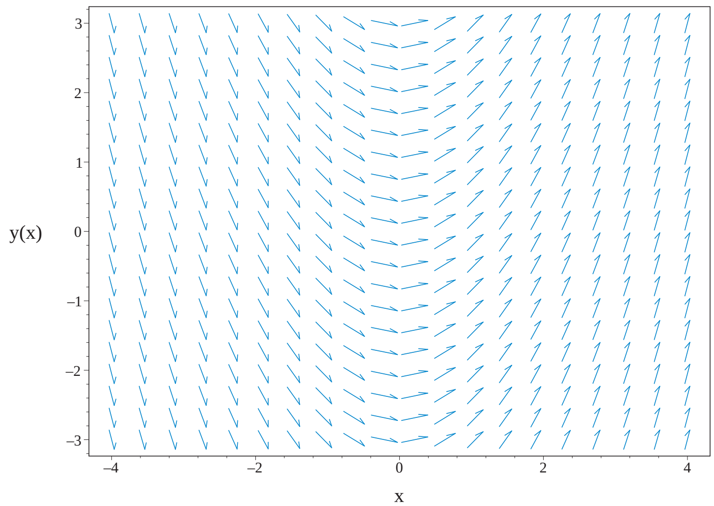


Figure 2.15: Slope field plot
$y' = x$

Summary of solutions found

$$y = \frac{x^2}{2} + c_1$$

Solved as first order Exact ode

Time used: 0.053 (sec)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y&=(x)\,\mathrm{d}x\\(-x)\,\mathrm{d}x+\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x\\N(x,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-x)\\&=0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(1)\\&=0\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \tag{1}$$

$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -x \mathrm{d}x \\
\phi &= -\frac{x^2}{2} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 1$. Therefore equation (4) becomes

$$
1 = 0 + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 1
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \mathrm{d}y &= \int (1) \mathrm{d}y \\
f(y) &= y + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = -\frac{x^2}{2} + y + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$
c_1 = -\frac{x^2}{2} + y
$$

Solving for y gives

$$y = \frac{x^2}{2} + c_1$$

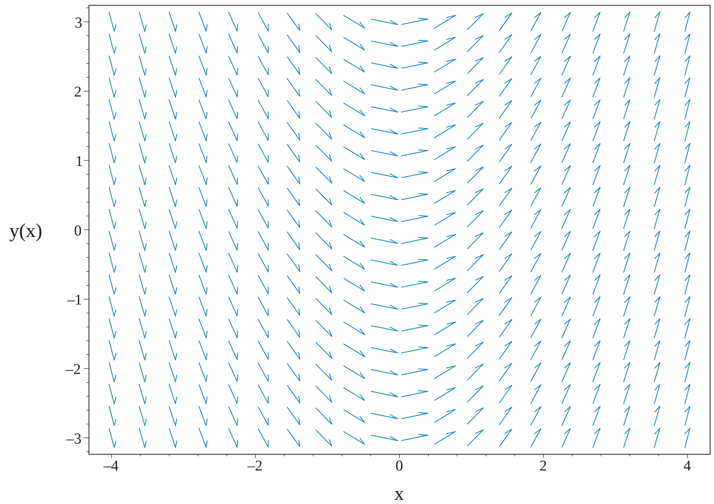


Figure 2.16: Slope field plot
$y' = x$

Summary of solutions found

$$y = \frac{x^2}{2} + c_1$$

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) = x$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}y(x)\right) dx = \int x dx + C1$

- Evaluate integral

  $y(x) = \frac{x^2}{2} + C1$

- Solve for $y(x)$

  $y(x) = \frac{x^2}{2} + C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 11

```
dsolve(diff(y(x),x) = x,
       y(x),singsol=all)
```

$$y = \frac{x^2}{2} + c_1$$

Mathematica DSolve solution

Solving time : 0.002 (sec)
Leaf size : 15

```
DSolve[{D[y[x],x] == x,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x^2}{2} + c_1$$

2.1.8 problem 8

Internal problem ID [8396]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 8
Date solved : `Tuesday, December 17, 2024 at 12:25:50 PM`
CAS classification : `[_quadrature]`

Solve

$$y' = y$$

Solved as first order autonomous ode

Time used: 0.087 (sec)

Integrating gives

$$\begin{aligned}\int \frac{1}{y} dy &= 1\\ \ln(y) &= x + c_1\\ e^{\ln(y)} &= \mathrm{e}^{x+c_1}\\ y &= c_1 \,\mathrm{e}^{x}\end{aligned}$$

The following diagram is the phase line diagram. It classifies each of the above equilibrium points as stable or not stable or semi-stable.

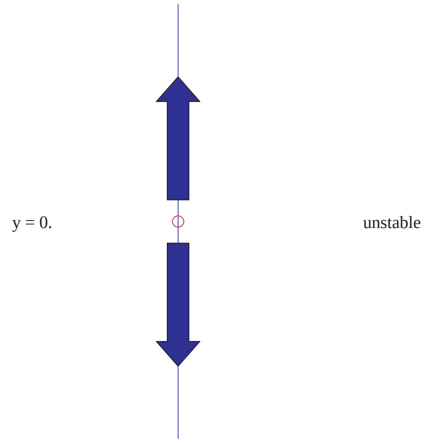


Figure 2.17: Phase line diagram

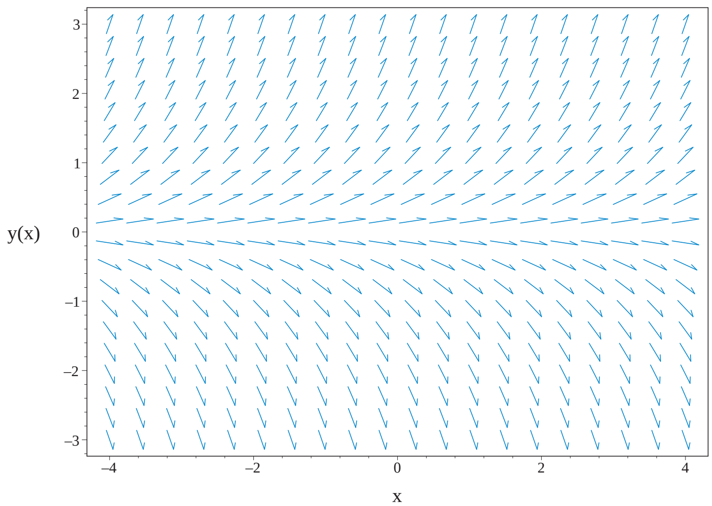


Figure 2.18: Slope field plot
$y' = y$

Summary of solutions found

$$y = 0$$

$$y = c_1 \, \mathrm{e}^{x}$$

Solved as first order homogeneous class D2 ode

Time used: 0.145 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$u'(x)\,x + u(x) = u(x)\,x$$

Which is now solved The ode $u'(x) = \frac{u(x)(x-1)}{x}$ is separable as it can be written as

$$\begin{aligned}
u'(x) &= \frac{u(x)\,(x-1)}{x} \\
&= f(x)g(u)
\end{aligned}$$

Where

$$\begin{aligned}
f(x) &= \frac{x-1}{x} \\
g(u) &= u
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\
\int \frac{1}{u}\,du &= \int \frac{x-1}{x}\,dx \\
\ln\left(u(x)\right) &= x + \ln\left(\frac{1}{x}\right) + c_1
\end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned}
\ln\left(u(x)\right) &= x + \ln\left(\frac{1}{x}\right) + c_1 \\
u(x) &= 0
\end{aligned}$$

Solving for $u(x)$ gives

$$\begin{aligned}
u(x) &= 0 \\
u(x) &= \frac{e^{x+c_1}}{x}
\end{aligned}$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = \frac{e^{x+c_1}}{x}$ back to y gives

$$y = e^{x+c_1}$$

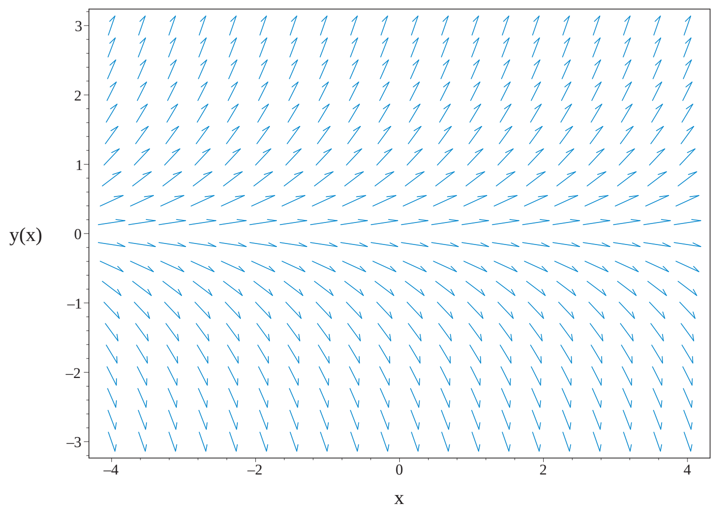


Figure 2.19: Slope field plot
$y' = y$

<u>Summary of solutions found</u>

$$y = 0$$

$$y = e^{x+c_1}$$

Solved as first order Exact ode

Time used: 0.100 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y&=(y)\,\mathrm{d}x\\ (-y)\,\mathrm{d}x+\mathrm{d}y&=0 \qquad\qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-y\\ N(x,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-y)\\ &=-1\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((-1) - (0)) \\
&= -1
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -1 \, \mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-x} \\
&= \mathrm{e}^{-x}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-x}(-y) \\
&= -y\,\mathrm{e}^{-x}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-x}(1) \\
&= \mathrm{e}^{-x}
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-y\,\mathrm{e}^{-x}\right) + \left(\mathrm{e}^{-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\int \frac{\partial \phi}{\partial y} \, \mathrm{d}y = \int \overline{N} \, \mathrm{d}y$$

$$\int \frac{\partial \phi}{\partial y} \, \mathrm{d}y = \int \mathrm{e}^{-x} \, \mathrm{d}y$$

$$\phi = y \, \mathrm{e}^{-x} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = -y \, \mathrm{e}^{-x} + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = -y \, \mathrm{e}^{-x}$. Therefore equation (4) becomes

$$-y \, \mathrm{e}^{-x} = -y \, \mathrm{e}^{-x} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 0$$

Therefore

$$f(x) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = y \, \mathrm{e}^{-x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = y \, \mathrm{e}^{-x}$$

Solving for y gives

$$y = c_1 \, \mathrm{e}^{x}$$

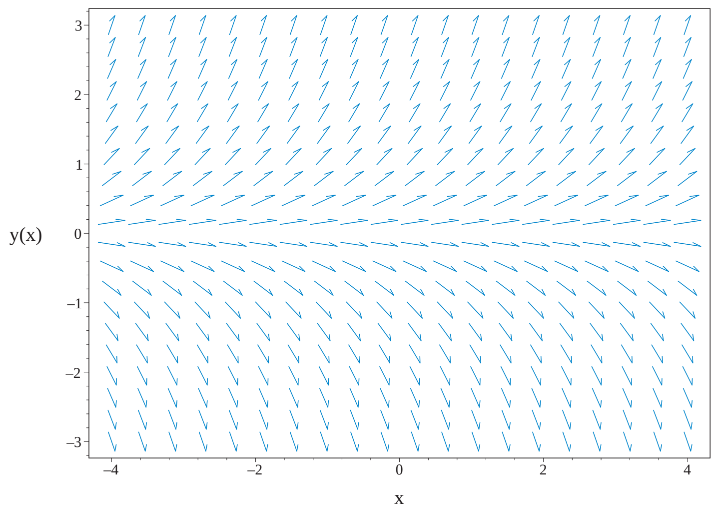


Figure 2.20: Slope field plot
$y' = y$

Summary of solutions found

$$y = c_1 \, \mathrm{e}^{x}$$

Solved using Lie symmetry for first order ode

Time used: 0.431 (sec)

Writing the ode as

$$y' = y$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + y(b_3 - a_2) - y^2 a_3 - x b_2 - y b_3 - b_1 = 0 \tag{5E}$$

Putting the above in normal form gives

$$-y^2 a_3 - x b_2 - y a_2 - b_1 + b_2 = 0$$

Setting the numerator to zero gives

$$-y^2 a_3 - x b_2 - y a_2 - b_1 + b_2 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_3 v_2^2 - a_2 v_2 - b_2 v_1 - b_1 + b_2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-a_3 v_2^2 - a_2 v_2 - b_2 v_1 - b_1 + b_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-a_2 &= 0 \\
-a_3 &= 0 \\
-b_2 &= 0 \\
-b_1 + b_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= a_1 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 0 \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{y} dy
\end{aligned}
$$

Which results in

$$S = \ln(y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = y$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 0 \\ S_y &= \frac{1}{y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned} \int dS &= \int 1\, dR \\ S(R) &= R + c_2 \end{aligned}$$

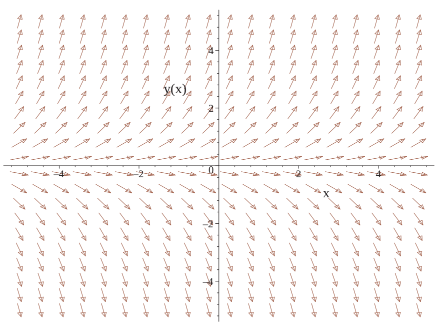
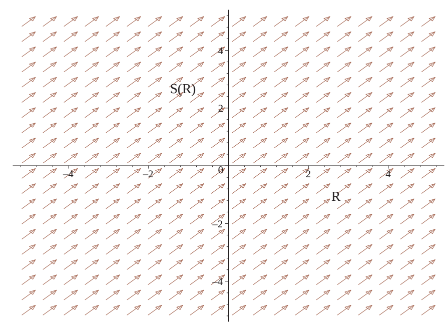
To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\ln(y) = x + c_2$$

Which gives

$$y = e^{x+c_2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = y$ 	$R = x$ $S = \ln(y)$	$\frac{dS}{dR} = 1$ 

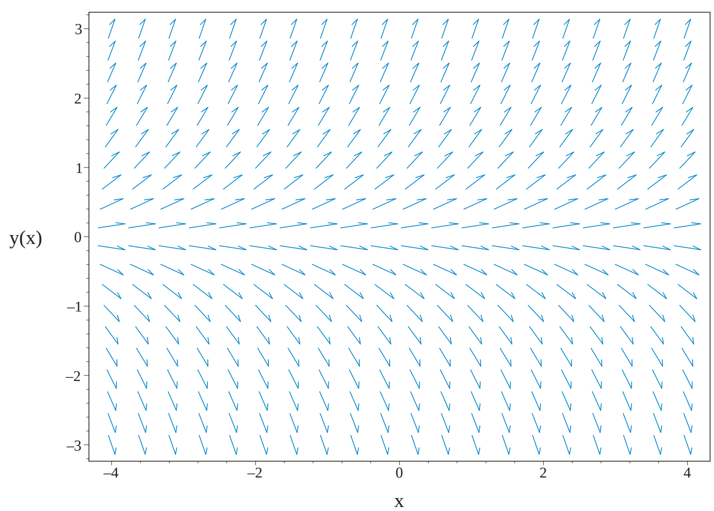


Figure 2.21: Slope field plot
$y' = y$

Summary of solutions found

$$y = e^{x+c_2}$$

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) = y(x)$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = y(x)$

- Separate variables

  $\frac{\frac{d}{dx}y(x)}{y(x)} = 1$

- Integrate both sides with respect to x

  $\int \frac{\frac{d}{dx}y(x)}{y(x)}dx = \int 1dx + C1$

- Evaluate integral

  $\ln(y(x)) = x + C1$

- Solve for $y(x)$

  $y(x) = e^{x+C1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : `0.001` (sec)
Leaf size : 8

```
dsolve(diff(y(x),x) = y(x),
       y(x),singsol=all)
```

$$y = e^x c_1$$

Mathematica DSolve solution

Solving time : 0.023 (sec)
Leaf size : 16

```
DSolve[{D[y[x],x] == y[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^x$$
$$y(x) \to 0$$

2.1.9 problem 9

Internal problem ID [8397]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 9
Date solved : `Tuesday, December 17, 2024 at 12:25:51 PM`
CAS classification : `[_quadrature]`

Solve

$$y' = 0$$

Solved as first order quadrature ode

Time used: 0.025 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\, dx + c_1$$
$$y = c_1$$

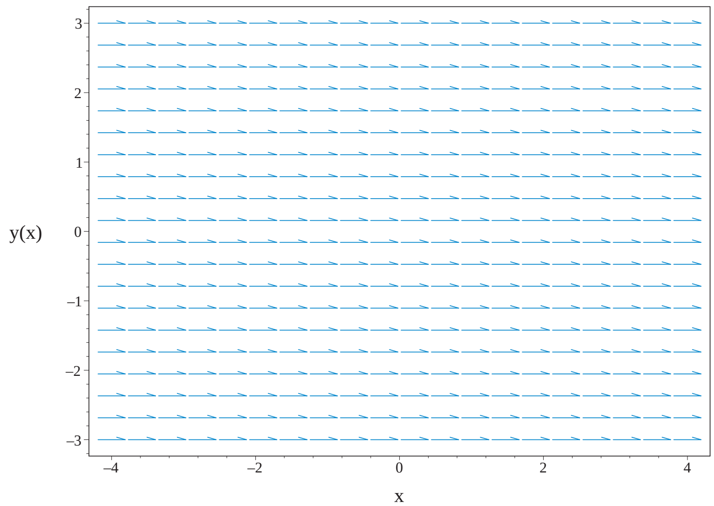


Figure 2.22: Slope field plot
$y' = 0$

Summary of solutions found

$$y = c_1$$

Solved as first order homogeneous class D2 ode

Time used: 0.213 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$u'(x)\,x + u(x) = 0$$

Which is now solved The ode $u'(x) = -\frac{u(x)}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(u) &= u \end{aligned}$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(u)}\, du &= \int f(x)\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{1}{x}\, dx \\
\ln (u(x)) &= \ln \left(\frac{1}{x}\right) + c_1
\end{aligned}
$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$
u(x) = 0
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
\ln (u(x)) = \ln \left(\frac{1}{x}\right) + c_1 \\
u(x) = 0
\end{aligned}
$$

Solving for $u(x)$ gives

$$
\begin{aligned}
u(x) &= 0 \\
u(x) &= \frac{e^{c_1}}{x}
\end{aligned}
$$

Converting $u(x) = 0$ back to y gives

$$
y = 0
$$

Converting $u(x) = \frac{e^{c_1}}{x}$ back to y gives

$$
y = e^{c_1}
$$

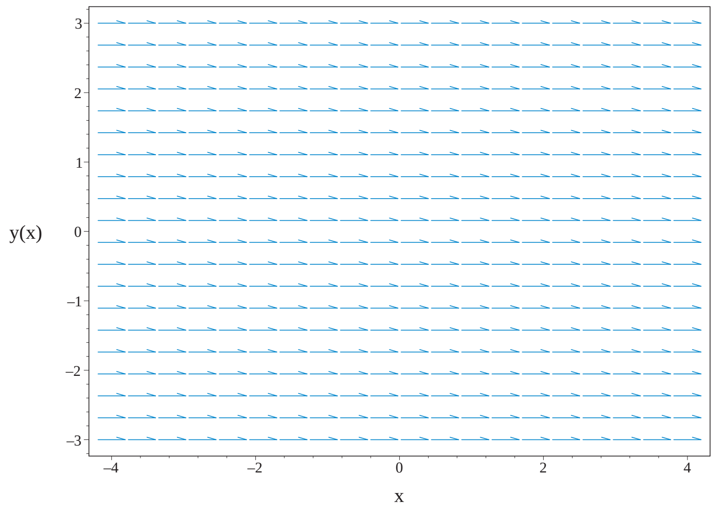


Figure 2.23: Slope field plot
$y' = 0$

Summary of solutions found

$$y = 0$$
$$y = e^{c_1}$$

Solved as first order ode of type differential

Time used: 0.011 (sec)

Writing the ode as

$$y' = 0 \tag{1}$$

Which becomes

$$(1)\,dy = (0)\,dx \tag{2}$$

But the RHS is complete differential because

$$(0)\,dx = d(0)$$

Hence (2) becomes

$$(1)\,dy = d(0)$$

Integrating gives

$$y = c_1$$

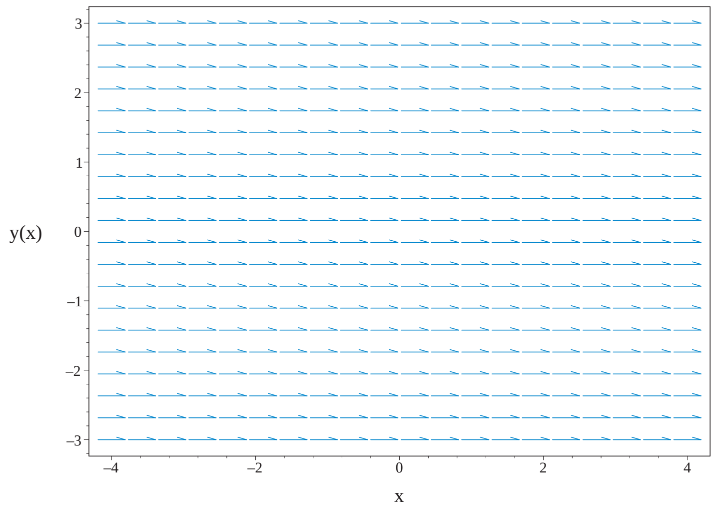


Figure 2.24: Slope field plot
$y' = 0$

Summary of solutions found

$$y = c_1$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) = 0$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Integrate both sides with respect to x
  $\int \left(\frac{d}{dx}y(x)\right) dx = \int 0 dx + C1$
- Evaluate integral
  $y(x) = C1$
- Solve for $y(x)$
  $y(x) = C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 5

```
dsolve(diff(y(x),x) = 0,
       y(x),singsol=all)
```

$$y = c_1$$

Mathematica DSolve solution

Solving time : 0.002 (sec)
Leaf size : 7

```
DSolve[{D[y[x],x] == 0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1$$

2.1.10 problem 10

Internal problem ID [8398]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 10
Date solved : `Tuesday, December 17, 2024 at 12:25:52 PM`
CAS classification : `[_quadrature]`

Solve

$$y' = 1 + \frac{\sec(x)}{x}$$

Solved as first order quadrature ode

Time used: 0.247 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int \frac{x + \sec(x)}{x}\, dx$$
$$y = \int \frac{x + \sec(x)}{x} dx + c_1$$

$$y = \int \frac{x + \sec(x)}{x} dx + c_1$$

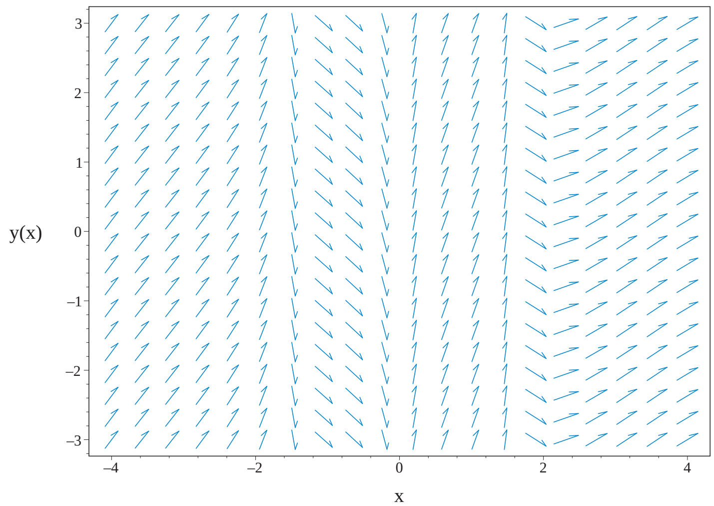


Figure 2.25: Slope field plot
$y' = 1 + \frac{\sec(x)}{x}$

Summary of solutions found

$$y = \int \frac{x + \sec(x)}{x} dx + c_1$$

Solved as first order Exact ode

Time used: 0.408 (sec)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(1 + \frac{\sec(x)}{x}\right)\mathrm{d}x\\ \left(-1 - \frac{\sec(x)}{x}\right)\mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -1 - \frac{\sec(x)}{x}\\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-1 - \frac{\sec(x)}{x}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -1 - \frac{\sec(x)}{x} \mathrm{d}x$$

$$\phi = \int \left(-1 - \frac{\sec(x)}{x}\right) dx + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 1$. Therefore equation (4) becomes

$$1 = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 1$$

Integrating the above w.r.t y gives

$$\int f'(y) \mathrm{d}y = \int (1) \mathrm{d}y$$

$$f(y) = y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \int \left(-1 - \frac{\sec(x)}{x}\right) dx + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \int \left(-1 - \frac{\sec(x)}{x}\right) dx + y$$

Solving for y gives

$$y = c_1 - \int \left(-1 - \frac{\sec(x)}{x}\right) dx$$

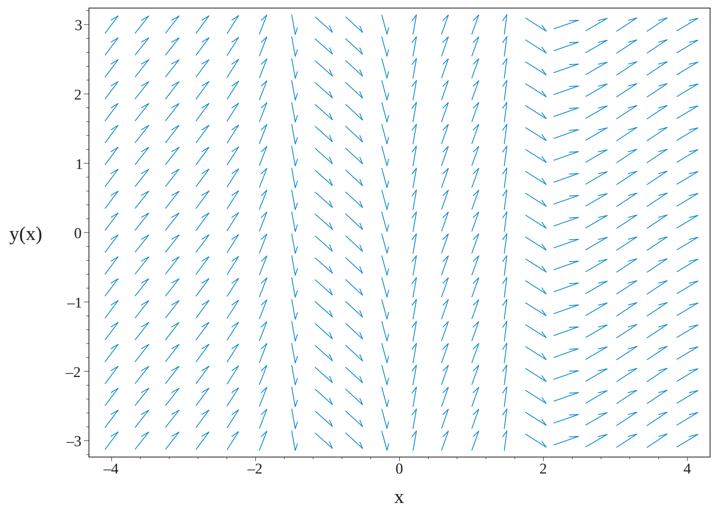


Figure 2.26: Slope field plot
$y' = 1 + \frac{\sec(x)}{x}$

Summary of solutions found

$$y = c_1 - \int \left(-1 - \frac{\sec(x)}{x}\right) dx$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) = 1 + \frac{\sec(x)}{x}$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}y(x)\right) dx = \int \left(1 + \frac{\sec(x)}{x}\right) dx + C1$$

- Evaluate integral

$$y(x) = x + \int \frac{2\,\mathrm{e}^{\mathrm{I}x}}{\left((\mathrm{e}^{\mathrm{I}x})^2+1\right)x} dx + C1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 15

```
dsolve(diff(y(x),x) = 1+sec(x)/x,
       y(x),singsol=all)
```

$$y = \int \frac{\sec(x)}{x} dx + x + c_1$$

Mathematica DSolve solution

Solving time : 0.75 (sec)
Leaf size : 25

```
DSolve[{D[y[x],x] == 1+Sec[x]/x,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \int_1^x \left(\frac{\sec(K[1])}{K[1]} + 1\right) dK[1] + c_1$$

2.1.11 problem 11

Internal problem ID [8399]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 11
Date solved : `Tuesday, December 17, 2024 at 12:25:53 PM`
CAS classification : `[_linear]`

Solve

$$y' = x + \frac{\sec(x)\, y}{x}$$

Solved as first order linear ode

Time used: 0.398 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = -\frac{\sec(x)}{x}$$
$$p(x) = x$$

The integrating factor μ is

$$\mu = \mathrm{e}^{\int -\frac{\sec(x)}{x}dx}$$

Therefore the solution is

$$y = \left(\int x\, \mathrm{e}^{\int -\frac{\sec(x)}{x}dx}dx + c_1\right)\mathrm{e}^{-\int -\frac{\sec(x)}{x}dx}$$

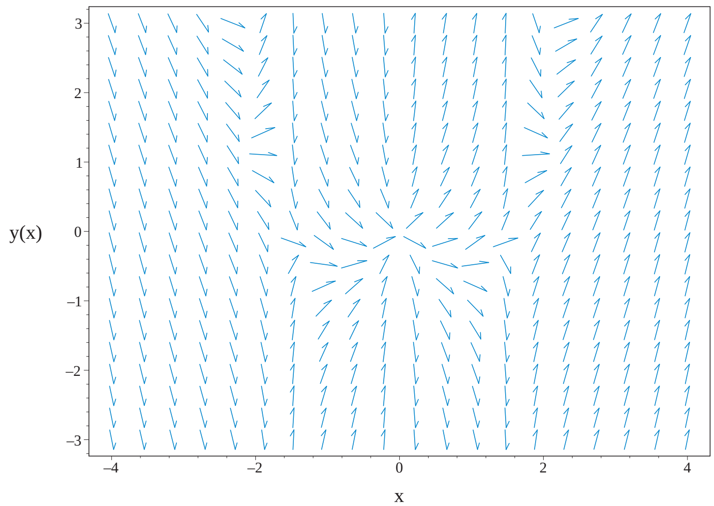


Figure 2.27: Slope field plot
$y' = x + \frac{\sec(x) y}{x}$

Summary of solutions found

$$y = \left(\int x\, \mathrm{e}^{\int -\frac{\sec(x)}{x}dx} dx + c_1\right) \mathrm{e}^{-\int -\frac{\sec(x)}{x}dx}$$

Solved as first order Exact ode

Time used: 0.564 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y&=\left(\frac{\sec(x)\,y}{x}+x\right)\mathrm{d}x\\ \left(-x-\frac{\sec(x)\,y}{x}\right)\mathrm{d}x+\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x-\frac{\sec(x)\,y}{x}\\ N(x,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-x-\frac{\sec(x)\,y}{x}\right)\\ &=-\frac{\sec(x)}{x}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(1)\\ &=0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(-\frac{\sec(x)}{x}\right) - (0)\right) \\
&= -\frac{\sec(x)}{x}
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{\sec(x)}{x} \, \mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{\int -\frac{\sec(x)}{x} dx} \\
&= \mathrm{e}^{-\int \frac{\sec(x)}{x} dx}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-\int \frac{\sec(x)}{x} dx}\left(-x - \frac{\sec(x) y}{x}\right) \\
&= -\frac{(\sec(x) y + x^2) \mathrm{e}^{-\int \frac{\sec(x)}{x} dx}}{x}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-\int \frac{\sec(x)}{x} dx}(1) \\
&= \mathrm{e}^{-\int \frac{\sec(x)}{x} dx}
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\frac{(\sec(x) y + x^2) \mathrm{e}^{-\int \frac{\sec(x)}{x} dx}}{x}\right) + \left(\mathrm{e}^{-\int \frac{\sec(x)}{x} dx}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\int \frac{\partial\phi}{\partial y}\,\mathrm{d}y = \int \overline{N}\,\mathrm{d}y$$
$$\int \frac{\partial\phi}{\partial y}\,\mathrm{d}y = \int \mathrm{e}^{-\int \frac{\sec(x)}{x}dx}\,\mathrm{d}y$$
$$\phi = \mathrm{e}^{-\int \frac{\sec(x)}{x}dx}y + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial\phi}{\partial x} = -\frac{\sec(x)\,\mathrm{e}^{-\int \frac{\sec(x)}{x}dx}y}{x} + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial\phi}{\partial x} = -\frac{(\sec(x)y+x^2)\mathrm{e}^{-\int \frac{\sec(x)}{x}dx}}{x}$. Therefore equation (4) becomes

$$-\frac{(\sec(x)\,y + x^2)\,\mathrm{e}^{-\int \frac{\sec(x)}{x}dx}}{x} = -\frac{\sec(x)\,\mathrm{e}^{-\int \frac{\sec(x)}{x}dx}y}{x} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -\mathrm{e}^{-\int \frac{\sec(x)}{x}dx}x$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int \left(-\mathrm{e}^{-\int \frac{\sec(x)}{x}dx}x\right)\mathrm{d}x$$
$$f(x) = \int_0^x -\mathrm{e}^{-\int \frac{\sec(\tau)}{\tau}d\tau}\tau d\tau + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{-\int \frac{\sec(x)}{x}dx} y + \int_0^x -\mathrm{e}^{-\int \frac{\sec(\tau)}{\tau}d\tau} \tau d\tau + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{-\int \frac{\sec(x)}{x}dx} y + \int_0^x -\mathrm{e}^{-\int \frac{\sec(\tau)}{\tau}d\tau} \tau d\tau$$

Solving for y gives

$$y = -\left(\int_0^x -\mathrm{e}^{-\int \frac{\sec(\tau)}{\tau}d\tau} \tau d\tau - c_1\right) \mathrm{e}^{\int \frac{\sec(x)}{x}dx}$$

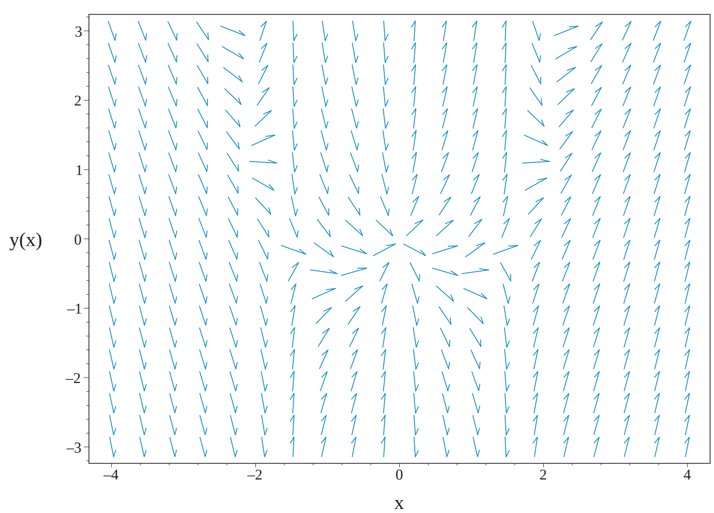


Figure 2.28: Slope field plot
$y' = x + \frac{\sec(x)y}{x}$

Summary of solutions found

$$y = -\left(\int_0^x -\mathrm{e}^{-\int \frac{\sec(\tau)}{\tau}d\tau} \tau d\tau - c_1\right) \mathrm{e}^{\int \frac{\sec(x)}{x}dx}$$

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) = x + \frac{\sec(x)y(x)}{x}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$
- Solve for the highest derivative

  $\frac{d}{dx}y(x) = x + \frac{\sec(x)y(x)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $\frac{d}{dx}y(x) - \frac{\sec(x)y(x)}{x} = x$
- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(\frac{d}{dx}y(x) - \frac{\sec(x)y(x)}{x}\right) = \mu(x)\,x$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\,\mu(x))$

  $\mu(x)\left(\frac{d}{dx}y(x) - \frac{\sec(x)y(x)}{x}\right) = \left(\frac{d}{dx}y(x)\right)\mu(x) + y(x)\left(\frac{d}{dx}\mu(x)\right)$
- Isolate $\frac{d}{dx}\mu(x)$

  $\frac{d}{dx}\mu(x) = -\frac{\mu(x)\sec(x)}{x}$
- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{\int -\frac{\sec(x)}{x}dx}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(y(x)\,\mu(x))\right)dx = \int \mu(x)\,x dx + C1$
- Evaluate the integral on the lhs

  $y(x)\,\mu(x) = \int \mu(x)\,x dx + C1$
- Solve for $y(x)$

  $y(x) = \frac{\int \mu(x)x dx + C1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{\int -\frac{\sec(x)}{x}dx}$

  $y(x) = \frac{\int \mathrm{e}^{\int -\frac{\sec(x)}{x}dx} x dx + C1}{\mathrm{e}^{\int -\frac{\sec(x)}{x}dx}}$
- Simplify

  $y(x) = \left(\int \mathrm{e}^{-\left(\int \frac{\sec(x)}{x}dx\right)} x dx + C1\right)\mathrm{e}^{\int \frac{\sec(x)}{x}dx}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 31

```
dsolve(diff(y(x),x) = x+sec(x)*y(x)/x,
       y(x),singsol=all)
```

$$y = \left(\int x\,\mathrm{e}^{-\left(\int \frac{\sec(x)}{x}dx\right)}dx + c_1\right)\mathrm{e}^{\int \frac{\sec(x)}{x}dx}$$

Mathematica DSolve solution

Solving time : 0.441 (sec)
Leaf size : 56

```
DSolve[{D[y[x],x] == x+Sec[x]*y[x]/x,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \exp\left(\int_1^x \frac{\sec(K[1])}{K[1]}dK[1]\right)\left(\int_1^x \exp\left(-\int_1^{K[2]} \frac{\sec(K[1])}{K[1]}dK[1]\right)K[2]dK[2] + c_1\right)$$

2.1.12 problem 12

Internal problem ID [8400]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 12
Date solved : `Tuesday, December 17, 2024 at 12:26:01 PM`
CAS classification : `[_separable]`

Solve

$$y' = \frac{2y}{x}$$

With initial conditions

$$y(0) = 0$$

Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + q(x)y = p(x)$$

Where here

$$q(x) = -\frac{2}{x}$$
$$p(x) = 0$$

Hence the ode is

$$y' - \frac{2y}{x} = 0$$

The domain of $q(x) = -\frac{2}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

But the point $x_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

Solved as first order linear ode

Time used: 0.071 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= -\frac{2}{x} \\ p(x) &= 0 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= e^{\int -\frac{2}{x}dx} \\ &= \frac{1}{x^2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x^2}\right) &= 0 \end{aligned}$$

Integrating gives

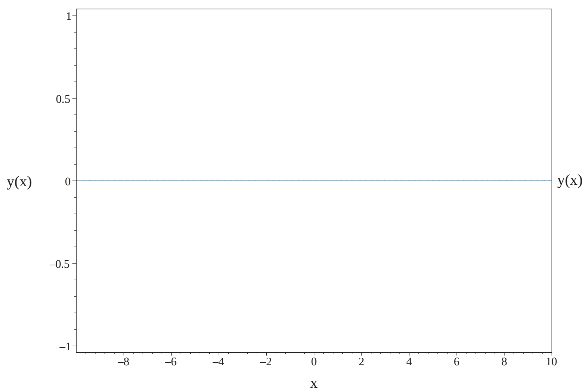
$$\begin{aligned} \frac{y}{x^2} &= \int 0\,dx + c_1 \\ &= c_1 \end{aligned}$$

Dividing throughout by the integrating factor $\frac{1}{x^2}$ gives the final solution

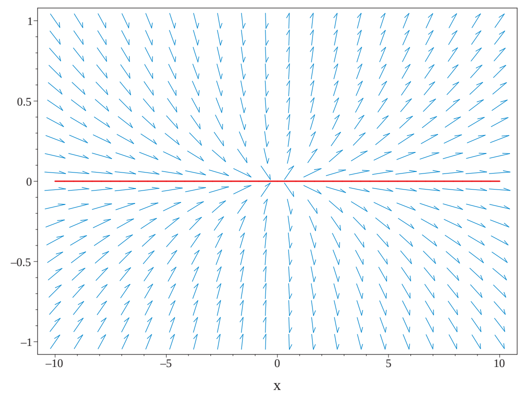
$$y = c_1 x^2$$

Solving for the constant of integration from initial conditions, the solution becomes

$$y = 0$$



(a) Solution plot
$y = 0$



(b) Slope field plot
$y' = \frac{2y}{x}$

Summary of solutions found

$$y = 0$$

Solved as first order separable ode

Time used: 0.203 (sec)

The ode $y' = \frac{2y}{x}$ is separable as it can be written as

$$\begin{aligned} y' &= \frac{2y}{x} \\ &= f(x)g(y) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{2}{x} \\ g(y) &= y \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$
$$\int \frac{1}{y}\,dy = \int \frac{2}{x}\,dx$$
$$\ln(y) = \ln(x^2) + c_1$$

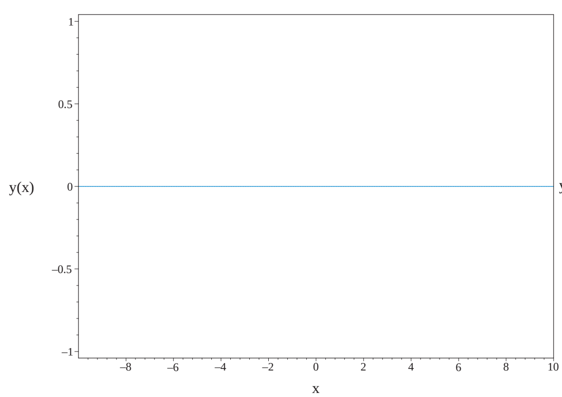
We now need to find the singular solutions, these are found by finding for what values $g(y)$ is zero, since we had to divide by this above. Solving $g(y) = 0$ or $y = 0$ for y gives

$$y = 0$$

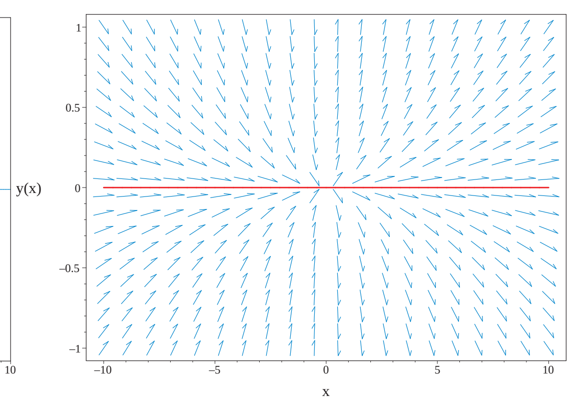
Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\ln(y) = \ln(x^2) + c_1$$
$$y = 0$$



(a) Solution plot
$y = 0$



(b) Slope field plot
$y' = \frac{2y}{x}$

Summary of solutions found

$$y = 0$$

Solved as first order homogeneous class A ode

Time used: 0.239 (sec)

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{2y}{x} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = 2y$ and $N = x$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}x}x + u &= 2u \\ \frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{u(x)}{x} \end{aligned}$$

Or

$$u'(x) - \frac{u(x)}{x} = 0$$

Or

$$u'(x)\,x - u(x) = 0$$

Which is now solved as separable in $u(x)$.

The ode $u'(x) = \frac{u(x)}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= \frac{u(x)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{1}{x} \\ g(u) &= u \end{aligned}$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\
\int \frac{1}{u}\,du &= \int \frac{1}{x}\,dx \\
\ln\left(u(x)\right) &= \ln\left(x\right) + c_1
\end{aligned}
$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$
u(x) = 0
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
\ln\left(u(x)\right) &= \ln\left(x\right) + c_1 \\
u(x) &= 0
\end{aligned}
$$

Solving for $u(x)$ gives

$$
u(x) = 0
$$

$$
u(x) = x\,\mathrm{e}^{c_1}
$$

Converting $u(x) = 0$ back to y gives

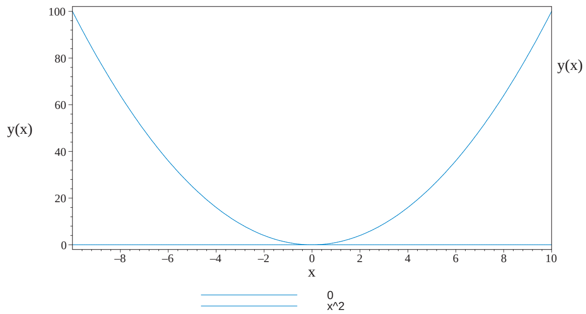
$$
y = 0
$$

Converting $u(x) = x\,\mathrm{e}^{c_1}$ back to y gives

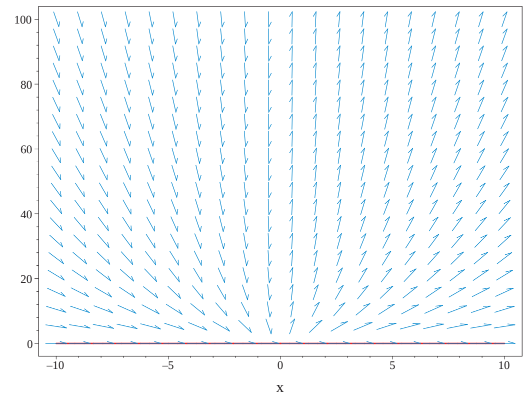
$$
y = x^2\mathrm{e}^{c_1}
$$

Solving for the constant of integration from initial conditions, the solution becomes

$$
y = x^2
$$



(a) Solutions plot



(b) Slope field plot
$y' = \frac{2y}{x}$

Summary of solutions found

$$y = 0$$

$$y = x^2$$

Solved as first order homogeneous class D2 ode

Time used: 0.056 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$u'(x)\,x + u(x) = 2u(x)$$

Which is now solved The ode $u'(x) = \frac{u(x)}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= \frac{u(x)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{1}{x} \\ g(u) &= u \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\ \int \frac{1}{u}\,du &= \int \frac{1}{x}\,dx \\ \ln(u(x)) &= \ln(x) + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\ln(u(x)) = \ln(x) + c_1$$
$$u(x) = 0$$

Solving for $u(x)$ gives

$$u(x) = 0$$
$$u(x) = x\,\mathrm{e}^{c_1}$$

Converting $u(x) = 0$ back to y gives

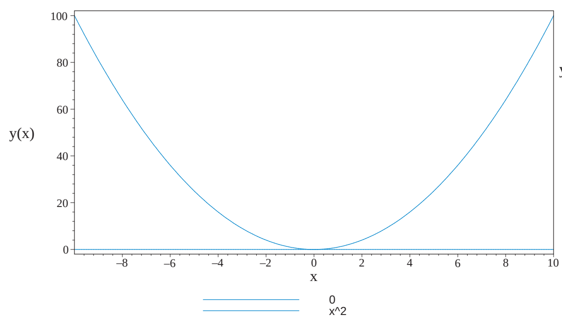
$$y = 0$$

Converting $u(x) = x\,\mathrm{e}^{c_1}$ back to y gives

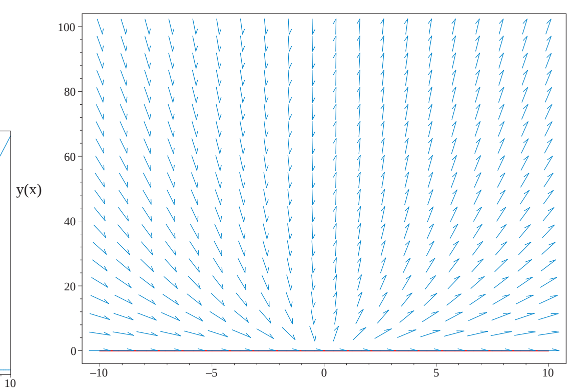
$$y = x^2\mathrm{e}^{c_1}$$

Solving for the constant of integration from initial conditions, the solution becomes

$$y = x^2$$



(a) Solutions plot



(b) Slope field plot
$y' = \frac{2y}{x}$

Summary of solutions found

$$y = 0$$
$$y = x^2$$

Summary of solutions found

$$y = 0$$
$$y = x^2$$

Solved as first order homogeneous class Maple C ode

Time used: 0.290 (sec)

Let $Y = y - y_0$ and $X = x - x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{2Y(X) + 2y_0}{x_0 + X}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 0$$
$$y_0 = 0$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{2Y(X)}{X}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= \frac{2Y}{X} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = 2Y$ and $N = X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= 2u \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{u(X)}{X}\end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)X - u(X) = 0$$

Which is now solved as separable in $u(X)$.

The ode $\frac{d}{dX}u(X) = \frac{u(X)}{X}$ is separable as it can be written as

$$\begin{aligned}\frac{d}{dX}u(X) &= \frac{u(X)}{X} \\ &= f(X)g(u)\end{aligned}$$

Where

$$\begin{aligned}f(X) &= \frac{1}{X} \\ g(u) &= u\end{aligned}$$

Integrating gives

$$\begin{aligned}\int \frac{1}{g(u)}\,du &= \int f(X)\,dX \\ \int \frac{1}{u}\,du &= \int \frac{1}{X}\,dX \\ \ln(u(X)) &= \ln(X) + c_1\end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(X)$ gives

$$u(X) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\ln(u(X)) = \ln(X) + c_1$$
$$u(X) = 0$$

Solving for $u(X)$ gives

$$u(X) = 0$$
$$u(X) = e^{c_1} X$$

Converting $u(X) = 0$ back to $Y(X)$ gives

$$Y(X) = 0$$

Converting $u(X) = e^{c_1} X$ back to $Y(X)$ gives

$$Y(X) = X^2 e^{c_1}$$

Using the solution for $Y(X)$

$$Y(X) = 0 \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y$$
$$X = x$$

Then the solution in y becomes using EQ (A)

$$y = 0$$

Using the solution for $Y(X)$

$$Y(X) = X^2 e^{c_1} \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

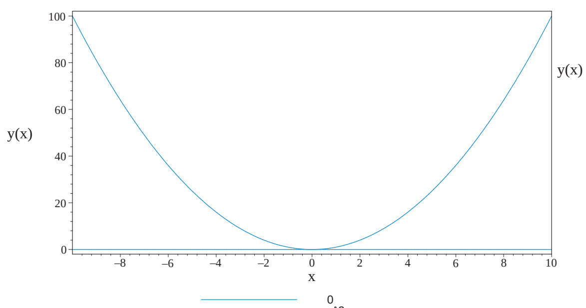
$$Y = y$$
$$X = x$$

Then the solution in y becomes using EQ (A)

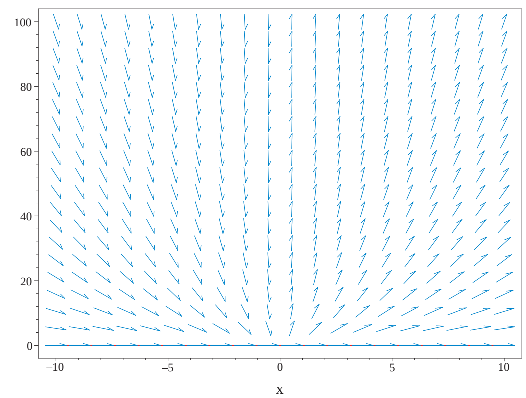
$$y = x^2 e^{c_1}$$

Solving for the constant of integration from initial conditions, the solution becomes

$$y = x^2$$



(a) Solutions plot



(b) Slope field plot
$y' = \frac{2y}{x}$

Solved as first order Exact ode

Time used: 0.180 (sec)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(\frac{2y}{x}\right)\mathrm{d}x\\ \left(-\frac{2y}{x}\right)\mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\frac{2y}{x}\\ N(x,y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{2y}{x}\right)\\ &= -\frac{2}{x}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1\left(\left(-\frac{2}{x}\right) - (0)\right)\\ &= -\frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\frac{2}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(x)}\\ &= \frac{1}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^2}\left(-\frac{2y}{x}\right)\\ &= -\frac{2y}{x^3}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x^2}(1)\\ &= \frac{1}{x^2}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\frac{2y}{x^3}\right) + \left(\frac{1}{x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (2) w.r.t. y gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial y}\,\mathrm{d}y &= \int \overline{N}\,\mathrm{d}y \\
\int \frac{\partial \phi}{\partial y}\,\mathrm{d}y &= \int \frac{1}{x^2}\,\mathrm{d}y \\
\phi &= \frac{y}{x^2} + f(x) \qquad (3)
\end{aligned}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$
\frac{\partial \phi}{\partial x} = -\frac{2y}{x^3} + f'(x) \tag{4}
$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = -\frac{2y}{x^3}$. Therefore equation (4) becomes

$$
-\frac{2y}{x^3} = -\frac{2y}{x^3} + f'(x) \tag{5}
$$

Solving equation (5) for $f'(x)$ gives

$$
f'(x) = 0
$$

Therefore

$$
f(x) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = \frac{y}{x^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

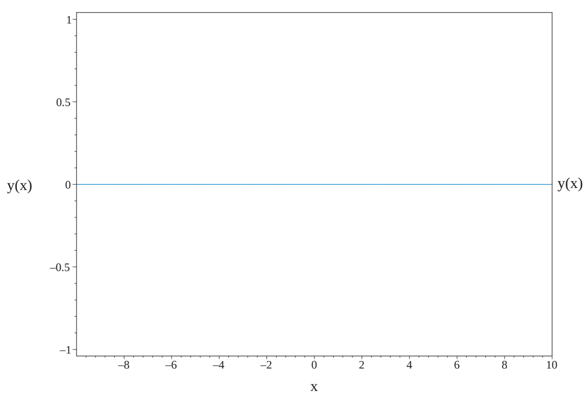
$$c_1 = \frac{y}{x^2}$$

Solving for the constant of integration from initial conditions, the solution becomes

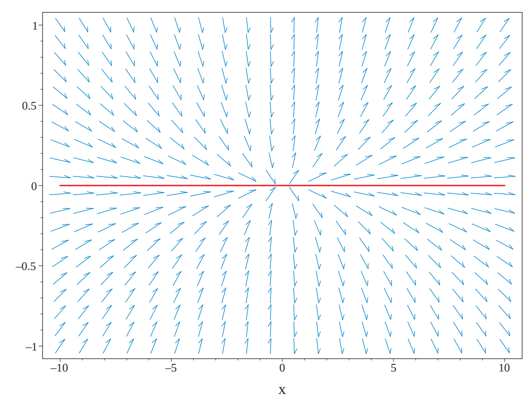
$$\frac{y}{x^2} = 0$$

Solving for y gives

$$y = 0$$



(a) Solution plot
$y = 0$



(b) Slope field plot
$y' = \frac{2y}{x}$

Summary of solutions found

$$y = 0$$

Solved using Lie symmetry for first order ode

Time used: 0.375 (sec)

Writing the ode as

$$
\begin{aligned}
y' &= \frac{2y}{x} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
b_2 + \frac{2y(b_3 - a_2)}{x} - \frac{4y^2 a_3}{x^2} + \frac{2y(x a_2 + y a_3 + a_1)}{x^2} - \frac{2(x b_2 + y b_3 + b_1)}{x} = 0 \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{b_2 x^2 + 2y^2 a_3 + 2x b_1 - 2y a_1}{x^2} = 0
$$

Setting the numerator to zero gives

$$
-b_2 x^2 - 2y^2 a_3 - 2x b_1 + 2y a_1 = 0 \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-2a_3v_2^2 - b_2v_1^2 + 2a_1v_2 - 2b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-2a_3v_2^2 - b_2v_1^2 + 2a_1v_2 - 2b_1v_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
2a_1 &= 0 \\
-2a_3 &= 0 \\
-2b_1 &= 0 \\
-b_2 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= 0 \\
\eta &= y
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y} dy \end{aligned}$$

Which results in

$$S = \ln (y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{2y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 0 \\ S_y &= \frac{1}{y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{2}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{2}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int \frac{2}{R}\, dR$$
$$S(R) = 2\ln(R) + c_2$$

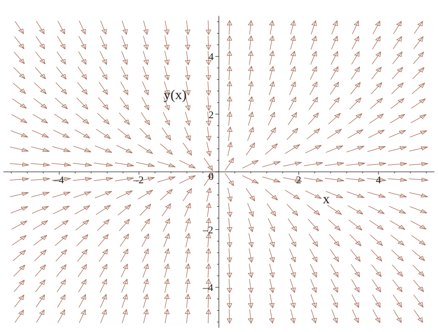
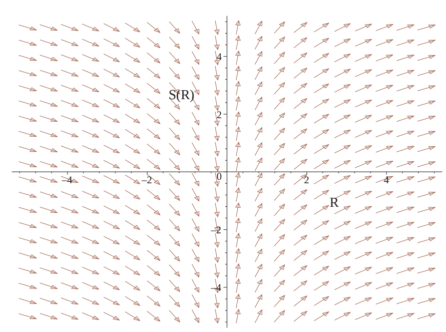
To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\ln(y) = 2\ln(x) + c_2$$

Which gives

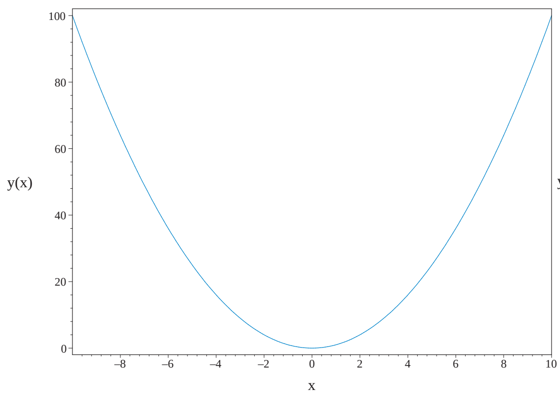
$$y = e^{c_2} x^2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

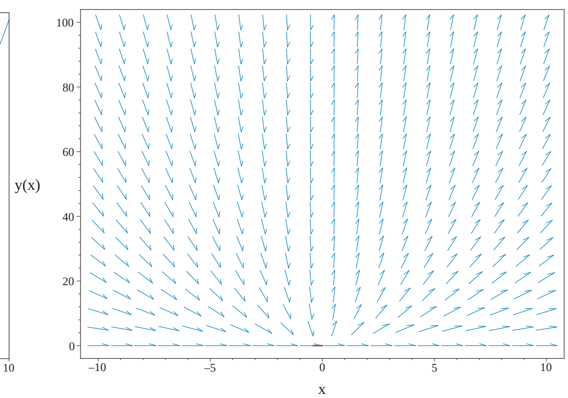
Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2y}{x}$ 	$R = x$ $S = \ln(y)$	$\frac{dS}{dR} = \frac{2}{R}$ 

Solving for the constant of integration from initial conditions, the solution becomes

$$y = x^2$$



(a) Solution plot
$y = x^2$



(b) Slope field plot
$y' = \frac{2y}{x}$

Summary of solutions found

$$y = x^2$$

Maple step by step solution

Let's solve
$\left[\frac{d}{dx}y(x) = \frac{2y(x)}{x}, y(0) = 0\right]$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = \frac{2y(x)}{x}$
- Separate variables
  $\frac{\frac{d}{dx}y(x)}{y(x)} = \frac{2}{x}$
- Integrate both sides with respect to x
  $\int \frac{\frac{d}{dx}y(x)}{y(x)}dx = \int \frac{2}{x}dx + C1$
- Evaluate integral
  $\ln(y(x)) = 2\ln(x) + C1$
- Solve for $y(x)$

$y(x) = e^{C1} x^2$

- Use initial condition $y(0) = 0$

  $0 = 0$

- Solve for $\_C1$

  $C1 = C1$

- Substitute $\_C1 = \_C1$ into general solution and simplify

  $y(x) = e^{C1} x^2$

- Solution to the IVP

  $y(x) = e^{C1} x^2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.013 (sec)
Leaf size : 9

```
dsolve([diff(y(x),x) = 2*y(x)/x,
       op([y(0) = 0])],y(x),singsol=all)
```

$$y = c_1 x^2$$

Mathematica DSolve solution

Solving time : 0.001 (sec)
Leaf size : 6

```
DSolve[{D[y[x],x] == 2*y[x]/x,y[0]==0},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to 0$$

2.1.13 problem 13

Internal problem ID [8401]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 13
Date solved : `Tuesday, December 17, 2024 at 12:26:03 PM`
CAS classification : `[_separable]`

Solve

$$y' = \frac{2y}{x}$$

Solved as first order linear ode

Time used: 0.046 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= -\frac{2}{x} \\ p(x) &= 0 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= \mathrm{e}^{\int -\frac{2}{x}dx} \\ &= \frac{1}{x^2} \end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\mu y = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x^2}\right) = 0$$

Integrating gives

$$\frac{y}{x^2} = \int 0\, dx + c_1$$
$$= c_1$$

Dividing throughout by the integrating factor $\frac{1}{x^2}$ gives the final solution

$$y = c_1 x^2$$

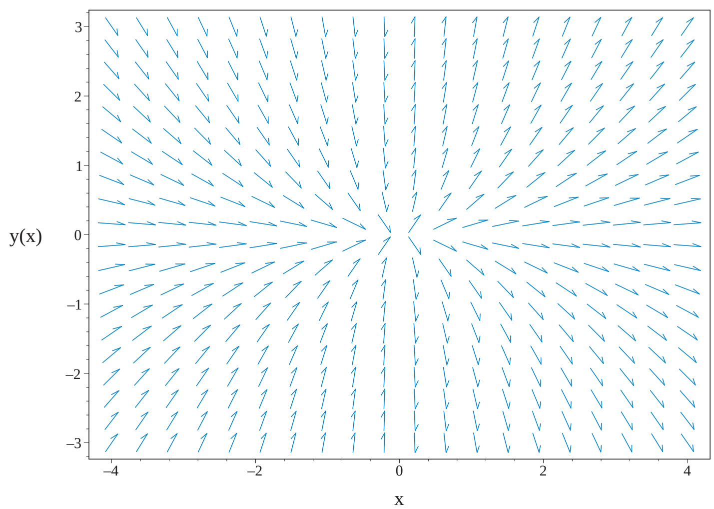


Figure 2.36: Slope field plot
$y' = \frac{2y}{x}$

Summary of solutions found

$$y = c_1 x^2$$

Solved as first order separable ode

Time used: 0.107 (sec)

The ode $y' = \frac{2y}{x}$ is separable as it can be written as

$$\begin{aligned} y' &= \frac{2y}{x} \\ &= f(x)g(y) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{2}{x} \\ g(y) &= y \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(y)}\, dy &= \int f(x)\, dx \\ \int \frac{1}{y}\, dy &= \int \frac{2}{x}\, dx \\ \ln(y) &= \ln\left(x^2\right) + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(y)$ is zero, since we had to divide by this above. Solving $g(y) = 0$ or $y = 0$ for y gives

$$y = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} \ln(y) &= \ln\left(x^2\right) + c_1 \\ y &= 0 \end{aligned}$$

Solving for y gives

$$\begin{aligned} y &= 0 \\ y &= \mathrm{e}^{c_1} x^2 \end{aligned}$$

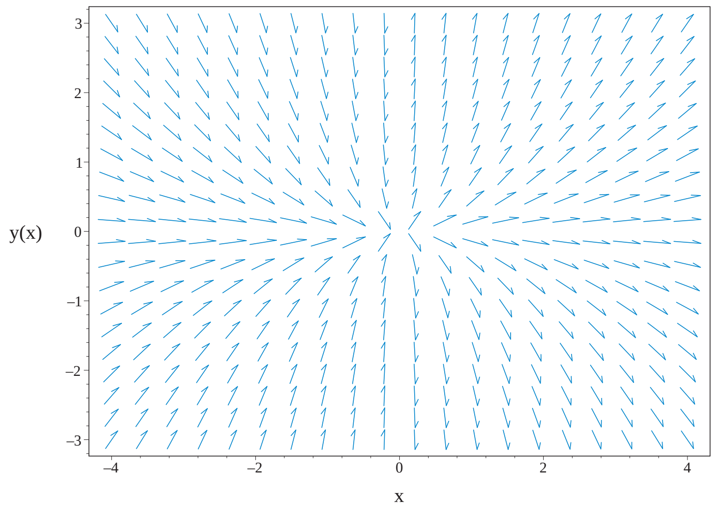


Figure 2.37: Slope field plot
$y' = \frac{2y}{x}$

Summary of solutions found

$$y = 0$$

$$y = e^{c_1} x^2$$

Solved as first order homogeneous class A ode

Time used: 0.178 (sec)

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{2y}{x} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x, y)$ and $N(x, y)$ are both homogeneous functions and of the same order. Recall that a function $f(x, y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x, y)$$

In this case, it can be seen that both $M = 2y$ and $N = x$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{dy}{dx} = \frac{du}{dx} x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\frac{\mathrm{d}u}{\mathrm{d}x}x + u = 2u$$
$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{u(x)}{x}$$

Or

$$u'(x) - \frac{u(x)}{x} = 0$$

Or

$$u'(x)\,x - u(x) = 0$$

Which is now solved as separable in $u(x)$.

The ode $u'(x) = \frac{u(x)}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= \frac{u(x)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{1}{x} \\ g(u) &= u \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\ \int \frac{1}{u}\,du &= \int \frac{1}{x}\,dx \\ \ln(u(x)) &= \ln(x) + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} \ln(u(x)) &= \ln(x) + c_1 \\ u(x) &= 0 \end{aligned}$$

Solving for $u(x)$ gives

$$u(x) = 0$$
$$u(x) = e^{c_1} x$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = e^{c_1} x$ back to y gives

$$y = e^{c_1} x^2$$

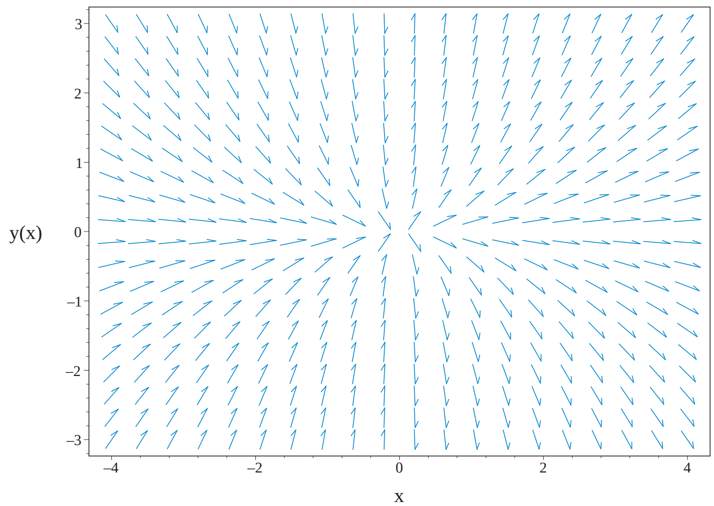


Figure 2.38: Slope field plot
$y' = \frac{2y}{x}$

<u>Summary of solutions found</u>

$$y = 0$$
$$y = e^{c_1} x^2$$

Solved as first order homogeneous class D2 ode

Time used: 0.066 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$u'(x)\,x + u(x) = 2u(x)$$

Which is now solved The ode $u'(x) = \frac{u(x)}{x}$ is separable as it can be written as

$$\begin{aligned}
u'(x) &= \frac{u(x)}{x} \\
&= f(x)g(u)
\end{aligned}$$

Where

$$\begin{aligned}
f(x) &= \frac{1}{x} \\
g(u) &= u
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\
\int \frac{1}{u}\,du &= \int \frac{1}{x}\,dx \\
\ln\left(u(x)\right) &= \ln\left(x\right) + c_1
\end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned}
\ln\left(u(x)\right) &= \ln\left(x\right) + c_1 \\
u(x) &= 0
\end{aligned}$$

Solving for $u(x)$ gives

$$\begin{aligned}
u(x) &= 0 \\
u(x) &= \mathrm{e}^{c_1} x
\end{aligned}$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = e^{c_1}x$ back to y gives

$$y = e^{c_1}x^2$$

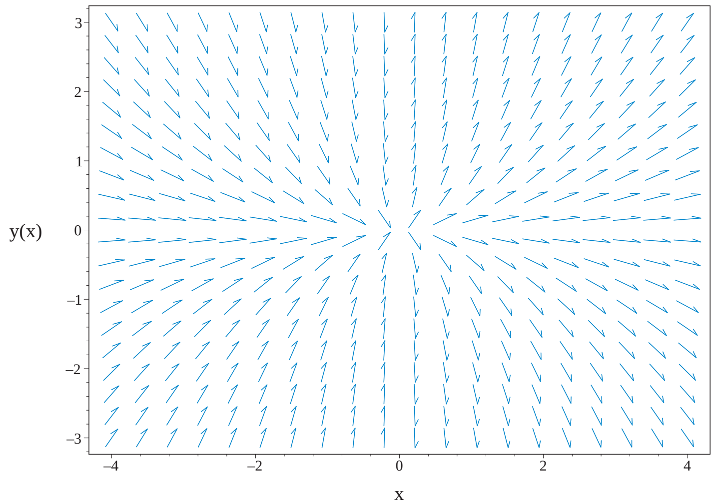


Figure 2.39: Slope field plot
$y' = \frac{2y}{x}$

Summary of solutions found

$$y = 0$$

$$y = e^{c_1}x^2$$

Summary of solutions found

$$y = 0$$

$$y = e^{c_1}x^2$$

Solved as first order homogeneous class Maple C ode

Time used: 0.223 (sec)

Let $Y = y - y_0$ and $X = x - x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{2Y(X) + 2y_0}{x_0 + X}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 0$$
$$y_0 = 0$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{2Y(X)}{X}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= \frac{2Y}{X} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = 2Y$ and $N = X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\frac{\mathrm{d}u}{\mathrm{d}X}X + u = 2u$$
$$\frac{\mathrm{d}u}{\mathrm{d}X} = \frac{u(X)}{X}$$

Or

$$\frac{d}{dX}u(X) - \frac{u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)X - u(X) = 0$$

Which is now solved as separable in $u(X)$.

The ode $\frac{d}{dX}u(X) = \frac{u(X)}{X}$ is separable as it can be written as

$$\begin{aligned}\frac{d}{dX}u(X) &= \frac{u(X)}{X}\\ &= f(X)g(u)\end{aligned}$$

Where

$$\begin{aligned}f(X) &= \frac{1}{X}\\ g(u) &= u\end{aligned}$$

Integrating gives

$$\begin{aligned}\int \frac{1}{g(u)}\,du &= \int f(X)\,dX\\ \int \frac{1}{u}\,du &= \int \frac{1}{X}\,dX\\ \ln(u(X)) &= \ln(X) + c_1\end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(X)$ gives

$$u(X) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned}\ln(u(X)) &= \ln(X) + c_1\\ u(X) &= 0\end{aligned}$$

Solving for $u(X)$ gives

$$\begin{aligned}u(X) &= 0\\ u(X) &= \mathrm{e}^{c_1}X\end{aligned}$$

Converting $u(X) = 0$ back to $Y(X)$ gives

$$Y(X) = 0$$

Converting $u(X) = e^{c_1}X$ back to $Y(X)$ gives

$$Y(X) = X^2 e^{c_1}$$

Using the solution for $Y(X)$

$$Y(X) = 0 \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y$$
$$X = x$$

Then the solution in y becomes using EQ (A)

$$y = 0$$

Using the solution for $Y(X)$

$$Y(X) = X^2 e^{c_1} \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y$$
$$X = x$$

Then the solution in y becomes using EQ (A)

$$y = e^{c_1} x^2$$

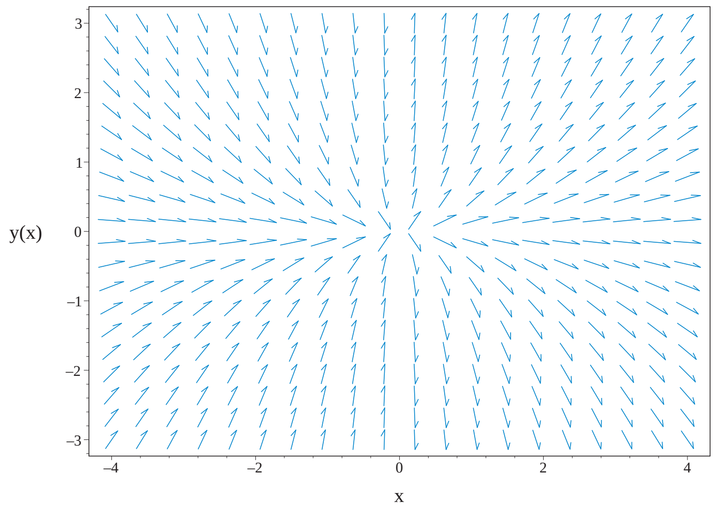


Figure 2.40: Slope field plot
$y' = \frac{2y}{x}$

Solved as first order Exact ode

Time used: 0.068 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(\frac{2y}{x}\right)\mathrm{d}x$$

$$\left(-\frac{2y}{x}\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -\frac{2y}{x} \\ N(x,y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{2y}{x}\right) \\ &= -\frac{2}{x} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1\left(\left(-\frac{2}{x}\right) - (0)\right) \\ &= -\frac{2}{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\frac{2}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(x)} \\ &= \frac{1}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{x^2}\left(-\frac{2y}{x}\right) \\ &= -\frac{2y}{x^3}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{x^2}(1) \\ &= \frac{1}{x^2}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-\frac{2y}{x^3}\right) + \left(\frac{1}{x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\int \frac{\partial \phi}{\partial y}\,\mathrm{d}y = \int \overline{N}\,\mathrm{d}y$$

$$\int \frac{\partial \phi}{\partial y}\,\mathrm{d}y = \int \frac{1}{x^2}\,\mathrm{d}y$$

$$\phi = \frac{y}{x^2} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = -\frac{2y}{x^3} + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = -\frac{2y}{x^3}$. Therefore equation (4) becomes

$$-\frac{2y}{x^3} = -\frac{2y}{x^3} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 0$$

Therefore

$$f(x) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = \frac{y}{x^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \frac{y}{x^2}$$

Solving for y gives

$$y = c_1 x^2$$

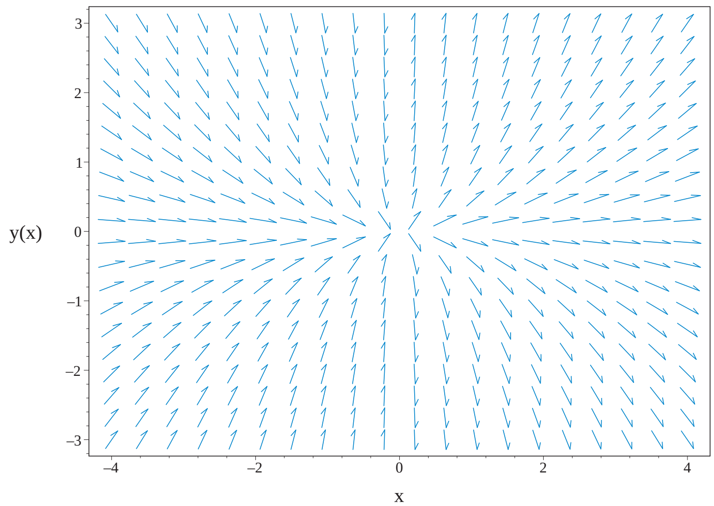


Figure 2.41: Slope field plot
$y' = \frac{2y}{x}$

Summary of solutions found

$$y = c_1 x^2$$

Solved using Lie symmetry for first order ode

Time used: 0.342 (sec)

Writing the ode as

$$y' = \frac{2y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{2y(b_3 - a_2)}{x} - \frac{4y^2 a_3}{x^2} + \frac{2y(xa_2 + ya_3 + a_1)}{x^2} - \frac{2(xb_2 + yb_3 + b_1)}{x} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{b_2 x^2 + 2y^2 a_3 + 2xb_1 - 2ya_1}{x^2} = 0$$

Setting the numerator to zero gives

$$-b_2 x^2 - 2y^2 a_3 - 2xb_1 + 2ya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-2a_3 v_2^2 - b_2 v_1^2 + 2a_1 v_2 - 2b_1 v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-2a_3 v_2^2 - b_2 v_1^2 + 2a_1 v_2 - 2b_1 v_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 2a_1 &= 0 \\ -2a_3 &= 0 \\ -2b_1 &= 0 \\ -b_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 0 \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{y} dy
\end{aligned}
$$

Which results in

$$
S = \ln(y)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{2y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= 0 \\
S_y &= \frac{1}{y}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{2}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{2}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned}
\int dS &= \int \frac{2}{R}\, dR \\
S(R) &= 2\ln(R) + c_2
\end{aligned}$$

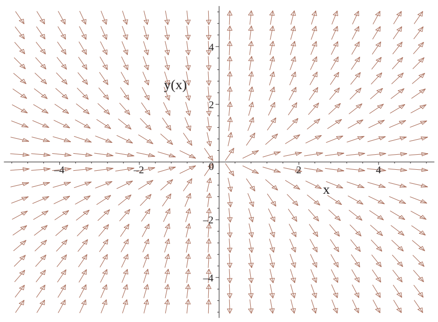
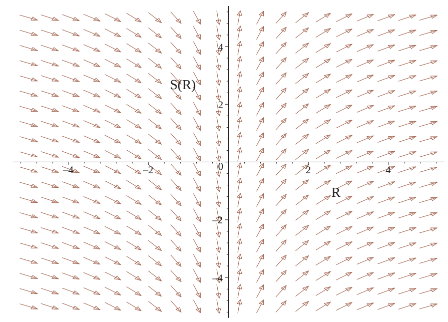
To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\ln(y) = 2\ln(x) + c_2$$

Which gives

$$y = e^{c_2} x^2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2y}{x}$ 	$R = x$ $S = \ln(y)$	$\frac{dS}{dR} = \frac{2}{R}$ 

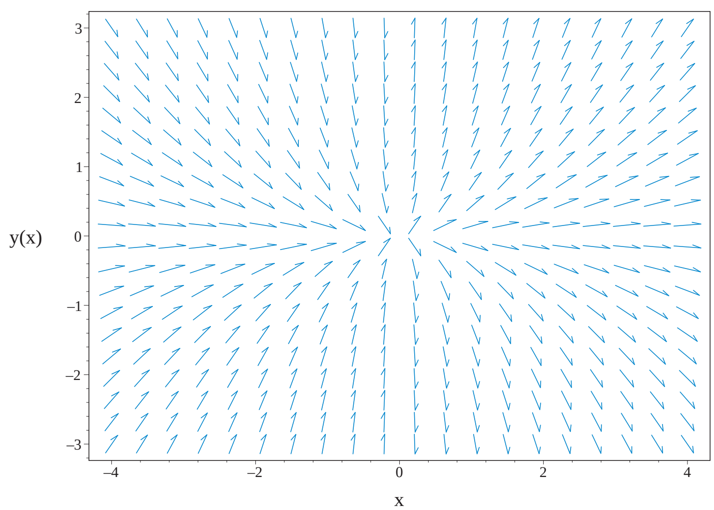


Figure 2.42: Slope field plot
$y' = \frac{2y}{x}$

Summary of solutions found

$$y = e^{c_2} x^2$$

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) = \frac{2y(x)}{x}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = \frac{2y(x)}{x}$

- Separate variables

  $\frac{\frac{d}{dx}y(x)}{y(x)} = \frac{2}{x}$

- Integrate both sides with respect to x

  $\int \frac{\frac{d}{dx}y(x)}{y(x)}dx = \int \frac{2}{x}dx + C1$

- Evaluate integral

  $\ln(y(x)) = 2\ln(x) + C1$

- Solve for $y(x)$

  $y(x) = e^{C1}x^2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 9

```
dsolve(diff(y(x),x) = 2*y(x)/x,
       y(x),singsol=all)
```

$$y = c_1 x^2$$

Mathematica DSolve solution

Solving time : 0.026 (sec)
Leaf size : 16

```
DSolve[{D[y[x],x] == 2*y[x]/x,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 x^2$$
$$y(x) \to 0$$

2.1.14 problem 14

Internal problem ID [8402]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 14
Date solved : `Tuesday, December 17, 2024 at 12:26:05 PM`
CAS classification : `[_separable]`

Solve

$$y' = \frac{\ln(y^2+1)}{\ln(x^2+1)}$$

Solved as first order separable ode

Time used: 0.296 (sec)

The ode $y' = \frac{\ln(y^2+1)}{\ln(x^2+1)}$ is separable as it can be written as

$$\begin{aligned} y' &= \frac{\ln(y^2+1)}{\ln(x^2+1)} \\ &= f(x)g(y) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{1}{\ln(x^2+1)} \\ g(y) &= \ln\left(y^2+1\right) \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(y)}\,dy &= \int f(x)\,dx \\ \int \frac{1}{\ln(y^2+1)}\,dy &= \int \frac{1}{\ln(x^2+1)}\,dx \end{aligned}$$

$$\int^{y} \frac{1}{\ln(\tau^2+1)}d\tau = \int \frac{1}{\ln(x^2+1)}dx + 2c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(y)$ is zero, since we had to divide by this above. Solving $g(y) = 0$ or $\ln(y^2 + 1) = 0$ for y gives

$$y = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\int^{y} \frac{1}{\ln(\tau^2 + 1)} d\tau = \int \frac{1}{\ln(x^2 + 1)} dx + 2c_1$$
$$y = 0$$

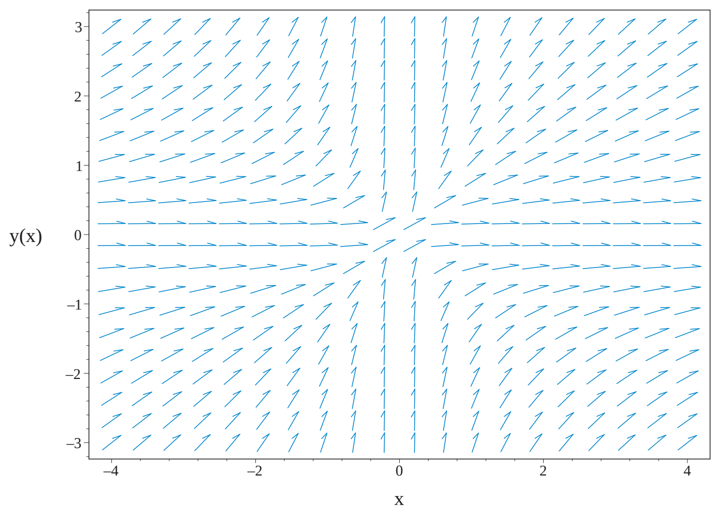


Figure 2.43: Slope field plot
$y' = \frac{\ln(y^2+1)}{\ln(x^2+1)}$

Summary of solutions found

$$\int^{y} \frac{1}{\ln(\tau^2 + 1)} d\tau = \int \frac{1}{\ln(x^2 + 1)} dx + 2c_1$$
$$y = 0$$

Solved as first order Exact ode

Time used: 0.147 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(\frac{\ln(y^2+1)}{\ln(x^2+1)}\right)\mathrm{d}x$$

$$\left(-\frac{\ln(y^2+1)}{\ln(x^2+1)}\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{\ln(y^2+1)}{\ln(x^2+1)}\\
N(x,y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{\ln(y^2+1)}{\ln(x^2+1)}\right)\\
&= -\frac{2y}{(y^2+1)\ln(x^2+1)}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\
&= 1\left(\left(-\frac{2y}{(y^2+1)\ln(x^2+1)}\right) - (0)\right)\\
&= -\frac{2y}{(y^2+1)\ln(x^2+1)}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\
&= -\frac{\ln(x^2+1)}{\ln(y^2+1)}\left((0) - \left(-\frac{2y}{(y^2+1)\ln(x^2+1)}\right)\right)\\
&= -\frac{2y}{\ln(y^2+1)(y^2+1)}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \,\mathrm{d}y} \\
&= e^{\int -\frac{2y}{\ln\left(y^2+1\right)\left(y^2+1\right)} \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\ln\left(\ln\left(y^2+1\right)\right)} \\
&= \frac{1}{\ln\left(y^2+1\right)}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{\ln\left(y^2+1\right)}\left(-\frac{\ln\left(y^2+1\right)}{\ln\left(x^2+1\right)}\right) \\
&= -\frac{1}{\ln\left(x^2+1\right)}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{\ln\left(y^2+1\right)}(1) \\
&= \frac{1}{\ln\left(y^2+1\right)}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\frac{1}{\ln\left(x^2+1\right)}\right) + \left(\frac{1}{\ln\left(y^2+1\right)}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{1}{\ln\left(x^2+1\right)}\,\mathrm{d}x$$
$$\phi = \int -\frac{1}{\ln\left(x^2+1\right)} dx + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\ln(y^2+1)}$. Therefore equation (4) becomes

$$\frac{1}{\ln\left(y^2+1\right)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{\ln\left(y^2+1\right)}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{\ln\left(y^2+1\right)}\right)\mathrm{d}y$$
$$f(y) = \int^y \frac{1}{\ln\left(\tau^2+1\right)} d\tau + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \int -\frac{1}{\ln\left(x^2+1\right)} dx + \int^y \frac{1}{\ln\left(\tau^2+1\right)} d\tau + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \int -\frac{1}{\ln(x^2+1)}dx + \int^y \frac{1}{\ln(\tau^2+1)}d\tau$$

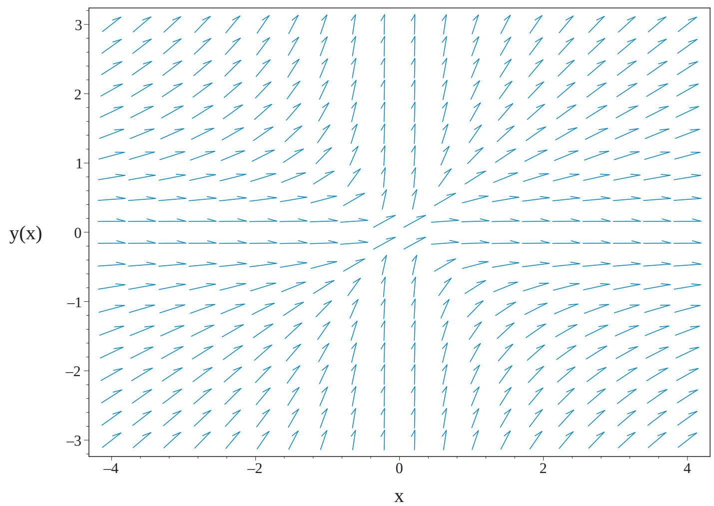


Figure 2.44: Slope field plot
$y' = \frac{\ln(y^2+1)}{\ln(x^2+1)}$

Summary of solutions found

$$\int -\frac{1}{\ln(x^2+1)}dx + \int^y \frac{1}{\ln(\tau^2+1)}d\tau = c_1$$

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) = \frac{\ln(1+y(x)^2)}{\ln(x^2+1)}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = \frac{\ln(1+y(x)^2)}{\ln(x^2+1)}$

- Separate variables

$$\frac{\frac{d}{dx}y(x)}{\ln\left(1+y(x)^2\right)} = \frac{1}{\ln(x^2+1)}$$

- Integrate both sides with respect to x

$$\int \frac{\frac{d}{dx}y(x)}{\ln\left(1+y(x)^2\right)}dx = \int \frac{1}{\ln(x^2+1)}dx + C1$$

- Cannot compute integral

$$\int \frac{\frac{d}{dx}y(x)}{\ln\left(1+y(x)^2\right)}dx = \int \frac{1}{\ln(x^2+1)}dx + C1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 30

```
dsolve(diff(y(x),x) = ln(1+y(x)^2)/ln(x^2+1),
       y(x),singsol=all)
```

$$\int \frac{1}{\ln(x^2+1)}dx - \left(\int^{y} \frac{1}{\ln(\_a^2+1)}d\_a\right) + c_1 = 0$$

Mathematica DSolve solution

Solving time : 0.669 (sec)
Leaf size : 48

```
DSolve[{D[y[x],x] == Log[1+y[x]^2]/Log[1+x^2],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \text{InverseFunction}\left[\int_1^{\#1} \frac{1}{\log\left(K[1]^2+1\right)} dK[1]\&\right]\left[\int_1^x \frac{1}{\log\left(K[2]^2+1\right)} dK[2]+c_1\right]$$
$$y(x) \to 0$$

2.1.15 problem 15

Internal problem ID [8403]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 15
Date solved : `Tuesday, December 17, 2024 at 12:26:07 PM`
CAS classification : `[_quadrature]`

Solve

$$y' = \frac{1}{x}$$

Solved as first order quadrature ode

Time used: 0.037 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int \frac{1}{x}\, dx$$
$$y = \ln(x) + c_1$$

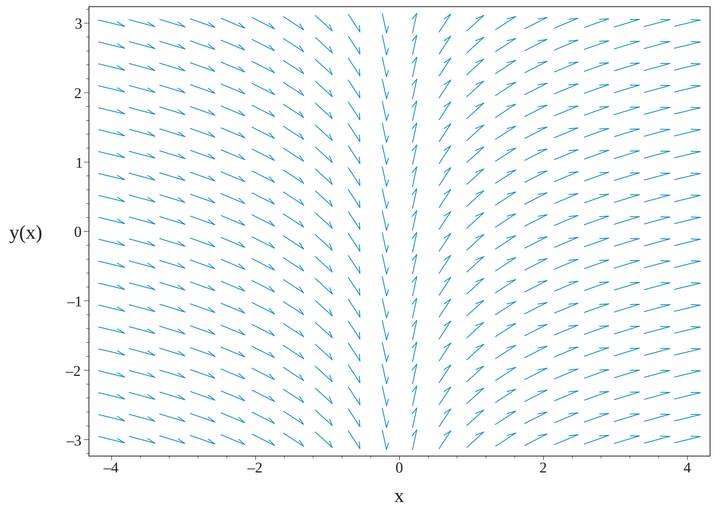


Figure 2.45: Slope field plot
$y' = \frac{1}{x}$

Summary of solutions found

$$y = \ln(x) + c_1$$

Solved as first order Exact ode

Time used: 0.085 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(\frac{1}{x}\right)\mathrm{d}x\\ \left(-\frac{1}{x}\right)\mathrm{d}x + \mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{1}{x}\\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int -\frac{1}{x}\,\mathrm{d}x$$
$$\phi=-\ln(x)+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=0+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=1$. Therefore equation (4) becomes

$$1=0+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=1$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y=\int(1)\,\mathrm{d}y$$
$$f(y)=y+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi=-\ln(x)+y+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -\ln(x) + y$$

Solving for y gives

$$y = \ln(x) + c_1$$

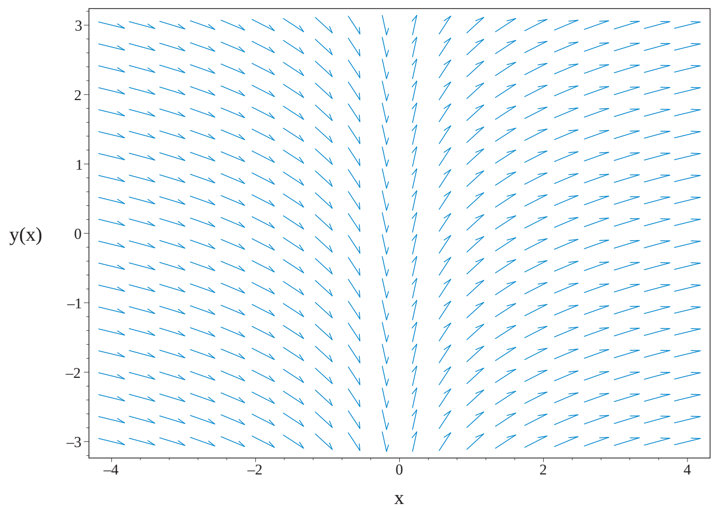


Figure 2.46: Slope field plot
$y' = \frac{1}{x}$

Summary of solutions found

$$y = \ln(x) + c_1$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) = \frac{1}{x}$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Integrate both sides with respect to x
  $\int \left(\frac{d}{dx}y(x)\right) dx = \int \frac{1}{x} dx + C1$
- Evaluate integral
  $y(x) = \ln(x) + C1$

- Solve for $y(x)$

  $y(x) = \ln(x) + C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 8

```
dsolve(diff(y(x),x) = 1/x,
       y(x),singsol=all)
```

$$y = \ln(x) + c_1$$

Mathematica DSolve solution

Solving time : 0.002 (sec)
Leaf size : 10

```
DSolve[{D[y[x],x] == 1/x,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \log(x) + c_1$$

2.1.16 problem 16

Internal problem ID [8404]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 16
Date solved : `Tuesday, December 17, 2024 at 12:26:08 PM`
CAS classification :
`[[_homogeneous, 'class G'], _rational, [_Abel, '2nd type', 'class B']]`

Solve

$$y' = \frac{-xy - 1}{4x^3y - 2x^2}$$

Solved as first order Exact ode

Time used: 0.459 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(\frac{-xy-1}{4y\,x^3-2x^2}\right)\mathrm{d}x$$

$$\left(-\frac{-xy-1}{4y\,x^3-2x^2}\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -\frac{-xy-1}{4y\,x^3-2x^2} \\ N(x,y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{-xy-1}{4y\,x^3-2x^2}\right) \\ &= -\frac{3}{2x\,(2xy-1)^2} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(\frac{x}{4y\,x^3 - 2x^2} + \frac{4(-xy-1)\,x^3}{(4y\,x^3 - 2x^2)^2}\right) - (0)\right) \\
&= -\frac{3}{2x\,(2xy-1)^2}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{4y\,x^3 - 2x^2}{xy+1}\left((0) - \left(\frac{x}{4y\,x^3 - 2x^2} + \frac{4(-xy-1)\,x^3}{(4y\,x^3 - 2x^2)^2}\right)\right) \\
&= \frac{3x}{2x^2y^2 + xy - 1}
\end{aligned}
$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$
R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}
$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$
\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(0) - \left(\frac{x}{4y\,x^3-2x^2} + \frac{4(-xy-1)x^3}{(4y\,x^3-2x^2)^2}\right)}{x\left(-\frac{-xy-1}{4y\,x^3-2x^2}\right) - y\,(1)} \\
&= -\frac{3}{8x^3y^3 - 10x^2y^2 + xy + 1}
\end{aligned}
$$

Replacing all powers of terms xy by t gives

$$
R = -\frac{3}{8t^3 - 10t^2 + t + 1}
$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$
\begin{aligned}
\mu &= e^{\int R\,\mathrm{d}t} \\
&= e^{\int \left(-\frac{3}{8t^3-10t^2+t+1}\right)\mathrm{d}t}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{2\ln(4t+1)}{5}-\frac{3\ln(t-1)}{5}+\ln(2t-1)} \\
&= \frac{2t-1}{(4t+1)^{2/5}(t-1)^{3/5}}
\end{aligned}
$$

Now t is replaced back with xy giving

$$
\mu = \frac{2xy-1}{(4xy+1)^{2/5}(xy-1)^{3/5}}
$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{2xy-1}{(4xy+1)^{2/5}(xy-1)^{3/5}}\left(-\frac{-xy-1}{4y\,x^3-2x^2}\right) \\
&= \frac{xy+1}{2(xy-1)^{3/5}(4xy+1)^{2/5}x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{2xy-1}{(4xy+1)^{2/5}(xy-1)^{3/5}}(1) \\
&= \frac{2xy-1}{(4xy+1)^{2/5}(xy-1)^{3/5}}
\end{aligned}
$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$
\begin{aligned}
\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{xy+1}{2(xy-1)^{3/5}(4xy+1)^{2/5}x^2}\right)+\left(\frac{2xy-1}{(4xy+1)^{2/5}(xy-1)^{3/5}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \frac{xy+1}{2\left(xy-1\right)^{3/5}\left(4xy+1\right)^{2/5}x^2}\,\mathrm{d}x$$
$$\phi = \frac{\left(xy-1\right)^{2/5}\left(4xy+1\right)^{3/5}}{2x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= \frac{\left(4xy+1\right)^{3/5}}{5\left(xy-1\right)^{3/5}} + \frac{6(xy-1)^{2/5}}{5\left(4xy+1\right)^{2/5}} + f'(y) \qquad (4)\\ &= \frac{2xy-1}{\left(4xy+1\right)^{2/5}\left(xy-1\right)^{3/5}} + f'(y)\end{aligned}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2xy-1}{(4xy+1)^{2/5}(xy-1)^{3/5}}$. Therefore equation (4) becomes

$$\frac{2xy-1}{\left(4xy+1\right)^{2/5}\left(xy-1\right)^{3/5}} = \frac{2xy-1}{\left(4xy+1\right)^{2/5}\left(xy-1\right)^{3/5}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{\left(xy-1\right)^{2/5}\left(4xy+1\right)^{3/5}}{2x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \frac{\left(xy-1\right)^{2/5}\left(4xy+1\right)^{3/5}}{2x}$$

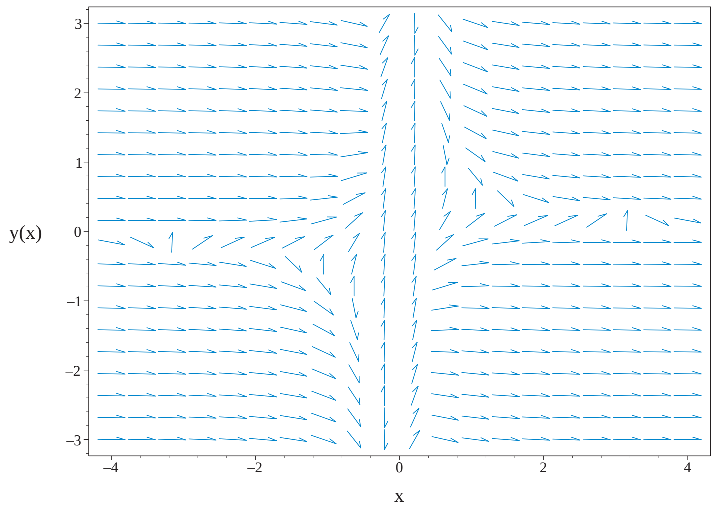


Figure 2.47: Slope field plot
$y' = \frac{-xy-1}{4x^3y-2x^2}$

Summary of solutions found

$$\frac{(xy-1)^{2/5}(4xy+1)^{3/5}}{2x} = c_1$$

Solved as first order isobaric ode

Time used: 0.695 (sec)

Solving for y' gives

$$y' = -\frac{xy+1}{2x^2(2xy-1)} \tag{1}$$

Each of the above ode's is now solved An ode $y' = f(x,y)$ is isobaric if

$$f(tx, t^m y) = t^{m-1} f(x,y) \tag{1}$$

Where here

$$f(x,y) = -\frac{xy+1}{2x^2(2xy-1)} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = -1$$

Since the ode is isobaric of order $m = -1$, then the substitution

$$\begin{aligned} y &= ux^m \\ &= \frac{u}{x} \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$-\frac{u(x)}{x^2}+\frac{u'(x)}{x}=-\frac{u(x)+1}{2x^2\left(2u\left(x\right)-1\right)}$$

The ode $u'(x)=\frac{4u(x)^2-3u(x)-1}{2x(2u(x)-1)}$ is separable as it can be written as

$$\begin{aligned}u'(x)&=\frac{4u(x)^2-3u(x)-1}{2x\left(2u\left(x\right)-1\right)}\\&=f(x)g(u)\end{aligned}$$

Where

$$\begin{aligned}f(x)&=\frac{1}{2x}\\g(u)&=\frac{4u^2-3u-1}{2u-1}\end{aligned}$$

Integrating gives

$$\begin{aligned}\int\frac{1}{g(u)}\,du&=\int f(x)\,dx\\\int\frac{2u-1}{4u^2-3u-1}\,du&=\int\frac{1}{2x}\,dx\\\ln\left(\left(u(x)-1\right)^{1/5}\left(4u(x)+1\right)^{3/10}\right)&=\ln\left(\sqrt{x}\right)+c_1\end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u)=0$ or $\frac{4u^2-3u-1}{2u-1}=0$ for $u(x)$ gives

$$\begin{aligned}u(x)&=1\\u(x)&=-\frac{1}{4}\end{aligned}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned}\ln\left(\left(u(x)-1\right)^{1/5}\left(4u(x)+1\right)^{3/10}\right)&=\ln\left(\sqrt{x}\right)+c_1\\u(x)&=1\\u(x)&=-\frac{1}{4}\end{aligned}$$

Converting $\ln\left((u(x)-1)^{1/5}(4u(x)+1)^{3/10}\right) = \frac{\ln(x)}{2} + c_1$ back to y gives

$$\ln\left((xy-1)^{1/5}(4xy+1)^{3/10}\right) = \frac{\ln(x)}{2} + c_1$$

Converting $u(x) = 1$ back to y gives

$$xy = 1$$

Converting $u(x) = -\frac{1}{4}$ back to y gives

$$xy = -\frac{1}{4}$$

Solving for y gives

$$\ln\left((xy-1)^{1/5}(4xy+1)^{3/10}\right) = \frac{\ln(x)}{2} + c_1$$
$$y = \frac{1}{x}$$
$$y = -\frac{1}{4x}$$

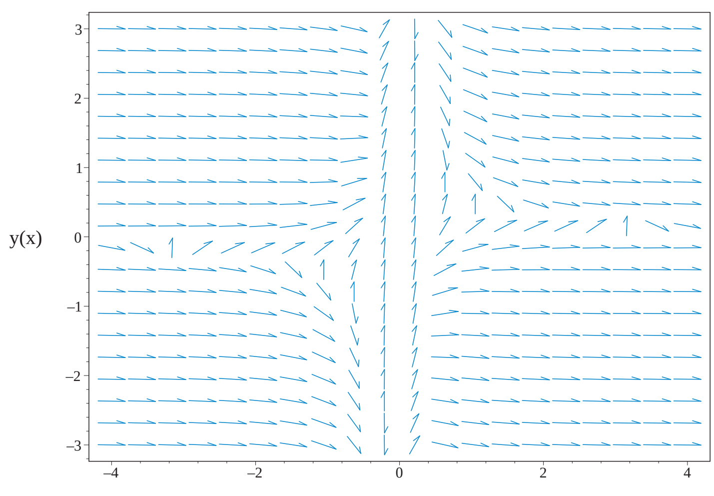


Figure 2.48: Slope field plot
$y' = \frac{-xy-1}{4x^3y-2x^2}$

Summary of solutions found

$$
\begin{aligned}
&\ln\left((xy-1)^{1/5}(4xy+1)^{3/10}\right)=\frac{\ln(x)}{2}+c_1\\
&y=\frac{1}{x}\\
&y=-\frac{1}{4x}
\end{aligned}
$$

Solved using Lie symmetry for first order ode

Time used: 1.054 (sec)

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{xy+1}{2x^2(2xy-1)}\\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}
$$

To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = xa_2+ya_3+a_1 \tag{1E}
$$

$$
\eta = xb_2+yb_3+b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1,a_2,a_3,b_1,b_2,b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2-\frac{(xy+1)(b_3-a_2)}{2x^2(2xy-1)}-\frac{(xy+1)^2a_3}{4x^4(2xy-1)^2}\\
&-\left(-\frac{y}{2x^2(2xy-1)}+\frac{xy+1}{x^3(2xy-1)}+\frac{(xy+1)y}{x^2(2xy-1)^2}\right)(xa_2+ya_3+a_1)\\
&-\left(-\frac{1}{2x(2xy-1)}+\frac{xy+1}{x(2xy-1)^2}\right)(xb_2+yb_3+b_1)=0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$\frac{16x^6y^2b_2 - 16x^5yb_2 - 4x^4y^2a_2 - 4x^4y^2b_3 - 8x^3y^3a_3 - 8x^3y^2a_1 - 2b_2x^4 - 8x^3ya_2 - 8x^3yb_3 - 11x^2y^2a_3 - 6}{4x^4(2xy-1)^2}$$
$$= 0$$

Setting the numerator to zero gives

$$16x^6y^2b_2 - 16x^5yb_2 - 4x^4y^2a_2 - 4x^4y^2b_3 - 8x^3y^3a_3 - 8x^3y^2a_1 - 2b_2x^4 - 8x^3ya_2 - 8x^3yb_3 - 11x^2y^2a_3 - 6x^3b_1 - 10x^2ya_1 + 2x^2a_2 + 2x^2b_3 + 2xya_3 + 4xa_1 - a_3 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}16b_2v_1^6v_2^2 - 4a_2v_1^4v_2^2 - 8a_3v_1^3v_2^3 - 16b_2v_1^5v_2 - 4b_3v_1^4v_2^2 - 8a_1v_1^3v_2^2 \\ - 8a_2v_1^3v_2 - 11a_3v_1^2v_2^2 - 2b_2v_1^4 - 8b_3v_1^3v_2 - 10a_1v_1^2v_2 \\ - 6b_1v_1^3 + 2a_2v_1^2 + 2a_3v_1v_2 + 2b_3v_1^2 + 4a_1v_1 - a_3 = 0\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}16b_2v_1^6v_2^2 - 16b_2v_1^5v_2 + (-4a_2 - 4b_3)\,v_1^4v_2^2 - 2b_2v_1^4 - 8a_3v_1^3v_2^3 \\ - 8a_1v_1^3v_2^2 + (-8a_2 - 8b_3)\,v_1^3v_2 - 6b_1v_1^3 - 11a_3v_1^2v_2^2 \\ - 10a_1v_1^2v_2 + (2a_2 + 2b_3)\,v_1^2 + 2a_3v_1v_2 + 4a_1v_1 - a_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-10a_1 &= 0\\
-8a_1 &= 0\\
4a_1 &= 0\\
-11a_3 &= 0\\
-8a_3 &= 0\\
-a_3 &= 0\\
2a_3 &= 0\\
-6b_1 &= 0\\
-16b_2 &= 0\\
-2b_2 &= 0\\
16b_2 &= 0\\
-8a_2 - 8b_3 &= 0\\
-4a_2 - 4b_3 &= 0\\
2a_2 + 2b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0\\
a_2 &= -b_3\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -x\\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{xy+1}{2x^2\,(2xy-1)}\right)(-x) \\
&= \frac{4x^2y^2 - 3xy - 1}{4y\,x^2 - 2x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{4x^2y^2 - 3xy - 1}{4y\,x^2 - 2x}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{3\ln(4xy+1)}{5} + \frac{2\ln(xy-1)}{5}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = -\frac{xy+1}{2x^2\,(2xy-1)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{4y^2x - 2y}{4x^2y^2 - 3xy - 1} \\
S_y &= \frac{4y\,x^2 - 2x}{4x^2y^2 - 3xy - 1}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

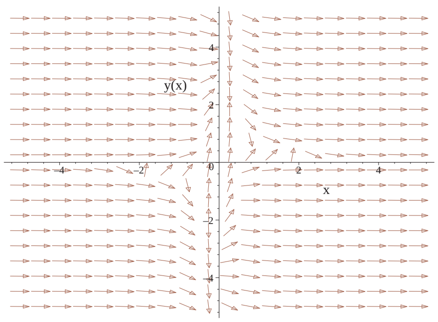
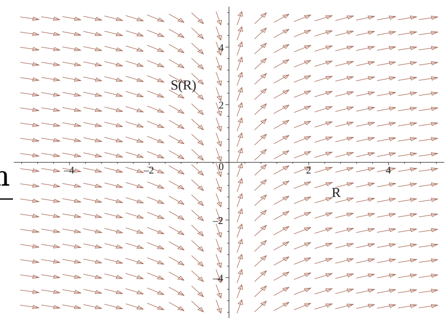
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$
\begin{aligned}
\int dS &= \int \frac{1}{R}\,dR \\
S(R) &= \ln(R) + c_2
\end{aligned}
$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$
\frac{3\ln(4xy+1)}{5} + \frac{2\ln(xy-1)}{5} = \ln(x) + c_2
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{xy+1}{2x^2(2xy-1)}$ 	$R = x$ $S = \frac{3\ln(4xy+1)}{5} + \frac{2\ln}{}$	$\frac{dS}{dR} = \frac{1}{R}$ 

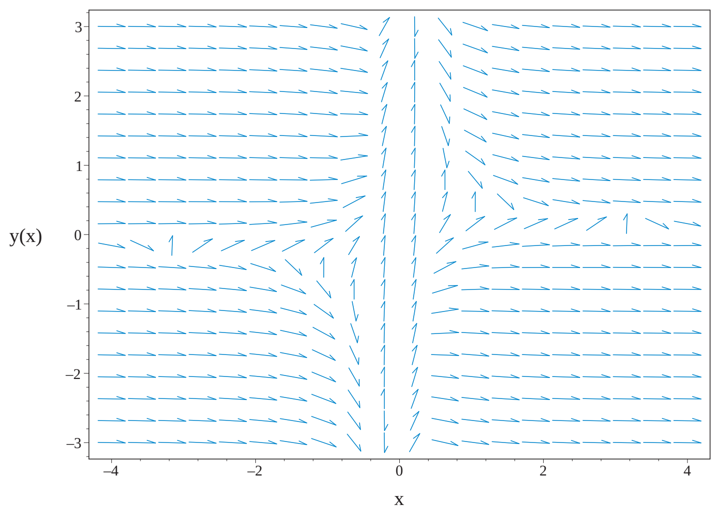


Figure 2.49: Slope field plot
$y' = \frac{-xy-1}{4x^3y-2x^2}$

Summary of solutions found

$$\frac{3\ln(4xy+1)}{5} + \frac{2\ln(xy-1)}{5} = \ln(x) + c_2$$

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) = \frac{-xy(x)-1}{4x^3y(x)-2x^2}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$
- Solve for the highest derivative

  $\frac{d}{dx}y(x) = \frac{-xy(x)-1}{4x^3y(x)-2x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

Maple dsolve solution

Solving time : `0.683` (sec)
Leaf size : 37

```
dsolve(diff(y(x),x) = (-x*y(x)-1)/(4*y(x)*x^3-2*x^2),
       y(x),singsol=all)
```

$$y = \frac{\operatorname{RootOf}\left(\_Z^{25}c_1 - 10\_Z^{20}c_1 + 25\_Z^{15}c_1 - 16x^5\right)^5 - 1}{4x}$$

Mathematica DSolve solution

Solving time : 15.164 (sec)
Leaf size : 391

```
DSolve[{D[y[x],x] == (-x*y[x]-1)/(4*x^3*y[x]-2*x^2),{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \text{Root}\left[64\#1^5 c_1{}^5 x^5 - 80\#1^4 c_1{}^5 x^4 - 20\#1^3 c_1{}^5 x^3 + 25\#1^2 c_1{}^5 x^2 + 10\#1 c_1{}^5 x - x^5 + c_1{}^5\&, 1\right]$$
$$y(x) \to \text{Root}\left[64\#1^5 c_1{}^5 x^5 - 80\#1^4 c_1{}^5 x^4 - 20\#1^3 c_1{}^5 x^3 + 25\#1^2 c_1{}^5 x^2 + 10\#1 c_1{}^5 x - x^5 + c_1{}^5\&, 2\right]$$
$$y(x) \to \text{Root}\left[64\#1^5 c_1{}^5 x^5 - 80\#1^4 c_1{}^5 x^4 - 20\#1^3 c_1{}^5 x^3 + 25\#1^2 c_1{}^5 x^2 + 10\#1 c_1{}^5 x - x^5 + c_1{}^5\&, 3\right]$$
$$y(x) \to \text{Root}\left[64\#1^5 c_1{}^5 x^5 - 80\#1^4 c_1{}^5 x^4 - 20\#1^3 c_1{}^5 x^3 + 25\#1^2 c_1{}^5 x^2 + 10\#1 c_1{}^5 x - x^5 + c_1{}^5\&, 4\right]$$
$$y(x) \to \text{Root}\left[64\#1^5 c_1{}^5 x^5 - 80\#1^4 c_1{}^5 x^4 - 20\#1^3 c_1{}^5 x^3 + 25\#1^2 c_1{}^5 x^2 + 10\#1 c_1{}^5 x - x^5 + c_1{}^5\&, 5\right]$$

2.1.17 problem 17

Internal problem ID [8405]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 17
Date solved : `Tuesday, December 17, 2024 at 12:26:11 PM`
CAS classification : `[[_1st_order, _with_linear_symmetries], _Clairaut]`

Solve

$$\frac{y'^2}{4} - xy' + y = 0$$

Solved as first order Clairaut ode

Time used: 0.076 (sec)

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$\frac{1}{4}p^2 - xp + y = 0$$

Solving for y from the above results in

$$y = -\frac{1}{4}p^2 + xp \tag{1A}$$

The above ode is a Clairaut ode which is now solved.

We start by replacing y' by p which gives

$$\begin{aligned} y &= -\frac{1}{4}p^2 + xp \\ &= -\frac{1}{4}p^2 + xp \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = -\frac{p^2}{4}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p.

The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1 x - \frac{1}{4}c_1^2$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = -\frac{p^2}{4}$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x - \frac{p}{2} \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$p_1 = y = x^2$$

Substituting the above back in (1) results in

$$y = x^2$$

Summary of solutions found

$$y = x^2$$

$$y = c_1 x - \frac{1}{4} c_1^2$$

Maple step by step solution

Let's solve

$\frac{\left(\frac{d}{dx}y(x)\right)^2}{4} - x\left(\frac{d}{dx}y(x)\right) + y(x) = 0$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\left[\frac{d}{dx}y(x) = 2x - 2\sqrt{x^2 - y(x)}, \frac{d}{dx}y(x) = 2x + 2\sqrt{x^2 - y(x)}\right]$
- Solve the equation $\frac{d}{dx}y(x) = 2x - 2\sqrt{x^2 - y(x)}$
- Solve the equation $\frac{d}{dx}y(x) = 2x + 2\sqrt{x^2 - y(x)}$
- Set of solutions
  $\{workingODE, workingODE\}$

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
```

```
trying dAlembert
<- dAlembert successful`
```

Maple dsolve solution

Solving time : 0.034 (sec)
Leaf size : 18

```
dsolve(1/4*diff(y(x),x)^2-diff(y(x),x)*x+y(x) = 0,
       y(x),singsol=all)
```

$$y = x^2$$
$$y = -\frac{c_1(c_1 - 4x)}{4}$$

Mathematica DSolve solution

Solving time : 0.008 (sec)
Leaf size : 25

```
DSolve[{(1/4)*(D[y[x],x])^2-x*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \rightarrow c_1 x - \frac{c_1{}^2}{4}$$
$$y(x) \rightarrow x^2$$

2.1.18 problem 18

Internal problem ID [8406]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 18
Date solved : `Tuesday, December 17, 2024 at 12:26:12 PM`
CAS classification : `[_quadrature]`

Solve

$$y' = \sqrt{\frac{1+y}{y^2}}$$

With initial conditions

$$y(0) = 1$$

Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \sqrt{\frac{1+y}{y^2}} \end{aligned}$$

The y domain of $f(x,y)$ when $x=0$ is

$$\{-1 \le y < 0, 0 < y \le \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\sqrt{\frac{1+y}{y^2}}\right) \\ &= \frac{\frac{1}{y^2} - \frac{2(1+y)}{y^3}}{2\sqrt{\frac{1+y}{y^2}}} \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x=0$ is

$$\{-\infty \leq y < -1, -1 < y < 0, 0 < y \leq \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

Solved as first order autonomous ode

Time used: 489.566 (sec)

Integrating gives

$$\int \frac{1}{\sqrt{\frac{1+y}{y^2}}} dy = dx$$

$$\frac{2\sqrt{\frac{1+y}{y^2}}\, y(y-2)}{3} = x + c_1$$

The following diagram is the phase line diagram. It classifies each of the above equilibrium points as stable or not stable or semi-stable.

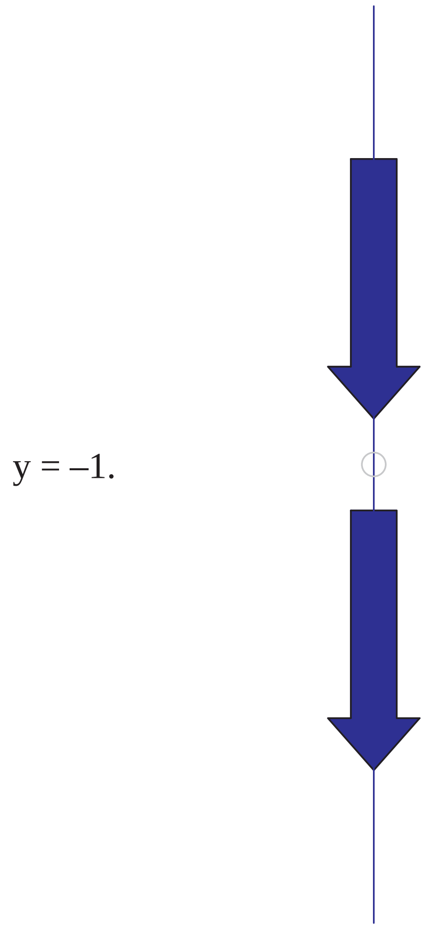


Figure 2.50: Phase line diagram

Solving for the constant of integration from initial conditions, the solution becomes

$$\frac{2\sqrt{\frac{1+y}{y^2}}\,y(y-2)}{3} = x - \frac{2\sqrt{2}}{3}$$

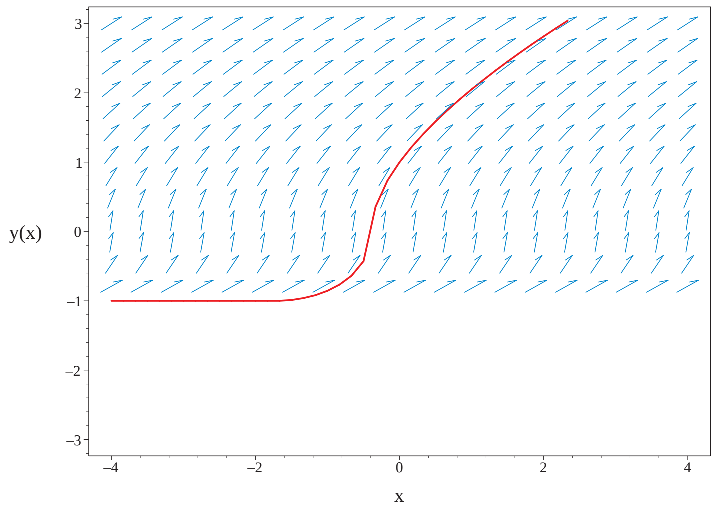


Figure 2.51: Slope field plot
$y' = \sqrt{\frac{1+y}{y^2}}$

Summary of solutions found

$$\frac{2\sqrt{\frac{1+y}{y^2}}\, y(y-2)}{3} = x - \frac{2\sqrt{2}}{3}$$

Solved as first order Exact ode

Time used: 620.792 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(\sqrt{\frac{1+y}{y^2}}\right)\mathrm{d}x$$

$$\left(-\sqrt{\frac{1+y}{y^2}}\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -\sqrt{\frac{1+y}{y^2}}\\ N(x,y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\sqrt{\frac{1+y}{y^2}}\right)\\ &= \frac{y+2}{2\sqrt{\frac{1+y}{y^2}}\,y^3}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(-\frac{\frac{1}{y^2} - \frac{2(1+y)}{y^3}}{2\sqrt{\frac{1+y}{y^2}}}\right) - (0)\right) \\
&= \frac{y+2}{2\sqrt{\frac{1+y}{y^2}}\, y^3}
\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= -\frac{1}{\sqrt{\frac{1+y}{y^2}}}\left((0) - \left(-\frac{\frac{1}{y^2} - \frac{2(1+y)}{y^3}}{2\sqrt{\frac{1+y}{y^2}}}\right)\right) \\
&= \frac{y+2}{2y\,(1+y)}
\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}
\mu &= e^{\int B \,\mathrm{d}y} \\
&= e^{\int \frac{y+2}{2y(1+y)}\,\mathrm{d}y}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{\ln(y) - \frac{\ln(1+y)}{2}} \\
&= \frac{y}{\sqrt{1+y}}
\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{y}{\sqrt{1+y}}\left(-\sqrt{\frac{1+y}{y^2}}\right) \\
&= -\frac{\sqrt{\frac{1+y}{y^2}}\, y}{\sqrt{1+y}}
\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{y}{\sqrt{1+y}}(1) \\ &= \frac{y}{\sqrt{1+y}}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-\frac{\sqrt{\frac{1+y}{y^2}}\, y}{\sqrt{1+y}}\right) + \left(\frac{y}{\sqrt{1+y}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{\sqrt{\frac{1+y}{y^2}}\, y}{\sqrt{1+y}}\,\mathrm{d}x \\ \phi &= -\frac{\sqrt{\frac{1+y}{y^2}}\, yx}{\sqrt{1+y}} + f(y) \end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= -\frac{yx\left(\frac{1}{y^2} - \frac{2(1+y)}{y^3}\right)}{2\sqrt{\frac{1+y}{y^2}}\,\sqrt{1+y}} - \frac{\sqrt{\frac{1+y}{y^2}}\, x}{\sqrt{1+y}} + \frac{\sqrt{\frac{1+y}{y^2}}\, yx}{2\,(1+y)^{3/2}} + f'(y) \\ &= 0 + f'(y)\end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\frac{y}{\sqrt{1+y}}$. Therefore equation (4) becomes

$$\frac{y}{\sqrt{1+y}}=0+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=\frac{y}{\sqrt{1+y}}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int\left(\frac{y}{\sqrt{1+y}}\right)\mathrm{d}y\\ f(y) &= \frac{2\sqrt{1+y}\,(y-2)}{3}+c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi=-\frac{\sqrt{\frac{1+y}{y^2}}\,yx}{\sqrt{1+y}}+\frac{2\sqrt{1+y}\,(y-2)}{3}+c_1$$

But since ϕ itself is a constant function, then let $\phi=c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1=-\frac{\sqrt{\frac{1+y}{y^2}}\,yx}{\sqrt{1+y}}+\frac{2\sqrt{1+y}\,(y-2)}{3}$$

Solving for the constant of integration from initial conditions, the solution becomes

$$-\frac{\sqrt{\frac{1+y}{y^2}}\,yx}{\sqrt{1+y}}+\frac{2\sqrt{1+y}\,(y-2)}{3}=-\frac{2\sqrt{2}}{3}$$

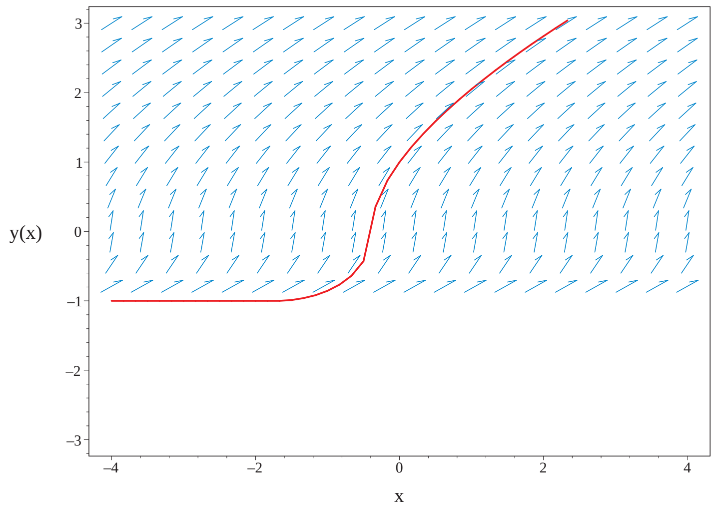


Figure 2.52: Slope field plot
$y' = \sqrt{\frac{1+y}{y^2}}$

Summary of solutions found

$$-\frac{\sqrt{\frac{1+y}{y^2}}\, yx}{\sqrt{1+y}} + \frac{2\sqrt{1+y}\,(y-2)}{3} = -\frac{2\sqrt{2}}{3}$$

Maple step by step solution

Let's solve

$\left[\frac{d}{dx}y(x) = \sqrt{\frac{1+y(x)}{y(x)^2}}, y(0) = 1\right]$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$
- Solve for the highest derivative

  $\frac{d}{dx}y(x) = \sqrt{\frac{1+y(x)}{y(x)^2}}$
- Separate variables

  $\frac{\frac{d}{dx}y(x)}{\sqrt{\frac{1+y(x)}{y(x)^2}}} = 1$
- Integrate both sides with respect to x

  $\int \frac{\frac{d}{dx}y(x)}{\sqrt{\frac{1+y(x)}{y(x)^2}}}dx = \int 1dx + C1$
- Evaluate integral

$$\frac{2(1+y(x))(-2+y(x))}{3\sqrt{\frac{1+y(x)}{y(x)^2}}\, y(x)} = x + C1$$

- Solve for $y(x)$

$$y(x) = \frac{\left(-8+9C1^2+18C1x+9x^2+3\sqrt{9C1^4+36C1^3x+54C1^2x^2+36C1\,x^3+9x^4-16C1^2-32C1x-16x^2}\right)^{1/3}}{2} + \frac{}{\left(-8+9C1^2+18C1\right.}$$

- Use initial condition $y(0) = 1$

$$1 = \frac{\left(-8+9C1^2+3\sqrt{9C1^4-16C1^2}\right)^{1/3}}{2} + \frac{2}{\left(-8+9C1^2+3\sqrt{9C1^4-16C1^2}\right)^{1/3}} + 1$$

- Solve for $\_C1$

$$C1 = RootOf\left(\left(-8+9\_Z^2+3\sqrt{9\_Z^4-16\_Z^2}\right)^{2/3}+4\right)$$

- Substitute $\_C1 = RootOf\left(\left(-8+9\_Z^2+3\sqrt{9\_Z^4-16\_Z^2}\right)^{2/3}+4\right)$ into general solution and

$$y(x) = \frac{\left(-8+9RootOf\left(\left(-8+9\_Z^2+3\sqrt{\_Z^2\left(9\_Z^2-16\right)}\right)^{2/3}+4\right)^2+18RootOf\left(\left(-8+9\_Z^2+3\sqrt{\_Z^2\left(9\_Z^2-16\right)}\right)^{2/3}+\right.\right.}{}$$

- Solution to the IVP

$$y(x) = \frac{\left(-8+9RootOf\left(\left(-8+9\_Z^2+3\sqrt{\_Z^2\left(9\_Z^2-16\right)}\right)^{2/3}+4\right)^2+18RootOf\left(\left(-8+9\_Z^2+3\sqrt{\_Z^2\left(9\_Z^2-16\right)}\right)^{2/3}+\right.\right.}{}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

Maple dsolve solution

Solving time : `0.690` (sec)
Leaf size : 146

```
dsolve([diff(y(x),x) = ((y(x)+1)/y(x)^2)^(1/2),
       op([y(0) = 1])],y(x),singsol=all)
```

$$y = -\frac{\left(1+i\sqrt{3}\right)\left(-12\sqrt{2}\,x+9x^2+\sqrt{\left(-12\sqrt{2}\,x+9x^2-8\right)\left(3x-2\sqrt{2}\right)^2}\right)^{2/3}-4i\sqrt{3}-4\left(-12\sqrt{2}\,x+9x \cdots\right.}{4\left(-12\sqrt{2}\,x+9x^2+\sqrt{\left(-12\sqrt{2}\,x+9x^2-8\right)\left(3x-2\sqrt{2} \cdots\right.}\right.}$$

Mathematica DSolve solution

Solving time : `0.077` (sec)
Leaf size : 123

```
DSolve[{D[y[x],x]==Sqrt[ (1+y[x])/y[x]^2],y[0]==1},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{1}{4}\left(1+i\sqrt{3}\right)\sqrt[3]{9x^2+\sqrt{81x^4-216\sqrt{2}x^3+288x^2-64}-12\sqrt{2}x} + \frac{i\left(\sqrt{3}+i\right)}{\sqrt[3]{9x^2+\sqrt{81x^4-216\sqrt{2}x^3+288x^2-64}-12\sqrt{2}x}}+1$$

2.1.19 problem 19

Internal problem ID [8407]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 19
Date solved : `Tuesday, December 17, 2024 at 12:50:38 PM`
CAS classification : `[‘y=_G(x,y’)‘]`

Solve

$$y' = \sqrt{1 - x^2 - y^2}$$

Unknown ode type.

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) = \sqrt{1 - x^2 - y(x)^2}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = \sqrt{1 - x^2 - y(x)^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying an equivalence to an Abel ODE
trying 1st order ODE linearizable_by_differentiation
```

```
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

Maple dsolve solution

Solving time : `0.019` (sec)
Leaf size : maple_leaf_size

```
dsolve(diff(y(x),x) = (1-x^2-y(x)^2)^(1/2),
       y(x),singsol=all)
```

No solution found

Mathematica DSolve solution

Solving time : `0.0` (sec)
Leaf size : 0

```
DSolve[{D[y[x],x]==Sqrt[ 1-x^2-y[x]^2],{}},
       y[x],x,IncludeSingularSolutions->True]
```

Not solved

2.1.20 problem 20

Internal problem ID [8408]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 20
Date solved : `Tuesday, December 17, 2024 at 12:50:40 PM`
CAS classification : `[_Bernoulli]`

Solve

$$y' + \frac{y}{3} = \frac{(1-2x)\, y^4}{3}$$

Solved as first order Bernoulli ode

Time used: 0.459 (sec)

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{1}{3}y - \frac{2}{3}y^4x + \frac{1}{3}y^4 \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \left(-\frac{1}{3}\right) y + \left(-\frac{2x}{3} + \frac{1}{3}\right) y^4 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

Comparing this to (1) shows that

$$\begin{aligned} f_0 &= -\frac{1}{3} \\ f_1 &= -\frac{2x}{3} + \frac{1}{3} \end{aligned}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $v = y^{1-n}$ in equation (3) which generates a new ODE in $v(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{3} \\ f_1(x) &= -\frac{2x}{3} + \frac{1}{3} \\ n &= 4 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^4$ gives

$$y'\frac{1}{y^4} = -\frac{1}{3y^3} - \frac{2x}{3} + \frac{1}{3} \tag{4}$$

Let

$$\begin{aligned} v &= y^{1-n} \\ &= \frac{1}{y^3} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$v' = -\frac{3}{y^4}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -\frac{v'(x)}{3} &= -\frac{v(x)}{3} - \frac{2x}{3} + \frac{1}{3} \\ v' &= v + 2x - 1 \end{aligned} \tag{7}$$

The above now is a linear ODE in $v(x)$ which is now solved.

In canonical form a linear first order is

$$v'(x) + q(x)v(x) = p(x)$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= -1 \\
p(x) &= 2x - 1
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= \mathrm{e}^{\int (-1)dx} \\
&= \mathrm{e}^{-x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu v) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mu v) &= (\mu)\,(2x-1) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(v\,\mathrm{e}^{-x}\right) &= \left(\mathrm{e}^{-x}\right)(2x-1) \\
\mathrm{d}\left(v\,\mathrm{e}^{-x}\right) &= \left((2x-1)\,\mathrm{e}^{-x}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
v\,\mathrm{e}^{-x} &= \int (2x-1)\,\mathrm{e}^{-x}\,dx \\
&= -(2x+1)\,\mathrm{e}^{-x} + c_1
\end{aligned}
$$

Dividing throughout by the integrating factor e^{-x} gives the final solution

$$
v(x) = c_1\,\mathrm{e}^{x} - 2x - 1
$$

The substitution $v = y^{1-n}$ is now used to convert the above solution back to y which results in

$$
\frac{1}{y^3} = c_1\,\mathrm{e}^{x} - 2x - 1
$$

Solving for y gives

$$
\begin{aligned}
y &= \frac{1}{\left(c_1\,\mathrm{e}^{x} - 2x - 1\right)^{1/3}} \\
y &= -\frac{1}{2\left(c_1\,\mathrm{e}^{x} - 2x - 1\right)^{1/3}} - \frac{i\sqrt{3}}{2\left(c_1\,\mathrm{e}^{x} - 2x - 1\right)^{1/3}} \\
y &= -\frac{1}{2\left(c_1\,\mathrm{e}^{x} - 2x - 1\right)^{1/3}} + \frac{i\sqrt{3}}{2\left(c_1\,\mathrm{e}^{x} - 2x - 1\right)^{1/3}}
\end{aligned}
$$

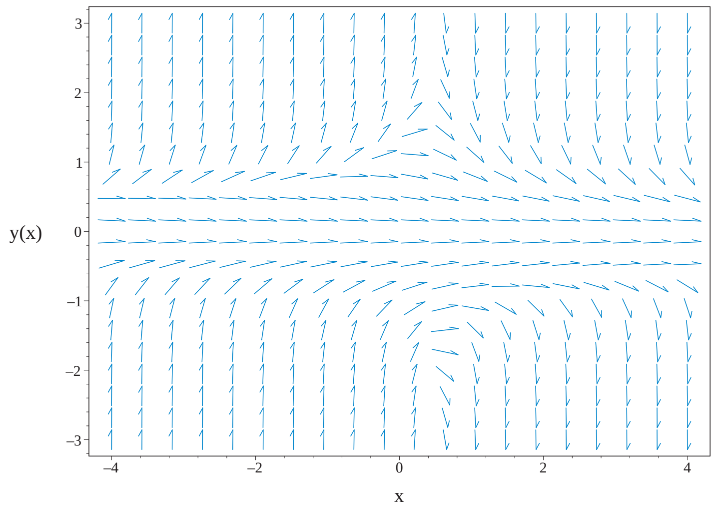


Figure 2.53: Slope field plot
$y' + \frac{y}{3} = \frac{(1-2x)y^4}{3}$

Summary of solutions found

$$y = \frac{1}{(c_1\,e^{x} - 2x - 1)^{1/3}}$$

$$y = -\frac{1}{2\,(c_1\,e^{x} - 2x - 1)^{1/3}} - \frac{i\sqrt{3}}{2\,(c_1\,e^{x} - 2x - 1)^{1/3}}$$

$$y = -\frac{1}{2\,(c_1\,e^{x} - 2x - 1)^{1/3}} + \frac{i\sqrt{3}}{2\,(c_1\,e^{x} - 2x - 1)^{1/3}}$$

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) + \frac{y(x)}{3} = \frac{(-2x+1)y(x)^4}{3}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = -\frac{y(x)}{3} + \frac{(-2x+1)y(x)^4}{3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 61

```
dsolve(diff(y(x),x)+1/3*y(x) = 1/3*(-2*x+1)*y(x)^4,
       y(x),singsol=all)
```

$$y = \frac{1}{\left(\mathrm{e}^{x} c_1 - 2x - 1\right)^{1/3}}$$
$$y = -\frac{1 + i\sqrt{3}}{2\left(\mathrm{e}^{x} c_1 - 2x - 1\right)^{1/3}}$$
$$y = \frac{i\sqrt{3} - 1}{2\left(\mathrm{e}^{x} c_1 - 2x - 1\right)^{1/3}}$$

Mathematica DSolve solution

Solving time : 4.787 (sec)
Leaf size : 76

```
DSolve[{D[y[x],x]+y[x]/3== (1-2*x)/3*y[x]^4,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{\sqrt[3]{-2x + c_1 e^x - 1}}$$
$$y(x) \to -\frac{\sqrt[3]{-1}}{\sqrt[3]{-2x + c_1 e^x - 1}}$$
$$y(x) \to \frac{(-1)^{2/3}}{\sqrt[3]{-2x + c_1 e^x - 1}}$$
$$y(x) \to 0$$

2.1.21 problem 21

Internal problem ID [8409]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 21
Date solved : `Tuesday, December 17, 2024 at 12:50:42 PM`
CAS classification : `[[_1st_order, _with_linear_symmetries], _Chini]`

Solve

$$y' = \sqrt{y} + x$$

Solved as first order isobaric ode

Time used: 2.647 (sec)

Solving for y' gives

$$y' = \sqrt{y} + x \tag{1}$$

Each of the above ode's is now solved An ode $y' = f(x, y)$ is isobaric if

$$f(tx, t^m y) = t^{m-1} f(x, y) \tag{1}$$

Where here

$$f(x, y) = \sqrt{y} + x \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = 2$$

Since the ode is isobaric of order $m = 2$, then the substitution

$$\begin{aligned} y &= u x^m \\ &= u\, x^2 \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$2xu(x) + x^2u'(x) = \sqrt{x^2u(x)} + x$$

The ode $u'(x) = -\frac{2u(x)-\sqrt{u(x)}-1}{x}$ is separable as it can be written as

$$\begin{aligned}
u'(x) &= -\frac{2u(x) - \sqrt{u(x)} - 1}{x} \\
&= f(x)g(u)
\end{aligned}$$

Where

$$\begin{aligned}
f(x) &= \frac{1}{x} \\
g(u) &= -2u + \sqrt{u} + 1
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\
\int \frac{1}{-2u + \sqrt{u} + 1}\,du &= \int \frac{1}{x}\,dx \\
\ln\left(\frac{1}{\left(2\sqrt{u(x)}+1\right)^{1/3}\left(\sqrt{u(x)}-1\right)^{2/3}}\right) &= \ln(x) + c_1
\end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $-2u+\sqrt{u}+1 = 0$ for $u(x)$ gives

$$u(x) = 1$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned}
\ln\left(\frac{1}{\left(2\sqrt{u(x)}+1\right)^{1/3}\left(\sqrt{u(x)}-1\right)^{2/3}}\right) &= \ln(x) + c_1 \\
u(x) &= 1
\end{aligned}$$

Converting $\ln\left(\frac{1}{\left(2\sqrt{u(x)}+1\right)^{1/3}\left(\sqrt{u(x)}-1\right)^{2/3}}\right) = \ln(x) + c_1$ back to y gives

$$\ln\left(\frac{1}{\left(2\sqrt{\frac{y}{x^2}}+1\right)^{1/3}\left(\sqrt{\frac{y}{x^2}}-1\right)^{2/3}}\right) = \ln(x) + c_1$$

Converting $u(x) = 1$ back to y gives

$$\frac{y}{x^2} = 1$$

Solving for y gives

$$\ln\left(\frac{1}{\left(2\sqrt{\frac{y}{x^2}}+1\right)^{1/3}\left(\sqrt{\frac{y}{x^2}}-1\right)^{2/3}}\right) = \ln(x) + c_1$$

$$y = x^2$$

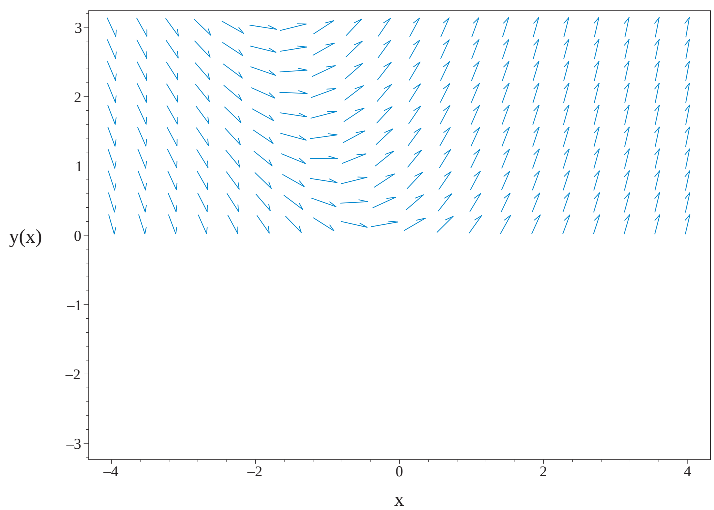


Figure 2.54: Slope field plot
$y' = \sqrt{y} + x$

Summary of solutions found

$$\ln\left(\frac{1}{\left(2\sqrt{\frac{y}{x^2}}+1\right)^{1/3}\left(\sqrt{\frac{y}{x^2}}-1\right)^{2/3}}\right) = \ln(x) + c_1$$

$$y = x^2$$

Solved using Lie symmetry for first order ode

Time used: 1.714 (sec)

Writing the ode as

$$
\begin{aligned}
y' &= \sqrt{y} + x \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
b_2 + \left(\sqrt{y} + x\right)(b_3 - a_2) - \left(\sqrt{y} + x\right)^2 a_3 - x a_2 - y a_3 - a_1 - \frac{x b_2 + y b_3 + b_1}{2\sqrt{y}} = 0 \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{4y^{3/2} a_3 + 4yx a_3 + 2\sqrt{y}\, x^2 a_3 + 2y a_2 - y b_3 + 4x a_2 \sqrt{y} - 2\sqrt{y}\, x b_3 + 2a_1 \sqrt{y} - 2b_2 \sqrt{y} + x b_2 + b_1}{2\sqrt{y}} = 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-4y^{3/2} a_3 - 2\sqrt{y}\, x^2 a_3 - 4x a_2 \sqrt{y} + 2\sqrt{y}\, x b_3 - 4yx a_3 \\
&- 2a_1 \sqrt{y} + 2b_2 \sqrt{y} - x b_2 - 2y a_2 + y b_3 - b_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{y}, y^{3/2}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \sqrt{y} = v_3, y^{3/2} = v_4\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-2v_3v_1^2a_3 - 4v_1a_2v_3 - 4v_2v_1a_3 + 2v_3v_1b_3 - 2a_1v_3 \\ &- 2v_2a_2 - 4v_4a_3 - v_1b_2 + 2b_2v_3 + v_2b_3 - b_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$\begin{aligned} &-2v_3v_1^2a_3 - 4v_2v_1a_3 + (-4a_2 + 2b_3)\, v_1v_3 - v_1b_2 \\ &+ (-2a_2 + b_3)\, v_2 + (-2a_1 + 2b_2)\, v_3 - 4v_4a_3 - b_1 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -4a_3 &= 0 \\ -2a_3 &= 0 \\ -b_1 &= 0 \\ -b_2 &= 0 \\ -2a_1 + 2b_2 &= 0 \\ -4a_2 + 2b_3 &= 0 \\ -2a_2 + b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= a_2 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= 2a_2 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = 2y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= 2y - \left(\sqrt{y}+x\right)(x) \\ &= -\sqrt{y}\,x - x^2 + 2y \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\sqrt{y}\,x - x^2 + 2y} dy\end{aligned}$$

Which results in

$$S = \frac{\ln\left(x+2\sqrt{y}\right)}{6} - \frac{\ln\left(-x+2\sqrt{y}\right)}{6} - \frac{\ln\left(\sqrt{y}+x\right)}{3} + \frac{\ln\left(\sqrt{y}-x\right)}{3} + \frac{\ln\left(-x^2+y\right)}{3} + \frac{\ln\left(-x^2+4y\right)}{6}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \sqrt{y} + x$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x^3 - 3xy - 2y^{3/2}}{(x^2 - 4y)(x^2 - y)} \\
S_y &= \frac{-x^2 + \sqrt{y}\,x + 2y}{(x^2 - 4y)(x^2 - y)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

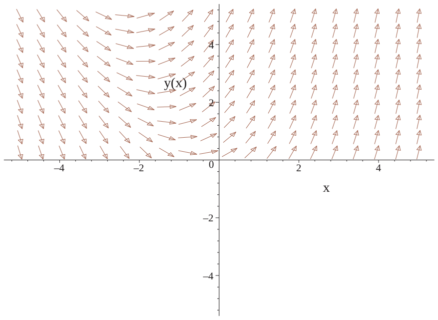
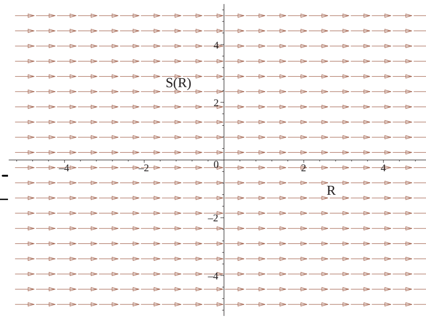
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned}
\int dS &= \int 0\, dR + c_2 \\
S(R) &= c_2
\end{aligned}$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\frac{\ln\left(x + 2\sqrt{y}\right)}{6} - \frac{\ln\left(-x + 2\sqrt{y}\right)}{6} - \frac{\ln\left(\sqrt{y} + x\right)}{3} + \frac{\ln\left(\sqrt{y} - x\right)}{3} + \frac{\ln\left(-x^2 + y\right)}{3} + \frac{\ln\left(-x^2 + 4y\right)}{6} = c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \sqrt{y} + x$ 	$R = x$ $S = \frac{\ln\left(x+2\sqrt{y}\right)}{6} - \frac{\ln\left(-\right.}{} \ldots \frac{\left(\sqrt{y} - \right.}{3}$	$\frac{dS}{dR} = 0$ 

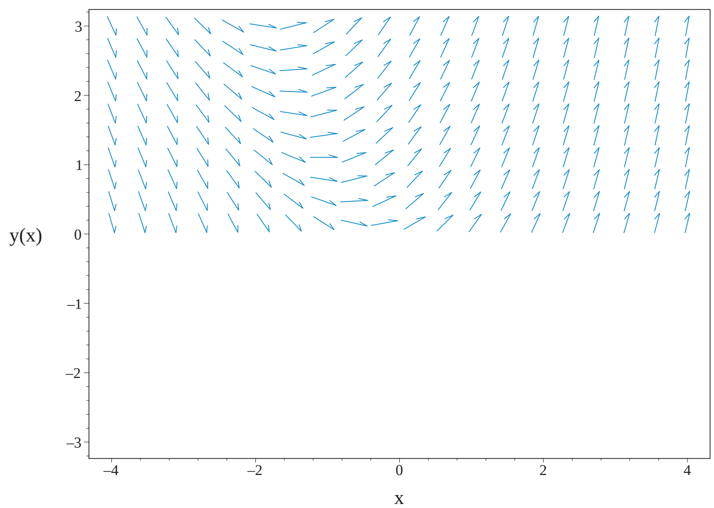


Figure 2.55: Slope field plot
$y' = \sqrt{y} + x$

Summary of solutions found

$$\frac{\ln\left(x+2\sqrt{y}\right)}{6} - \frac{\ln\left(-x+2\sqrt{y}\right)}{6} - \frac{\ln\left(\sqrt{y}+x\right)}{3} + \frac{\ln\left(\sqrt{y}-x\right)}{3} + \frac{\ln\left(-x^2+y\right)}{3} + \frac{\ln\left(-x^2+4y\right)}{6} = c_2$$

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) = \sqrt{y(x)} + x$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = \sqrt{y(x)} + x$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 69

```
dsolve(diff(y(x),x) = y(x)^(1/2)+x,
       y(x),singsol=all)
```

$$-\frac{2\,\mathrm{arctanh}\left(2\sqrt{\frac{y}{x^2}}\right)}{3} + \frac{4\,\mathrm{arctanh}\left(\sqrt{\frac{y}{x^2}}\right)}{3} - \frac{\ln\left(\frac{-x^2+4y}{x^2}\right)}{3} - \frac{2\ln(2)}{3} - \frac{2\ln\left(\frac{y-x^2}{x^2}\right)}{3} - 2\ln(x) + c_1 = 0$$

Mathematica DSolve solution

Solving time : 45.725 (sec)
Leaf size : 716

```
DSolve[{D[y[x],x]==Sqrt[y[x]]+x,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{4}\left(3x^2 + \frac{e^{3c_1}x(8+e^{3c_1}x^3)}{\sqrt[3]{-e^{18c_1}x^6 + 20e^{15c_1}x^3 + 8\sqrt{-e^{24c_1}(-1+e^{3c_1}x^3)^3} + 8e^{12c_1}}} + e^{-6c_1}\sqrt[3]{-e^{18c_1}x^6 + 20e^{15c_1}x^3 + 8\sqrt{-e^{24c_1}(-1+e^{3c_1}x^3)^3} + 8e^{12c_1}}\right)$$

$$y(x) \to \frac{1}{72}\left(54x^2 - \frac{9i(\sqrt{3}-i)\,e^{3c_1}x(8+e^{3c_1}x^3)}{\sqrt[3]{-e^{18c_1}x^6 + 20e^{15c_1}x^3 + 8\sqrt{-e^{24c_1}(-1+e^{3c_1}x^3)^3} + 8e^{12c_1}}} + 9i\left(\sqrt{3}+i\right)e^{-6c_1}\sqrt[3]{-e^{18c_1}x^6 + 20e^{15c_1}x^3 + 8\sqrt{-e^{24c_1}(-1+e^{3c_1}x^3)^3} + 8e^{12c_1}}\right)$$

$$y(x) \to \frac{1}{72}\left(54x^2 + \frac{9i(\sqrt{3}+i)\,e^{3c_1}x(8+e^{3c_1}x^3)}{\sqrt[3]{-e^{18c_1}x^6 + 20e^{15c_1}x^3 + 8\sqrt{-e^{24c_1}(-1+e^{3c_1}x^3)^3} + 8e^{12c_1}}} - 9\left(1 + i\sqrt{3}\right)e^{-6c_1}\sqrt[3]{-e^{18c_1}x^6 + 20e^{15c_1}x^3 + 8\sqrt{-e^{24c_1}(-1+e^{3c_1}x^3)^3} + 8e^{12c_1}}\right)$$

$$y(x) \to \frac{-(-x^6)^{2/3} + 3x^4 + \sqrt[3]{-x^6}x^2}{4x^2}$$

$$y(x) \to \frac{(1+i\sqrt{3})(-x^6)^{2/3} + 6x^4 + i(\sqrt{3}+i)\sqrt[3]{-x^6}x^2}{8x^2}$$

$$y(x) \to \frac{1}{8}x^2\left(\frac{(1+i\sqrt{3})x^4}{(-x^6)^{2/3}} + \frac{i(\sqrt{3}+i)x^2}{\sqrt[3]{-x^6}} + 6\right)$$

2.1.22 problem 23

Internal problem ID [8410]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 23
Date solved : `Tuesday, December 17, 2024 at 12:50:47 PM`
CAS classification :
`[[_homogeneous, ‘class A‘], _rational, [_Abel, ‘2nd type‘, ‘class B‘]]`

Solve

$$x^2y' + y^2 = xyy'$$

Solved as first order homogeneous class A ode

Time used: 0.333 (sec)

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y^2}{x(-x+y)} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = -y^2$ and $N = x(x-y)$ are both homogeneous and of the same order $n = 2$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}x}x + u &= \frac{u^2}{u-1} \\ \frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{\frac{u(x)^2}{u(x)-1} - u(x)}{x}\end{aligned}$$

Or

$$u'(x) - \frac{\frac{u(x)^2}{u(x)-1} - u(x)}{x} = 0$$

Or

$$u'(x)\,xu(x) - u'(x)\,x - u(x) = 0$$

Or

$$x(u(x) - 1)\,u'(x) - u(x) = 0$$

Which is now solved as separable in $u(x)$.

The ode $u'(x) = \frac{u(x)}{x(u(x)-1)}$ is separable as it can be written as

$$\begin{aligned}u'(x) &= \frac{u(x)}{x\,(u\,(x) - 1)} \\ &= f(x)g(u)\end{aligned}$$

Where

$$\begin{aligned}f(x) &= \frac{1}{x} \\ g(u) &= \frac{u}{u-1}\end{aligned}$$

Integrating gives

$$\begin{aligned}\int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\ \int \frac{u-1}{u}\,du &= \int \frac{1}{x}\,dx \\ u(x) + \ln\left(\frac{1}{u\,(x)}\right) &= \ln\,(x) + c_1\end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $\frac{u}{u-1} = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$u(x) + \ln\left(\frac{1}{u(x)}\right) = \ln(x) + c_1$$
$$u(x) = 0$$

Solving for $u(x)$ gives

$$u(x) = 0$$
$$u(x) = -\operatorname{LambertW}\left(-\frac{e^{-c_1}}{x}\right)$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = -\operatorname{LambertW}\left(-\frac{e^{-c_1}}{x}\right)$ back to y gives

$$y = -x \operatorname{LambertW}\left(-\frac{e^{-c_1}}{x}\right)$$

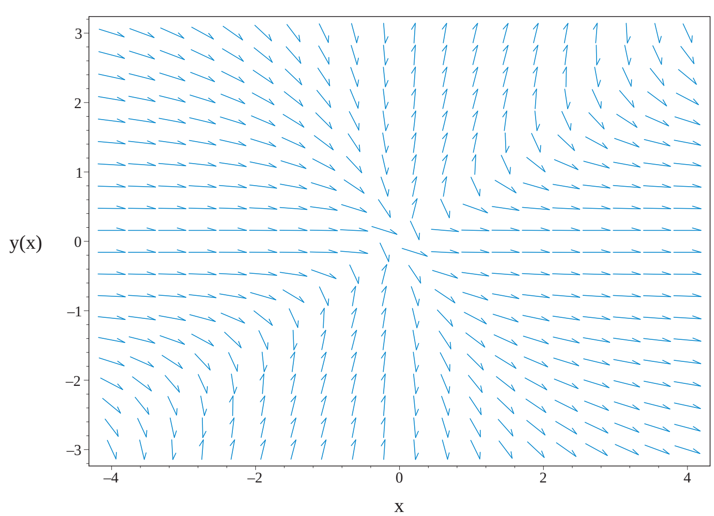


Figure 2.56: Slope field plot
$x^2y' + y^2 = xyy'$

Summary of solutions found

$$
\begin{aligned}
y &= 0 \\
y &= -x \operatorname{LambertW}\left(-\frac{\mathrm{e}^{-c_1}}{x}\right)
\end{aligned}
$$

Solved as first order homogeneous class D2 ode

Time used: 0.161 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$
x^2(u'(x)\,x + u(x)) + u(x)^2\,x^2 = x^2u(x)\,(u'(x)\,x + u(x))
$$

Which is now solved The ode $u'(x) = \frac{u(x)}{(u(x)-1)x}$ is separable as it can be written as

$$
\begin{aligned}
u'(x) &= \frac{u(x)}{(u\,(x) - 1)\,x} \\
&= f(x)g(u)
\end{aligned}
$$

Where

$$
\begin{aligned}
f(x) &= \frac{1}{x} \\
g(u) &= \frac{u}{u - 1}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\
\int \frac{u - 1}{u}\,du &= \int \frac{1}{x}\,dx \\
u(x) + \ln\left(\frac{1}{u\,(x)}\right) &= \ln\,(x) + c_1
\end{aligned}
$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $\frac{u}{u-1} = 0$ for $u(x)$ gives

$$
u(x) = 0
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$u(x)+\ln\left(\frac{1}{u(x)}\right)=\ln(x)+c_1$$
$$u(x)=0$$

Solving for $u(x)$ gives

$$u(x)=0$$
$$u(x)=-\operatorname{LambertW}\left(-\frac{\mathrm{e}^{-c_1}}{x}\right)$$

Converting $u(x)=0$ back to y gives

$$y=0$$

Converting $u(x)=-\operatorname{LambertW}\left(-\frac{\mathrm{e}^{-c_1}}{x}\right)$ back to y gives

$$y=-x\operatorname{LambertW}\left(-\frac{\mathrm{e}^{-c_1}}{x}\right)$$

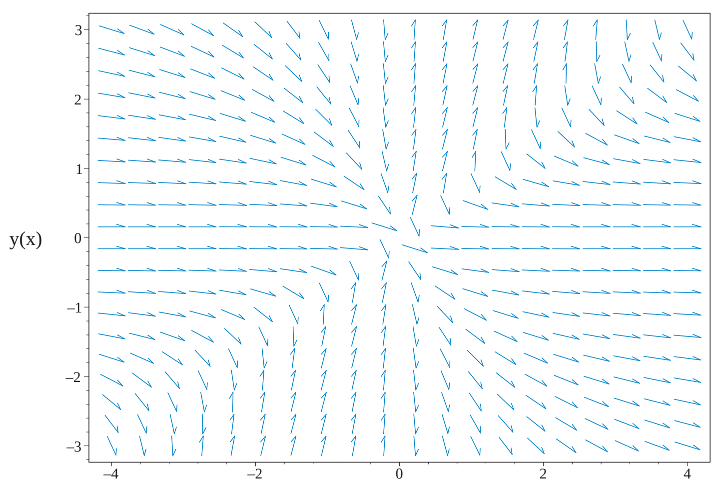


Figure 2.57: Slope field plot
$x^2y'+y^2=xyy'$

Summary of solutions found

$$y=0$$
$$y=-x\operatorname{LambertW}\left(-\frac{\mathrm{e}^{-c_1}}{x}\right)$$

Summary of solutions found

$$
\begin{aligned}
y &= 0 \\
y &= -x \,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{-c_1}}{x}\right)
\end{aligned}
$$

Solved as first order homogeneous class Maple C ode

Time used: 0.381 (sec)

Let $Y = y - y_0$ and $X = x - x_0$ then the above is transformed to new ode in $Y(X)$

$$
\frac{d}{dX}Y(X) = \frac{(Y(X) + y_0)^2}{(x_0 + X)(-x_0 - X + Y(X) + y_0)}
$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$
\begin{aligned}
x_0 &= 0 \\
y_0 &= 0
\end{aligned}
$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$
\frac{d}{dX}Y(X) = \frac{Y(X)^2}{-X^2 + XY(X)}
$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= \frac{Y^2}{X(Y - X)} \qquad (1)
\end{aligned}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = -Y^2$ and $N = X(X - Y)$ are both homogeneous and of the same order $n = 2$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{u^2}{u-1}\\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{u(X)^2}{u(X)-1} - u(X)}{X}\end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{u(X)^2}{u(X)-1} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right) Xu(X) - \left(\frac{d}{dX}u(X)\right) X - u(X) = 0$$

Or

$$X(u(X)-1)\left(\frac{d}{dX}u(X)\right) - u(X) = 0$$

Which is now solved as separable in $u(X)$.

The ode $\frac{d}{dX}u(X) = \frac{u(X)}{X(u(X)-1)}$ is separable as it can be written as

$$\begin{aligned}\frac{d}{dX}u(X) &= \frac{u(X)}{X\left(u\left(X\right)-1\right)}\\ &= f(X)g(u)\end{aligned}$$

Where

$$\begin{aligned}f(X) &= \frac{1}{X}\\ g(u) &= \frac{u}{u-1}\end{aligned}$$

Integrating gives

$$\begin{aligned}\int \frac{1}{g(u)}\,du &= \int f(X)\,dX\\ \int \frac{u-1}{u}\,du &= \int \frac{1}{X}\,dX\\ u(X) + \ln\left(\frac{1}{u\left(X\right)}\right) &= \ln\left(X\right) + c_1\end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $\frac{u}{u-1} = 0$ for $u(X)$ gives

$$u(X) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$u(X) + \ln\left(\frac{1}{u(X)}\right) = \ln(X) + c_1$$
$$u(X) = 0$$

Solving for $u(X)$ gives

$$u(X) = 0$$
$$u(X) = -\operatorname{LambertW}\left(-\frac{e^{-c_1}}{X}\right)$$

Converting $u(X) = 0$ back to $Y(X)$ gives

$$Y(X) = 0$$

Converting $u(X) = -\operatorname{LambertW}\left(-\frac{e^{-c_1}}{X}\right)$ back to $Y(X)$ gives

$$Y(X) = -X \operatorname{LambertW}\left(-\frac{e^{-c_1}}{X}\right)$$

Using the solution for $Y(X)$

$$Y(X) = 0 \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y$$
$$X = x$$

Then the solution in y becomes using EQ (A)

$$y = 0$$

Using the solution for $Y(X)$

$$Y(X) = -X \operatorname{LambertW}\left(-\frac{e^{-c_1}}{X}\right) \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y$$
$$X = x$$

Then the solution in y becomes using EQ (A)

$$y = -x \operatorname{LambertW}\left(-\frac{e^{-c_1}}{x}\right)$$

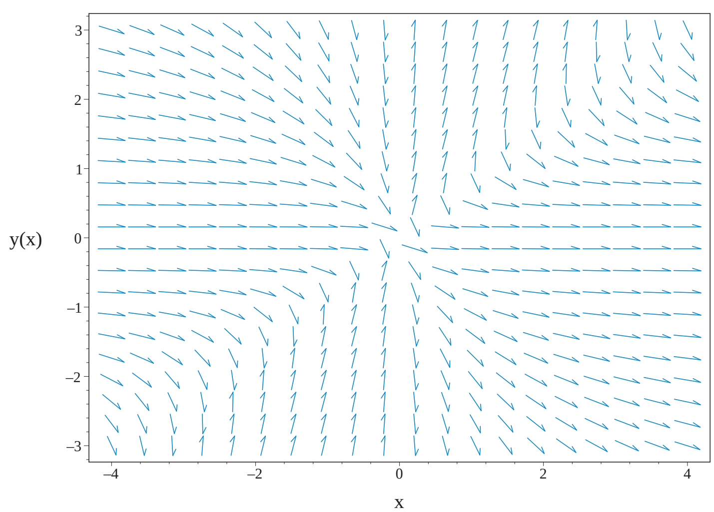


Figure 2.58: Slope field plot
$x^2 y' + y^2 = x y y'$

Solved as first order Exact ode

Time used: 0.222 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x^2-yx\right)\mathrm{d}y&=\left(-y^2\right)\mathrm{d}x\\ \left(y^2\right)\mathrm{d}x+\left(x^2-yx\right)\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=y^2\\ N(x,y)&=x^2-yx\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y^2)\\ &= 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2 - yx)\\ &= 2x - y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. By inspection $\frac{1}{x^2 y}$ is an integrating factor. Therefore by multiplying $M = y^2$ and $N = x^2 - yx$ by this integrating factor the ode becomes exact. The new M, N are

$$\begin{aligned}M &= \frac{y}{x^2}\\ N &= \frac{x^2 - yx}{x^2 y}\end{aligned}$$

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{x^2-yx}{x^2y}\right)\mathrm{d}y&=\left(-\frac{y}{x^2}\right)\mathrm{d}x\\ \left(\frac{y}{x^2}\right)\mathrm{d}x+\left(\frac{x^2-yx}{x^2y}\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=\frac{y}{x^2}\\ N(x,y)&=\frac{x^2-yx}{x^2y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(\frac{y}{x^2}\right)\\ &=\frac{1}{x^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}\left(\frac{x^2-yx}{x^2y}\right)\\ &=\frac{1}{x^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{y}{x^2} \, \mathrm{d}x$$

$$\phi = -\frac{y}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{1}{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x^2 - yx}{x^2 y}$. Therefore equation (4) becomes

$$\frac{x^2 - yx}{x^2 y} = -\frac{1}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$

$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{y}{x} + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -\frac{y}{x} + \ln(y)$$

Solving for y gives

$$y = \mathrm{e}^{-\mathrm{LambertW}\left(-\frac{\mathrm{e}^{c_1}}{x}\right)+c_1}$$

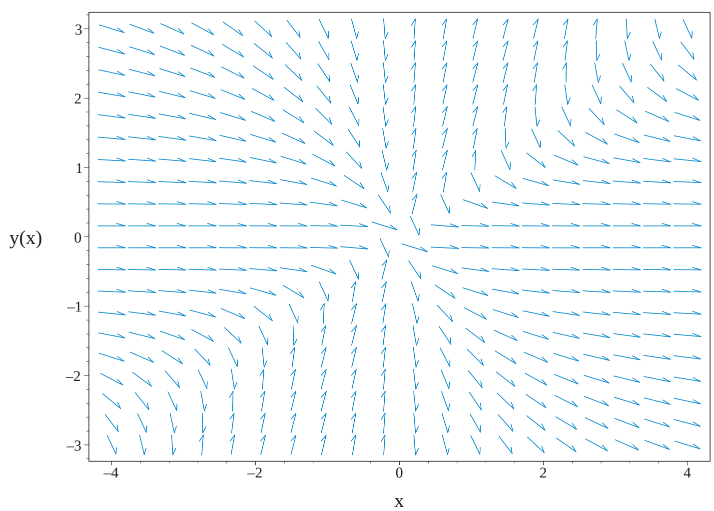


Figure 2.59: Slope field plot
$x^2y' + y^2 = xyy'$

Summary of solutions found

$$y = \mathrm{e}^{-\mathrm{LambertW}\left(-\frac{\mathrm{e}^{c_1}}{x}\right)+c_1}$$

Solved as first order isobaric ode

Time used: 0.122 (sec)

Solving for y' gives

$$y' = \frac{y^2}{x(-x+y)} \tag{1}$$

Each of the above ode's is now solved An ode $y' = f(x, y)$ is isobaric if

$$f(tx, t^m y) = t^{m-1} f(x, y) \tag{1}$$

Where here

$$f(x, y) = \frac{y^2}{x(-x+y)} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = 1$$

Since the ode is isobaric of order $m = 1$, then the substitution

$$\begin{aligned} y &= ux^m \\ &= ux \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$u(x) + xu'(x) = \frac{xu(x)^2}{-x + xu(x)}$$

The ode $u'(x) = \frac{u(x)}{(u(x)-1)x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= \frac{u(x)}{(u(x)-1)x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{1}{x} \\ g(u) &= \frac{u}{u-1} \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(u)}\, du = \int f(x)\, dx$$
$$\int \frac{u-1}{u}\, du = \int \frac{1}{x}\, dx$$
$$u(x) + \ln\left(\frac{1}{u(x)}\right) = \ln(x) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $\frac{u}{u-1} = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$u(x) + \ln\left(\frac{1}{u(x)}\right) = \ln(x) + c_1$$
$$u(x) = 0$$

Solving for $u(x)$ gives

$$u(x) = 0$$
$$u(x) = -\operatorname{LambertW}\left(-\frac{e^{-c_1}}{x}\right)$$

Converting $u(x) = 0$ back to y gives

$$\frac{y}{x} = 0$$

Converting $u(x) = -\operatorname{LambertW}\left(-\frac{e^{-c_1}}{x}\right)$ back to y gives

$$\frac{y}{x} = -\operatorname{LambertW}\left(-\frac{e^{-c_1}}{x}\right)$$

Solving for y gives

$$y = 0$$
$$y = -x \operatorname{LambertW}\left(-\frac{e^{-c_1}}{x}\right)$$

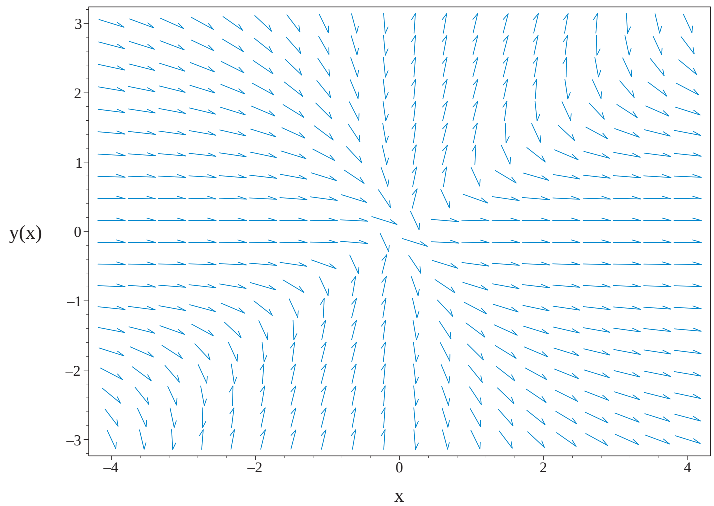


Figure 2.60: Slope field plot
$x^2y' + y^2 = xyy'$

Summary of solutions found

$$y = 0$$

$$y = -x \operatorname{LambertW}\left(-\frac{e^{-c_1}}{x}\right)$$

Solved using Lie symmetry for first order ode

Time used: 0.515 (sec)

Writing the ode as

$$y' = \frac{y^2}{x(-x+y)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{y^2(b_3 - a_2)}{x\,(-x+y)} - \frac{y^4 a_3}{x^2\,(-x+y)^2} \\ & - \left(-\frac{y^2}{x^2\,(-x+y)} + \frac{y^2}{x\,(-x+y)^2}\right)(x a_2 + y a_3 + a_1) \\ & - \left(\frac{2y}{x\,(-x+y)} - \frac{y^2}{x\,(-x+y)^2}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{x^4 b_2 - x^2 y^2 a_2 + x^2 y^2 b_3 - 2x\,y^3 a_3 + 2x^2 y b_1 - 2x\,y^2 a_1 - x\,y^2 b_1 + y^3 a_1}{x^2\,(x-y)^2} = 0$$

Setting the numerator to zero gives

$$x^4 b_2 - x^2 y^2 a_2 + x^2 y^2 b_3 - 2x\,y^3 a_3 + 2x^2 y b_1 - 2x\,y^2 a_1 - x\,y^2 b_1 + y^3 a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_2 v_1^2 v_2^2 - 2a_3 v_1 v_2^3 + b_2 v_1^4 + b_3 v_1^2 v_2^2 - 2a_1 v_1 v_2^2 + a_1 v_2^3 + 2b_1 v_1^2 v_2 - b_1 v_1 v_2^2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$b_2v_1^4 + (b_3 - a_2)\, v_1^2v_2^2 + 2b_1v_1^2v_2 - 2a_3v_1v_2^3 + (-2a_1 - b_1)\, v_1v_2^2 + a_1v_2^3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ b_2 &= 0 \\ -2a_3 &= 0 \\ 2b_1 &= 0 \\ -2a_1 - b_1 &= 0 \\ b_3 - a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= y - \left(\frac{y^2}{x\,(-x + y)}\right)(x) \\ &= \frac{xy}{x - y} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{xy}{x-y}} dy \end{aligned}$$

Which results in

$$S = -\frac{y}{x} + \ln(y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y^2}{x(-x+y)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y}{x^2} \\ S_y &= \frac{x-y}{xy} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int 0\, dR + c_2$$
$$S(R) = c_2$$

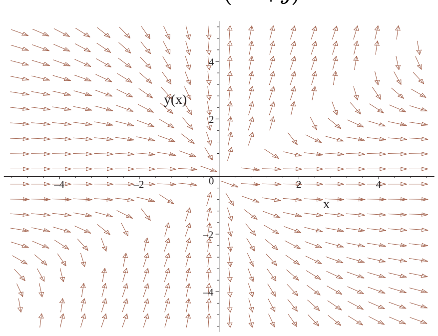
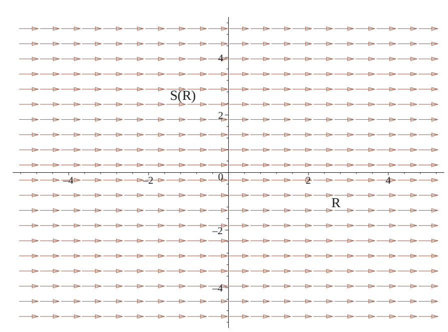
To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\frac{\ln(y)\, x - y}{x} = c_2$$

Which gives

$$y = \mathrm{e}^{-\operatorname{LambertW}\left(-\frac{\mathrm{e}^{c_2}}{x}\right)+c_2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2}{x(-x+y)}$ 	$R = x$ $S = \frac{\ln(y)\, x - y}{x}$	$\frac{dS}{dR} = 0$ 

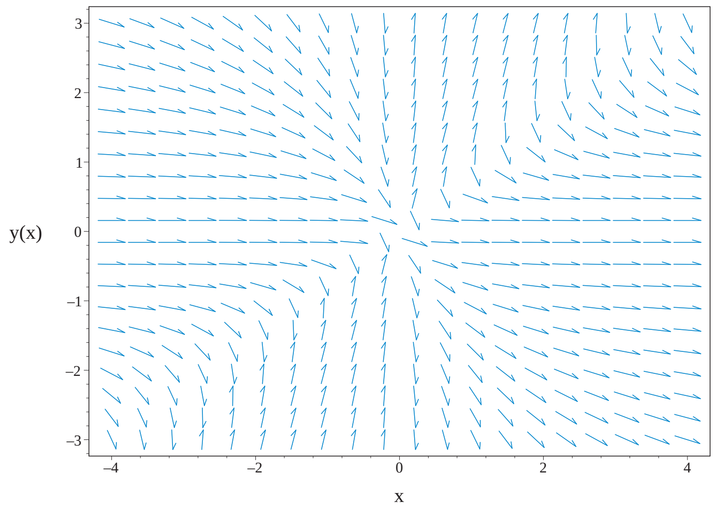


Figure 2.61: Slope field plot
$x^2y' + y^2 = xyy'$

Summary of solutions found

$$y = \mathrm{e}^{-\operatorname{LambertW}\left(-\frac{\mathrm{e}^{c_2}}{x}\right)+c_2}$$

Solved as first order ode of type dAlembert

Time used: 4.532 (sec)

Let $p = y'$ the ode becomes

$$x^2p + y^2 = xyp$$

Solving for y from the above results in

$$y = \left(\frac{p}{2} + \frac{\sqrt{p^2 - 4p}}{2}\right)x \tag{1}$$

$$y = \left(\frac{p}{2} - \frac{\sqrt{p^2 - 4p}}{2}\right)x \tag{2}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. Each of the above ode's is dAlembert ode which is now solved.

Solving ode 1A

Taking derivative of (*) w.r.t. x gives

$$
\begin{aligned}
p &= f + (xf' + g')\frac{dp}{dx} \\
p - f &= (xf' + g')\frac{dp}{dx} \qquad (2)
\end{aligned}
$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{p}{2} + \frac{\sqrt{p\,(p-4)}}{2} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
\frac{p}{2} - \frac{\sqrt{p\,(p-4)}}{2} = \left(\frac{x}{2} + \frac{xp}{2\sqrt{p^2-4p}} - \frac{x}{\sqrt{p^2-4p}}\right) p'(x) \qquad (2A)
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
\frac{p}{2} - \frac{\sqrt{p\,(p-4)}}{2} = 0
$$

Solving the above for p results in

$$
p_1 = 0
$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$
y = 0
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{\frac{p(x)}{2} - \frac{\sqrt{p(x)(p(x)-4)}}{2}}{\frac{x}{2} + \frac{xp(x)}{2\sqrt{p(x)^2-4p(x)}} - \frac{x}{\sqrt{p(x)^2-4p(x)}}} \qquad (3)
$$

This ODE is now solved for $p(x)$. No inversion is needed. The ode $p'(x) = \frac{\left(p(x)-\sqrt{p(x)(p(x)-4)}\right)\sqrt{p(x)(p(x)-4)}}{x\left(\sqrt{p(x)(p(x)-4)}+p(x)-2\right)}$ is separable as it can be written as

$$
\begin{aligned}
p'(x) &= \frac{\left(p(x) - \sqrt{p\,(x)\,(p\,(x)-4)}\right)\sqrt{p\,(x)\,(p\,(x)-4)}}{x\left(\sqrt{p\,(x)\,(p\,(x)-4)} + p\,(x) - 2\right)} \\
&= f(x)g(p)
\end{aligned}
$$

Where

$$
\begin{aligned}
f(x) &= \frac{1}{x} \\
g(p) &= \frac{\left(p-\sqrt{p\left(p-4\right)}\right)\sqrt{p\left(p-4\right)}}{\sqrt{p\left(p-4\right)}+p-2}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(p)}\, dp &= \int f(x)\, dx \\
\int \frac{\sqrt{p\left(p-4\right)}+p-2}{\left(p-\sqrt{p\left(p-4\right)}\right)\sqrt{p\left(p-4\right)}}\, dp &= \int \frac{1}{x}\, dx \\
\ln\left(\frac{1}{\sqrt{\sqrt{p\left(x\right)\left(p\left(x\right)-4\right)}+p\left(x\right)-2}\,\sqrt{p\left(x\right)}}\right)+\frac{\sqrt{p\left(x\right)\left(p\left(x\right)-4\right)}}{2}+\frac{p(x)}{2} &= \ln\left(x\right)+c_1
\end{aligned}
$$

We now need to find the singular solutions, these are found by finding for what values $g(p)$ is zero, since we had to divide by this above. Solving $g(p)=0$ or $\frac{\left(p-\sqrt{p(p-4)}\right)\sqrt{p(p-4)}}{\sqrt{p(p-4)}+p-2}=0$ for $p(x)$ gives

$$
\begin{aligned}
p(x) &= 0 \\
p(x) &= 4
\end{aligned}
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
\ln\left(\frac{1}{\sqrt{\sqrt{p\left(x\right)\left(p\left(x\right)-4\right)}+p\left(x\right)-2}\,\sqrt{p\left(x\right)}}\right)+\frac{\sqrt{p\left(x\right)\left(p\left(x\right)-4\right)}}{2}+\frac{p(x)}{2} &= \ln\left(x\right)+c_1 \\
p(x) &= 0 \\
p(x) &= 4
\end{aligned}
$$

Solving for $p(x)$ gives

$$
\begin{aligned}
p(x) &= 0 \\
p(x) &= 4 \\
p(x) &= -\frac{\operatorname{LambertW}\left(-\frac{\sqrt{2}\,\mathrm{e}^{-c_1}}{2x}\right)^2}{\operatorname{LambertW}\left(-\frac{\sqrt{2}\,\mathrm{e}^{-c_1}}{2x}\right)+1}
\end{aligned}
$$

$$p(x) = -\frac{\operatorname{LambertW}\left(\frac{\sqrt{2}\,e^{-c_1}}{2x}\right)^2}{\operatorname{LambertW}\left(\frac{\sqrt{2}\,e^{-c_1}}{2x}\right)+1}$$

Substituing the above solution for p in (2A) gives

$$y = 0$$
$$y = 2x$$
$$y = x\left(-\frac{\operatorname{LambertW}\left(-\frac{\sqrt{2}\,e^{-c_1}}{2x}\right)^2}{2\left(\operatorname{LambertW}\left(-\frac{\sqrt{2}\,e^{-c_1}}{2x}\right)+1\right)} + \frac{\sqrt{-\frac{\operatorname{LambertW}\left(-\frac{\sqrt{2}\,e^{-c_1}}{2x}\right)^2\left(-\frac{\operatorname{LambertW}\left(-\frac{\sqrt{2}\,e^{-c_1}}{2x}\right)^2}{\operatorname{LambertW}\left(-\frac{\sqrt{2}\,e^{-c_1}}{2x}\right)+1}-4\right)}{\operatorname{LambertW}\left(-\frac{\sqrt{2}\,e^{-c_1}}{2x}\right)+1}}}{2}\right)$$
$$y = x\left(-\frac{\operatorname{LambertW}\left(\frac{\sqrt{2}\,e^{-c_1}}{2x}\right)^2}{2\left(\operatorname{LambertW}\left(\frac{\sqrt{2}\,e^{-c_1}}{2x}\right)+1\right)} + \frac{\sqrt{-\frac{\operatorname{LambertW}\left(\frac{\sqrt{2}\,e^{-c_1}}{2x}\right)^2\left(-\frac{\operatorname{LambertW}\left(\frac{\sqrt{2}\,e^{-c_1}}{2x}\right)^2}{\operatorname{LambertW}\left(\frac{\sqrt{2}\,e^{-c_1}}{2x}\right)+1}-4\right)}{\operatorname{LambertW}\left(\frac{\sqrt{2}\,e^{-c_1}}{2x}\right)+1}}}{2}\right)$$

<u>Solving ode 2A</u>

Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = \frac{p}{2} - \frac{\sqrt{p\,(p-4)}}{2}$$
$$g = 0$$

Hence (2) becomes

$$\frac{p}{2} + \frac{\sqrt{p\,(p-4)}}{2} = \left(\frac{x}{2} - \frac{xp}{2\sqrt{p^2-4p}} + \frac{x}{\sqrt{p^2-4p}}\right)p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$\frac{p}{2} + \frac{\sqrt{p\,(p-4)}}{2} = 0$$

Solving the above for p results in

$$p_1 = 0$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$y = 0$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{\frac{p(x)}{2} + \frac{\sqrt{p(x)(p(x)-4)}}{2}}{\frac{x}{2} - \frac{xp(x)}{2\sqrt{p(x)^2-4p(x)}} + \frac{x}{\sqrt{p(x)^2-4p(x)}}} \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed. The ode $p'(x) = -\frac{\left(\sqrt{p(x)(p(x)-4)}+p(x)\right)\sqrt{p(x)(p(x)-4)}}{x\left(-\sqrt{p(x)(p(x)-4)}+p(x)-2\right)}$ is separable as it can be written as

$$\begin{aligned} p'(x) &= -\frac{\left(\sqrt{p\,(x)\,(p\,(x)-4)}+p(x)\right)\sqrt{p\,(x)\,(p\,(x)-4)}}{x\left(-\sqrt{p\,(x)\,(p\,(x)-4)}+p\,(x)-2\right)} \\ &= f(x)g(p) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(p) &= \frac{\left(p+\sqrt{p\,(p-4)}\right)\sqrt{p\,(p-4)}}{-\sqrt{p\,(p-4)}+p-2} \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(p)}\,dp &= \int f(x)\,dx \\ \int \frac{-\sqrt{p\,(p-4)}+p-2}{\left(p+\sqrt{p\,(p-4)}\right)\sqrt{p\,(p-4)}}\,dp &= \int -\frac{1}{x}\,dx \\ \ln\left(\frac{\sqrt{p\,(x)}}{\sqrt{\sqrt{p\,(x)\,(p\,(x)-4)}+p\,(x)-2}}\right) + \frac{\sqrt{p\,(x)\,(p\,(x)-4)}}{2} - \frac{p(x)}{2} &= \ln\left(\frac{1}{x}\right) + c_2 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(p)$ is zero, since we had to divide by this above. Solving $g(p) = 0$ or $\frac{\left(p+\sqrt{p(p-4)}\right)\sqrt{p(p-4)}}{-\sqrt{p(p-4)}+p-2} = 0$ for $p(x)$ gives

$$p(x) = 0$$
$$p(x) = 4$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\ln\left(\frac{\sqrt{p(x)}}{\sqrt{\sqrt{p(x)(p(x)-4)}+p(x)-2}}\right)+\frac{\sqrt{p(x)(p(x)-4)}}{2}-\frac{p(x)}{2} = \ln\left(\frac{1}{x}\right)+c_2$$
$$p(x) = 0$$
$$p(x) = 4$$

Substituing the above solution for p in (2A) gives

$$y = x\left(\frac{\left(\mathrm{RootOf}\left(\_Z^2x^2-2\_Z^2\mathrm{e}^{\frac{2c_2\_Z+2\_Z+4}{\_Z}}+4\_Zx^2+4x^2\right)+2\right)^2}{4\,\mathrm{RootOf}\left(\_Z^2x^2-2\_Z^2\mathrm{e}^{\frac{2c_2\_Z+2\_Z+4}{\_Z}}+4\_Zx^2+4x^2\right)}-\frac{\sqrt{2}\sqrt{\frac{\left(\mathrm{RootOf}\left(\_Z^2x^2-2\_Z^2\mathrm{e}^{\frac{2c_2\_}{}}\right.\right.}{}}}{}\right.$$

The solution

$$y = 2x$$

was found not to satisfy the ode or the IC. Hence it is removed.

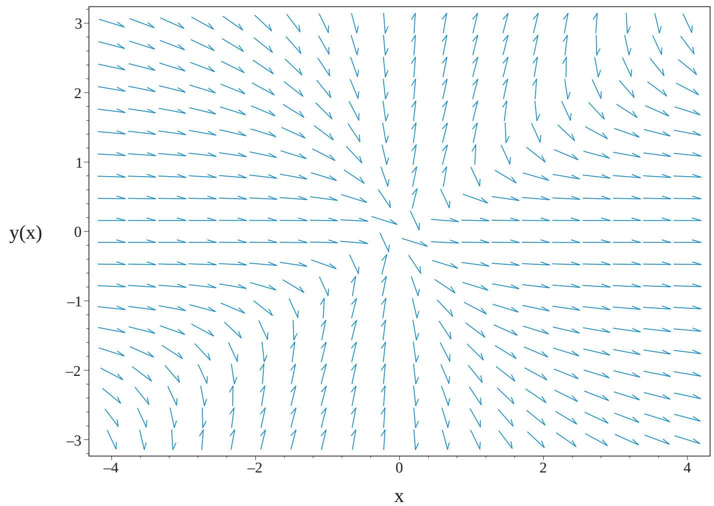


Figure 2.62: Slope field plot
$x^2y' + y^2 = xyy'$

Summary of solutions found

$$y = 0$$

$$y = x \left(\frac{\left(\operatorname{RootOf}\left(\_Z^2 x^2 - 2\_Z^2 \mathrm{e}^{\frac{2c_2\_Z + 2\_Z + 4}{\_Z}} + 4\_Z x^2 + 4x^2\right) + 2\right)^2}{4 \operatorname{RootOf}\left(\_Z^2 x^2 - 2\_Z^2 \mathrm{e}^{\frac{2c_2\_Z + 2\_Z + 4}{\_Z}} + 4\_Z x^2 + 4x^2\right)} \right.$$

$$- \frac{\sqrt{2}\sqrt{\frac{\left(\operatorname{RootOf}\left(\_Z^2 x^2 - 2\_Z^2 \mathrm{e}^{\frac{2c_2\_Z+2\_Z+4}{\_Z}} + 4\_Z x^2 + 4x^2\right) + 2\right)^2 \left(\frac{\left(\operatorname{RootOf}\left(\_Z^2 x^2 - 2\_Z^2 \mathrm{e}^{\frac{2c_2\_Z+2\_Z+4}{\_Z}} + 4\_Z x^2 + 4x^2\right) \cdots\right.}{2 \operatorname{RootOf}\left(\_Z^2 x^2 - 2\_Z^2 \mathrm{e}^{\frac{2c_2\_Z+2\_Z+4}{\_Z}} + 4\_Z x^2 + 4x^2 \cdots\right.} \cdots \right.}{\operatorname{RootOf}\left(\_Z^2 x^2 - 2\_Z^2 \mathrm{e}^{\frac{2c_2\_Z+2\_Z+4}{\_Z}} + 4\_Z x^2 + 4x^2\right)}}}{4}$$

$$y = x \left(-\frac{\operatorname{LambertW}\left(-\frac{\sqrt{2}\,\mathrm{e}^{-c_1}}{2x}\right)^2}{2\left(\operatorname{LambertW}\left(-\frac{\sqrt{2}\,\mathrm{e}^{-c_1}}{2x}\right) + 1\right)} + \frac{\sqrt{-\frac{\operatorname{LambertW}\left(-\frac{\sqrt{2}\,\mathrm{e}^{-c_1}}{2x}\right)^2 \left(-\frac{\operatorname{LambertW}\left(-\frac{\sqrt{2}\,\mathrm{e}^{-c_1}}{2x}\right)^2}{\operatorname{LambertW}\left(-\frac{\sqrt{2}\,\mathrm{e}^{-c_1}}{2x}\right)+1} - 4\right)}{\operatorname{LambertW}\left(-\frac{\sqrt{2}\,\mathrm{e}^{-c_1}}{2x}\right)+1}}}{2} \right)$$

$$y = x\left(-\frac{\mathrm{LambertW}\left(\frac{\sqrt{2}\,\mathrm{e}^{-c_1}}{2x}\right)^2}{2\left(\mathrm{LambertW}\left(\frac{\sqrt{2}\,\mathrm{e}^{-c_1}}{2x}\right)+1\right)} + \frac{\sqrt{-\frac{\mathrm{LambertW}\left(\frac{\sqrt{2}\,\mathrm{e}^{-c_1}}{2x}\right)^2\left(-\frac{\mathrm{LambertW}\left(\frac{\sqrt{2}\,\mathrm{e}^{-c_1}}{2x}\right)^2}{\mathrm{LambertW}\left(\frac{\sqrt{2}\,\mathrm{e}^{-c_1}}{2x}\right)+1}-4\right)}{\mathrm{LambertW}\left(\frac{\sqrt{2}\,\mathrm{e}^{-c_1}}{2x}\right)+1}}}{2}\right)$$

Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y(x)\right) + y(x)^2 = xy(x)\left(\frac{d}{dx}y(x)\right)$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = -\frac{y(x)^2}{-xy(x)+x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

Maple dsolve solution

Solving time : 0.042 (sec)
Leaf size : 17

```
dsolve(diff(y(x),x)*x^2+y(x)^2 = x*y(x)*diff(y(x),x),
       y(x),singsol=all)
```

$$
y = -x \operatorname{LambertW}\left(-\frac{e^{-c_1}}{x}\right)
$$

Mathematica DSolve solution

Solving time : 2.129 (sec)
Leaf size : 25

```
DSolve[{x^2*D[y[x],x]+y[x]^2==x*y[x]*D[y[x],x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
& y(x) \rightarrow -xW\left(-\frac{e^{-c_1}}{x}\right) \\
& y(x) \rightarrow 0
\end{aligned}
$$

2.1.23 problem 24

Internal problem ID [8411]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 24
Date solved : `Tuesday, December 17, 2024 at 12:50:55 PM`
CAS classification : `[_separable]`

Solve

$$y = xy' + x^2{y'}^2$$

Solving for the derivative gives these ODE's to solve

$$y' = \frac{-\frac{1}{2} + \frac{\sqrt{1+4y}}{2}}{x} \tag{1}$$

$$y' = \frac{-\frac{1}{2} - \frac{\sqrt{1+4y}}{2}}{x} \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

The ode $y' = \frac{-1+\sqrt{1+4y}}{2x}$ is separable as it can be written as

$$\begin{aligned} y' &= \frac{-1+\sqrt{1+4y}}{2x} \\ &= f(x)g(y) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{1}{x} \\ g(y) &= -\frac{1}{2} + \frac{\sqrt{1+4y}}{2} \end{aligned}$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(y)}\,dy &= \int f(x)\,dx \\
\int \frac{1}{-\frac{1}{2}+\frac{\sqrt{1+4y}}{2}}\,dy &= \int \frac{1}{x}\,dx \\
\sqrt{1+4y}+\ln\left(-1+\sqrt{1+4y}\right) &= \ln(x)+c_1
\end{aligned}
$$

We now need to find the singular solutions, these are found by finding for what values $g(y)$ is zero, since we had to divide by this above. Solving $g(y)=0$ or $-\frac{1}{2}+\frac{\sqrt{1+4y}}{2}=0$ for y gives

$$
y = 0
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
\sqrt{1+4y}+\ln\left(-1+\sqrt{1+4y}\right) &= \ln(x)+c_1 \\
y &= 0
\end{aligned}
$$

Solving for y gives

$$
\begin{aligned}
y &= 0 \\
y &= \frac{\operatorname{LambertW}\left(x\,\mathrm{e}^{c_1-1}\right)^2}{4}+\frac{\operatorname{LambertW}\left(x\,\mathrm{e}^{c_1-1}\right)}{2}
\end{aligned}
$$

We now need to find the singular solutions, these are found by finding for what values $\left(\frac{-\frac{1}{2}+\frac{\sqrt{1+4y}}{2}}{x}\right)$ is zero. These give

$$
y = 0
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

The solution $y=0$ satisfies the ode and initial conditions.

Solving Eq. (2)

The ode $y'=-\frac{1+\sqrt{1+4y}}{2x}$ is separable as it can be written as

$$
\begin{aligned}
y' &= -\frac{1+\sqrt{1+4y}}{2x} \\
&= f(x)g(y)
\end{aligned}
$$

Where

$$f(x) = \frac{1}{x}$$
$$g(y) = -\frac{\sqrt{1+4y}}{2} - \frac{1}{2}$$

Integrating gives

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$
$$\int \frac{1}{-\frac{\sqrt{1+4y}}{2} - \frac{1}{2}}\,dy = \int \frac{1}{x}\,dx$$
$$-\sqrt{1+4y} + \ln\left(1+\sqrt{1+4y}\right) = \ln(x) + c_2$$

Solving for y gives

$$y = \frac{\mathrm{LambertW}\left(-x\,\mathrm{e}^{c_2-1}\right)^2}{4} + \frac{\mathrm{LambertW}\left(-x\,\mathrm{e}^{c_2-1}\right)}{2}$$

Maple step by step solution

Let's solve

$y(x) = x\left(\frac{d}{dx}y(x)\right) + x^2\left(\frac{d}{dx}y(x)\right)^2$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\left[\frac{d}{dx}y(x) = \frac{-\frac{1}{2}-\frac{\sqrt{1+4y(x)}}{2}}{x}, \frac{d}{dx}y(x) = \frac{-\frac{1}{2}+\frac{\sqrt{1+4y(x)}}{2}}{x}\right]$

□ Solve the equation $\frac{d}{dx}y(x) = \frac{-\frac{1}{2}-\frac{\sqrt{1+4y(x)}}{2}}{x}$

  ◦ Separate variables

  $\frac{\frac{d}{dx}y(x)}{-\frac{1}{2}-\frac{\sqrt{1+4y(x)}}{2}} = \frac{1}{x}$

  ◦ Integrate both sides with respect to x

  $\int \frac{\frac{d}{dx}y(x)}{-\frac{1}{2}-\frac{\sqrt{1+4y(x)}}{2}}dx = \int \frac{1}{x}dx + \_C1$

  ◦ Evaluate integral

  $\frac{\ln(y(x))}{2} - \sqrt{1+4y(x)} - \frac{\ln\left(-1+\sqrt{1+4y(x)}\right)}{2} + \frac{\ln\left(1+\sqrt{1+4y(x)}\right)}{2} = \ln(x) + \_C1$

- ☐ Solve the equation $\frac{d}{dx}y(x) = \frac{-\frac{1}{2}+\frac{\sqrt{1+4y(x)}}{2}}{x}$
  - Separate variables
    $\frac{\frac{d}{dx}y(x)}{-\frac{1}{2}+\frac{\sqrt{1+4y(x)}}{2}} = \frac{1}{x}$
  - Integrate both sides with respect to x
    $\int \frac{\frac{d}{dx}y(x)}{-\frac{1}{2}+\frac{\sqrt{1+4y(x)}}{2}}dx = \int \frac{1}{x}dx + \_C1$
  - Evaluate integral
    $\frac{\ln(y(x))}{2} + \sqrt{1+4y(x)} + \frac{\ln\left(-1+\sqrt{1+4y(x)}\right)}{2} - \frac{\ln\left(1+\sqrt{1+4y(x)}\right)}{2} = \ln(x) + \_C1$
- Set of solutions
  $\left\{\frac{\ln(y(x))}{2} - \sqrt{1+4y(x)} - \frac{\ln\left(-1+\sqrt{1+4y(x)}\right)}{2} + \frac{\ln\left(1+\sqrt{1+4y(x)}\right)}{2} = \ln(x) + C1, \frac{\ln(y(x))}{2} + \sqrt{1+4y(x)}\right.$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
trying simple symmetries for implicit equations
<- symmetries for implicit equations successful`
```

Maple dsolve solution

Solving time : 0.048 (sec)
Leaf size : 97

```
dsolve(y(x) = diff(y(x),x)*x+x^2*diff(y(x),x)^2,
       y(x),singsol=all)
```

$$\ln(x) - \sqrt{4y+1} - \frac{\ln\left(-1+\sqrt{4y+1}\right)}{2} + \frac{\ln\left(1+\sqrt{4y+1}\right)}{2} - \frac{\ln(y)}{2} - c_1 = 0$$
$$\ln(x) + \sqrt{4y+1} + \frac{\ln\left(-1+\sqrt{4y+1}\right)}{2} - \frac{\ln\left(1+\sqrt{4y+1}\right)}{2} - \frac{\ln(y)}{2} - c_1 = 0$$

Mathematica DSolve solution

Solving time : 18.695 (sec)
Leaf size : 72

```
DSolve[{y[x]==x*D[y[x],x]+x^2*(D[y[x],x])^2,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
y(x) &\to \frac{1}{4} W\left(-e^{-1-2c_1} x\right) \left(2 + W\left(-e^{-1-2c_1} x\right)\right) \\
y(x) &\to \frac{1}{4} W\left(e^{-1+2c_1} x\right) \left(2 + W\left(e^{-1+2c_1} x\right)\right) \\
y(x) &\to 0
\end{aligned}
$$

2.1.24 problem 25

Internal problem ID [8412]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 25
Date solved : `Tuesday, December 17, 2024 at 12:50:56 PM`
CAS classification : `[_quadrature]`

Solve

$$(x+y)\,y' = 0$$

Factoring the ode gives these factors

$$x+y = 0 \tag{1}$$

$$y' = 0 \tag{2}$$

Now each of the above equations is solved in turn.

Solving equation (1)

Solving for y from

$$x+y = 0$$

Solving gives $y = -x$

Solving equation (2)

Solved as first order quadrature ode

Time used: 0.013 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\, dx + c_1$$
$$y = c_1$$

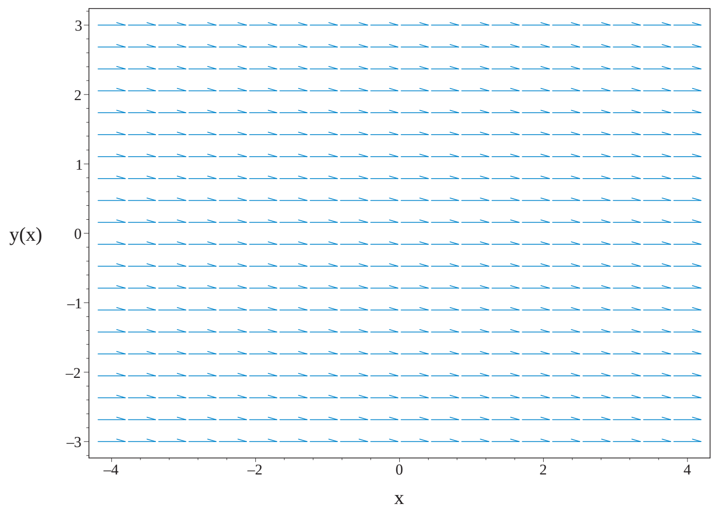


Figure 2.63: Slope field plot
$y' = 0$

Summary of solutions found

$$y = c_1$$

Solved as first order homogeneous class D2 ode

Time used: 0.151 (sec)

Applying change of variables $y = u(x)\, x$, then the ode becomes

$$u'(x)\, x + u(x) = 0$$

Which is now solved The ode $u'(x) = -\frac{u(x)}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$
\begin{aligned}
f(x) &= -\frac{1}{x} \\
g(u) &= u
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(u)}\, du &= \int f(x)\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{1}{x}\, dx \\
\ln(u(x)) &= \ln\left(\frac{1}{x}\right) + c_1
\end{aligned}
$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$
u(x) = 0
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
\ln(u(x)) &= \ln\left(\frac{1}{x}\right) + c_1 \\
u(x) &= 0
\end{aligned}
$$

Solving for $u(x)$ gives

$$
\begin{aligned}
u(x) &= 0 \\
u(x) &= \frac{e^{c_1}}{x}
\end{aligned}
$$

Converting $u(x) = 0$ back to y gives

$$
y = 0
$$

Converting $u(x) = \frac{e^{c_1}}{x}$ back to y gives

$$
y = e^{c_1}
$$

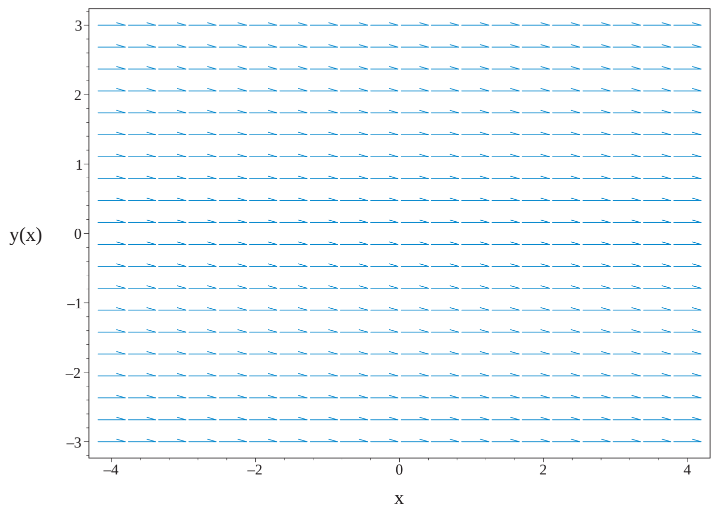


Figure 2.64: Slope field plot
$y' = 0$

Summary of solutions found

$$y = 0$$

$$y = e^{c_1}$$

Solved as first order ode of type differential

Time used: 0.011 (sec)

Writing the ode as

$$y' = 0 \tag{1}$$

Which becomes

$$(1)\, dy = (0)\, dx \tag{2}$$

But the RHS is complete differential because

$$(0)\, dx = d(0)$$

Hence (2) becomes

$$(1)\, dy = d(0)$$

Integrating gives

$$y = c_1$$

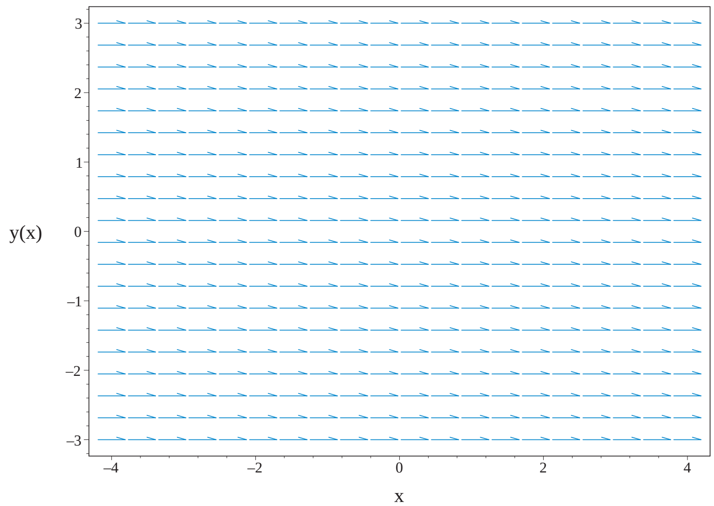


Figure 2.65: Slope field plot
$y' = 0$

Summary of solutions found

$$y = c_1$$

Maple step by step solution

Let's solve

$(x + y(x))\left(\frac{d}{dx}y(x)\right) = 0$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = 0$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}y(x)\right) dx = \int 0 dx + C1$

- Evaluate integral

  $y(x) = C1$

- Solve for $y(x)$

  $y(x) = C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 11

```
dsolve((x+y(x))*diff(y(x),x) = 0,
       y(x),singsol=all)
```

$$y = -x$$
$$y = c_1$$

Mathematica DSolve solution

Solving time : 0.002 (sec)
Leaf size : 14

```
DSolve[{(x+y[x])*D[y[x],x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -x$$
$$y(x) \to c_1$$

2.1.25 problem 26

Internal problem ID [8413]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 26
Date solved : `Tuesday, December 17, 2024 at 12:50:57 PM`
CAS classification : `[_quadrature]`

Solve

$$xy' = 0$$

Solved as first order quadrature ode

Time used: 0.024 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\,dx + c_1$$
$$y = c_1$$

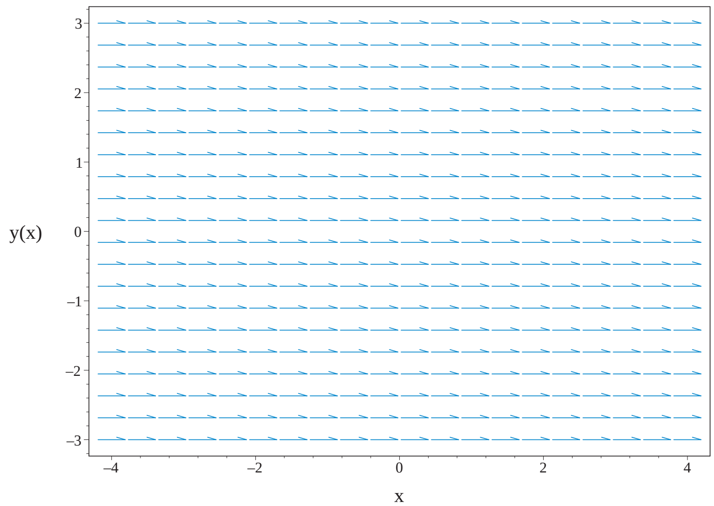


Figure 2.66: Slope field plot
$xy' = 0$

Summary of solutions found

$$y = c_1$$

Solved as first order homogeneous class D2 ode

Time used: 0.132 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$x(u'(x)\,x + u(x)) = 0$$

Which is now solved The ode $u'(x) = -\frac{u(x)}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(u) &= u \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(u)}\,du = \int f(x)\,dx$$
$$\int \frac{1}{u}\,du = \int -\frac{1}{x}\,dx$$
$$\ln(u(x)) = \ln\left(\frac{1}{x}\right) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\ln(u(x)) = \ln\left(\frac{1}{x}\right) + c_1$$
$$u(x) = 0$$

Solving for $u(x)$ gives

$$u(x) = 0$$
$$u(x) = \frac{e^{c_1}}{x}$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = \frac{e^{c_1}}{x}$ back to y gives

$$y = e^{c_1}$$

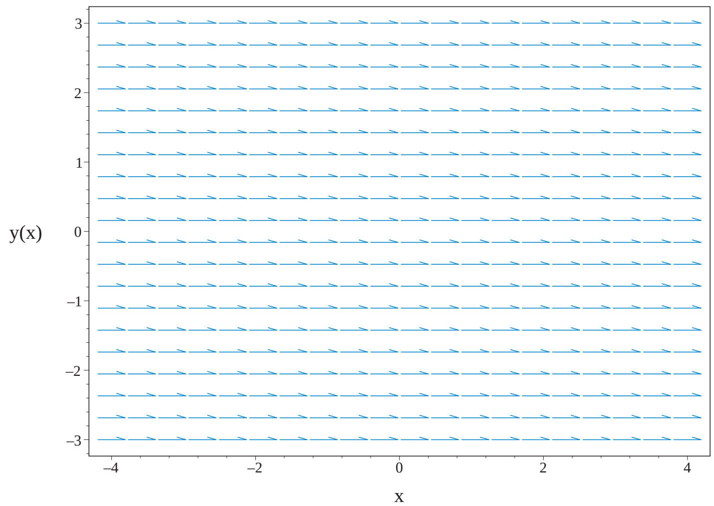


Figure 2.67: Slope field plot
$xy' = 0$

Summary of solutions found

$$y = 0$$

$$y = e^{c_1}$$

Solved as first order ode of type differential

Time used: 0.010 (sec)

Writing the ode as

$$y' = 0 \tag{1}$$

Which becomes

$$(1)\, dy = (0)\, dx \tag{2}$$

But the RHS is complete differential because

$$(0)\, dx = d(0)$$

Hence (2) becomes

$$(1)\, dy = d(0)$$

Integrating gives

$$y = c_1$$

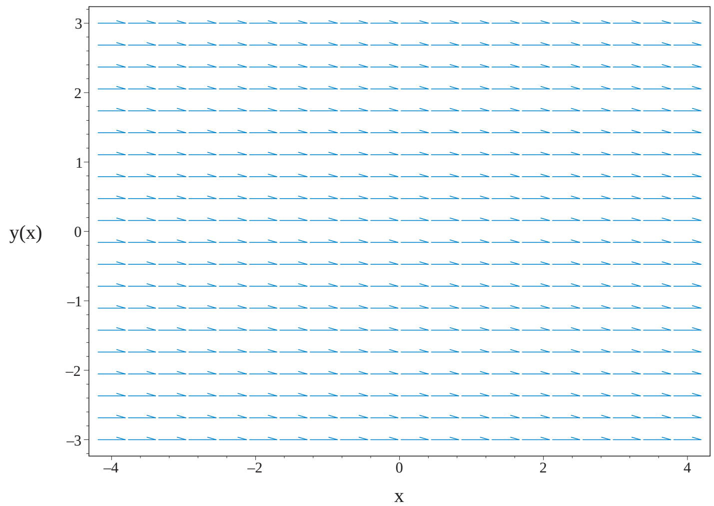


Figure 2.68: Slope field plot
$xy' = 0$

Summary of solutions found

$$y = c_1$$

Maple step by step solution

Let's solve
$x\left(\frac{d}{dx}y(x)\right) = 0$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = 0$
- Integrate both sides with respect to x
  $\int \left(\frac{d}{dx}y(x)\right) dx = \int 0dx + C1$
- Evaluate integral
  $y(x) = C1$
- Solve for $y(x)$
  $y(x) = C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 5

```
dsolve(diff(y(x),x)*x = 0,
       y(x),singsol=all)
```

$$y = c_1$$

Mathematica DSolve solution

Solving time : 0.002 (sec)
Leaf size : 7

```
DSolve[{x*D[y[x],x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1$$

2.1.26 problem 27

Internal problem ID [8414]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 27
Date solved : `Tuesday, December 17, 2024 at 12:50:57 PM`
CAS classification : `[_quadrature]`

Solve

$$\frac{y'}{x+y} = 0$$

Factoring the ode gives these factors

$$\frac{1}{x+y} = 0 \tag{1}$$

$$y' = 0 \tag{2}$$

Now each of the above equations is solved in turn.

Solving equation (1)

Solving for y from

$$\frac{1}{x+y} = 0$$

Unable to solve for y from the above.

Solving equation (2)

Solved as first order quadrature ode

Time used: 0.017 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\, dx + c_1$$
$$y = c_1$$

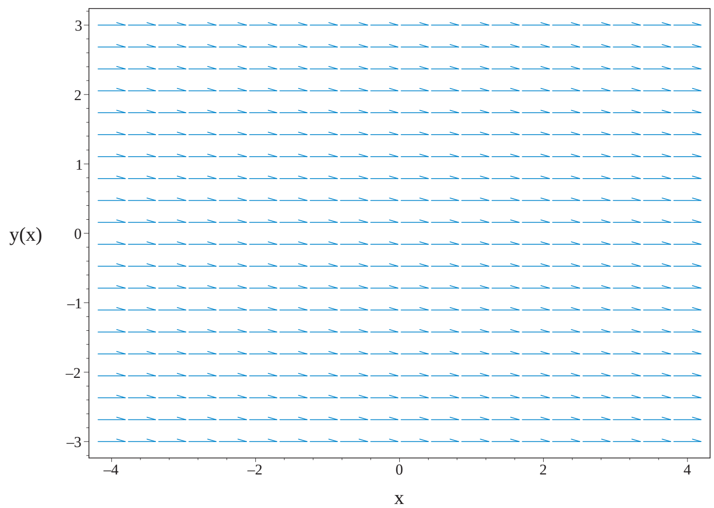


Figure 2.69: Slope field plot
$y' = 0$

Summary of solutions found

$$y = c_1$$

Solved as first order homogeneous class D2 ode

Time used: 0.150 (sec)

Applying change of variables $y = u(x)\, x$, then the ode becomes

$$u'(x)\, x + u(x) = 0$$

Which is now solved The ode $u'(x) = -\frac{u(x)}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$
\begin{aligned}
f(x) &= -\frac{1}{x} \\
g(u) &= u
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(u)}\, du &= \int f(x)\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{1}{x}\, dx \\
\ln(u(x)) &= \ln\left(\frac{1}{x}\right) + c_1
\end{aligned}
$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$
u(x) = 0
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
\ln(u(x)) &= \ln\left(\frac{1}{x}\right) + c_1 \\
u(x) &= 0
\end{aligned}
$$

Solving for $u(x)$ gives

$$
\begin{aligned}
u(x) &= 0 \\
u(x) &= \frac{e^{c_1}}{x}
\end{aligned}
$$

Converting $u(x) = 0$ back to y gives

$$
y = 0
$$

Converting $u(x) = \frac{e^{c_1}}{x}$ back to y gives

$$
y = e^{c_1}
$$

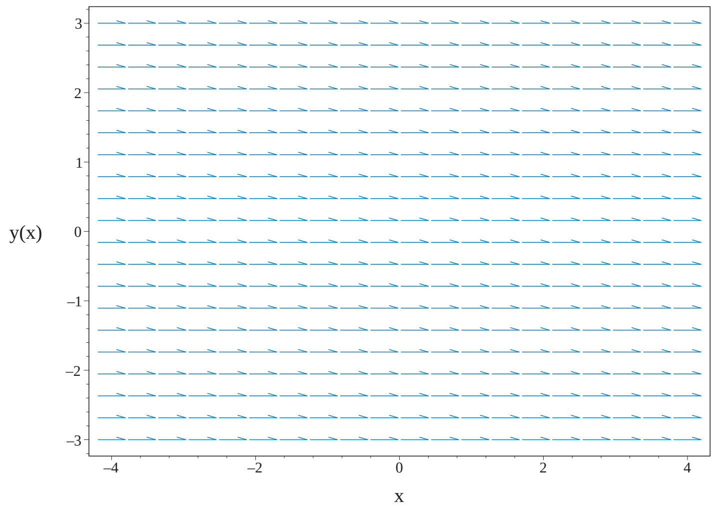


Figure 2.70: Slope field plot
$y' = 0$

Summary of solutions found

$$y = 0$$

$$y = e^{c_1}$$

Solved as first order ode of type differential

Time used: 0.010 (sec)

Writing the ode as

$$y' = 0 \tag{1}$$

Which becomes

$$(1)\, dy = (0)\, dx \tag{2}$$

But the RHS is complete differential because

$$(0)\, dx = d(0)$$

Hence (2) becomes

$$(1)\, dy = d(0)$$

Integrating gives

$$y = c_1$$

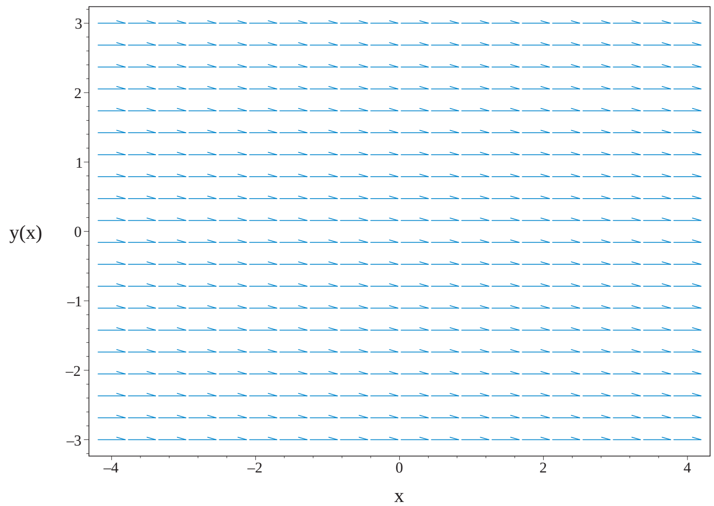


Figure 2.71: Slope field plot
$y' = 0$

Summary of solutions found

$$y = c_1$$

Maple step by step solution

Let's solve
$\frac{\frac{d}{dx}y(x)}{x+y(x)} = 0$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = 0$
- Integrate both sides with respect to x
  $\int \left(\frac{d}{dx}y(x)\right) dx = \int 0 dx + C1$
- Evaluate integral
  $y(x) = C1$
- Solve for $y(x)$
  $y(x) = C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 5

```
dsolve(1/(x+y(x))*diff(y(x),x) = 0,
       y(x),singsol=all)
```

$$y = c_1$$

Mathematica DSolve solution

Solving time : 0.002 (sec)
Leaf size : 7

```
DSolve[{1/(x+y[x])*D[y[x],x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1$$

2.1.27 problem 28

Internal problem ID [8415]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 28
Date solved : `Tuesday, December 17, 2024 at 12:50:58 PM`
CAS classification : `[_quadrature]`

Solve

$$\frac{y'}{x} = 0$$

Solved as first order quadrature ode

Time used: 0.024 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\, dx + c_1$$
$$y = c_1$$

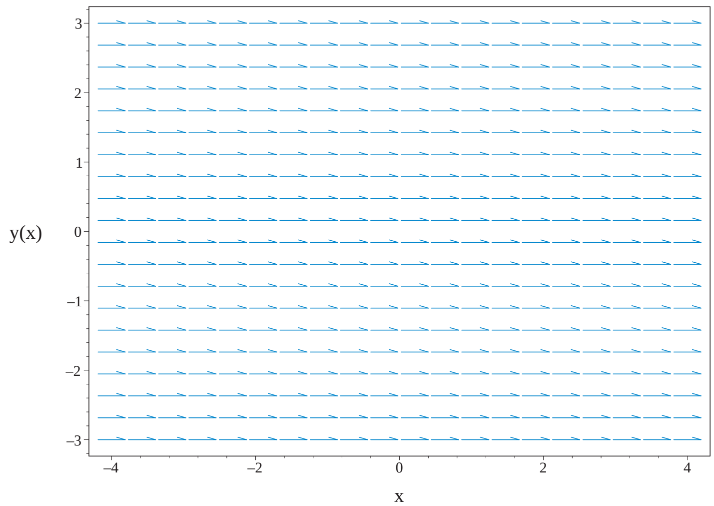


Figure 2.72: Slope field plot
$\frac{y'}{x} = 0$

Summary of solutions found

$$y = c_1$$

Solved as first order homogeneous class D2 ode

Time used: 0.132 (sec)

Applying change of variables $y = u(x)\, x$, then the ode becomes

$$\frac{u'(x)\, x + u(x)}{x} = 0$$

Which is now solved The ode $u'(x) = -\frac{u(x)}{x}$ is separable as it can be written as

$$\begin{aligned}u'(x) &= -\frac{u(x)}{x}\\ &= f(x)g(u)\end{aligned}$$

Where

$$\begin{aligned}f(x) &= -\frac{1}{x}\\ g(u) &= u\end{aligned}$$

Integrating gives

$$\int \frac{1}{g(u)}\,du = \int f(x)\,dx$$
$$\int \frac{1}{u}\,du = \int -\frac{1}{x}\,dx$$
$$\ln(u(x)) = \ln\left(\frac{1}{x}\right) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\ln(u(x)) = \ln\left(\frac{1}{x}\right) + c_1$$
$$u(x) = 0$$

Solving for $u(x)$ gives

$$u(x) = 0$$
$$u(x) = \frac{e^{c_1}}{x}$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = \frac{e^{c_1}}{x}$ back to y gives

$$y = e^{c_1}$$

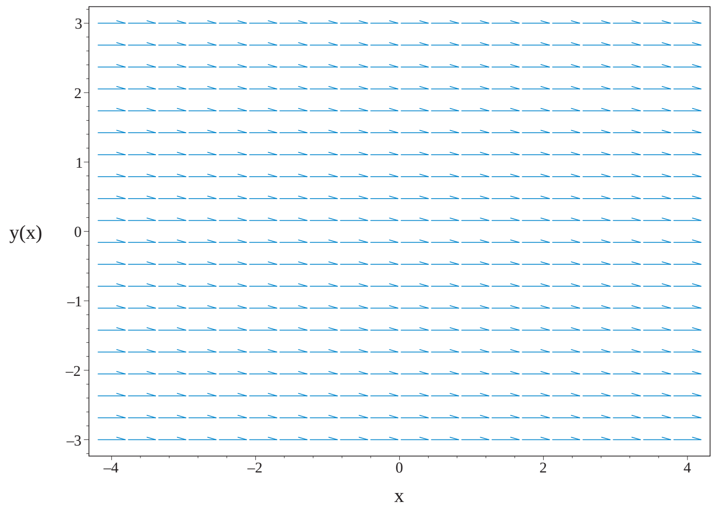


Figure 2.73: Slope field plot
$\frac{y'}{x} = 0$

Summary of solutions found

$$y = 0$$
$$y = e^{c_1}$$

Solved as first order ode of type differential

Time used: 0.010 (sec)

Writing the ode as

$$y' = 0 \tag{1}$$

Which becomes

$$(1)\, dy = (0)\, dx \tag{2}$$

But the RHS is complete differential because

$$(0)\, dx = d(0)$$

Hence (2) becomes

$$(1)\, dy = d(0)$$

Integrating gives

$$y = c_1$$

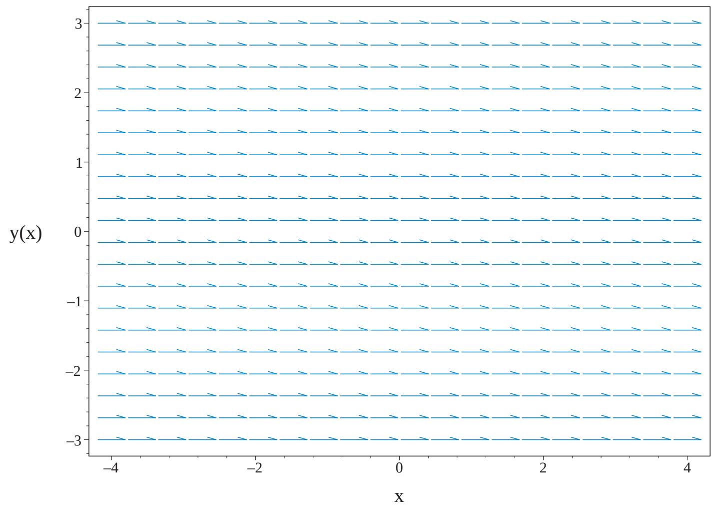


Figure 2.74: Slope field plot
$\frac{y'}{x} = 0$

Summary of solutions found

$$y = c_1$$

Maple step by step solution

Let's solve
$\frac{\frac{d}{dx}y(x)}{x} = 0$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = 0$
- Integrate both sides with respect to x
  $\int \left(\frac{d}{dx}y(x)\right) dx = \int 0dx + C1$
- Evaluate integral
  $y(x) = C1$
- Solve for $y(x)$
  $y(x) = C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 5

```
dsolve(1/x*diff(y(x),x) = 0,
       y(x),singsol=all)
```

$$y = c_1$$

Mathematica DSolve solution

Solving time : 0.002 (sec)
Leaf size : 7

```
DSolve[{1/x*D[y[x],x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1$$

2.1.28 problem 29

Internal problem ID [8416]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 29
Date solved : `Tuesday, December 17, 2024 at 12:50:59 PM`
CAS classification : `[_quadrature]`

Solve

$$y' = 0$$

Solved as first order quadrature ode

Time used: 0.026 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\,dx + c_1$$
$$y = c_1$$

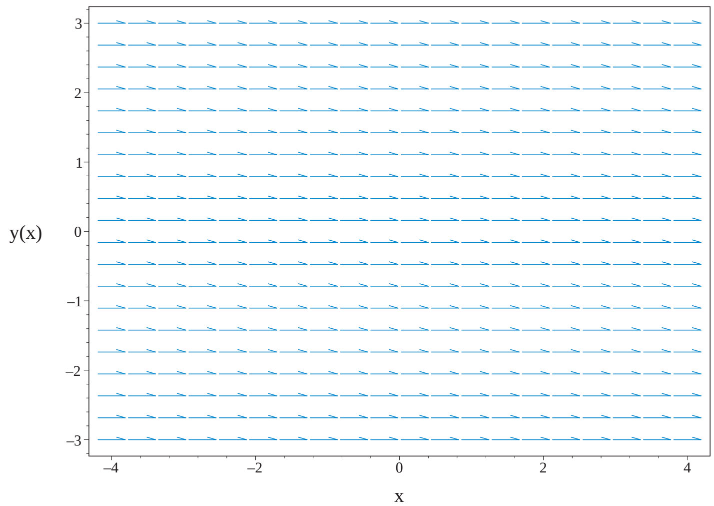


Figure 2.75: Slope field plot
$y' = 0$

Summary of solutions found

$$y = c_1$$

Solved as first order homogeneous class D2 ode

Time used: 0.153 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$u'(x)\,x + u(x) = 0$$

Which is now solved The ode $u'(x) = -\frac{u(x)}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(u) &= u \end{aligned}$$

Integrating gives

$$\begin{aligned}\int \frac{1}{g(u)} du &= \int f(x) dx \\ \int \frac{1}{u} du &= \int -\frac{1}{x} dx \\ \ln(u(x)) &= \ln\left(\frac{1}{x}\right) + c_1\end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned}\ln(u(x)) = \ln\left(\frac{1}{x}\right) + c_1 \\ u(x) = 0\end{aligned}$$

Solving for $u(x)$ gives

$$\begin{aligned}u(x) &= 0 \\ u(x) &= \frac{e^{c_1}}{x}\end{aligned}$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = \frac{e^{c_1}}{x}$ back to y gives

$$y = e^{c_1}$$

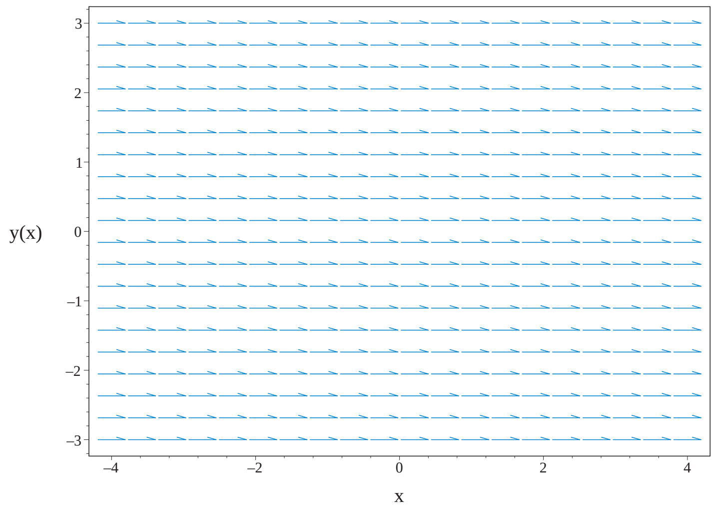


Figure 2.76: Slope field plot
$y' = 0$

Summary of solutions found

$$y = 0$$

$$y = e^{c_1}$$

Solved as first order ode of type differential

Time used: 0.010 (sec)

Writing the ode as

$$y' = 0 \tag{1}$$

Which becomes

$$(1)\, dy = (0)\, dx \tag{2}$$

But the RHS is complete differential because

$$(0)\, dx = d(0)$$

Hence (2) becomes

$$(1)\, dy = d(0)$$

Integrating gives

$$y = c_1$$

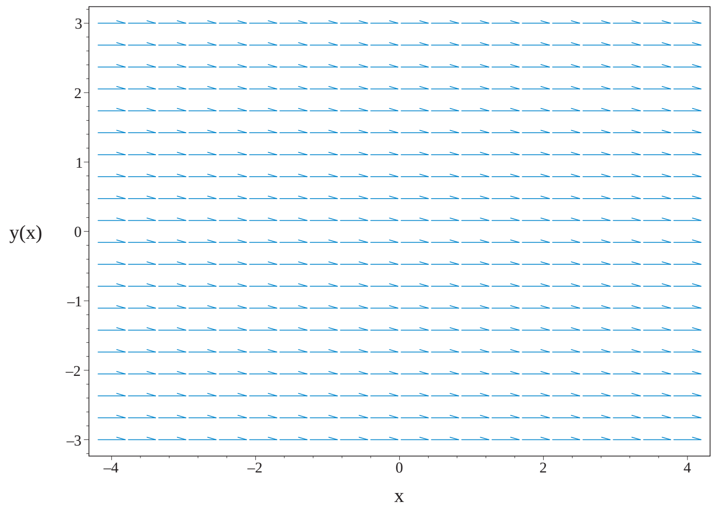


Figure 2.77: Slope field plot
$y' = 0$

Summary of solutions found

$$y = c_1$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) = 0$

- Highest derivative means the order of the ODE is 1
$\frac{d}{dx}y(x)$
- Integrate both sides with respect to x
$\int \left(\frac{d}{dx}y(x)\right) dx = \int 0 dx + C1$
- Evaluate integral
$y(x) = C1$
- Solve for $y(x)$
$y(x) = C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.000 (sec)
Leaf size : 5

```
dsolve(diff(y(x),x) = 0,
       y(x),singsol=all)
```

$$y = c_1$$

Mathematica DSolve solution

Solving time : 0.002 (sec)
Leaf size : 7

```
DSolve[{D[y[x],x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1$$

2.1.29 problem 30

Internal problem ID [8417]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 30
Date solved : `Tuesday, December 17, 2024 at 12:50:59 PM`
CAS classification : `[[_homogeneous, ‘class C‘], _rational, _dAlembert]`

Solve

$$y = x{y'}^2 + {y'}^2$$

Solved as first order ode of type dAlembert

Time used: 0.179 (sec)

Let $p = y'$ the ode becomes

$$y = x\,p^2 + p^2$$

Solving for y from the above results in

$$y = x\,p^2 + p^2 \tag{1}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= p^2 \\
g &= p^2
\end{aligned}
$$

Hence (2) becomes

$$
-p^2 + p = (2xp + 2p)\, p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
-p^2 + p = 0
$$

Solving the above for p results in

$$
\begin{aligned}
p_1 &= 0 \\
p_2 &= 1
\end{aligned}
$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$
\begin{aligned}
y &= 0 \\
y &= x + 1
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{-p(x)^2 + p(x)}{2p(x)\, x + 2p(x)} \tag{3}
$$

This ODE is now solved for $p(x)$. No inversion is needed. In canonical form a linear first order is

$$
p'(x) + q(x)p(x) = p(x)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= \frac{1}{2x + 2} \\
p(x) &= \frac{1}{2x + 2}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\, dx} \\
&= \mathrm{e}^{\int \frac{1}{2x+2} dx} \\
&= \sqrt{x + 1}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu p) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mu p) &= (\mu)\left(\frac{1}{2x+2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(p\sqrt{x+1}\right) &= \left(\sqrt{x+1}\right)\left(\frac{1}{2x+2}\right) \\
\mathrm{d}\left(p\sqrt{x+1}\right) &= \left(\frac{\sqrt{x+1}}{2x+2}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
p\sqrt{x+1} &= \int \frac{\sqrt{x+1}}{2x+2}\,dx \\
&= \sqrt{x+1} + c_1
\end{aligned}
$$

Dividing throughout by the integrating factor $\sqrt{x+1}$ gives the final solution

$$
p(x) = \frac{\sqrt{x+1} + c_1}{\sqrt{x+1}}
$$

Substituing the above solution for p in (2A) gives

$$
y = \frac{x\left(\sqrt{x+1} + c_1\right)^2}{x+1} + \frac{\left(\sqrt{x+1} + c_1\right)^2}{x+1}
$$

<u>Summary of solutions found</u>

$$
\begin{aligned}
y &= 0 \\
y &= x+1 \\
y &= \frac{x\left(\sqrt{x+1} + c_1\right)^2}{x+1} + \frac{\left(\sqrt{x+1} + c_1\right)^2}{x+1}
\end{aligned}
$$

Maple step by step solution

Let's solve

$y(x) = x\left(\frac{d}{dx}y(x)\right)^2 + \left(\frac{d}{dx}y(x)\right)^2$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\left[\frac{d}{dx}y(x) = \frac{\sqrt{(x+1)\,y(x)}}{x+1}, \frac{d}{dx}y(x) = -\frac{\sqrt{(x+1)\,y(x)}}{x+1}\right]$

- Solve the equation $\frac{d}{dx}y(x) = \frac{\sqrt{(x+1)\,y(x)}}{x+1}$
- Solve the equation $\frac{d}{dx}y(x) = -\frac{\sqrt{(x+1)\,y(x)}}{x+1}$
- Set of solutions

  $\{workingODE, workingODE\}$

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

Maple dsolve solution

Solving time : 0.048 (sec)
Leaf size : 53

```
dsolve(y(x) = x*diff(y(x),x)^2+diff(y(x),x)^2,
       y(x),singsol=all)
```

$$
\begin{aligned}
y &= 0 \\
y &= \frac{\left(x+1+\sqrt{(x+1)\,(c_1+1)}\right)^2}{x+1} \\
y &= \frac{\left(-x-1+\sqrt{(x+1)\,(c_1+1)}\right)^2}{x+1}
\end{aligned}
$$

Mathematica DSolve solution

Solving time : 0.065 (sec)
Leaf size : 57

```
DSolve[{y[x]==x*(D[y[x],x])^2+(D[y[x],x])^2,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
y(x) &\to x - c_1\sqrt{x+1} + 1 + \frac{c_1{}^2}{4} \\
y(x) &\to x + c_1\sqrt{x+1} + 1 + \frac{c_1{}^2}{4} \\
y(x) &\to 0
\end{aligned}
$$

2.1.30 problem 31

Internal problem ID [8418]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 31
Date solved : `Tuesday, December 17, 2024 at 12:51:00 PM`
CAS classification : `[[_homogeneous, 'class A'], _rational, _Riccati]`

Solve

$$y' = \frac{5x^2 - xy + y^2}{x^2}$$

Solved as first order homogeneous class A ode

Time used: 0.350 (sec)

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{5x^2 - xy + y^2}{x^2} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = 5x^2 - xy + y^2$ and $N = x^2$ are both homogeneous and of the same order $n = 2$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\frac{\mathrm{d}u}{\mathrm{d}x}x + u = u^2 - u + 5$$
$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{u(x)^2 - 2u(x) + 5}{x}$$

Or

$$u'(x) - \frac{u(x)^2 - 2u(x) + 5}{x} = 0$$

Or

$$u'(x)\,x - u(x)^2 + 2u(x) - 5 = 0$$

Which is now solved as separable in $u(x)$.

The ode $u'(x) = \frac{u(x)^2 - 2u(x) + 5}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= \frac{u(x)^2 - 2u(x) + 5}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{1}{x} \\ g(u) &= u^2 - 2u + 5 \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(u)}\,du = \int f(x)\,dx$$
$$\int \frac{1}{u^2 - 2u + 5}\,du = \int \frac{1}{x}\,dx$$
$$\frac{\arctan\left(\frac{u(x)}{2} - \frac{1}{2}\right)}{2} = \ln(x) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u^2 - 2u + 5 = 0$ for $u(x)$ gives

$$\begin{aligned} u(x) &= 1 - 2i \\ u(x) &= 1 + 2i \end{aligned}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
\frac{\arctan\left(\frac{u(x)}{2}-\frac{1}{2}\right)}{2} &= \ln(x)+c_1 \\
u(x) &= 1-2i \\
u(x) &= 1+2i
\end{aligned}
$$

Solving for $u(x)$ gives

$$
\begin{aligned}
u(x) &= 1-2i \\
u(x) &= 1+2i \\
u(x) &= 1+2\tan(2\ln(x)+2c_1)
\end{aligned}
$$

Converting $u(x)=1-2i$ back to y gives

$$
y=(1-2i)\,x
$$

Converting $u(x)=1+2i$ back to y gives

$$
y=(1+2i)\,x
$$

Converting $u(x)=1+2\tan(2\ln(x)+2c_1)$ back to y gives

$$
y=x(1+2\tan(2\ln(x)+2c_1))
$$

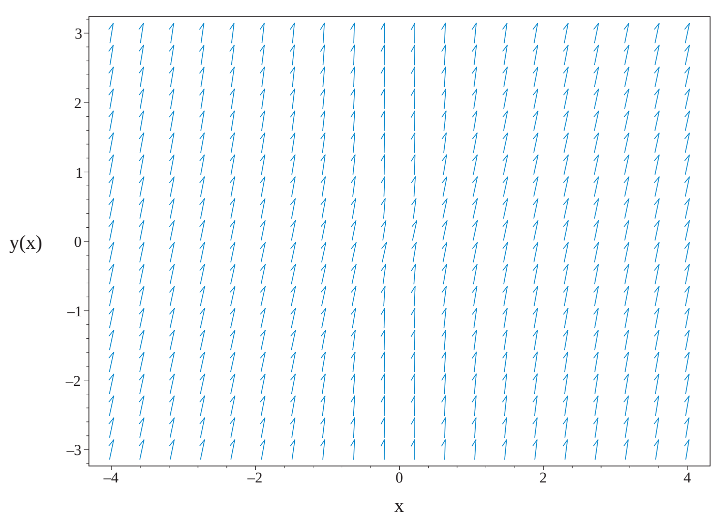


Figure 2.78: Slope field plot
$y'=\frac{5x^2-xy+y^2}{x^2}$

Summary of solutions found

$$
\begin{aligned}
y &= (1-2i)\,x \\
y &= (1+2i)\,x \\
y &= x(1+2\tan(2\ln(x)+2c_1))
\end{aligned}
$$

Solved as first order homogeneous class D2 ode

Time used: 0.190 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$
u'(x)\,x + u(x) = \frac{5x^2 - x^2 u(x) + u(x)^2\,x^2}{x^2}
$$

Which is now solved The ode $u'(x) = \frac{u(x)^2-2u(x)+5}{x}$ is separable as it can be written as

$$
\begin{aligned}
u'(x) &= \frac{u(x)^2 - 2u(x) + 5}{x} \\
&= f(x)g(u)
\end{aligned}
$$

Where

$$
\begin{aligned}
f(x) &= \frac{1}{x} \\
g(u) &= u^2 - 2u + 5
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\
\int \frac{1}{u^2-2u+5}\,du &= \int \frac{1}{x}\,dx \\
\frac{\arctan\left(\frac{u(x)}{2}-\frac{1}{2}\right)}{2} &= \ln(x) + c_1
\end{aligned}
$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u^2 - 2u + 5 = 0$ for $u(x)$ gives

$$
\begin{aligned}
u(x) &= 1-2i \\
u(x) &= 1+2i
\end{aligned}
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\frac{\arctan\left(\frac{u(x)}{2}-\frac{1}{2}\right)}{2} = \ln(x) + c_1$$
$$u(x) = 1-2i$$
$$u(x) = 1+2i$$

Solving for $u(x)$ gives

$$u(x) = 1-2i$$
$$u(x) = 1+2i$$
$$u(x) = 1+2\tan(2\ln(x)+2c_1)$$

Converting $u(x) = 1-2i$ back to y gives

$$y = (1-2i)\,x$$

Converting $u(x) = 1+2i$ back to y gives

$$y = (1+2i)\,x$$

Converting $u(x) = 1+2\tan(2\ln(x)+2c_1)$ back to y gives

$$y = x(1+2\tan(2\ln(x)+2c_1))$$

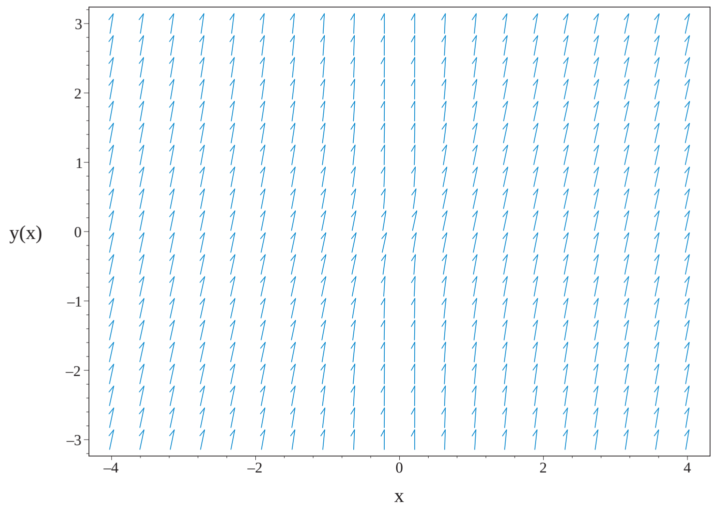


Figure 2.79: Slope field plot
$y' = \frac{5x^2-xy+y^2}{x^2}$

Summary of solutions found

$$y = (1 - 2i)\, x$$
$$y = (1 + 2i)\, x$$
$$y = x(1 + 2\tan(2\ln(x) + 2c_1))$$

Summary of solutions found

$$y = (1 - 2i)\, x$$
$$y = (1 + 2i)\, x$$
$$y = x(1 + 2\tan(2\ln(x) + 2c_1))$$

Solved as first order homogeneous class Maple C ode

Time used: 0.383 (sec)

Let $Y = y - y_0$ and $X = x - x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{5(x_0 + X)^2 - (x_0 + X)(Y(X) + y_0) + (Y(X) + y_0)^2}{(x_0 + X)^2}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 0$$
$$y_0 = 0$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{5X^2 - XY(X) + Y(X)^2}{X^2}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= \frac{5X^2 - XY + Y^2}{X^2} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = 5X^2 - XY + Y^2$ and $N = X^2$ are both homogeneous and of the same order $n = 2$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= u^2 - u + 5\\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{u(X)^2 - 2u(X) + 5}{X}\end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{u(X)^2 - 2u(X) + 5}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)X - u(X)^2 + 2u(X) - 5 = 0$$

Which is now solved as separable in $u(X)$.

The ode $\frac{d}{dX}u(X) = \frac{u(X)^2 - 2u(X) + 5}{X}$ is separable as it can be written as

$$\begin{aligned}\frac{d}{dX}u(X) &= \frac{u(X)^2 - 2u(X) + 5}{X}\\ &= f(X)g(u)\end{aligned}$$

Where

$$\begin{aligned}f(X) &= \frac{1}{X}\\ g(u) &= u^2 - 2u + 5\end{aligned}$$

Integrating gives

$$\begin{aligned}\int \frac{1}{g(u)}\,du &= \int f(X)\,dX\\ \int \frac{1}{u^2 - 2u + 5}\,du &= \int \frac{1}{X}\,dX\\ \frac{\arctan\left(\frac{u(X)}{2} - \frac{1}{2}\right)}{2} &= \ln(X) + c_1\end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u^2 - 2u + 5 = 0$ for $u(X)$ gives

$$
\begin{aligned}
u(X) &= 1 - 2i \\
u(X) &= 1 + 2i
\end{aligned}
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
\frac{\arctan\left(\frac{u(X)}{2} - \frac{1}{2}\right)}{2} &= \ln(X) + c_1 \\
u(X) &= 1 - 2i \\
u(X) &= 1 + 2i
\end{aligned}
$$

Solving for $u(X)$ gives

$$
\begin{aligned}
u(X) &= 1 - 2i \\
u(X) &= 1 + 2i \\
u(X) &= 1 + 2\tan(2\ln(X) + 2c_1)
\end{aligned}
$$

Converting $u(X) = 1 - 2i$ back to $Y(X)$ gives

$$
Y(X) = (1 - 2i) X
$$

Converting $u(X) = 1 + 2i$ back to $Y(X)$ gives

$$
Y(X) = (1 + 2i) X
$$

Converting $u(X) = 1 + 2\tan(2\ln(X) + 2c_1)$ back to $Y(X)$ gives

$$
Y(X) = X(1 + 2\tan(2\ln(X) + 2c_1))
$$

Using the solution for $Y(X)$

$$
Y(X) = (1 - 2i) X \tag{A}
$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y + y_0 \\
X &= x + x_0
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= y \\
X &= x
\end{aligned}
$$

Then the solution in y becomes using EQ (A)

$$
y = (1 - 2i)\, x
$$

Using the solution for $Y(X)$

$$
Y(X) = (1 + 2i)\, X \tag{A}
$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y + y_0 \\
X &= x + x_0
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= y \\
X &= x
\end{aligned}
$$

Then the solution in y becomes using EQ (A)

$$
y = (1 + 2i)\, x
$$

Using the solution for $Y(X)$

$$
Y(X) = X(1 + 2\tan(2\ln(X) + 2c_1)) \tag{A}
$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y + y_0 \\
X &= x + x_0
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= y \\
X &= x
\end{aligned}
$$

Then the solution in y becomes using EQ (A)

$$
y = x(1 + 2\tan(2\ln(x) + 2c_1))
$$

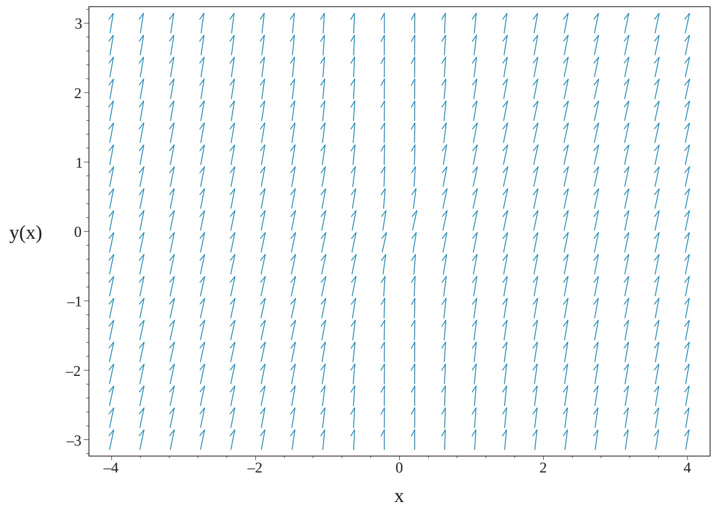


Figure 2.80: Slope field plot
$y' = \frac{5x^2-xy+y^2}{x^2}$

Solved as first order isobaric ode

Time used: 0.168 (sec)

Solving for y' gives

$$y' = \frac{5x^2 - xy + y^2}{x^2} \tag{1}$$

Each of the above ode's is now solved An ode $y' = f(x,y)$ is isobaric if

$$f(tx, t^m y) = t^{m-1} f(x,y) \tag{1}$$

Where here

$$f(x,y) = \frac{5x^2 - xy + y^2}{x^2} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = 1$$

Since the ode is isobaric of order $m = 1$, then the substitution

$$\begin{aligned} y &= u x^m \\ &= ux \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$u(x) + xu'(x) = \frac{5x^2 - x^2 u(x) + x^2 u(x)^2}{x^2}$$

The ode $u'(x) = \frac{u(x)^2-2u(x)+5}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= \frac{u(x)^2 - 2u(x) + 5}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{1}{x} \\ g(u) &= u^2 - 2u + 5 \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)} \, du &= \int f(x) \, dx \\ \int \frac{1}{u^2 - 2u + 5} \, du &= \int \frac{1}{x} \, dx \\ \frac{\arctan\left(\frac{u(x)}{2} - \frac{1}{2}\right)}{2} &= \ln(x) + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u^2 - 2u + 5 = 0$ for $u(x)$ gives

$$\begin{aligned} u(x) &= 1 - 2i \\ u(x) &= 1 + 2i \end{aligned}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} \frac{\arctan\left(\frac{u(x)}{2} - \frac{1}{2}\right)}{2} &= \ln(x) + c_1 \\ u(x) &= 1 - 2i \\ u(x) &= 1 + 2i \end{aligned}$$

Solving for $u(x)$ gives

$$u(x) = 1 - 2i$$

$$u(x) = 1 + 2i$$

$$u(x) = 1 + 2\tan(2\ln(x) + 2c_1)$$

Converting $u(x) = 1 - 2i$ back to y gives

$$\frac{y}{x} = 1 - 2i$$

Converting $u(x) = 1 + 2i$ back to y gives

$$\frac{y}{x} = 1 + 2i$$

Converting $u(x) = 1 + 2\tan(2\ln(x) + 2c_1)$ back to y gives

$$\frac{y}{x} = 1 + 2\tan(2\ln(x) + 2c_1)$$

Solving for y gives

$$y = (1 - 2i)x$$
$$y = (1 + 2i)x$$
$$y = x(1 + 2\tan(2\ln(x) + 2c_1))$$

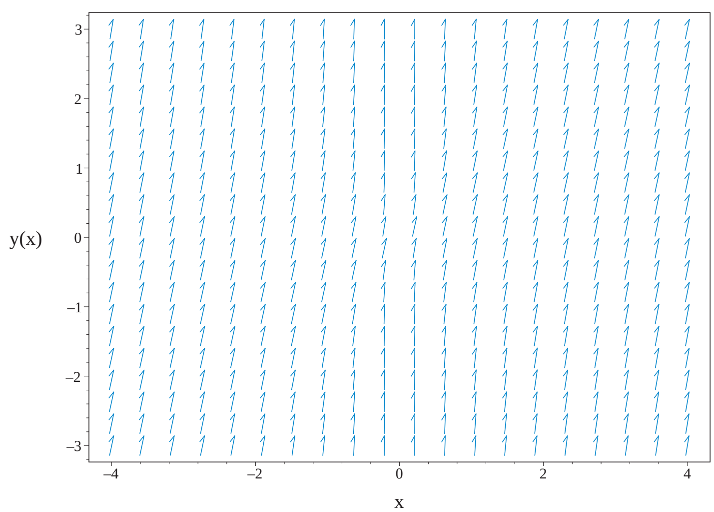


Figure 2.81: Slope field plot
$y' = \frac{5x^2 - xy + y^2}{x^2}$

Summary of solutions found

$$y = (1 - 2i)x$$
$$y = (1 + 2i)x$$
$$y = x(1 + 2\tan(2\ln(x) + 2c_1))$$

Solved using Lie symmetry for first order ode

Time used: 0.547 (sec)

Writing the ode as

$$\begin{aligned} y' &= \frac{5x^2 - xy + y^2}{x^2} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{(5x^2 - xy + y^2)(b_3 - a_2)}{x^2} - \frac{(5x^2 - xy + y^2)^2 a_3}{x^4} \\ & - \left(\frac{10x - y}{x^2} - \frac{2(5x^2 - xy + y^2)}{x^3}\right)(x a_2 + y a_3 + a_1) \\ & - \frac{(2y - x)(x b_2 + y b_3 + b_1)}{x^2} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{5x^4 a_2 + 25x^4 a_3 - 2b_2 x^4 - 5x^4 b_3 - 10x^3 y a_3 + 2x^3 y b_2 - x^2 y^2 a_2 + 12x^2 y^2 a_3 + x^2 y^2 b_3 - 4x\, y^3 a_3 + y^4 a_3 -}{x^4}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & -5x^4 a_2 - 25x^4 a_3 + 2b_2 x^4 + 5x^4 b_3 + 10x^3 y a_3 - 2x^3 y b_2 + x^2 y^2 a_2 - 12x^2 y^2 a_3 \\ & - x^2 y^2 b_3 + 4x\, y^3 a_3 - y^4 a_3 + x^3 b_1 - x^2 y a_1 - 2x^2 y b_1 + 2x\, y^2 a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$\{x,y\}$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$\{x=v_1, y=v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}&-5a_2v_1^4+a_2v_1^2v_2^2-25a_3v_1^4+10a_3v_1^3v_2-12a_3v_1^2v_2^2+4a_3v_1v_2^3-a_3v_2^4+2b_2v_1^4\\&-2b_2v_1^3v_2+5b_3v_1^4-b_3v_1^2v_2^2-a_1v_1^2v_2+2a_1v_1v_2^2+b_1v_1^3-2b_1v_1^2v_2=0\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1,v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}&(-5a_2-25a_3+2b_2+5b_3)\,v_1^4+(10a_3-2b_2)\,v_1^3v_2+b_1v_1^3\\&+(a_2-12a_3-b_3)\,v_1^2v_2^2+(-a_1-2b_1)\,v_1^2v_2+4a_3v_1v_2^3+2a_1v_1v_2^2-a_3v_2^4=0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}b_1&=0\\2a_1&=0\\-a_3&=0\\4a_3&=0\\-a_1-2b_1&=0\\10a_3-2b_2&=0\\a_2-12a_3-b_3&=0\\-5a_2-25a_3+2b_2+5b_3&=0\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{5x^2 - xy + y^2}{x^2}\right)(x) \\
&= \frac{-5x^2 + 2xy - y^2}{x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-5x^2+2xy-y^2}{x}} dy
\end{aligned}
$$

Which results in

$$S = -\frac{\arctan\left(\frac{2y-2x}{4x}\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{5x^2 - xy + y^2}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{y}{5x^2 - 2xy + y^2} \\
S_y &= -\frac{x}{5x^2 - 2xy + y^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned}
\int dS &= \int -\frac{1}{R}\, dR \\
S(R) &= -\ln(R) + c_2
\end{aligned}$$

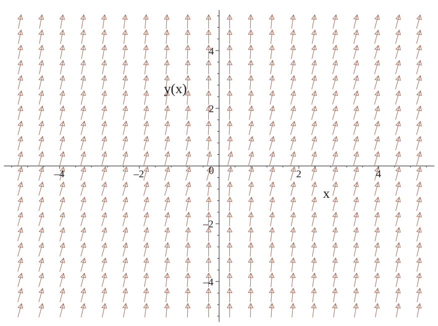
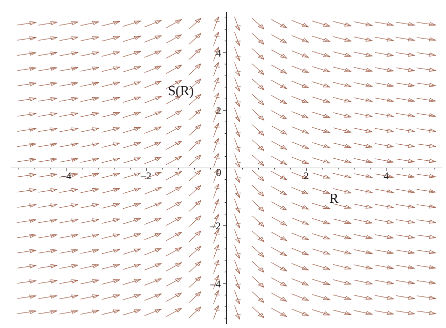
To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\frac{\arctan\left(\frac{x-y}{2x}\right)}{2} = -\ln(x) + c_2$$

Which gives

$$y = -2\tan(-2\ln(x) + 2c_2)\,x + x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{5x^2-xy+y^2}{x^2}$ 	$R = x$ $S = \frac{\arctan\left(\frac{x-y}{2x}\right)}{2}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

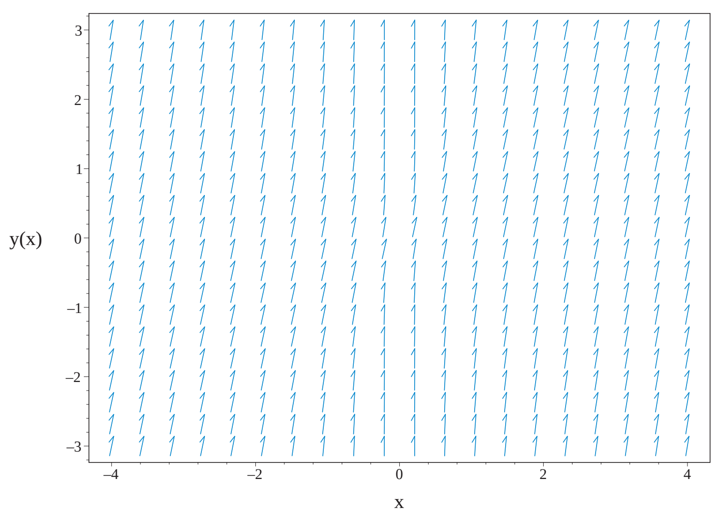


Figure 2.82: Slope field plot
$y' = \frac{5x^2-xy+y^2}{x^2}$

Summary of solutions found

$$y = -2\tan\left(-2\ln\left(x\right) + 2c_2\right)x + x$$

Solved as first order ode of type Riccati

Time used: 0.140 (sec)

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{5x^2 - xy + y^2}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = 5 - \frac{y}{x} + \frac{y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 5$, $f_1(x) = -\frac{1}{x}$ and $f_2(x) = \frac{1}{x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - \left(f_2' + f_1f_2\right)u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2}{x^3} \\ f_1f_2 &= -\frac{1}{x^3} \\ f_2^2f_0 &= \frac{5}{x^4} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x^2} + \frac{3u'(x)}{x^3} + \frac{5u(x)}{x^4} = 0$$

This is Euler second order ODE. Let the solution be $u = x^r$, then $u' = rx^{r-1}$ and $u'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 3xrx^{r-1} + 5x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 3r\,x^r + 5x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 3r + 5 = 0$$

Or

$$r^2 + 2r + 5 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -1 - 2i \\ r_2 &= -1 + 2i \end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

Where in this case $\alpha = -1$ and $\beta = -2$. Hence the solution becomes

$$\begin{aligned} u &= c_1 x^{r_1} + c_2 x^{r_2} \\ &= c_1 x^{\alpha+i\beta} + c_2 x^{\alpha-i\beta} \\ &= x^{\alpha}\left(c_1 x^{i\beta} + c_2 x^{-i\beta}\right) \\ &= x^{\alpha}\left(c_1 e^{\ln\left(x^{i\beta}\right)} + c_2 e^{\ln\left(x^{-i\beta}\right)}\right) \\ &= x^{\alpha}\left(c_1 e^{i(\beta\ln x)} + c_2 e^{-i(\beta\ln x)}\right) \end{aligned}$$

Using the values for $\alpha = -1, \beta = -2$, the above becomes

$$u = x^{-1}\left(c_1 e^{-2i\ln(x)} + c_2 e^{2i\ln(x)}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$u = \frac{1}{x}\left(c_1 \cos\left(2\ln\left(x\right)\right) + c_2 \sin\left(2\ln\left(x\right)\right)\right)$$

Will add steps showing solving for IC soon.

Taking derivative gives

$$u'(x) = -\frac{c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x))}{x^2} + \frac{-\frac{2c_1 \sin(2\ln(x))}{x} + \frac{2c_2 \cos(2\ln(x))}{x}}{x}$$

Doing change of constants, the solution becomes

$$y = -\frac{\left(-\frac{c_3 \cos(2\ln(x))+\sin(2\ln(x))}{x^2} + \frac{-\frac{2c_3 \sin(2\ln(x))}{x} + \frac{2\cos(2\ln(x))}{x}}{x}\right)x^3}{c_3 \cos(2\ln(x)) + \sin(2\ln(x))}$$

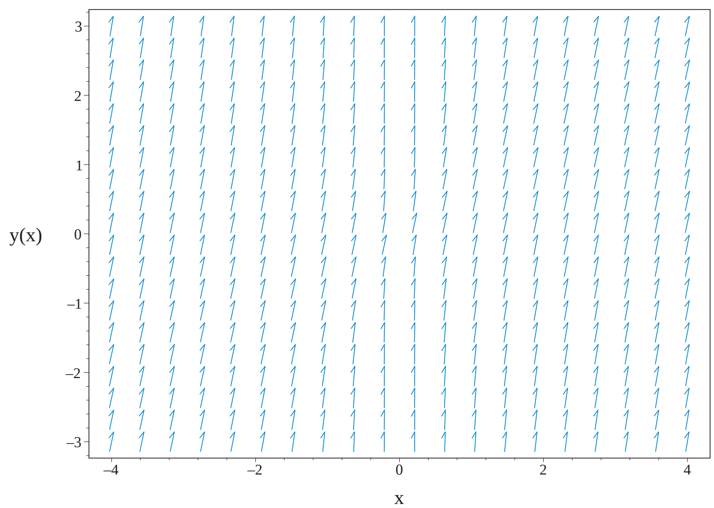


Figure 2.83: Slope field plot
$y' = \frac{5x^2-xy+y^2}{x^2}$

Summary of solutions found

$$y = -\frac{\left(-\frac{c_3 \cos(2\ln(x))+\sin(2\ln(x))}{x^2} + \frac{-\frac{2c_3 \sin(2\ln(x))}{x} + \frac{2\cos(2\ln(x))}{x}}{x}\right)x^3}{c_3 \cos(2\ln(x)) + \sin(2\ln(x))}$$

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) = \frac{5x^2 - xy(x) + y(x)^2}{x^2}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = \frac{5x^2 - xy(x) + y(x)^2}{x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

Maple dsolve solution

Solving time : `0.010` (sec)
Leaf size : 19

```
dsolve(diff(y(x),x) = (5*x^2-x*y(x)+y(x)^2)/x^2,
       y(x),singsol=all)
```

$$y = x(1 + 2\tan(2\ln(x) + 2c_1))$$

Mathematica DSolve solution

Solving time : 0.881 (sec)
Leaf size : 18

```
DSolve[{D[y[x],x]==(5*x^2-x*y[x]+y[x]^2)/x^2,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x + 2x\tan(2(\log(x) + c_1))$$

2.1.31 problem 32

Internal problem ID [8419]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 32
Date solved : `Tuesday, December 17, 2024 at 12:51:03 PM`
CAS classification :
`[[_homogeneous, 'class C'], _rational, [_Abel, '2nd type', 'class A']]`

Solve

$$2t + 3x + (x + 2)\, x' = 0$$

Summary of solutions found

$$
\begin{aligned}
x &= -2t + 4 \\
x &= -t + 1 \\
x &= -2t + 4 + \frac{e^{c_1}}{2} - \frac{\sqrt{e^{2c_1} - 4\,(t-3)\, e^{c_1}}}{2} \\
x &= -2t + 4 + \frac{e^{c_1}}{2} + \frac{\sqrt{e^{2c_1} - 4\,(t-3)\, e^{c_1}}}{2}
\end{aligned}
$$

Solved as first order homogeneous class Maple C ode

Time used: 0.710 (sec)

Let $Y = x - y_0$ and $X = t - x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{3Y(X) + 3y_0 + 2x_0 + 2X}{Y(X) + y_0 + 2}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$
\begin{aligned}
x_0 &= 3 \\
y_0 &= -2
\end{aligned}
$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$
\frac{d}{dX}Y(X) = -\frac{3Y(X) + 2X}{Y(X)}
$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= -\frac{3Y + 2X}{Y}
\end{aligned} \tag{1}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = -3Y - 2X$ and $N = Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= -3 - \frac{2}{u} \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{-3 - \frac{2}{u(X)} - u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX}u(X) - \frac{-3 - \frac{2}{u(X)} - u(X)}{X} = 0
$$

Or

$$
\left(\frac{d}{dX}u(X)\right)u(X)\,X + u(X)^2 + 3u(X) + 2 = 0
$$

Which is now solved as separable in $u(X)$.

The ode $\frac{d}{dX}u(X) = -\frac{u(X)^2+3u(X)+2}{u(X)X}$ is separable as it can be written as

$$\begin{aligned}
\frac{d}{dX}u(X) &= -\frac{u(X)^2 + 3u(X) + 2}{u(X)X} \\
&= f(X)g(u)
\end{aligned}$$

Where

$$\begin{aligned}
f(X) &= -\frac{1}{X} \\
g(u) &= \frac{u^2 + 3u + 2}{u}
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\int \frac{1}{g(u)} du &= \int f(X) \, dX \\
\int \frac{u}{u^2 + 3u + 2} du &= \int -\frac{1}{X} dX \\
\ln\left(\frac{(u(X) + 2)^2}{u(X) + 1}\right) &= \ln\left(\frac{1}{X}\right) + c_1
\end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $\frac{u^2+3u+2}{u} = 0$ for $u(X)$ gives

$$\begin{aligned}
u(X) &= -2 \\
u(X) &= -1
\end{aligned}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned}
\ln\left(\frac{(u(X) + 2)^2}{u(X) + 1}\right) &= \ln\left(\frac{1}{X}\right) + c_1 \\
u(X) &= -2 \\
u(X) &= -1
\end{aligned}$$

Solving for $u(X)$ gives

$$
\begin{aligned}
u(X) &= -2 \\
u(X) &= -1 \\
u(X) &= \frac{-4X + e^{c_1} + \sqrt{e^{2c_1} - 4X\,e^{c_1}}}{2X} \\
u(X) &= -\frac{4X - e^{c_1} + \sqrt{e^{2c_1} - 4X\,e^{c_1}}}{2X}
\end{aligned}
$$

Converting $u(X) = -2$ back to $Y(X)$ gives

$$Y(X) = -2X$$

Converting $u(X) = -1$ back to $Y(X)$ gives

$$Y(X) = -X$$

Converting $u(X) = \frac{-4X+e^{c_1}+\sqrt{e^{2c_1}-4X\,e^{c_1}}}{2X}$ back to $Y(X)$ gives

$$Y(X) = -2X + \frac{e^{c_1}}{2} + \frac{\sqrt{e^{2c_1} - 4X\,e^{c_1}}}{2}$$

Converting $u(X) = -\frac{4X-e^{c_1}+\sqrt{e^{2c_1}-4X\,e^{c_1}}}{2X}$ back to $Y(X)$ gives

$$Y(X) = -2X + \frac{e^{c_1}}{2} - \frac{\sqrt{e^{2c_1} - 4X\,e^{c_1}}}{2}$$

Using the solution for $Y(X)$

$$Y(X) = -2X \tag{A}$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= x + y_0 \\
X &= t + x_0
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= x - 2 \\
X &= t + 3
\end{aligned}
$$

Then the solution in x becomes using EQ (A)

$$x + 2 = -2t + 6$$

Using the solution for $Y(X)$

$$Y(X) = -X \tag{A}$$

And replacing back terms in the above solution using

$$Y = x + y_0$$
$$X = t + x_0$$

Or

$$Y = x - 2$$
$$X = t + 3$$

Then the solution in x becomes using EQ (A)

$$x + 2 = -t + 3$$

Using the solution for $Y(X)$

$$Y(X) = -2X + \frac{e^{c_1}}{2} - \frac{\sqrt{e^{2c_1} - 4X\, e^{c_1}}}{2} \tag{A}$$

And replacing back terms in the above solution using

$$Y = x + y_0$$
$$X = t + x_0$$

Or

$$Y = x - 2$$
$$X = t + 3$$

Then the solution in x becomes using EQ (A)

$$x + 2 = -2t + 6 + \frac{e^{c_1}}{2} - \frac{\sqrt{e^{2c_1} - 4(t-3)\, e^{c_1}}}{2}$$

Using the solution for $Y(X)$

$$Y(X) = -2X + \frac{e^{c_1}}{2} + \frac{\sqrt{e^{2c_1} - 4X\, e^{c_1}}}{2} \tag{A}$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= x + y_0 \\
X &= t + x_0
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= x - 2 \\
X &= t + 3
\end{aligned}
$$

Then the solution in x becomes using EQ (A)

$$
x + 2 = -2t + 6 + \frac{e^{c_1}}{2} + \frac{\sqrt{e^{2c_1} - 4(t-3)e^{c_1}}}{2}
$$

Solving for x gives

$$
\begin{aligned}
x &= -2t + 4 \\
x &= -t + 1 \\
x &= -2t + 4 + \frac{e^{c_1}}{2} - \frac{\sqrt{e^{2c_1} - 4(t-3)e^{c_1}}}{2} \\
x &= -2t + 4 + \frac{e^{c_1}}{2} + \frac{\sqrt{e^{2c_1} - 4(t-3)e^{c_1}}}{2}
\end{aligned}
$$

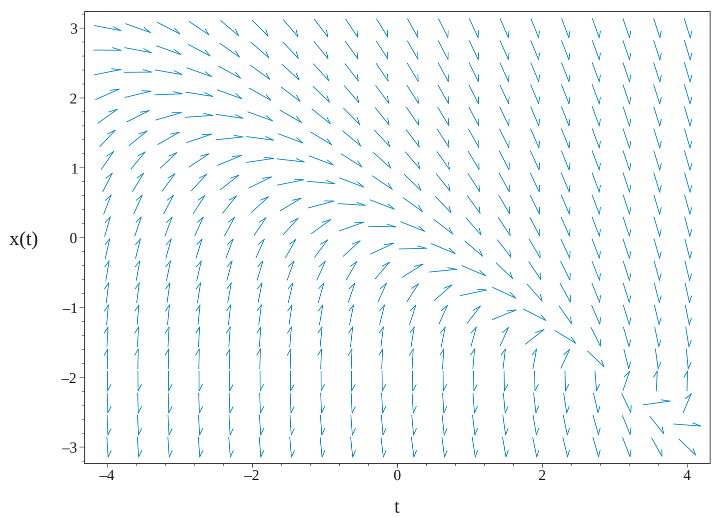


Figure 2.84: Slope field plot
$2t + 3x + (x+2)\,x' = 0$

Solved using Lie symmetry for first order ode

Time used: 0.886 (sec)

Writing the ode as

$$
\begin{aligned}
x' &= -\frac{3x+2t}{x+2} \\
x' &= \omega(t,x)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_t + \omega(\eta_x - \xi_t) - \omega^2 \xi_x - \omega_t \xi - \omega_x \eta = 0 \tag{A}
$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = t a_2 + x a_3 + a_1 \tag{1E}
$$

$$
\eta = t b_2 + x b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2 - \frac{(3x+2t)\,(b_3 - a_2)}{x+2} - \frac{(3x+2t)^2 a_3}{(x+2)^2} + \frac{2ta_2 + 2xa_3 + 2a_1}{x+2} \\
& - \left(-\frac{3}{x+2} + \frac{3x+2t}{(x+2)^2}\right)(tb_2 + xb_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{4t^2a_3 + 2t^2b_2 - 4txa_2 + 12txa_3 + 4txb_3 - 3x^2a_2 + 7x^2a_3 - x^2b_2 + 3x^2b_3 - 8ta_2 + 2tb_1 - 6tb_2 + 4tb_3 - 2}{(x+2)^2}
$$

$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
& -4t^2a_3 - 2t^2b_2 + 4txa_2 - 12txa_3 - 4txb_3 + 3x^2a_2 - 7x^2a_3 + x^2b_2 - 3x^2b_3 \\
& + 8ta_2 - 2tb_1 + 6tb_2 - 4tb_3 + 2xa_1 + 6xa_2 + 4xa_3 + 4xb_2 + 4a_1 + 6b_1 + 4b_2 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{t, x\}$ in them.

$$\{t, x\}$$

The following substitution is now made to be able to collect on all terms with $\{t, x\}$ in them

$$\{t = v_1, x = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &4a_2v_1v_2 + 3a_2v_2^2 - 4a_3v_1^2 - 12a_3v_1v_2 - 7a_3v_2^2 - 2b_2v_1^2 + b_2v_2^2 \\ &- 4b_3v_1v_2 - 3b_3v_2^2 + 2a_1v_2 + 8a_2v_1 + 6a_2v_2 + 4a_3v_2 \\ &- 2b_1v_1 + 6b_2v_1 + 4b_2v_2 - 4b_3v_1 + 4a_1 + 6b_1 + 4b_2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &(-4a_3 - 2b_2)\, v_1^2 + (4a_2 - 12a_3 - 4b_3)\, v_1v_2 + (8a_2 - 2b_1 + 6b_2 - 4b_3)\, v_1 \\ &+ (3a_2 - 7a_3 + b_2 - 3b_3)\, v_2^2 + (2a_1 + 6a_2 + 4a_3 + 4b_2)\, v_2 + 4a_1 + 6b_1 + 4b_2 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -4a_3 - 2b_2 &= 0 \\ 4a_1 + 6b_1 + 4b_2 &= 0 \\ 4a_2 - 12a_3 - 4b_3 &= 0 \\ 2a_1 + 6a_2 + 4a_3 + 4b_2 &= 0 \\ 3a_2 - 7a_3 + b_2 - 3b_3 &= 0 \\ 8a_2 - 2b_1 + 6b_2 - 4b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= -7a_3 - 3b_3 \\ a_2 &= 3a_3 + b_3 \\ a_3 &= a_3 \\ b_1 &= 6a_3 + 2b_3 \\ b_2 &= -2a_3 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = t - 3$$
$$\eta = x + 2$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(t,x)\,\xi \\ &= x + 2 - \left(-\frac{3x+2t}{x+2}\right)(t-3) \\ &= \frac{2t^2 + 3tx + x^2 - 6t - 5x + 4}{x+2} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,x) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial x}\right) S(t,x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{2t^2+3tx+x^2-6t-5x+4}{x+2}} dy\end{aligned}$$

Which results in

$$S = 2\ln(x + 2t - 4) - \ln(x + t - 1)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,x)S_x}{R_t + \omega(t,x)R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t,x)$ is the right hand side of the original ode given by

$$\omega(t,x) = -\frac{3x+2t}{x+2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_t &= 1\\
R_x &= 0\\
S_t &= \frac{4}{x+2t-4} - \frac{1}{x+t-1}\\
S_x &= \frac{x+2}{(x+t-1)(x+2t-4)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

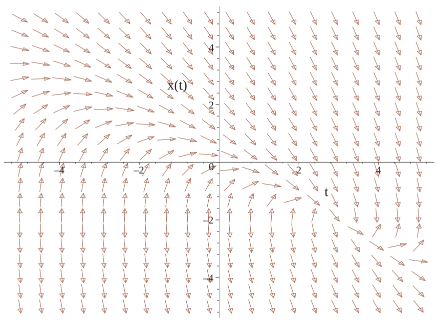
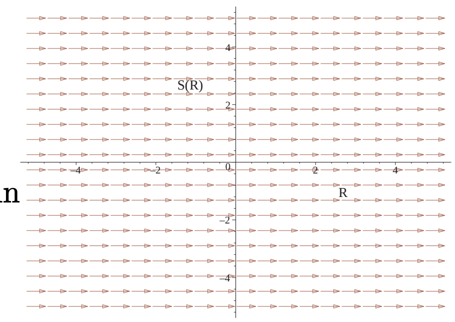
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned}
\int dS &= \int 0\,dR + c_2\\
S(R) &= c_2
\end{aligned}$$

To complete the solution, we just need to transform the above back to t, x coordinates. This results in

$$2\ln(x+2t-4) - \ln(x+t-1) = c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in t, x coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dx}{dt} = -\frac{3x+2t}{x+2}$ 	$R = t$ $S = 2\ln(x+2t-4) - \ln$	$\frac{dS}{dR} = 0$ 

Solving for x gives

$$x = \frac{e^{c_2}}{2} - \frac{\sqrt{e^{2c_2} - 4t\,e^{c_2} + 12\,e^{c_2}}}{2} - 2t + 4$$

$$x = \frac{e^{c_2}}{2} + \frac{\sqrt{e^{2c_2} - 4t\,e^{c_2} + 12\,e^{c_2}}}{2} - 2t + 4$$

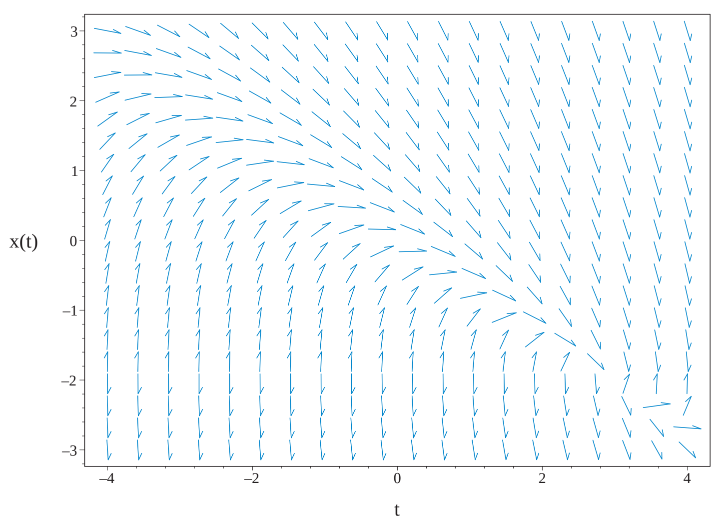


Figure 2.85: Slope field plot
$2t + 3x + (x+2)\,x' = 0$

Summary of solutions found

$$x = \frac{e^{c_2}}{2} - \frac{\sqrt{e^{2c_2} - 4t\,e^{c_2} + 12\,e^{c_2}}}{2} - 2t + 4$$

$$x = \frac{e^{c_2}}{2} + \frac{\sqrt{e^{2c_2} - 4t\,e^{c_2} + 12\,e^{c_2}}}{2} - 2t + 4$$

Solved as first order ode of type dAlembert

Time used: 0.360 (sec)

Let $p = x'$ the ode becomes

$$2t + 3x + (x + 2)\,p = 0$$

Solving for x from the above results in

$$x = -\frac{2t}{3+p} - \frac{2p}{3+p} \tag{1}$$

This has the form

$$x = tf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = x'(t)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. t gives

$$\begin{aligned} p &= f + (tf' + g')\frac{dp}{dt} \\ p - f &= (tf' + g')\frac{dp}{dt} \end{aligned} \tag{2}$$

Comparing the form $x = tf + g$ to (1A) shows that

$$\begin{aligned} f &= -\frac{2}{3+p} \\ g &= -\frac{2p}{3+p} \end{aligned}$$

Hence (2) becomes

$$p + \frac{2}{3+p} = \left(\frac{2t}{(3+p)^2} - \frac{2}{3+p} + \frac{2p}{(3+p)^2}\right) p'(t) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dt} = 0$ in the above which gives

$$p + \frac{2}{3 + p} = 0$$

Solving the above for p results in

$$\begin{aligned} p_1 &= -1 \\ p_2 &= -2 \end{aligned}$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$\begin{aligned} x &= -t + 1 \\ x &= -2t + 4 \end{aligned}$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}t} \neq 0$. From eq. (2A). This results in

$$p'(t) = \frac{p(t) + \frac{2}{3+p(t)}}{\frac{2t}{(3+p(t))^2} - \frac{2}{3+p(t)} + \frac{2p(t)}{(3+p(t))^2}} \tag{3}$$

This ODE is now solved for $p(t)$. No inversion is needed. The ode $p'(t) = \frac{p(t)^3+6p(t)^2+11p(t)+6}{2t-6}$ is separable as it can be written as

$$\begin{aligned} p'(t) &= \frac{p(t)^3 + 6p(t)^2 + 11p(t) + 6}{2t - 6} \\ &= f(t)g(p) \end{aligned}$$

Where

$$\begin{aligned} f(t) &= \frac{1}{t - 3} \\ g(p) &= \frac{1}{2}p^3 + 3p^2 + \frac{11}{2}p + 3 \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(p)}\, dp &= \int f(t)\, dt \\ \int \frac{1}{\frac{1}{2}p^3 + 3p^2 + \frac{11}{2}p + 3}\, dp &= \int \frac{1}{t - 3}\, dt \\ \ln\left(\frac{(3 + p(t))\,(p(t) + 1)}{(p\,(t) + 2)^2}\right) &= \ln\,(t - 3) + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(p)$ is zero, since we had to divide by this above. Solving $g(p) = 0$ or $\frac{1}{2}p^3+3p^2+\frac{11}{2}p+3 = 0$ for $p(t)$ gives

$$
\begin{aligned}
p(t) &= -3 \\
p(t) &= -2 \\
p(t) &= -1
\end{aligned}
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
\ln\left(\frac{(3+p(t))\,(p(t)+1)}{(p\,(t)+2)^2}\right) &= \ln\,(t-3)+c_1 \\
p(t) &= -3 \\
p(t) &= -2 \\
p(t) &= -1
\end{aligned}
$$

Solving for $p(t)$ gives

$$
\begin{aligned}
p(t) &= -3 \\
p(t) &= -2 \\
p(t) &= -1 \\
p(t) &= -\frac{2\sqrt{-t\,\mathrm{e}^{c_1}+3\,\mathrm{e}^{c_1}+1}-1}{\sqrt{-t\,\mathrm{e}^{c_1}+3\,\mathrm{e}^{c_1}+1}} \\
p(t) &= -\frac{2\sqrt{-t\,\mathrm{e}^{c_1}+3\,\mathrm{e}^{c_1}+1}+1}{\sqrt{-t\,\mathrm{e}^{c_1}+3\,\mathrm{e}^{c_1}+1}}
\end{aligned}
$$

Substituing the above solution for p in (2A) gives

$$
\begin{aligned}
x &= -2t+4 \\
x &= -t+1 \\
x &= -\frac{2t}{3-\frac{2\sqrt{-t\,\mathrm{e}^{c_1}+3\,\mathrm{e}^{c_1}+1}-1}{\sqrt{-t\,\mathrm{e}^{c_1}+3\,\mathrm{e}^{c_1}+1}}}+\frac{4\sqrt{-t\,\mathrm{e}^{c_1}+3\,\mathrm{e}^{c_1}+1}-2}{\sqrt{-t\,\mathrm{e}^{c_1}+3\,\mathrm{e}^{c_1}+1}\left(3-\frac{2\sqrt{-t\,\mathrm{e}^{c_1}+3\,\mathrm{e}^{c_1}+1}-1}{\sqrt{-t\,\mathrm{e}^{c_1}+3\,\mathrm{e}^{c_1}+1}}\right)} \\
x &= -\frac{2t}{3-\frac{2\sqrt{-t\,\mathrm{e}^{c_1}+3\,\mathrm{e}^{c_1}+1}+1}{\sqrt{-t\,\mathrm{e}^{c_1}+3\,\mathrm{e}^{c_1}+1}}}+\frac{4\sqrt{-t\,\mathrm{e}^{c_1}+3\,\mathrm{e}^{c_1}+1}+2}{\sqrt{-t\,\mathrm{e}^{c_1}+3\,\mathrm{e}^{c_1}+1}\left(3-\frac{2\sqrt{-t\,\mathrm{e}^{c_1}+3\,\mathrm{e}^{c_1}+1}+1}{\sqrt{-t\,\mathrm{e}^{c_1}+3\,\mathrm{e}^{c_1}+1}}\right)}
\end{aligned}
$$

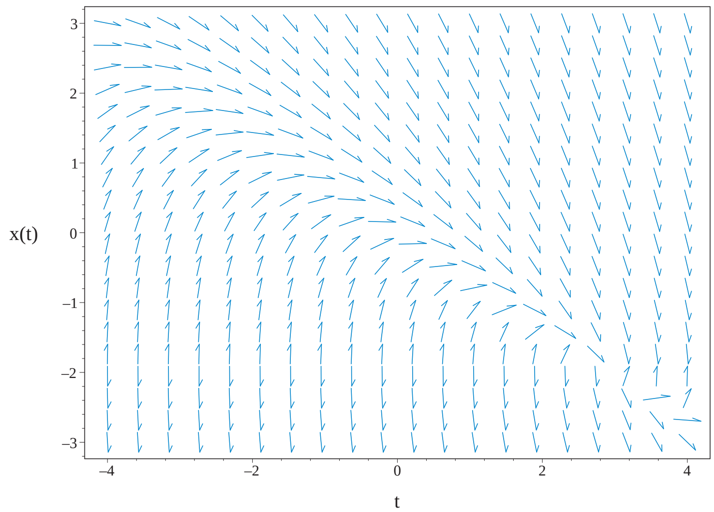


Figure 2.86: Slope field plot
$2t + 3x + (x+2)\,x' = 0$

Summary of solutions found

$$x = -2t + 4$$

$$x = -t + 1$$

$$x = -\frac{2t}{3 - \frac{2\sqrt{-t\,\mathrm{e}^{c_1}+3\,\mathrm{e}^{c_1}+1}-1}{\sqrt{-t\,\mathrm{e}^{c_1}+3\,\mathrm{e}^{c_1}+1}}} + \frac{4\sqrt{-t\,\mathrm{e}^{c_1}+3\,\mathrm{e}^{c_1}+1}-2}{\sqrt{-t\,\mathrm{e}^{c_1}+3\,\mathrm{e}^{c_1}+1}\left(3 - \frac{2\sqrt{-t\,\mathrm{e}^{c_1}+3\,\mathrm{e}^{c_1}+1}-1}{\sqrt{-t\,\mathrm{e}^{c_1}+3\,\mathrm{e}^{c_1}+1}}\right)}$$

$$x = -\frac{2t}{3 - \frac{2\sqrt{-t\,\mathrm{e}^{c_1}+3\,\mathrm{e}^{c_1}+1}+1}{\sqrt{-t\,\mathrm{e}^{c_1}+3\,\mathrm{e}^{c_1}+1}}} + \frac{4\sqrt{-t\,\mathrm{e}^{c_1}+3\,\mathrm{e}^{c_1}+1}+2}{\sqrt{-t\,\mathrm{e}^{c_1}+3\,\mathrm{e}^{c_1}+1}\left(3 - \frac{2\sqrt{-t\,\mathrm{e}^{c_1}+3\,\mathrm{e}^{c_1}+1}+1}{\sqrt{-t\,\mathrm{e}^{c_1}+3\,\mathrm{e}^{c_1}+1}}\right)}$$

Maple step by step solution

Let's solve

$2t + 3x(t) + (x(t)+2)\left(\frac{d}{dt}x(t)\right) = 0$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dt}x(t)$

- Solve for the highest derivative

  $\frac{d}{dt}x(t) = \frac{-2t-3x(t)}{x(t)+2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

Maple dsolve solution

Solving time : `4.428` (sec)
Leaf size : 30

```
dsolve(2*t+3*x(t)+(x(t)+2)*diff(x(t),t) = 0,
       x(t),singsol=all)
```

$$x(t) = \frac{-\sqrt{4\,(t-3)\,c_1+1}-1+(-4t+8)\,c_1}{2c_1}$$

Mathematica DSolve solution

Solving time : `60.105` (sec)
Leaf size : 1165

```
DSolve[{2*t+3*x[t]+(x[t]+2)*D[x[t],t]==0,{}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to -2 - \frac{2(t-3\cdots}{t\sqrt{\frac{3}{(t-3)^2} - \frac{3(t-3)^2\cosh\left(\frac{4c_1}{9}\right)+3(t-3)^2\sinh\left(\frac{4c_1}{9}\right)+2}{(t-3)^2\left((t-3)^2\cosh\left(\frac{4c_1}{9}\right)+(t-3)^2\sinh\left(\frac{4c_1}{9}\right)+1\right)} - \sqrt{-\frac{\cosh\left(\frac{4c_1}{9}\right)+\sinh\left(\frac{4c_1}{9}\right)}{(t-3)^2\left((t-3)^2\cosh\left(\frac{4c_1}{9}\right)+(t-3)^2\sinh\left(\frac{4c_1}{9}\right)+1\right)^2} - 3\sqrt{\cdots}}}}$$

$$x(t) \to -2 + \frac{2(t-3\cdots}{t\sqrt{\frac{3}{(t-3)^2} - \frac{3(t-3)^2\cosh\left(\frac{4c_1}{9}\right)+3(t-3)^2\sinh\left(\frac{4c_1}{9}\right)+2}{(t-3)^2\left((t-3)^2\cosh\left(\frac{4c_1}{9}\right)+(t-3)^2\sinh\left(\frac{4c_1}{9}\right)+1\right)} - \sqrt{-\frac{\cosh\left(\frac{4c_1}{9}\right)+\sinh\left(\frac{4c_1}{9}\right)}{(t-3)^2\left((t-3)^2\cosh\left(\frac{4c_1}{9}\right)+(t-3)^2\sinh\left(\frac{4c_1}{9}\right)+1\right)^2} - 3\sqrt{\cdots}}}}$$

$$x(t) \to -2 - \frac{2(t-3\cdots}{t\sqrt{\frac{3}{(t-3)^2} - \frac{3(t-3)^2\cosh\left(\frac{4c_1}{9}\right)+3(t-3)^2\sinh\left(\frac{4c_1}{9}\right)+2}{(t-3)^2\left((t-3)^2\cosh\left(\frac{4c_1}{9}\right)+(t-3)^2\sinh\left(\frac{4c_1}{9}\right)+1\right)} + \sqrt{-\frac{\cosh\left(\frac{4c_1}{9}\right)+\sinh\left(\frac{4c_1}{9}\right)}{(t-3)^2\left((t-3)^2\cosh\left(\frac{4c_1}{9}\right)+(t-3)^2\sinh\left(\frac{4c_1}{9}\right)+1\right)^2} - 3\sqrt{\cdots}}}}$$

$$x(t) \to -2 + \frac{2(t-3\cdots}{t\sqrt{\frac{3}{(t-3)^2} - \frac{3(t-3)^2\cosh\left(\frac{4c_1}{9}\right)+3(t-3)^2\sinh\left(\frac{4c_1}{9}\right)+2}{(t-3)^2\left((t-3)^2\cosh\left(\frac{4c_1}{9}\right)+(t-3)^2\sinh\left(\frac{4c_1}{9}\right)+1\right)} + \sqrt{-\frac{\cosh\left(\frac{4c_1}{9}\right)+\sinh\left(\frac{4c_1}{9}\right)}{(t-3)^2\left((t-3)^2\cosh\left(\frac{4c_1}{9}\right)+(t-3)^2\sinh\left(\frac{4c_1}{9}\right)+1\right)^2} - 3\sqrt{\cdots}}}}$$

2.1.32 problem 33

Internal problem ID [8420]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 33
Date solved : `Tuesday, December 17, 2024 at 12:51:06 PM`
CAS classification : `[_quadrature]`

Solve

$$y' = \frac{1}{1-y}$$

With initial conditions

$$y(0) = 2$$

Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(t,y) \\ &= -\frac{1}{-1+y} \end{aligned}$$

The y domain of $f(t,y)$ when $t=0$ is

$$\{y < 1 \vee 1 < y\}$$

And the point $y_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{-1+y}\right) \\ &= \frac{1}{(-1+y)^2} \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $t = 0$ is

$$\{y < 1 \vee 1 < y\}$$

And the point $y_0 = 2$ is inside this domain. Therefore solution exists and is unique.

Solved as first order autonomous ode

Time used: 0.236 (sec)

Integrating gives

$$\int (1 - y)\, dy = dt$$

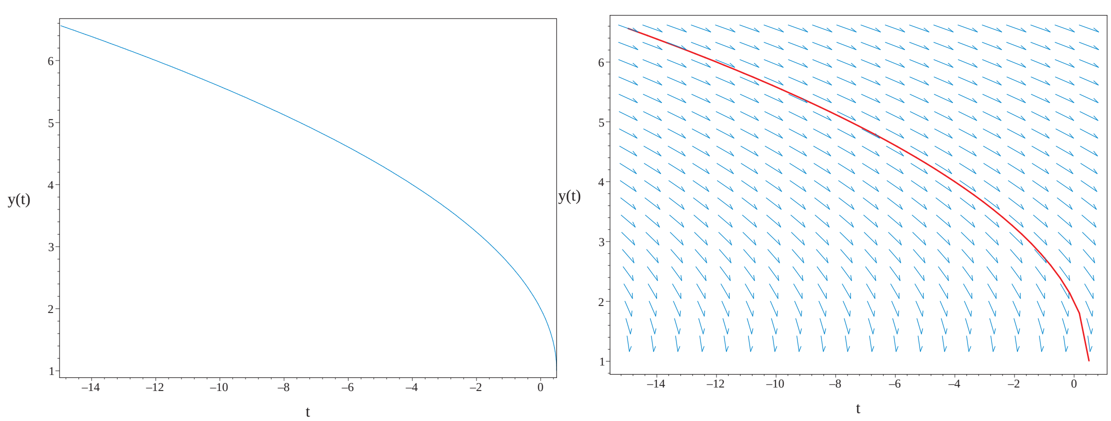
$$y - \frac{1}{2}y^2 = t + c_1$$

Solving for the constant of integration from initial conditions, the solution becomes

$$y - \frac{y^2}{2} = t$$

Solving for y gives

$$y = 1 + \sqrt{1 - 2t}$$



(a) Solution plot
$y = 1 + \sqrt{1 - 2t}$

(b) Slope field plot
$y' = \frac{1}{1-y}$

Summary of solutions found

$$y = 1 + \sqrt{1 - 2t}$$

Solved as first order Exact ode

Time used: 0.218 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,y)\,\mathrm{d}t+N(t,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(-1+y)\,\mathrm{d}y&=(-1)\,\mathrm{d}t\\ (1)\,\mathrm{d}t+(-1+y)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,y)&=1\\ N(t,y)&=-1+y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(1)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(-1+y)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(t,y)$

$$\frac{\partial \phi}{\partial t} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int M\,\mathrm{d}t\\ \int \frac{\partial \phi}{\partial t}\,\mathrm{d}t &= \int 1\,\mathrm{d}t\\ \phi &= t + f(y) \qquad (3)\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both t and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -1 + y$. Therefore equation (4) becomes

$$-1 + y = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -1 + y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (-1 + y)\,\mathrm{d}y$$

$$f(y) = -y + \frac{1}{2}y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = t - y + \frac{1}{2}y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

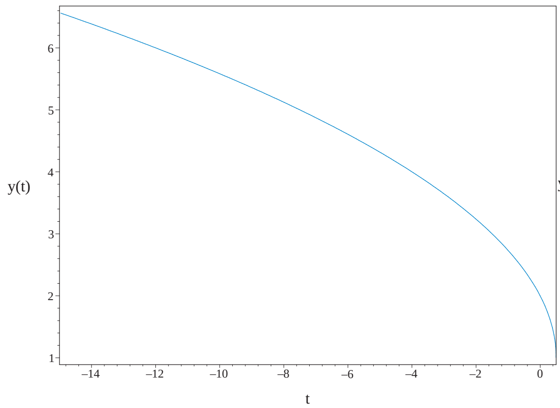
$$c_1 = t - y + \frac{1}{2}y^2$$

Solving for the constant of integration from initial conditions, the solution becomes

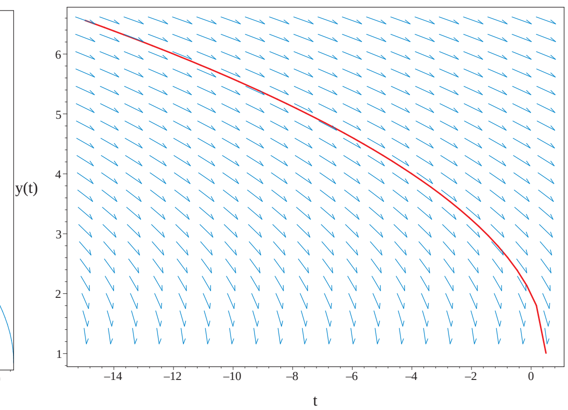
$$t - y + \frac{y^2}{2} = 0$$

Solving for y gives

$$y = 1 + \sqrt{1 - 2t}$$



(a) Solution plot
$y = 1 + \sqrt{1 - 2t}$



(b) Slope field plot
$y' = \frac{1}{1-y}$

Summary of solutions found

$$y = 1 + \sqrt{1 - 2t}$$

Solved using Lie symmetry for first order ode

Time used: 0.494 (sec)

Writing the ode as

$$y' = -\frac{1}{-1 + y}$$
$$y' = \omega(t, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_y - \xi_t) - \omega^2 \xi_y - \omega_t \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = t a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = t b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{b_3 - a_2}{-1 + y} - \frac{a_3}{(-1 + y)^2} - \frac{t b_2 + y b_3 + b_1}{(-1 + y)^2} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{-y^2 b_2 + t b_2 - a_2 y + 2 y b_2 + 2 y b_3 + a_2 + a_3 + b_1 - b_2 - b_3}{(-1 + y)^2} = 0$$

Setting the numerator to zero gives

$$y^2 b_2 - t b_2 + a_2 y - 2 y b_2 - 2 y b_3 - a_2 - a_3 - b_1 + b_2 + b_3 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{t, y\}$ in them.

$$\{t, y\}$$

The following substitution is now made to be able to collect on all terms with $\{t, y\}$ in them

$$\{t = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$b_2 v_2^2 + a_2 v_2 - b_2 v_1 - 2b_2 v_2 - 2b_3 v_2 - a_2 - a_3 - b_1 + b_2 + b_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-b_2 v_1 + b_2 v_2^2 + (a_2 - 2b_2 - 2b_3) v_2 - a_2 - a_3 - b_1 + b_2 + b_3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_2 &= 0 \\ -b_2 &= 0 \\ a_2 - 2b_2 - 2b_3 &= 0 \\ -a_2 - a_3 - b_1 + b_2 + b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= a_1 \\ a_2 &= 2b_3 \\ a_3 &= -b_3 - b_1 \\ b_1 &= b_1 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 1 \\ \eta &= 0 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(t,y)\,\xi \\
&= 0 - \left(-\frac{1}{-1+y}\right)(1) \\
&= \frac{1}{-1+y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial y}\right) S(t,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{1}{-1+y}} dy
\end{aligned}$$

Which results in

$$S = -y + \frac{1}{2}y^2$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,y)S_y}{R_t + \omega(t,y)R_y} \tag{2}$$

Where in the above R_t, R_y, S_t, S_y are all partial derivatives and $\omega(t,y)$ is the right hand side of the original ode given by

$$\omega(t,y) = -\frac{1}{-1+y}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_t &= 1 \\
R_y &= 0 \\
S_t &= 0 \\
S_y &= -1 + y
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -1 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for t, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -1
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

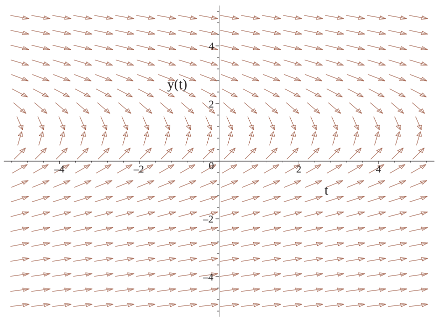
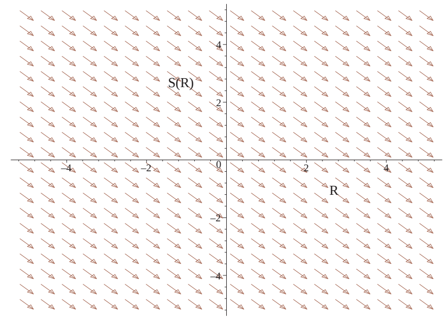
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$
\begin{aligned}
\int dS &= \int -1 \, dR \\
S(R) &= -R + c_2
\end{aligned}
$$

To complete the solution, we just need to transform the above back to t, y coordinates. This results in

$$
-y + \frac{y^2}{2} = -t + c_2
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

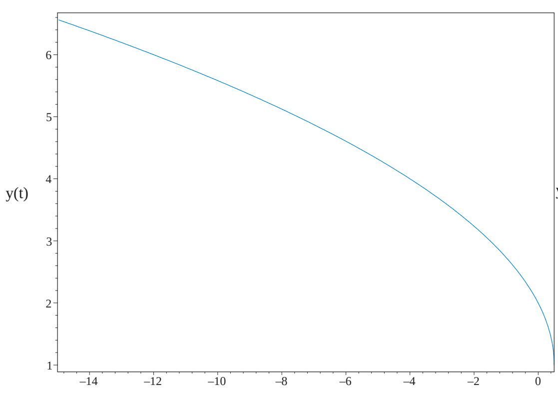
Original ode in t, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dt} = -\frac{1}{-1+y}$ 	$R = t$ $S = -y + \frac{1}{2}y^2$	$\frac{dS}{dR} = -1$ 

Solving for the constant of integration from initial conditions, the solution becomes

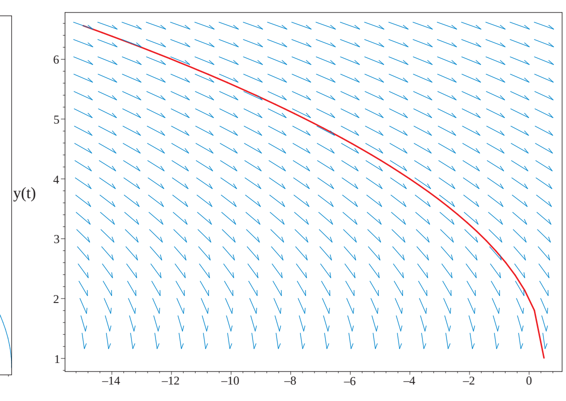
$$-y + \frac{y^2}{2} = -t$$

Solving for y gives

$$y = 1 + \sqrt{1 - 2t}$$



(a) Solution plot
$y = 1 + \sqrt{1 - 2t}$



(b) Slope field plot
$y' = \frac{1}{1-y}$

<u>Summary of solutions found</u>

$$y = 1 + \sqrt{1 - 2t}$$

Maple step by step solution

Let's solve

$\left[\frac{d}{dt}y(t) = \frac{1}{1-y(t)}, y(0) = 2\right]$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dt}y(t)$
- Solve for the highest derivative

  $\frac{d}{dt}y(t) = \frac{1}{1-y(t)}$
- Separate variables

  $\left(\frac{d}{dt}y(t)\right)(1-y(t)) = 1$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}y(t)\right)(1-y(t))\,dt = \int 1dt + C1$
- Evaluate integral

  $-\frac{y(t)^2}{2} + y(t) = t + C1$
- Solve for $y(t)$

  $\left\{y(t) = 1-\sqrt{1-2t-2C1}, y(t) = 1+\sqrt{1-2t-2C1}\right\}$
- Use initial condition $y(0) = 2$

  $2 = 1-\sqrt{1-2C1}$
- Solve for $\_C1$

  $C1 = ()$
- Solution does not satisfy initial condition
- Use initial condition $y(0) = 2$

  $2 = 1+\sqrt{1-2C1}$
- Solve for $\_C1$

  $C1 = 0$
- Substitute $\_C1 = 0$ into general solution and simplify

  $y(t) = 1+\sqrt{1-2t}$
- Solution to the IVP

  $y(t) = 1+\sqrt{1-2t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

Maple dsolve solution

Solving time : 0.036 (sec)
Leaf size : 13

```
dsolve([diff(y(t),t) = 1/(1-y(t)),
       op([y(0) = 2])],y(t),singsol=all)
```

$$y = 1 + \sqrt{-2t + 1}$$

Mathematica DSolve solution

Solving time : 0.003 (sec)
Leaf size : 16

```
DSolve[{D[y[t],t]==1/(1-y[t]),y[0]==2},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \sqrt{1 - 2t} + 1$$

2.1.33 problem 34

Internal problem ID [8421]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 34
Date solved : `Tuesday, December 17, 2024 at 12:51:08 PM`
CAS classification : `[_quadrature]`

Solve

$$p' = ap - bp^2$$

With initial conditions

$$p(\mathrm{t0}) = \mathrm{p0}$$

Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} p' &= f(t,p) \\ &= -b\,p^2 + ap \end{aligned}$$

The p domain of $f(t,p)$ when $t = \mathrm{t0}$ is

$$\{-\infty < p < \infty\}$$

But the point $p_0 = \mathrm{p0}$ is not inside this domain. Hence existence and uniqueness theorem does not apply. There could be infinite number of solutions, or one solution or no solution at all.

Solved as first order autonomous ode

Time used: 0.647 (sec)

Integrating gives

$$
\begin{aligned}
\int \frac{1}{-b\,p^2 + ap} dp &= dt \\
\frac{\ln(p) - \ln(bp - a)}{a} &= t + c_1
\end{aligned}
$$

The following diagram is the phase line diagram. It classifies each of the above equilibrium points as stable or not stable or semi-stable.

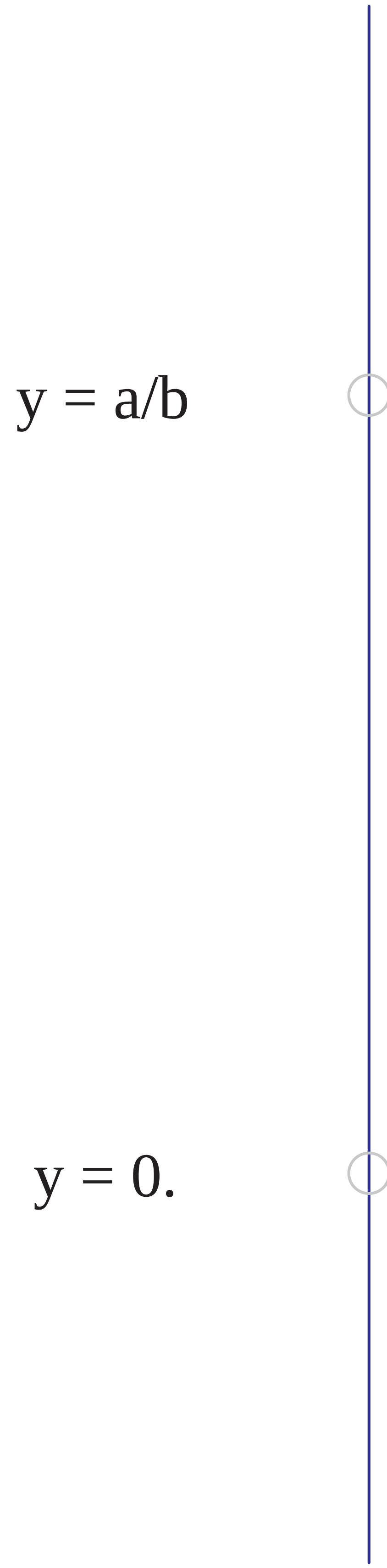


Figure 2.90: Phase line diagram

Solving for the constant of integration from initial conditions, the solution becomes

$$\frac{\ln(p)-\ln(bp-a)}{a}=t+\frac{-\mathrm{t0}\,a+\ln(\mathrm{p0})-\ln(\mathrm{p0}\,b-a)}{a}$$

Solving for p gives

$$p=\frac{a\,\mathrm{p0}}{-\mathrm{e}^{-ta+\mathrm{t0}\,a}b\,\mathrm{p0}+\mathrm{e}^{-ta+\mathrm{t0}\,a}a+\mathrm{p0}\,b}$$

Summary of solutions found

$$p=\frac{a\,\mathrm{p0}}{-\mathrm{e}^{-ta+\mathrm{t0}\,a}b\,\mathrm{p0}+\mathrm{e}^{-ta+\mathrm{t0}\,a}a+\mathrm{p0}\,b}$$

Solved as first order Bernoulli ode

Time used: 0.284 (sec)

In canonical form, the ODE is

$$\begin{aligned}p'&=F(t,p)\\&=-b\,p^2+ap\end{aligned}$$

This is a Bernoulli ODE.

$$p'=(a)\,p+(-b)\,p^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$p'=f_0(t)p+f_1(t)p^n \tag{2}$$

Comparing this to (1) shows that

$$\begin{aligned}f_0&=a\\f_1&=-b\end{aligned}$$

The first step is to divide the above equation by p^n which gives

$$\frac{p'}{p^n}=f_0(t)p^{1-n}+f_1(t) \tag{3}$$

The next step is use the substitution $v=p^{1-n}$ in equation (3) which generates a new ODE in $v(t)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $p(t)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(t) &= a \\
f_1(t) &= -b \\
n &= 2
\end{aligned}
$$

Dividing both sides of ODE (1) by $p^n = p^2$ gives

$$
p'\frac{1}{p^2} = \frac{a}{p} - b \tag{4}
$$

Let

$$
\begin{aligned}
v &= p^{1-n} \\
&= \frac{1}{p} \tag{5}
\end{aligned}
$$

Taking derivative of equation (5) w.r.t t gives

$$
v' = -\frac{1}{p^2}p' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-v'(t) &= av(t) - b \\
v' &= -av + b \tag{7}
\end{aligned}
$$

The above now is a linear ODE in $v(t)$ which is now solved.

Integrating gives

$$
\begin{aligned}
\int \frac{1}{-av+b}dv &= dt \\
-\frac{\ln(av-b)}{a} &= t + c_1
\end{aligned}
$$

Singular solutions are found by solving

$$
-av + b = 0
$$

for $v(t)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$
v(t) = \frac{b}{a}
$$

Solving for $v(t)$ gives

$$v(t) = \frac{b}{a}$$
$$v(t) = \frac{\mathrm{e}^{-c_1 a - ta} + b}{a}$$

The substitution $v = p^{1-n}$ is now used to convert the above solution back to p which results in

$$\frac{1}{p} = \frac{b}{a}$$

$$\frac{1}{p} = \frac{\mathrm{e}^{-c_1 a - ta} + b}{a}$$

Solving for the constant of integration from initial conditions, the solution becomes

$$\frac{1}{p} = \frac{\mathrm{e}^{\mathrm{t0}\, a + \ln\left(\frac{-\mathrm{p0}\, b + a}{\mathrm{p0}}\right) - ta} + b}{a}$$

Solving for p gives

$$\frac{1}{p} = \frac{b}{a}$$
$$p = \frac{a}{\mathrm{e}^{\mathrm{t0}\, a + \ln\left(\frac{-\mathrm{p0}\, b + a}{\mathrm{p0}}\right) - ta} + b}$$

The solution

$$\frac{1}{p} = \frac{b}{a}$$

was found not to satisfy the ode or the IC. Hence it is removed.

Summary of solutions found

$$p = \frac{a}{\mathrm{e}^{\mathrm{t0}\, a + \ln\left(\frac{-\mathrm{p0}\, b + a}{\mathrm{p0}}\right) - ta} + b}$$

Solved as first order Exact ode

Time used: 0.331 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(t,p)\,\mathrm{d}t + N(t,p)\,\mathrm{d}p = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}p &= \left(-b\,p^2 + ap\right)\mathrm{d}t\\ \left(b\,p^2 - ap\right)\mathrm{d}t + \mathrm{d}p &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(t,p) &= b\,p^2 - ap\\ N(t,p) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial p} = \frac{\partial N}{\partial t}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial p} &= \frac{\partial}{\partial p}\left(b\,p^2 - ap\right)\\ &= 2bp - a\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial p} \neq \frac{\partial N}{\partial t}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial p} - \frac{\partial N}{\partial t}\right)\\ &= 1((2bp - a) - (0))\\ &= 2bp - a\end{aligned}$$

Since A depends on p, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial t} - \frac{\partial M}{\partial p}\right)\\ &= -\frac{1}{p\,(-bp + a)}((0) - (2bp - a))\\ &= \frac{2bp - a}{p\,(-bp + a)}\end{aligned}$$

Since B does not depend on t, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \,\mathrm{d}p}\\ &= e^{\int \frac{2bp - a}{p(-bp+a)}\,\mathrm{d}p}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(p(bp-a))}\\ &= \frac{1}{p\,(bp - a)}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{p\,(bp-a)}\left(b\,p^2-ap\right)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{p\,(bp-a)}(1)\\ &= \frac{1}{p\,(bp-a)}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}p}{\mathrm{d}t} &= 0\\ (1)+\left(\frac{1}{p\,(bp-a)}\right)\frac{\mathrm{d}p}{\mathrm{d}t} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(t,p)$

$$\frac{\partial\phi}{\partial t} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial p} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial t}\,\mathrm{d}t &= \int\overline{M}\,\mathrm{d}t\\ \int\frac{\partial\phi}{\partial t}\,\mathrm{d}t &= \int 1\,\mathrm{d}t\end{aligned}$$

$$\phi = t+f(p) \tag{3}$$

Where $f(p)$ is used for the constant of integration since ϕ is a function of both t and p. Taking derivative of equation (3) w.r.t p gives

$$\frac{\partial\phi}{\partial p} = 0+f'(p) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial p} = \frac{1}{p(bp-a)}$. Therefore equation (4) becomes

$$\frac{1}{p(bp-a)} = 0 + f'(p) \tag{5}$$

Solving equation (5) for $f'(p)$ gives

$$f'(p) = -\frac{1}{p(-bp+a)}$$

Integrating the above w.r.t p gives

$$\int f'(p)\,\mathrm{d}p = \int \left(-\frac{1}{p(-bp+a)}\right) \mathrm{d}p$$
$$f(p) = -\frac{\ln(p)}{a} + \frac{\ln(bp-a)}{a} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(p)$ into equation (3) gives ϕ

$$\phi = t - \frac{\ln(p)}{a} + \frac{\ln(bp-a)}{a} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = t - \frac{\ln(p)}{a} + \frac{\ln(bp-a)}{a}$$

Solving for the constant of integration from initial conditions, the solution becomes

$$t - \frac{\ln(p)}{a} + \frac{\ln(bp-a)}{a} = \frac{\mathrm{t0}\, a - \ln(\mathrm{p0}) + \ln(\mathrm{p0}\, b - a)}{a}$$

Solving for p gives

$$p = \frac{a\,\mathrm{p0}}{-\mathrm{e}^{-ta+\mathrm{t0}\,a} b\,\mathrm{p0} + \mathrm{e}^{-ta+\mathrm{t0}\,a} a + \mathrm{p0}\, b}$$

<u>Summary of solutions found</u>

$$p = \frac{a\,\mathrm{p0}}{-\mathrm{e}^{-ta+\mathrm{t0}\,a} b\,\mathrm{p0} + \mathrm{e}^{-ta+\mathrm{t0}\,a} a + \mathrm{p0}\, b}$$

Solved using Lie symmetry for first order ode

Time used: 0.520 (sec)

Writing the ode as

$$
\begin{aligned}
p' &= -b\,p^2 + ap \\
p' &= \omega(t,p)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_p - \xi_t) - \omega^2\xi_p - \omega_t\xi - \omega_p\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = pa_3 + ta_2 + a_1 \tag{1E}$$

$$\eta = pb_3 + tb_2 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \left(-b\,p^2 + ap\right)(b_3 - a_2) - \left(-b\,p^2 + ap\right)^2 a_3 - (-2bp + a)(pb_3 + tb_2 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-b^2p^4a_3 + 2ab\,p^3a_3 - a^2p^2a_3 + b\,p^2a_2 + b\,p^2b_3 + 2bptb_2 - apa_2 - atb_2 + 2bpb_1 - ab_1 + b_2 = 0$$

Setting the numerator to zero gives

$$-b^2p^4a_3 + 2ab\,p^3a_3 - a^2p^2a_3 + b\,p^2a_2 + b\,p^2b_3 + 2bptb_2 - apa_2 - atb_2 + 2bpb_1 - ab_1 + b_2 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{p, t\}$ in them.

$$\{p, t\}$$

The following substitution is now made to be able to collect on all terms with $\{p, t\}$ in them

$$\{p = v_1, t = v_2\}$$

The above PDE (6E) now becomes

$$-b^2 a_3 v_1^4 + 2aba_3 v_1^3 - a^2 a_3 v_1^2 + ba_2 v_1^2 + 2bb_2 v_1 v_2 + bb_3 v_1^2 - aa_2 v_1 - ab_2 v_2 + 2bb_1 v_1 - ab_1 + b_2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-b^2 a_3 v_1^4 + 2aba_3 v_1^3 + (-a^2 a_3 + ba_2 + bb_3) v_1^2 + 2bb_2 v_1 v_2 + (-aa_2 + 2bb_1) v_1 - ab_2 v_2 - ab_1 + b_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-ab_2 &= 0 \\
2bb_2 &= 0 \\
-b^2 a_3 &= 0 \\
2aba_3 &= 0 \\
-ab_1 + b_2 &= 0 \\
-aa_2 + 2bb_1 &= 0 \\
-a^2 a_3 + ba_2 + bb_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= a_1 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = 1$$
$$\eta = 0$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(t,p)\,\xi \\
&= 0 - \left(-b\,p^2 + ap\right)(1) \\
&= b\,p^2 - ap \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t,p) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dp}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial p}\right)S(t,p) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = t$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{b\,p^2 - ap} dy
\end{aligned}$$

Which results in

$$S = -\frac{\ln(p)}{a} + \frac{\ln(bp - a)}{a}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t,p)S_p}{R_t + \omega(t,p)R_p} \tag{2}$$

Where in the above R_t, R_p, S_t, S_p are all partial derivatives and $\omega(t,p)$ is the right hand side of the original ode given by

$$\omega(t,p) = -b\,p^2 + ap$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_t &= 1\\
R_p &= 0\\
S_t &= 0\\
S_p &= -\frac{1}{p\,(-bp+a)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, p in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned}
\int dS &= \int -1\,dR\\
S(R) &= -R + c_2
\end{aligned}$$

To complete the solution, we just need to transform the above back to t, p coordinates. This results in

$$\frac{-\ln(p) + \ln(bp-a)}{a} = -t + c_2$$

Which gives

$$p = -\frac{a}{\mathrm{e}^{c_2 a - ta} - b}$$

Solving for the constant of integration from initial conditions, the solution becomes

$$p = -\frac{a}{\mathrm{e}^{\mathrm{t0}\,a + \ln\left(\frac{\mathrm{p0}\,b-a}{\mathrm{p0}}\right) - ta} - b}$$

Summary of solutions found

$$p = -\frac{a}{\mathrm{e}^{\mathrm{t0}\,a+\ln\left(\frac{\mathrm{p0}\,b-a}{\mathrm{p0}}\right)-ta} - b}$$

Maple step by step solution

Let's solve
$[p' = ap - bp^2, p(t0) = p0]$

- Highest derivative means the order of the ODE is 1
p'
- Solve for the highest derivative
$p' = ap - bp^2$
- Separate variables
$\frac{p'}{ap-bp^2} = 1$
- Integrate both sides with respect to t
$\int \frac{p'}{ap-bp^2}dt = \int 1dt + C1$
- Evaluate integral
$-\frac{\ln(bp-a)}{a} + \frac{\ln(p)}{a} = t + C1$
- Solve for p
$p = \frac{\mathrm{e}^{C1\,a+ta}a}{-1+b\,\mathrm{e}^{C1\,a+ta}}$
- Use initial condition $p(t0) = p0$
$p0 = \frac{\mathrm{e}^{C1\,a+t0\,a}a}{-1+b\,\mathrm{e}^{C1\,a+t0\,a}}$
- Solve for $\_C1$
$C1 = \frac{-t0a+\ln\left(-\frac{p0}{-p0b+a}\right)}{a}$
- Substitute $\_C1 = \frac{-t0a+\ln\left(-\frac{p0}{-p0b+a}\right)}{a}$ into general solution and simplify
$p = \frac{a\,\mathrm{e}^{a(t-t0)}p0}{bp0\,\mathrm{e}^{a(t-t0)}-p0b+a}$
- Solution to the IVP
$p = \frac{a\,\mathrm{e}^{a(t-t0)}p0}{bp0\,\mathrm{e}^{a(t-t0)}-p0b+a}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

Maple dsolve solution

Solving time : 0.013 (sec)
Leaf size : 29

```
dsolve([diff(p(t),t) = a*p(t)-b*p(t)^2,
       op([p(t0) = p0])],p(t),singsol=all)
```

$$p = \frac{\mathrm{p0}\, a}{\left(-\mathrm{p0}\, b + a\right) \mathrm{e}^{-a(t-\mathrm{t0})} + \mathrm{p0}\, b}$$

Mathematica DSolve solution

Solving time : 0.801 (sec)
Leaf size : 39

```
DSolve[{D[p[t],t]==a*p[t]-b*p[t]^2,p[t0]==p0},
       p[t],t,IncludeSingularSolutions->True]
```

$$p(t) \rightarrow \frac{a\mathrm{p0}e^{at}}{b\mathrm{p0}\left(e^{at} - e^{a\mathrm{t0}}\right) + ae^{a\mathrm{t0}}}$$

2.1.34 problem 35

Internal problem ID [8422]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 35
Date solved : `Tuesday, December 17, 2024 at 12:51:10 PM`
CAS classification :
`[[_homogeneous, ‘class G‘], _exact, _rational, _Bernoulli]`

Solve

$$y^2 + \frac{2}{x} + 2yxy' = 0$$

Solved as first order Bernoulli ode

Time used: 0.261 (sec)

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y^2 x + 2}{2y\, x^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \left(-\frac{1}{2x}\right) y + \left(-\frac{1}{x^2}\right) \frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

Comparing this to (1) shows that

$$
\begin{aligned}
f_0 &= -\frac{1}{2x} \\
f_1 &= -\frac{1}{x^2}
\end{aligned}
$$

The first step is to divide the above equation by y^n which gives

$$
\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}
$$

The next step is use the substitution $v = y^{1-n}$ in equation (3) which generates a new ODE in $v(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{1}{2x} \\
f_1(x) &= -\frac{1}{x^2} \\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$
y'y = -\frac{y^2}{2x} - \frac{1}{x^2} \tag{4}
$$

Let

$$
\begin{aligned}
v &= y^{1-n} \\
&= y^2
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
v' = 2yy' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{v'(x)}{2} &= -\frac{v(x)}{2x} - \frac{1}{x^2} \\
v' &= -\frac{v}{x} - \frac{2}{x^2}
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $v(x)$ which is now solved.

In canonical form a linear first order is

$$v'(x) + q(x)v(x) = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= \frac{1}{x} \\ p(x) &= -\frac{2}{x^2} \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= \mathrm{e}^{\int \frac{1}{x}dx} \\ &= x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu v) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu v) &= (\mu)\left(-\frac{2}{x^2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}(vx) &= (x)\left(-\frac{2}{x^2}\right) \\ \mathrm{d}(vx) &= \left(-\frac{2}{x}\right)\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} vx &= \int -\frac{2}{x}\,dx \\ &= -2\ln(x) + c_1 \end{aligned}$$

Dividing throughout by the integrating factor x gives the final solution

$$v(x) = \frac{-2\ln(x) + c_1}{x}$$

The substitution $v = y^{1-n}$ is now used to convert the above solution back to y which results in

$$y^2 = \frac{-2\ln(x) + c_1}{x}$$

Solving for y gives

$$y = \frac{\sqrt{-x\left(2\ln\left(x\right)-c_1\right)}}{x}$$
$$y = -\frac{\sqrt{-x\left(2\ln\left(x\right)-c_1\right)}}{x}$$

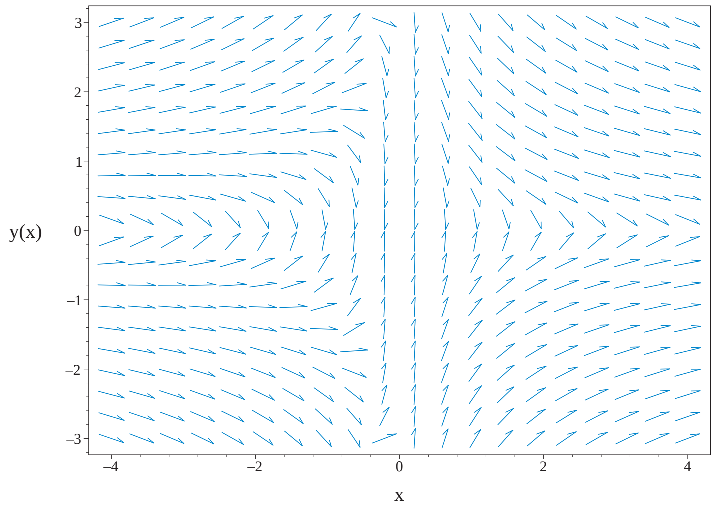


Figure 2.91: Slope field plot
$y^2 + \frac{2}{x} + 2yxy' = 0$

Summary of solutions found

$$y = \frac{\sqrt{-x\left(2\ln\left(x\right)-c_1\right)}}{x}$$
$$y = -\frac{\sqrt{-x\left(2\ln\left(x\right)-c_1\right)}}{x}$$

Solved as first order Exact ode

Time used: 0.076 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(2yx)\,\mathrm{d}y&=\left(-y^2-\frac{2}{x}\right)\mathrm{d}x\\ \left(y^2+\frac{2}{x}\right)\mathrm{d}x+(2yx)\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=y^2+\frac{2}{x}\\ N(x,y)&=2yx\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(y^2+\frac{2}{x}\right)\\ &=2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2yx)\\ &= 2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int y^2 + \frac{2}{x}\, \mathrm{d}x\end{aligned}$$

$$\phi = y^2 x + 2\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2yx + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2yx$. Therefore equation (4) becomes

$$2yx = 2yx + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = y^2 x + 2\ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = y^2 x + 2\ln(x)$$

Solving for y gives

$$y = \frac{\sqrt{-x(2\ln(x) - c_1)}}{x}$$

$$y = -\frac{\sqrt{-x(2\ln(x) - c_1)}}{x}$$

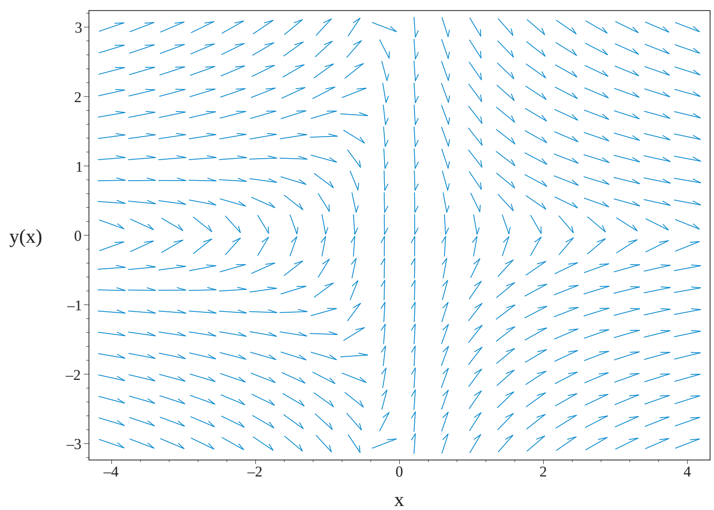


Figure 2.92: Slope field plot
$y^2 + \frac{2}{x} + 2yxy' = 0$

Summary of solutions found

$$y = \frac{\sqrt{-x(2\ln(x) - c_1)}}{x}$$

$$y = -\frac{\sqrt{-x(2\ln(x) - c_1)}}{x}$$

Solved as first order isobaric ode

Time used: 0.324 (sec)

Solving for y' gives

$$y' = -\frac{y^2x+2}{2yx^2} \tag{1}$$

Each of the above ode's is now solved An ode $y' = f(x, y)$ is isobaric if

$$f(tx, t^m y) = t^{m-1} f(x, y) \tag{1}$$

Where here

$$f(x, y) = -\frac{y^2x+2}{2yx^2} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = -\frac{1}{2}$$

Since the ode is isobaric of order $m = -\frac{1}{2}$, then the substitution

$$\begin{aligned} y &= ux^m \\ &= \frac{u}{\sqrt{x}} \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$-\frac{u(x)}{2x^{3/2}} + \frac{u'(x)}{\sqrt{x}} = -\frac{u(x)^2+2}{2x^{3/2}u(x)}$$

The ode $u'(x) = -\frac{1}{u(x)x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{1}{u(x)\, x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(u) &= \frac{1}{u} \end{aligned}$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\
\int u\,du &= \int -\frac{1}{x}\,dx \\
\frac{u(x)^2}{2} &= \ln\left(\frac{1}{x}\right) + c_1
\end{aligned}
$$

Solving for $u(x)$ gives

$$
\begin{aligned}
u(x) &= \sqrt{2\ln\left(\frac{1}{x}\right) + 2c_1} \\
u(x) &= -\sqrt{2\ln\left(\frac{1}{x}\right) + 2c_1}
\end{aligned}
$$

Converting $u(x) = \sqrt{2\ln\left(\frac{1}{x}\right) + 2c_1}$ back to y gives

$$
y\sqrt{x} = \sqrt{2\ln\left(\frac{1}{x}\right) + 2c_1}
$$

Converting $u(x) = -\sqrt{2\ln\left(\frac{1}{x}\right) + 2c_1}$ back to y gives

$$
y\sqrt{x} = -\sqrt{2\ln\left(\frac{1}{x}\right) + 2c_1}
$$

Solving for y gives

$$
\begin{aligned}
y &= \frac{\sqrt{2\ln\left(\frac{1}{x}\right) + 2c_1}}{\sqrt{x}} \\
y &= -\frac{\sqrt{2\ln\left(\frac{1}{x}\right) + 2c_1}}{\sqrt{x}}
\end{aligned}
$$

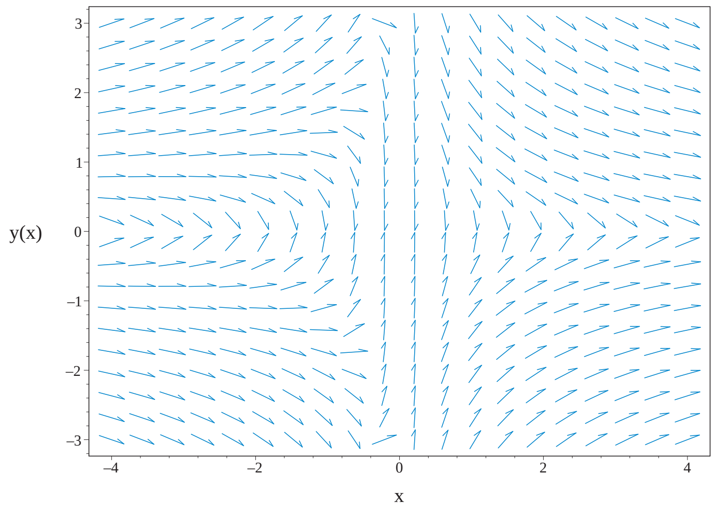


Figure 2.93: Slope field plot
$y^2 + \frac{2}{x} + 2yxy' = 0$

Summary of solutions found

$$y = \frac{\sqrt{2\ln\left(\frac{1}{x}\right) + 2c_1}}{\sqrt{x}}$$

$$y = -\frac{\sqrt{2\ln\left(\frac{1}{x}\right) + 2c_1}}{\sqrt{x}}$$

Solved using Lie symmetry for first order ode

Time used: 0.486 (sec)

Writing the ode as

$$y' = -\frac{y^2 x + 2}{2y\,x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(y^2x+2)(b_3-a_2)}{2y\,x^2} - \frac{(y^2x+2)^2 a_3}{4y^2x^4} - \left(-\frac{y}{2x^2} + \frac{y^2x+2}{y\,x^3}\right)(xa_2+ya_3+a_1) - \left(-\frac{1}{x} + \frac{y^2x+2}{2y^2x^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{6b_2y^2x^4 - 3x^2y^4a_3 + 2x^3y^2b_1 - 2x^2y^3a_1 - 4x^3b_2 - 4x^2ya_2 - 8x^2yb_3 - 12x\,y^2a_3 - 4x^2b_1 - 8xya_1 - 4a_3}{4y^2x^4} = 0$$

Setting the numerator to zero gives

$$6b_2y^2x^4 - 3x^2y^4a_3 + 2x^3y^2b_1 - 2x^2y^3a_1 - 4x^3b_2 - 4x^2ya_2 - 8x^2yb_3 - 12x\,y^2a_3 - 4x^2b_1 - 8xya_1 - 4a_3 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-3a_3v_1^2v_2^4 + 6b_2v_1^4v_2^2 - 2a_1v_1^2v_2^3 + 2b_1v_1^3v_2^2 - 4a_2v_1^2v_2 - 12a_3v_1v_2^2 - 4b_2v_1^3 - 8b_3v_1^2v_2 - 8a_1v_1v_2 - 4b_1v_1^2 - 4a_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}&6b_2v_1^4v_2^2 + 2b_1v_1^3v_2^2 - 4b_2v_1^3 - 3a_3v_1^2v_2^4 - 2a_1v_1^2v_2^3\\&+(-4a_2 - 8b_3)\,v_1^2v_2 - 4b_1v_1^2 - 12a_3v_1v_2^2 - 8a_1v_1v_2 - 4a_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-8a_1 &= 0\\ -2a_1 &= 0\\ -12a_3 &= 0\\ -4a_3 &= 0\\ -3a_3 &= 0\\ -4b_1 &= 0\\ 2b_1 &= 0\\ -4b_2 &= 0\\ 6b_2 &= 0\\ -4a_2 - 8b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0\\ a_2 &= -2b_3\\ a_3 &= 0\\ b_1 &= 0\\ b_2 &= 0\\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= -2x\\ \eta &= y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{y^2 x + 2}{2y\,x^2}\right)(-2x) \\
&= -\frac{2}{xy} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\frac{2}{xy}} dy
\end{aligned}$$

Which results in

$$S = -\frac{y^2 x}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y^2 x + 2}{2y\,x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{y^2}{4} \\
S_y &= -\frac{yx}{2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

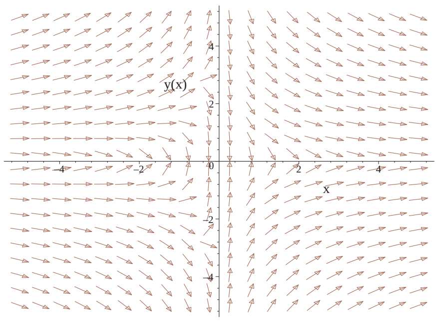
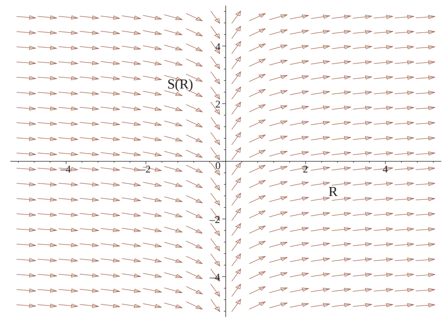
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned}
\int dS &= \int \frac{1}{2R}\, dR \\
S(R) &= \frac{\ln(R)}{2} + c_2
\end{aligned}$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$-\frac{y^2 x}{4} = \frac{\ln(x)}{2} + c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y^2x+2}{2y\,x^2}$  	$R = x$ $S = -\frac{y^2x}{4}$	$\frac{dS}{dR} = \frac{1}{2R}$ 

Solving for y gives

$$y = \frac{\sqrt{-2x\left(\ln\left(x\right)+2c_2\right)}}{x}$$

$$y = -\frac{\sqrt{-2x\left(\ln\left(x\right)+2c_2\right)}}{x}$$

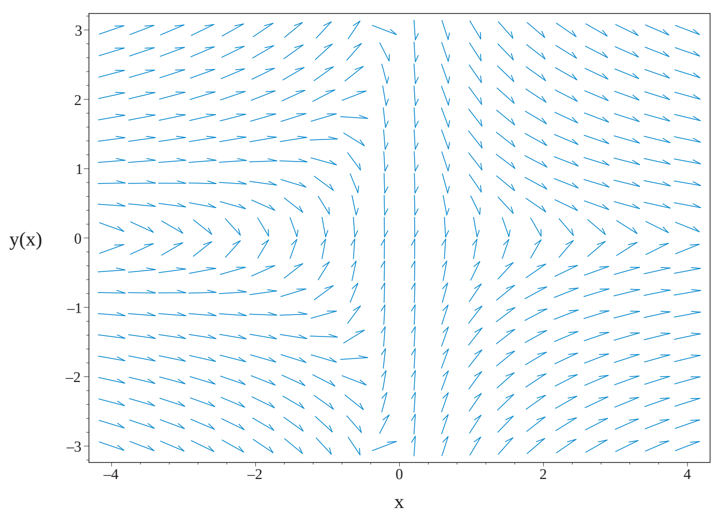


Figure 2.94: Slope field plot
$y^2 + \frac{2}{x} + 2yxy' = 0$

Summary of solutions found

$$y = \frac{\sqrt{-2x\left(\ln\left(x\right)+2c_2\right)}}{x}$$

$$y = -\frac{\sqrt{-2x\left(\ln\left(x\right)+2c_2\right)}}{x}$$

Maple step by step solution

Let's solve

$y(x)^2 + \frac{2}{x} + 2xy(x)\left(\frac{d}{dx}y(x)\right) = 0$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $\frac{d}{dx}F(x, y(x)) = 0$

  ◦ Compute derivative of lhs

    $\frac{\partial}{\partial x}F(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right)\left(\frac{d}{dx}y(x)\right) = 0$

  ◦ Evaluate derivatives

    $2y = 2y$

  ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = C1, M(x, y) = \frac{\partial}{\partial x}F(x, y), N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int \left(y^2 + \frac{2}{x}\right)dx + \_F1(y)$

- Evaluate integral

  $F(x, y) = x\,y^2 + 2\ln\left(x\right) + \_F1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$

- Compute derivative

  $2xy = 2xy + \frac{d}{dy}\_F1(y)$

- Isolate for $\frac{d}{dy}\_F1(y)$

  $\frac{d}{dy}\_F1(y) = 0$

- Solve for $\_F1(y)$
  $\_F1(y) = 0$
- Substitute $\_F1(y)$ into equation for $F(x,y)$
  $F(x,y) = x\,y^2 + 2\ln(x)$
- Substitute $F(x,y)$ into the solution of the ODE
  $x\,y^2 + 2\ln(x) = C1$
- Solve for $y(x)$
  $\left\{y(x) = \frac{\sqrt{-x(2\ln(x)-C1)}}{x}, y(x) = -\frac{\sqrt{-x(2\ln(x)-C1)}}{x}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

Maple dsolve solution

Solving time : `0.023` (sec)
Leaf size : 36

```
dsolve(y(x)^2+2/x+2*x*y(x)*diff(y(x),x) = 0,
       y(x),singsol=all)
```

$$y = \frac{\sqrt{x\left(-2\ln(x) + c_1\right)}}{x}$$
$$y = -\frac{\sqrt{x\left(-2\ln(x) + c_1\right)}}{x}$$

Mathematica DSolve solution

Solving time : 0.205 (sec)
Leaf size : 44

```
DSolve[{(y[x]^2+2/x)+2*y[x]*x*D[y[x],x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{\sqrt{-2\log(x) + c_1}}{\sqrt{x}}$$
$$y(x) \to \frac{\sqrt{-2\log(x) + c_1}}{\sqrt{x}}$$

2.1.35 problem 36

Internal problem ID [8423]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 36
Date solved : `Tuesday, December 17, 2024 at 12:51:12 PM`
CAS classification : `[_Clairaut]`

Solve

$$xf' - f = \frac{f'^2\left(1 - f'^\lambda\right)^2}{\lambda^2}$$

Solved as first order Clairaut ode

Time used: 0.075 (sec)

This is Clairaut ODE. It has the form

$$f = xf' + g(f')$$

Where g is function of $f'(x)$. Let $p = f'$ the ode becomes

$$xp - f = \frac{p^2(1 - p^\lambda)^2}{\lambda^2}$$

Solving for f from the above results in

$$f = -\frac{p\left(p^{2\lambda}p - x\,\lambda^2 - 2p^\lambda p + p\right)}{\lambda^2} \tag{1A}$$

The above ode is a Clairaut ode which is now solved.

We start by replacing f' by p which gives

$$\begin{aligned} f &= xp - \frac{p^2(p^{2\lambda} - 2p^\lambda + 1)}{\lambda^2} \\ &= xp - \frac{p^2(p^{2\lambda} - 2p^\lambda + 1)}{\lambda^2} \end{aligned}$$

Writing the ode as

$$f = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$f = xp + g \tag{1}$$

Then we see that

$$g = -\frac{p^2(p^{2\lambda} - 2p^\lambda + 1)}{\lambda^2}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p.

The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$f = c_2x - \frac{c_2^2(c_2^{2\lambda} - 2c_2^\lambda + 1)}{\lambda^2}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = -\frac{p^2(p^{2\lambda}-2p^\lambda+1)}{\lambda^2}$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x - \frac{2p(p^{2\lambda} - 2p^\lambda + 1)}{\lambda^2} - \frac{p^2\left(\frac{2p^{2\lambda}\lambda}{p} - \frac{2p^\lambda\lambda}{p}\right)}{\lambda^2} \\
&= 0
\end{aligned}$$

Unable to solve for p. No singular solutions can be found.

Summary of solutions found

$$f = c_2 x - \frac{c_2^2 \left(c_2^{2\lambda} - 2c_2^{\lambda} + 1\right)}{\lambda^2}$$

Maple step by step solution

Let's solve

$$x\left(\frac{d}{dx}f(x)\right) - f(x) = \frac{\left(\frac{d}{dx}f(x)\right)^2 \left(1-\left(\frac{d}{dx}f(x)\right)^{\lambda}\right)^2}{\lambda^2}$$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}f(x)$

- Solve for the highest derivative

  $\frac{d}{dx}f(x) = \mathit{RootOf}\left(\left(\_Z^{\lambda}\right)^2 \_Z^2 - \_Zx\,\lambda^2 + f(x)\,\lambda^2 - 2\_Z^{\lambda}\_Z^2 + \_Z^2\right)$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- 1st order, parametric methods successful
<- dAlembert successful`
```

Maple dsolve solution

Solving time : 0.677 (sec)
Leaf size : 318

```
dsolve(x*diff(f(x),x)-f(x) = diff(f(x),x)^2/lambda^2*(1-diff(f(x),x)^lambda)^2,
       f(x),singsol=all)
```

$$f = 0$$

$$f = \frac{\lambda^2 x^2 \left(2\lambda\, \mathrm{e}^{\mathrm{RootOf}\left(2\lambda\, \mathrm{e}^{\_Z(2\lambda+1)}+2\, \mathrm{e}^{\_Z(2\lambda+1)}-2\lambda\, \mathrm{e}^{\_Z(\lambda+1)}-x\,\lambda^2-4\, \mathrm{e}^{\_Z(\lambda+1)}+2\, \mathrm{e}^{\_Z}\right)\lambda} + \mathrm{e}^{\mathrm{Roo}} \right.}{4\left(\lambda\, \mathrm{e}^{\mathrm{RootOf}\left(2\lambda\, \mathrm{e}^{\_Z(2\lambda+1)}+2\, \mathrm{e}^{\_Z(2\lambda+1)}-2\lambda\, \mathrm{e}^{\_Z(\lambda+1)}-x\,\lambda^2-4\, \mathrm{e}^{\_Z(\lambda+1)}+2\, \mathrm{e}^{\_Z}\right)\lambda} + \mathrm{e}^{\mathrm{RootOf}\left(2\lambda\, \mathrm{e}^{\_Z(2\lambda+1)}+2\, \mathrm{e}^{\_Z(2\lambda+1)}-2\lambda\, \mathrm{e}^{\_Z(\lambda+1)}-\right.}\right.}$$

$$f = c_1 x - \frac{c_1^2 \left(-1 + c_1^\lambda\right)^2}{\lambda^2}$$

Mathematica DSolve solution

Solving time : 14.489 (sec)
Leaf size : 30

```
DSolve[{x*D[ f[x],x]-f[x]==D[ f[x],x]^2/\[Lambda]^2*(1-D[ f[x],x]^\[Lambda])^2,{}},
       f[x],x,IncludeSingularSolutions->True]
```

$$f(x) \to c_1 \left(x - \frac{c_1 \left(-1 + c_1{}^\lambda\right)^2}{\lambda^2}\right)$$
$$f(x) \to 0$$

2.1.36 problem 37

Internal problem ID [8424]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 37
Date solved : `Tuesday, December 17, 2024 at 12:51:16 PM`
CAS classification : `[_rational, _Riccati]`

Solve

$$xy' - 2y + by^2 = c\,x^4$$

Solved as first order ode of type Riccati

Time used: 0.537 (sec)

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{-c\,x^4 + b\,y^2 - 2y}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = c\,x^3 - \frac{b\,y^2}{x} + \frac{2y}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = c\,x^3$, $f_1(x) = \frac{2}{x}$ and $f_2(x) = -\frac{b}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{ub}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{b}{x^2} \\ f_1f_2 &= -\frac{2b}{x^2} \\ f_2^2 f_0 &= b^2xc \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{bu''(x)}{x} + \frac{bu'(x)}{x^2} + b^2xcu(x) = 0$$

In normal form the ode

$$-\frac{b\left(\frac{d^2u}{dx^2}\right)}{x} + \frac{b\left(\frac{du}{dx}\right)}{x^2} + b^2xcu = 0 \tag{1}$$

Becomes

$$\frac{d^2u}{dx^2} + p(x)\left(\frac{du}{dx}\right) + q(x)\, u = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= -x^2bc \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}u(\tau) + p_1\left(\frac{d}{d\tau}u(\tau)\right) + q_1u(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\frac{d^2}{dx^2}\tau(x) + p(x)\left(\frac{d}{dx}\tau(x)\right)}{\left(\frac{d}{dx}\tau(x)\right)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\left(\frac{d}{dx}\tau(x)\right)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\frac{d^2}{dx^2}\tau(x) + p(x)\left(\frac{d}{dx}\tau(x)\right) = 0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int \mathrm{e}^{-\int p(x)dx}dx\\ &= \int \mathrm{e}^{-\int -\frac{1}{x}dx}dx\\ &= \int e^{\ln(x)}\,dx\\ &= \int x dx\\ &= \frac{x^2}{2}\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}q_1(\tau) &= \frac{q(x)}{\left(\frac{d}{dx}\tau\left(x\right)\right)^2}\\ &= \frac{-x^2bc}{x^2}\\ &= -bc\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}\frac{d^2}{d\tau^2}u(\tau) + q_1u(\tau) = 0\\ \frac{d^2}{d\tau^2}u(\tau) - bcu(\tau) = 0\end{aligned}$$

The above ode is now solved for $u(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Au''(\tau) + Bu'(\tau) + Cu(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -bc$. Let the solution be $u(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2\mathrm{e}^{\tau\lambda} - bc\,\mathrm{e}^{\tau\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$-bc + \lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -bc$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-bc)}\\ &= \pm\sqrt{bc}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\sqrt{bc}\\ \lambda_2 &= -\sqrt{bc}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \sqrt{bc}\\ \lambda_2 &= -\sqrt{bc}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}u(\tau) &= c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau}\\ u(\tau) &= c_1 e^{\left(\sqrt{bc}\right)\tau} + c_2 e^{\left(-\sqrt{bc}\right)\tau}\end{aligned}$$

Or

$$u(\tau) = c_1\, \mathrm{e}^{\tau\sqrt{bc}} + c_2\, \mathrm{e}^{-\tau\sqrt{bc}}$$

Will add steps showing solving for IC soon.

The above solution is now transformed back to u using (6) which results in

$$u = c_1\, \mathrm{e}^{\frac{x^2\sqrt{bc}}{2}} + c_2\, \mathrm{e}^{-\frac{x^2\sqrt{bc}}{2}}$$

Will add steps showing solving for IC soon.

Taking derivative gives

$$u'(x) = c_1 x\sqrt{bc}\,\mathrm{e}^{\frac{x^2\sqrt{bc}}{2}} - c_2 x\sqrt{bc}\,\mathrm{e}^{-\frac{x^2\sqrt{bc}}{2}}$$

Doing change of constants, the solution becomes

$$y = \frac{\left(c_3 x\sqrt{bc}\,\mathrm{e}^{\frac{x^2\sqrt{bc}}{2}} - x\sqrt{bc}\,\mathrm{e}^{-\frac{x^2\sqrt{bc}}{2}}\right)x}{b\left(c_3\,\mathrm{e}^{\frac{x^2\sqrt{bc}}{2}} + \mathrm{e}^{-\frac{x^2\sqrt{bc}}{2}}\right)}$$

Summary of solutions found

$$y = \frac{\left(c_3 x\sqrt{bc}\,\mathrm{e}^{\frac{x^2\sqrt{bc}}{2}} - x\sqrt{bc}\,\mathrm{e}^{-\frac{x^2\sqrt{bc}}{2}}\right)x}{b\left(c_3\,\mathrm{e}^{\frac{x^2\sqrt{bc}}{2}} + \mathrm{e}^{-\frac{x^2\sqrt{bc}}{2}}\right)}$$

Maple step by step solution

Let's solve
$x\left(\frac{d}{dx}y(x)\right) - 2y(x) + by(x)^2 = c\,x^4$

- Highest derivative means the order of the ODE is 1
$\frac{d}{dx}y(x)$
- Solve for the highest derivative
$\frac{d}{dx}y(x) = \frac{2y(x) - by(x)^2 + c\,x^4}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
<- Chini successful`
```

Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 29

```
dsolve(diff(y(x),x)*x-2*y(x)+b*y(x)^2 = c*x^4,
       y(x),singsol=all)
```

$$y = \frac{i \tan\left(-\frac{i x^2 \sqrt{b}\,\sqrt{c}}{2} + c_1\right) x^2 \sqrt{c}}{\sqrt{b}}$$

Mathematica DSolve solution

Solving time : 0.235 (sec)
Leaf size : 153

```
DSolve[{x*D[y[x],x]-2*y[x]+b*y[x]^2==c*x^4,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt{c}x^2\left(-\cos\left(\frac{1}{2}\sqrt{-b}\sqrt{c}x^2\right) + c_1 \sin\left(\frac{1}{2}\sqrt{-b}\sqrt{c}x^2\right)\right)}{\sqrt{-b}\left(\sin\left(\frac{1}{2}\sqrt{-b}\sqrt{c}x^2\right) + c_1 \cos\left(\frac{1}{2}\sqrt{-b}\sqrt{c}x^2\right)\right)}$$

$$y(x) \to \frac{\sqrt{c}x^2 \tan\left(\frac{1}{2}\sqrt{-b}\sqrt{c}x^2\right)}{\sqrt{-b}}$$

2.1.37 problem 38

Internal problem ID [8425]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 38
Date solved : `Tuesday, December 17, 2024 at 12:51:18 PM`
CAS classification : `[_rational, _Riccati]`

Solve

$$xy' - y + y^2 = x^{2/3}$$

Solved as first order ode of type Riccati

Time used: 0.365 (sec)

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y^2 - x^{2/3} - y}{x} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{y^2}{x} + \frac{1}{x^{1/3}} + \frac{y}{x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{1}{x^{1/3}}$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = -\frac{1}{x}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{u}{x}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{1}{x^2} \\ f_1 f_2 &= -\frac{1}{x^2} \\ f_2^2 f_0 &= \frac{1}{x^{7/3}} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{u''(x)}{x} + \frac{u(x)}{x^{7/3}} = 0$$

Writing the ode as

$$x^2 \left(\frac{d^2 u}{dx^2}\right) - x^{2/3} u = 0 \tag{1}$$

Bessel ode has the form

$$x^2 \left(\frac{d^2 u}{dx^2}\right) + \left(\frac{du}{dx}\right) x + \left(-n^2 + x^2\right) u = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 \left(\frac{d^2 u}{dx^2}\right) + (1 - 2\alpha)\, x \left(\frac{du}{dx}\right) + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) u = 0 \tag{3}$$

With the standard solution

$$u = x^{\alpha} (c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned} \alpha &= \frac{1}{2} \\ \beta &= 3i \\ n &= \frac{3}{2} \\ \gamma &= \frac{1}{3} \end{aligned}$$

Substituting all the above into (4) gives the solution as

$$u=-\frac{c_1\,x^{1/6}\sqrt{2}\,\sqrt{3}\,\left(3\cosh\left(3x^{1/3}\right)x^{1/3}-\sinh\left(3x^{1/3}\right)\right)}{9\sqrt{\pi}\,\sqrt{i x^{1/3}}}-\frac{i c_2\,x^{1/6}\sqrt{2}\,\sqrt{3}\,\left(3\sinh\left(3x^{1/3}\right)x^{1/3}-\cosh\left(3x^{1/3}\right.\right.}{9\sqrt{\pi}\,\sqrt{i x^{1/3}}}$$

Will add steps showing solving for IC soon.

Taking derivative gives

$$\begin{aligned}u'(x)=&-\frac{c_1\sqrt{2}\,\sqrt{3}\,\left(3\cosh\left(3x^{1/3}\right)x^{1/3}-\sinh\left(3x^{1/3}\right)\right)}{54x^{5/6}\sqrt{\pi}\,\sqrt{i x^{1/3}}}-\frac{c_1\sqrt{2}\,\sqrt{3}\,\sinh\left(3x^{1/3}\right)}{3x^{1/6}\sqrt{\pi}\,\sqrt{i x^{1/3}}}\\&+\frac{i c_1\sqrt{2}\,\sqrt{3}\,\left(3\cosh\left(3x^{1/3}\right)x^{1/3}-\sinh\left(3x^{1/3}\right)\right)}{54\sqrt{x}\,\sqrt{\pi}\,\left(i x^{1/3}\right)^{3/2}}\\&-\frac{i c_2\sqrt{2}\,\sqrt{3}\,\left(3\sinh\left(3x^{1/3}\right)x^{1/3}-\cosh\left(3x^{1/3}\right)\right)}{54x^{5/6}\sqrt{\pi}\,\sqrt{i x^{1/3}}}\\&-\frac{i c_2\sqrt{2}\,\sqrt{3}\,\cosh\left(3x^{1/3}\right)}{3x^{1/6}\sqrt{\pi}\,\sqrt{i x^{1/3}}}-\frac{c_2\sqrt{2}\,\sqrt{3}\,\left(3\sinh\left(3x^{1/3}\right)x^{1/3}-\cosh\left(3x^{1/3}\right)\right)}{54\sqrt{x}\,\sqrt{\pi}\,\left(i x^{1/3}\right)^{3/2}}\end{aligned}$$

Doing change of constants, the solution becomes

$$y=\frac{\left(-\frac{c_3\sqrt{2}\,\sqrt{3}\,\left(3\cosh\left(3x^{1/3}\right)x^{1/3}-\sinh\left(3x^{1/3}\right)\right)}{54x^{5/6}\sqrt{\pi}\,\sqrt{i x^{1/3}}}-\frac{c_3\sqrt{2}\,\sqrt{3}\,\sinh\left(3x^{1/3}\right)}{3x^{1/6}\sqrt{\pi}\,\sqrt{i x^{1/3}}}+\frac{i c_3\sqrt{2}\,\sqrt{3}\,\left(3\cosh\left(3x^{1/3}\right)x^{1/3}-\sinh\left(3x^{1/3}\right)\right)}{54\sqrt{x}\,\sqrt{\pi}\,\left(i x^{1/3}\right)^{3/2}}-\frac{i\sqrt{2}\,\sqrt{3}\,\left(3\sin\right.}{5}\right.}{-\frac{c_3\,x^{1/6}\sqrt{2}\,\sqrt{3}\,\left(3\cosh\left(3x^{1/3}\right)x^{1/3}-\sinh\left(3x^{1/3}\right)\right)}{9\sqrt{\pi}\,\sqrt{i x^{1/3}}}-\frac{i x^{1/6}\sqrt{2}\,\sqrt{3}\,\left(\right.}{}}$$

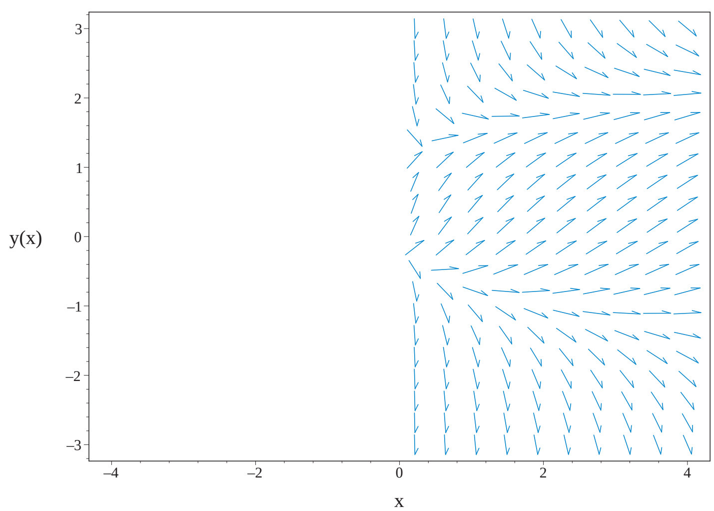


Figure 2.95: Slope field plot
$xy'-y+y^2=x^{2/3}$

Summary of solutions found

$$y = \frac{\left(-\frac{c_3\sqrt{2}\,\sqrt{3}\,(3\cosh(3x^{1/3})x^{1/3}-\sinh(3x^{1/3}))}{54x^{5/6}\sqrt{\pi}\,\sqrt{ix^{1/3}}} - \frac{c_3\sqrt{2}\,\sqrt{3}\,\sinh(3x^{1/3})}{3x^{1/6}\sqrt{\pi}\,\sqrt{ix^{1/3}}} + \frac{ic_3\sqrt{2}\,\sqrt{3}\,(3\cosh(3x^{1/3})x^{1/3}-\sinh(3x^{1/3}))}{54\sqrt{x}\,\sqrt{\pi}\,(ix^{1/3})^{3/2}} - \frac{i\sqrt{2}\,\sqrt{3}\,(3\sin}{5}\right.}{-\frac{c_3\,x^{1/6}\sqrt{2}\,\sqrt{3}\,(3\cosh(3x^{1/3})x^{1/3}-\sinh(3x^{1/3}))}{9\sqrt{\pi}\,\sqrt{ix^{1/3}}} - \frac{ix^{1/6}\sqrt{2}\,\sqrt{3}\,(}{}}$$

Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y(x)\right) - y(x) + y(x)^2 = x^{2/3}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = \frac{y(x)-y(x)^2+x^{2/3}}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = y(x)/x^(4/3), y(x)`
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
```

```
    checking if the LODE is missing y
    -> Trying an equivalence, under non-integer power transformations,
       to LODEs admitting Liouvillian solutions.
       -> Trying a Liouvillian solution using Kovacics algorithm
          A Liouvillian solution exists
          Group is reducible or imprimitive
       <- Kovacics algorithm successful
    <- Equivalence, under non-integer power transformations successful
 <- Riccati to 2nd Order successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 72

```
dsolve(diff(y(x),x)*x-y(x)+y(x)^2 = x^(2/3),
       y(x),singsol=all)
```

$$y = \frac{x^{1/3}\left(c_1 e^{6x^{1/3}} \operatorname{abs}\left(1, 3x^{1/3}-1\right) + c_1 e^{6x^{1/3}} |3x^{1/3}-1| - 3x^{1/3}\right)}{c_1 e^{6x^{1/3}} |3x^{1/3}-1| + 3x^{1/3} + 1}$$

Mathematica DSolve solution

Solving time : 0.193 (sec)
Leaf size : 131

```
DSolve[{x*D[y[x],x]-y[x]+y[x]^2==x^(2/3),{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{3x^{2/3}\left(c_1 \cosh\left(3\sqrt[3]{x}\right) - i \sinh\left(3\sqrt[3]{x}\right)\right)}{\left(-3i\sqrt[3]{x} - c_1\right)\cosh\left(3\sqrt[3]{x}\right) + \left(3c_1\sqrt[3]{x} + i\right)\sinh\left(3\sqrt[3]{x}\right)}$$

$$y(x) \to \frac{3x^{2/3}\cosh\left(3\sqrt[3]{x}\right)}{3\sqrt[3]{x}\sinh\left(3\sqrt[3]{x}\right) - \cosh\left(3\sqrt[3]{x}\right)}$$

2.1.38 problem 39

Internal problem ID [8426]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 39
Date solved : `Tuesday, December 17, 2024 at 12:51:30 PM`
CAS classification : `[_rational, _Riccati]`

Solve

$$u' + u^2 = \frac{1}{x^{4/5}}$$

Solved as first order ode of type Riccati

Time used: 0.260 (sec)

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= -\frac{u^2 x^{4/5} - 1}{x^{4/5}} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$u' = -u^2 + \frac{1}{x^{4/5}}$$

With Riccati ODE standard form

$$u' = f_0(x) + f_1(x)u + f_2(x)u^2$$

Shows that $f_0(x) = \frac{1}{x^{4/5}}$, $f_1(x) = 0$ and $f_2(x) = -1$. Let

$$\begin{aligned} u &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \frac{1}{x^{4/5}} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-u''(x) + \frac{u(x)}{x^{4/5}} = 0$$

Writing the ode as

$$x^2 \left(\frac{d^2 u}{dx^2}\right) - x^{6/5} u = 0 \tag{1}$$

Bessel ode has the form

$$x^2 \left(\frac{d^2 u}{dx^2}\right) + \left(\frac{du}{dx}\right) x + \left(-n^2 + x^2\right) u = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 \left(\frac{d^2 u}{dx^2}\right) + (1 - 2\alpha)\, x \left(\frac{du}{dx}\right) + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) u = 0 \tag{3}$$

With the standard solution

$$u = x^{\alpha} (c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned} \alpha &= \frac{1}{2} \\ \beta &= \frac{5i}{3} \\ n &= \frac{5}{6} \\ \gamma &= \frac{3}{5} \end{aligned}$$

Substituting all the above into (4) gives the solution as

$$u = c_1\sqrt{x}\,\mathrm{BesselJ}\left(\frac{5}{6}, \frac{5ix^{3/5}}{3}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\frac{5}{6}, \frac{5ix^{3/5}}{3}\right)$$

Will add steps showing solving for IC soon.

Taking derivative gives

$$u'(x) = \frac{c_1\,\mathrm{BesselJ}\left(\frac{5}{6}, \frac{5ix^{3/5}}{3}\right)}{2\sqrt{x}} + ic_1\,x^{1/10}\left(\mathrm{BesselJ}\left(-\frac{1}{6}, \frac{5ix^{3/5}}{3}\right) + \frac{i\,\mathrm{BesselJ}\left(\frac{5}{6}, \frac{5ix^{3/5}}{3}\right)}{2x^{3/5}}\right) + \frac{c_2\,\mathrm{BesselY}\left(\frac{5}{6}, \frac{5ix^{3/5}}{3}\right)}{2\sqrt{x}} + ic_2\,x^{1/10}\left(\mathrm{BesselY}\right.$$

Doing change of constants, the solution becomes

$$u = \frac{\frac{c_3\,\mathrm{BesselJ}\left(\frac{5}{6}, \frac{5ix^{3/5}}{3}\right)}{2\sqrt{x}} + ic_3\,x^{1/10}\left(\mathrm{BesselJ}\left(-\frac{1}{6}, \frac{5ix^{3/5}}{3}\right) + \frac{i\,\mathrm{BesselJ}\left(\frac{5}{6}, \frac{5ix^{3/5}}{3}\right)}{2x^{3/5}}\right) + \frac{\mathrm{BesselY}\left(\frac{5}{6}, \frac{5ix^{3/5}}{3}\right)}{2\sqrt{x}} + ix^{1/10}\left(\mathrm{Bess}\right.}{c_3\sqrt{x}\,\mathrm{BesselJ}\left(\frac{5}{6}, \frac{5ix^{3/5}}{3}\right) + \sqrt{x}\,\mathrm{BesselY}\left(\frac{5}{6}, \frac{5ix^{3/5}}{3}\right)}$$

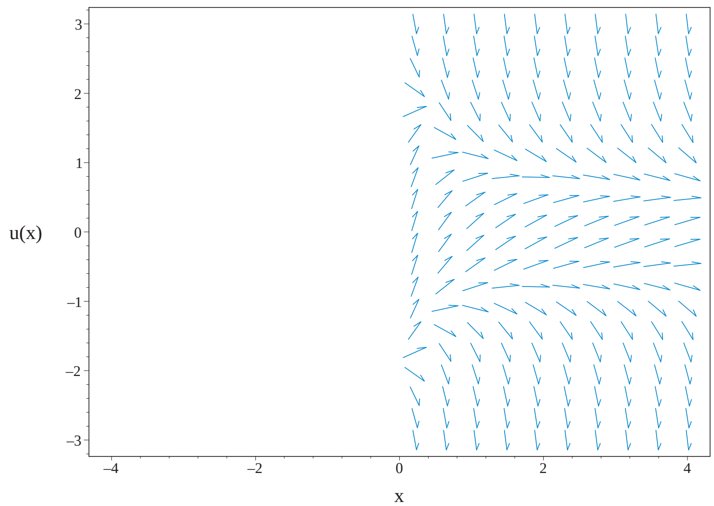


Figure 2.96: Slope field plot
$u' + u^2 = \frac{1}{x^{4/5}}$

Summary of solutions found

$$u = \frac{\frac{c_3 \operatorname{BesselJ}\left(\frac{5}{6}, \frac{5 i x^{3/5}}{3}\right)}{2\sqrt{x}} + i c_3 x^{1/10}\left(\operatorname{BesselJ}\left(-\frac{1}{6}, \frac{5 i x^{3/5}}{3}\right) + \frac{i \operatorname{BesselJ}\left(\frac{5}{6}, \frac{5 i x^{3/5}}{3}\right)}{2 x^{3/5}}\right) + \frac{\operatorname{BesselY}\left(\frac{5}{6}, \frac{5 i x^{3/5}}{3}\right)}{2\sqrt{x}} + i x^{1/10}\left(\operatorname{Bess}\right.}{c_3 \sqrt{x} \operatorname{BesselJ}\left(\frac{5}{6}, \frac{5 i x^{3/5}}{3}\right) + \sqrt{x} \operatorname{BesselY}\left(\frac{5}{6}, \frac{5 i x^{3/5}}{3}\right)}$$

Maple step by step solution

Let's solve

$\frac{d}{dx}u(x) + u(x)^2 = \frac{1}{x^{4/5}}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}u(x)$

- Solve for the highest derivative

  $\frac{d}{dx}u(x) = -u(x)^2 + \frac{1}{x^{4/5}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
<- Riccati Special successful`
```

Maple dsolve solution

Solving time : 0.014 (sec)
Leaf size : 46

```
dsolve(diff(u(x),x)+u(x)^2 = 1/x^(4/5),
       u(x),singsol=all)
```

$$u(x) = \frac{\text{BesselI}\left(-\frac{1}{6}, \frac{5x^{3/5}}{3}\right) c_1 - \text{BesselK}\left(\frac{1}{6}, \frac{5x^{3/5}}{3}\right)}{x^{2/5}\left(c_1 \text{BesselI}\left(\frac{5}{6}, \frac{5x^{3/5}}{3}\right) + \text{BesselK}\left(\frac{5}{6}, \frac{5x^{3/5}}{3}\right)\right)}$$

Mathematica DSolve solution

Solving time : 0.267 (sec)
Leaf size : 286

```
DSolve[{D[u[x],x]+u[x]^2==x^(-4/5),{}},
       u[x],x,IncludeSingularSolutions->True]
```

$$u(x) \to \frac{(-1)^{5/6}x^{3/5}\,\text{Gamma}\left(\frac{11}{6}\right)\text{BesselI}\left(-\frac{1}{6}, \frac{5x^{3/5}}{3}\right) + (-1)^{5/6}\,\text{Gamma}\left(\frac{11}{6}\right)\text{BesselI}\left(\frac{5}{6}, \frac{5x^{3/5}}{3}\right) + (-1)^{5/6}x^{3/5}\,\text{G}}{2x\left((-1)^{5/6}\,\text{Gamm}\right.}$$

$$u(x) \to \frac{x^{3/5}\,\text{BesselI}\left(-\frac{11}{6}, \frac{5x^{3/5}}{3}\right) + \text{BesselI}\left(-\frac{5}{6}, \frac{5x^{3/5}}{3}\right) + x^{3/5}\,\text{BesselI}\left(\frac{1}{6}, \frac{5x^{3/5}}{3}\right)}{2x\,\text{BesselI}\left(-\frac{5}{6}, \frac{5x^{3/5}}{3}\right)}$$

2.1.39 problem 40

Internal problem ID [8427]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 40
Date solved : `Tuesday, December 17, 2024 at 12:51:30 PM`
CAS classification :
`[[_homogeneous, ‘class A‘], _rational, [_Abel, ‘2nd type‘, ‘class A‘]]`

Solve

$$yy' - y = x$$

Solved as first order homogeneous class A ode

Time used: 0.708 (sec)

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y+x}{y} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = y + x$ and $N = y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\begin{aligned}\frac{du}{dx}x + u &= 1 + \frac{1}{u}\\ \frac{du}{dx} &= \frac{1 + \frac{1}{u(x)} - u(x)}{x}\end{aligned}$$

Or

$$u'(x) - \frac{1 + \frac{1}{u(x)} - u(x)}{x} = 0$$

Or

$$u'(x)\, u(x)\, x + u(x)^2 - u(x) - 1 = 0$$

Which is now solved as separable in $u(x)$.

The ode $u'(x) = -\frac{u(x)^2 - u(x) - 1}{u(x)x}$ is separable as it can be written as

$$\begin{aligned}u'(x) &= -\frac{u(x)^2 - u(x) - 1}{u\,(x)\, x}\\ &= f(x)g(u)\end{aligned}$$

Where

$$\begin{aligned}f(x) &= -\frac{1}{x}\\ g(u) &= \frac{u^2 - u - 1}{u}\end{aligned}$$

Integrating gives

$$\begin{aligned}\int \frac{1}{g(u)}\, du &= \int f(x)\, dx\\ \int \frac{u}{u^2 - u - 1}\, du &= \int -\frac{1}{x}\, dx\\ \frac{\ln\left(u(x)^2 - u(x) - 1\right)}{2} - \frac{\sqrt{5}\, \operatorname{arctanh}\left(\frac{(2u(x)-1)\sqrt{5}}{5}\right)}{5} &= \ln\left(\frac{1}{x}\right) + c_1\end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $\frac{u^2-u-1}{u} = 0$ for

$u(x)$ gives

$$
\begin{aligned}
u(x) &= -\frac{\sqrt{5}}{2}+\frac{1}{2} \\
u(x) &= \frac{\sqrt{5}}{2}+\frac{1}{2}
\end{aligned}
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
\frac{\ln\left(u(x)^2-u(x)-1\right)}{2}-\frac{\sqrt{5}\,\operatorname{arctanh}\left(\frac{(2u(x)-1)\sqrt{5}}{5}\right)}{5} &= \ln\left(\frac{1}{x}\right)+c_1 \\
u(x) &= -\frac{\sqrt{5}}{2}+\frac{1}{2} \\
u(x) &= \frac{\sqrt{5}}{2}+\frac{1}{2}
\end{aligned}
$$

Converting $\frac{\ln\left(u(x)^2-u(x)-1\right)}{2}-\frac{\sqrt{5}\,\operatorname{arctanh}\left(\frac{(2u(x)-1)\sqrt{5}}{5}\right)}{5} = \ln\left(\frac{1}{x}\right)+c_1$ back to y gives

$$
\frac{\sqrt{5}\left(\sqrt{5}\,\ln\left(\frac{y^2-yx-x^2}{x^2}\right)+2\,\operatorname{arctanh}\left(\frac{(-2y+x)\sqrt{5}}{5x}\right)\right)}{10} = \ln\left(\frac{1}{x}\right)+c_1
$$

Converting $u(x) = -\frac{\sqrt{5}}{2}+\frac{1}{2}$ back to y gives

$$
y = x\left(-\frac{\sqrt{5}}{2}+\frac{1}{2}\right)
$$

Converting $u(x) = \frac{\sqrt{5}}{2}+\frac{1}{2}$ back to y gives

$$
y = x\left(\frac{\sqrt{5}}{2}+\frac{1}{2}\right)
$$

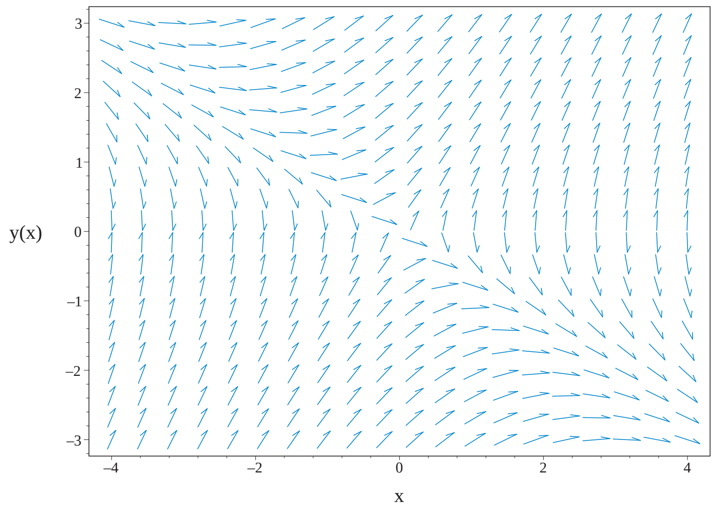


Figure 2.97: Slope field plot
$yy' - y = x$

Summary of solutions found

$$\frac{\sqrt{5}\left(\sqrt{5}\,\ln\left(\frac{y^2-yx-x^2}{x^2}\right)+2\,\operatorname{arctanh}\left(\frac{(-2y+x)\sqrt{5}}{5x}\right)\right)}{10} = \ln\left(\frac{1}{x}\right)+c_1$$

$$y = x\left(-\frac{\sqrt{5}}{2}+\frac{1}{2}\right)$$

$$y = x\left(\frac{\sqrt{5}}{2}+\frac{1}{2}\right)$$

Solved as first order homogeneous class D2 ode

Time used: 0.281 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$u(x)\,x(u'(x)\,x+u(x)) - u(x)\,x = x$$

Which is now solved The ode $u'(x) = -\frac{u(x)^2-u(x)-1}{u(x)x}$ is separable as it can be written as

$$\begin{aligned}u'(x) &= -\frac{u(x)^2-u(x)-1}{u\left(x\right)x}\\ &= f(x)g(u)\end{aligned}$$

Where

$$
\begin{aligned}
f(x) &= -\frac{1}{x} \\
g(u) &= \frac{u^2 - u - 1}{u}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(u)}\, du &= \int f(x)\, dx \\
\int \frac{u}{u^2 - u - 1}\, du &= \int -\frac{1}{x}\, dx \\
\frac{\ln\left(u(x)^2 - u(x) - 1\right)}{2} - \frac{\sqrt{5}\, \operatorname{arctanh}\left(\frac{(2u(x)-1)\sqrt{5}}{5}\right)}{5} &= \ln\left(\frac{1}{x}\right) + c_1
\end{aligned}
$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $\frac{u^2-u-1}{u} = 0$ for $u(x)$ gives

$$
\begin{aligned}
u(x) &= -\frac{\sqrt{5}}{2} + \frac{1}{2} \\
u(x) &= \frac{\sqrt{5}}{2} + \frac{1}{2}
\end{aligned}
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
\frac{\ln\left(u(x)^2 - u(x) - 1\right)}{2} - \frac{\sqrt{5}\, \operatorname{arctanh}\left(\frac{(2u(x)-1)\sqrt{5}}{5}\right)}{5} &= \ln\left(\frac{1}{x}\right) + c_1 \\
u(x) &= -\frac{\sqrt{5}}{2} + \frac{1}{2} \\
u(x) &= \frac{\sqrt{5}}{2} + \frac{1}{2}
\end{aligned}
$$

Converting $\frac{\ln\left(u(x)^2-u(x)-1\right)}{2} - \frac{\sqrt{5}\, \operatorname{arctanh}\left(\frac{(2u(x)-1)\sqrt{5}}{5}\right)}{5} = \ln\left(\frac{1}{x}\right) + c_1$ back to y gives

$$
\frac{\ln\left(\frac{y^2}{x^2} - \frac{y}{x} - 1\right)}{2} - \frac{\sqrt{5}\, \operatorname{arctanh}\left(\frac{\left(\frac{2y}{x}-1\right)\sqrt{5}}{5}\right)}{5} = \ln\left(\frac{1}{x}\right) + c_1
$$

Converting $u(x) = -\frac{\sqrt{5}}{2} + \frac{1}{2}$ back to y gives

$$y = x\left(-\frac{\sqrt{5}}{2} + \frac{1}{2}\right)$$

Converting $u(x) = \frac{\sqrt{5}}{2} + \frac{1}{2}$ back to y gives

$$y = x\left(\frac{\sqrt{5}}{2} + \frac{1}{2}\right)$$

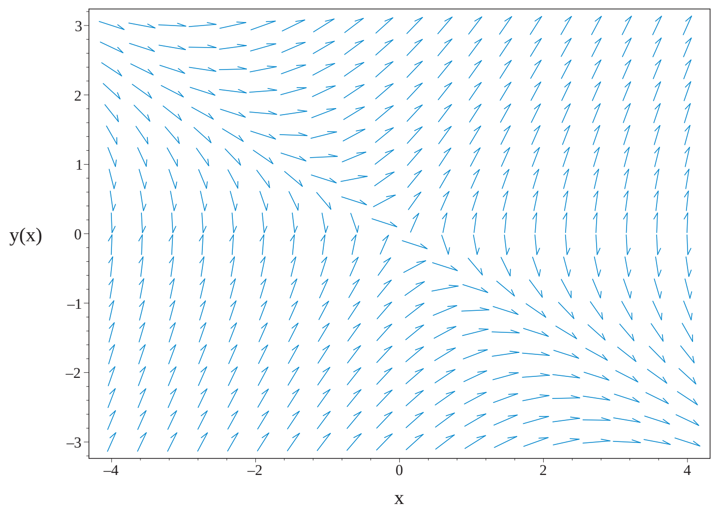


Figure 2.98: Slope field plot
$yy' - y = x$

Summary of solutions found

$$\frac{\ln\left(\frac{y^2}{x^2} - \frac{y}{x} - 1\right)}{2} - \frac{\sqrt{5}\,\operatorname{arctanh}\left(\frac{\left(\frac{2y}{x}-1\right)\sqrt{5}}{5}\right)}{5} = \ln\left(\frac{1}{x}\right) + c_1$$

$$y = x\left(-\frac{\sqrt{5}}{2} + \frac{1}{2}\right)$$

$$y = x\left(\frac{\sqrt{5}}{2} + \frac{1}{2}\right)$$

Summary of solutions found

$$\frac{\sqrt{5}\left(\sqrt{5}\,\ln\left(\frac{y^{2}-yx-x^{2}}{x^{2}}\right)+2\,\operatorname{arctanh}\left(\frac{(-2y+x)\sqrt{5}}{5x}\right)\right)}{10}=\ln\left(\frac{1}{x}\right)+c_1$$

$$y=x\left(-\frac{\sqrt{5}}{2}+\frac{1}{2}\right)$$

$$y=x\left(\frac{\sqrt{5}}{2}+\frac{1}{2}\right)$$

Solved as first order homogeneous class Maple C ode

Time used: 0.681 (sec)

Let $Y=y-y_0$ and $X=x-x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X)=\frac{Y(X)+y_0+x_0+X}{Y(X)+y_0}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0=0$$
$$y_0=0$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X)=\frac{Y(X)+X}{Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y)\\ &= \frac{Y+X}{Y} \end{aligned} \tag{1}$$

An ode of the form $Y'=\frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^nX,t^nY)=t^nf(X,Y)$$

In this case, it can be seen that both $M = Y + X$ and $N = Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= 1 + \frac{1}{u}\\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{1 + \frac{1}{u(X)} - u(X)}{X}\end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{1 + \frac{1}{u(X)} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right) u(X)\, X + u(X)^2 - u(X) - 1 = 0$$

Which is now solved as separable in $u(X)$.

The ode $\frac{d}{dX}u(X) = -\frac{u(X)^2 - u(X) - 1}{u(X)X}$ is separable as it can be written as

$$\begin{aligned}\frac{d}{dX}u(X) &= -\frac{u(X)^2 - u(X) - 1}{u\,(X)\, X}\\ &= f(X)g(u)\end{aligned}$$

Where

$$\begin{aligned}f(X) &= -\frac{1}{X}\\ g(u) &= \frac{u^2 - u - 1}{u}\end{aligned}$$

Integrating gives

$$\begin{aligned}\int \frac{1}{g(u)}\, du &= \int f(X)\, dX\\ \int \frac{u}{u^2 - u - 1}\, du &= \int -\frac{1}{X}\, dX\end{aligned}$$

$$\frac{\ln\left(u(X)^2 - u(X) - 1\right)}{2} - \frac{\sqrt{5}\,\operatorname{arctanh}\left(\frac{(2u(X)-1)\sqrt{5}}{5}\right)}{5} = \ln\left(\frac{1}{X}\right) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $\frac{u^2-u-1}{u} = 0$ for $u(X)$ gives

$$\begin{aligned} u(X) &= -\frac{\sqrt{5}}{2} + \frac{1}{2} \\ u(X) &= \frac{\sqrt{5}}{2} + \frac{1}{2} \end{aligned}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} \frac{\ln\left(u(X)^2 - u(X) - 1\right)}{2} - \frac{\sqrt{5}\,\operatorname{arctanh}\left(\frac{(2u(X)-1)\sqrt{5}}{5}\right)}{5} &= \ln\left(\frac{1}{X}\right) + c_1 \\ u(X) &= -\frac{\sqrt{5}}{2} + \frac{1}{2} \\ u(X) &= \frac{\sqrt{5}}{2} + \frac{1}{2} \end{aligned}$$

Converting $\frac{\ln\left(u(X)^2-u(X)-1\right)}{2} - \frac{\sqrt{5}\,\operatorname{arctanh}\left(\frac{(2u(X)-1)\sqrt{5}}{5}\right)}{5} = \ln\left(\frac{1}{X}\right) + c_1$ back to $Y(X)$ gives

$$\frac{\sqrt{5}\left(\sqrt{5}\,\ln\left(\frac{Y(X)^2-Y(X)X-X^2}{X^2}\right) + 2\,\operatorname{arctanh}\left(\frac{(-2Y(X)+X)\sqrt{5}}{5X}\right)\right)}{10} = \ln\left(\frac{1}{X}\right) + c_1$$

Converting $u(X) = -\frac{\sqrt{5}}{2} + \frac{1}{2}$ back to $Y(X)$ gives

$$Y(X) = X\left(-\frac{\sqrt{5}}{2} + \frac{1}{2}\right)$$

Converting $u(X) = \frac{\sqrt{5}}{2} + \frac{1}{2}$ back to $Y(X)$ gives

$$Y(X) = X\left(\frac{\sqrt{5}}{2} + \frac{1}{2}\right)$$

Using the solution for $Y(X)$

$$\frac{\sqrt{5}\left(\sqrt{5}\,\ln\left(\frac{Y(X)^2-Y(X)X-X^2}{X^2}\right) + 2\,\operatorname{arctanh}\left(\frac{(-2Y(X)+X)\sqrt{5}}{5X}\right)\right)}{10} = \ln\left(\frac{1}{X}\right) + c_1 \quad \text{(A)}$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y + y_0 \\
X &= x + x_0
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= y \\
X &= x
\end{aligned}
$$

Then the solution in y becomes using EQ (A)

$$
\frac{\sqrt{5}\left(\sqrt{5}\,\ln\left(\frac{y^2 - yx - x^2}{x^2}\right) + 2\,\operatorname{arctanh}\left(\frac{(-2y+x)\sqrt{5}}{5x}\right)\right)}{10} = \ln\left(\frac{1}{x}\right) + c_1
$$

Using the solution for $Y(X)$

$$
Y(X) = X\left(-\frac{\sqrt{5}}{2} + \frac{1}{2}\right) \tag{A}
$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y + y_0 \\
X &= x + x_0
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= y \\
X &= x
\end{aligned}
$$

Then the solution in y becomes using EQ (A)

$$
y = x\left(-\frac{\sqrt{5}}{2} + \frac{1}{2}\right)
$$

Using the solution for $Y(X)$

$$
Y(X) = X\left(\frac{\sqrt{5}}{2} + \frac{1}{2}\right) \tag{A}
$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y + y_0 \\
X &= x + x_0
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= y \\
X &= x
\end{aligned}
$$

Then the solution in y becomes using EQ (A)

$$
y = x\left(\frac{\sqrt{5}}{2} + \frac{1}{2}\right)
$$

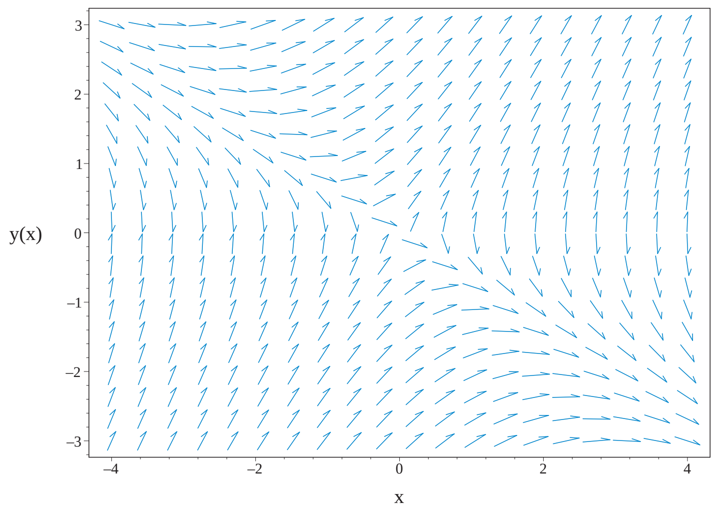


Figure 2.99: Slope field plot
$yy' - y = x$

Solved as first order isobaric ode

Time used: 0.250 (sec)

Solving for y' gives

$$
y' = \frac{y + x}{y} \tag{1}
$$

Each of the above ode's is now solved An ode $y' = f(x, y)$ is isobaric if

$$
f(tx, t^m y) = t^{m-1} f(x, y) \tag{1}
$$

Where here

$$
f(x, y) = \frac{y + x}{y} \tag{2}
$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = 1$$

Since the ode is isobaric of order $m = 1$, then the substitution

$$\begin{aligned} y &= ux^m \\ &= ux \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$u(x) + xu'(x) = \frac{xu(x) + x}{xu(x)}$$

The ode $u'(x) = -\frac{u(x)^2 - u(x) - 1}{u(x)x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)^2 - u(x) - 1}{u(x)x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(u) &= \frac{u^2 - u - 1}{u} \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\, du &= \int f(x)\, dx \\ \int \frac{u}{u^2 - u - 1}\, du &= \int -\frac{1}{x}\, dx \\ \frac{\ln\left(u(x)^2 - u(x) - 1\right)}{2} - \frac{\sqrt{5}\,\operatorname{arctanh}\left(\frac{(2u(x)-1)\sqrt{5}}{5}\right)}{5} &= \ln\left(\frac{1}{x}\right) + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $\frac{u^2-u-1}{u} = 0$ for $u(x)$ gives

$$\begin{aligned} u(x) &= -\frac{\sqrt{5}}{2} + \frac{1}{2} \\ u(x) &= \frac{\sqrt{5}}{2} + \frac{1}{2} \end{aligned}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\frac{\ln\left(u(x)^2-u(x)-1\right)}{2}-\frac{\sqrt{5}\,\operatorname{arctanh}\left(\frac{(2u(x)-1)\sqrt{5}}{5}\right)}{5}=\ln\left(\frac{1}{x}\right)+c_1$$
$$u(x)=-\frac{\sqrt{5}}{2}+\frac{1}{2}$$
$$u(x)=\frac{\sqrt{5}}{2}+\frac{1}{2}$$

Converting $\frac{\ln\left(u(x)^2-u(x)-1\right)}{2}-\frac{\sqrt{5}\,\operatorname{arctanh}\left(\frac{(2u(x)-1)\sqrt{5}}{5}\right)}{5}=\ln\left(\frac{1}{x}\right)+c_1$ back to y gives

$$\frac{\ln\left(\frac{y^2}{x^2}-\frac{y}{x}-1\right)}{2}-\frac{\sqrt{5}\,\operatorname{arctanh}\left(\frac{\left(\frac{2y}{x}-1\right)\sqrt{5}}{5}\right)}{5}=\ln\left(\frac{1}{x}\right)+c_1$$

Converting $u(x)=-\frac{\sqrt{5}}{2}+\frac{1}{2}$ back to y gives

$$\frac{y}{x}=-\frac{\sqrt{5}}{2}+\frac{1}{2}$$

Converting $u(x)=\frac{\sqrt{5}}{2}+\frac{1}{2}$ back to y gives

$$\frac{y}{x}=\frac{\sqrt{5}}{2}+\frac{1}{2}$$

Solving for y gives

$$\frac{\ln\left(\frac{y^2}{x^2}-\frac{y}{x}-1\right)}{2}-\frac{\sqrt{5}\,\operatorname{arctanh}\left(\frac{\left(\frac{2y}{x}-1\right)\sqrt{5}}{5}\right)}{5}=\ln\left(\frac{1}{x}\right)+c_1$$
$$y=-\frac{x\left(\sqrt{5}-1\right)}{2}$$
$$y=\frac{x\left(\sqrt{5}+1\right)}{2}$$

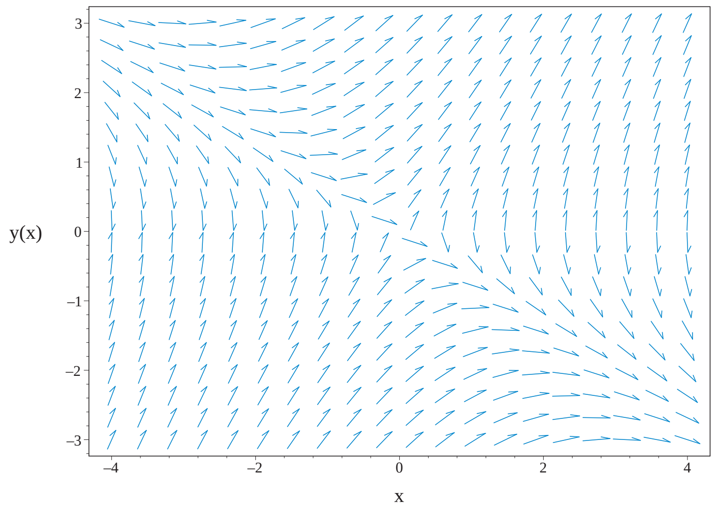


Figure 2.100: Slope field plot
$yy' - y = x$

Summary of solutions found

$$
\begin{aligned}
\frac{\ln\left(\frac{y^2}{x^2} - \frac{y}{x} - 1\right)}{2} - \frac{\sqrt{5}\,\operatorname{arctanh}\left(\frac{\left(\frac{2y}{x}-1\right)\sqrt{5}}{5}\right)}{5} &= \ln\left(\frac{1}{x}\right) + c_1 \\
y &= -\frac{x\left(\sqrt{5}-1\right)}{2} \\
y &= \frac{x\left(\sqrt{5}+1\right)}{2}
\end{aligned}
$$

Solved using Lie symmetry for first order ode

Time used: 1.520 (sec)

Writing the ode as

$$
\begin{aligned}
y' &= \frac{y+x}{y} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{(y+x)(b_3 - a_2)}{y} - \frac{(y+x)^2 a_3}{y^2} - \frac{xa_2 + ya_3 + a_1}{y} - \left(\frac{1}{y} - \frac{y+x}{y^2}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^2a_3 - x^2b_2 + 2xya_2 + 2xya_3 - 2xyb_3 + y^2a_2 + 2y^2a_3 - b_2y^2 - y^2b_3 - xb_1 + ya_1}{y^2} = 0$$

Setting the numerator to zero gives

$$-x^2a_3 + x^2b_2 - 2xya_2 - 2xya_3 + 2xyb_3 - y^2a_2 - 2y^2a_3 + b_2y^2 + y^2b_3 + xb_1 - ya_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-2a_2v_1v_2 - a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 - 2a_3v_2^2 + b_2v_1^2 + b_2v_2^2 + 2b_3v_1v_2 + b_3v_2^2 - a_1v_2 + b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(-a_3+b_2)\, v_1^2+(-2a_2-2a_3+2b_3)\, v_1 v_2+b_1 v_1+(-a_2-2a_3+b_2+b_3)\, v_2^2-a_1 v_2=0 \quad \text{(8E)}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_1 &= 0 \\ -a_1 &= 0 \\ -a_3+b_2 &= 0 \\ -2a_2-2a_3+2b_3 &= 0 \\ -a_2-2a_3+b_2+b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= -b_2+b_3 \\ a_3 &= b_2 \\ b_1 &= 0 \\ b_2 &= b_2 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x,y)\, \xi \\ &= y - \left(\frac{y+x}{y}\right)(x) \\ &= \frac{-x^2-yx+y^2}{y} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-x^2 - yx + y^2}{y}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln(-x^2 - yx + y^2)}{2} - \frac{\sqrt{5}\, \mathrm{arctanh}\left(\frac{(-x+2y)\sqrt{5}}{5x}\right)}{5}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y + x}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y + x}{x^2 + yx - y^2} \\ S_y &= -\frac{y}{x^2 + yx - y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

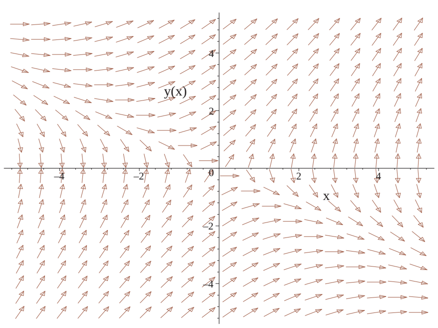
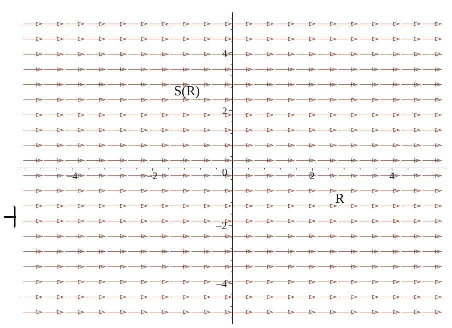
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int 0\, dR + c_2$$
$$S(R) = c_2$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\frac{\ln\left(y^2 - yx - x^2\right)}{2} + \frac{\sqrt{5}\,\operatorname{arctanh}\left(\frac{(-2y+x)\sqrt{5}}{5x}\right)}{5} = c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+x}{y}$ 	$R = x$ $S = \frac{\ln\left(-x^2 - yx + y^2\right)}{2} +$	$\frac{dS}{dR} = 0$ 

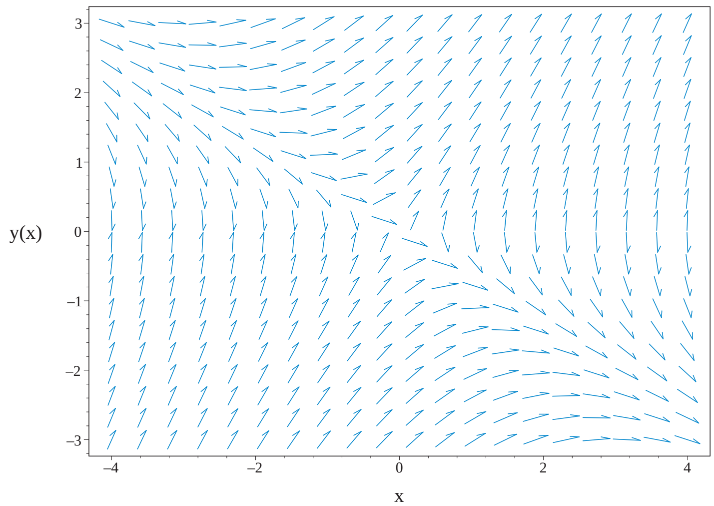


Figure 2.101: Slope field plot
$yy' - y = x$

Summary of solutions found

$$\frac{\ln \left(y^{2}-yx-x^{2}\right)}{2}+\frac{\sqrt{5}\, \operatorname{arctanh}\left(\frac{\left(-2y+x\right)\sqrt{5}}{5x}\right)}{5} = c_{2}$$

Maple step by step solution

Let's solve
$y(x)\left(\frac{d}{dx}y(x)\right)-y(x)=x$

- Highest derivative means the order of the ODE is 1
$\frac{d}{dx}y(x)$
- Solve for the highest derivative
$\frac{d}{dx}y(x)=\frac{x+y(x)}{y(x)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
```

```
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

Maple dsolve solution

Solving time : 0.516 (sec)
Leaf size : 53

```
dsolve(y(x)*diff(y(x),x)-y(x) = x,
       y(x),singsol=all)
```

$$-\frac{\ln\left(\frac{-x^2-xy+y^2}{x^2}\right)}{2}-\frac{\sqrt{5}\,\mathrm{arctanh}\left(\frac{(x-2y)\sqrt{5}}{5x}\right)}{5}-\ln(x)-c_1=0$$

Mathematica DSolve solution

Solving time : 0.064 (sec)
Leaf size : 63

```
DSolve[{y[x]*D[y[x],x] - y[x] == x,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$\mathrm{Solve}\left[\frac{1}{10}\left(\left(5+\sqrt{5}\right)\log\left(-\frac{2y(x)}{x}+\sqrt{5}+1\right)-\left(\sqrt{5}-5\right)\log\left(\frac{2y(x)}{x}+\sqrt{5}-1\right)\right)=-\log(x)+c_1,y(x)\right]$$

2.1.40 problem 41

Internal problem ID [8428]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 41
Date solved : `Thursday, December 12, 2024 at 09:06:23 AM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$y'' + 2y' + y = 0$$

Solved as second order linear constant coeff ode

Time used: 0.063 (sec)

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} + 2\lambda\,\mathrm{e}^{x\lambda} + \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(2)^2 - (4)(1)(1)}\\ &= -1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$y = c_1 \mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{-x} \tag{1}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1\,\mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{-x}$$

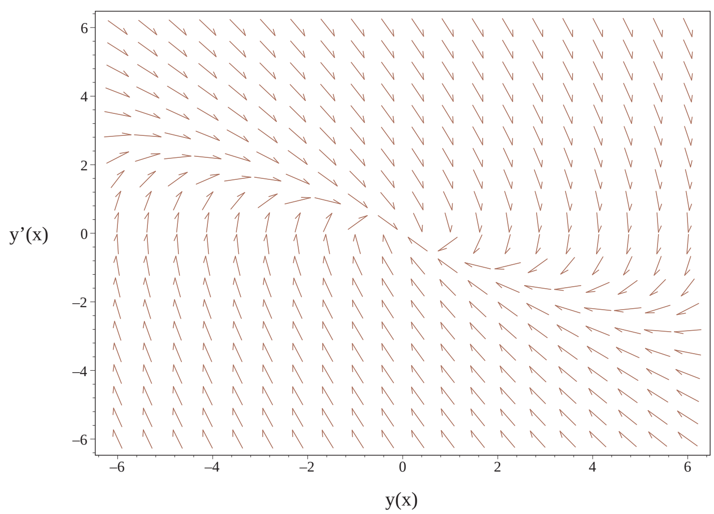


Figure 2.102: Slope field plot
$y'' + 2y' + y = 0$

Solved as second order solved by an integrating factor

Time used: 0.030 (sec)

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = 2$. Therefore, there is an integrating factor given by

$$\begin{aligned}M(x) &= e^{\frac{1}{2}\int p\,dx}\\ &= e^{\int 2\,dx}\\ &= \mathrm{e}^{x}\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)y)'' &= 0 \\
(e^x y)'' &= 0
\end{aligned}
$$

Integrating once gives

$$
(e^x y)' = c_1
$$

Integrating again gives

$$
(e^x y) = c_1 x + c_2
$$

Hence the solution is

$$
y = \frac{c_1 x + c_2}{e^x}
$$

Or

$$
y = c_1 x \, e^{-x} + c_2 \, e^{-x}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = c_1 x \, e^{-x} + c_2 \, e^{-x}
$$

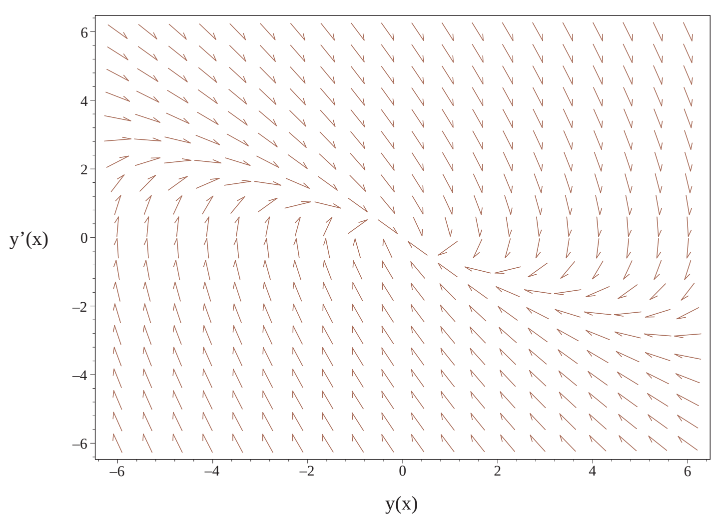


Figure 2.103: Slope field plot
$y'' + 2y' + y = 0$

Solved as second order ode using Kovacic algorithm

Time used: 0.050 (sec)

Writing the ode as

$$y'' + 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.40: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\
&= z_1 e^{-x} \\
&= z_1 \left(\mathrm{e}^{-x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\, dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}(x)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1\,\mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{-x}$$

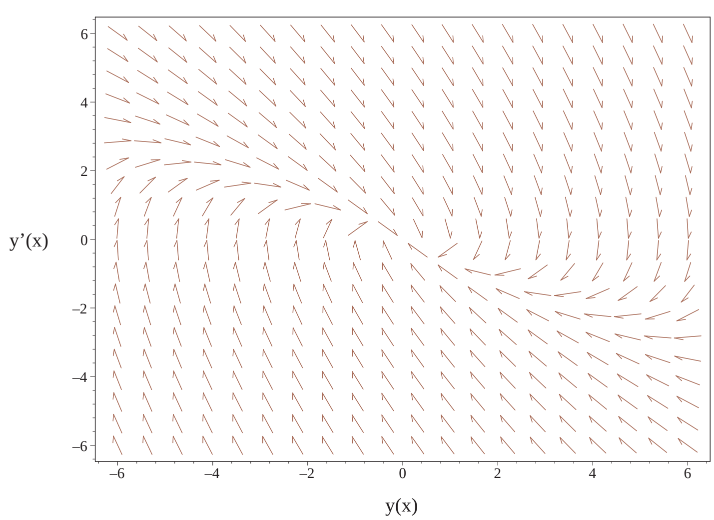


Figure 2.104: Slope field plot
$y'' + 2y' + y = 0$

Solved as second order ode adjoint method

Time used: 0.415 (sec)

In normal form the ode

$$y'' + 2y' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 2 \\ q(x) &= 1 \\ r(x) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (2\xi(x))' + (\xi(x)) &= 0 \\ \xi''(x) - 2\xi'(x) + \xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = -2, C = 1$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} - 2\lambda\, \mathrm{e}^{x\lambda} + \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(-2)^2 - (4)\,(1)\,(1)} \\ &= 1 \end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -1$. Therefore the solution is

$$\xi = c_1 \mathrm{e}^{x} + c_2 x \, \mathrm{e}^{x} \tag{1}$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}
\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi\,(x)}
\end{aligned}$$

Or

$$y' + y\left(2 - \frac{c_1\, \mathrm{e}^{x} + c_2\, \mathrm{e}^{x} + c_2 x\, \mathrm{e}^{x}}{c_1\, \mathrm{e}^{x} + c_2 x\, \mathrm{e}^{x}}\right) = 0$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned}
q(x) &= -\frac{-c_2 x - c_1 + c_2}{c_2 x + c_1} \\
p(x) &= 0
\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int q\, dx} \\
&= \mathrm{e}^{\int -\frac{-c_2 x - c_1 + c_2}{c_2 x + c_1} dx} \\
&= \frac{\mathrm{e}^{x}}{c_2 x + c_1}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} \mu y &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y\, \mathrm{e}^{x}}{c_2 x + c_1}\right) &= 0
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{y\, \mathrm{e}^{x}}{c_2 x + c_1} &= \int 0\, dx + c_3 \\
&= c_3
\end{aligned}$$

Dividing throughout by the integrating factor $\frac{e^x}{c_2 x+c_1}$ gives the final solution

$$y = (c_2 x + c_1) e^{-x} c_3$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = (c_2 x + c_1) e^{-x} c_3$$

The constants can be merged to give

$$y = (c_2 x + c_1) e^{-x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = (c_2 x + c_1) e^{-x}$$

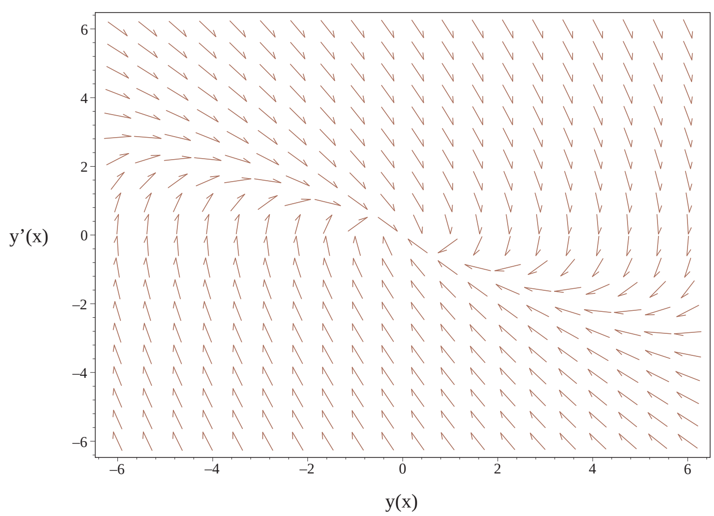


Figure 2.105: Slope field plot
$y'' + 2y' + y = 0$

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) + 2\frac{d}{dx}y(x) + y(x) = 0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Characteristic polynomial of ODE
  $r^2 + 2r + 1 = 0$
- Factor the characteristic polynomial
  $(r+1)^2 = 0$
- Root of the characteristic polynomial
  $r = -1$
- 1st solution of the ODE
  $y_1(x) = \mathrm{e}^{-x}$
- Repeated root, multiply $y_1(x)$ by x to ensure linear independence
  $y_2(x) = x\,\mathrm{e}^{-x}$
- General solution of the ODE
  $y(x) = C1y_1(x) + C2y_2(x)$
- Substitute in solutions
  $y(x) = C1\,\mathrm{e}^{-x} + C2\,\mathrm{e}^{-x}x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 14

```
dsolve(diff(diff(y(x),x),x)+2*diff(y(x),x)+y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-x}(c_2x + c_1)$$

Mathematica DSolve solution

Solving time : 0.014 (sec)
Leaf size : 18

```
DSolve[{D[y[x],{x,2}]+2*D[y[x],x]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \rightarrow e^{-x}(c_2x + c_1)$$

2.1.41 problem 41

Internal problem ID [8429]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 41
Date solved : `Thursday, December 12, 2024 at 09:06:24 AM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$5y'' + 2y' + 4y = 0$$

With initial conditions

$$\begin{aligned} y(0) &= 0 \\ y'(0) &= 5 \end{aligned}$$

Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= \frac{2}{5} \\ q(x) &= \frac{4}{5} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{2y'}{5} + \frac{4y}{5} = 0$$

The domain of $p(x) = \frac{2}{5}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{4}{5}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solved as second order linear constant coeff ode

Time used: 0.165 (sec)

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 5, B = 2, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$5\lambda^2 \mathrm{e}^{x\lambda} + 2\lambda\,\mathrm{e}^{x\lambda} + 4\,\mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$5\lambda^2 + 2\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 5, B = 2, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(5)} \pm \frac{1}{(2)\,(5)}\sqrt{2^2 - (4)\,(5)\,(4)}\\ &= -\frac{1}{5} \pm \frac{i\sqrt{19}}{5}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{5} + \frac{i\sqrt{19}}{5}\\ \lambda_2 &= -\frac{1}{5} - \frac{i\sqrt{19}}{5}\end{aligned}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{5} + \frac{i\sqrt{19}}{5}$$
$$\lambda_2 = -\frac{1}{5} - \frac{i\sqrt{19}}{5}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{5}$ and $\beta = \frac{\sqrt{19}}{5}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

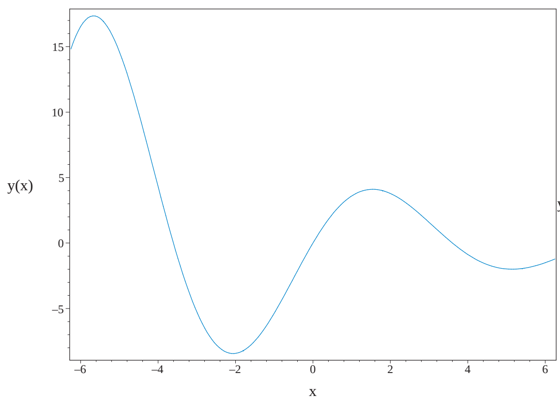
Which becomes

$$y = e^{-\frac{x}{5}}\left(c_1 \cos\left(\frac{\sqrt{19}\,x}{5}\right) + c_2 \sin\left(\frac{\sqrt{19}\,x}{5}\right)\right)$$

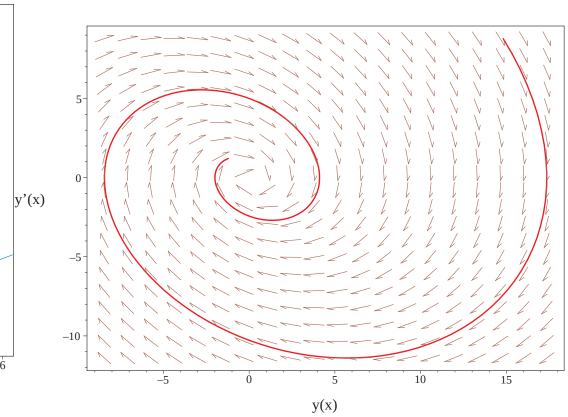
Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{25\,\mathrm{e}^{-\frac{x}{5}}\sqrt{19}\,\sin\left(\frac{\sqrt{19}\,x}{5}\right)}{19}$$



(a) Solution plot
$y = \frac{25\,\mathrm{e}^{-\frac{x}{5}}\sqrt{19}\,\sin\left(\frac{\sqrt{19}\,x}{5}\right)}{19}$



(b) Slope field plot
$5y'' + 2y' + 4y = 0$

Solved as second order ode using Kovacic algorithm

Time used: 0.221 (sec)

Writing the ode as

$$
\begin{align*}
5y'' + 2y' + 4y &= 0 \tag{1}\\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= 5\\
B &= 2 \tag{3}\\
C &= 4
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5}\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-19}{25} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= -19\\
t &= 25
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(x) = -\frac{19 z(x)}{25} \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.42: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{19}{25}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{19}\, x}{5}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{5}\,dx} \\
&= z_1 e^{-\frac{x}{5}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{5}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x}{5}} \cos\left(\frac{\sqrt{19}\,x}{5}\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2}{5}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\frac{2x}{5}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{5\sqrt{19}\,\tan\left(\frac{\sqrt{19}\,x}{5}\right)}{19}\right)
\end{aligned}
$$

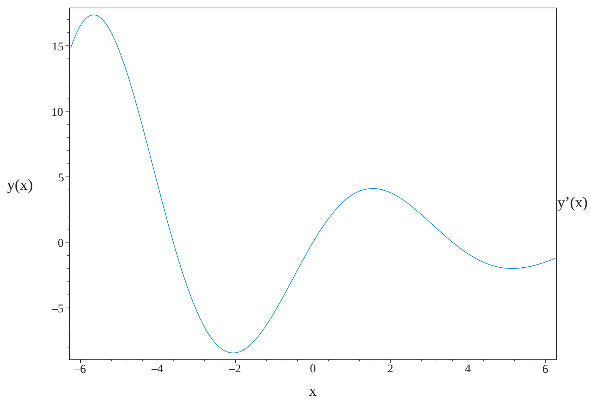
Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x}{5}} \cos\left(\frac{\sqrt{19}\,x}{5}\right)\right) + c_2\left(\mathrm{e}^{-\frac{x}{5}} \cos\left(\frac{\sqrt{19}\,x}{5}\right)\left(\frac{5\sqrt{19}\,\tan\left(\frac{\sqrt{19}\,x}{5}\right)}{19}\right)\right)
\end{aligned}
$$

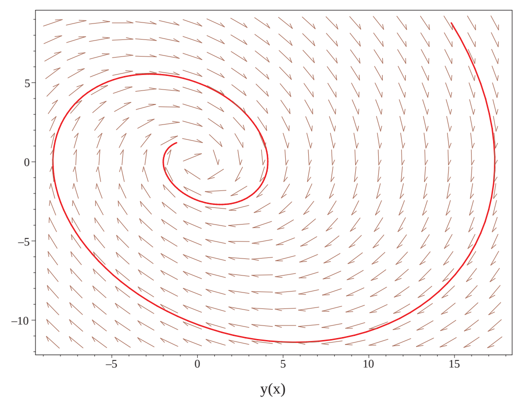
Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{25\,\mathrm{e}^{-\frac{x}{5}}\sqrt{19}\,\sin\left(\frac{\sqrt{19}\,x}{5}\right)}{19}$$



(a) Solution plot
$y = \frac{25\,\mathrm{e}^{-\frac{x}{5}}\sqrt{19}\,\sin\left(\frac{\sqrt{19}\,x}{5}\right)}{19}$



(b) Slope field plot
$5y'' + 2y' + 4y = 0$

Solved as second order ode adjoint method

Time used: 1.472 (sec)

In normal form the ode

$$5y'' + 2y' + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{2}{5} \\ q(x) &= \frac{4}{5} \\ r(x) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\,p)' + \xi q &= 0 \\ \xi'' - \left(\frac{2\xi(x)}{5}\right)' + \left(\frac{4\xi(x)}{5}\right) &= 0 \\ \xi''(x) - \frac{2\xi'(x)}{5} + \frac{4\xi(x)}{5} &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = -\frac{2}{5}, C = \frac{4}{5}$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} - \frac{2\lambda\,\mathrm{e}^{x\lambda}}{5} + \frac{4\,\mathrm{e}^{x\lambda}}{5} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \frac{2}{5}\lambda + \frac{4}{5} = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -\frac{2}{5}, C = \frac{4}{5}$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{\frac{2}{5}}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-\frac{2}{5}^2 - (4)\,(1)\left(\frac{4}{5}\right)} \\
&= \frac{1}{5} \pm \frac{i\sqrt{19}}{5}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= \frac{1}{5} + \frac{i\sqrt{19}}{5} \\
\lambda_2 &= \frac{1}{5} - \frac{i\sqrt{19}}{5}
\end{aligned}$$

Which simplifies to

$$\lambda_1 = \frac{1}{5} + \frac{i\sqrt{19}}{5}$$

$$\lambda_2 = \frac{1}{5} - \frac{i\sqrt{19}}{5}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = \frac{1}{5}$ and $\beta = \frac{\sqrt{19}}{5}$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$\xi = e^{\frac{x}{5}}\left(c_1 \cos\left(\frac{\sqrt{19}\,x}{5}\right) + c_2 \sin\left(\frac{\sqrt{19}\,x}{5}\right)\right)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}
\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y &= \int \xi(x)\,r(x)\,dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi(x)}
\end{aligned}$$

Or

$$y' + y\left(\frac{2}{5} - \frac{\left(\frac{e^{\frac{x}{5}}\left(c_1 \cos\left(\frac{\sqrt{19}\,x}{5}\right) + c_2 \sin\left(\frac{\sqrt{19}\,x}{5}\right)\right)}{5} + e^{\frac{x}{5}}\left(-\frac{c_1 \sin\left(\frac{\sqrt{19}\,x}{5}\right)\sqrt{19}}{5} + \frac{c_2\sqrt{19}\,\cos\left(\frac{\sqrt{19}\,x}{5}\right)}{5}\right)\right) e^{-\frac{x}{5}}}{c_1 \cos\left(\frac{\sqrt{19}\,x}{5}\right) + c_2 \sin\left(\frac{\sqrt{19}\,x}{5}\right)}\right) = 0$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned}
q(x) &= -\frac{(c_2\sqrt{19} - c_1)\cos\left(\frac{\sqrt{19}\,x}{5}\right) - \sin\left(\frac{\sqrt{19}\,x}{5}\right)(c_1\sqrt{19} + c_2)}{5c_1 \cos\left(\frac{\sqrt{19}\,x}{5}\right) + 5c_2 \sin\left(\frac{\sqrt{19}\,x}{5}\right)} \\
p(x) &= 0
\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= e^{\int -\frac{(c_2\sqrt{19}-c_1)\cos\left(\frac{\sqrt{19}\,x}{5}\right) - \sin\left(\frac{\sqrt{19}\,x}{5}\right)(c_1\sqrt{19}+c_2)}{5c_1\cos\left(\frac{\sqrt{19}\,x}{5}\right) + 5c_2\sin\left(\frac{\sqrt{19}\,x}{5}\right)}dx} \\
&= \frac{\sqrt{\sec\left(\frac{\sqrt{19}\,x}{5}\right)^2}\, e^{\frac{\sqrt{19}\,\arctan\left(\tan\left(\frac{\sqrt{19}\,x}{5}\right)\right)}{19}}}{\tan\left(\frac{\sqrt{19}\,x}{5}\right)c_2 + c_1}
\end{aligned}$$

The ode becomes

$$\frac{d}{dx}\mu y = 0$$

$$\frac{d}{dx}\left(\frac{y\sqrt{\sec\left(\frac{\sqrt{19}\,x}{5}\right)^2}\,\mathrm{e}^{\frac{\sqrt{19}\,\arctan\left(\tan\left(\frac{\sqrt{19}\,x}{5}\right)\right)}{19}}}{\tan\left(\frac{\sqrt{19}\,x}{5}\right)c_2 + c_1}\right) = 0$$

Integrating gives

$$\begin{aligned}\frac{y\sqrt{\sec\left(\frac{\sqrt{19}\,x}{5}\right)^2}\,\mathrm{e}^{\frac{\sqrt{19}\,\arctan\left(\tan\left(\frac{\sqrt{19}\,x}{5}\right)\right)}{19}}}{\tan\left(\frac{\sqrt{19}\,x}{5}\right)c_2 + c_1} &= \int 0\,dx + c_3 \\ &= c_3\end{aligned}$$

Dividing throughout by the integrating factor $\frac{\sqrt{\sec\left(\frac{\sqrt{19}\,x}{5}\right)^2}\,\mathrm{e}^{\frac{\sqrt{19}\,\arctan\left(\tan\left(\frac{\sqrt{19}\,x}{5}\right)\right)}{19}}}{\tan\left(\frac{\sqrt{19}\,x}{5}\right)c_2+c_1}$ gives the final solution

$$y = \frac{\left(\tan\left(\frac{\sqrt{19}\,x}{5}\right)c_2 + c_1\right)\mathrm{e}^{-\frac{\sqrt{19}\,\arctan\left(\tan\left(\frac{\sqrt{19}\,x}{5}\right)\right)}{19}}c_3}{\sqrt{\sec\left(\frac{\sqrt{19}\,x}{5}\right)^2}}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{\left(\tan\left(\frac{\sqrt{19}\,x}{5}\right)c_2 + c_1\right)\mathrm{e}^{-\frac{\sqrt{19}\,\arctan\left(\tan\left(\frac{\sqrt{19}\,x}{5}\right)\right)}{19}}c_3}{\sqrt{\sec\left(\frac{\sqrt{19}\,x}{5}\right)^2}}$$

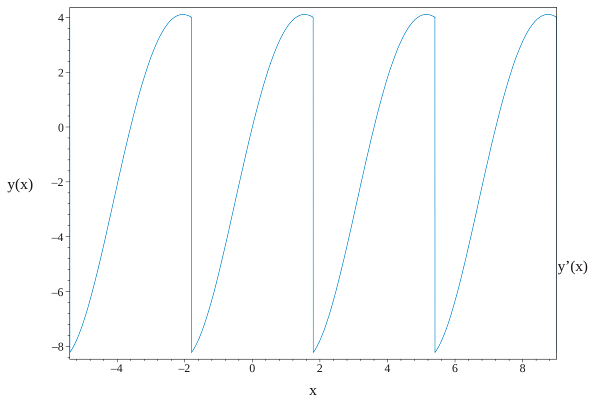
The constants can be merged to give

$$y = \frac{\left(\tan\left(\frac{\sqrt{19}\,x}{5}\right)c_2 + c_1\right)\mathrm{e}^{-\frac{\sqrt{19}\,\arctan\left(\tan\left(\frac{\sqrt{19}\,x}{5}\right)\right)}{19}}}{\sqrt{\sec\left(\frac{\sqrt{19}\,x}{5}\right)^2}}$$

Will add steps showing solving for IC soon.

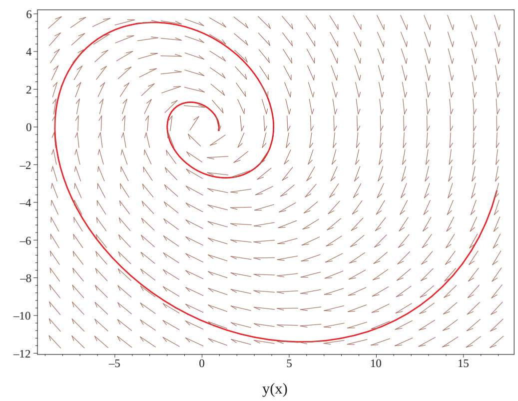
Summary of solutions found

$$y = \frac{25\sqrt{19}\,\tan\left(\frac{\sqrt{19}\,x}{5}\right) \mathrm{e}^{-\frac{\sqrt{19}\,\arctan\left(\tan\left(\frac{\sqrt{19}\,x}{5}\right)\right)}{19}}}{19\sqrt{\sec\left(\frac{\sqrt{19}\,x}{5}\right)^{2}}}$$



(a) Solution plot

$$y = \frac{25\sqrt{19}\,\tan\left(\frac{\sqrt{19}\,x}{5}\right) \mathrm{e}^{-\frac{\sqrt{19}\,\arctan\left(\tan\left(\frac{\sqrt{19}\,x}{5}\right)\right)}{19}}}{19\sqrt{\sec\left(\frac{\sqrt{19}\,x}{5}\right)^{2}}}$$



(b) Slope field plot
$5y'' + 2y' + 4y = 0$

Maple step by step solution

Let's solve

$$\left[5\frac{d^2}{dx^2}y(x) + 2\frac{d}{dx}y(x) + 4y(x) = 0, y(0) = 0, \left(\frac{d}{dx}y(x)\right)\bigg|_{\{x=0\}} = 5\right]$$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{2\frac{d}{dx}y(x)}{5} - \frac{4y(x)}{5}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{2\frac{d}{dx}y(x)}{5} + \frac{4y(x)}{5} = 0$

- Characteristic polynomial of ODE

  $r^2 + \frac{2}{5}r + \frac{4}{5} = 0$

- Use quadratic formula to solve for r

$r = \frac{\left(-\frac{2}{5}\right)\pm\left(\sqrt{-\frac{76}{25}}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{5} - \frac{I\sqrt{19}}{5}, -\frac{1}{5} + \frac{I\sqrt{19}}{5}\right)$

- 1st solution of the ODE

  $y_1(x) = e^{-\frac{x}{5}} \cos\left(\frac{\sqrt{19}\, x}{5}\right)$

- 2nd solution of the ODE

  $y_2(x) = e^{-\frac{x}{5}} \sin\left(\frac{\sqrt{19}\, x}{5}\right)$

- General solution of the ODE

  $y(x) = C1 y_1(x) + C2 y_2(x)$

- Substitute in solutions

  $y(x) = C1\, e^{-\frac{x}{5}} \cos\left(\frac{\sqrt{19}\, x}{5}\right) + C2\, e^{-\frac{x}{5}} \sin\left(\frac{\sqrt{19}\, x}{5}\right)$

□ Check validity of solution $y(x) = \_C1 e^{-\frac{x}{5}} \cos\left(\frac{\sqrt{19}\, x}{5}\right) + \_C2 e^{-\frac{x}{5}} \sin\left(\frac{\sqrt{19}\, x}{5}\right)$

- Use initial condition $y(0) = 0$

  $0 = \_C1$

- Compute derivative of the solution

  $\frac{d}{dx} y(x) = -\frac{\_C1 e^{-\frac{x}{5}} \cos\left(\frac{\sqrt{19}\, x}{5}\right)}{5} - \frac{\_C1 e^{-\frac{x}{5}} \sqrt{19} \sin\left(\frac{\sqrt{19}\, x}{5}\right)}{5} - \frac{\_C2 e^{-\frac{x}{5}} \sin\left(\frac{\sqrt{19}\, x}{5}\right)}{5} + \frac{\_C2 e^{-\frac{x}{5}} \sqrt{19} \cos\left(\frac{\sqrt{19}\, x}{5}\right)}{5}$

- Use the initial condition $\left(\frac{d}{dx} y(x)\right)\Big|_{\{x=0\}} = 5$

  $5 = -\frac{\_C1}{5} + \frac{\_C2\sqrt{19}}{5}$

- Solve for $\_C1$ and $\_C2$

  $\left\{\_C1 = 0, \_C2 = \frac{25\sqrt{19}}{19}\right\}$

- Substitute constant values into general solution and simplify

  $y(x) = \frac{25\sqrt{19}\, e^{-\frac{x}{5}} \sin\left(\frac{\sqrt{19}\, x}{5}\right)}{19}$

- Solution to the IVP

  $y(x) = \frac{25\sqrt{19}\, e^{-\frac{x}{5}} \sin\left(\frac{\sqrt{19}\, x}{5}\right)}{19}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

Maple dsolve solution

Solving time : 0.046 (sec)
Leaf size : 20

```
dsolve([5*diff(diff(y(x),x),x)+2*diff(y(x),x)+4*y(x) = 0,
       op([y(0) = 0, D(y)(0) = 5])],y(x),singsol=all)
```

$$y = \frac{25\sqrt{19}\,\mathrm{e}^{-\frac{x}{5}} \sin\left(\frac{\sqrt{19}\,x}{5}\right)}{19}$$

Mathematica DSolve solution

Solving time : 0.029 (sec)
Leaf size : 6

```
DSolve[{5*D[y[x],{x,2}]+2*D[y[x],x]+4*y[x]==0,{y[0]==0,Derivative[1][y][0] ==0}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to 0$$

2.1.42 problem 42

Internal problem ID [8430]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 42
Date solved : `Thursday, December 12, 2024 at 09:06:27 AM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$y'' + y' + 4y = 1$$

Solved as second order linear constant coeff ode

Time used: 0.163 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 4, f(x) = 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + \lambda\, e^{x\lambda} + 4\, e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(4)} \\ &= -\frac{1}{2} \pm \frac{i\sqrt{15}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{15}}{2} \\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{15}}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{15}}{2} \\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{15}}{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{15}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{15}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{15}\, x}{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{15}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{15}\, x}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{15}\, x}{2}\right), e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{15}\, x}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{15}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{15}\, x}{2}\right)\right)\right) + \left(\frac{1}{4}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{1}{4} + \mathrm{e}^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{15}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{15}\, x}{2}\right)\right)$$

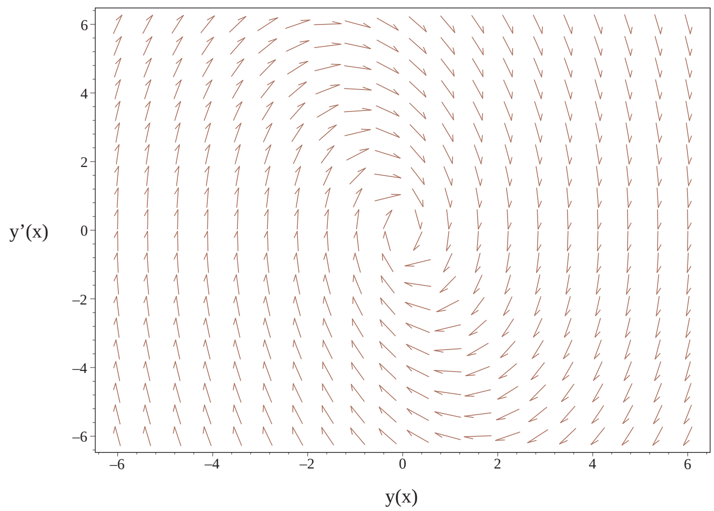


Figure 2.109: Slope field plot
$y'' + y' + 4y = 1$

Solved as second order ode using Kovacic algorithm

Time used: 0.208 (sec)

Writing the ode as

$$y'' + y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-15}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -15 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{15z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.44: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{15}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{15}\,x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{15}\,x}{2}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\sqrt{15}\,\tan\left(\frac{\sqrt{15}\,x}{2}\right)}{15}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{15}\,x}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{15}\,x}{2}\right) \left(\frac{2\sqrt{15}\,\tan\left(\frac{\sqrt{15}\,x}{2}\right)}{15}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{15}\,x}{2}\right) + \frac{2c_2\,\mathrm{e}^{-\frac{x}{2}}\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,x}{2}\right)}{15}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
1
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{15}\, x}{2}\right), \frac{2\, e^{-\frac{x}{2}} \sqrt{15}\, \sin\left(\frac{\sqrt{15}\, x}{2}\right)}{15} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{4}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\, e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{15}\, x}{2}\right) + \frac{2c_2\, e^{-\frac{x}{2}} \sqrt{15}\, \sin\left(\frac{\sqrt{15}\, x}{2}\right)}{15}\right) + \left(\frac{1}{4}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1\, e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{15}\, x}{2}\right) + \frac{2c_2\, e^{-\frac{x}{2}} \sqrt{15}\, \sin\left(\frac{\sqrt{15}\, x}{2}\right)}{15} + \frac{1}{4}$$

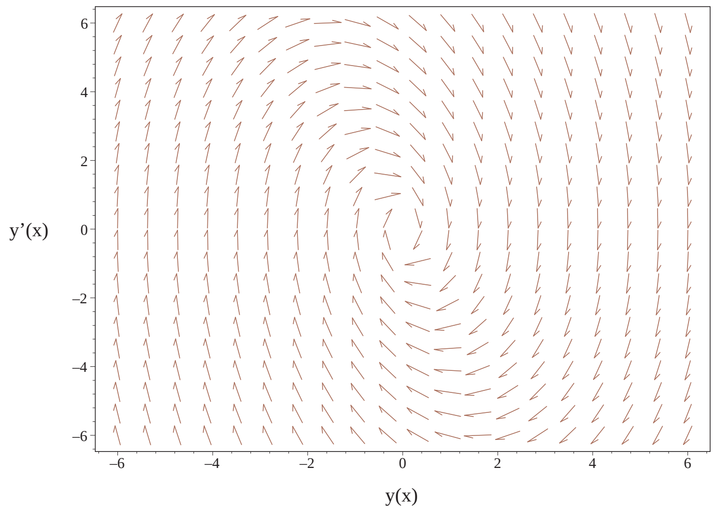


Figure 2.110: Slope field plot
$y'' + y' + 4y = 1$

Solved as second order ode adjoint method

Time used: 12.940 (sec)

In normal form the ode

$$y'' + y' + 4y = 1 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 4 \\ r(x) &= 1 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (\xi(x))' + (4\xi(x)) &= 0 \\ \xi''(x) - \xi'(x) + 4\xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = -1, C = 4$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} - \lambda\,\mathrm{e}^{x\lambda} + 4\,\mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = 4$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2 - (4)\,(1)\,(4)} \\
&= \frac{1}{2} \pm \frac{i\sqrt{15}}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= \frac{1}{2} + \frac{i\sqrt{15}}{2} \\
\lambda_2 &= \frac{1}{2} - \frac{i\sqrt{15}}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= \frac{1}{2} + \frac{i\sqrt{15}}{2} \\
\lambda_2 &= \frac{1}{2} - \frac{i\sqrt{15}}{2}
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = \frac{1}{2}$ and $\beta = \frac{\sqrt{15}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$\xi = e^{\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{15}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{15}\,x}{2}\right)\right)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$
\begin{aligned}
\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y &= \int \xi(x)\,r(x)\,dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi(x)}
\end{aligned}
$$

Or

$$
y' + y\left(1 - \frac{\left(\frac{e^{\frac{x}{2}}\left(c_1\cos\left(\frac{\sqrt{15}\,x}{2}\right)+c_2\sin\left(\frac{\sqrt{15}\,x}{2}\right)\right)}{2} + e^{\frac{x}{2}}\left(-\frac{c_1\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,x}{2}\right)}{2} + \frac{c_2\sqrt{15}\,\cos\left(\frac{\sqrt{15}\,x}{2}\right)}{2}\right)\right)e^{-\frac{x}{2}}}{c_1\cos\left(\frac{\sqrt{15}\,x}{2}\right) + c_2\sin\left(\frac{\sqrt{15}\,x}{2}\right)}\right) = \frac{e^{-\frac{x}{2}}\left(c_1\left(\cdots\right.\right.}{\cdots}
$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$
y' + q(x)y = p(x)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= -\frac{\left(c_2\sqrt{15}-c_1\right)\cos\left(\frac{\sqrt{15}\,x}{2}\right) - \sin\left(\frac{\sqrt{15}\,x}{2}\right)\left(c_1\sqrt{15}+c_2\right)}{2c_1\cos\left(\frac{\sqrt{15}\,x}{2}\right) + 2c_2\sin\left(\frac{\sqrt{15}\,x}{2}\right)} \\
p(x) &= \frac{\left(-c_2\sqrt{15}+c_1\right)\cos\left(\frac{\sqrt{15}\,x}{2}\right) + \sin\left(\frac{\sqrt{15}\,x}{2}\right)\left(c_1\sqrt{15}+c_2\right)}{8c_1\cos\left(\frac{\sqrt{15}\,x}{2}\right) + 8c_2\sin\left(\frac{\sqrt{15}\,x}{2}\right)}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= e^{\int -\frac{\left(c_2\sqrt{15}-c_1\right)\cos\left(\frac{\sqrt{15}\,x}{2}\right)-\sin\left(\frac{\sqrt{15}\,x}{2}\right)\left(c_1\sqrt{15}+c_2\right)}{2c_1\cos\left(\frac{\sqrt{15}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{15}\,x}{2}\right)}dx} \\
&= \frac{\sqrt{\sec\left(\frac{\sqrt{15}\,x}{2}\right)^2}\, e^{\frac{\sqrt{15}\,\arctan\left(\tan\left(\frac{\sqrt{15}\,x}{2}\right)\right)}{15}}}{c_1 + c_2\tan\left(\frac{\sqrt{15}\,x}{2}\right)}
\end{aligned}
$$

The ode becomes

$$\frac{d}{dx}(\mu y) = \mu p$$

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{\left(-c_2\sqrt{15}+c_1\right)\cos\left(\frac{\sqrt{15}\,x}{2}\right)+\sin\left(\frac{\sqrt{15}\,x}{2}\right)\left(c_1\sqrt{15}+c_2\right)}{8c_1\cos\left(\frac{\sqrt{15}\,x}{2}\right)+8c_2\sin\left(\frac{\sqrt{15}\,x}{2}\right)}\right)$$

$$\frac{d}{dx}\left(\frac{y\sqrt{\sec\left(\frac{\sqrt{15}\,x}{2}\right)^2}\,e^{\frac{\sqrt{15}\,\arctan\left(\tan\left(\frac{\sqrt{15}\,x}{2}\right)\right)}{15}}}{c_1+c_2\tan\left(\frac{\sqrt{15}\,x}{2}\right)}\right)$$
$$= \left(\frac{\sqrt{\sec\left(\frac{\sqrt{15}\,x}{2}\right)^2}\,e^{\frac{\sqrt{15}\,\arctan\left(\tan\left(\frac{\sqrt{15}\,x}{2}\right)\right)}{15}}}{c_1+c_2\tan\left(\frac{\sqrt{15}\,x}{2}\right)}\right)\left(\frac{\left(-c_2\sqrt{15}+c_1\right)\cos\left(\frac{\sqrt{15}\,x}{2}\right)+\sin\left(\frac{\sqrt{15}\,x}{2}\right)\left(c_1\sqrt{15}+c_2\right)}{8c_1\cos\left(\frac{\sqrt{15}\,x}{2}\right)+8c_2\sin\left(\frac{\sqrt{15}\,x}{2}\right)}\right)$$

$$d\left(\frac{y\sqrt{\sec\left(\frac{\sqrt{15}\,x}{2}\right)^2}\,e^{\frac{\sqrt{15}\,\arctan\left(\tan\left(\frac{\sqrt{15}\,x}{2}\right)\right)}{15}}}{c_1+c_2\tan\left(\frac{\sqrt{15}\,x}{2}\right)}\right)$$
$$= \left(\frac{\left(\left(-c_2\sqrt{15}+c_1\right)\cos\left(\frac{\sqrt{15}\,x}{2}\right)+\sin\left(\frac{\sqrt{15}\,x}{2}\right)\left(c_1\sqrt{15}+c_2\right)\right)\sqrt{\sec\left(\frac{\sqrt{15}\,x}{2}\right)^2}\,e^{\frac{\sqrt{15}\,\arctan\left(\tan\left(\frac{\sqrt{15}\,x}{2}\right)\right)}{15}}}{\left(8c_1\cos\left(\frac{\sqrt{15}\,x}{2}\right)+8c_2\sin\left(\frac{\sqrt{15}\,x}{2}\right)\right)\left(c_1+c_2\tan\left(\frac{\sqrt{15}\,x}{2}\right)\right)}\right)dx$$

Integrating gives

$$\frac{y\sqrt{\sec\left(\frac{\sqrt{15}\,x}{2}\right)^2}\,e^{\frac{\sqrt{15}\,\arctan\left(\tan\left(\frac{\sqrt{15}\,x}{2}\right)\right)}{15}}}{c_1+c_2\tan\left(\frac{\sqrt{15}\,x}{2}\right)} = \int\frac{\left(\left(-c_2\sqrt{15}+c_1\right)\cos\left(\frac{\sqrt{15}\,x}{2}\right)+\sin\left(\frac{\sqrt{15}\,x}{2}\right)\left(c_1\sqrt{15}+c_2\right)\right)\sqrt{\sec}}{\left(8c_1\cos\left(\frac{\sqrt{15}\,x}{2}\right)+8c_2\sin\left(\frac{\sqrt{15}\,x}{2}\right)\right)\left(c_1+c_2\,\mathrm{ta}\right.}$$
$$= \int\frac{\left(\left(-c_2\sqrt{15}+c_1\right)\cos\left(\frac{\sqrt{15}\,x}{2}\right)+\sin\left(\frac{\sqrt{15}\,x}{2}\right)\left(c_1\sqrt{15}+c_2\right)\right)\sqrt{\sec}}{\left(8c_1\cos\left(\frac{\sqrt{15}\,x}{2}\right)+8c_2\sin\left(\frac{\sqrt{15}\,x}{2}\right)\right)\left(c_1+c_2\,\mathrm{ta}\right.}$$

Dividing throughout by the integrating factor $\frac{\sqrt{\sec\left(\frac{\sqrt{15}\,x}{2}\right)^2}\,e^{\frac{\sqrt{15}\,\arctan\left(\tan\left(\frac{\sqrt{15}\,x}{2}\right)\right)}{15}}}{c_1+c_2\tan\left(\frac{\sqrt{15}\,x}{2}\right)}$ gives the final solution

$$y = \frac{\left(c_1 + c_2 \tan\left(\frac{\sqrt{15}\,x}{2}\right)\right) e^{-\frac{\sqrt{15}\,\arctan\left(\tan\left(\frac{\sqrt{15}\,x}{2}\right)\right)}{15}} \left(\int \frac{\left(\left(-c_2\sqrt{15}+c_1\right)\cos\left(\frac{\sqrt{15}\,x}{2}\right)+\sin\left(\frac{\sqrt{15}\,x}{2}\right)\left(c_1\sqrt{15}+c_2\right)\right)\sqrt{\sec\left(\frac{\sqrt{15}\,x}{2}\right)^2}\, e^{-}}{\left(8c_1\cos\left(\frac{\sqrt{15}\,x}{2}\right)+8c_2\sin\left(\frac{\sqrt{15}\,x}{2}\right)\right)\left(c_1+c_2\tan\left(\frac{\sqrt{15}\,x}{2}\right.\right.}}{\sqrt{\sec\left(\frac{\sqrt{15}\,x}{2}\right)^2}}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{\left(c_1 + c_2 \tan\left(\frac{\sqrt{15}\,x}{2}\right)\right) e^{-\frac{\sqrt{15}\,\arctan\left(\tan\left(\frac{\sqrt{15}\,x}{2}\right)\right)}{15}} \left(\int \frac{\left(\left(-c_2\sqrt{15}+c_1\right)\cos\left(\frac{\sqrt{15}\,x}{2}\right)+\sin\left(\frac{\sqrt{15}\,x}{2}\right)\left(c_1\sqrt{15}+c_2\right)\right)\sqrt{\sec\left(\frac{\sqrt{15}\,x}{2}\right)^2}\, e^{\frac{\sqrt{15}}{}}}{\left(8c_1\cos\left(\frac{\sqrt{15}\,x}{2}\right)+8c_2\sin\left(\frac{\sqrt{15}\,x}{2}\right)\right)\left(c_1+c_2\tan\left(\frac{\sqrt{15}\,x}{2}\right)\right)}}{\sqrt{\sec\left(\frac{\sqrt{15}\,x}{2}\right)^2}}$$

The constants can be merged to give

$$y = \frac{\left(c_1 + c_2 \tan\left(\frac{\sqrt{15}\,x}{2}\right)\right) e^{-\frac{\sqrt{15}\,\arctan\left(\tan\left(\frac{\sqrt{15}\,x}{2}\right)\right)}{15}} \left(\int \frac{\left(\left(-c_2\sqrt{15}+c_1\right)\cos\left(\frac{\sqrt{15}\,x}{2}\right)+\sin\left(\frac{\sqrt{15}\,x}{2}\right)\left(c_1\sqrt{15}+c_2\right)\right)\sqrt{\sec\left(\frac{\sqrt{15}\,x}{2}\right)^2}\, e^{\frac{\sqrt{15}}{}}}{\left(8c_1\cos\left(\frac{\sqrt{15}\,x}{2}\right)+8c_2\sin\left(\frac{\sqrt{15}\,x}{2}\right)\right)\left(c_1+c_2\tan\left(\frac{\sqrt{15}\,x}{2}\right)\right)}}{\sqrt{\sec\left(\frac{\sqrt{15}\,x}{2}\right)^2}}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{\left(c_1 + c_2 \tan\left(\frac{\sqrt{15}\,x}{2}\right)\right) e^{-\frac{\sqrt{15}\,\arctan\left(\tan\left(\frac{\sqrt{15}\,x}{2}\right)\right)}{15}} \left(\int \frac{\left(\left(-c_2\sqrt{15}+c_1\right)\cos\left(\frac{\sqrt{15}\,x}{2}\right)+\sin\left(\frac{\sqrt{15}\,x}{2}\right)\left(c_1\sqrt{15}+c_2\right)\right)\sqrt{\sec\left(\frac{\sqrt{15}\,x}{2}\right)^2}\, e^{\frac{\sqrt{15}}{}}}{\left(8c_1\cos\left(\frac{\sqrt{15}\,x}{2}\right)+8c_2\sin\left(\frac{\sqrt{15}\,x}{2}\right)\right)\left(c_1+c_2\tan\left(\frac{\sqrt{15}\,x}{2}\right)\right)}}{\sqrt{\sec\left(\frac{\sqrt{15}\,x}{2}\right)^2}}$$

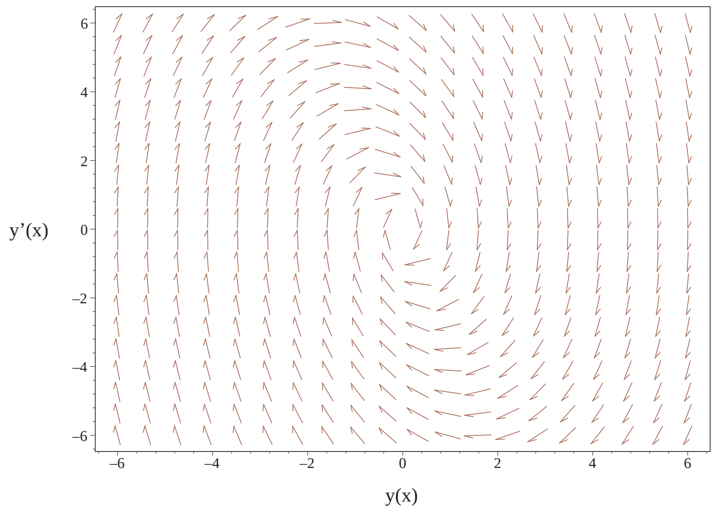


Figure 2.111: Slope field plot
$y'' + y' + 4y = 1$

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + \frac{d}{dx}y(x) + 4y(x) = 1$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-15})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{\mathrm{I}\sqrt{15}}{2}, -\frac{1}{2} + \frac{\mathrm{I}\sqrt{15}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{15}\,x}{2}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{15}\,x}{2}\right)$

- General solution of the ODE

  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

$$y(x) = C1\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{15}\,x}{2}\right) + C2\,\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{15}\,x}{2}\right) + y_p(x)$$

- □ Find a particular solution $y_p(x)$ of the ODE
  - ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $$\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 1\right]$$
  - ◦ Wronskian of solutions of the homogeneous equation

    $$W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{15}\,x}{2}\right) & \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{15}\,x}{2}\right) \\ -\frac{\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{15}\,x}{2}\right)}{2} - \frac{\mathrm{e}^{-\frac{x}{2}} \sqrt{15}\, \sin\left(\frac{\sqrt{15}\,x}{2}\right)}{2} & -\frac{\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{15}\,x}{2}\right)}{2} + \frac{\mathrm{e}^{-\frac{x}{2}} \sqrt{15}\, \cos\left(\frac{\sqrt{15}\,x}{2}\right)}{2} \end{bmatrix}$$
  - ◦ Compute Wronskian

    $$W(y_1(x), y_2(x)) = \frac{\sqrt{15}\,\mathrm{e}^{-x}}{2}$$
  - ◦ Substitute functions into equation for $y_p(x)$

    $$y_p(x) = -\frac{2\sqrt{15}\,\mathrm{e}^{-\frac{x}{2}}\left(\cos\left(\frac{\sqrt{15}\,x}{2}\right)\left(\int \mathrm{e}^{\frac{x}{2}} \sin\left(\frac{\sqrt{15}\,x}{2}\right) dx\right) - \sin\left(\frac{\sqrt{15}\,x}{2}\right)\left(\int \mathrm{e}^{\frac{x}{2}} \cos\left(\frac{\sqrt{15}\,x}{2}\right) dx\right)\right)}{15}$$
  - ◦ Compute integrals

    $$y_p(x) = \frac{1}{4}$$
- • Substitute particular solution into general solution to ODE

  $$y(x) = C1\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{15}\,x}{2}\right) + C2\,\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{15}\,x}{2}\right) + \frac{1}{4}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 32

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)+4*y(x) = 1,
       y(x),singsol=all)
```

$$y = e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{15}\, x}{2}\right) c_2 + e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{15}\, x}{2}\right) c_1 + \frac{1}{4}$$

Mathematica DSolve solution

Solving time : 0.026 (sec)
Leaf size : 51

```
DSolve[{D[y[x],{x,2}]+D[y[x],x]+4*y[x]==1,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2 e^{-x/2} \cos\left(\frac{\sqrt{15}x}{2}\right) + c_1 e^{-x/2} \sin\left(\frac{\sqrt{15}x}{2}\right) + \frac{1}{4}$$

2.1.43 problem 43

Internal problem ID [8431]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 43
Date solved : `Thursday, December 12, 2024 at 09:06:41 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' + y' + 4y = \sin(x)$$

Solved as second order linear constant coeff ode

Time used: 0.185 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 4, f(x) = \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + \lambda e^{x\lambda} + 4 e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(4)} \\ &= -\frac{1}{2} \pm \frac{i\sqrt{15}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{15}}{2} \\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{15}}{2}\end{aligned}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{15}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{15}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{15}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{15}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{15}\,x}{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{15}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{15}\, x}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin\left(x\right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos\left(x\right), \sin\left(x\right)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{15}\, x}{2}\right), \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{15}\, x}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos\left(x\right) + A_2 \sin\left(x\right)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 \cos\left(x\right) + 3A_2 \sin\left(x\right) - A_1 \sin\left(x\right) + A_2 \cos\left(x\right) = \sin\left(x\right)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{10}, A_2 = \frac{3}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos\left(x\right)}{10} + \frac{3 \sin\left(x\right)}{10}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\mathrm{e}^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{15}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{15}\, x}{2}\right)\right)\right) + \left(-\frac{\cos(x)}{10} + \frac{3\sin(x)}{10}\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = -\frac{\cos(x)}{10} + \frac{3\sin(x)}{10} + \mathrm{e}^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{15}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{15}\, x}{2}\right)\right)
$$

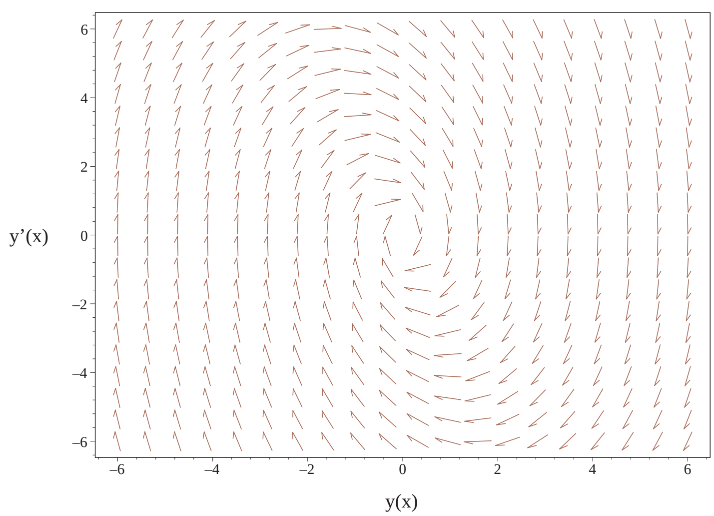


Figure 2.112: Slope field plot
$y'' + y' + 4y = \sin(x)$

Solved as second order ode using Kovacic algorithm

Time used: 0.277 (sec)

Writing the ode as

$$
y'' + y' + 4y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-15}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -15 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{15z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.46: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -\frac{15}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos\left(\frac{\sqrt{15}\, x}{2}\right)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\, dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{15}\, x}{2}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{2\sqrt{15}\, \tan\left(\frac{\sqrt{15}\, x}{2}\right)}{15}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{15}\, x}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{15}\, x}{2}\right) \left(\frac{2\sqrt{15}\, \tan\left(\frac{\sqrt{15}\, x}{2}\right)}{15}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1\, \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{15}\, x}{2}\right) + \frac{2c_2\, \mathrm{e}^{-\frac{x}{2}} \sqrt{15}\, \sin\left(\frac{\sqrt{15}\, x}{2}\right)}{15}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{15}\, x}{2}\right), \frac{2\, e^{-\frac{x}{2}} \sqrt{15}\, \sin\left(\frac{\sqrt{15}\, x}{2}\right)}{15} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$3A_1 \cos(x) + 3A_2 \sin(x) - A_1 \sin(x) + A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{10}, A_2 = \frac{3}{10}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)}{10} + \frac{3\sin(x)}{10}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\, e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{15}\, x}{2}\right) + \frac{2c_2\, e^{-\frac{x}{2}} \sqrt{15}\, \sin\left(\frac{\sqrt{15}\, x}{2}\right)}{15}\right) + \left(-\frac{\cos(x)}{10} + \frac{3\sin(x)}{10}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{15}\,x}{2}\right)+\frac{2 c_2 \,\mathrm{e}^{-\frac{x}{2}} \sqrt{15}\, \sin\left(\frac{\sqrt{15}\,x}{2}\right)}{15}-\frac{\cos\left(x\right)}{10}+\frac{3 \sin\left(x\right)}{10}$$

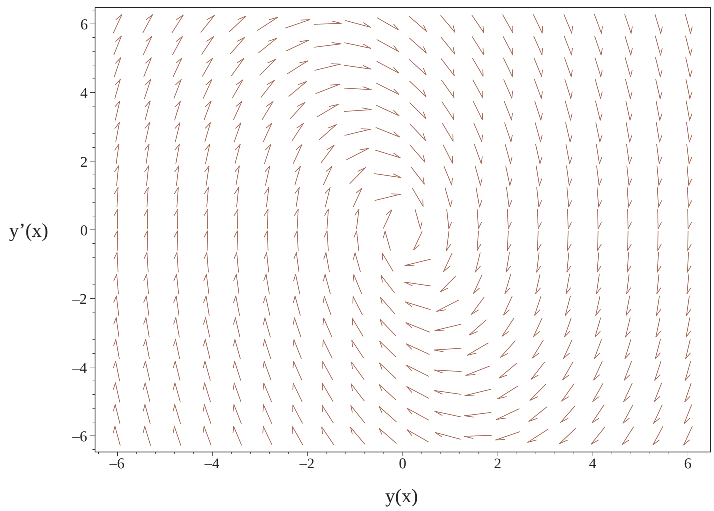


Figure 2.113: Slope field plot
$y''+y'+4y = \sin\left(x\right)$

Solved as second order ode adjoint method

Time used: 72.330 (sec)

In normal form the ode

$$y''+y'+4y = \sin\left(x\right) \tag{1}$$

Becomes

$$y''+p(x)\, y'+q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 1\\ q(x) &= 4\\ r(x) &= \sin\left(x\right) \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned}\xi'' - (\xi\, p)' + \xi q &= 0\\ \xi'' - (\xi(x))' + (4\xi(x)) &= 0\\ \xi''(x) - \xi'(x) + 4\xi(x) &= 0\end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = -1, C = 4$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} - \lambda\, \mathrm{e}^{x\lambda} + 4\, \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2 - (4)\,(1)\,(4)}\\ &= \frac{1}{2} \pm \frac{i\sqrt{15}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{1}{2} + \frac{i\sqrt{15}}{2}\\ \lambda_2 &= \frac{1}{2} - \frac{i\sqrt{15}}{2}\end{aligned}$$

Which simplifies to

$$\lambda_1 = \frac{1}{2} + \frac{i\sqrt{15}}{2}$$

$$\lambda_2 = \frac{1}{2} - \frac{i\sqrt{15}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = \frac{1}{2}$ and $\beta = \frac{\sqrt{15}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$\xi = e^{\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{15}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{15}\,x}{2}\right)\right)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y = \int \xi(x)\,r(x)\,dx$$

$$y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) = \frac{\int \xi(x)\,r(x)\,dx}{\xi(x)}$$

Or

$$y' + y\left(1 - \frac{\left(\frac{e^{\frac{x}{2}}\left(c_1\cos\left(\frac{\sqrt{15}\,x}{2}\right)+c_2\sin\left(\frac{\sqrt{15}\,x}{2}\right)\right)}{2} + e^{\frac{x}{2}}\left(-\frac{c_1\sqrt{15}\,\sin\left(\frac{\sqrt{15}\,x}{2}\right)}{2} + \frac{c_2\sqrt{15}\,\cos\left(\frac{\sqrt{15}\,x}{2}\right)}{2}\right)\right)e^{-\frac{x}{2}}}{c_1\cos\left(\frac{\sqrt{15}\,x}{2}\right) + c_2\sin\left(\frac{\sqrt{15}\,x}{2}\right)}\right) = \frac{e^{-\frac{x}{2}}\left(c_1\left(\left(-\right.\right.\right.}{}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = -\frac{\left(c_2\sqrt{15} - c_1\right)\cos\left(\frac{\sqrt{15}\,x}{2}\right) - \sin\left(\frac{\sqrt{15}\,x}{2}\right)\left(c_1\sqrt{15} + c_2\right)}{2c_1\cos\left(\frac{\sqrt{15}\,x}{2}\right) + 2c_2\sin\left(\frac{\sqrt{15}\,x}{2}\right)}$$

$$p(x) = \frac{\left(c_2(\cos(x) - 3\sin(x))\sqrt{15} + 5c_1(\sin(x) + \cos(x))\right)\cos\left(\frac{\sqrt{15}\,x}{2}\right) - \left(c_1(\cos(x) - 3\sin(x))\sqrt{15} - 5\right.}{20c_1\cos\left(\frac{\sqrt{15}\,x}{2}\right) + 20c_2\sin\left(\frac{\sqrt{15}\,x}{2}\right)}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q \, dx} \\
&= \mathrm{e}^{\int -\frac{\left(c_2\sqrt{15}-c_1\right)\cos\left(\frac{\sqrt{15}\,x}{2}\right)-\sin\left(\frac{\sqrt{15}\,x}{2}\right)\left(c_1\sqrt{15}+c_2\right)}{2c_1\cos\left(\frac{\sqrt{15}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{15}\,x}{2}\right)}dx} \\
&= \frac{\sqrt{\sec\left(\frac{\sqrt{15}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{15}\,\arctan\left(\tan\left(\frac{\sqrt{15}\,x}{2}\right)\right)}{15}}}{c_1+c_2\tan\left(\frac{\sqrt{15}\,x}{2}\right)}
\end{aligned}
$$

The ode becomes

$$
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu p
$$

$$
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{\left(c_2(\cos(x)-3\sin(x))\sqrt{15}+5c_1(\sin(x)+\cos(x))\right)\cos\left(\frac{\sqrt{15}\,x}{2}\right)-\left(c_1(\cos(x)-3\sin(x))\sqrt{15}-\right.}{20c_1\cos\left(\frac{\sqrt{15}\,x}{2}\right)+20c_2\sin\left(\frac{\sqrt{15}\,x}{2}\right)}\right.
$$

$$
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y\sqrt{\sec\left(\frac{\sqrt{15}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{15}\,\arctan\left(\tan\left(\frac{\sqrt{15}\,x}{2}\right)\right)}{15}}}{c_1+c_2\tan\left(\frac{\sqrt{15}\,x}{2}\right)}\right) = \left(\frac{\sqrt{\sec\left(\frac{\sqrt{15}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{15}\,\arctan\left(\tan\left(\frac{\sqrt{15}\,x}{2}\right)\right)}{15}}}{c_1+c_2\tan\left(\frac{\sqrt{15}\,x}{2}\right)}\right)\left(\frac{\left(c_2(\cos(x)-3\sin(x))\sqrt{15}+5c_1(\sin(x)+\cos(x))\right)\cos\left(\frac{\sqrt{15}}{2}\right.}{20c_1\cos\left(\frac{\sqrt{15}}{2}\right.}\right.
$$

$$
\mathrm{d}\left(\frac{y\sqrt{\sec\left(\frac{\sqrt{15}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{15}\,\arctan\left(\tan\left(\frac{\sqrt{15}\,x}{2}\right)\right)}{15}}}{c_1+c_2\tan\left(\frac{\sqrt{15}\,x}{2}\right)}\right) = \left(\frac{\left(\left(c_2(\cos(x)-3\sin(x))\sqrt{15}+5c_1(\sin(x)+\cos(x))\right)\cos\left(\frac{\sqrt{15}\,x}{2}\right)-\left(c_1(\cos(x)-3\sin(x))\sqrt{15}-5c\right.\right.}{\left(20c_1\cos\left(\frac{\sqrt{15}\,x}{2}\right)+20c_2\sin\left(\frac{\sqrt{15}\,x}{2}\right)\right)\left(c_1+c_2\right.}\right.
$$

Integrating gives

$$\frac{y\sqrt{\sec\left(\frac{\sqrt{15}\,x}{2}\right)^{2}}\,\mathrm{e}^{\frac{\sqrt{15}\,\arctan\left(\tan\left(\frac{\sqrt{15}\,x}{2}\right)\right)}{15}}}{c_1+c_2\tan\left(\frac{\sqrt{15}\,x}{2}\right)} = \int \frac{\left(\left(c_2(\cos(x)-3\sin(x))\sqrt{15}+5c_1(\sin(x)+\cos(x))\right)\cos\left(\frac{\sqrt{15}\,x}{2}\right.\right.}{\left(20c_1\cos\right.}$$

$$= \int \frac{\left(\left(c_2(\cos(x)-3\sin(x))\sqrt{15}+5c_1(\sin(x)+\cos(x))\right)\cos\left(\frac{\sqrt{15}\,x}{2}\right.\right.}{\left(20c_1\cos\right.}$$

Dividing throughout by the integrating factor $\frac{\sqrt{\sec\left(\frac{\sqrt{15}\,x}{2}\right)^{2}}\,\mathrm{e}^{\frac{\sqrt{15}\,\arctan\left(\tan\left(\frac{\sqrt{15}\,x}{2}\right)\right)}{15}}}{c_1+c_2\tan\left(\frac{\sqrt{15}\,x}{2}\right)}$ gives the final solution

$$y = \frac{\left(c_1+c_2\tan\left(\frac{\sqrt{15}\,x}{2}\right)\right)\mathrm{e}^{-\frac{\sqrt{15}\,\arctan\left(\tan\left(\frac{\sqrt{15}\,x}{2}\right)\right)}{15}}\left(\int \frac{\left(\left(c_2(\cos(x)-3\sin(x))\sqrt{15}+5c_1(\sin(x)+\cos(x))\right)\cos\left(\frac{\sqrt{15}\,x}{2}\right)-\left(c_1(\cos(x)\right.\right.}{\left(20c_1\cos\left(\frac{\sqrt{15}\,x}{2}\right)+2\right.}\right.}{\sqrt{\sec\left(\frac{\sqrt{15}\,x}{2}\right)^{2}}}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{\left(c_1+c_2\tan\left(\frac{\sqrt{15}\,x}{2}\right)\right)\mathrm{e}^{-\frac{\sqrt{15}\,\arctan\left(\tan\left(\frac{\sqrt{15}\,x}{2}\right)\right)}{15}}\left(\int \frac{\left(\left(c_2(\cos(x)-3\sin(x))\sqrt{15}+5c_1(\sin(x)+\cos(x))\right)\cos\left(\frac{\sqrt{15}\,x}{2}\right)-\left(c_1(\cos(x)-3\right.\right.}{\left(20c_1\cos\left(\frac{\sqrt{15}\,x}{2}\right)+20c_2\right.}\right.}{\sqrt{\sec\left(\frac{\sqrt{15}\,x}{2}\right)^{2}}}$$

The constants can be merged to give

$$y = \frac{\left(c_1+c_2\tan\left(\frac{\sqrt{15}\,x}{2}\right)\right)\mathrm{e}^{-\frac{\sqrt{15}\,\arctan\left(\tan\left(\frac{\sqrt{15}\,x}{2}\right)\right)}{15}}\left(\int \frac{\left(\left(c_2(\cos(x)-3\sin(x))\sqrt{15}+5c_1(\sin(x)+\cos(x))\right)\cos\left(\frac{\sqrt{15}\,x}{2}\right)-\left(c_1(\cos(x)-3\right.\right.}{\left(20c_1\cos\left(\frac{\sqrt{15}\,x}{2}\right)+20c_2\right.}\right.}{\sqrt{\sec\left(\frac{\sqrt{15}\,x}{2}\right)^{2}}}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{\left(c_1 + c_2 \tan\left(\frac{\sqrt{15}\, x}{2}\right)\right) \mathrm{e}^{-\frac{\sqrt{15}\, \arctan\left(\tan\left(\frac{\sqrt{15}\, x}{2}\right)\right)}{15}} \left(\int \frac{\left(\left(c_2(\cos(x)-3\sin(x))\sqrt{15}+5c_1(\sin(x)+\cos(x))\right)\cos\left(\frac{\sqrt{15}\, x}{2}\right)-\left(c_1(\cos(x)-3\right.\right.}{\left(20c_1\cos\left(\frac{\sqrt{15}\, x}{2}\right)+20c_2\right.}\right.}{\sqrt{\sec\left(\frac{\sqrt{15}\, x}{2}\right)^2}}$$

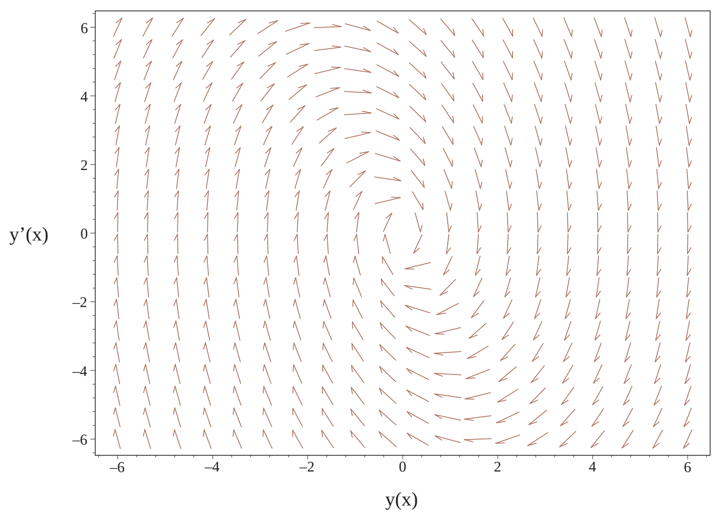


Figure 2.114: Slope field plot
$y'' + y' + 4y = \sin(x)$

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + \frac{d}{dx}y(x) + 4y(x) = \sin(x)$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1) \pm (\sqrt{-15})}{2}$

- Roots of the characteristic polynomial

$r = \left(-\frac{1}{2} - \frac{\mathrm{I}\sqrt{15}}{2}, -\frac{1}{2} + \frac{\mathrm{I}\sqrt{15}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{15}\,x}{2}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{15}\,x}{2}\right)$

- General solution of the ODE

  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y(x) = C1\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{15}\,x}{2}\right) + C2\,\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{15}\,x}{2}\right) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \sin(x)\right]$

- Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{15}\,x}{2}\right) & \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{15}\,x}{2}\right) \\ -\frac{\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{15}\,x}{2}\right)}{2} - \frac{\mathrm{e}^{-\frac{x}{2}} \sqrt{15}\,\sin\left(\frac{\sqrt{15}\,x}{2}\right)}{2} & -\frac{\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{15}\,x}{2}\right)}{2} + \frac{\mathrm{e}^{-\frac{x}{2}} \sqrt{15}\,\cos\left(\frac{\sqrt{15}\,x}{2}\right)}{2} \end{array}\right]$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = \frac{\sqrt{15}\,\mathrm{e}^{-x}}{2}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{2\sqrt{15}\,\mathrm{e}^{-\frac{x}{2}}\left(\cos\left(\frac{\sqrt{15}\,x}{2}\right)\left(\int \mathrm{e}^{\frac{x}{2}} \sin(x) \sin\left(\frac{\sqrt{15}\,x}{2}\right)dx\right) - \sin\left(\frac{\sqrt{15}\,x}{2}\right)\left(\int \mathrm{e}^{\frac{x}{2}} \sin(x) \cos\left(\frac{\sqrt{15}\,x}{2}\right)dx\right)\right)}{15}$

- Compute integrals

  $y_p(x) = -\frac{\cos(x)}{10} + \frac{3\sin(x)}{10}$

- Substitute particular solution into general solution to ODE

  $y(x) = C1\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{15}\,x}{2}\right) + C2\,\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{15}\,x}{2}\right) - \frac{\cos(x)}{10} + \frac{3\sin(x)}{10}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 39

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)+4*y(x) = sin(x),
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{15}\, x}{2}\right) c_2 + \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{15}\, x}{2}\right) c_1 + \frac{3\sin(x)}{10} - \frac{\cos(x)}{10}$$

Mathematica DSolve solution

Solving time : 1.604 (sec)
Leaf size : 60

```
DSolve[{D[y[x],{x,2}]+D[y[x],x]+4*y[x]==Sin[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{3\sin(x)}{10} - \frac{\cos(x)}{10} + c_2 e^{-x/2} \cos\left(\frac{\sqrt{15}x}{2}\right) + c_1 e^{-x/2} \sin\left(\frac{\sqrt{15}x}{2}\right)$$

2.1.44 problem 44

Internal problem ID [8432]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 44
Date solved : `Tuesday, December 17, 2024 at 12:51:34 PM`
CAS classification : `[[_homogeneous, ‘class A‘], _rational, _dAlembert]`

Solve

$$y = x{y'}^2$$

Solved as first order homogeneous class A ode

Time used: 0.823 (sec)

Solving for y' gives

$$y' = \frac{\sqrt{xy}}{x} \tag{1}$$

$$y' = -\frac{\sqrt{xy}}{x} \tag{2}$$

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{\sqrt{xy}}{x} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = \sqrt{xy}$ and $N = x$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\frac{\mathrm{d}u}{\mathrm{d}x}x + u = \sqrt{u}$$
$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\sqrt{u(x)} - u(x)}{x}$$

Or

$$u'(x) - \frac{\sqrt{u(x)} - u(x)}{x} = 0$$

Or

$$u'(x)\,x - \sqrt{u(x)} + u(x) = 0$$

Which is now solved as separable in $u(x)$.

The ode $u'(x) = \frac{\sqrt{u(x)}-u(x)}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= \frac{\sqrt{u(x)} - u(x)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{1}{x} \\ g(u) &= \sqrt{u} - u \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(u)}\,du = \int f(x)\,dx$$
$$\int \frac{1}{\sqrt{u} - u}\,du = \int \frac{1}{x}\,dx$$
$$-2\ln\left(\sqrt{u(x)} - 1\right) = \ln(x) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $\sqrt{u} - u = 0$ for $u(x)$ gives

$$\begin{aligned} u(x) &= 0 \\ u(x) &= 1 \end{aligned}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
-2\ln\left(\sqrt{u(x)}-1\right) &= \ln(x)+c_1 \\
u(x) &= 0 \\
u(x) &= 1
\end{aligned}
$$

Converting $-2\ln\left(\sqrt{u(x)}-1\right) = \ln(x)+c_1$ back to y gives

$$
-2\ln\left(\sqrt{\frac{y}{x}}-1\right) = \ln(x)+c_1
$$

Converting $u(x)=0$ back to y gives

$$
y = 0
$$

Converting $u(x)=1$ back to y gives

$$
y = x
$$

In canonical form, the ODE is

$$
\begin{aligned}
y' &= F(x,y) \\
&= -\frac{\sqrt{xy}}{x} \qquad (1)
\end{aligned}
$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$
f(t^n x, t^n y) = t^n f(x,y)
$$

In this case, it can be seen that both $M = -\sqrt{xy}$ and $N = x$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$
\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x + u
$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}x}x + u &= -\sqrt{u} \\
\frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{-\sqrt{u(x)}-u(x)}{x}
\end{aligned}
$$

Or

$$u'(x) - \frac{-\sqrt{u\left(x\right)} - u(x)}{x} = 0$$

Or

$$u'(x)\,x + \sqrt{u\left(x\right)} + u(x) = 0$$

Which is now solved as separable in $u(x)$.

The ode $u'(x) = -\frac{\sqrt{u(x)}+u(x)}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{\sqrt{u\left(x\right)} + u(x)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{1}{x} \\ g(u) &= -\sqrt{u} - u \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\ \int \frac{1}{-\sqrt{u} - u}\,du &= \int \frac{1}{x}\,dx \\ \ln\left(\frac{1}{\left(\sqrt{u\left(x\right)}+1\right)^{2}}\right) &= \ln\left(x\right) + c_2 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $-\sqrt{u} - u = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} \ln\left(\frac{1}{\left(\sqrt{u\left(x\right)}+1\right)^{2}}\right) &= \ln\left(x\right) + c_2 \\ u(x) &= 0 \end{aligned}$$

Converting $\ln\left(\frac{1}{\left(\sqrt{u(x)}+1\right)^2}\right) = \ln(x) + c_2$ back to y gives

$$\ln\left(\frac{1}{\left(\sqrt{\frac{y}{x}}+1\right)^2}\right) = \ln(x) + c_2$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Solving for y gives

$$\begin{aligned}
y &= 0\\
y &= x\\
y &= \left(\frac{2x\,\mathrm{e}^{c_1}\left(\sqrt{x\,\mathrm{e}^{c_1}}-1\right)}{\sqrt{x\,\mathrm{e}^{c_1}}} - x\,\mathrm{e}^{c_1} + 1\right)\mathrm{e}^{-c_1}\\
y &= \left(\frac{2x\,\mathrm{e}^{c_1}\left(\sqrt{x\,\mathrm{e}^{c_1}}+1\right)}{\sqrt{x\,\mathrm{e}^{c_1}}} - x\,\mathrm{e}^{c_1} + 1\right)\mathrm{e}^{-c_1}\\
y &= -\left(-\frac{2x\,\mathrm{e}^{c_2}\left(\sqrt{x\,\mathrm{e}^{c_2}}-1\right)}{\sqrt{x\,\mathrm{e}^{c_2}}} + x\,\mathrm{e}^{c_2} - 1\right)\mathrm{e}^{-c_2}\\
y &= -\left(-\frac{2x\,\mathrm{e}^{c_2}\left(\sqrt{x\,\mathrm{e}^{c_2}}+1\right)}{\sqrt{x\,\mathrm{e}^{c_2}}} + x\,\mathrm{e}^{c_2} - 1\right)\mathrm{e}^{-c_2}
\end{aligned}$$

Summary of solutions found

$$\begin{aligned}
y &= 0\\
y &= x\\
y &= \left(\frac{2x\,\mathrm{e}^{c_1}\left(\sqrt{x\,\mathrm{e}^{c_1}}-1\right)}{\sqrt{x\,\mathrm{e}^{c_1}}} - x\,\mathrm{e}^{c_1} + 1\right)\mathrm{e}^{-c_1}\\
y &= \left(\frac{2x\,\mathrm{e}^{c_1}\left(\sqrt{x\,\mathrm{e}^{c_1}}+1\right)}{\sqrt{x\,\mathrm{e}^{c_1}}} - x\,\mathrm{e}^{c_1} + 1\right)\mathrm{e}^{-c_1}\\
y &= -\left(-\frac{2x\,\mathrm{e}^{c_2}\left(\sqrt{x\,\mathrm{e}^{c_2}}-1\right)}{\sqrt{x\,\mathrm{e}^{c_2}}} + x\,\mathrm{e}^{c_2} - 1\right)\mathrm{e}^{-c_2}\\
y &= -\left(-\frac{2x\,\mathrm{e}^{c_2}\left(\sqrt{x\,\mathrm{e}^{c_2}}+1\right)}{\sqrt{x\,\mathrm{e}^{c_2}}} + x\,\mathrm{e}^{c_2} - 1\right)\mathrm{e}^{-c_2}
\end{aligned}$$

Solved as first order ode of type nonlinear p but separable

Time used: 0.240 (sec)

The ode has the form

$$(y')^{\frac{n}{m}} = f(x)g(y) \tag{1}$$

Where $n = 2, m = 1, f = \frac{1}{x}, g = y$. Hence the ode is

$$(y')^2 = \frac{y}{x}$$

Solving for y' from (1) gives

$$\begin{aligned} y' &= \sqrt{fg} \\ y' &= -\sqrt{fg} \end{aligned}$$

To be able to solve as separable ode, we have to now assume that $f > 0, g > 0$.

$$\begin{aligned} \frac{1}{x} &> 0 \\ y &> 0 \end{aligned}$$

Under the above assumption the differential equations become separable and can be written as

$$\begin{aligned} y' &= \sqrt{f}\,\sqrt{g} \\ y' &= -\sqrt{f}\,\sqrt{g} \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{\sqrt{g}}\,dy &= \left(\sqrt{f}\right)\,dx \\ -\frac{1}{\sqrt{g}}\,dy &= \left(\sqrt{f}\right)\,dx \end{aligned}$$

Replacing $f(x), g(y)$ by their values gives

$$\begin{aligned} \frac{1}{\sqrt{y}}\,dy &= \left(\sqrt{\frac{1}{x}}\right)\,dx \\ -\frac{1}{\sqrt{y}}\,dy &= \left(\sqrt{\frac{1}{x}}\right)\,dx \end{aligned}$$

Integrating now gives the following solutions

$$
\begin{aligned}
\int \frac{1}{\sqrt{y}}dy &= \int \sqrt{\frac{1}{x}}dx + c_1 \\
2\sqrt{y} &= 2x\sqrt{\frac{1}{x}} \\
\int -\frac{1}{\sqrt{y}}dy &= \int \sqrt{\frac{1}{x}}dx + c_1 \\
-2\sqrt{y} &= 2x\sqrt{\frac{1}{x}}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
y &= x\sqrt{\frac{1}{x}}\, c_1 + \frac{c_1^2}{4} + x \\
y &= x\sqrt{\frac{1}{x}}\, c_1 + \frac{c_1^2}{4} + x
\end{aligned}
$$

Summary of solutions found

$$
y = x\sqrt{\frac{1}{x}}\, c_1 + \frac{c_1^2}{4} + x
$$

Solved as first order ode of type dAlembert

Time used: 0.076 (sec)

Let $p = y'$ the ode becomes

$$
y = x\, p^2
$$

Solving for y from the above results in

$$
y = x\, p^2 \tag{1}
$$

This has the form

$$
y = xf(p) + g(p) \tag{*}
$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. x gives

$$
\begin{aligned}
p &= f + (xf' + g')\frac{dp}{dx} \\
p - f &= (xf' + g')\frac{dp}{dx} \qquad (2)
\end{aligned}
$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= p^2 \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
-p^2 + p = 2xpp'(x) \qquad (2A)
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
-p^2 + p = 0
$$

Solving the above for p results in

$$
\begin{aligned}
p_1 &= 0 \\
p_2 &= 1
\end{aligned}
$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$
\begin{aligned}
y &= 0 \\
y &= x
\end{aligned}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{-p(x)^2 + p(x)}{2xp\,(x)} \qquad (3)
$$

This ODE is now solved for $p(x)$. No inversion is needed. In canonical form a linear first order is

$$
p'(x) + q(x)p(x) = p(x)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= \frac{1}{2x} \\
p(x) &= \frac{1}{2x}
\end{aligned}
$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dx}\\ &= \mathrm{e}^{\int \frac{1}{2x}dx}\\ &= \sqrt{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu p) &= \mu p\\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu p) &= (\mu)\left(\frac{1}{2x}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(p\sqrt{x}\right) &= \left(\sqrt{x}\right)\left(\frac{1}{2x}\right)\\ \mathrm{d}\left(p\sqrt{x}\right) &= \left(\frac{1}{2\sqrt{x}}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}p\sqrt{x} &= \int \frac{1}{2\sqrt{x}}\,dx\\ &= \sqrt{x}+c_1\end{aligned}$$

Dividing throughout by the integrating factor $\sqrt{x}$ gives the final solution

$$p(x) = \frac{\sqrt{x}+c_1}{\sqrt{x}}$$

Substituing the above solution for p in (2A) gives

$$y = \left(\sqrt{x}+c_1\right)^2$$

Summary of solutions found

$$\begin{aligned}y &= 0\\ y &= x\\ y &= \left(\sqrt{x}+c_1\right)^2\end{aligned}$$

Maple step by step solution

Let's solve

$y(x) = x\left(\frac{d}{dx}y(x)\right)^2$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\left[\frac{d}{dx}y(x) = \frac{\sqrt{xy(x)}}{x}, \frac{d}{dx}y(x) = -\frac{\sqrt{xy(x)}}{x}\right]$

- Solve the equation $\frac{d}{dx}y(x) = \frac{\sqrt{xy(x)}}{x}$
- Solve the equation $\frac{d}{dx}y(x) = -\frac{\sqrt{xy(x)}}{x}$
- Set of solutions

  $\{workingODE, workingODE\}$

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

Maple dsolve solution

Solving time : `0.044` (sec)
Leaf size : 39

```
dsolve(y(x) = x*diff(y(x),x)^2,
       y(x),singsol=all)
```

$$y = 0$$
$$y = \frac{\left(x + \sqrt{c_1 x}\right)^2}{x}$$
$$y = \frac{\left(-x + \sqrt{c_1 x}\right)^2}{x}$$

Mathematica DSolve solution

Solving time : `0.049` (sec)
Leaf size : 46

```
DSolve[{y[x]==x*(D[y[x],x])^2,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{4}\left(-2\sqrt{x} + c_1\right)^2$$
$$y(x) \to \frac{1}{4}\left(2\sqrt{x} + c_1\right)^2$$
$$y(x) \to 0$$

2.1.45 problem 45

Internal problem ID [8433]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 45
Date solved : `Tuesday, December 17, 2024 at 12:51:36 PM`
CAS classification : `[_dAlembert]`

Solve

$$y'y = 1 - x{y'}^3$$

Solved as first order dAlembert ode

Time used: 0.173 (sec)

Let $p = y'$ the ode becomes

$$py = -x\,p^3 + 1$$

Solving for y from the above results in

$$y = -p^2 x + \frac{1}{p} \tag{1}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = -p^2$$
$$g = \frac{1}{p}$$

Hence (2) becomes

$$p^2 + p = \left(-2xp - \frac{1}{p^2}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p^2 + p = 0$$

Solving the above for p results in

$$p_1 = -1$$
$$p_2 = 0$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$y = -x - 1$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x)^2 + p(x)}{-2p(x)\,x - \frac{1}{p(x)^2}} \tag{3}$$

Inverting the above ode gives

$$\frac{d}{dp}x(p) = \frac{-2x(p)\,p - \frac{1}{p^2}}{p^2 + p} \tag{4}$$

This ODE is now solved for $x(p)$. The integrating factor is

$$\mu = \mathrm{e}^{\int \frac{2}{p+1}dp}$$
$$\mu = (p+1)^2$$
$$\mu = (p+1)^2 \tag{5}$$

Integrating gives

$$\begin{aligned} x(p) &= \frac{1}{\mu}\left(\int \mu\left(-\frac{1}{p^3\,(p+1)}\right)\,dp + c_1\right) \\ &= \frac{1}{\mu}\left(\frac{\frac{1}{2p^2} + \frac{1}{p} + c_1}{(p+1)^2} + c_1\right) \\ &= \frac{\frac{1}{2p^2} + \frac{1}{p} + c_1}{(p+1)^2} \end{aligned} \tag{5}$$

Now we need to eliminate p between the above solution and (1A). The first method is to solve for p from Eq. (1A) and substitute the result into Eq. (5). The Second method is to solve for p from Eq. (5) and substitute the result into (1A).

Eliminating p from the following two equations

$$x = \frac{\frac{1}{2p^2} + \frac{1}{p} + c_1}{(p+1)^2}$$

$$y = -p^2 x + \frac{1}{p}$$

results in

$$p = \text{RootOf}\left(-1 + 2x\_Z^4 + 4x\_Z^3 + (-2c_1 + 2x)\_Z^2 - 2\_Z\right)$$

Substituting the above into Eq (1A) and simplifying gives

$$y = -\frac{x \,\text{RootOf}\left(-1 + 2x\_Z^4 + 4x\_Z^3 + (-2c_1 + 2x)\_Z^2 - 2\_Z\right)^3 - 1}{\text{RootOf}\left(-1 + 2x\_Z^4 + 4x\_Z^3 + (-2c_1 + 2x)\_Z^2 - 2\_Z\right)}$$

Summary of solutions found

$$y = -\frac{x \,\text{RootOf}\left(-1 + 2x\_Z^4 + 4x\_Z^3 + (-2c_1 + 2x)\_Z^2 - 2\_Z\right)^3 - 1}{\text{RootOf}\left(-1 + 2x\_Z^4 + 4x\_Z^3 + (-2c_1 + 2x)\_Z^2 - 2\_Z\right)}$$

$$y = -x - 1$$

Maple step by step solution

Let's solve

$y(x) \left(\frac{d}{dx}y(x)\right) = 1 - x\left(\frac{d}{dx}y(x)\right)^3$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

$$\left[\frac{d}{dx}y(x) = \frac{\left(\left(12\sqrt{3}\,\sqrt{\frac{4y(x)^3+27x}{x}}+108\right)x^2\right)^{1/3}}{6x} - \frac{2y(x)}{\left(\left(12\sqrt{3}\,\sqrt{\frac{4y(x)^3+27x}{x}}+108\right)x^2\right)^{1/3}}, \frac{d}{dx}y(x) = -\frac{\left(\left(12\sqrt{3}\,\sqrt{\frac{4y(x)^3+27}{x}}\right.\right.}{12x}\right.$$

- Solve the equation $\frac{d}{dx}y(x) = \frac{\left(\left(12\sqrt{3}\,\sqrt{\frac{4y(x)^3+27x}{x}}+108\right)x^2\right)^{1/3}}{6x} - \frac{2y(x)}{\left(\left(12\sqrt{3}\,\sqrt{\frac{4y(x)^3+27x}{x}}+108\right)x^2\right)^{1/3}}$

- Solve the equation $\frac{d}{dx}y(x) = -\frac{\left(\left(12\sqrt{3}\,\sqrt{\frac{4y(x)^3+27x}{x}}+108\right)x^2\right)^{1/3}}{12x} + \frac{y(x)}{\left(\left(12\sqrt{3}\,\sqrt{\frac{4y(x)^3+27x}{x}}+108\right)x^2\right)^{1/3}} - \frac{\mathrm{I}\sqrt{3}\left(\right.}{}$

- Solve the equation $\frac{d}{dx}y(x) = -\frac{\left(\left(12\sqrt{3}\,\sqrt{\frac{4y(x)^3+27x}{x}}+108\right)x^2\right)^{1/3}}{12x} + \frac{y(x)}{\left(\left(12\sqrt{3}\,\sqrt{\frac{4y(x)^3+27x}{x}}+108\right)x^2\right)^{1/3}} + \frac{\mathrm{I}\sqrt{3}\left(\right.}{}$

- Set of solutions
  $\{workingODE, workingODE, workingODE\}$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

Maple dsolve solution

Solving time : `0.048` (sec)
Leaf size : 1792

```
dsolve(y(x)*diff(y(x),x) = 1-x*diff(y(x),x)^3,
       y(x),singsol=all)
```

$$\frac{12\left(-2\left(\left(\sqrt{3}\,\sqrt{\frac{4y^3+27x}{x}}+9\right)x^2\right)^{2/3}y+x\left(\frac{3^{2/3}2^{1/3}\left(\sqrt{3}\,\sqrt{\frac{4y^3+27x}{x}}+9\right)\left(\left(\sqrt{3}\,\sqrt{\frac{4y^3+27x}{x}}+9\right)x^2\right)^{1/3}}{6}+2^{2/3}3^{1/3}y^2\right)\cdots}{\left(2^{2/3}3^{1/3}\left(\left(\sqrt{3}\,\sqrt{\frac{4y^3+27x}{x}}+9\right)^2x^4\right)^{1/3}-2x\left(y3^{2/3}2^{1/3}-3\left(\left(\sqrt{3}\,\sqrt{\frac{4y^3+27x}{x}}+9\right)x^2\right)^{1/3}\right)\right)^2\left(y2^{2/3}3^{1/\cdots}\right.}$$

$$+x$$

$$+\frac{9\left(\left(\sqrt{3}\,\sqrt{\frac{4y^3+27x}{x}}+9\right)^2x^4\right)^{1/3}x^4 3^{1/3}2^{2/3}\left(y\left(\left(\sqrt{3}\,\sqrt{\frac{4y^3+27x}{x}}+9\right)x^2\right)^{1/3}3^{2/3}2^{1/3}-\frac{\sqrt{\frac{4y^3+27x}{x}}\,2^{2/3}3^{5/6}x}{2}-\cdots\right.}{\left(-\frac{2^{2/3}3^{1/3}\left(\left(\sqrt{3}\,\sqrt{\frac{4y^3+27x}{x}}+9\right)^2x^4\right)^{1/3}}{2}+x\left(y3^{2/3}2^{1/3}-3\left(\left(\sqrt{3}\,\sqrt{\frac{4y^3+27x}{x}}+9\right)x^2\right)^{1/3}\right)\right)^2\left(-y2^{2/3}3^{1/3}x\cdots\right.}$$

$$-\frac{\left(-8\left(\left(\sqrt{3}\,\sqrt{\frac{4y^3+27x}{x}}+9\right)x^2\right)^{2/3}y+x\left(\left(i3^{1/6}-\frac{3^{2/3}}{3}\right)2^{1/3}\left(\sqrt{3}\,\sqrt{\frac{4y^3+27x}{x}}+9\right)\left(\left(\sqrt{3}\,\sqrt{\frac{4y^3+27x}{x}}+9\cdots\right.\right.\right.\right.}{6\left(\left(-i+\sqrt{3}\right)\left(\left(\sqrt{3}\,\sqrt{\frac{4y^3+27x}{x}}+9\right)x^2\right)^{2/3}+\left(i3^{1/3}+3^{5/6}\right)yx2^{2/3}\right)^2\left(\frac{\left(i3^{5/6}+3^{1/3}\right)2^{2/3}\left(\left(\sqrt{3}\,\sqrt{\frac{4y^3+27x}{x}}+9\right)^2x\cdots\right.}{6}\cdots\right.}$$

$$+x$$

$$+\frac{24x^4\left(\left(\sqrt{3}\,\sqrt{\frac{4y^3+27x}{x}}+9\right)^2x^4\right)^{1/3}3^{1/3}2^{2/3}\left(-\left(\left(\sqrt{3}\,\sqrt{\frac{4y^3+27x}{x}}+9\right)x^2\right)^{2/3}+y\left(i3^{1/6}-\frac{3^{2/3}}{3}\right)2^{1/3}\left(\left(\cdots\right.\right.\right.}{\left(\left(-i\sqrt{3}-1\right)\left(\left(\sqrt{3}\,\sqrt{\frac{4y^3+27x}{x}}+9\right)x^2\right)^{2/3}+\left(-i3^{5/6}+3^{1/3}\right)xy2^{2/3}\right)^2\left(\frac{\left(i3^{5/6}+3^{1/3}\right)2^{2/3}\left(\left(\sqrt{3}\,\sqrt{\frac{4y^3+27x}{x}}+9\right)^{2}\cdots\right.}{6}\cdots\right.}$$

$$\frac{x^3\left(\left(\sqrt{3}\,\sqrt{\frac{4y^3+27x}{x}}+9\right)x^2\right)^{2/3}\left(8\left(\left(\sqrt{3}\,\sqrt{\frac{4y^3+27x}{x}}+9\right)x^2\right)^{2/3}y+x\left(\left(i3^{1/6}+\frac{3^{2/3}}{3}\right)2^{1/3}\left(\sqrt{3}\,\sqrt{\frac{4y^3+27x}{x}}\cdots\right.\right.\right.}{6\left(\left(i\sqrt{3}-1\right)\left(\left(\sqrt{3}\,\sqrt{\frac{4y^3+27x}{x}}+9\right)x^2\right)^{2/3}+yx2^{2/3}\left(i3^{5/6}+3^{1/3}\right)\right)^2\left(\frac{\left(-3^{1/3}+i3^{5/6}\right)2^{2/3}\left(\left(\sqrt{3}\,\sqrt{\frac{4y^3+27x}{x}}+9\right)^2x^4\cdots\right.}{6}\cdots\right.}$$

$$+x$$

$$+\frac{24x^4\left(\left(\sqrt{3}\,\sqrt{\frac{4y^3+27x}{x}}+9\right)^2x^4\right)^{1/3}3^{1/3}2^{2/3}\left(\left(\left(\sqrt{3}\,\sqrt{\frac{4y^3+27x}{x}}+9\right)x^2\right)^{2/3}+y\left(i3^{1/6}+\frac{3^{2/3}}{3}\right)2^{1/3}\left(\left(\sqrt{3}\cdots\right.\right.\right.}{\left(\cdots\right)^{2/3}\cdots\right)^2\left(\frac{\left(-3^{1/3}+i3^{5/6}\right)2^{2/3}\left(\left(\sqrt{3}\,\sqrt{\frac{4y^3+27x}{x}}+9\right)^2x^4\cdots\right.}{\cdots}\right.}$$

Mathematica DSolve solution

Solving time : 85.048 (sec)
Leaf size : 20717

```
DSolve[{D[y[x],x]*y[x]==1-x*(D[y[x],x])^3,{}},
       y[x],x,IncludeSingularSolutions->True]
```

Too large to display

2.1.46 problem 46

Internal problem ID [8434]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 46
Date solved : `Tuesday, December 17, 2024 at 12:51:37 PM`
CAS classification : `[_quadrature]`

Solve

$$f' = \frac{1}{f}$$

Solved as first order autonomous ode

Time used: 0.142 (sec)

Integrating gives

$$\int f df = dx$$
$$\frac{f^2}{2} = x + c_1$$

Solving for f gives

$$f = \sqrt{2c_1 + 2x}$$
$$f = -\sqrt{2c_1 + 2x}$$

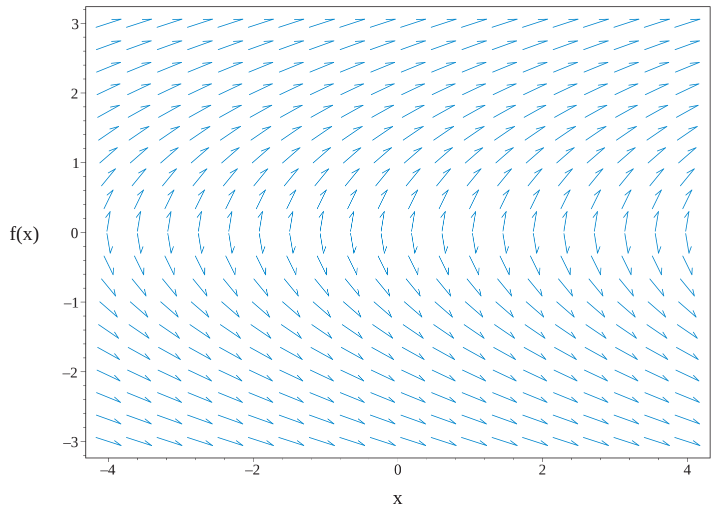


Figure 2.115: Slope field plot
$f' = \frac{1}{f}$

Summary of solutions found

$$f = \sqrt{2c_1 + 2x}$$
$$f = -\sqrt{2c_1 + 2x}$$

Solved as first order Bernoulli ode

Time used: 0.081 (sec)

In canonical form, the ODE is

$$\begin{aligned} f' &= F(x, f) \\ &= \frac{1}{f} \end{aligned}$$

This is a Bernoulli ODE.

$$f' = (1)\frac{1}{f} \tag{1}$$

The standard Bernoulli ODE has the form

$$f' = f_0(x)f + f_1(x)f^n \tag{2}$$

Comparing this to (1) shows that

$$f_0 = 0$$
$$f_1 = 1$$

The first step is to divide the above equation by f^n which gives

$$\frac{f'}{f^n} = f_1(x) \tag{3}$$

The next step is use the substitution $v = f^{1-n}$ in equation (3) which generates a new ODE in $v(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $f(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= 0 \\ f_1(x) &= 1 \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $f^n = \frac{1}{f}$ gives

$$f'f = 0 + 1 \tag{4}$$

Let

$$\begin{aligned} v &= f^{1-n} \\ &= f^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$v' = 2ff' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{v'(x)}{2} &= 1 \\ v' &= 2 \end{aligned} \tag{7}$$

The above now is a linear ODE in $v(x)$ which is now solved.

Since the ode has the form $v'(x) = f(x)$, then we only need to integrate $f(x)$.

$$\begin{aligned} \int dv &= \int 2\,dx \\ v(x) &= 2x + c_1 \end{aligned}$$

The substitution $v = f^{1-n}$ is now used to convert the above solution back to f which results in

$$f^2 = 2x + c_1$$

Solving for f gives

$$f = \sqrt{2x + c_1}$$
$$f = -\sqrt{2x + c_1}$$

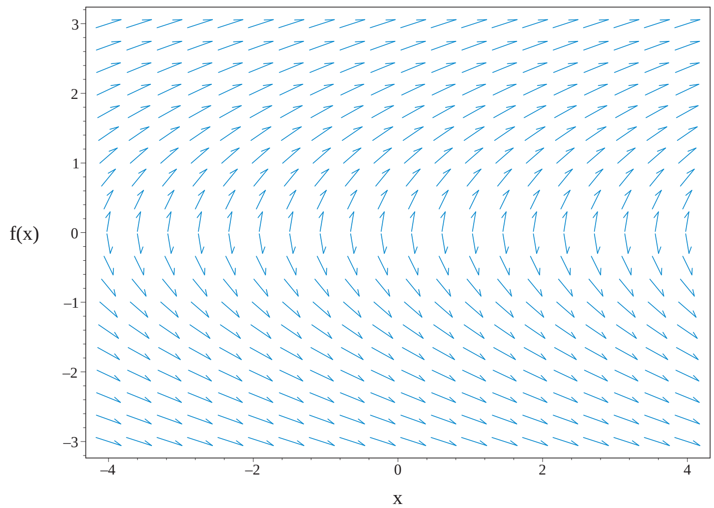


Figure 2.116: Slope field plot
$f' = \frac{1}{f}$

Summary of solutions found

$$f = \sqrt{2x + c_1}$$
$$f = -\sqrt{2x + c_1}$$

Solved as first order Exact ode

Time used: 0.066 (sec)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, f)\,\mathrm{d}x + N(x, f)\,\mathrm{d}f = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(f)\,\mathrm{d}f &= \mathrm{d}x \\
-\mathrm{d}x + (f)\,\mathrm{d}f &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, f) &= -1 \\
N(x, f) &= f
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial f} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial f} &= \frac{\partial}{\partial f}(-1) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(f) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial f} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, f)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial f} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -1 \, \mathrm{d}x$$

$$\phi = -x + f(f) \tag{3}$$

Where $f(f)$ is used for the constant of integration since ϕ is a function of both x and f. Taking derivative of equation (3) w.r.t f gives

$$\frac{\partial \phi}{\partial f} = 0 + f'(f) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial f} = f$. Therefore equation (4) becomes

$$f = 0 + f'(f) \tag{5}$$

Solving equation (5) for $f'(f)$ gives

$$f'(f) = f$$

Integrating the above w.r.t f gives

$$\int f'(f) \, \mathrm{d}f = \int (f) \, \mathrm{d}f$$

$$f(f) = \frac{f^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(f)$ into equation (3) gives ϕ

$$\phi = -x + \frac{f^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -x + \frac{f^2}{2}$$

Solving for f gives

$$f = \sqrt{2c_1 + 2x}$$
$$f = -\sqrt{2c_1 + 2x}$$

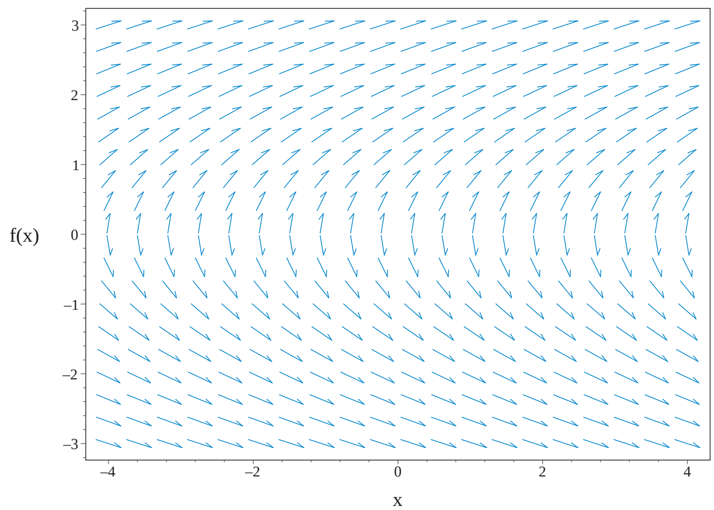


Figure 2.117: Slope field plot
$f' = \frac{1}{f}$

Summary of solutions found

$$f = \sqrt{2c_1 + 2x}$$
$$f = -\sqrt{2c_1 + 2x}$$

Solved using Lie symmetry for first order ode

Time used: 0.376 (sec)

Writing the ode as

$$
\begin{aligned}
f' &= \frac{1}{f} \\
f' &= \omega(x, f)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_f - \xi_x) - \omega^2 \xi_f - \omega_x \xi - \omega_f \eta = 0 \tag{A}
$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = f a_3 + x a_2 + a_1 \tag{1E}
$$

$$
\eta = f b_3 + x b_2 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
b_2 + \frac{b_3 - a_2}{f} - \frac{a_3}{f^2} + \frac{f b_3 + x b_2 + b_1}{f^2} = 0 \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{b_2 f^2 - f a_2 + 2 f b_3 + x b_2 - a_3 + b_1}{f^2} = 0
$$

Setting the numerator to zero gives

$$
b_2 f^2 - f a_2 + 2 f b_3 + x b_2 - a_3 + b_1 = 0 \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{f, x\}$ in them.

$$
\{f, x\}
$$

The following substitution is now made to be able to collect on all terms with $\{f, x\}$ in them

$$\{f = v_1, x = v_2\}$$

The above PDE (6E) now becomes

$$b_2 v_1^2 - a_2 v_1 + b_2 v_2 + 2b_3 v_1 - a_3 + b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$b_2 v_1^2 + (-a_2 + 2b_3) v_1 + b_2 v_2 - a_3 + b_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_2 &= 0 \\ -a_2 + 2b_3 &= 0 \\ -a_3 + b_1 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= a_1 \\ a_2 &= 2b_3 \\ a_3 &= b_1 \\ b_1 &= b_1 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 1 \\ \eta &= 0 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,f)\,\xi \\
&= 0 - \left(\frac{1}{f}\right)(1) \\
&= -\frac{1}{f} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,f) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{df}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial f}\right) S(x,f) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\frac{1}{f}} dy
\end{aligned}
$$

Which results in

$$
S = -\frac{f^2}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,f)S_f}{R_x + \omega(x,f)R_f} \tag{2}
$$

Where in the above R_x, R_f, S_x, S_f are all partial derivatives and $\omega(x,f)$ is the right hand side of the original ode given by

$$
\omega(x,f) = \frac{1}{f}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_f &= 0 \\
S_x &= 0 \\
S_f &= -f
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -1 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, f in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -1
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

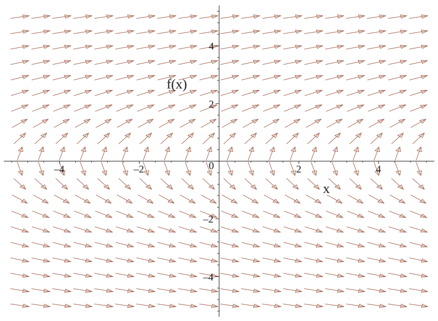
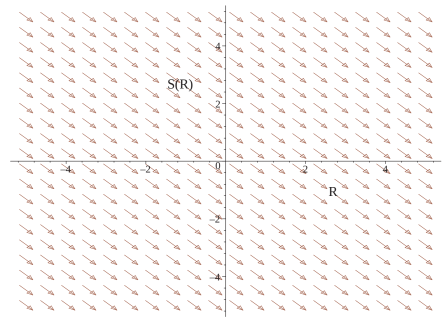
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$
\begin{aligned}
\int dS &= \int -1 \, dR \\
S(R) &= -R + c_2
\end{aligned}
$$

To complete the solution, we just need to transform the above back to x, f coordinates. This results in

$$
-\frac{f^2}{2} = -x + c_2
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, f coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{df}{dx} = \frac{1}{f}$ 	$R = x$ $S = -\frac{f^2}{2}$	$\frac{dS}{dR} = -1$ 

Solving for f gives

$$f = \sqrt{-2c_2 + 2x}$$

$$f = -\sqrt{-2c_2 + 2x}$$

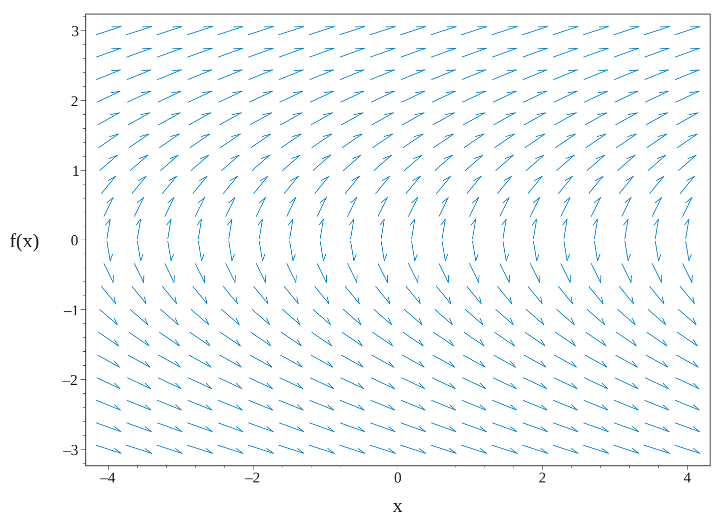


Figure 2.118: Slope field plot
$f' = \frac{1}{f}$

Summary of solutions found

$$f = \sqrt{-2c_2 + 2x}$$

$$f = -\sqrt{-2c_2 + 2x}$$

Solved as first order ode of type dAlembert

Time used: 0.272 (sec)

Let $p = f'$ the ode becomes

$$p = \frac{1}{f}$$

Solving for f from the above results in

$$f = \frac{1}{p} \tag{1}$$

This has the form

$$f = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = f'(x)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $f = xf + g$ to (1A) shows that

$$\begin{aligned} f &= 0 \\ g &= \frac{1}{p} \end{aligned}$$

Hence (2) becomes

$$p = -\frac{p'(x)}{p^2} \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p = 0$$

No valid singular solutions found.

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$p'(x) = -p(x)^3 \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed. Integrating gives

$$\begin{aligned}\int -\frac{1}{p^3} dp &= dx \\ \frac{1}{2p^2} &= x + c_1\end{aligned}$$

Singular solutions are found by solving

$$-p^3 = 0$$

for $p(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$p(x) = 0$$

Solving for $p(x)$ gives

$$\begin{aligned}p(x) &= 0 \\ p(x) &= \frac{1}{\sqrt{2c_1 + 2x}} \\ p(x) &= -\frac{1}{\sqrt{2c_1 + 2x}}\end{aligned}$$

Substituing the above solution for p in (2A) gives

$$\begin{aligned}f &= \sqrt{2c_1 + 2x} \\ f &= -\sqrt{2c_1 + 2x}\end{aligned}$$

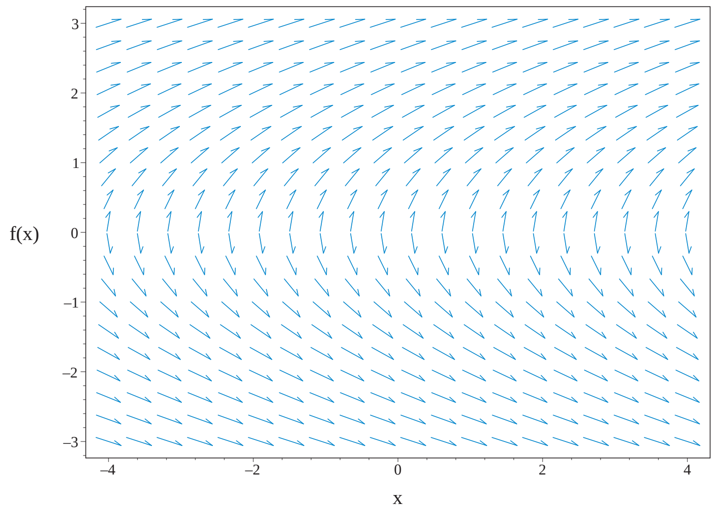


Figure 2.119: Slope field plot
$f' = \frac{1}{f}$

Summary of solutions found

$$f = \sqrt{2c_1 + 2x}$$

$$f = -\sqrt{2c_1 + 2x}$$

Maple step by step solution

Let's solve
$\frac{d}{dx}f(x) = \frac{1}{f(x)}$

- Highest derivative means the order of the ODE is 1
$\frac{d}{dx}f(x)$
- Solve for the highest derivative
$\frac{d}{dx}f(x) = \frac{1}{f(x)}$
- Separate variables
$f(x)\left(\frac{d}{dx}f(x)\right) = 1$
- Integrate both sides with respect to x
$\int f(x)\left(\frac{d}{dx}f(x)\right)dx = \int 1dx + C1$
- Evaluate integral
$\frac{f(x)^2}{2} = x + C1$
- Solve for $f(x)$
$\left\{f(x) = \sqrt{2x + 2C1}, f(x) = -\sqrt{2x + 2C1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 23

```
dsolve(diff(f(x),x) = 1/f(x),
       f(x),singsol=all)
```

$$f = \sqrt{c_1 + 2x}$$
$$f = -\sqrt{c_1 + 2x}$$

Mathematica DSolve solution

Solving time : 0.063 (sec)
Leaf size : 38

```
DSolve[{D[ f[x],x]==f[x]^(-1),{}},
       f[x],x,IncludeSingularSolutions->True]
```

$$f(x) \to -\sqrt{2}\sqrt{x + c_1}$$
$$f(x) \to \sqrt{2}\sqrt{x + c_1}$$

2.1.47 problem 47

Internal problem ID [8435]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 47
Date solved : `Thursday, December 12, 2024 at 09:07:59 AM`
CAS classification : `[[_2nd_order, _missing_y]]`

Solve

$$ty'' + 4y' = t^2$$

Solved as second order linear exact ode

Time used: 0.158 (sec)

An ode of the form

$$p(t)\,y'' + q(t)\,y' + r(t)\,y = s(t)$$

is exact if

$$p''(t) - q'(t) + r(t) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}p(x) &= t\\ q(x) &= 4\\ r(x) &= 0\\ s(x) &= t^2\end{aligned}$$

Hence

$$
\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 0
\end{aligned}
$$

Therefore (1) becomes

$$
0-(0)+(0)=0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(t)\,y' + (q(t)-p'(t))\,y)' = s(x)
$$

Integrating gives

$$
p(t)\,y' + (q(t)-p'(t))\,y = \int s(t)\;dt
$$

Substituting the above values for p, q, r, s gives

$$
ty' + 3y = \int t^2\,dt
$$

We now have a first order ode to solve which is

$$
ty' + 3y = \frac{t^3}{3} + c_1
$$

In canonical form a linear first order is

$$
y' + q(t)y = p(t)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(t) &= \frac{3}{t} \\
p(t) &= \frac{t^3+3c_1}{3t}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\,dt} \\
&= \mathrm{e}^{\int \frac{3}{t}dt} \\
&= t^3
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(\frac{t^3+3c_1}{3t}\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}(y\,t^3) &= (t^3)\left(\frac{t^3+3c_1}{3t}\right) \\
\mathrm{d}(y\,t^3) &= \left(\frac{(t^3+3c_1)\,t^2}{3}\right)\mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
y\,t^3 &= \int \frac{(t^3+3c_1)\,t^2}{3}\,dt \\
&= \frac{(t^3+3c_1)^2}{18} + c_2
\end{aligned}
$$

Dividing throughout by the integrating factor t^3 gives the final solution

$$
y = \frac{\frac{(t^3+3c_1)^2}{18} + c_2}{t^3}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = \frac{\frac{(t^3+3c_1)^2}{18} + c_2}{t^3}
$$

Solved as second order missing y ode

Time used: 0.079 (sec)

This is second order ode with missing dependent variable y. Let

$$
p(t) = y'
$$

Then

$$
p'(t) = y''
$$

Hence the ode becomes

$$
tp'(t) + 4p(t) - t^2 = 0
$$

Which is now solve for $p(t)$ as first order ode. In canonical form a linear first order is

$$p'(t) + q(t)p(t) = p(t)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(t) &= \frac{4}{t} \\ p(t) &= t \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q \, dt} \\ &= \mathrm{e}^{\int \frac{4}{t} dt} \\ &= t^4 \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mu p) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}t}(\mu p) &= (\mu)(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(p\, t^4\right) &= \left(t^4\right)(t) \\ \mathrm{d}\left(p\, t^4\right) &= t^5 \, \mathrm{d}t \end{aligned}$$

Integrating gives

$$\begin{aligned} p\, t^4 &= \int t^5 \, dt \\ &= \frac{t^6}{6} + c_1 \end{aligned}$$

Dividing throughout by the integrating factor t^4 gives the final solution

$$p(t) = \frac{t^6 + 6c_1}{6t^4}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{t^6 + 6c_1}{6t^4}$$

Since the ode has the form $y' = f(t)$, then we only need to integrate $f(t)$.

$$\int dy = \int \frac{t^6 + 6c_1}{6t^4}\, dt$$
$$y = \frac{t^3}{18} - \frac{c_1}{3t^3} + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{t^3}{18} - \frac{c_1}{3t^3} + c_2$$

Solved as second order integrable as is ode

Time used: 0.070 (sec)

Integrating both sides of the ODE w.r.t t gives

$$\int (ty'' + 4y')\, dt = \int t^2 dt$$
$$ty' + 3y = \frac{t^3}{3} + c_1$$

Which is now solved for y. In canonical form a linear first order is

$$y' + q(t)y = p(t)$$

Comparing the above to the given ode shows that

$$q(t) = \frac{3}{t}$$
$$p(t) = \frac{t^3 + 3c_1}{3t}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\, dt} \\ &= e^{\int \frac{3}{t} dt} \\ &= t^3\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dt}(\mu y) &= \mu p \\
\frac{d}{dt}(\mu y) &= (\mu)\left(\frac{t^3+3c_1}{3t}\right) \\
\frac{d}{dt}(y\,t^3) &= (t^3)\left(\frac{t^3+3c_1}{3t}\right) \\
d(y\,t^3) &= \left(\frac{(t^3+3c_1)\,t^2}{3}\right) dt
\end{aligned}$$

Integrating gives

$$\begin{aligned}
y\,t^3 &= \int \frac{(t^3+3c_1)\,t^2}{3}\,dt \\
&= \frac{(t^3+3c_1)^2}{18} + c_2
\end{aligned}$$

Dividing throughout by the integrating factor t^3 gives the final solution

$$y = \frac{\frac{(t^3+3c_1)^2}{18} + c_2}{t^3}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{\frac{(t^3+3c_1)^2}{18} + c_2}{t^3}$$

Solved as second order integrable as is ode (ABC method)

Time used: 0.066 (sec)

Writing the ode as

$$ty'' + 4y' = t^2$$

Integrating both sides of the ODE w.r.t t gives

$$\begin{aligned}
\int (ty'' + 4y')\,dt &= \int t^2 dt \\
ty' + 3y &= \frac{t^3}{3} + c_1
\end{aligned}$$

Which is now solved for y. In canonical form a linear first order is

$$y' + q(t)y = p(t)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(t) &= \frac{3}{t} \\ p(t) &= \frac{t^3 + 3c_1}{3t} \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dt} \\ &= \mathrm{e}^{\int \frac{3}{t}dt} \\ &= t^3 \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(\frac{t^3 + 3c_1}{3t}\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(y\,t^3\right) &= \left(t^3\right)\left(\frac{t^3 + 3c_1}{3t}\right) \\ \mathrm{d}\left(y\,t^3\right) &= \left(\frac{(t^3 + 3c_1)\,t^2}{3}\right)\mathrm{d}t \end{aligned}$$

Integrating gives

$$\begin{aligned} y\,t^3 &= \int \frac{(t^3 + 3c_1)\,t^2}{3}\,dt \\ &= \frac{\left(t^3 + 3c_1\right)^2}{18} + c_2 \end{aligned}$$

Dividing throughout by the integrating factor t^3 gives the final solution

$$y = \frac{\frac{\left(t^3+3c_1\right)^2}{18} + c_2}{t^3}$$

Will add steps showing solving for IC soon.

Solved as second order ode using non constant coeff transformation on B method

Time used: 0.165 (sec)

Given an ode of the form

$$Ay'' + By' + Cy = F(t)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= t \\ B &= 4 \\ C &= 0 \\ F &= t^2 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned}AB'' + BB' + CB &= (t)(0) + (4)(0) + (0)(4) \\ &= 0\end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$4tv'' + (16)v' = 0$$

Now by applying $v' = u$ the above becomes

$$4tu'(t) + 16u(t) = 0$$

Which is now solved for u. In canonical form a linear first order is

$$u'(t) + q(t)u(t) = p(t)$$

Comparing the above to the given ode shows that

$$\begin{aligned}q(t) &= \frac{4}{t} \\ p(t) &= 0\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dt} \\ &= \mathrm{e}^{\int \frac{4}{t}dt} \\ &= t^4\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mu u &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(u\,t^4\right) &= 0\end{aligned}$$

Integrating gives

$$\begin{aligned}u\,t^4 &= \int 0\,dt + c_1 \\ &= c_1\end{aligned}$$

Dividing throughout by the integrating factor t^4 gives the final solution

$$u(t) = \frac{c_1}{t^4}$$

The ode for v now becomes

$$v'(t) = \frac{c_1}{t^4}$$

Which is now solved for v. Since the ode has the form $v'(t) = f(t)$, then we only need to integrate $f(t)$.

$$\begin{aligned}
\int dv &= \int \frac{c_1}{t^4}\,dt \\
v(t) &= -\frac{c_1}{3t^3} + c_2
\end{aligned}$$

Replacing $v(t)$ above by $\frac{y}{4}$, then the homogeneous solution is

$$\begin{aligned}
y_h(t) &= Bv \\
&= -\frac{4(-3c_2\,t^3 + c_1)}{3t^3}
\end{aligned}$$

And now the particular solution $y_p(t)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= 4 \\
y_2 &= \frac{1}{t^3}
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 4 & \frac{1}{t^3} \\ \frac{d}{dt}(4) & \frac{d}{dt}\left(\frac{1}{t^3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 4 & \frac{1}{t^3} \\ 0 & -\frac{3}{t^4} \end{vmatrix}$$

Therefore

$$W = (4)\left(-\frac{3}{t^4}\right) - \left(\frac{1}{t^3}\right)(0)$$

Which simplifies to

$$W = -\frac{12}{t^4}$$

Which simplifies to

$$W = -\frac{12}{t^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{1}{t}}{-\frac{12}{t^3}}\, dt$$

Which simplifies to

$$u_1 = -\int -\frac{t^2}{12} dt$$

Hence

$$u_1 = \frac{t^3}{36}$$

And Eq. (3) becomes

$$u_2 = \int \frac{4t^2}{-\frac{12}{t^3}}\, dt$$

Which simplifies to

$$u_2 = \int -\frac{t^5}{3} dt$$

Hence

$$u_2 = -\frac{t^6}{18}$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = \frac{t^3}{18}$$

Hence the complete solution is

$$\begin{aligned} y(t) &= y_h + y_p \\ &= \left(-\frac{4(-3c_2 t^3 + c_1)}{3t^3}\right) + \left(\frac{t^3}{18}\right) \\ &= \frac{t^6 + 72c_2 t^3 - 24c_1}{18t^3} \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{t^6 + 72c_2 t^3 - 24c_1}{18t^3}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.169 (sec)

Writing the ode as

$$ty'' + 4y' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= 4 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{2}{t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A} dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.51: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{t^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{1}{t} + (-)\,(0) \\
&= -\frac{1}{t} \\
&= -\frac{1}{t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{t}\right)(0) + \left(\left(\frac{1}{t^2}\right) + \left(-\frac{1}{t}\right)^2 - \left(\frac{2}{t^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int -\frac{1}{t} dt} \\
&= \frac{1}{t}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{t}\, dt} \\
&= z_1 e^{-2\ln(t)} \\
&= z_1 \left(\frac{1}{t^2}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{t^3}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dt}}{y_1^2} dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{t} dt}}{(y_1)^2} dt \\
&= y_1 \int \frac{e^{-4\ln(t)}}{(y_1)^2} dt \\
&= y_1 \left(\frac{t^3}{3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{t^3}\right) + c_2 \left(\frac{1}{t^3} \left(\frac{t^3}{3}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$ty'' + 4y' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1}{t^3} + \frac{c_2}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{t^3}$$
$$y_2 = \frac{1}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{a W(t)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(t)}{a W(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{t^3} & \frac{1}{3} \\ \frac{d}{dt}\left(\frac{1}{t^3}\right) & \frac{d}{dt}\left(\frac{1}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{t^3} & \frac{1}{3} \\ -\frac{3}{t^4} & 0 \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{t^3}\right)(0) - \left(\frac{1}{3}\right)\left(-\frac{3}{t^4}\right)$$

Which simplifies to

$$W = \frac{1}{t^4}$$

Which simplifies to

$$W = \frac{1}{t^4}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{t^2}{3}}{\frac{1}{t^3}}\, dt$$

Which simplifies to

$$u_1 = -\int \frac{t^5}{3} dt$$

Hence

$$u_1 = -\frac{t^6}{18}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1}{t}}{\frac{1}{t^3}}\, dt$$

Which simplifies to

$$u_2 = \int t^2 dt$$

Hence

$$u_2 = \frac{t^3}{3}$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = \frac{t^3}{18}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{t^3} + \frac{c_2}{3}\right) + \left(\frac{t^3}{18}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{c_1}{t^3} + \frac{c_2}{3} + \frac{t^3}{18}$$

Solved as second order ode adjoint method

Time used: 0.411 (sec)

In normal form the ode

$$ty'' + 4y' = t^2 \tag{1}$$

Becomes

$$y'' + p(t)\,y' + q(t)\,y = r(t) \tag{2}$$

Where

$$\begin{aligned} p(t) &= \frac{4}{t} \\ q(t) &= 0 \\ r(t) &= t \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\,p)' + \xi q &= 0 \\ \xi'' - \left(\frac{4\xi(t)}{t}\right)' + (0) &= 0 \\ \xi''(t) + \frac{4\xi(t)}{t^2} - \frac{4\xi'(t)}{t} &= 0 \end{aligned}$$

Which is solved for $\xi(t)$. This is Euler second order ODE. Let the solution be $\xi = t^r$, then $\xi' = rt^{r-1}$ and $\xi'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} - 4trt^{r-1} + 4t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r - 4r\,t^r + 4t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) - 4r + 4 = 0$$

Or

$$r^2 - 5r + 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= 1 \\ r_2 &= 4 \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$\xi = c_1\xi_1 + c_2\xi_2$$

Where $\xi_1 = t^{r_1}$ and $\xi_2 = t^{r_2}$. Hence

$$\xi = c_2\, t^4 + c_1 t$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned} \xi(t)\, y' - y\xi'(t) + \xi(t)\, p(t)\, y &= \int \xi(t)\, r(t)\, dt \\ y' + y\left(p(t) - \frac{\xi'(t)}{\xi\,(t)}\right) &= \frac{\int \xi(t)\, r(t)\, dt}{\xi\,(t)} \end{aligned}$$

Or

$$y' + y\left(\frac{4}{t} - \frac{4c_2\, t^3 + c_1}{c_2\, t^4 + c_1 t}\right) = \frac{\frac{1}{6}c_2\, t^6 + \frac{1}{3}c_1\, t^3}{c_2\, t^4 + c_1 t}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(t)y = p(t)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(t) &= \frac{3c_1}{t\,(c_2\, t^3 + c_1)} \\ p(t) &= \frac{t^2(c_2\, t^3 + 2c_1)}{6c_2\, t^3 + 6c_1} \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\, dt} \\ &= \mathrm{e}^{\int \frac{3c_1}{t\left(c_2\, t^3 + c_1\right)}dt} \\ &= \frac{t^3}{c_2\, t^3 + c_1} \end{aligned}$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(\frac{t^2(c_2\,t^3+2c_1)}{6c_2\,t^3+6c_1}\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y\,t^3}{c_2\,t^3+c_1}\right) &= \left(\frac{t^3}{c_2\,t^3+c_1}\right)\left(\frac{t^2(c_2\,t^3+2c_1)}{6c_2\,t^3+6c_1}\right) \\
\mathrm{d}\left(\frac{y\,t^3}{c_2\,t^3+c_1}\right) &= \left(\frac{t^5(c_2\,t^3+2c_1)}{(6c_2\,t^3+6c_1)\,(c_2\,t^3+c_1)}\right)\mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y\,t^3}{c_2\,t^3+c_1} &= \int \frac{t^5(c_2\,t^3+2c_1)}{(6c_2\,t^3+6c_1)\,(c_2\,t^3+c_1)}\,dt \\
&= \frac{t^3}{18c_2} + \frac{c_1^2}{18c_2^2\,(c_2\,t^3+c_1)} + c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{t^3}{c_2\,t^3+c_1}$ gives the final solution

$$
y = \frac{(c_2\,t^3+c_1)\left(\frac{t^3}{18c_2}+\frac{c_1^2}{18c_2^2(c_2\,t^3+c_1)}+c_3\right)}{t^3}
$$

Hence, the solution found using Lagrange adjoint equation method is

$$
y = \frac{(c_2\,t^3+c_1)\left(\frac{t^3}{18c_2}+\frac{c_1^2}{18c_2^2(c_2\,t^3+c_1)}+c_3\right)}{t^3}
$$

The constants can be merged to give

$$
y = \frac{(c_2\,t^3+c_1)\left(\frac{t^3}{18c_2}+\frac{c_1^2}{18c_2^2(c_2\,t^3+c_1)}+1\right)}{t^3}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = \frac{(c_2\,t^3+c_1)\left(\frac{t^3}{18c_2}+\frac{c_1^2}{18c_2^2(c_2\,t^3+c_1)}+1\right)}{t^3}
$$

Maple step by step solution

Let's solve

$t\left(\frac{d^2}{dt^2}y(t)\right) + 4\frac{d}{dt}y(t) = t^2$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$
- Make substitution $u = \frac{d}{dt}y(t)$ to reduce order of ODE

  $t\left(\frac{d}{dt}u(t)\right) + 4u(t) = t^2$
- Solve for the highest derivative

  $\frac{d}{dt}u(t) = \frac{-4u(t)+t^2}{t}$
- Collect w.r.t. $u(t)$ and simplify

  $\frac{d}{dt}u(t) = -\frac{4u(t)}{t} + t$
- Group terms with $u(t)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $\frac{d}{dt}u(t) + \frac{4u(t)}{t} = t$
- The ODE is linear; multiply by an integrating factor $\mu(t)$

  $\mu(t)\left(\frac{d}{dt}u(t) + \frac{4u(t)}{t}\right) = \mu(t)\, t$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(u(t)\,\mu(t))$

  $\mu(t)\left(\frac{d}{dt}u(t) + \frac{4u(t)}{t}\right) = \left(\frac{d}{dt}u(t)\right)\mu(t) + u(t)\left(\frac{d}{dt}\mu(t)\right)$
- Isolate $\frac{d}{dt}\mu(t)$

  $\frac{d}{dt}\mu(t) = \frac{4\mu(t)}{t}$
- Solve to find the integrating factor

  $\mu(t) = t^4$
- Integrate both sides with respect to t

  $\int \left(\frac{d}{dt}(u(t)\,\mu(t))\right) dt = \int \mu(t)\, t dt + C1$
- Evaluate the integral on the lhs

  $u(t)\,\mu(t) = \int \mu(t)\, t dt + C1$
- Solve for $u(t)$

  $u(t) = \frac{\int \mu(t) t dt + C1}{\mu(t)}$
- Substitute $\mu(t) = t^4$

  $u(t) = \frac{\int t^5 dt + C1}{t^4}$
- Evaluate the integrals on the rhs

$u(t) = \frac{\frac{t^6}{6}+C1}{t^4}$

- Simplify

  $u(t) = \frac{t^6+6C1}{6t^4}$

- Solve 1st ODE for $u(t)$

  $u(t) = \frac{t^6+6C1}{6t^4}$

- Make substitution $u = \frac{d}{dt}y(t)$

  $\frac{d}{dt}y(t) = \frac{t^6+6C1}{6t^4}$

- Integrate both sides to solve for $y(t)$

  $\int \left(\frac{d}{dt}y(t)\right) dt = \int \frac{t^6+6C1}{6t^4} dt + C2$

- Compute integrals

  $y(t) = \frac{t^3}{18} - \frac{C1}{3t^3} + C2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(-_a^2+4*_b(_a))/_a, _b(_a)`   *
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 17

```
dsolve(t*diff(diff(y(t),t),t)+4*diff(y(t),t) = t^2,
       y(t),singsol=all)
```

$$y = \frac{t^3}{18} - \frac{c_1}{3t^3} + c_2$$

Mathematica DSolve solution

Solving time : 0.032 (sec)
Leaf size : 24

```
DSolve[{t*D[y[t],{t,2}]+4*D[y[t],t]==t^2,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{t^3}{18} - \frac{c_1}{3t^3} + c_2$$

2.1.48 problem 48

Internal problem ID [8436]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 48
Date solved : `Thursday, December 12, 2024 at 09:08:00 AM`
CAS classification : `[[_2nd_order, _missing_y]]`

Solve

$$\left(t^2+9\right) y''+2ty'=0$$

With initial conditions

$$y(3)=2\pi$$
$$y'(3)=\frac{2}{3}$$

Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y''+p(t)y'+q(t)y=F$$

Where here

$$p(t)=\frac{2t}{t^2+9}$$
$$q(t)=0$$
$$F=0$$

Hence the ode is

$$y'' + \frac{2ty'}{t^2+9} = 0$$

The domain of $p(t) = \frac{2t}{t^2+9}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 3$ is inside this domain. Hence solution exists and is unique.

Solved as second order linear exact ode

Time used: 0.184 (sec)

An ode of the form

$$p(t)\, y'' + q(t)\, y' + r(t)\, y = s(t)$$

is exact if

$$p''(t) - q'(t) + r(t) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= t^2 + 9 \\ q(x) &= 2t \\ r(x) &= 0 \\ s(x) &= 0 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 2 \\ q'(x) &= 2 \end{aligned}$$

Therefore (1) becomes

$$2 - (2) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(t)\, y' + (q(t) - p'(t))\, y\right)' = s(x)$$

Integrating gives

$$p(t)\, y' + (q(t) - p'(t))\, y = \int s(t)\; dt$$

Substituting the above values for p, q, r, s gives

$$(t^2+9)\,y' = c_1$$

We now have a first order ode to solve which is

$$(t^2+9)\,y' = c_1$$

Since the ode has the form $y' = f(t)$, then we only need to integrate $f(t)$.

$$\int dy = \int \frac{c_1}{t^2+9}\,dt$$
$$y = \frac{c_1 \arctan\left(\frac{t}{3}\right)}{3} + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = 4\arctan\left(\frac{t}{3}\right) + \pi$$

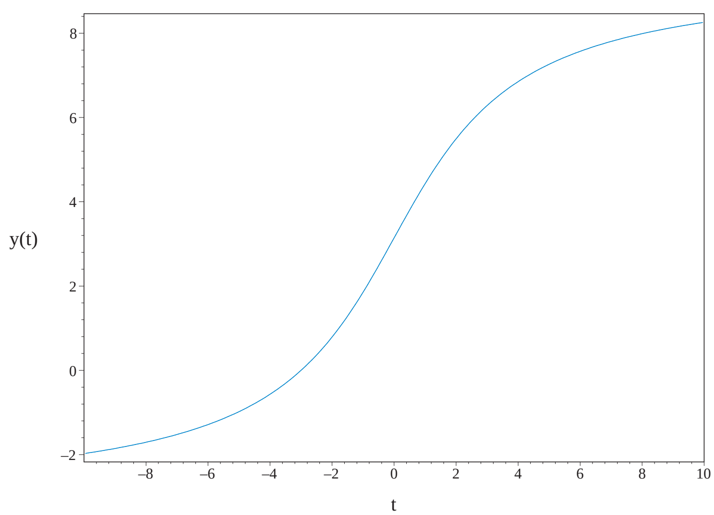


Figure 2.120: Solution plot
$y = 4\arctan\left(\frac{t}{3}\right) + \pi$

Solved as second order missing y ode

Time used: 0.075 (sec)

This is second order ode with missing dependent variable y. Let

$$p(t) = y'$$

Then

$$p'(t) = y''$$

Hence the ode becomes

$$(t^2 + 9)\, p'(t) + 2tp(t) = 0$$

Which is now solve for $p(t)$ as first order ode. In canonical form a linear first order is

$$p'(t) + q(t)p(t) = p(t)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(t) &= \frac{2t}{t^2 + 9} \\ p(t) &= 0 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dt} \\ &= \mathrm{e}^{\int \frac{2t}{t^2+9}dt} \\ &= t^2 + 9 \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mu p &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(p(t^2 + 9)\right) &= 0 \end{aligned}$$

Integrating gives

$$\begin{aligned} p(t^2 + 9) &= \int 0\,dt + c_1 \\ &= c_1 \end{aligned}$$

Dividing throughout by the integrating factor t^2+9 gives the final solution

$$p(t) = \frac{c_1}{t^2+9}$$

Solving for the constant of integration from initial conditions, the solution becomes

$$p(t) = \frac{12}{t^2+9}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{12}{t^2+9}$$

Since the ode has the form $y' = f(t)$, then we only need to integrate $f(t)$.

$$\int dy = \int \frac{12}{t^2+9}\, dt$$

$$y = 4\arctan\left(\frac{t}{3}\right) + c_2$$

Solving for the constant of integration from initial conditions, the solution becomes

$$y = 4\arctan\left(\frac{t}{3}\right) + \pi$$

Summary of solutions found

$$y = 4\arctan\left(\frac{t}{3}\right) + \pi$$

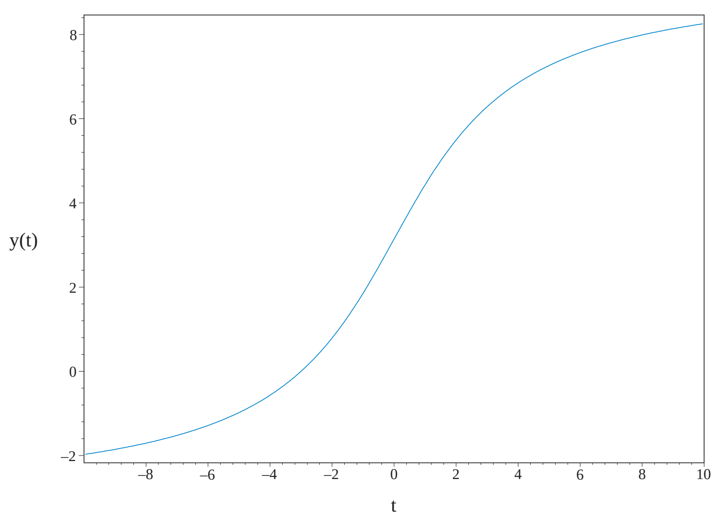


Figure 2.121: Solution plot
$y = 4\arctan\left(\frac{t}{3}\right) + \pi$

Solved as second order integrable as is ode

Time used: 0.036 (sec)

Integrating both sides of the ODE w.r.t t gives

$$\int \left(\left(t^2+9\right) y''+2ty'\right) dt = 0$$
$$\left(t^2+9\right) y' = c_1$$

Which is now solved for y. Since the ode has the form $y' = f(t)$, then we only need to integrate $f(t)$.

$$\int dy = \int \frac{c_1}{t^2+9}\, dt$$
$$y = \frac{c_1 \arctan\left(\frac{t}{3}\right)}{3} + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = 4\arctan\left(\frac{t}{3}\right) + \pi$$

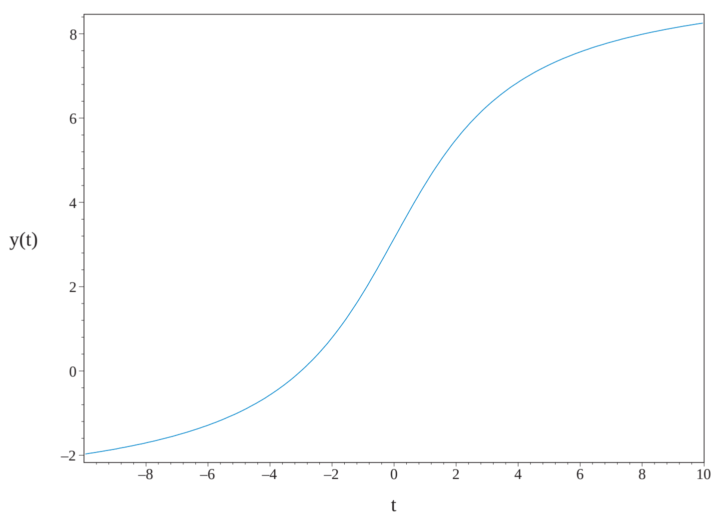


Figure 2.122: Solution plot
$y = 4\arctan\left(\frac{t}{3}\right) + \pi$

Solved as second order integrable as is ode (ABC method)

Time used: 0.033 (sec)

Writing the ode as

$$\left(t^2+9\right) y''+2ty'=0$$

Integrating both sides of the ODE w.r.t t gives

$$\int \left(\left(t^2+9\right) y''+2ty'\right) dt = 0$$
$$\left(t^2+9\right) y' = c_1$$

Which is now solved for y. Since the ode has the form $y'=f(t)$, then we only need to integrate $f(t)$.

$$\int dy = \int \frac{c_1}{t^2+9}\, dt$$
$$y = \frac{c_1 \arctan\left(\frac{t}{3}\right)}{3}+c_2$$

Will add steps showing solving for IC soon.

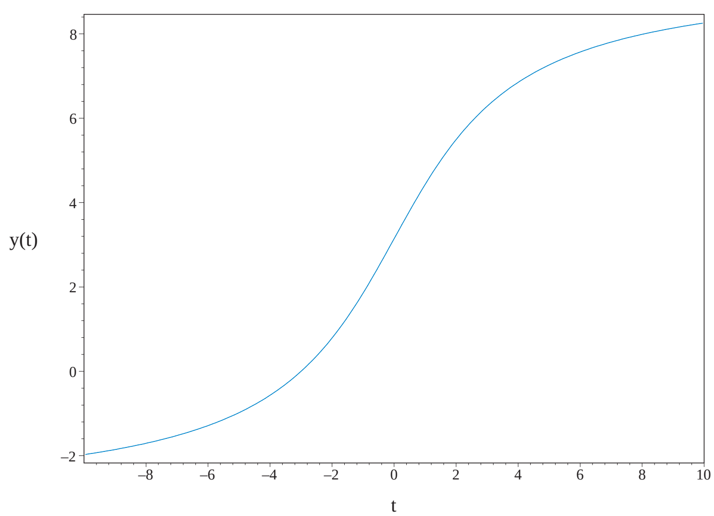


Figure 2.123: Solution plot
$y=4\arctan\left(\frac{t}{3}\right)+\pi$

Solved as second order ode using Kovacic algorithm

Time used: 0.233 (sec)

Writing the ode as

$$
\begin{align*}
\left(t^2+9\right) y''+2ty' &= 0 \tag{1}\\
Ay''+By'+Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= t^2+9 \\
B &= 2t \tag{3}\\
C &= 0
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(t) = y e^{\int \frac{B}{2A}\, dt}
$$

Then (2) becomes

$$
z''(t) = r z(t) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5}\\
&= \frac{2AB'-2BA'+B^2-4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{9}{\left(t^2+9\right)^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= 9\\
t &= \left(t^2+9\right)^2
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(t) = \left(\frac{9}{\left(t^2+9\right)^2}\right) z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.53: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = (t^2+9)^2$. There is a pole at $t = 3i$ of order 2. There is a pole at $t = -3i$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4(t-3i)^2} - \frac{1}{4(t+3i)^2} - \frac{i}{12(t-3i)} + \frac{i}{12t+36i}$$

For the pole at $t = 3i$ let b be the coefficient of $\frac{1}{(t-3i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

For the pole at $t = -3i$ let b be the coefficient of $\frac{1}{(t+3i)^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= 0 \\
\alpha_\infty^- &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{9}{(t^2+9)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
$3i$	2	0	$\frac{1}{2}$	$\frac{1}{2}$
$-3i$	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^+ + \alpha_{c_2}^+\right) \\
&= 1 - (1) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((+)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^+}{t-c_1}\right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^+}{t-c_2}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2t-6i} + \frac{1}{2t+6i} + (-)(0) \\
&= \frac{1}{2t-6i} + \frac{1}{2t+6i} \\
&= \frac{t}{t^2+9}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d=0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(t) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
(0) + 2\left(\frac{1}{2t-6i} + \frac{1}{2t+6i}\right)(0) + \left(\left(-\frac{1}{2(t-3i)^2} - \frac{1}{2(t+3i)^2}\right) + \left(\frac{1}{2t-6i} + \frac{1}{2t+6i}\right)^2 - \left(\frac{9}{(t^2+9)^2}\right.\right.
$$

$$
0 =
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(t) &= p e^{\int \omega\, dt} \\
&= e^{\int \left(\frac{1}{2t-6i} + \frac{1}{2t+6i}\right) dt} \\
&= \sqrt{-t^2-9}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2t}{t^2+9}\,dt} \\
&= z_1 e^{-\frac{\ln\left(t^2+9\right)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{t^2+9}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = i
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2t}{t^2+9}\,dt}}{\left(y_1\right)^2}\,dt \\
&= y_1 \int \frac{e^{-\ln\left(t^2+9\right)}}{\left(y_1\right)^2}\,dt \\
&= y_1 \left(-\frac{\arctan\left(\frac{t}{3}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(i) + c_2\left(i\left(-\frac{\arctan\left(\frac{t}{3}\right)}{3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = 4\arctan\left(\frac{t}{3}\right) + \pi
$$

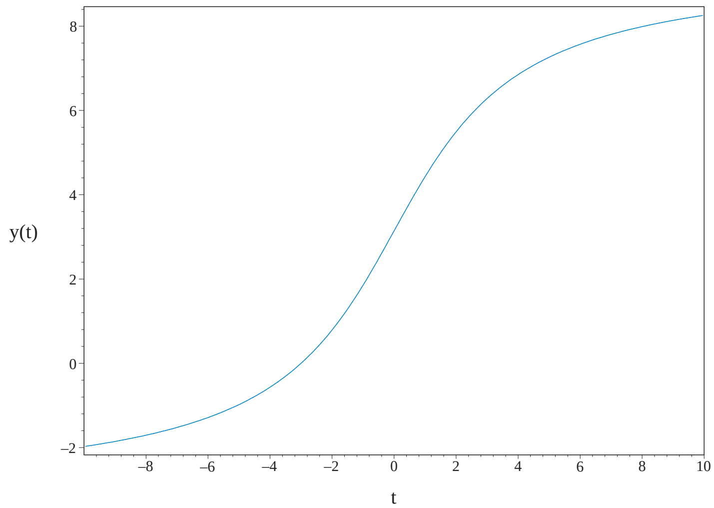


Figure 2.124: Solution plot
$y = 4\arctan\left(\frac{t}{3}\right) + \pi$

Solved as second order ode adjoint method

Time used: 0.487 (sec)

In normal form the ode

$$\left(t^2+9\right) y'' + 2ty' = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = r(t) \tag{2}$$

Where

$$\begin{aligned} p(t) &= \frac{2t}{t^2+9} \\ q(t) &= 0 \\ r(t) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - \left(\frac{2t\xi(t)}{t^2+9}\right)' + (0) &= 0 \\ \xi''(t) - \frac{2t\xi'(t)}{t^2+9} + \frac{(2t^2-18)\, \xi(t)}{\left(t^2+9\right)^2} &= 0 \end{aligned}$$

Which is solved for $\xi(t)$. An ode of the form

$$p(t)\,\xi'' + q(t)\,\xi' + r(t)\,\xi = s(t)$$

is exact if

$$p''(t) - q'(t) + r(t) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}p(x) &= 1\\ q(x) &= -\frac{2t}{t^2+9}\\ r(x) &= \frac{2t^2-18}{\left(t^2+9\right)^2}\\ s(x) &= 0\end{aligned}$$

Hence

$$\begin{aligned}p''(x) &= 0\\ q'(x) &= -\frac{2}{t^2+9} + \frac{4t^2}{\left(t^2+9\right)^2}\end{aligned}$$

Therefore (1) becomes

$$0 - \left(-\frac{2}{t^2+9} + \frac{4t^2}{\left(t^2+9\right)^2}\right) + \left(\frac{2t^2-18}{\left(t^2+9\right)^2}\right) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(t)\,\xi' + \left(q(t) - p'(t)\right)\xi\right)' = s(x)$$

Integrating gives

$$p(t)\,\xi' + \left(q(t) - p'(t)\right)\xi = \int s(t)\;dt$$

Substituting the above values for p, q, r, s gives

$$\xi' - \frac{2t\xi}{t^2+9} = c_1$$

We now have a first order ode to solve which is

$$\xi' - \frac{2t\xi}{t^2+9} = c_1$$

In canonical form a linear first order is

$$\xi' + q(t)\xi = p(t)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(t) &= -\frac{2t}{t^2+9} \\ p(t) &= c_1 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dt} \\ &= \mathrm{e}^{\int -\frac{2t}{t^2+9}dt} \\ &= \frac{1}{t^2+9} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mu\xi) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}t}(\mu\xi) &= (\mu)(c_1) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\xi}{t^2+9}\right) &= \left(\frac{1}{t^2+9}\right)(c_1) \\ \mathrm{d}\left(\frac{\xi}{t^2+9}\right) &= \left(\frac{c_1}{t^2+9}\right)\mathrm{d}t \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{\xi}{t^2+9} &= \int \frac{c_1}{t^2+9}\,dt \\ &= \frac{c_1 \arctan\left(\frac{t}{3}\right)}{3} + c_2 \end{aligned}$$

Dividing throughout by the integrating factor $\frac{1}{t^2+9}$ gives the final solution

$$\xi = (t^2+9)\left(\frac{c_1 \arctan\left(\frac{t}{3}\right)}{3} + c_2\right)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$
\begin{aligned}
\xi(t)\, y' - y\xi'(t) + \xi(t)\, p(t)\, y &= \int \xi(t)\, r(t)\, dt \\
y' + y\left(p(t) - \frac{\xi'(t)}{\xi\,(t)}\right) &= \frac{\int \xi(t)\, r(t)\, dt}{\xi\,(t)}
\end{aligned}
$$

Or

$$
y' + y\left(\frac{2t}{t^2+9} - \frac{2t\left(\frac{c_1 \arctan\left(\frac{t}{3}\right)}{3} + c_2\right) + c_1}{(t^2+9)\left(\frac{c_1 \arctan\left(\frac{t}{3}\right)}{3} + c_2\right)}\right) = 0
$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$
y' + q(t)y = p(t)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(t) &= -\frac{3c_1}{\left(c_1 \arctan\left(\frac{t}{3}\right) + 3c_2\right)(t^2+9)} \\
p(t) &= 0
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\, dt} \\
&= \mathrm{e}^{\int -\frac{3c_1}{\left(c_1 \arctan\left(\frac{t}{3}\right)+3c_2\right)\left(t^2+9\right)} dt} \\
&= \frac{1}{c_1 \arctan\left(\frac{t}{3}\right) + 3c_2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mu y &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y}{c_1 \arctan\left(\frac{t}{3}\right) + 3c_2}\right) &= 0
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{c_1 \arctan\left(\frac{t}{3}\right) + 3c_2} &= \int 0\, dt + c_3 \\
&= c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{1}{c_1 \arctan\left(\frac{t}{3}\right)+3c_2}$ gives the final solution

$$y = \left(c_1 \arctan\left(\frac{t}{3}\right) + 3c_2\right) c_3$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \left(c_1 \arctan\left(\frac{t}{3}\right) + 3c_2\right) c_3$$

The constants can be merged to give

$$y = c_1 \arctan\left(\frac{t}{3}\right) + 3c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = 4 \arctan\left(\frac{t}{3}\right) + \pi$$

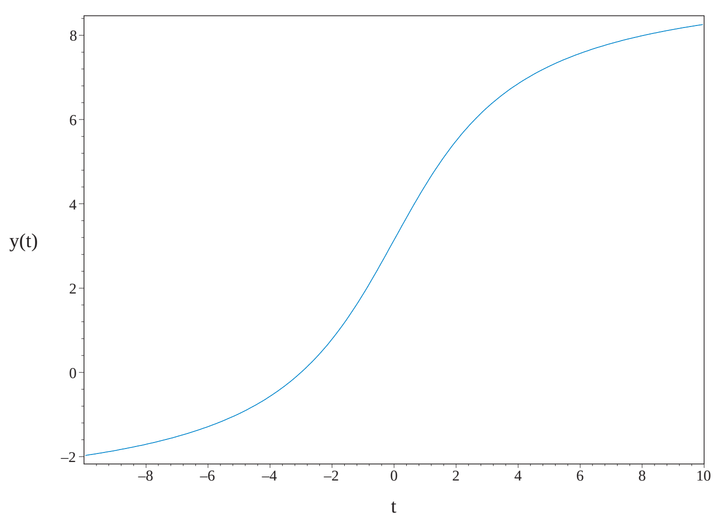


Figure 2.125: Solution plot
$y = 4 \arctan\left(\frac{t}{3}\right) + \pi$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
<- LODE missing y successful`
```

Maple dsolve solution

Solving time : 0.075 (sec)
Leaf size : 12

```
dsolve([(t^2+9)*diff(diff(y(t),t),t)+2*t*diff(y(t),t) = 0,
       op([y(3) = 2*Pi, D(y)(3) = 2/3])],y(t),singsol=all)
```

$$y = \pi + 4 \arctan\left(\frac{t}{3}\right)$$

Mathematica DSolve solution

Solving time : 0.02 (sec)
Leaf size : 15

```
DSolve[{(t^2+9)*D[y[t],{t,2}]+2*t*D[y[t],t]==0,{y[3]==2*Pi,Derivative[1][y][3]==2/3}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to 4 \arctan\left(\frac{t}{3}\right) + \pi$$

2.1.49 problem 49

Internal problem ID [8437]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 49
Date solved : `Thursday, December 12, 2024 at 09:08:02 AM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$t^2y'' - 3ty' + 5y = 0$$

Solved as second order Euler type ode

Time used: 0.108 (sec)

This is Euler second order ODE. Let the solution be $y = t^r$, then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} - 3trt^{r-1} + 5t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r - 3r\,t^r + 5t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) - 3r + 5 = 0$$

Or

$$r^2 - 4r + 5 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= 2 - i \\ r_2 &= 2 + i \end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

Where in this case $\alpha = 2$ and $\beta = -1$. Hence the solution becomes

$$\begin{aligned} y &= c_1 t^{r_1} + c_2 t^{r_2} \\ &= c_1 t^{\alpha + i\beta} + c_2 t^{\alpha - i\beta} \\ &= t^{\alpha}\left(c_1 t^{i\beta} + c_2 t^{-i\beta}\right) \\ &= t^{\alpha}\left(c_1 e^{\ln\left(t^{i\beta}\right)} + c_2 e^{\ln\left(t^{-i\beta}\right)}\right) \\ &= t^{\alpha}\left(c_1 e^{i(\beta \ln t)} + c_2 e^{-i(\beta \ln t)}\right) \end{aligned}$$

Using the values for $\alpha = 2, \beta = -1$, the above becomes

$$y = t^2\left(c_1 e^{-i\ln(t)} + c_2 e^{i\ln(t)}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y = t^2(c_1 \cos(\ln(t)) + c_2 \sin(\ln(t)))$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = t^2(c_1 \cos(\ln(t)) + c_2 \sin(\ln(t)))$$

Solved as second order ode using change of variable on x method 2

Time used: 0.483 (sec)

In normal form the ode

$$t^2y'' - 3ty' + 5y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= -\frac{3}{t} \\ q(t) &= \frac{5}{t^2} \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\, \tau'(t)}{\tau'\,(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'\,(t)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t) + p(t)\, \tau'(t) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\int p(t)dt}dt \\ &= \int \mathrm{e}^{-\int -\frac{3}{t}dt}dt \\ &= \int e^{3\ln(t)}\, dt \\ &= \int t^3dt \\ &= \frac{t^4}{4} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\ &= \frac{\frac{5}{t^2}}{t^6} \\ &= \frac{5}{t^8} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned} \frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + \frac{5y(\tau)}{t^8} &= 0 \end{aligned}$$

But in terms of τ

$$\frac{5}{t^8} = \frac{5}{16\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{5y(\tau)}{16\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. Writing the ode as

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{5y(\tau)}{16\tau^2} = 0 \tag{1}$$

$$A\frac{d^2}{d\tau^2}y(\tau) + B\frac{d}{d\tau}y(\tau) + Cy(\tau) = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= \frac{5}{16\tau^2} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(\tau) = y(\tau)\, e^{\int \frac{B}{2A}\, d\tau}$$

Then (2) becomes

$$z''(\tau) = rz(\tau) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5}{16\tau^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -5 \\ t &= 16\tau^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(\tau) = \left(-\frac{5}{16\tau^2}\right) z(\tau) \tag{7}$$

Equation (7) is now solved. After finding $z(\tau)$ then $y(\tau)$ is found using the inverse transformation

$$y(\tau) = z(\tau)\, e^{-\int \frac{B}{2A}\, d\tau}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.54: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 16\tau^2$. There is a pole at $\tau = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{16\tau^2}$$

For the pole at $\tau = 0$ let b be the coefficient of $\frac{1}{\tau^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{i}{4} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{i}{4} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{\tau^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{5}{16\tau^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{5}{16}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + \frac{i}{4} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - \frac{i}{4} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{5}{16\tau^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}+\frac{i}{4}$	$\frac{1}{2}-\frac{i}{4}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}+\frac{i}{4}$	$\frac{1}{2}-\frac{i}{4}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2} - \frac{i}{4}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - \frac{i}{4} - \left(\frac{1}{2} - \frac{i}{4}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{\tau - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{\tau - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{\frac{1}{2} - \frac{i}{4}}{\tau} + (-)(0) \\
&= \frac{\frac{1}{2} - \frac{i}{4}}{\tau} \\
&= \frac{\frac{1}{2} - \frac{i}{4}}{\tau}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(\tau)$ of degree $d = 0$ to solve the ode. The polynomial $p(\tau)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(\tau) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered}
(0) + 2\left(\frac{\frac{1}{2} - \frac{i}{4}}{\tau}\right)(0) + \left(\left(\frac{-\frac{1}{2} + \frac{i}{4}}{\tau^2}\right) + \left(\frac{\frac{1}{2} - \frac{i}{4}}{\tau}\right)^2 - \left(-\frac{5}{16\tau^2}\right)\right) = 0 \\
0 = 0
\end{gathered}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(\tau) &= p e^{\int \omega \, d\tau} \\
&= \mathrm{e}^{\int \frac{\frac{1}{2} - \frac{i}{4}}{\tau} d\tau} \\
&= \tau^{\frac{1}{2} - \frac{i}{4}}
\end{aligned}$$

The first solution to the original ode in $y(\tau)$ is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, d\tau}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= \tau^{\frac{1}{2} - \frac{i}{4}}
\end{aligned}$$

Which simplifies to

$$y_1 = \tau^{\frac{1}{2} - \frac{i}{4}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, d\tau}}{y_1^2} \, d\tau$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, d\tau \\
&= \tau^{\frac{1}{2}-\frac{i}{4}} \int \frac{1}{\tau^{1-\frac{i}{2}}}\, d\tau \\
&= \tau^{\frac{1}{2}-\frac{i}{4}} \left(-2i\tau\, \tau^{-1+\frac{i}{2}}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y(\tau) &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\tau^{\frac{1}{2}-\frac{i}{4}}\right) + c_2 \left(\tau^{\frac{1}{2}-\frac{i}{4}} \left(-2i\tau\, \tau^{-1+\frac{i}{2}}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \left(\frac{t^4}{4}\right)^{\frac{1}{2}-\frac{i}{4}} - 2ic_2 \left(\frac{t^4}{4}\right)^{\frac{1}{2}+\frac{i}{4}}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \left(\frac{t^4}{4}\right)^{\frac{1}{2}-\frac{i}{4}} - 2ic_2 \left(\frac{t^4}{4}\right)^{\frac{1}{2}+\frac{i}{4}}$$

Solved as second order ode using change of variable on x method 1

Time used: 0.176 (sec)

In normal form the ode

$$t^2 y'' - 3ty' + 5y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(t) &= -\frac{3}{t} \\
q(t) &= \frac{5}{t^2}
\end{aligned}
$$

Applying change of variables $\tau = g(t)$ to (2) results

$$
\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}
$$

Where τ is the new independent variable, and

$$
p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}
$$

$$
q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}
$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$
\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{5}\sqrt{\frac{1}{t^2}}}{c} \\
\tau'' &= -\frac{\sqrt{5}}{c\sqrt{\frac{1}{t^2}}\,t^3}
\end{aligned} \tag{6}
$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \\
&= \frac{-\frac{\sqrt{5}}{c\sqrt{\frac{1}{t^2}}\,t^3} - \frac{3}{t}\frac{\sqrt{5}\sqrt{\frac{1}{t^2}}}{c}}{\left(\frac{\sqrt{5}\sqrt{\frac{1}{t^2}}}{c}\right)^2} \\
&= -\frac{4c\sqrt{5}}{5}
\end{aligned}
$$

Therefore ode (3) now becomes

$$
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) = 0
$$

$$
\frac{d^2}{d\tau^2}y(\tau) - \frac{4c\sqrt{5}\left(\frac{d}{d\tau}y(\tau)\right)}{5} + c^2 y(\tau) = 0 \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{\frac{2\sqrt{5}\, c\tau}{5}} \left(c_1 \cos\left(\frac{\sqrt{5}\, c\tau}{5}\right) + c_2 \sin\left(\frac{\sqrt{5}\, c\tau}{5}\right)\right)$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dt \\
&= \frac{\int \sqrt{5}\, \sqrt{\frac{1}{t^2}} dt}{c} \\
&= \frac{\sqrt{5}\, \ln(t)}{c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = t^2 (c_1 \cos(\ln(t)) + c_2 \sin(\ln(t)))$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = t^2 (c_1 \cos(\ln(t)) + c_2 \sin(\ln(t)))$$

Solved as second order ode using change of variable on y method 2

Time used: 0.110 (sec)

In normal form the ode

$$t^2 y'' - 3ty' + 5y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$\begin{aligned}
p(t) &= -\frac{3}{t} \\
q(t) &= \frac{5}{t^2}
\end{aligned}$$

Applying change of variables on the depndent variable $y = v(t)\, t^n$ to (2) gives the following ode where the dependent variables is $v(t)$ and not y.

$$v''(t) + \left(\frac{2n}{t} + p\right) v'(t) + \left(\frac{n(n-1)}{t^2} + \frac{np}{t} + q\right) v(t) = 0 \tag{3}$$

Let the coefficient of $v(t)$ above be zero. Hence

$$\frac{n(n-1)}{t^2} + \frac{np}{t} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(t)$ and $q(t)$ into (4) gives

$$\frac{n(n-1)}{t^2} - \frac{3n}{t^2} + \frac{5}{t^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 + i \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(t) + \left(\frac{4+2i}{t} - \frac{3}{t}\right) v'(t) &= 0 \\ v''(t) + \frac{(1+2i)\, v'(t)}{t} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(t) = v'(t)$$

Then (7) becomes

$$u'(t) + \frac{(1+2i)\, u(t)}{t} = 0 \tag{8}$$

The above is now solved for $u(t)$. In canonical form a linear first order is

$$u'(t) + q(t)u(t) = p(t)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(t) &= \frac{1+2i}{t} \\ p(t) &= 0 \end{aligned}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\,dt} \\
&= \mathrm{e}^{\int \frac{1+2i}{t}dt} \\
&= t^{1+2i}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mu u &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(u\,t^{1+2i}\right) &= 0
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
u\,t^{1+2i} &= \int 0\,dt + c_1 \\
&= c_1
\end{aligned}
$$

Dividing throughout by the integrating factor t^{1+2i} gives the final solution

$$
u(t) = t^{-1-2i}c_1
$$

Now that $u(t)$ is known, then

$$
\begin{aligned}
v'(t) &= u(t) \\
v(t) &= \int u(t)\,dt + c_2 \\
&= \frac{it^{-2i}c_1}{2} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(t)\,t^n \\
&= \left(\frac{it^{-2i}c_1}{2} + c_2\right)t^{2+i} \\
&= \frac{\left(it^{-2i}c_1 + 2c_2\right)t^{2+i}}{2}
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = \left(\frac{it^{-2i}c_1}{2} + c_2\right)t^{2+i}
$$

Solved as second order ode using Kovacic algorithm

Time used: 0.187 (sec)

Writing the ode as

$$t^2y'' - 3ty' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= -3t \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -5 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\frac{5}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.55: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{5}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + i \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - i
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{5}{4t^2}
$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{5}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + i \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - i
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{5}{4t^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + i$	$\frac{1}{2} - i$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2} + i$	$\frac{1}{2} - i$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = \frac{1}{2} - i$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - i - \left(\frac{1}{2} - i\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{\frac{1}{2} - i}{t} + (-)(0) \\ &= \frac{\frac{1}{2} - i}{t} \\ &= \frac{\frac{1}{2} - i}{t} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{\frac{1}{2} - i}{t}\right)(0) + \left(\left(\frac{-\frac{1}{2} + i}{t^2} \right) + \left(\frac{\frac{1}{2} - i}{t} \right)^2 - \left(-\frac{5}{4t^2} \right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(t) &= p e^{\int \omega \, dt} \\ &= e^{\int \frac{\frac{1}{2} - i}{t} dt} \\ &= t^{\frac{1}{2} - i} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3t}{t^2}\,dt} \\
&= z_1 e^{\frac{3\ln(t)}{2}} \\
&= z_1 \left(t^{3/2}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = t^{2-i}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3t}{t^2}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{3\ln(t)}}{(y_1)^2}\,dt \\
&= y_1 \left(-\frac{i t^4 t^{-4+2i}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(t^{2-i}\right) + c_2 \left(t^{2-i}\left(-\frac{i t^4 t^{-4+2i}}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = c_1\, t^{2-i} - \frac{i c_2\, t^{2+i}}{2}
$$

Solved as second order ode adjoint method

Time used: 0.879 (sec)

In normal form the ode

$$t^2y'' - 3ty' + 5y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\,y' + q(t)\,y = r(t) \tag{2}$$

Where

$$\begin{aligned} p(t) &= -\frac{3}{t} \\ q(t) &= \frac{5}{t^2} \\ r(t) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - \left(-\frac{3\xi(t)}{t}\right)' + \left(\frac{5\xi(t)}{t^2}\right) &= 0 \\ \xi''(t) + \frac{2\xi(t)}{t^2} + \frac{3\xi'(t)}{t} &= 0 \end{aligned}$$

Which is solved for $\xi(t)$. This is Euler second order ODE. Let the solution be $\xi = t^r$, then $\xi' = rt^{r-1}$ and $\xi'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} + 3trt^{r-1} + 2t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r + 3r\,t^r + 2t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) + 3r + 2 = 0$$

Or

$$r^2 + 2r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -1 - i \\ r_2 &= -1 + i \end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$
\begin{aligned}
r_1 &= \alpha + i\beta \\
r_2 &= \alpha - i\beta
\end{aligned}
$$

Where in this case $\alpha = -1$ and $\beta = -1$. Hence the solution becomes

$$
\begin{aligned}
\xi &= c_1 t^{r_1} + c_2 t^{r_2} \\
&= c_1 t^{\alpha+i\beta} + c_2 t^{\alpha-i\beta} \\
&= t^{\alpha}\left(c_1 t^{i\beta} + c_2 t^{-i\beta}\right) \\
&= t^{\alpha}\left(c_1 e^{\ln\left(t^{i\beta}\right)} + c_2 e^{\ln\left(t^{-i\beta}\right)}\right) \\
&= t^{\alpha}\left(c_1 e^{i(\beta \ln t)} + c_2 e^{-i(\beta \ln t)}\right)
\end{aligned}
$$

Using the values for $\alpha = -1, \beta = -1$, the above becomes

$$
\xi = t^{-1}\left(c_1 e^{-i\ln(t)} + c_2 e^{i\ln(t)}\right)
$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$
\xi = \frac{1}{t}\left(c_1 \cos\left(\ln\left(t\right)\right) + c_2 \sin\left(\ln\left(t\right)\right)\right)
$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$
\begin{aligned}
\xi(t)\, y' - y\xi'(t) + \xi(t)\, p(t)\, y &= \int \xi(t)\, r(t)\, dt \\
y' + y\left(p(t) - \frac{\xi'(t)}{\xi\left(t\right)}\right) &= \frac{\int \xi(t)\, r(t)\, dt}{\xi\left(t\right)}
\end{aligned}
$$

Or

$$
y' + y\left(-\frac{3}{t} - \frac{\left(-\frac{c_1 \cos(\ln(t)) + c_2 \sin(\ln(t))}{t^2} + \frac{-\frac{c_1 \sin(\ln(t))}{t} + \frac{c_2 \cos(\ln(t))}{t}}{t}\right) t}{c_1 \cos\left(\ln\left(t\right)\right) + c_2 \sin\left(\ln\left(t\right)\right)}\right) = 0
$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$
y' + q(t) y = p(t)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(t) &= -\frac{(2c_1+c_2)\cos(\ln(t)) - \sin(\ln(t))(c_1-2c_2)}{t(c_1\cos(\ln(t)) + c_2\sin(\ln(t)))} \\
p(t) &= 0
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\,dt} \\
&= \mathrm{e}^{\int -\frac{(2c_1+c_2)\cos(\ln(t))-\sin(\ln(t))(c_1-2c_2)}{t(c_1\cos(\ln(t))+c_2\sin(\ln(t)))}dt} \\
&= \mathrm{e}^{\frac{\ln\left(\tan(\ln(t))^2+1\right)}{2}-\ln(c_2\tan(\ln(t))+c_1)-2\ln(t)}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mu y &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(y\,\mathrm{e}^{\frac{\ln\left(\tan(\ln(t))^2+1\right)}{2}-\ln(c_2\tan(\ln(t))+c_1)-2\ln(t)}\right) &= 0
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
y\,\mathrm{e}^{\frac{\ln\left(\tan(\ln(t))^2+1\right)}{2}-\ln(c_2\tan(\ln(t))+c_1)-2\ln(t)} &= \int 0\,dt + c_3 \\
&= c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\mathrm{e}^{\frac{\ln\left(\tan(\ln(t))^2+1\right)}{2}-\ln(c_2\tan(\ln(t))+c_1)-2\ln(t)}$ gives the final solution

$$
y = \frac{(c_2\tan(\ln(t)) + c_1)\,t^2 c_3}{\sqrt{\tan(\ln(t))^2+1}}
$$

Hence, the solution found using Lagrange adjoint equation method is

$$
y = \frac{(c_2\tan(\ln(t)) + c_1)\,t^2 c_3}{\sqrt{\tan(\ln(t))^2+1}}
$$

The constants can be merged to give

$$
y = \frac{(c_2\tan(\ln(t)) + c_1)\,t^2}{\sqrt{\tan(\ln(t))^2+1}}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{(c_2 \tan(\ln(t)) + c_1)\, t^2}{\sqrt{\tan(\ln(t))^2 + 1}}$$

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dt^2}y(t)\right)t^2 - 3t\left(\frac{d}{dt}y(t)\right) + 5y(t) = 0$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dt^2}y(t)$
- Isolate 2nd derivative
$\frac{d^2}{dt^2}y(t) = -\frac{5y(t)}{t^2} + \frac{3\left(\frac{d}{dt}y(t)\right)}{t}$
- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
$\frac{d^2}{dt^2}y(t) - \frac{3\left(\frac{d}{dt}y(t)\right)}{t} + \frac{5y(t)}{t^2} = 0$
- Multiply by denominators of the ODE
$\left(\frac{d^2}{dt^2}y(t)\right)t^2 - 3t\left(\frac{d}{dt}y(t)\right) + 5y(t) = 0$
- Make a change of variables
$s = \ln(t)$
- □ Substitute the change of variables back into the ODE
  - ○ Calculate the 1st derivative of y with respect to t , using the chain rule
  $\frac{d}{dt}y(t) = \left(\frac{d}{ds}y(s)\right)\left(\frac{d}{dt}s(t)\right)$
  - ○ Compute derivative
  $\frac{d}{dt}y(t) = \frac{\frac{d}{ds}y(s)}{t}$
  - ○ Calculate the 2nd derivative of y with respect to t , using the chain rule
  $\frac{d^2}{dt^2}y(t) = \left(\frac{d^2}{ds^2}y(s)\right)\left(\frac{d}{dt}s(t)\right)^2 + \left(\frac{d^2}{dt^2}s(t)\right)\left(\frac{d}{ds}y(s)\right)$
  - ○ Compute derivative
  $\frac{d^2}{dt^2}y(t) = \frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}$

  Substitute the change of variables back into the ODE
  $\left(\frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}\right)t^2 - 3\frac{d}{ds}y(s) + 5y(s) = 0$
- Simplify

$\frac{d^2}{ds^2}y(s) - 4\frac{d}{ds}y(s) + 5y(s) = 0$

- Characteristic polynomial of ODE
  $r^2 - 4r + 5 = 0$
- Use quadratic formula to solve for r
  $r = \frac{4 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial
  $r = (2 - \mathrm{I}, 2 + \mathrm{I})$
- 1st solution of the ODE
  $y_1(s) = \mathrm{e}^{2s}\cos(s)$
- 2nd solution of the ODE
  $y_2(s) = \mathrm{e}^{2s}\sin(s)$
- General solution of the ODE
  $y(s) = C1y_1(s) + C2y_2(s)$
- Substitute in solutions
  $y(s) = C1\,\mathrm{e}^{2s}\cos(s) + C2\,\mathrm{e}^{2s}\sin(s)$
- Change variables back using $s = \ln(t)$
  $y(t) = C1\,t^2\cos(\ln(t)) + C2\,t^2\sin(\ln(t))$
- Simplify
  $y(t) = t^2(C1\cos(\ln(t)) + C2\sin(\ln(t)))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 19

```
dsolve(diff(diff(y(t),t),t)*t^2-3*t*diff(y(t),t)+5*y(t) = 0,
       y(t),singsol=all)
```

$$y = t^2(c_1 \sin(\ln(t)) + c_2 \cos(\ln(t)))$$

Mathematica DSolve solution

Solving time : 0.027 (sec)
Leaf size : 22

```
DSolve[{t^2*D[y[t],{t,2}]-3*t*D[y[t],t]+5*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to t^2(c_2 \cos(\log(t)) + c_1 \sin(\log(t)))$$

2.1.50 problem 50

Internal problem ID [8438]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 50
Date solved : `Thursday, December 12, 2024 at 09:08:04 AM`
CAS classification : `[[_2nd_order, _missing_y]]`

Solve

$$ty'' + y' = 0$$

Solved as second order linear exact ode

Time used: 0.095 (sec)

An ode of the form

$$p(t)\, y'' + q(t)\, y' + r(t)\, y = s(t)$$

is exact if

$$p''(t) - q'(t) + r(t) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}
p(x) &= t\\
q(x) &= 1\\
r(x) &= 0\\
s(x) &= 0
\end{aligned}$$

Hence

$$p''(x) = 0$$
$$q'(x) = 0$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(t)\, y' + (q(t) - p'(t))\, y)' = s(x)$$

Integrating gives

$$p(t)\, y' + (q(t) - p'(t))\, y = \int s(t)\, dt$$

Substituting the above values for p, q, r, s gives

$$ty' = c_1$$

We now have a first order ode to solve which is

$$ty' = c_1$$

Since the ode has the form $y' = f(t)$, then we only need to integrate $f(t)$.

$$\int dy = \int \frac{c_1}{t}\, dt$$
$$y = c_1 \ln(t) + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \ln(t) + c_2$$

Solved as second order missing y ode

Time used: 0.058 (sec)

This is second order ode with missing dependent variable y. Let

$$p(t) = y'$$

Then

$$p'(t) = y''$$

Hence the ode becomes

$$tp'(t) + p(t) = 0$$

Which is now solve for $p(t)$ as first order ode. In canonical form a linear first order is

$$p'(t) + q(t)p(t) = p(t)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(t) &= \frac{1}{t} \\ p(t) &= 0 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dt} \\ &= \mathrm{e}^{\int \frac{1}{t}dt} \\ &= t \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mu p &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}(pt) &= 0 \end{aligned}$$

Integrating gives

$$\begin{aligned} pt &= \int 0\,dt + c_1 \\ &= c_1 \end{aligned}$$

Dividing throughout by the integrating factor t gives the final solution

$$p(t) = \frac{c_1}{t}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{c_1}{t}$$

Since the ode has the form $y' = f(t)$, then we only need to integrate $f(t)$.

$$\int dy = \int \frac{c_1}{t}\, dt$$
$$y = c_1 \ln(t) + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \ln(t) + c_2$$

Solved as second order integrable as is ode

Time used: 0.036 (sec)

Integrating both sides of the ODE w.r.t t gives

$$\int (ty'' + y')\, dt = 0$$
$$ty' = c_1$$

Which is now solved for y. Since the ode has the form $y' = f(t)$, then we only need to integrate $f(t)$.

$$\int dy = \int \frac{c_1}{t}\, dt$$
$$y = c_1 \ln(t) + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \ln(t) + c_2$$

Solved as second order integrable as is ode (ABC method)

Time used: 0.026 (sec)

Writing the ode as

$$ty'' + y' = 0$$

Integrating both sides of the ODE w.r.t t gives

$$\int (ty'' + y')\, dt = 0$$
$$ty' = c_1$$

Which is now solved for y. Since the ode has the form $y' = f(t)$, then we only need to integrate $f(t)$.

$$
\begin{aligned}
\int dy &= \int \frac{c_1}{t}\, dt \\
y &= c_1 \ln(t) + c_2
\end{aligned}
$$

Will add steps showing solving for IC soon.

Solved as second order ode using non constant coeff transformation on B method

Time used: 0.062 (sec)

Given an ode of the form

$$
Ay'' + By' + Cy = F(t)
$$

This method reduces the order ode the ODE by one by applying the transformation

$$
y = Bv
$$

This results in

$$
\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\
ABv'' + \left(2AB' + B^2\right) v' + (AB'' + BB' + CB)\, v &= 0 \qquad (1)
\end{aligned}
$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$
ABv'' + \left(2AB' + B^2\right) v' = 0
$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$
ABu' + \left(2AB' + B^2\right) u = 0
$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A &= t \\
B &= 1 \\
C &= 0 \\
F &= 0
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= (t)(0) + (1)(0) + (0)(1) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
tv'' + (1) v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
tu'(t) + u(t) = 0
$$

Which is now solved for u. In canonical form a linear first order is

$$
u'(t) + q(t)u(t) = p(t)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(t) &= \frac{1}{t} \\
p(t) &= 0
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q \, dt} \\
&= e^{\int \frac{1}{t} dt} \\
&= t
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \mu u &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}t} (ut) &= 0
\end{aligned}
$$

Integrating gives

$$\begin{aligned} ut &= \int 0 \, dt + c_1 \\ &= c_1 \end{aligned}$$

Dividing throughout by the integrating factor t gives the final solution

$$u(t) = \frac{c_1}{t}$$

The ode for v now becomes

$$v'(t) = \frac{c_1}{t}$$

Which is now solved for v. Since the ode has the form $v'(t) = f(t)$, then we only need to integrate $f(t)$.

$$\begin{aligned} \int dv &= \int \frac{c_1}{t} \, dt \\ v(t) &= c_1 \ln(t) + c_2 \end{aligned}$$

Replacing $v(t)$ above by y, then the solution becomes

$$\begin{aligned} y(t) &= Bv \\ &= c_1 \ln(t) + c_2 \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \ln(t) + c_2$$

Solved as second order ode using Kovacic algorithm

Time used: 0.210 (sec)

Writing the ode as

$$ty'' + y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4t^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4t^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\frac{1}{4t^2}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A} dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.57: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4t^2}$$

For the pole at $t = 0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{t^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4t^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{t-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2t} + (-)\,(0) \\
&= \frac{1}{2t} \\
&= \frac{1}{2t}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{1}{2t}\right)(0) + \left(\left(-\frac{1}{2t^2}\right) + \left(\frac{1}{2t}\right)^2 - \left(-\frac{1}{4t^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega \, dt} \\
&= \mathrm{e}^{\int \frac{1}{2t} dt} \\
&= \sqrt{t}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{t}\, dt} \\
&= z_1 e^{-\frac{\ln(t)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{t}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dt}}{y_1^2} dt$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{1}{t} dt}}{(y_1)^2} dt \\ &= y_1 \int \frac{e^{-\ln(t)}}{(y_1)^2} dt \\ &= y_1(\ln(t)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2(1(\ln(t))) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 + c_2 \ln(t)$$

Solved as second order ode adjoint method

Time used: 0.347 (sec)

In normal form the ode

$$ty'' + y' = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = r(t) \tag{2}$$

Where

$$\begin{aligned} p(t) &= \frac{1}{t} \\ q(t) &= 0 \\ r(t) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - \left(\frac{\xi(t)}{t}\right)' + (0) &= 0 \\
\xi''(t) + \frac{\xi(t)}{t^2} - \frac{\xi'(t)}{t} &= 0
\end{aligned}$$

Which is solved for $\xi(t)$. This is Euler second order ODE. Let the solution be $\xi = t^r$, then $\xi' = rt^{r-1}$ and $\xi'' = r(r-1)t^{r-2}$. Substituting these back into the given ODE gives

$$t^2(r(r-1))t^{r-2} - trt^{r-1} + t^r = 0$$

Simplifying gives

$$r(r-1)\,t^r - r\,t^r + t^r = 0$$

Since $t^r \neq 0$ then dividing throughout by t^r gives

$$r(r-1) - r + 1 = 0$$

Or

$$r^2 - 2r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned}
r_1 &= 1 \\
r_2 &= 1
\end{aligned}$$

Since the roots are equal, then the general solution is

$$\xi = c_1\xi_1 + c_2\xi_2$$

Where $\xi_1 = t^r$ and $\xi_2 = t^r \ln(t)$. Hence

$$\xi = c_1 t + c_2 t \ln(t)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}
\xi(t)\, y' - y\xi'(t) + \xi(t)\, p(t)\, y &= \int \xi(t)\, r(t)\, dt \\
y' + y\left(p(t) - \frac{\xi'(t)}{\xi(t)}\right) &= \frac{\int \xi(t)\, r(t)\, dt}{\xi(t)}
\end{aligned}$$

Or

$$y' + y\left(\frac{1}{t} - \frac{c_1 + c_2 \ln(t) + c_2}{c_1 t + c_2 t \ln(t)}\right) = 0$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(t)y = p(t)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(t) &= -\frac{c_2}{t\left(c_1 + c_2 \ln(t)\right)} \\ p(t) &= 0 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dt} \\ &= \mathrm{e}^{\int -\frac{c_2}{t(c_1 + c_2 \ln(t))}dt} \\ &= \frac{1}{c_1 + c_2 \ln(t)} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mu y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y}{c_1 + c_2 \ln(t)}\right) &= 0 \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{y}{c_1 + c_2 \ln(t)} &= \int 0\,dt + c_3 \\ &= c_3 \end{aligned}$$

Dividing throughout by the integrating factor $\frac{1}{c_1 + c_2 \ln(t)}$ gives the final solution

$$y = (c_1 + c_2 \ln(t))\, c_3$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = (c_1 + c_2 \ln(t))\, c_3$$

The constants can be merged to give

$$y = c_1 + c_2 \ln(t)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 + c_2 \ln(t)$$

Maple step by step solution

Let's solve

$t\left(\frac{d^2}{dt^2}y(t)\right) + \frac{d}{dt}y(t) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$

- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = -\frac{\frac{d}{dt}y(t)}{t}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t) + \frac{\frac{d}{dt}y(t)}{t} = 0$

- Multiply by denominators of the ODE

  $t\left(\frac{d^2}{dt^2}y(t)\right) + \frac{d}{dt}y(t) = 0$

- Make a change of variables

  $s = \ln(t)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to t , using the chain rule

    $\frac{d}{dt}y(t) = \left(\frac{d}{ds}y(s)\right)\left(\frac{d}{dt}s(t)\right)$

  - Compute derivative

    $\frac{d}{dt}y(t) = \frac{\frac{d}{ds}y(s)}{t}$

  - Calculate the 2nd derivative of y with respect to t , using the chain rule

    $\frac{d^2}{dt^2}y(t) = \left(\frac{d^2}{ds^2}y(s)\right)\left(\frac{d}{dt}s(t)\right)^2 + \left(\frac{d^2}{dt^2}s(t)\right)\left(\frac{d}{ds}y(s)\right)$

  - Compute derivative

    $\frac{d^2}{dt^2}y(t) = \frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}$

  Substitute the change of variables back into the ODE

  $t\left(\frac{\frac{d^2}{ds^2}y(s)}{t^2} - \frac{\frac{d}{ds}y(s)}{t^2}\right) + \frac{\frac{d}{ds}y(s)}{t} = 0$

- Simplify

$\frac{\frac{d^2}{ds^2}y(s)}{t} = 0$

- Isolate 2nd derivative

  $\frac{d^2}{ds^2}y(s) = 0$
- Characteristic polynomial of ODE

  $r^2 = 0$
- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{0})}{2}$
- Roots of the characteristic polynomial

  $r = 0$
- 1st solution of the ODE

  $y_1(s) = 1$
- Repeated root, multiply $y_1(s)$ by s to ensure linear independence

  $y_2(s) = s$
- General solution of the ODE

  $y(s) = C1y_1(s) + C2y_2(s)$
- Substitute in solutions

  $y(s) = C2s + C1$
- Change variables back using $s = \ln(t)$

  $y(t) = C2\ln(t) + C1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 10

```
dsolve(t*diff(diff(y(t),t),t)+diff(y(t),t) = 0,
       y(t),singsol=all)
```

$$y = c_2 \ln(t) + c_1$$

Mathematica DSolve solution

Solving time : 0.01 (sec)
Leaf size : 13

```
DSolve[{t*D[y[t],{t,2}]+D[y[t],t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to c_1 \log(t) + c_2$$

2.1.51 problem 51

Internal problem ID [8439]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 51
Date solved : `Thursday, December 12, 2024 at 09:08:06 AM`
CAS classification : `[[_2nd_order, _missing_y]]`

Solve

$$t^2y'' - 2y' = 0$$

Solved as second order missing y ode

Time used: 0.162 (sec)

This is second order ode with missing dependent variable y. Let

$$p(t) = y'$$

Then

$$p'(t) = y''$$

Hence the ode becomes

$$t^2p'(t) - 2p(t) = 0$$

Which is now solve for $p(t)$ as first order ode. In canonical form a linear first order is

$$p'(t) + q(t)p(t) = p(t)$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(t) &= -\frac{2}{t^2} \\
p(t) &= 0
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q \, dt} \\
&= \mathrm{e}^{\int -\frac{2}{t^2} dt} \\
&= \mathrm{e}^{\frac{2}{t}}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mu p &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(p\,\mathrm{e}^{\frac{2}{t}}\right) &= 0
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
p\,\mathrm{e}^{\frac{2}{t}} &= \int 0 \, dt + c_1 \\
&= c_1
\end{aligned}
$$

Dividing throughout by the integrating factor $\mathrm{e}^{\frac{2}{t}}$ gives the final solution

$$
p(t) = \mathrm{e}^{-\frac{2}{t}} c_1
$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
y' = \mathrm{e}^{-\frac{2}{t}} c_1
$$

Since the ode has the form $y' = f(t)$, then we only need to integrate $f(t)$.

$$
\begin{aligned}
\int dy &= \int \mathrm{e}^{-\frac{2}{t}} c_1 \, dt \\
y &= c_1 \left(t\,\mathrm{e}^{-\frac{2}{t}} - 2 \,\mathrm{Ei}_1\left(\frac{2}{t}\right)\right) + c_2
\end{aligned}
$$

$$
y = \mathrm{e}^{-\frac{2}{t}} c_1 t - 2 \,\mathrm{Ei}_1\left(\frac{2}{t}\right) c_1 + c_2
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = \mathrm{e}^{-\frac{2}{t}} c_1 t - 2 \,\mathrm{Ei}_1\left(\frac{2}{t}\right) c_1 + c_2
$$

Solved as second order ode using non constant coeff transformation on B method

Time used: 0.079 (sec)

Given an ode of the form

$$Ay'' + By' + Cy = F(t)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + (AB'' + BB' + CB) v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= t^2 \\ B &= -2 \\ C &= 0 \\ F &= 0 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= (t^2)(0) + (-2)(0) + (0)(-2) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-2t^2v'' + (4)v' = 0$$

Now by applying $v' = u$ the above becomes

$$-2t^2u'(t) + 4u(t) = 0$$

Which is now solved for u. In canonical form a linear first order is

$$u'(t) + q(t)u(t) = p(t)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(t) &= -\frac{2}{t^2} \\ p(t) &= 0 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dt} \\ &= \mathrm{e}^{\int -\frac{2}{t^2}dt} \\ &= \mathrm{e}^{\frac{2}{t}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mu u &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(u\,\mathrm{e}^{\frac{2}{t}}\right) &= 0 \end{aligned}$$

Integrating gives

$$\begin{aligned} u\,\mathrm{e}^{\frac{2}{t}} &= \int 0\,dt + c_1 \\ &= c_1 \end{aligned}$$

Dividing throughout by the integrating factor $\mathrm{e}^{\frac{2}{t}}$ gives the final solution

$$u(t) = \mathrm{e}^{-\frac{2}{t}}c_1$$

The ode for v now becomes

$$v'(t) = \mathrm{e}^{-\frac{2}{t}} c_1$$

Which is now solved for v. Since the ode has the form $v'(t) = f(t)$, then we only need to integrate $f(t)$.

$$\int dv = \int \mathrm{e}^{-\frac{2}{t}} c_1 \, dt$$
$$v(t) = c_1 \left(t\,\mathrm{e}^{-\frac{2}{t}} - 2 \,\mathrm{Ei}_1\left(\frac{2}{t}\right)\right) + c_2$$

$$v(t) = \mathrm{e}^{-\frac{2}{t}} c_1 t - 2 \,\mathrm{Ei}_1\left(\frac{2}{t}\right) c_1 + c_2$$

Replacing $v(t)$ above by $-\frac{y}{2}$, then the solution becomes

$$\begin{aligned} y(t) &= Bv \\ &= -2\,\mathrm{e}^{-\frac{2}{t}} c_1 t + 4 \,\mathrm{Ei}_1\left(\frac{2}{t}\right) c_1 - 2c_2 \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -2\,\mathrm{e}^{-\frac{2}{t}} c_1 t + 4 \,\mathrm{Ei}_1\left(\frac{2}{t}\right) c_1 - 2c_2$$

Solved as second order ode using Kovacic algorithm

Time used: 0.144 (sec)

Writing the ode as

$$t^2 y'' - 2y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= t^2 \\ B &= -2 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1 + 2t}{t^4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 + 2t \\ t &= t^4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \left(\frac{1 + 2t}{t^4}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A} dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.59: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 1 \\ &= 3 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = t^4$. There is a pole at $t = 0$ of order 4. Since there is no odd order pole larger than 2 and the order at ∞ is 3 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Looking at higher order poles of order $2v \geq 4$ (must be even order for case one).Then for each pole c, $[\sqrt{r}]_c$ is the sum of terms $\frac{1}{(t-c)^i}$ for $2 \leq i \leq v$ in the Laurent series expansion of $\sqrt{r}$ expanded around each pole c. Hence

$$[\sqrt{r}]_c = \sum_{2}^{v} \frac{a_i}{(t-c)^i} \tag{1B}$$

Let a be the coefficient of the term $\frac{1}{(t-c)^v}$ in the above where v is the pole order divided by 2. Let b be the coefficient of $\frac{1}{(t-c)^{v+1}}$ in r minus the coefficient of $\frac{1}{(t-c)^{v+1}}$ in $[\sqrt{r}]_c$. Then

$$\begin{aligned} \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) \end{aligned}$$

The partial fraction decomposition of r is

$$r = \frac{1}{t^4} + \frac{2}{t^3}$$

There is pole in r at $t = 0$ of order 4, hence $v = 2$. Expanding $\sqrt{r}$ as Laurent series about this pole $c = 0$ gives

$$[\sqrt{r}]_c \approx \frac{1}{t^2} + \frac{1}{t} - \frac{1}{2} + \frac{t}{2} - \frac{5t^2}{8} + \frac{7t^3}{8} + \dots \tag{2B}$$

Using eq. (1B), taking the sum up to $v = 2$ the above becomes

$$[\sqrt{r}]_c = \frac{1}{t^2} \tag{3B}$$

The above shows that the coefficient of $\frac{1}{(t-0)^2}$ is

$$a = 1$$

Now we need to find b. let b be the coefficient of the term $\frac{1}{(t-c)^{v+1}}$ in r minus the coefficient of the same term but in the sum $[\sqrt{r}]_c$ found in eq. (3B). Here c is current pole which is $c = 0$. This term becomes $\frac{1}{t^3}$. The coefficient of this term in the sum $[\sqrt{r}]_c$ is seen to be 0 and the coefficient of this term r is found from the partial fraction decomposition from above to be 2. Therefore

$$\begin{aligned} b &= (2) - (0) \\ &= 2 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_c &= \frac{1}{t^2} \\ \alpha_c^+ &= \frac{1}{2}\left(\frac{b}{a} + v\right) &&= \frac{1}{2}\left(\frac{2}{1} + 2\right) &&= 2 \\ \alpha_c^- &= \frac{1}{2}\left(-\frac{b}{a} + v\right) &&= \frac{1}{2}\left(-\frac{2}{1} + 2\right) &&= 0 \end{aligned}$$

Since the order of r at ∞ is $3 > 2$ then

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1 + 2t}{t^4}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	4	$\frac{1}{t^2}$	2	0

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
3	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 0$ then

$$\begin{aligned} d &= \alpha_\infty^+ - \left(\alpha_{c_1}^-\right) \\ &= 0 - (0) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1} \right) + (+)[\sqrt{r}]_\infty \\ &= -\frac{1}{t^2} + (0) \\ &= -\frac{1}{t^2} \\ &= -\frac{1}{t^2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0)+2\left(-\frac{1}{t^2}\right)(0)+\left(\left(\frac{2}{t^3}\right)+\left(-\frac{1}{t^2}\right)^2-\left(\frac{1+2t}{t^4}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$
\begin{aligned}
z_1(t) &= pe^{\int \omega \, dt} \\
&= \mathrm{e}^{\int -\frac{1}{t^2} dt} \\
&= \mathrm{e}^{\frac{1}{t}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{t^2} \, dt} \\
&= z_1 e^{-\frac{1}{t}} \\
&= z_1\left(\mathrm{e}^{-\frac{1}{t}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{t^2} \, dt}}{(y_1)^2} \, dt \\
&= y_1 \int \frac{e^{-\frac{2}{t}}}{(y_1)^2} \, dt \\
&= y_1\left(t\,\mathrm{e}^{-\frac{2}{t}} - 2\,\mathrm{Ei}_1\left(\frac{2}{t}\right)\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2\left(1\left(t\,\mathrm{e}^{-\frac{2}{t}} - 2\ \mathrm{Ei}_1\left(\frac{2}{t}\right)\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

<u>Summary of solutions found</u>

$$
y = c_1 + c_2\left(t\,\mathrm{e}^{-\frac{2}{t}} - 2\ \mathrm{Ei}_1\left(\frac{2}{t}\right)\right)
$$

Solved as second order ode adjoint method

Time used: 0.522 (sec)

In normal form the ode

$$
t^2 y'' - 2y' = 0 \tag{1}
$$

Becomes

$$
y'' + p(t)\,y' + q(t)\,y = r(t) \tag{2}
$$

Where

$$
\begin{aligned}
p(t) &= -\frac{2}{t^2} \\
q(t) &= 0 \\
r(t) &= 0
\end{aligned}
$$

The Lagrange adjoint ode is given by

$$
\begin{aligned}
\xi'' - (\xi\,p)' + \xi q &= 0 \\
\xi'' - \left(-\frac{2\xi(t)}{t^2}\right)' + (0) &= 0 \\
\xi''(t) - \frac{4\xi(t)}{t^3} + \frac{2\xi'(t)}{t^2} &= 0
\end{aligned}
$$

Which is solved for $\xi(t)$. An ode of the form

$$
p(t)\,\xi'' + q(t)\,\xi' + r(t)\,\xi = s(t)
$$

is exact if

$$p''(t) - q'(t) + r(t) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}
p(x) &= 1\\
q(x) &= \frac{2}{t^2}\\
r(x) &= -\frac{4}{t^3}\\
s(x) &= 0
\end{aligned}$$

Hence

$$\begin{aligned}
p''(x) &= 0\\
q'(x) &= -\frac{4}{t^3}
\end{aligned}$$

Therefore (1) becomes

$$0 - \left(-\frac{4}{t^3}\right) + \left(-\frac{4}{t^3}\right) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$\left(p(t)\, \xi' + (q(t) - p'(t))\, \xi\right)' = s(x)$$

Integrating gives

$$p(t)\, \xi' + (q(t) - p'(t))\, \xi = \int s(t)\, dt$$

Substituting the above values for p, q, r, s gives

$$\xi' + \frac{2\xi}{t^2} = c_1$$

We now have a first order ode to solve which is

$$\xi' + \frac{2\xi}{t^2} = c_1$$

In canonical form a linear first order is

$$\xi' + q(t)\xi = p(t)$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(t) &= \frac{2}{t^2} \\
p(t) &= c_1
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\, dt} \\
&= \mathrm{e}^{\int \frac{2}{t^2} dt} \\
&= \mathrm{e}^{-\frac{2}{t}}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu \xi) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}t}(\mu \xi) &= (\mu)\,(c_1) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\xi\, \mathrm{e}^{-\frac{2}{t}}\right) &= \left(\mathrm{e}^{-\frac{2}{t}}\right)(c_1) \\
\mathrm{d}\left(\xi\, \mathrm{e}^{-\frac{2}{t}}\right) &= \left(\mathrm{e}^{-\frac{2}{t}} c_1\right)\, \mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\xi\, \mathrm{e}^{-\frac{2}{t}} &= \int \mathrm{e}^{-\frac{2}{t}} c_1\, dt \\
&= c_1 \left(t\, \mathrm{e}^{-\frac{2}{t}} - 2\, \mathrm{Ei}_1\left(\frac{2}{t}\right)\right) + c_2
\end{aligned}
$$

Dividing throughout by the integrating factor $\mathrm{e}^{-\frac{2}{t}}$ gives the final solution

$$
\xi = -2\, \mathrm{e}^{\frac{2}{t}}\, \mathrm{Ei}_1\left(\frac{2}{t}\right) c_1 + c_2\, \mathrm{e}^{\frac{2}{t}} + c_1 t
$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$
\begin{aligned}
\xi(t)\, y' - y\xi'(t) + \xi(t)\, p(t)\, y &= \int \xi(t)\, r(t)\, dt \\
y' + y\left(p(t) - \frac{\xi'(t)}{\xi\,(t)}\right) &= \frac{\int \xi(t)\, r(t)\, dt}{\xi\,(t)}
\end{aligned}
$$

Or

$$y' + y \left(-\frac{2}{t^2} - \frac{\frac{4 e^{\frac{2}{t}} \mathrm{Ei}_1\left(\frac{2}{t}\right) c_1}{t^2} - \frac{2 e^{\frac{2}{t}} e^{-\frac{2}{t}} c_1}{t} - \frac{2 c_2 e^{\frac{2}{t}}}{t^2} + c_1}{-2 e^{\frac{2}{t}} \mathrm{Ei}_1\left(\frac{2}{t}\right) c_1 + c_2 e^{\frac{2}{t}} + c_1 t} \right) = 0$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(t)y = p(t)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(t) &= -\frac{c_1}{-2 e^{\frac{2}{t}} \mathrm{Ei}_1\left(\frac{2}{t}\right) c_1 + c_2 e^{\frac{2}{t}} + c_1 t} \\ p(t) &= 0 \end{aligned}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{c_1}{-2 e^{\frac{2}{t}} \mathrm{Ei}_1\left(\frac{2}{t}\right) c_1 + c_2 e^{\frac{2}{t}} + c_1 t} dt}$$

Therefore the solution is

$$y = c_3 e^{-\int -\frac{c_1}{-2 e^{\frac{2}{t}} \mathrm{Ei}_1\left(\frac{2}{t}\right) c_1 + c_2 e^{\frac{2}{t}} + c_1 t} dt}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = c_3 e^{-\int -\frac{c_1}{-2 e^{\frac{2}{t}} \mathrm{Ei}_1\left(\frac{2}{t}\right) c_1 + c_2 e^{\frac{2}{t}} + c_1 t} dt}$$

The constants can be merged to give

$$y = e^{-\int -\frac{c_1}{-2 e^{\frac{2}{t}} \mathrm{Ei}_1\left(\frac{2}{t}\right) c_1 + c_2 e^{\frac{2}{t}} + c_1 t} dt}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = e^{-\int -\frac{c_1}{-2 e^{\frac{2}{t}} \mathrm{Ei}_1\left(\frac{2}{t}\right) c_1 + c_2 e^{\frac{2}{t}} + c_1 t} dt}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
<- LODE missing y successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 25

```
dsolve(diff(diff(y(t),t),t)*t^2-2*diff(y(t),t) = 0,
       y(t),singsol=all)
```

$$y = \mathrm{e}^{-\frac{2}{t}} c_2 t - 2\,\mathrm{Ei}_1\left(\frac{2}{t}\right) c_2 + c_1$$

Mathematica DSolve solution

Solving time : 0.018 (sec)
Leaf size : 29

```
DSolve[{t^2*D[y[t],{t,2}]-2*D[y[t],t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to 2c_1 \,\mathrm{ExpIntegralEi}\left(-\frac{2}{t}\right) + c_1 e^{-2/t} t + c_2$$

2.1.52 problem 52

Internal problem ID [8440]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 52
Date solved : `Wednesday, December 18, 2024 at 01:53:06 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + \frac{(t^2-1)\,y'}{t} + \frac{t^2 y}{\left(1+\mathrm{e}^{\frac{t^2}{2}}\right)^2} = 0$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
-> Trying changes of variables to rationalize or make the ODE simpler
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying 2nd order exact linear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   <- to_const_coeffs successful: conversion to a linear ODE with constant coefficient
```

Maple dsolve solution

Solving time : 0.056 (sec)
Leaf size : 80

```
dsolve(diff(diff(y(t),t),t)+(t^2-1)/t*diff(y(t),t)+t^2/(1+exp(1/2*t^2))^2*y(t) = 0,
       y(t),singsol=all)
```

$$y = \frac{\left(c_1 \left(1+\mathrm{e}^{\frac{t^2}{2}}\right)^{-\frac{i\sqrt{3}}{2}} \left(\mathrm{e}^{\frac{t^2}{2}}\right)^{\frac{i\sqrt{3}}{2}} + c_2 \left(1+\mathrm{e}^{\frac{t^2}{2}}\right)^{\frac{i\sqrt{3}}{2}} \left(\mathrm{e}^{\frac{t^2}{2}}\right)^{-\frac{i\sqrt{3}}{2}}\right) \sqrt{1+\mathrm{e}^{\frac{t^2}{2}}}}{\sqrt{\mathrm{e}^{\frac{t^2}{2}}}}$$

Mathematica DSolve solution

Solving time : 0.141 (sec)
Leaf size : 72

```
DSolve[{D[y[t],{t,2}]+(t^2-1)/t*D[y[t],t]+t^2/(1 + Exp[t^2/2])^2*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to e^{\operatorname{arctanh}\left(2e^{\frac{t^2}{2}}+1\right)} \left(c_2 \cos\left(\sqrt{3}\operatorname{arctanh}\left(2e^{\frac{t^2}{2}}+1\right)\right) - c_1 \sin\left(\sqrt{3}\operatorname{arctanh}\left(2e^{\frac{t^2}{2}}+1\right)\right)\right)$$

2.1.53 problem 53

Internal problem ID [8441]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 53
Date solved : Thursday, December 12, 2024 at 09:08:08 AM
CAS classification :
`[[_Emden, _Fowler], [_2nd_order, _linear, '_with_symmetry_[0,F(x)]']]`

Solve

$$ty'' - y' + 4t^3y = 0$$

Solved as second order ode using change of variable on x method 2

Time used: 0.324 (sec)

In normal form the ode

$$ty'' - y' + 4t^3y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$p(t) = -\frac{1}{t}$$
$$q(t) = 4t^2$$

Applying change of variables $\tau = g(t)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(t) + p(t)\,\tau'(t) = 0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int \mathrm{e}^{-\int p(t)dt}dt \\ &= \int \mathrm{e}^{-\int -\frac{1}{t}dt}dt \\ &= \int e^{\ln(t)}\,dt \\ &= \int tdt \\ &= \frac{t^2}{2}\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}q_1(\tau) &= \frac{q(t)}{\tau'(t)^2} \\ &= \frac{4t^2}{t^2} \\ &= 4\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}\frac{d^2}{d\tau^2}y(\tau) + q_1y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + 4y(\tau) &= 0\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\tau\lambda} + 4\,\mathrm{e}^{\tau\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)} \\
&= \pm 2i
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +2i \\
\lambda_2 &= -2i
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 2i \\
\lambda_2 &= -2i
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^0(c_1 \cos(2\tau) + c_2 \sin(2\tau))$$

Or

$$y(\tau) = c_1 \cos(2\tau) + c_2 \sin(2\tau)$$

Will add steps showing solving for IC soon.

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos\left(t^2\right) + c_2 \sin\left(t^2\right)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \cos\left(t^2\right) + c_2 \sin\left(t^2\right)$$

Solved as second order ode using change of variable on x method 1

Time used: 0.099 (sec)

In normal form the ode

$$ty'' - y' + 4t^3 y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(t) &= -\frac{1}{t} \\ q(t) &= 4t^2 \end{aligned}$$

Applying change of variables $\tau = g(t)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(t) + p(t)\, \tau'(t)}{\tau'(t)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(t)}{\tau'(t)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{t^2}}{c} \\ \tau'' &= \frac{2t}{c\sqrt{t^2}}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(t) + p(t)\,\tau'(t)}{\tau'(t)^2} \\ &= \frac{\frac{2t}{c\sqrt{t^2}} - \frac{1}{t}\frac{2\sqrt{t^2}}{c}}{\left(\frac{2\sqrt{t^2}}{c}\right)^2} \\ &= 0\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\,dt \\ &= \frac{\int 2\sqrt{t^2}dt}{c} \\ &= \frac{t\sqrt{t^2}}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos\left(t\sqrt{t^2}\right) + c_2 \sin\left(t\sqrt{t^2}\right)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \cos\left(t\sqrt{t^2}\right) + c_2 \sin\left(t\sqrt{t^2}\right)$$

Solved as second order Bessel ode

Time used: 0.061 (sec)

Writing the ode as

$$t^2y'' - y't + 4t^4y = 0 \tag{1}$$

Bessel ode has the form

$$t^2y'' + y't + \left(-n^2 + t^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$t^2y'' + (1 - 2\alpha)\, ty' + \left(\beta^2\gamma^2 t^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = t^{\alpha}(c_1 \,\text{BesselJ}\,(n, \beta\, t^{\gamma}) + c_2 \,\text{BesselY}\,(n, \beta\, t^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= 1\\ \beta &= 1\\ n &= \frac{1}{2}\\ \gamma &= 2\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{c_1 t\sqrt{2}\, \sin\left(t^2\right)}{\sqrt{\pi}\, \sqrt{t^2}} - \frac{c_2 t\sqrt{2}\, \cos\left(t^2\right)}{\sqrt{\pi}\, \sqrt{t^2}}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{c_1 t\sqrt{2}\, \sin\left(t^2\right)}{\sqrt{\pi}\, \sqrt{t^2}} - \frac{c_2 t\sqrt{2}\, \cos\left(t^2\right)}{\sqrt{\pi}\, \sqrt{t^2}}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.299 (sec)

Writing the ode as

$$
\begin{aligned}
ty'' - y' + 4t^3y &= 0 && (1)\\
Ay'' + By' + Cy &= 0 && (2)
\end{aligned}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= t\\
B &= -1 \qquad (3)\\
C &= 4t^3
\end{aligned}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(t) = ye^{\int \frac{B}{2A}\,dt}
$$

Then (2) becomes

$$
z''(t) = rz(t) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \qquad (5)\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-16t^4 + 3}{4t^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -16t^4 + 3\\
t &= 4t^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(t) = \left(\frac{-16t^4 + 3}{4t^2}\right) z(t) \tag{7}
$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.60: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4t^2$. There is a pole at $t = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -4t^2 + \frac{3}{4t^2}$$

For the pole at $t=0$ let b be the coefficient of $\frac{1}{t^2}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{3}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0\\
\alpha_c^+ &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{2}\\
\alpha_c^- &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is $O_r(\infty)=-2$ then

$$
v=\frac{-O_r(\infty)}{2}=\frac{2}{2}=1
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving t^i for $0\leq i\leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i t^i\\
&= \sum_{i=0}^{1} a_i t^i \qquad (8)
\end{aligned}
$$

Let a be the coefficient of $t^v=t^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r}\approx 2it-\frac{3i}{16t^3}-\frac{9i}{1024t^7}-\frac{27i}{32768t^{11}}-\frac{405i}{4194304t^{15}}-\frac{1701i}{134217728t^{19}}-\frac{15309i}{8589934592t^{23}}-\frac{72171i}{274877906944t^{27}}+\dots \qquad (9)
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a=2i
$$

From Eq. (9) the sum up to $v=1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i t^i\\
&= 2it \qquad (10)
\end{aligned}
$$

Now we need to find b, where b be the coefficient of $t^{v-1}=t^0=1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2=-4t^2
$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v=0$ or not. Since $v=1$ which is not zero, then starting $r=\frac{s}{t}$, we do long division and write this in the form

$$
r=Q+\frac{R}{t}
$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{-16t^4+3}{4t^2} \\
&= Q + \frac{R}{4t^2} \\
&= (-4t^2) + \left(\frac{3}{4t^2}\right) \\
&= -4t^2 + \frac{3}{4t^2}
\end{aligned}$$

We see that the coefficient of the term t in the quotient is 0. Now b can be found.

$$\begin{aligned}
b &= (0) - (0) \\
&= 0
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 2it \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{2i} - 1\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{2i} - 1\right) &&= -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{-16t^4+3}{4t^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$2it$	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{t-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{t-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2t} + (-)\,(2it) \\
&= -\frac{1}{2t} - 2it \\
&= -\frac{1}{2t} - 2it
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(t)$ of degree $d = 0$ to solve the ode. The polynomial $p(t)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(t) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{2t} - 2it\right)(0) + \left(\left(\frac{1}{2t^2} - 2i\right) + \left(-\frac{1}{2t} - 2it\right)^2 - \left(\frac{-16t^4+3}{4t^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(t) &= p e^{\int \omega\,dt} \\
&= e^{\int \left(-\frac{1}{2t} - 2it\right)dt} \\
&= \frac{e^{-it^2}}{\sqrt{t}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{t}\,dt} \\
&= z_1 e^{\frac{\ln(t)}{2}} \\
&= z_1\left(\sqrt{t}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-i t^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{t}\,dt}}{(y_1)^2}\,dt \\
&= y_1 \int \frac{e^{\ln(t)}}{(y_1)^2}\,dt \\
&= y_1\left(-\frac{i\,\mathrm{e}^{2 i t^2}}{4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-i t^2}\right) + c_2\left(\mathrm{e}^{-i t^2}\left(-\frac{i\,\mathrm{e}^{2 i t^2}}{4}\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1\,\mathrm{e}^{-i t^2} - \frac{i c_2\,\mathrm{e}^{i t^2}}{4}$$

Solved as second order ode adjoint method

Time used: 0.463 (sec)

In normal form the ode

$$ty'' - y' + 4t^3y = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = r(t) \tag{2}$$

Where

$$\begin{aligned} p(t) &= -\frac{1}{t} \\ q(t) &= 4t^2 \\ r(t) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - \left(-\frac{\xi(t)}{t}\right)' + \left(4t^2\xi(t)\right) &= 0 \\ \xi''(t) + \frac{\xi'(t)}{t} + \frac{(4t^4 - 1)\,\xi(t)}{t^2} &= 0 \end{aligned}$$

Which is solved for $\xi(t)$. Writing the ode as

$$\xi''t^2 + \xi't + \left(4t^4 - 1\right)\xi = 0 \tag{1}$$

Bessel ode has the form

$$\xi''t^2 + \xi't + \left(-n^2 + t^2\right)\xi = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$\xi''t^2 + (1 - 2\alpha)\, t\xi' + \left(\beta^2\gamma^2t^{2\gamma} - n^2\gamma^2 + \alpha^2\right)\xi = 0 \tag{3}$$

With the standard solution

$$\xi = t^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, t^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, t^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$
\begin{aligned}
\alpha &= 0 \\
\beta &= 1 \\
n &= -\frac{1}{2} \\
\gamma &= 2
\end{aligned}
$$

Substituting all the above into (4) gives the solution as

$$
\xi = \frac{c_1\sqrt{2}\,\cos\left(t^2\right)}{\sqrt{\pi}\,\sqrt{t^2}} + \frac{c_2\sqrt{2}\,\sin\left(t^2\right)}{\sqrt{\pi}\,\sqrt{t^2}}
$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$
\begin{aligned}
\xi(t)\,y' - y\xi'(t) + \xi(t)\,p(t)\,y &= \int \xi(t)\,r(t)\,dt \\
y' + y\left(p(t) - \frac{\xi'(t)}{\xi\,(t)}\right) &= \frac{\int \xi(t)\,r(t)\,dt}{\xi\,(t)}
\end{aligned}
$$

Or

$$
y' + y\left(-\frac{1}{t} - \frac{-\frac{c_1\sqrt{2}\,\cos(t^2)t}{\sqrt{\pi}\,(t^2)^{3/2}} - \frac{2c_1t\sqrt{2}\,\sin(t^2)}{\sqrt{\pi}\,\sqrt{t^2}} - \frac{c_2\sqrt{2}\,\sin(t^2)t}{\sqrt{\pi}\,(t^2)^{3/2}} + \frac{2c_2t\sqrt{2}\,\cos(t^2)}{\sqrt{\pi}\,\sqrt{t^2}}}{\frac{c_1\sqrt{2}\,\cos(t^2)}{\sqrt{\pi}\,\sqrt{t^2}} + \frac{c_2\sqrt{2}\,\sin(t^2)}{\sqrt{\pi}\,\sqrt{t^2}}}\right) = 0
$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$
y' + q(t)y = p(t)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(t) &= -\frac{2t(\cos\left(t^2\right)c_2 - \sin\left(t^2\right)c_1)}{c_1\cos\left(t^2\right) + c_2\sin\left(t^2\right)} \\
p(t) &= 0
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\,dt} \\
&= \mathrm{e}^{\int -\frac{2t\left(\cos\left(t^2\right)c_2 - \sin\left(t^2\right)c_1\right)}{c_1\cos\left(t^2\right) + c_2\sin\left(t^2\right)}dt} \\
&= \frac{1}{c_1\cos\left(t^2\right) + c_2\sin\left(t^2\right)}
\end{aligned}
$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu y = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y}{c_1\cos(t^2) + c_2\sin(t^2)}\right) = 0$$

Integrating gives

$$\frac{y}{c_1\cos(t^2) + c_2\sin(t^2)} = \int 0\,dt + c_3$$
$$= c_3$$

Dividing throughout by the integrating factor $\frac{1}{c_1\cos(t^2)+c_2\sin(t^2)}$ gives the final solution

$$y = \left(c_1\cos\left(t^2\right) + c_2\sin\left(t^2\right)\right)c_3$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \left(c_1\cos\left(t^2\right) + c_2\sin\left(t^2\right)\right)c_3$$

The constants can be merged to give

$$y = c_1\cos\left(t^2\right) + c_2\sin\left(t^2\right)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1\cos\left(t^2\right) + c_2\sin\left(t^2\right)$$

Maple step by step solution

Let's solve
$t\left(\frac{d^2}{dt^2}y(t)\right) - \frac{d}{dt}y(t) + 4t^3y(t) = 0$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dt^2}y(t)$
- Isolate 2nd derivative
$\frac{d^2}{dt^2}y(t) = -4t^2y(t) + \frac{\frac{d}{dt}y(t)}{t}$

- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dt^2}y(t) - \frac{\frac{d}{dt}y(t)}{t} + 4t^2y(t) = 0$

□ Check to see if $t_0 = 0$ is a regular singular point

- Define functions

$\left[P_2(t) = -\frac{1}{t}, P_3(t) = 4t^2\right]$

- $t \cdot P_2(t)$ is analytic at $t = 0$

$$\left.(t \cdot P_2(t))\right|_{t=0} = -1$$

- $t^2 \cdot P_3(t)$ is analytic at $t = 0$

$$\left.(t^2 \cdot P_3(t))\right|_{t=0} = 0$$

- $t = 0$is a regular singular point

Check to see if $t_0 = 0$ is a regular singular point

$t_0 = 0$

- Multiply by denominators

$t\left(\frac{d^2}{dt^2}y(t)\right) - \frac{d}{dt}y(t) + 4t^3y(t) = 0$

- Assume series solution for $y(t)$

$$y(t) = \sum_{k=0}^{\infty} a_k t^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $t^3 \cdot y(t)$ to series expansion

$$t^3 \cdot y(t) = \sum_{k=0}^{\infty} a_k t^{k+r+3}$$

- Shift index using $k- >k - 3$

$$t^3 \cdot y(t) = \sum_{k=3}^{\infty} a_{k-3} t^{k+r}$$

- Convert $\frac{d}{dt}y(t)$ to series expansion

$$\frac{d}{dt}y(t) = \sum_{k=0}^{\infty} a_k(k+r)\, t^{k+r-1}$$

- Shift index using $k- >k + 1$

$$\frac{d}{dt}y(t) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, t^{k+r}$$

- Convert $t \cdot \left(\frac{d^2}{dt^2}y(t)\right)$ to series expansion

$t \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, t^{k+r-1}$

- Shift index using $k- > k+1$

  $t \cdot \left(\frac{d^2}{dt^2}y(t)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, t^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r(-2+r)\, t^{-1+r} + a_1(1+r)(-1+r)\, t^r + a_2(2+r)\, r\, t^{1+r} + a_3(3+r)(1+r)\, t^{2+r} + \left(\sum_{k=3}^{\infty} (a_{k+1}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(-2+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{0, 2\}$
- The coefficients of each power of t must be 0

  $[a_1(1+r)(-1+r) = 0, a_2(2+r)\, r = 0, a_3(3+r)(1+r) = 0]$
- Solve for the dependent coefficient(s)

  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1+r)(k+r-1) + 4a_{k-3} = 0$
- Shift index using $k- > k+3$

  $a_{k+4}(k+4+r)(k+2+r) + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+4} = -\frac{4a_k}{(k+4+r)(k+2+r)}$
- Recursion relation for $r = 0$

  $a_{k+4} = -\frac{4a_k}{(k+4)(k+2)}$
- Solution for $r = 0$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^k, a_{k+4} = -\frac{4a_k}{(k+4)(k+2)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$
- Recursion relation for $r = 2$

  $a_{k+4} = -\frac{4a_k}{(k+6)(k+4)}$
- Solution for $r = 2$

  $\left[y(t) = \sum_{k=0}^{\infty} a_k t^{k+2}, a_{k+4} = -\frac{4a_k}{(k+6)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$
- Combine solutions and rename parameters

  $\left[y(t) = \left(\sum_{k=0}^{\infty} a_k t^k\right) + \left(\sum_{k=0}^{\infty} b_k t^{k+2}\right), a_{4+k} = -\frac{4a_k}{(4+k)(k+2)}, a_1 = 0, a_2 = 0, a_3 = 0, b_{4+k} = -\frac{4b_k}{(k+6)(4+\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 17

```
dsolve(t*diff(diff(y(t),t),t)-diff(y(t),t)+4*t^3*y(t) = 0,
       y(t),singsol=all)
```

$$y = c_1 \sin\left(t^2\right) + c_2 \cos\left(t^2\right)$$

Mathematica DSolve solution

Solving time : 0.02 (sec)
Leaf size : 20

```
DSolve[{t*D[y[t],{t,2}]-D[y[t],t]+4*t^3*y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to c_1 \cos\left(t^2\right) + c_2 \sin\left(t^2\right)$$

2.1.54 problem 54

Internal problem ID [8442]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 54
Date solved : `Thursday, December 12, 2024 at 09:08:10 AM`
CAS classification : `[[_2nd_order, _quadrature]]`

Solve

$$y'' = 0$$

Solved as second order ode quadrature

Time used: 0.020 (sec)

Integrating twice gives the solution

$$y = c_1 t + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 t + c_2$$

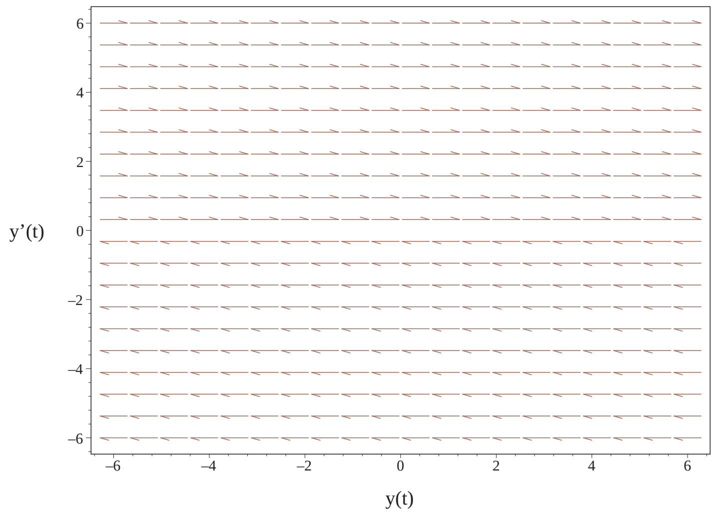


Figure 2.126: Slope field plot
$y'' = 0$

Solved as second order linear constant coeff ode

Time used: 0.043 (sec)

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 0, C = 0$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(0)^2 - (4)(1)(0)} \\ &= 0\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 0$. Therefore the solution is

$$y = c_1 1 + c_2 t \tag{1}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_2 t + c_1$$

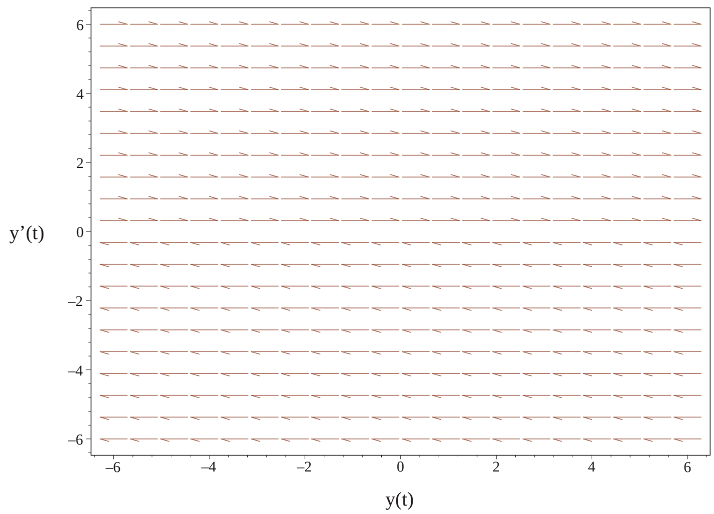


Figure 2.127: Slope field plot
$y'' = 0$

Solved as second order linear exact ode

Time used: 0.047 (sec)

An ode of the form

$$p(t)\, y'' + q(t)\, y' + r(t)\, y = s(t)$$

is exact if

$$p''(t) - q'(t) + r(t) = 0 \tag{1}$$

For the given ode we have

$$p(x) = 1$$
$$q(x) = 0$$
$$r(x) = 0$$
$$s(x) = 0$$

Hence

$$
\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 0
\end{aligned}
$$

Therefore (1) becomes

$$
0 - (0) + (0) = 0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(t)\, y' + (q(t) - p'(t))\, y)' = s(x)
$$

Integrating gives

$$
p(t)\, y' + (q(t) - p'(t))\, y = \int s(t)\, dt
$$

Substituting the above values for p, q, r, s gives

$$
y' = c_1
$$

We now have a first order ode to solve which is

$$
y' = c_1
$$

Since the ode has the form $y' = f(t)$, then we only need to integrate $f(t)$.

$$
\begin{aligned}
\int dy &= \int c_1\, dt \\
y &= c_1 t + c_2
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = c_1 t + c_2
$$

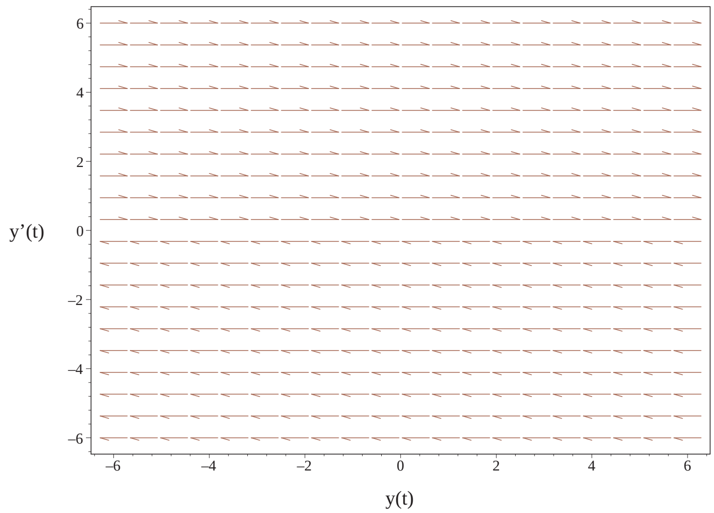


Figure 2.128: Slope field plot
$y'' = 0$

Solved as second order missing y ode

Time used: 0.039 (sec)

This is second order ode with missing dependent variable y. Let

$$p(t) = y'$$

Then

$$p'(t) = y''$$

Hence the ode becomes

$$p'(t) = 0$$

Which is now solve for $p(t)$ as first order ode. Since the ode has the form $p'(t) = f(t)$, then we only need to integrate $f(t)$.

$$\int dp = \int 0\, dt + c_1$$
$$p(t) = c_1$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = c_1$$

Since the ode has the form $y' = f(t)$, then we only need to integrate $f(t)$.

$$\int dy = \int c_1 \, dt$$
$$y = c_1 t + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 t + c_2$$

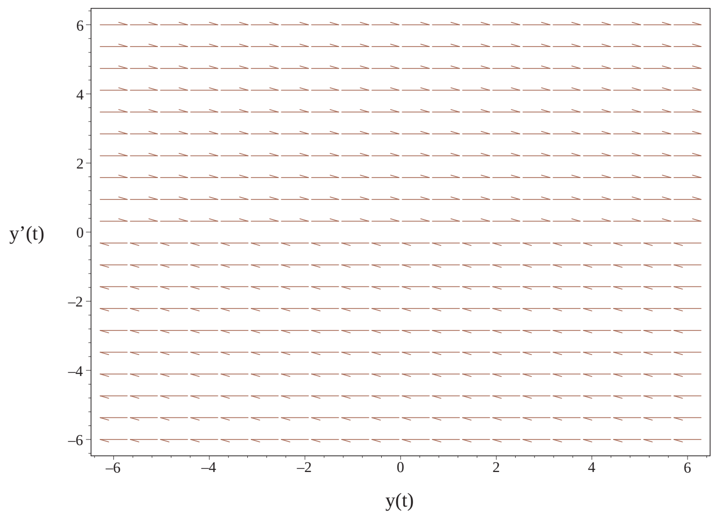


Figure 2.129: Slope field plot
$y'' = 0$

Solved as second order integrable as is ode

Time used: 0.024 (sec)

Integrating both sides of the ODE w.r.t t gives

$$\int y'' dt = 0$$
$$y' = c_1$$

Which is now solved for y. Since the ode has the form $y' = f(t)$, then we only need to integrate $f(t)$.

$$\int dy = \int c_1 \, dt$$
$$y = c_1 t + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 t + c_2$$

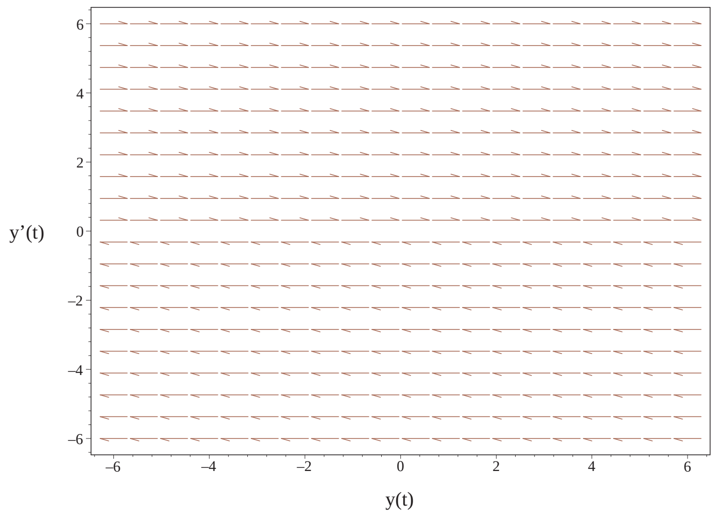


Figure 2.130: Slope field plot
$y'' = 0$

Solved as second order integrable as is ode (ABC method)

Time used: 0.024 (sec)

Writing the ode as

$$y'' = 0$$

Integrating both sides of the ODE w.r.t t gives

$$\int y'' dt = 0$$
$$y' = c_1$$

Which is now solved for y. Since the ode has the form $y' = f(t)$, then we only need to integrate $f(t)$.

$$\int dy = \int c_1 \, dt$$
$$y = c_1 t + c_2$$

Will add steps showing solving for IC soon.

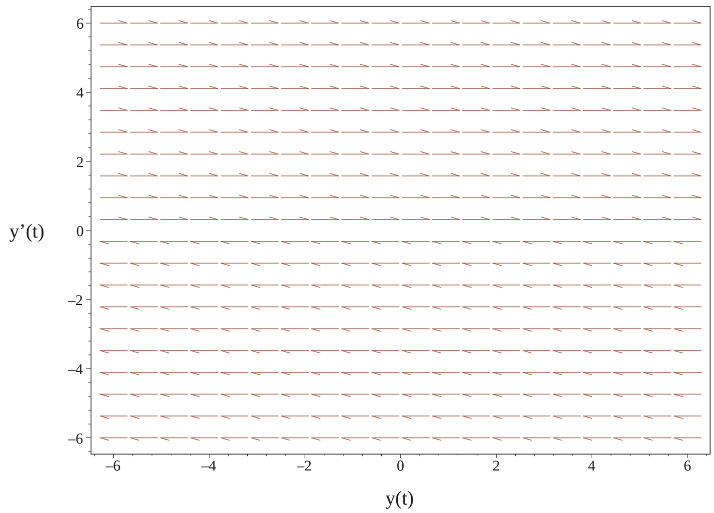


Figure 2.131: Slope field plot
$y'' = 0$

Solved as second order can be made integrable

Time used: 0.252 (sec)

Multiplying the ode by y' gives

$$y'y'' = 0$$

Integrating the above w.r.t t gives

$$\int y'y''dt = 0$$
$$\frac{y'^2}{2} = c_1$$

Which is now solved for y. Solving for the derivative gives these ODE's to solve

$$y' = \sqrt{2}\,\sqrt{c_1} \tag{1}$$
$$y' = -\sqrt{2}\,\sqrt{c_1} \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Since the ode has the form $y' = f(t)$, then we only need to integrate $f(t)$.

$$\int dy = \int \sqrt{2}\,\sqrt{c_1}\,dt$$
$$y = \sqrt{2}\,\sqrt{c_1}\,t + c_2$$

Solving Eq. (2)

Since the ode has the form $y' = f(t)$, then we only need to integrate $f(t)$.

$$\int dy = \int -\sqrt{2}\,\sqrt{c_1}\,dt$$
$$y = -\sqrt{2}\,\sqrt{c_1}\,t + c_3$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\sqrt{2}\,\sqrt{c_1}\,t + c_3$$
$$y = \sqrt{2}\,\sqrt{c_1}\,t + c_2$$

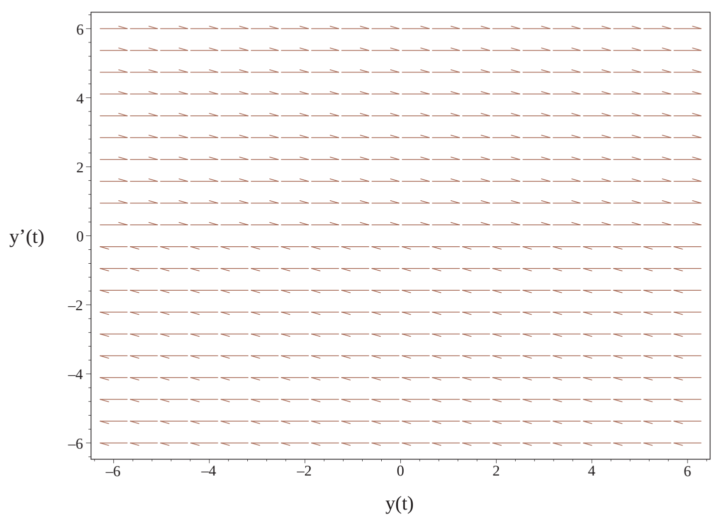


Figure 2.132: Slope field plot
$y'' = 0$

Solved as second order ode using Kovacic algorithm

Time used: 0.028 (sec)

Writing the ode as

$$y'' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t) \, e^{-\int \frac{B}{2A} \, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.62: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= 1
\end{aligned}
$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{y_1^2}\, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dt \\ &= 1 \int \frac{1}{1}\, dt \\ &= 1(t) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2(1(t)) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_2 t + c_1$$

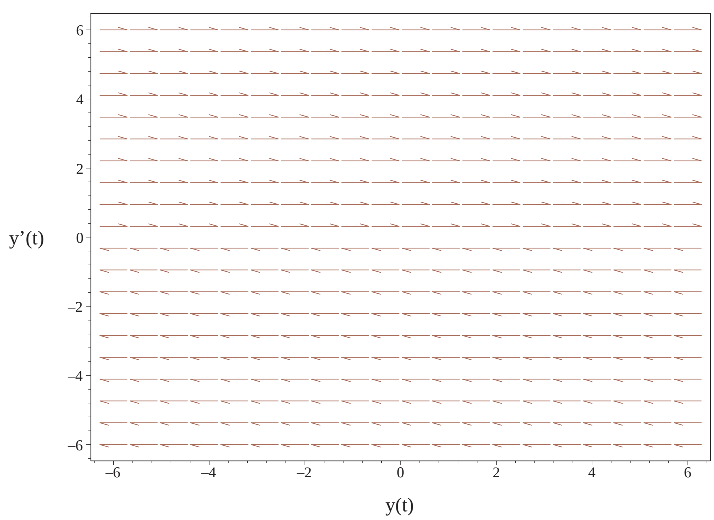


Figure 2.133: Slope field plot
$y'' = 0$

Solved as second order ode adjoint method

Time used: 0.363 (sec)

In normal form the ode

$$y'' = 0 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = r(t) \tag{2}$$

Where

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 0 \\ r(t) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi^{''} - (\xi\, p)' + \xi q &= 0 \\ \xi^{''} - (0)' + (0) &= 0 \\ \xi''(t) &= 0 \end{aligned}$$

Which is solved for $\xi(t)$. Integrating twice gives the solution

$$\xi = c_1 t + c_2$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned} \xi(t)\, y' - y\xi'(t) + \xi(t)\, p(t)\, y &= \int \xi(t)\, r(t)\, dt \\ y' + y\left(p(t) - \frac{\xi'(t)}{\xi\,(t)}\right) &= \frac{\int \xi(t)\, r(t)\, dt}{\xi\,(t)} \end{aligned}$$

Or

$$y' - \frac{y c_1}{c_1 t + c_2} = 0$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(t)y = p(t)$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(t) &= -\frac{c_1}{c_1 t + c_2} \\
p(t) &= 0
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q \, dt} \\
&= \mathrm{e}^{\int -\frac{c_1}{c_1 t + c_2} dt} \\
&= \frac{1}{c_1 t + c_2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mu y &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y}{c_1 t + c_2}\right) &= 0
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{c_1 t + c_2} &= \int 0 \, dt + c_3 \\
&= c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{1}{c_1 t + c_2}$ gives the final solution

$$
y = (c_1 t + c_2) c_3
$$

Hence, the solution found using Lagrange adjoint equation method is

$$
y = (c_1 t + c_2) c_3
$$

The constants can be merged to give

$$
y = c_1 t + c_2
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = c_1 t + c_2
$$

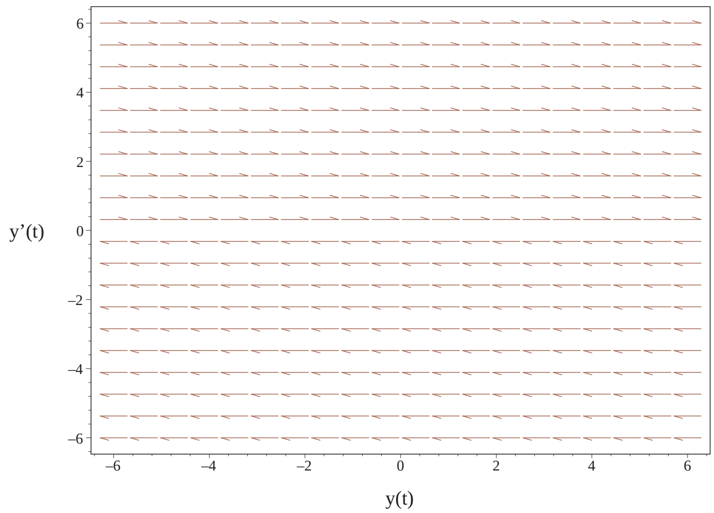


Figure 2.134: Slope field plot
$y'' = 0$

Maple step by step solution

Let's solve

$\frac{d^2}{dt^2}y(t) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$

- Characteristic polynomial of ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{0})}{2}$

- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the ODE

  $y_1(t) = 1$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t$

- General solution of the ODE

  $y(t) = C1y_1(t) + C2y_2(t)$

- Substitute in solutions

  $y(t) = C2t + C1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 9

```
dsolve(diff(diff(y(t),t),t) = 0,
       y(t),singsol=all)
```

$$y = c_1 t + c_2$$

Mathematica DSolve solution

Solving time : 0.002 (sec)
Leaf size : 12

```
DSolve[{D[y[t],{t,2}]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to c_2 t + c_1$$

2.1.55 problem 55

Internal problem ID [8443]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 55
Date solved : `Thursday, December 12, 2024 at 09:08:11 AM`
CAS classification : `[[_2nd_order, _quadrature]]`

Solve

$$y'' = 1$$

Solved as second order ode quadrature

Time used: 0.027 (sec)

The ODE can be written as

$$y'' = 1$$

Integrating once gives

$$y' = t + c_1$$

Integrating again gives

$$y = \frac{t^2}{2} + c_1 x + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{1}{2}t^2 + c_1 t + c_2$$

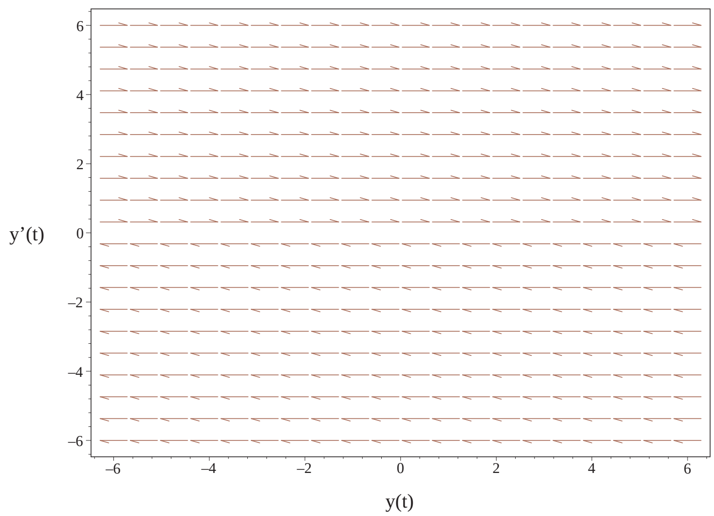


Figure 2.135: Slope field plot
$y'' = 1$

Solved as second order linear constant coeff ode

Time used: 0.091 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 0, C = 0, f(t) = 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 0, C = 0$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(0)^2 - (4)(1)(0)}\\ &= 0\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 0$. Therefore the solution is

$$y = c_1 1 + c_2 t \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_2 t + c_1$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, t\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t\}]$$

Since t is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 t^2$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{t^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 t + c_1) + \left(\frac{t^2}{2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{1}{2}t^2 + c_2 t + c_1$$

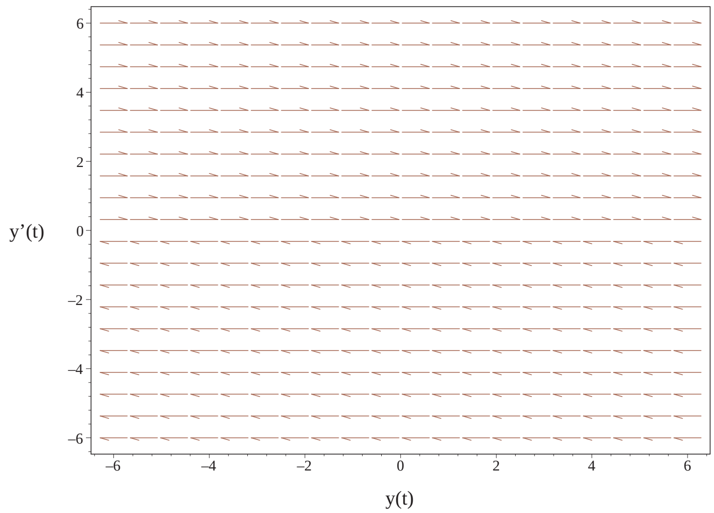


Figure 2.136: Slope field plot
$y'' = 1$

Solved as second order linear exact ode

Time used: 0.058 (sec)

An ode of the form

$$p(t)\,y'' + q(t)\,y' + r(t)\,y = s(t)$$

is exact if

$$p''(t) - q'(t) + r(t) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 0 \\ r(x) &= 0 \\ s(x) &= 1 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(t)\,y' + (q(t) - p'(t))\,y)' = s(x)$$

Integrating gives

$$p(t)\,y' + (q(t) - p'(t))\,y = \int s(t)\;dt$$

Substituting the above values for p, q, r, s gives

$$y' = \int 1\,dt$$

We now have a first order ode to solve which is

$$y' = t + c_1$$

Since the ode has the form $y' = f(t)$, then we only need to integrate $f(t)$.

$$\int dy = \int t + c_1\,dt$$

$$y = \frac{1}{2}t^2 + c_1 t + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{1}{2}t^2 + c_1 t + c_2$$

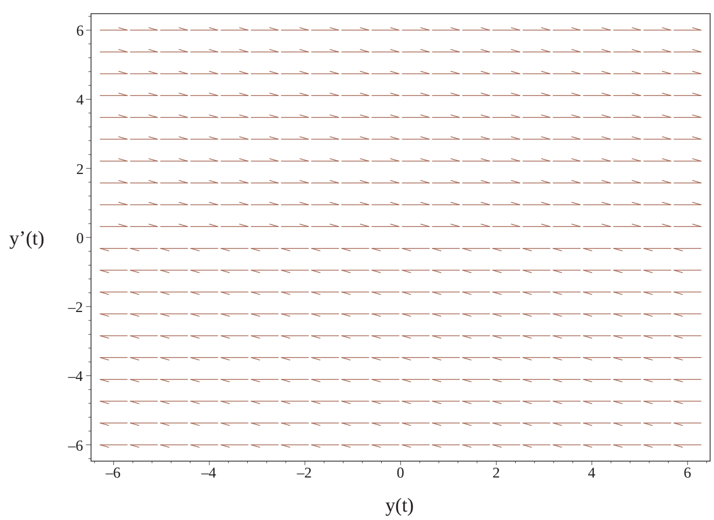


Figure 2.137: Slope field plot
$y'' = 1$

Solved as second order missing y ode

Time used: 0.043 (sec)

This is second order ode with missing dependent variable y. Let

$$p(t) = y'$$

Then

$$p'(t) = y''$$

Hence the ode becomes

$$p'(t) - 1 = 0$$

Which is now solve for $p(t)$ as first order ode. Since the ode has the form $p'(t) = f(t)$, then we only need to integrate $f(t)$.

$$\int dp = \int 1\, dt$$
$$p(t) = t + c_1$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = t + c_1$$

Since the ode has the form $y' = f(t)$, then we only need to integrate $f(t)$.

$$\int dy = \int t + c_1\, dt$$
$$y = \frac{1}{2}t^2 + c_1 t + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{1}{2}t^2 + c_1 t + c_2$$

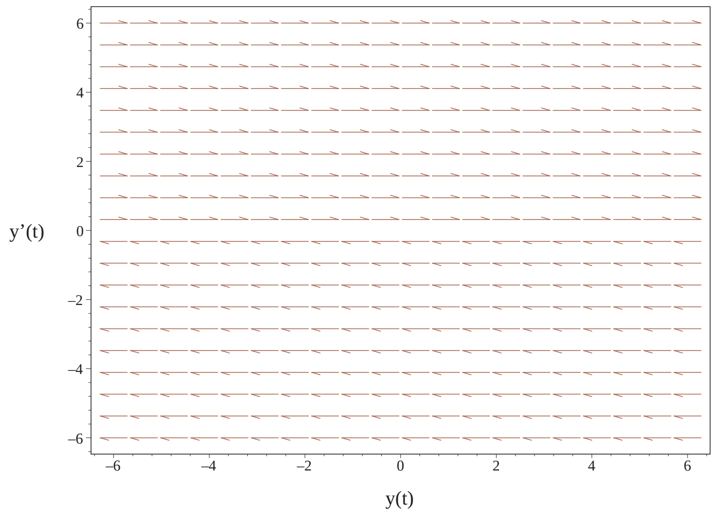


Figure 2.138: Slope field plot
$y'' = 1$

Solved as second order integrable as is ode

Time used: 0.028 (sec)

Integrating both sides of the ODE w.r.t t gives

$$\int y'' dt = \int 1 dt$$
$$y' = t + c_1$$

Which is now solved for y. Since the ode has the form $y' = f(t)$, then we only need to integrate $f(t)$.

$$\int dy = \int t + c_1 \, dt$$
$$y = \frac{1}{2}t^2 + c_1 t + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{1}{2}t^2 + c_1 t + c_2$$

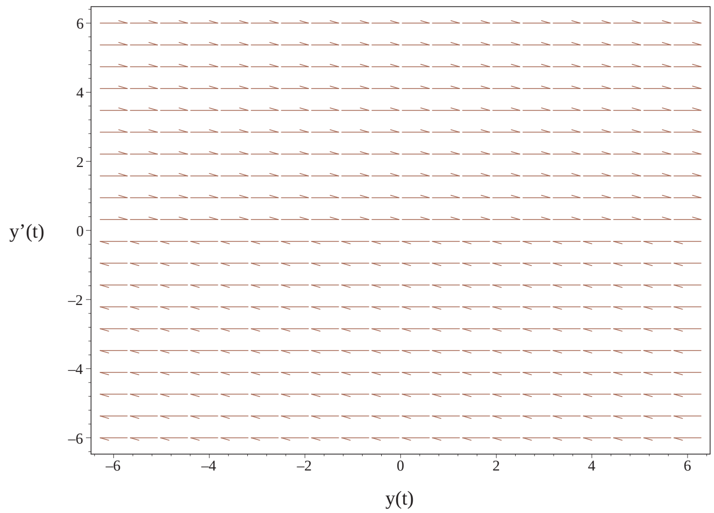


Figure 2.139: Slope field plot
$y'' = 1$

Solved as second order integrable as is ode (ABC method)

Time used: 0.031 (sec)

Writing the ode as

$$y'' = 1$$

Integrating both sides of the ODE w.r.t t gives

$$\int y''dt = \int 1dt$$
$$y' = t + c_1$$

Which is now solved for y. Since the ode has the form $y' = f(t)$, then we only need to integrate $f(t)$.

$$\int dy = \int t + c_1\, dt$$
$$y = \frac{1}{2}t^2 + c_1 t + c_2$$

Will add steps showing solving for IC soon.

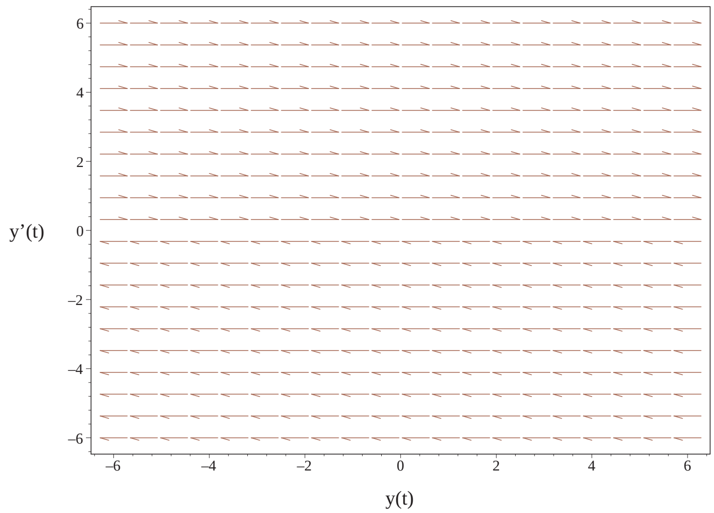


Figure 2.140: Slope field plot
$y'' = 1$

Solved as second order can be made integrable

Time used: 0.416 (sec)

Multiplying the ode by y' gives

$$y'y'' - y' = 0$$

Integrating the above w.r.t t gives

$$\int (y'y'' - y')\,dt = 0$$

$$\frac{y'^2}{2} - y = c_1$$

Which is now solved for y. Solving for the derivative gives these ODE's to solve

$$y' = \sqrt{2y + 2c_1} \tag{1}$$

$$y' = -\sqrt{2y + 2c_1} \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Integrating gives

$$\int \frac{1}{\sqrt{2y + 2c_1}} dy = dt$$

$$\sqrt{2y + 2c_1} = t + c_2$$

Singular solutions are found by solving

$$\sqrt{2y+2c_1} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = -c_1$$

Solving for y gives

$$y = \frac{1}{2}c_2^2 + c_2 t + \frac{1}{2}t^2 - c_1$$

Solving Eq. (2)

Integrating gives

$$\int -\frac{1}{\sqrt{2y+2c_1}} dy = dt$$
$$-\sqrt{2y+2c_1} = t + c_3$$

Singular solutions are found by solving

$$-\sqrt{2y+2c_1} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = -c_1$$

Solving for y gives

$$y = \frac{1}{2}c_3^2 + t c_3 + \frac{1}{2}t^2 - c_1$$

Will add steps showing solving for IC soon.

The solution

$$y = -c_1$$

was found not to satisfy the ode or the IC. Hence it is removed.

Summary of solutions found

$$y = \frac{1}{2}c_2^2 + c_2 t + \frac{1}{2}t^2 - c_1$$
$$y = \frac{1}{2}c_3^2 + t c_3 + \frac{1}{2}t^2 - c_1$$

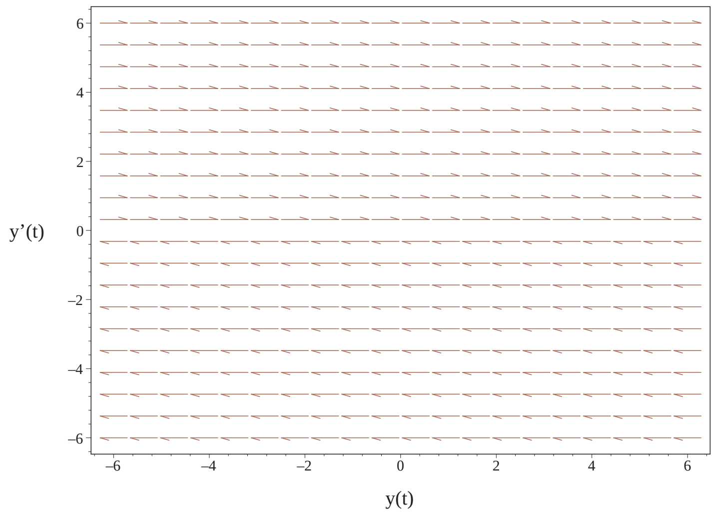


Figure 2.141: Slope field plot
$y'' = 1$

Solved as second order ode using Kovacic algorithm

Time used: 0.054 (sec)

Writing the ode as

$$y'' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = y e^{\int \frac{B}{2A} \, dt}$$

Then (2) becomes

$$z''(t) = r z(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.64: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= 1 \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dt \\ &= 1 \int \frac{1}{1}\,dt \\ &= 1(t) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2(1(t)) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_2 t + c_1$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, t\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t\}]$$

Since t is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 t^2$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 = 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{t^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 t + c_1) + \left(\frac{t^2}{2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

<u>Summary of solutions found</u>

$$y = \frac{1}{2}t^2 + c_2 t + c_1$$

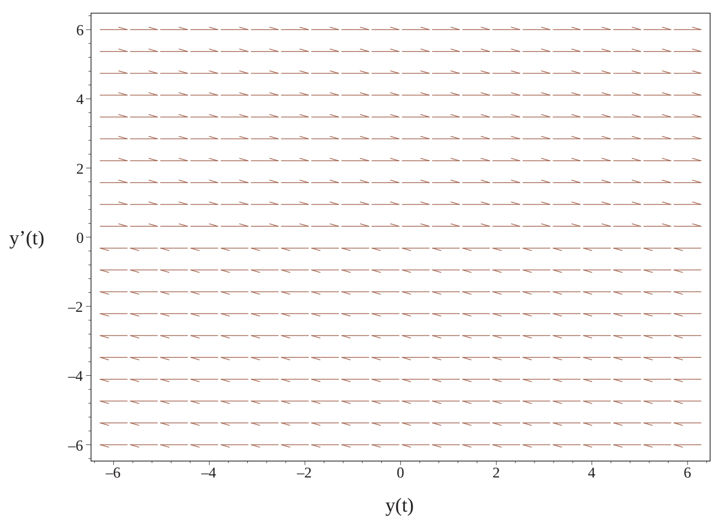


Figure 2.142: Slope field plot
$y'' = 1$

Solved as second order ode adjoint method

Time used: 0.431 (sec)

In normal form the ode

$$y'' = 1 \tag{1}$$

Becomes

$$y'' + p(t)\, y' + q(t)\, y = r(t) \tag{2}$$

Where

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 0 \\ r(t) &= 1 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (0)' + (0) &= 0 \\ \xi''(t) &= 0 \end{aligned}$$

Which is solved for $\xi(t)$. Integrating twice gives the solution

$$\xi = c_1 t + c_2$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned} \xi(t)\, y' - y\xi'(t) + \xi(t)\, p(t)\, y &= \int \xi(t)\, r(t)\, dt \\ y' + y\left(p(t) - \frac{\xi'(t)}{\xi\,(t)}\right) &= \frac{\int \xi(t)\, r(t)\, dt}{\xi\,(t)} \end{aligned}$$

Or

$$y' - \frac{y c_1}{c_1 t + c_2} = \frac{\frac{1}{2} c_1\, t^2 + c_2 t}{c_1 t + c_2}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(t) y = p(t)$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(t) &= -\frac{c_1}{c_1 t + c_2} \\
p(t) &= \frac{t(c_1 t + 2c_2)}{2c_1 t + 2c_2}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\,dt} \\
&= \mathrm{e}^{\int -\frac{c_1}{c_1 t + c_2} dt} \\
&= \frac{1}{c_1 t + c_2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(\frac{t(c_1 t + 2c_2)}{2c_1 t + 2c_2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y}{c_1 t + c_2}\right) &= \left(\frac{1}{c_1 t + c_2}\right)\left(\frac{t(c_1 t + 2c_2)}{2c_1 t + 2c_2}\right) \\
\mathrm{d}\left(\frac{y}{c_1 t + c_2}\right) &= \left(\frac{t(c_1 t + 2c_2)}{(2c_1 t + 2c_2)(c_1 t + c_2)}\right) \mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{c_1 t + c_2} &= \int \frac{t(c_1 t + 2c_2)}{(2c_1 t + 2c_2)(c_1 t + c_2)}\,dt \\
&= \frac{t}{2c_1} + \frac{c_2^2}{2c_1^2 (c_1 t + c_2)} + c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{1}{c_1 t + c_2}$ gives the final solution

$$
y = \frac{2c_1^3 c_3 t + (2c_2 c_3 + t^2)\,c_1^2 + c_1 c_2 t + c_2^2}{2c_1^2}
$$

Hence, the solution found using Lagrange adjoint equation method is

$$
y = \frac{2c_1^3 c_3 t + (2c_2 c_3 + t^2)\,c_1^2 + c_1 c_2 t + c_2^2}{2c_1^2}
$$

The constants can be merged to give

$$y = \frac{2c_1^3 t + (t^2 + 2c_2) c_1^2 + c_1 c_2 t + c_2^2}{2c_1^2}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{2c_1^3 t + (t^2 + 2c_2) c_1^2 + c_1 c_2 t + c_2^2}{2c_1^2}$$

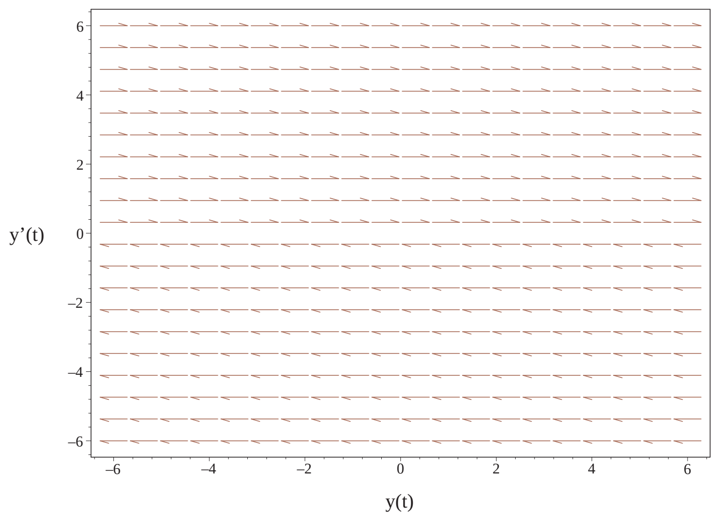


Figure 2.143: Slope field plot
$y'' = 1$

Maple step by step solution

Let's solve

$\frac{d^2}{dt^2}y(t) = 1$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dt^2}y(t)$
- Characteristic polynomial of homogeneous ODE

  $r^2 = 0$
- Use quadratic formula to solve for r

  $r = \frac{0 \pm \left(\sqrt{0}\right)}{2}$

- Roots of the characteristic polynomial
  $r = 0$
- 1st solution of the homogeneous ODE
  $y_1(t) = 1$
- Repeated root, multiply $y_1(t)$ by t to ensure linear independence
  $y_2(t) = t$
- General solution of the ODE
  $y(t) = C1y_1(t) + C2y_2(t) + y_p(t)$
- Substitute in solutions of the homogeneous ODE
  $y(t) = C1 + C2t + y_p(t)$
- ☐ Find a particular solution $y_p(t)$ of the ODE
  - Use variation of parameters to find y_p here $f(t)$ is the forcing function
    $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 1\right]$
  - Wronskian of solutions of the homogeneous equation
    $W(y_1(t), y_2(t)) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$
  - Compute Wronskian
    $W(y_1(t), y_2(t)) = 1$
  - Substitute functions into equation for $y_p(t)$
    $y_p(t) = -\left(\int t dt\right) + \left(\int 1 dt\right) t$
  - Compute integrals
    $y_p(t) = \frac{t^2}{2}$
- Substitute particular solution into general solution to ODE
  $y(t) = C1 + C2t + \frac{1}{2}t^2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 14

```
dsolve(diff(diff(y(t),t),t) = 1,
       y(t),singsol=all)
```

$$y = \frac{1}{2}t^2 + c_1 t + c_2$$

Mathematica DSolve solution

Solving time : 0.002 (sec)
Leaf size : 19

```
DSolve[{D[y[t],{t,2}]==1,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{t^2}{2} + c_2 t + c_1$$

2.1.56 problem 56

Internal problem ID [8444]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 56
Date solved : `Thursday, December 12, 2024 at 09:08:13 AM`
CAS classification : `[[_2nd_order, _quadrature]]`

Solve

$$y'' = f(t)$$

Solved as second order ode quadrature

Time used: 0.039 (sec)

Integrating once gives

$$y' = \int f(t)\,dt + c_1$$

Integrating again gives

$$y = \int \left(\int f(t)\,dt\right) dt + c_1 x + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \int \int f(t)\,dtdt + c_1 t + c_2$$

Solved as second order linear constant coeff ode

Time used: 0.355 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 0, C = 0, f(t) = f(t)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 0, C = 0$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(0)^2 - (4)(1)(0)} \\ &= 0\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 0$. Therefore the solution is

$$y = c_1 1 + c_2 t \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_2 t + c_1$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = 1$$
$$y_2 = t$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & t \\ \frac{d}{dt}(1) & \frac{d}{dt}(t) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & t \\ 0 & 1 \end{vmatrix}$$

Therefore

$$W = (1)(1) - (t)(0)$$

Which simplifies to

$$W = 1$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{tf(t)}{1}\,dt$$

Which simplifies to

$$u_1 = -\int tf(t)\,dt$$

Hence

$$u_1 = -\int_0^t \alpha f(\alpha)\,d\alpha$$

And Eq. (3) becomes

$$u_2 = \int \frac{f(t)}{1}\,dt$$

Which simplifies to

$$u_2 = \int f(t)\,dt$$

Hence

$$u_2 = \int_0^t f(\alpha)\,d\alpha$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -\int_0^t \alpha f(\alpha)\,d\alpha + t\int_0^t f(\alpha)\,d\alpha$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= (c_2 t + c_1) + \left(-\int_0^t \alpha f(\alpha)\,d\alpha + t\int_0^t f(\alpha)\,d\alpha\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\int_0^t \alpha f(\alpha)\,d\alpha + t\int_0^t f(\alpha)\,d\alpha + c_2 t + c_1$$

Solved as second order linear exact ode

Time used: 0.050 (sec)

An ode of the form

$$p(t)\,y'' + q(t)\,y' + r(t)\,y = s(t)$$

is exact if

$$p''(t) - q'(t) + r(t) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}
p(x) &= 1\\
q(x) &= 0\\
r(x) &= 0\\
s(x) &= f(t)
\end{aligned}$$

Hence

$$\begin{aligned}
p''(x) &= 0\\
q'(x) &= 0
\end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(t)\,y' + (q(t) - p'(t))\,y)' = s(x)$$

Integrating gives

$$p(t)\,y' + (q(t) - p'(t))\,y = \int s(t)\,dt$$

Substituting the above values for p, q, r, s gives

$$y' = \int f(t)\,dt$$

We now have a first order ode to solve which is

$$y' = \int f(t)\,dt + c_1$$

Since the ode has the form $y' = f(t)$, then we only need to integrate $f(t)$.

$$\begin{aligned}\int dy &= \int \int f(t)\,dt + c_1\,dt\\ y &= \int \left(\int f(t)\,dt + c_1\right) dt + c_2\end{aligned}$$

$$y = \int \left(\int f(t)\,dt + c_1\right) dt + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \int \left(\int f(t)\,dt + c_1\right) dt + c_2$$

Solved as second order missing y ode

Time used: 0.047 (sec)

This is second order ode with missing dependent variable y. Let

$$p(t) = y'$$

Then

$$p'(t) = y''$$

Hence the ode becomes

$$p'(t) - f(t) = 0$$

Which is now solve for $p(t)$ as first order ode. Since the ode has the form $p'(t) = f(t)$, then we only need to integrate $f(t)$.

$$\begin{aligned}\int dp &= \int f(t)\ dt\\ p(t) &= \int f(t)\,dt + c_1\end{aligned}$$

$$p(t) = \int f(t)\,dt + c_1$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \int f(t)\,dt + c_1$$

Since the ode has the form $y' = f(t)$, then we only need to integrate $f(t)$.

$$\int dy = \int \int f(t)\,dt + c_1\,dt$$
$$y = \int \left(\int f(t)\,dt + c_1\right) dt + c_2$$

$$y = \int \left(\int f(t)\,dt + c_1\right) dt + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \int \left(\int f(t)\,dt + c_1\right) dt + c_2$$

Solved as second order ode using Kovacic algorithm

Time used: 0.084 (sec)

Writing the ode as

$$y'' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.66: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= 1 \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{y_1^2}\,dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dt \\ &= 1 \int \frac{1}{1}\,dt \\ &= 1(t) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2(1(t))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_2 t + c_1$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on t as well. Let

$$y_p(t) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= 1 \\
y_2 &= t
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(t)}{aW(t)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(t)}{aW(t)} \tag{3}$$

Where $W(t)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} 1 & t \\ \frac{d}{dt}(1) & \frac{d}{dt}(t) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} 1 & t \\ 0 & 1 \end{vmatrix}$$

Therefore

$$W = (1)(1) - (t)(0)$$

Which simplifies to

$$W = 1$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{tf(t)}{1}\,dt$$

Which simplifies to

$$u_1 = -\int tf(t)\,dt$$

Hence

$$u_1 = -\int_0^t \alpha f(\alpha)\,d\alpha$$

And Eq. (3) becomes

$$u_2 = \int \frac{f(t)}{1}\,dt$$

Which simplifies to

$$u_2 = \int f(t)\,dt$$

Hence

$$u_2 = \int_0^t f(\alpha)\, d\alpha$$

Therefore the particular solution, from equation (1) is

$$y_p(t) = -\int_0^t \alpha f(\alpha)\, d\alpha + t \int_0^t f(\alpha)\, d\alpha$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 t + c_1) + \left(-\int_0^t \alpha f(\alpha)\, d\alpha + t \int_0^t f(\alpha)\, d\alpha\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\int_0^t \alpha f(\alpha)\, d\alpha + t \int_0^t f(\alpha)\, d\alpha + c_2 t + c_1$$

Maple step by step solution

Let's solve
$\frac{d^2}{dt^2} y(t) = f(t)$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dt^2} y(t)$
- Characteristic polynomial of homogeneous ODE
  $r^2 = 0$
- Use quadratic formula to solve for r
  $r = \frac{0 \pm \left(\sqrt{0}\right)}{2}$
- Roots of the characteristic polynomial
  $r = 0$
- 1st solution of the homogeneous ODE
  $y_1(t) = 1$
- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

$y_2(t) = t$

- General solution of the ODE

  $y(t) = C1 y_1(t) + C2 y_2(t) + y_p(t)$

- Substitute in solutions of the homogeneous ODE

  $y(t) = C1 + C2t + y_p(t)$

- ☐ Find a particular solution $y_p(t)$ of the ODE

  - Use variation of parameters to find y_p here $f(t)$ is the forcing function

    $\left[y_p(t) = -y_1(t) \left(\int \frac{y_2(t) f(t)}{W(y_1(t), y_2(t))} dt\right) + y_2(t) \left(\int \frac{y_1(t) f(t)}{W(y_1(t), y_2(t))} dt\right), f(t) = f(t)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(t), y_2(t)) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(t), y_2(t)) = 1$

  - Substitute functions into equation for $y_p(t)$

    $y_p(t) = -\left(\int t f(t) \, dt\right) + t\left(\int f(t) \, dt\right)$

  - Compute integrals

    $y_p(t) = -\left(\int t f(t) \, dt\right) + t\left(\int f(t) \, dt\right)$

- Substitute particular solution into general solution to ODE

  $y(t) = C1 + C2t - \left(\int t f(t) \, dt\right) + t\left(\int f(t) \, dt\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 15

```
dsolve(diff(diff(y(t),t),t) = f(t),
       y(t),singsol=all)
```

$$y = \int \left(\int f(t)\, dt\right) dt + c_1 t + c_2$$

Mathematica DSolve solution

Solving time : 0.011 (sec)
Leaf size : 30

```
DSolve[{D[y[t],{t,2}]==f[t],{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \int_1^t \int_1^{K[2]} f(K[1]) dK[1] dK[2] + c_2 t + c_1$$

2.1.57 problem 57

Internal problem ID [8445]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 57
Date solved : `Thursday, December 12, 2024 at 09:08:14 AM`
CAS classification : `[[_2nd_order, _quadrature]]`

Solve

$$y'' = k$$

Solved as second order ode quadrature

Time used: 0.029 (sec)

The ODE can be written as

$$y'' = k$$

Integrating once gives

$$y' = kt + c_1$$

Integrating again gives

$$y = \frac{k\,t^2}{2} + c_1 x + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{1}{2}k\,t^2 + c_1 t + c_2$$

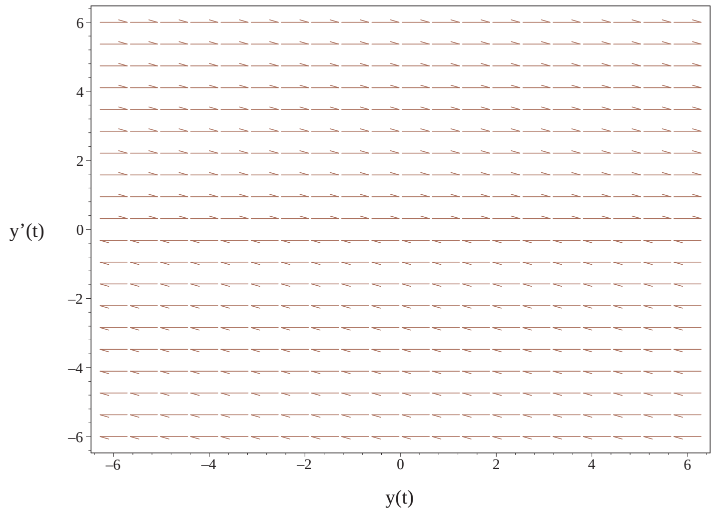


Figure 2.144: Slope field plot
$y'' = k$

Solved as second order linear constant coeff ode

Time used: 0.096 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = f(t)$$

Where $A = 1, B = 0, C = 0, f(t) = k$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(t) + By'(t) + Cy(t) = 0$$

Where in the above $A = 1, B = 0, C = 0$. Let the solution be $y = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(0)^2 - (4)(1)(0)}\\ &= 0\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 0$. Therefore the solution is

$$y = c_1 1 + c_2 t \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_2 t + c_1$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, t\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t\}]$$

Since t is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 t^2$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 = k$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{k}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{k\,t^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 t + c_1) + \left(\frac{k\,t^2}{2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{1}{2} k\,t^2 + c_2 t + c_1$$

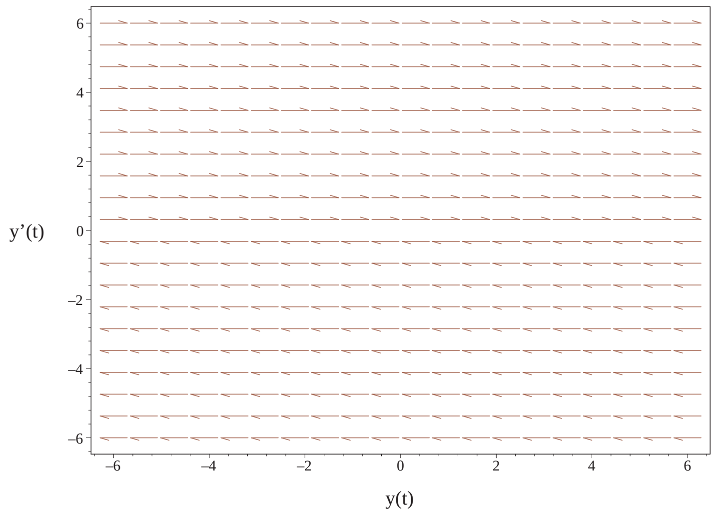


Figure 2.145: Slope field plot
$y'' = k$

Solved as second order linear exact ode

Time used: 0.058 (sec)

An ode of the form

$$p(t)\, y'' + q(t)\, y' + r(t)\, y = s(t)$$

is exact if

$$p''(t) - q'(t) + r(t) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 0 \\ r(x) &= 0 \\ s(x) &= k \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(t)\,y' + (q(t) - p'(t))\,y)' = s(x)$$

Integrating gives

$$p(t)\,y' + (q(t) - p'(t))\,y = \int s(t)\;dt$$

Substituting the above values for p, q, r, s gives

$$y' = \int k\,dt$$

We now have a first order ode to solve which is

$$y' = kt + c_1$$

Since the ode has the form $y' = f(t)$, then we only need to integrate $f(t)$.

$$\int dy = \int kt + c_1\,dt$$
$$y = \frac{1}{2}k\,t^2 + c_1 t + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{1}{2}k\,t^2 + c_1 t + c_2$$

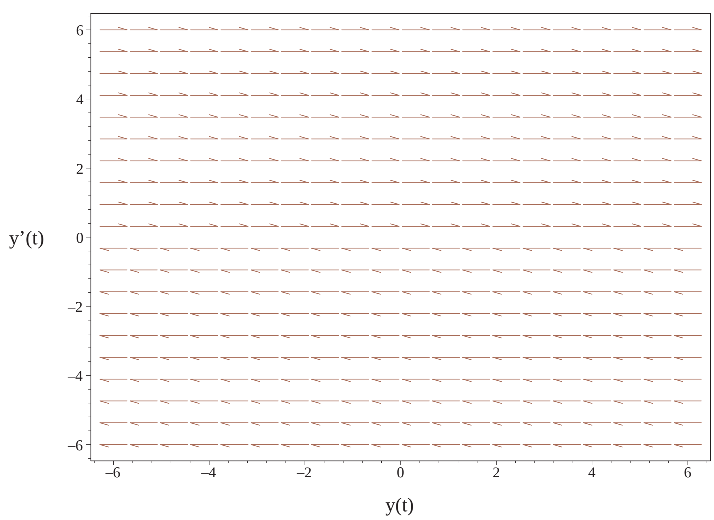


Figure 2.146: Slope field plot
$y'' = k$

Solved as second order missing y ode

Time used: 0.052 (sec)

This is second order ode with missing dependent variable y. Let

$$p(t) = y'$$

Then

$$p'(t) = y''$$

Hence the ode becomes

$$p'(t) - k = 0$$

Which is now solve for $p(t)$ as first order ode. Since the ode has the form $p'(t) = f(t)$, then we only need to integrate $f(t)$.

$$\int dp = \int k\,dt$$
$$p(t) = kt + c_1$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = kt + c_1$$

Since the ode has the form $y' = f(t)$, then we only need to integrate $f(t)$.

$$\int dy = \int kt + c_1\,dt$$
$$y = \frac{1}{2}k\,t^2 + c_1t + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{1}{2}k\,t^2 + c_1t + c_2$$

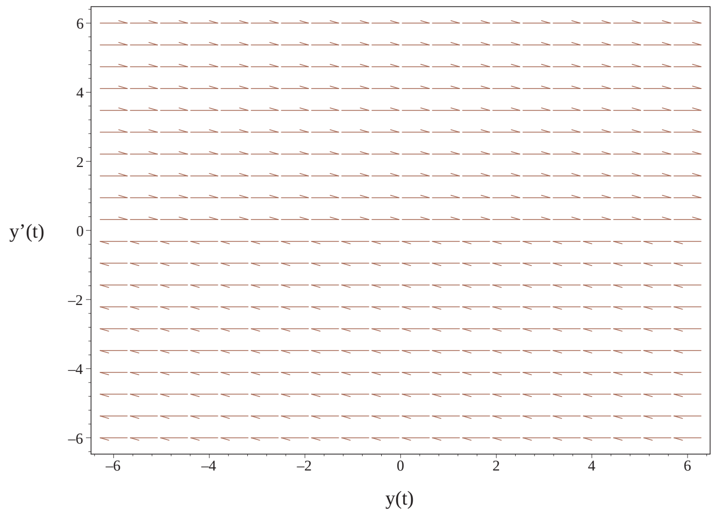


Figure 2.147: Slope field plot
$y'' = k$

Solved as second order integrable as is ode

Time used: 0.037 (sec)

Integrating both sides of the ODE w.r.t t gives

$$\int y'' dt = \int k dt$$
$$y' = kt + c_1$$

Which is now solved for y. Since the ode has the form $y' = f(t)$, then we only need to integrate $f(t)$.

$$\int dy = \int kt + c_1\, dt$$
$$y = \frac{1}{2}k\,t^2 + c_1 t + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{1}{2}k\,t^2 + c_1 t + c_2$$

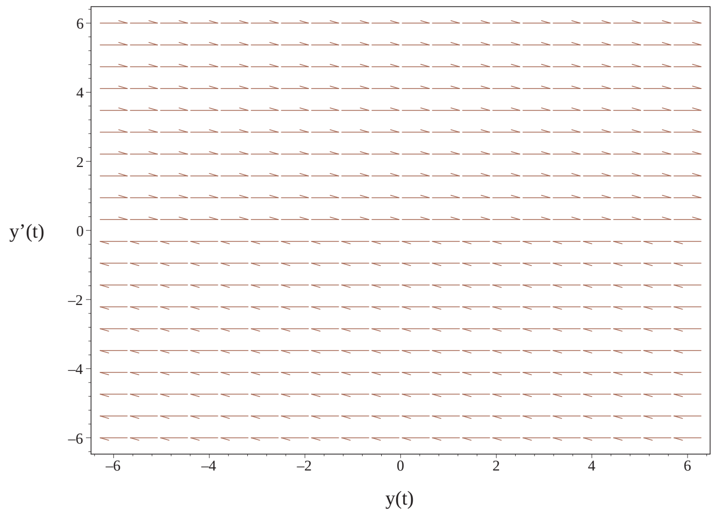


Figure 2.148: Slope field plot
$y'' = k$

Solved as second order integrable as is ode (ABC method)

Time used: 0.038 (sec)

Writing the ode as

$$y'' = k$$

Integrating both sides of the ODE w.r.t t gives

$$\int y''dt = \int kdt$$
$$y' = kt + c_1$$

Which is now solved for y. Since the ode has the form $y' = f(t)$, then we only need to integrate $f(t)$.

$$\int dy = \int kt + c_1\, dt$$
$$y = \frac{1}{2}k\,t^2 + c_1 t + c_2$$

Will add steps showing solving for IC soon.

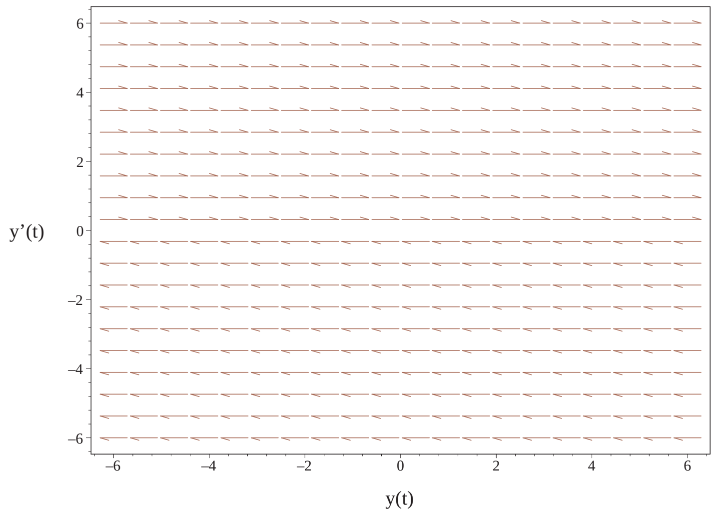


Figure 2.149: Slope field plot
$y'' = k$

Solved as second order can be made integrable

Time used: 0.494 (sec)

Multiplying the ode by y' gives

$$y'y'' - y'k = 0$$

Integrating the above w.r.t t gives

$$\int (y'y'' - y'k)\,dt = 0$$

$$\frac{y'^2}{2} - ky = c_1$$

Which is now solved for y. Solving for the derivative gives these ODE's to solve

$$y' = \sqrt{2ky + 2c_1} \tag{1}$$

$$y' = -\sqrt{2ky + 2c_1} \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Integrating gives

$$\int \frac{1}{\sqrt{2ky + 2c_1}} dy = dt$$

$$\frac{\sqrt{2ky + 2c_1}}{k} = t + c_2$$

Singular solutions are found by solving

$$\sqrt{2ky+2c_1} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = -\frac{c_1}{k}$$

Solving for y gives

$$y = -\frac{-c_2^2k^2 - 2c_2\,k^2t - k^2t^2 + 2c_1}{2k}$$

Solving Eq. (2)

Integrating gives

$$\int -\frac{1}{\sqrt{2ky+2c_1}}dy = dt$$
$$-\frac{\sqrt{2ky+2c_1}}{k} = t + c_3$$

Singular solutions are found by solving

$$-\sqrt{2ky+2c_1} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = -\frac{c_1}{k}$$

Solving for y gives

$$y = -\frac{-c_3^2k^2 - 2c_3\,k^2t - k^2t^2 + 2c_1}{2k}$$

Will add steps showing solving for IC soon.

The solution

$$y = -\frac{c_1}{k}$$

was found not to satisfy the ode or the IC. Hence it is removed.

Summary of solutions found

$$y = -\frac{-c_2^2k^2 - 2c_2\,k^2t - k^2t^2 + 2c_1}{2k}$$
$$y = -\frac{-c_3^2k^2 - 2c_3\,k^2t - k^2t^2 + 2c_1}{2k}$$

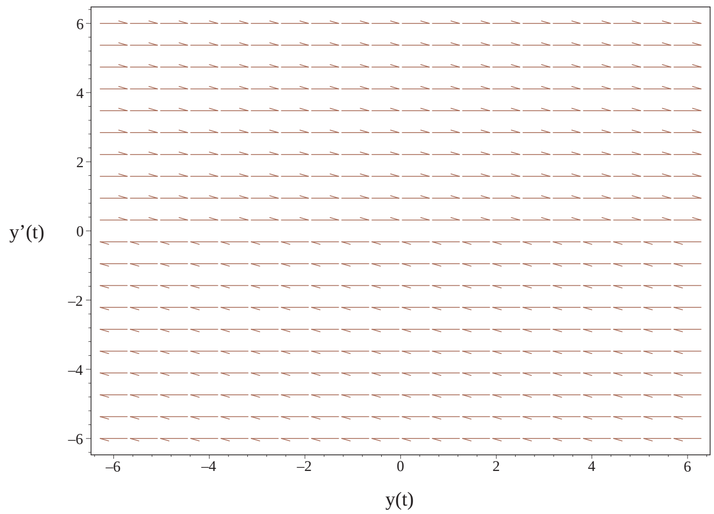


Figure 2.150: Slope field plot
$y'' = k$

Solved as second order ode using Kovacic algorithm

Time used: 0.121 (sec)

Writing the ode as

$$y'' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ye^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then y is found using the inverse transformation

$$y = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.68: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= 1
\end{aligned}
$$

Which simplifies to

$$
y_1 = 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{y_1^2} \, dt
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} \, dt \\
&= 1 \int \frac{1}{1} \, dt \\
&= 1(t)
\end{aligned}
$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2(1(t)) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(t) + By'(t) + Cy(t) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(t) + By'(t) + Cy(t) = f(t)$. y_h is the solution to

$$y'' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_2 t + c_1$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, t\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t\}]$$

Since t is duplicated in the UC_set, then this basis is multiplied by extra t. The UC_set becomes

$$[\{t^2\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 t^2$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 = k$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{k}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{k\,t^2}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 t + c_1) + \left(\frac{k\,t^2}{2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{1}{2} k\,t^2 + c_2 t + c_1$$

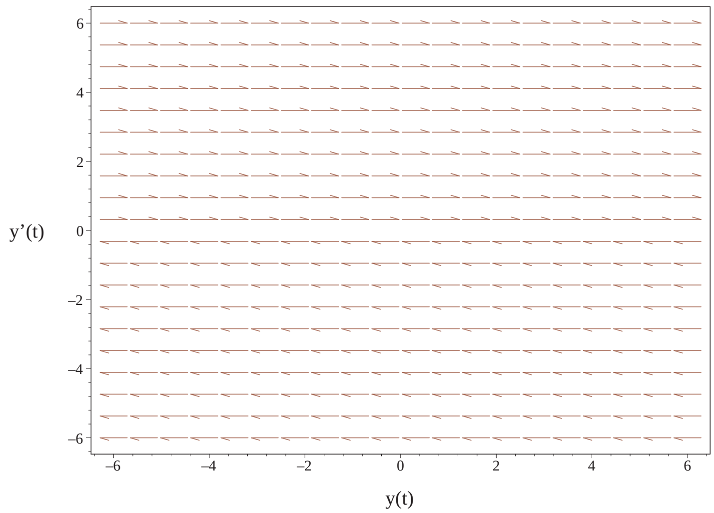


Figure 2.151: Slope field plot
$y'' = k$

Solved as second order ode adjoint method

Time used: 0.363 (sec)

In normal form the ode

$$y'' = k \tag{1}$$

Becomes

$$y'' + p(t)\,y' + q(t)\,y = r(t) \tag{2}$$

Where

$$
\begin{aligned}
p(t) &= 0\\
q(t) &= 0\\
r(t) &= k
\end{aligned}
$$

The Lagrange adjoint ode is given by

$$
\begin{aligned}
\xi'' - (\xi\,p)' + \xi q &= 0\\
\xi'' - (0)' + (0) &= 0\\
\xi''(t) &= 0
\end{aligned}
$$

Which is solved for $\xi(t)$. Integrating twice gives the solution

$$\xi = c_1 t + c_2$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$
\begin{aligned}
\xi(t)\, y' - y\xi'(t) + \xi(t)\, p(t)\, y &= \int \xi(t)\, r(t)\, dt \\
y' + y\left(p(t) - \frac{\xi'(t)}{\xi\,(t)}\right) &= \frac{\int \xi(t)\, r(t)\, dt}{\xi\,(t)}
\end{aligned}
$$

Or

$$
y' - \frac{yc_1}{c_1 t + c_2} = \frac{k\left(\frac{1}{2}c_1\, t^2 + c_2 t\right)}{c_1 t + c_2}
$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$
y' + q(t)y = p(t)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(t) &= -\frac{c_1}{c_1 t + c_2} \\
p(t) &= \frac{kt(c_1 t + 2c_2)}{2c_1 t + 2c_2}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\, dt} \\
&= e^{\int -\frac{c_1}{c_1 t + c_2} dt} \\
&= \frac{1}{c_1 t + c_2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}t}(\mu y) &= (\mu)\left(\frac{kt(c_1 t + 2c_2)}{2c_1 t + 2c_2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y}{c_1 t + c_2}\right) &= \left(\frac{1}{c_1 t + c_2}\right)\left(\frac{kt(c_1 t + 2c_2)}{2c_1 t + 2c_2}\right) \\
\mathrm{d}\left(\frac{y}{c_1 t + c_2}\right) &= \left(\frac{kt(c_1 t + 2c_2)}{(2c_1 t + 2c_2)\,(c_1 t + c_2)}\right) \mathrm{d}t
\end{aligned}
$$

Integrating gives

$$\frac{y}{c_1t+c_2} = \int \frac{kt(c_1t+2c_2)}{(2c_1t+2c_2)(c_1t+c_2)}\,dt$$
$$= \frac{k\left(\frac{t}{c_1}+\frac{c_2^2}{c_1^2(c_1t+c_2)}\right)}{2} + c_3$$

Dividing throughout by the integrating factor $\frac{1}{c_1t+c_2}$ gives the final solution

$$y = (c_1t+c_2)\left(\frac{k\left(\frac{t}{c_1}+\frac{c_2^2}{c_1^2(c_1t+c_2)}\right)}{2} + c_3\right)$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = (c_1t+c_2)\left(\frac{k\left(\frac{t}{c_1}+\frac{c_2^2}{c_1^2(c_1t+c_2)}\right)}{2} + c_3\right)$$

The constants can be merged to give

$$y = (c_1t+c_2)\left(\frac{k\left(\frac{t}{c_1}+\frac{c_2^2}{c_1^2(c_1t+c_2)}\right)}{2} + 1\right)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = (c_1t+c_2)\left(\frac{k\left(\frac{t}{c_1}+\frac{c_2^2}{c_1^2(c_1t+c_2)}\right)}{2} + 1\right)$$

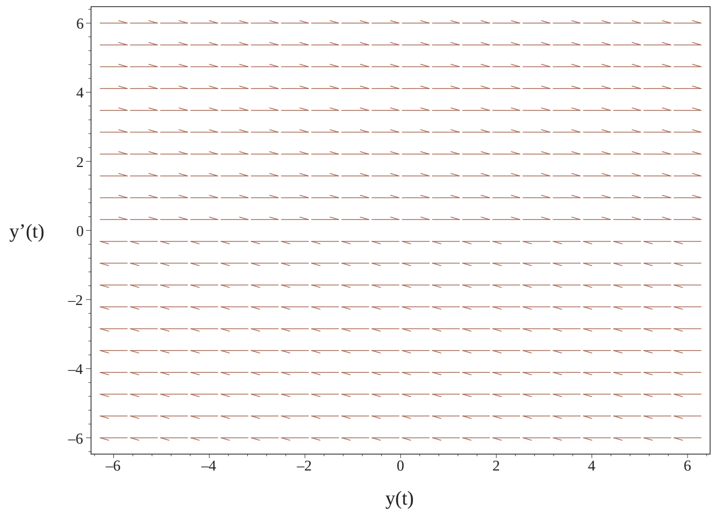


Figure 2.152: Slope field plot
$y'' = k$

Maple step by step solution

Let's solve
$\frac{d^2}{dt^2}y(t) = k$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dt^2}y(t)$
- Characteristic polynomial of homogeneous ODE
  $r^2 = 0$
- Use quadratic formula to solve for r
  $r = \frac{0 \pm \left(\sqrt{0}\right)}{2}$
- Roots of the characteristic polynomial
  $r = 0$
- 1st solution of the homogeneous ODE
  $y_1(t) = 1$
- Repeated root, multiply $y_1(t)$ by t to ensure linear independence
  $y_2(t) = t$
- General solution of the ODE
  $y(t) = C1 y_1(t) + C2 y_2(t) + y_p(t)$
- Substitute in solutions of the homogeneous ODE
  $y(t) = C1 + C2t + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- Use variation of parameters to find y_p here $f(t)$ is the forcing function
  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = k\right]$
- Wronskian of solutions of the homogeneous equation
  $W(y_1(t), y_2(t)) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$
- Compute Wronskian
  $W(y_1(t), y_2(t)) = 1$
- Substitute functions into equation for $y_p(t)$
  $y_p(t) = k\left(-\left(\int t dt\right) + \left(\int 1 dt\right) t\right)$
- Compute integrals
  $y_p(t) = \frac{k\,t^2}{2}$

• Substitute particular solution into general solution to ODE
  $y(t) = \mathit{C1} + \mathit{C2}t + \frac{1}{2}k\,t^2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : `0.002` (sec)
Leaf size : 15

```
dsolve(diff(diff(y(t),t),t) = k,
       y(t),singsol=all)
```

$$y = \frac{1}{2}k\,t^2 + c_1 t + c_2$$

Mathematica DSolve solution

Solving time : 0.003 (sec)
Leaf size : 20

```
DSolve[{D[y[t],{t,2}]==k,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{kt^2}{2} + c_2 t + c_1$$

2.1.58 problem 58

Internal problem ID [8446]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 58
Date solved : `Tuesday, December 17, 2024 at 12:51:38 PM`
CAS classification : `[[_homogeneous, 'class C'], _dAlembert]`

Solve

$$y' = -4\sin(-y + x) - 4$$

Solved as first order form A1 ode

Time used: 0.175 (sec)

The given ode has the general form

$$y' = B + Cf(ax + by + c) \tag{1}$$

Comparing (1) to the ode given shows the parameters in the ODE have these values

$$\begin{aligned} B &= -4 \\ C &= -4 \\ a &= 1 \\ b &= -1 \\ c &= 0 \end{aligned}$$

This form of ode can be solved by change of variables $u = ax + by + c$ which makes the ode separable.

$$u'(x) = a + by'$$

Or

$$y' = \frac{u'(x) - a}{b}$$

The ode becomes

$$
\begin{aligned}
\frac{u'-a}{b} &= B + Cf(u) \\
u' &= bB + bCf(u) + a \\
\frac{du}{bB + bCf(u) + a} &= dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{du}{bB + bCf(u) + a} &= x + c_1 \\
\int^{u} \frac{d\tau}{bB + bCf(\tau) + a} &= x + c_1
\end{aligned}
$$

Replacing back $u = ax + by + c$ the above becomes

$$
\int^{ax+by+c} \frac{d\tau}{bB + bCf(\tau) + a} = x + c_1 \tag{2}
$$

If initial conditions are given as $y(x_0) = y_0$, the above becomes

$$
\begin{aligned}
\int_0^{ax_0+by_0+c} \frac{d\tau}{bB + bCf(\tau) + a} &= x_0 + c_1 \\
c_1 &= \int_0^{ax+by_0+c} \frac{d\tau}{bB + bCf(\tau) + a} - x_0
\end{aligned}
$$

Substituting this into (2) gives

$$
\int^{ax+by+c} \frac{d\tau}{bB + bCf(\tau) + a} = x + \int_0^{ax+by_0+c} \frac{d\tau}{bB + bCf(\tau) + a} - x_0 \tag{3}
$$

Since no initial conditions are given, then using (2) and replacing the values of the parameters into (2) gives the solution as

$$
\int^{-y+x} \frac{1}{5 + 4\sin(\tau)} d\tau = x + c_1
$$

Which simplifies to

$$
\frac{2\arctan\left(\frac{5\tan\left(-\frac{y}{2}+\frac{x}{2}\right)}{3} + \frac{4}{3}\right)}{3} = x + c_1
$$

Solving for y gives

$$
y = x - 2\arctan\left(\frac{3\tan\left(\frac{3x}{2} + \frac{3c_1}{2}\right)}{5} - \frac{4}{5}\right)
$$

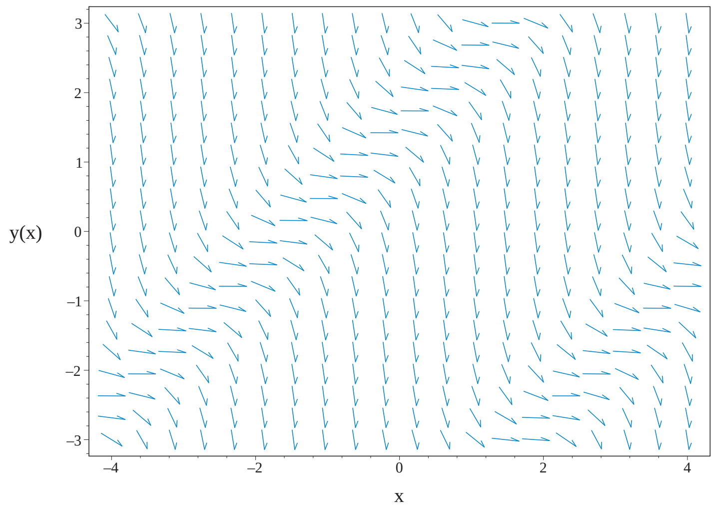


Figure 2.153: Slope field plot
$y' = -4\sin(-y+x) - 4$

Summary of solutions found

$$y = x - 2\arctan\left(\frac{3\tan\left(\frac{3x}{2}+\frac{3c_1}{2}\right)}{5} - \frac{4}{5}\right)$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) = -4\sin(x-y(x)) - 4$

- Highest derivative means the order of the ODE is 1
$\frac{d}{dx}y(x)$
- Solve for the highest derivative
$\frac{d}{dx}y(x) = -4\sin(x-y(x)) - 4$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
```

```
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

Maple dsolve solution

Solving time : 0.048 (sec)
Leaf size : 21

```
dsolve(diff(y(x),x) = -4*sin(x-y(x))-4,
       y(x),singsol=all)
```

$$y = x + 2\arctan\left(\frac{3\tan\left(-\frac{3x}{2} + \frac{3c_1}{2}\right)}{5} + \frac{4}{5}\right)$$

Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{D[y[x],x]==4*Sin[y[x]-x]-4,{}},
       y[x],x,IncludeSingularSolutions->True]
```

Timed out

2.1.59 problem 59

Internal problem ID [8447]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 59
Date solved : `Tuesday, December 17, 2024 at 12:52:52 PM`
CAS classification : `[[_homogeneous, ‘class C‘], _dAlembert]`

Solve

$$y' + \sin(-y + x) = 0$$

Solved as first order form A1 ode

Time used: 0.217 (sec)

The given ode has the general form

$$y' = B + Cf(ax + by + c) \tag{1}$$

Comparing (1) to the ode given shows the parameters in the ODE have these values

$$\begin{aligned} B &= 0 \\ C &= -1 \\ a &= 1 \\ b &= -1 \\ c &= 0 \end{aligned}$$

This form of ode can be solved by change of variables $u = ax + by + c$ which makes the ode separable.

$$u'(x) = a + by'$$

Or

$$y' = \frac{u'(x) - a}{b}$$

The ode becomes

$$\begin{aligned}\frac{u' - a}{b} &= B + Cf(u) \\ u' &= bB + bCf(u) + a \\ \frac{du}{bB + bCf(u) + a} &= dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\int \frac{du}{bB + bCf(u) + a} &= x + c_1 \\ \int^u \frac{d\tau}{bB + bCf(\tau) + a} &= x + c_1\end{aligned}$$

Replacing back $u = ax + by + c$ the above becomes

$$\int^{ax+by+c} \frac{d\tau}{bB + bCf(\tau) + a} = x + c_1 \tag{2}$$

If initial conditions are given as $y(x_0) = y_0$, the above becomes

$$\begin{aligned}\int_0^{ax_0+by_0+c} \frac{d\tau}{bB + bCf(\tau) + a} &= x_0 + c_1 \\ c_1 &= \int_0^{ax+by_0+c} \frac{d\tau}{bB + bCf(\tau) + a} - x_0\end{aligned}$$

Substituting this into (2) gives

$$\int^{ax+by+c} \frac{d\tau}{bB + bCf(\tau) + a} = x + \int_0^{ax+by_0+c} \frac{d\tau}{bB + bCf(\tau) + a} - x_0 \tag{3}$$

Since no initial conditions are given, then using (2) and replacing the values of the parameters into (2) gives the solution as

$$\int^{-y+x} \frac{1}{1 + \sin(\tau)} d\tau = x + c_1$$

Which simplifies to

$$-\frac{2}{\tan\left(-\frac{y}{2} + \frac{x}{2}\right) + 1} = x + c_1$$

Solving for y gives

$$y = x + 2\arctan\left(\frac{c_1 + x + 2}{x + c_1}\right)$$

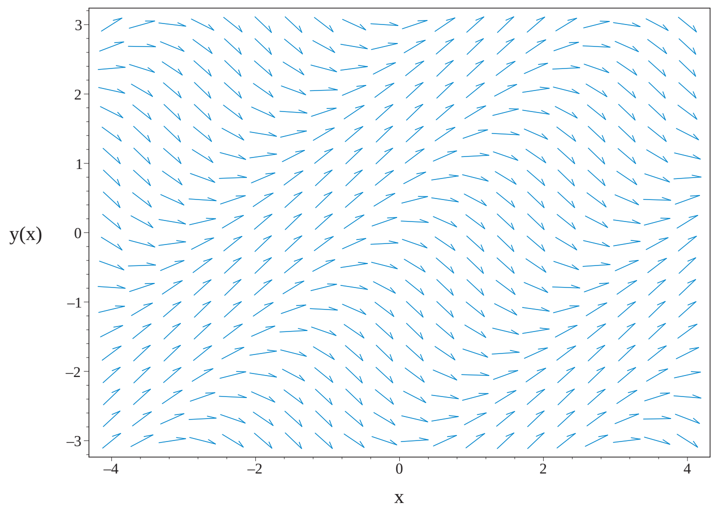


Figure 2.154: Slope field plot
$y' + \sin(-y + x) = 0$

Summary of solutions found

$$y = x + 2\arctan\left(\frac{c_1 + x + 2}{x + c_1}\right)$$

Solved using Lie symmetry for first order ode

Time used: 0.752 (sec)

Writing the ode as

$$y' = -\sin(-y + x)$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \sin(-y+x)(b_3 - a_2) - \sin(-y+x)^2 a_3 \\ & + \cos(-y+x)(xa_2 + ya_3 + a_1) - \cos(-y+x)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\begin{aligned} & -\sin(-y+x)^2 a_3 + \cos(-y+x) x a_2 - \cos(-y+x) x b_2 \\ & + \cos(-y+x) y a_3 - \cos(-y+x) y b_3 + \sin(-y+x) a_2 \\ & - \sin(-y+x) b_3 + \cos(-y+x) a_1 - \cos(-y+x) b_1 + b_2 = 0 \end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned} & -\sin(-y+x)^2 a_3 + \cos(-y+x) x a_2 - \cos(-y+x) x b_2 \\ & + \cos(-y+x) y a_3 - \cos(-y+x) y b_3 + \sin(-y+x) a_2 \\ & - \sin(-y+x) b_3 + \cos(-y+x) a_1 - \cos(-y+x) b_1 + b_2 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} & b_2 - \frac{a_3}{2} + \frac{a_3 \cos(-2y+2x)}{2} + \cos(-y+x) x a_2 - \cos(-y+x) x b_2 \\ & + \cos(-y+x) y a_3 - \cos(-y+x) y b_3 + \sin(-y+x) a_2 \\ & - \sin(-y+x) b_3 + \cos(-y+x) a_1 - \cos(-y+x) b_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \cos(-2y+2x), \cos(-y+x), \sin(-y+x)\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \cos(-2y+2x) = v_3, \cos(-y+x) = v_4, \sin(-y+x) = v_5\}$$

The above PDE (6E) now becomes

$$b_2 - \frac{1}{2}a_3 + \frac{1}{2}a_3v_3 + v_4v_1a_2 - v_4v_1b_2 + v_4v_2a_3 - v_4v_2b_3 + v_5a_2 - v_5b_3 + v_4a_1 - v_4b_1 = 0 \quad (7E)$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4, v_5\}$$

Equation (7E) now becomes

$$b_2 - \frac{a_3}{2} + (a_2 - b_2)\,v_1v_4 + (a_3 - b_3)\,v_2v_4 + \frac{a_3v_3}{2} + (a_1 - b_1)\,v_4 + (a_2 - b_3)\,v_5 = 0 \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
\frac{a_3}{2} &= 0 \\
a_1 - b_1 &= 0 \\
a_2 - b_2 &= 0 \\
a_2 - b_3 &= 0 \\
a_3 - b_3 &= 0 \\
b_2 - \frac{a_3}{2} &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= b_1 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= b_1 \\
b_2 &= 0 \\
b_3 &= 0
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= 1 \\
\eta &= 1
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= 1 - (-\sin(-y + x))\,(1) \\
&= 1 - \sin(y)\cos(x) + \cos(y)\sin(x) \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{1 - \sin(y)\cos(x) + \cos(y)\sin(x)} dy
\end{aligned}
$$

Which results in

$$
S = -\frac{2}{-\tan\left(-\frac{y}{2} + \frac{x}{2}\right) - 1}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = -\sin(-y + x)
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{1}{1+\sin(-y+x)} \\
S_y &= \frac{1}{1+\sin(-y+x)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -1 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -1
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$
\begin{aligned}
\int dS &= \int -1\, dR \\
S(R) &= -R + c_2
\end{aligned}
$$

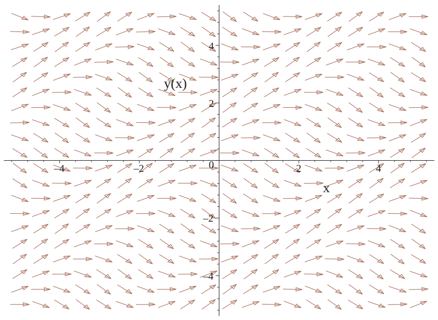
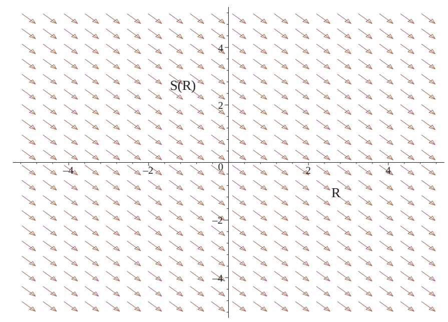
To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$
\frac{2}{\tan\left(-\frac{y}{2}+\frac{x}{2}\right)+1} = -x + c_2
$$

Which gives

$$
y = x + 2\arctan\left(\frac{c_2 - x - 2}{-x + c_2}\right)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\sin(-y+x)$ 	$R = x$ $S = \frac{2}{\tan\left(-\frac{y}{2}+\frac{x}{2}\right)+1}$	$\frac{dS}{dR} = -1$ 

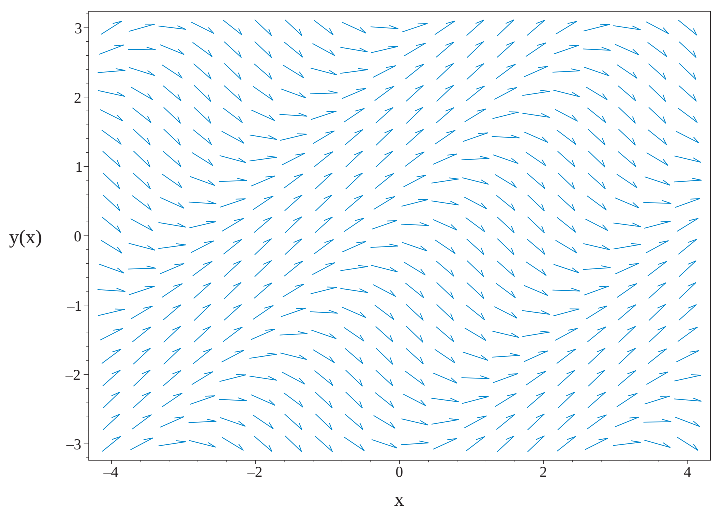


Figure 2.155: Slope field plot
$y' + \sin(-y+x) = 0$

Summary of solutions found

$$y = x + 2\arctan\left(\frac{c_2 - x - 2}{-x + c_2}\right)$$

Solved as first order ode of type dAlembert

Time used: 0.813 (sec)

Let $p = y'$ the ode becomes

$$p + \sin(-y + x) = 0$$

Solving for y from the above results in

$$y = x + \arcsin(p) \tag{1}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= 1 \\ g &= \arcsin(p) \end{aligned}$$

Hence (2) becomes

$$p - 1 = \frac{p'(x)}{\sqrt{-p^2 + 1}} \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - 1 = 0$$

Solving the above for p results in

$$p_1 = 1$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$y = x + \frac{\pi}{2}$$

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$p'(x) = (p(x) - 1)\sqrt{-p(x)^2 + 1} \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed. Integrating gives

$$\begin{aligned} \int \frac{1}{(p-1)\sqrt{-p^2+1}} dp &= dx \\ \frac{\sqrt{-p^2+1}}{p-1} &= x + c_1 \end{aligned}$$

Singular solutions are found by solving

$$(p-1)\sqrt{-p^2+1} = 0$$

for $p(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$\begin{aligned} p(x) &= -1 \\ p(x) &= 1 \end{aligned}$$

Substituing the above solution for p in (2A) gives

$$\begin{aligned} y = x + \arcsin\left(\frac{c_1^2 + 2c_1 x + x^2 - 1}{c_1^2 + 2c_1 x + x^2 + 1}\right) \\ y = x - \frac{\pi}{2} \\ y = x + \frac{\pi}{2} \end{aligned}$$

The solution

$$y = x - \frac{\pi}{2}$$

was found not to satisfy the ode or the IC. Hence it is removed.

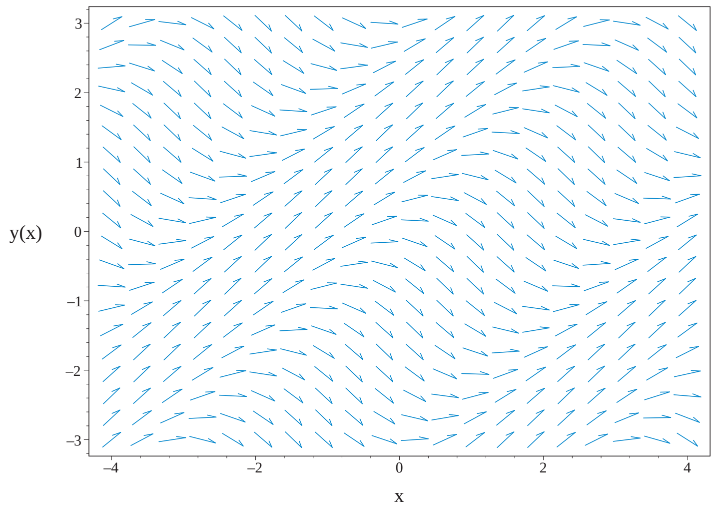


Figure 2.156: Slope field plot
$y' + \sin(-y + x) = 0$

Summary of solutions found

$$y = x + \frac{\pi}{2}$$

$$y = x + \arcsin\left(\frac{c_1^2 + 2c_1 x + x^2 - 1}{c_1^2 + 2c_1 x + x^2 + 1}\right)$$

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) + \sin(x - y(x)) = 0$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = -\sin(x - y(x))$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

Maple dsolve solution

Solving time : 0.046 (sec)
Leaf size : 23

```
dsolve(diff(y(x),x)+sin(x-y(x)) = 0,
       y(x),singsol=all)
```

$$y = x + 2 \arctan\left(\frac{c_1 - x - 2}{-x + c_1}\right)$$

Mathematica DSolve solution

Solving time : `37.899` (sec)
Leaf size : 553

```
DSolve[{D[y[x],x]-Sin[y[x]-x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -2\arccos\left(\frac{(-x+2+c_1)\cos\left(\frac{x}{2}\right)+(x-c_1)\sin\left(\frac{x}{2}\right)}{\sqrt{2}\sqrt{x^2-2(1+c_1)x+2+c_1{}^2+2c_1}}\right)$$
$$y(x) \to 2\arccos\left(\frac{(-x+2+c_1)\cos\left(\frac{x}{2}\right)+(x-c_1)\sin\left(\frac{x}{2}\right)}{\sqrt{2}\sqrt{x^2-2(1+c_1)x+2+c_1{}^2+2c_1}}\right)$$
$$y(x) \to -2\arccos\left(\frac{(x-2-c_1)\cos\left(\frac{x}{2}\right)+(-x+c_1)\sin\left(\frac{x}{2}\right)}{\sqrt{2}\sqrt{x^2-2(1+c_1)x+2+c_1{}^2+2c_1}}\right)$$
$$y(x) \to 2\arccos\left(\frac{(x-2-c_1)\cos\left(\frac{x}{2}\right)+(-x+c_1)\sin\left(\frac{x}{2}\right)}{\sqrt{2}\sqrt{x^2-2(1+c_1)x+2+c_1{}^2+2c_1}}\right)$$
$$y(x) \to -2\arccos\left(\frac{\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)}{\sqrt{2}}\right)$$
$$y(x) \to 2\arccos\left(\frac{\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)}{\sqrt{2}}\right)$$
$$y(x) \to -2\arccos\left(\frac{\sin\left(\frac{x}{2}\right)-\cos\left(\frac{x}{2}\right)}{\sqrt{2}}\right)$$
$$y(x) \to 2\arccos\left(\frac{\sin\left(\frac{x}{2}\right)-\cos\left(\frac{x}{2}\right)}{\sqrt{2}}\right)$$
$$y(x) \to -2\arccos\left(\frac{(x-2)\cos\left(\frac{x}{2}\right)-x\sin\left(\frac{x}{2}\right)}{\sqrt{2}\sqrt{x^2-2x+2}}\right)$$
$$y(x) \to 2\arccos\left(\frac{(x-2)\cos\left(\frac{x}{2}\right)-x\sin\left(\frac{x}{2}\right)}{\sqrt{2}\sqrt{x^2-2x+2}}\right)$$
$$y(x) \to -2\arccos\left(\frac{x\sin\left(\frac{x}{2}\right)-(x-2)\cos\left(\frac{x}{2}\right)}{\sqrt{2}\sqrt{x^2-2x+2}}\right)$$
$$y(x) \to 2\arccos\left(\frac{x\sin\left(\frac{x}{2}\right)-(x-2)\cos\left(\frac{x}{2}\right)}{\sqrt{2}\sqrt{x^2-2x+2}}\right)$$

2.1.60 problem 60

Internal problem ID [8448]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 60
Date solved : `Thursday, December 12, 2024 at 09:10:03 AM`
CAS classification : `[[_2nd_order, _quadrature]]`

Solve

$$y'' = 4\sin(x) - 4$$

Solved as second order ode quadrature

Time used: 0.053 (sec)

Integrating once gives

$$y' = -4x - 4\cos(x) + c_1$$

Integrating again gives

$$y = -2x^2 - 4\sin(x) + c_1 x + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -2x^2 - 4\sin(x) + c_1 x + c_2$$

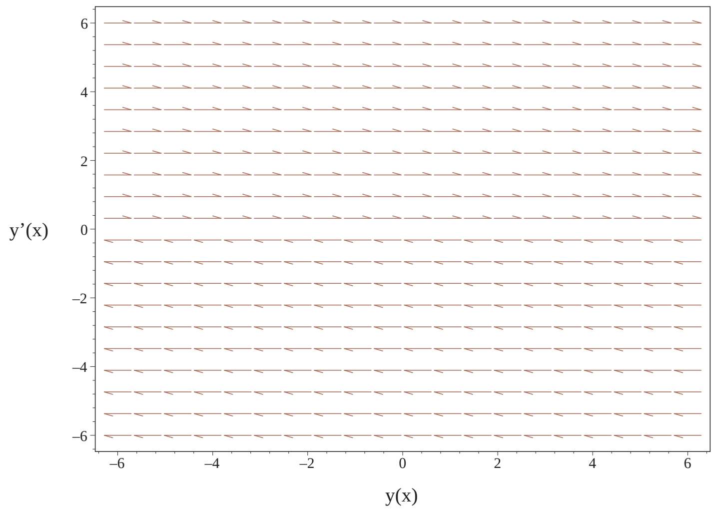


Figure 2.157: Slope field plot
$y'' = 4\sin(x) - 4$

Solved as second order linear constant coeff ode

Time used: 0.105 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 0, f(x) = 4\sin(x) - 4$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(0)^2 - (4)(1)(0)} \\ &= 0\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 0$. Therefore the solution is

$$y = c_1 1 + c_2 x \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_2 x + c_1$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\sin(x) - 4$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}, \{\cos(x), \sin(x)\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\}, \{\cos(x), \sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 + A_2 \cos(x) + A_3 \sin(x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 - A_2 \cos(x) - A_3 \sin(x) = 4\sin(x) - 4$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2, A_2 = 0, A_3 = -4]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2x^2 - 4\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 x + c_1) + \left(-2x^2 - 4\sin(x)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -2x^2 - 4\sin(x) + c_2 x + c_1$$

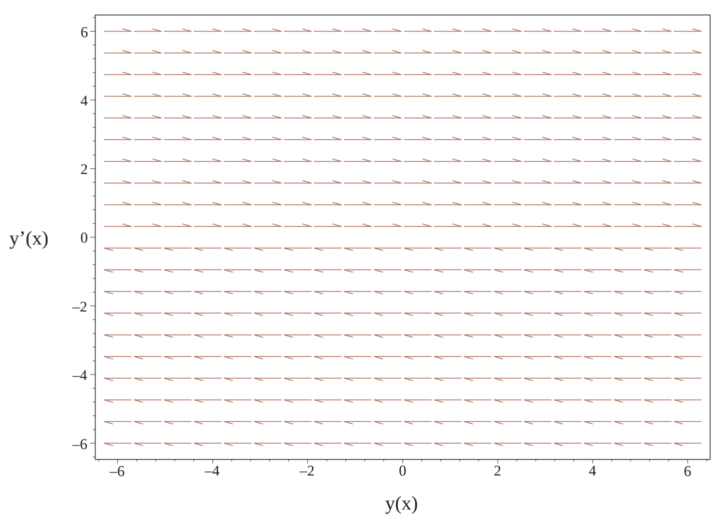


Figure 2.158: Slope field plot
$y'' = 4\sin(x) - 4$

Solved as second order linear exact ode

Time used: 0.087 (sec)

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned}p(x) &= 1\\ q(x) &= 0\\ r(x) &= 0\\ s(x) &= 4\sin(x) - 4\end{aligned}$$

Hence

$$\begin{aligned}p''(x) &= 0\\ q'(x) &= 0\end{aligned}$$

Therefore (1) becomes

$$0 - (0) + (0) = 0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)$$

Integrating gives

$$p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\;dx$$

Substituting the above values for p, q, r, s gives

$$y' = \int 4\sin(x) - 4\,dx$$

We now have a first order ode to solve which is

$$y' = -4x - 4\cos(x) + c_1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int -4x - 4\cos(x) + c_1 \, dx$$
$$y = -2x^2 + c_1 x - 4\sin(x) + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -2x^2 - 4\sin(x) + c_1 x + c_2$$

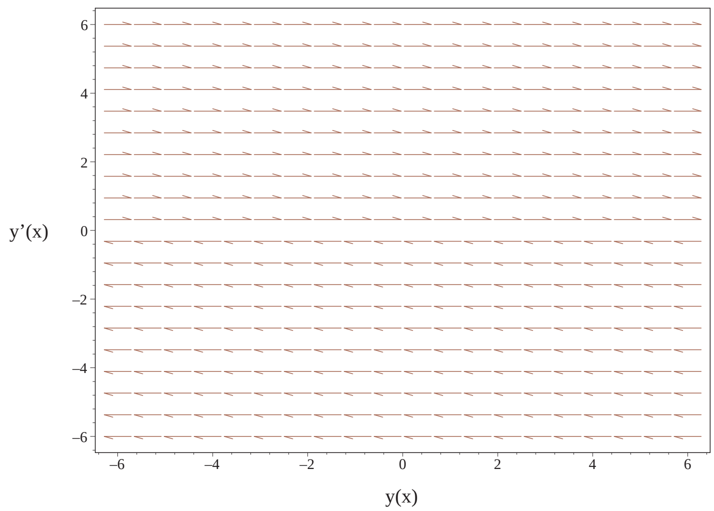


Figure 2.159: Slope field plot
$y'' = 4\sin(x) - 4$

Solved as second order missing y ode

Time used: 0.073 (sec)

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - 4\sin(x) + 4 = 0$$

Which is now solve for $p(x)$ as first order ode. Since the ode has the form $p'(x) = f(x)$, then we only need to integrate $f(x)$.

$$\int dp = \int 4\sin(x) - 4\,dx$$
$$p(x) = -4x - 4\cos(x) + c_1$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -4x - 4\cos(x) + c_1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int -4x - 4\cos(x) + c_1\,dx$$
$$y = -2x^2 + c_1 x - 4\sin(x) + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -2x^2 - 4\sin(x) + c_1 x + c_2$$

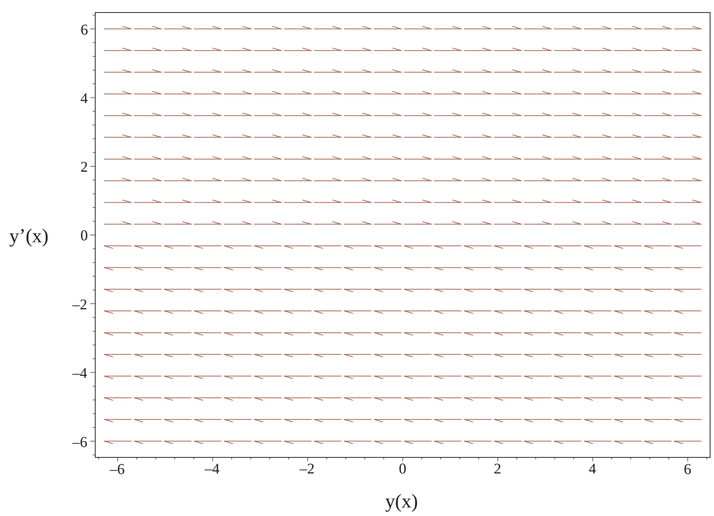


Figure 2.160: Slope field plot
$y'' = 4\sin(x) - 4$

Solved as second order integrable as is ode

Time used: 0.038 (sec)

Integrating both sides of the ODE w.r.t x gives

$$\int y'' dx = \int (4\sin(x) - 4)\, dx$$
$$y' = -4x - 4\cos(x) + c_1$$

Which is now solved for y. Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int -4x - 4\cos(x) + c_1\, dx$$
$$y = -2x^2 + c_1 x - 4\sin(x) + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -2x^2 - 4\sin(x) + c_1 x + c_2$$

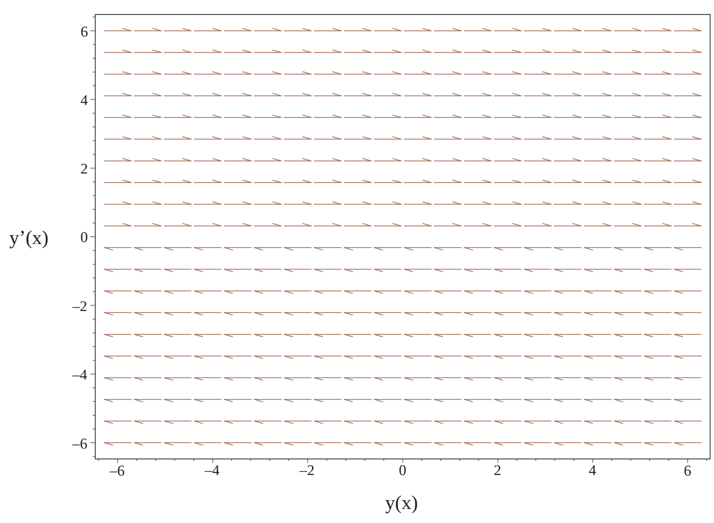


Figure 2.161: Slope field plot
$y'' = 4\sin(x) - 4$

Solved as second order integrable as is ode (ABC method)

Time used: 0.099 (sec)

Writing the ode as

$$y'' = 4\sin(x) - 4$$

Integrating both sides of the ODE w.r.t x gives

$$\int y'' dx = \int (4\sin(x) - 4)\, dx$$

$$y' = -4x - 4\cos(x) + c_1$$

Which is now solved for y. Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int -4x - 4\cos(x) + c_1\, dx$$

$$y = -2x^2 + c_1 x - 4\sin(x) + c_2$$

Will add steps showing solving for IC soon.

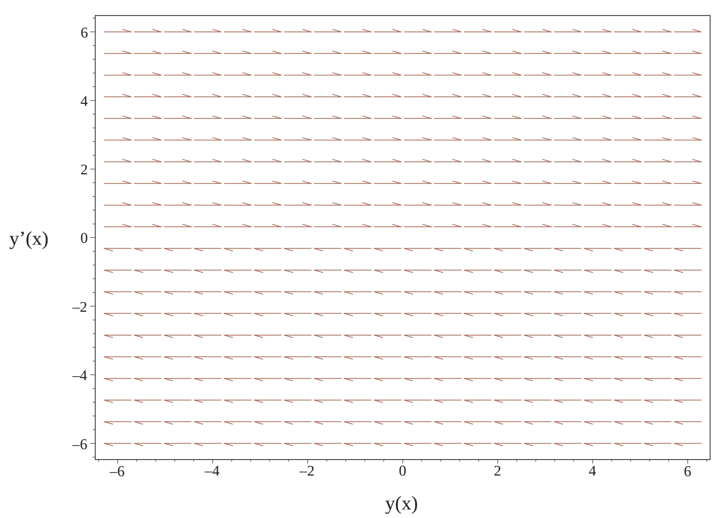


Figure 2.162: Slope field plot
$y'' = 4\sin(x) - 4$

Solved as second order ode using Kovacic algorithm

Time used: 0.071 (sec)

Writing the ode as

$$y'' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.72: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= 1 \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= 1 \int \frac{1}{1}\,dx \\ &= 1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2(1(x)) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_2 x + c_1$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\sin(x) - 4$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{1, x\}$$

Since 1 is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\}, \{\cos(x), \sin(x)\}]$$

Since x is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2\}, \{\cos(x), \sin(x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 + A_2 \cos(x) + A_3 \sin(x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 - A_2 \cos(x) - A_3 \sin(x) = 4\sin(x) - 4$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2, A_2 = 0, A_3 = -4]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2x^2 - 4\sin(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_2 x + c_1) + \left(-2x^2 - 4\sin(x)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -2x^2 - 4\sin(x) + c_2 x + c_1$$

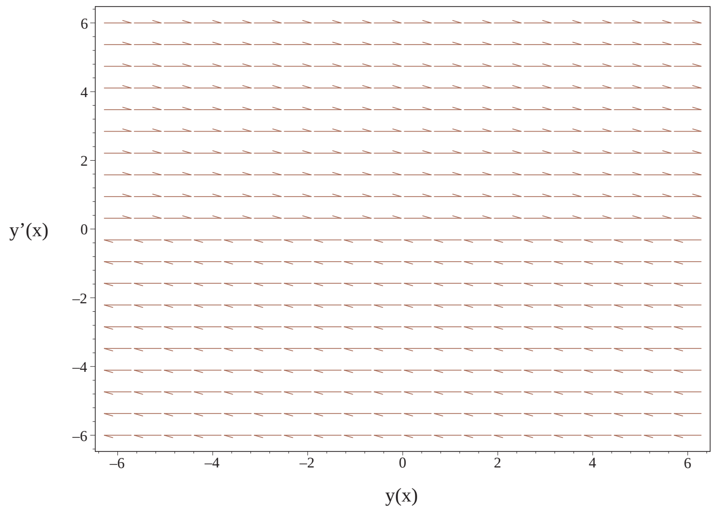


Figure 2.163: Slope field plot
$y'' = 4\sin(x) - 4$

Solved as second order ode adjoint method

Time used: 0.579 (sec)

In normal form the ode

$$y'' = 4\sin(x) - 4 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned}
p(x) &= 0 \\
q(x) &= 0 \\
r(x) &= 4\sin(x) - 4
\end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - (0)' + (0) &= 0 \\
\xi''(x) &= 0
\end{aligned}$$

Which is solved for $\xi(x)$. Integrating twice gives the solution

$$\xi = c_1 x + c_2$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\ y' + y\left(p(x) - \frac{\xi'(x)}{\xi\left(x\right)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi\left(x\right)}\end{aligned}$$

Or

$$y' - \frac{y c_1}{c_1 x + c_2} = \frac{4c_1(\sin\left(x\right) - x\cos\left(x\right)) - 4\cos\left(x\right) c_2 - 2c_1\, x^2 - 4c_2 x}{c_1 x + c_2}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned}q(x) &= -\frac{c_1}{c_1 x + c_2} \\ p(x) &= \frac{(-4c_1 x - 4c_2)\cos\left(x\right) - 2c_1\, x^2 - 4c_2 x + 4\sin\left(x\right) c_1}{c_1 x + c_2}\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\, dx} \\ &= \mathrm{e}^{\int -\frac{c_1}{c_1 x + c_2} dx} \\ &= \frac{1}{c_1 x + c_2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{(-4c_1 x - 4c_2)\cos\left(x\right) - 2c_1\, x^2 - 4c_2 x + 4\sin\left(x\right) c_1}{c_1 x + c_2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{c_1 x + c_2}\right) &= \left(\frac{1}{c_1 x + c_2}\right)\left(\frac{(-4c_1 x - 4c_2)\cos\left(x\right) - 2c_1\, x^2 - 4c_2 x + 4\sin\left(x\right) c_1}{c_1 x + c_2}\right) \\ \mathrm{d}\left(\frac{y}{c_1 x + c_2}\right) &= \left(\frac{(-4c_1 x - 4c_2)\cos\left(x\right) - 2c_1\, x^2 - 4c_2 x + 4\sin\left(x\right) c_1}{\left(c_1 x + c_2\right)^2}\right) \mathrm{d}x\end{aligned}$$

Integrating gives

$$\frac{y}{c_1x+c_2} = \int \frac{(-4c_1x-4c_2)\cos(x)-2c_1\,x^2-4c_2x+4\sin(x)\,c_1}{(c_1x+c_2)^2}\,dx$$

$$= -\frac{4\,\mathrm{Si}\left(x+\frac{c_2}{c_1}\right)\sin\left(\frac{c_2}{c_1}\right)}{c_1} - \frac{4\,\mathrm{Ci}\left(x+\frac{c_2}{c_1}\right)\cos\left(\frac{c_2}{c_1}\right)}{c_1} - \frac{2x}{c_1} - \frac{2c_2^2}{c_1^2\,(c_1x+c_2)} + 4c_1\left(-\frac{\sin(x)}{(c_1x+c_2)\,c\right.$$

Dividing throughout by the integrating factor $\frac{1}{c_1x+c_2}$ gives the final solution

$$y = \frac{c_1^3c_3x+c_1^2c_2c_3-2c_1^2x^2-4\sin(x)\,c_1^2-2c_1c_2x-2c_2^2}{c_1^2}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{c_1^3c_3x+c_1^2c_2c_3-2c_1^2x^2-4\sin(x)\,c_1^2-2c_1c_2x-2c_2^2}{c_1^2}$$

The constants can be merged to give

$$y = \frac{c_1^3x+c_1^2c_2-2c_1^2x^2-4\sin(x)\,c_1^2-2c_1c_2x-2c_2^2}{c_1^2}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{c_1^3x+c_1^2c_2-2c_1^2x^2-4\sin(x)\,c_1^2-2c_1c_2x-2c_2^2}{c_1^2}$$

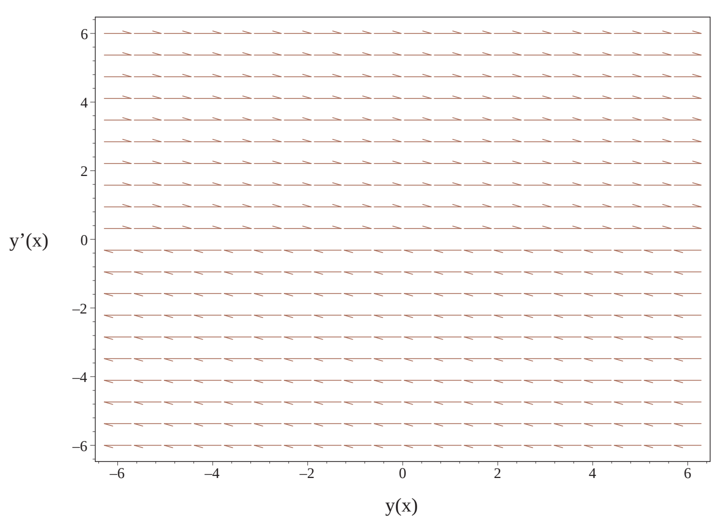


Figure 2.164: Slope field plot
$y'' = 4\sin(x) - 4$

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) = 4\sin(x) - 4$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Characteristic polynomial of homogeneous ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{0})}{2}$

- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the homogeneous ODE

  $y_1(x) = 1$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x$

- General solution of the ODE

  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y(x) = C1 + C2x + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 4\sin(x) - 4\right]$

  ◦ Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$

  ◦ Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  ◦ Substitute functions into equation for $y_p(x)$

    $y_p(x) = -4\left(\int x(\sin(x) - 1)\, dx\right) + 4x\left(\int (\sin(x) - 1)\, dx\right)$

  ◦ Compute integrals

    $y_p(x) = -2x^2 - 4\sin(x)$

- Substitute particular solution into general solution to ODE

  $y(x) = C1 + C2x - 2x^2 - 4\sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 18

```
dsolve(diff(diff(y(x),x),x) = 4*sin(x)-4,
       y(x),singsol=all)
```

$$y = -2x^2 - 4\sin(x) + c_1x + c_2$$

Mathematica DSolve solution

Solving time : 0.032 (sec)
Leaf size : 21

```
DSolve[{D[y[x],{x,2}]==4*Sin[x]-4,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -2x^2 - 4\sin(x) + c_2x + c_1$$

2.1.61 problem 61

Internal problem ID [8449]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 61
Date solved : `Thursday, December 12, 2024 at 09:10:05 AM`
CAS classification : `[[_2nd_order, _quadrature]]`

Solve

$$yy'' = 0$$

Solved as second order missing x ode

Time used: 0.098 (sec)

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dp}{dy}\frac{dy}{dx} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y)\left(\frac{d}{dy}p(y)\right) = 0$$

Which is now solved as first order ode for $p(y)$.

Since the ode has the form $p' = f(y)$, then we only need to integrate $f(y)$.

$$\int dp = \int 0\,dy + c_1$$
$$p = c_1$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = c_1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int c_1\,dx$$
$$y = c_1 x + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 x + c_2$$

Maple step by step solution

Let's solve
$y(x)\left(\frac{d^2}{dx^2}y(x)\right) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = 0$
- Characteristic polynomial of ODE
  $r^2 = 0$
- Use quadratic formula to solve for r
  $r = \frac{0\pm\left(\sqrt{0}\right)}{2}$
- Roots of the characteristic polynomial
  $r = 0$
- 1st solution of the ODE
  $y_1(x) = 1$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence
  $y_2(x) = x$
- General solution of the ODE
  $y(x) = C1y_1(x) + C2y_2(x)$
- Substitute in solutions
  $y(x) = C2x + C1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 13

```
dsolve(y(x)*diff(diff(y(x),x),x) = 0,
       y(x),singsol=all)
```

$$y = 0$$
$$y = c_1x + c_2$$

Mathematica DSolve solution

Solving time : 0.003 (sec)
Leaf size : 17

```
DSolve[{y[x]*D[y[x],{x,2}]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to 0$$
$$y(x) \to c_2x + c_1$$

2.1.62 problem 62

Internal problem ID [8450]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 62
Date solved : `Thursday, December 12, 2024 at 09:10:05 AM`
CAS classification :
`[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]`

Solve

$$yy'' = 1$$

Solved as second order missing x ode

Time used: 0.490 (sec)

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dp}{dy}\frac{dy}{dx} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$yp(y)\left(\frac{d}{dy}p(y)\right) = 1$$

Which is now solved as first order ode for $p(y)$.

The ode $p' = \frac{1}{py}$ is separable as it can be written as

$$\begin{aligned} p' &= \frac{1}{py} \\ &= f(y)g(p) \end{aligned}$$

Where

$$\begin{aligned} f(y) &= \frac{1}{y} \\ g(p) &= \frac{1}{p} \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(p)}\, dp &= \int f(y)\, dy \\ \int p\, dp &= \int \frac{1}{y}\, dy \\ \frac{p^2}{2} &= \ln(y) + c_1 \end{aligned}$$

Solving for p gives

$$\begin{aligned} p &= \sqrt{2\ln(y) + 2c_1} \\ p &= -\sqrt{2\ln(y) + 2c_1} \end{aligned}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \sqrt{2\ln(y) + 2c_1}$$

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} \frac{1}{\sqrt{2\ln(\tau) + 2c_1}} d\tau = x + c_2$$

Singular solutions are found by solving

$$\sqrt{2\ln(y) + 2c_1} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = e^{-c_1}$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\sqrt{2\ln(y) + 2c_1}$$

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} -\frac{1}{\sqrt{2\ln(\tau) + 2c_1}} d\tau = x + c_3$$

Singular solutions are found by solving

$$-\sqrt{2\ln(y) + 2c_1} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = e^{-c_1}$$

Will add steps showing solving for IC soon.

The solution

$$y = e^{-c_1}$$

was found not to satisfy the ode or the IC. Hence it is removed.

Summary of solutions found

$$\int^{y} \frac{1}{\sqrt{2\ln(\tau) + 2c_1}} d\tau = x + c_2$$

$$\int^{y} -\frac{1}{\sqrt{2\ln(\tau) + 2c_1}} d\tau = x + c_3$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-1/_a = 0, _b(_a), HINT = [
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, 0]
```

Maple dsolve solution

Solving time : 0.053 (sec)
Leaf size : 51

```
dsolve(y(x)*diff(diff(y(x),x),x) = 1,
       y(x),singsol=all)
```

$$\int^{y}\frac{1}{\sqrt{2\ln\left(\_a\right)-c_1}}d\_a-x-c_2=0$$
$$-\left(\int^{y}\frac{1}{\sqrt{2\ln\left(\_a\right)-c_1}}d\_a\right)-x-c_2=0$$

Mathematica DSolve solution

Solving time : 60.072 (sec)
Leaf size : 93

```
DSolve[{y[x]*D[y[x],{x,2}]==1,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \exp\left(-\text{erf}^{-1}\left(-i\sqrt{\frac{2}{\pi}}\sqrt{e^{c_1}(x+c_2)^2}\right)^2 - \frac{c_1}{2}\right)$$
$$y(x) \to \exp\left(-\text{erf}^{-1}\left(i\sqrt{\frac{2}{\pi}}\sqrt{e^{c_1}(x+c_2)^2}\right)^2 - \frac{c_1}{2}\right)$$

2.1.63 problem 63

Internal problem ID [8451]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 63
Date solved : `Thursday, December 12, 2024 at 09:10:06 AM`
CAS classification :
`[[_Emden, _Fowler], [_2nd_order, _with_linear_symmetries]]`

Solve

$$yy'' = x$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
-> trying 2nd order, dynamical_symmetries, fully reducible to Abel through one integra
trying 2nd order, integrating factors of the form mu(x,y)/(y)^n, only the singular cas
trying symmetries linear in x and y(x)
trying differential order: 2; exact nonlinear
trying 2nd order, integrating factor of the form mu(y)
trying 2nd order, integrating factor of the form mu(x,y)
trying 2nd order, integrating factor of the form mu(x,y)/(y)^n, only the general case
trying 2nd order, integrating factor of the form mu(y,y)
-> Calling odsolve with the ODE`, -(_y1^3*x-1)*y(x)/(_y1^3*x)+(1/3)*(3*(diff(y(x), x))
```

```
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
trying differential order: 2; mu polynomial in y
trying 2nd order, integrating factor of the form mu(x,y)
differential order: 2; looking for linear symmetries
differential order: 2; found: 1 linear symmetries. Trying reduction of order
`, `2nd order, trying reduction of order with given symmetries:`[x, 3/2*y]
```

Maple dsolve solution

Solving time : 0.128 (sec)
Leaf size : maple_leaf_size

```
dsolve(y(x)*diff(diff(y(x),x),x) = x,
       y(x),singsol=all)
```

No solution found

Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{y[x]*D[y[x],{x,2}]==x,{}},
       y[x],x,IncludeSingularSolutions->True]
```

Not solved

2.1.64 problem 64

Internal problem ID [8452]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 64
Date solved : `Wednesday, December 18, 2024 at 01:53:07 AM`
CAS classification :
`[[_Emden, _Fowler], [_2nd_order, _with_linear_symmetries]]`

Solve

$$y^2y'' = x$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
-> trying 2nd order, dynamical_symmetries, fully reducible to Abel through one integra
trying 2nd order, integrating factors of the form mu(x,y)/(y)^n, only the singular cas
trying symmetries linear in x and y(x)
trying differential order: 2; exact nonlinear
trying 2nd order, integrating factor of the form mu(y)
trying 2nd order, integrating factor of the form mu(x,y)
trying 2nd order, integrating factor of the form mu(x,y)/(y)^n, only the general case
trying 2nd order, integrating factor of the form mu(y,y)
-> Calling odsolve with the ODE`, -(_y1^3-4)*y(x)/_y1^3+2*((diff(y(x), x))*x+_y1)/_y1
```

```
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
trying differential order: 2; mu polynomial in y
trying 2nd order, integrating factor of the form mu(x,y)
differential order: 2; looking for linear symmetries
differential order: 2; found: 1 linear symmetries. Trying reduction of order
`, `2nd order, trying reduction of order with given symmetries:`[x, y]
```

Maple dsolve solution

Solving time : 0.089 (sec)
Leaf size : 106

```
dsolve(y(x)^2*diff(diff(y(x),x),x) = x,
       y(x),singsol=all)
```

$$y = \mathrm{RootOf}\left(\ln(x)\right.$$

$$\left.+2^{1/3}\left(\int^{\_Z}\frac{1}{2^{1/3}\_f + 2\,\mathrm{RootOf}\left(\mathrm{AiryBi}\left(\frac{2\_Z^2\_f+2^{2/3}}{2\_f}\right)c_1\_Z + \_Z\,\mathrm{AiryAi}\left(\frac{2\_Z^2\_f+2^{2/3}}{2\_f}\right) + \mathrm{AiryBi}\right.}\right.\right.$$

Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{y[x]^2*D[y[x],{x,2}]==x,{}},
       y[x],x,IncludeSingularSolutions->True]
```

Not solved

2.1.65 problem 65

Internal problem ID [8453]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 65
Date solved : `Thursday, December 12, 2024 at 09:10:07 AM`
CAS classification : `[[_2nd_order, _quadrature]]`

Solve

$$y^2y'' = 0$$

Solved as second order missing x ode

Time used: 0.100 (sec)

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dp}{dy}\frac{dy}{dx} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$y^2p(y)\left(\frac{d}{dy}p(y)\right) = 0$$

Which is now solved as first order ode for $p(y)$.

Since the ode has the form $p' = f(y)$, then we only need to integrate $f(y)$.

$$\int dp = \int 0\,dy + c_1$$
$$p = c_1$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = c_1$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int c_1\,dx$$
$$y = c_1 x + c_2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 x + c_2$$

Maple step by step solution

Let's solve

$y(x)^2 \left(\frac{d^2}{dx^2}y(x)\right) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = 0$
- Characteristic polynomial of ODE

  $r^2 = 0$
- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{0})}{2}$
- Roots of the characteristic polynomial

  $r = 0$
- 1st solution of the ODE

  $y_1(x) = 1$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence
  $y_2(x) = x$
- General solution of the ODE
  $y(x) = C1y_1(x) + C2y_2(x)$
- Substitute in solutions
  $y(x) = C2x + C1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 13

```
dsolve(y(x)^2*diff(diff(y(x),x),x) = 0,
       y(x),singsol=all)
```

$$y = 0$$
$$y = c_1 x + c_2$$

Mathematica DSolve solution

Solving time : 0.002 (sec)
Leaf size : 17

```
DSolve[{y[x]^2*D[y[x],{x,2}]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to 0$$
$$y(x) \to c_2 x + c_1$$

2.1.66 problem 66

Internal problem ID [8454]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 66
Date solved : `Thursday, December 12, 2024 at 09:10:08 AM`
CAS classification : `[NONE]`

Solve

$$3yy'' = \sin(x)$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
-> trying 2nd order, dynamical_symmetries, fully reducible to Abel through one integra
trying 2nd order, integrating factors of the form mu(x,y)/(y)^n, only the singular cas
trying symmetries linear in x and y(x)
trying differential order: 2; exact nonlinear
trying 2nd order, integrating factor of the form mu(y)
trying 2nd order, integrating factor of the form mu(x,y)
trying 2nd order, integrating factor of the form mu(x,y)/(y)^n, only the general case
trying 2nd order, integrating factor of the form mu(y,y)
trying differential order: 2; mu polynomial in y
trying 2nd order, integrating factor of the form mu(x,y)
```

```
differential order: 2; looking for linear symmetries
-> trying 2nd order, the S-function method
   -> trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods for
   -> trying 2nd order, the S-function method
   -> trying 2nd order, No Point Symmetries Class V
   -> trying 2nd order, No Point Symmetries Class V
   -> trying 2nd order, No Point Symmetries Class V
trying 2nd order, integrating factor of the form mu(x,y)/(y)^n, only the general case
-> trying 2nd order, dynamical_symmetries, only a reduction of order through one integ
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 5
`, `-> Computing symmetries using: way = formal`
```

Maple dsolve solution

Solving time : 0.048 (sec)
Leaf size : maple_leaf_size

```
dsolve(3*y(x)*diff(diff(y(x),x),x) = sin(x),
       y(x),singsol=all)
```

No solution found

Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{3*y[x]*D[y[x],{x,2}]==Sin[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

Not solved

2.1.67 problem 67

Internal problem ID [8455]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 67
Date solved : `Thursday, December 12, 2024 at 09:10:08 AM`
CAS classification :
`[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]`

Solve

$$3yy'' + y = 5$$

Solved as second order missing x ode

Time used: 483.685 (sec)

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dp}{dy}\frac{dy}{dx} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$3yp(y)\left(\frac{d}{dy}p(y)\right) + y = 5$$

Which is now solved as first order ode for $p(y)$.

The ode $p' = -\frac{y-5}{3py}$ is separable as it can be written as

$$\begin{aligned} p' &= -\frac{y-5}{3py} \\ &= f(y)g(p) \end{aligned}$$

Where

$$\begin{aligned} f(y) &= -\frac{y-5}{3y} \\ g(p) &= \frac{1}{p} \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(p)}\,dp &= \int f(y)\,dy \\ \int p\,dp &= \int -\frac{y-5}{3y}\,dy \\ \frac{p^2}{2} &= -\frac{y}{3} + \ln\left(y^{5/3}\right) + c_1 \end{aligned}$$

Solving for p gives

$$\begin{aligned} p &= -\frac{\sqrt{18\ln\left(y^{5/3}\right) + 18c_1 - 6y}}{3} \\ p &= \frac{\sqrt{18\ln\left(y^{5/3}\right) + 18c_1 - 6y}}{3} \end{aligned}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\frac{\sqrt{18\ln\left(y^{5/3}\right) + 18c_1 - 6y}}{3}$$

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^y -\frac{3}{\sqrt{18\ln\left(\tau^{5/3}\right) + 18c_1 - 6\tau}}d\tau = x + c_2$$

Singular solutions are found by solving

$$-\frac{\sqrt{18\ln\left(y^{5/3}\right) + 18c_1 - 6y}}{3} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = \left(\mathrm{e}^{-\frac{5\,\mathrm{LambertW}\left(-\frac{243^{4/5}\mathrm{e}^{-\frac{3c_1}{5}}}{405}\right)}{3} - c_1}\right)^{3/5}$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{\sqrt{18\ln\left(y^{5/3}\right) + 18c_1 - 6y}}{3}$$

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^y \frac{3}{\sqrt{18\ln\left(\tau^{5/3}\right) + 18c_1 - 6\tau}} d\tau = x + c_3$$

Singular solutions are found by solving

$$\frac{\sqrt{18\ln\left(y^{5/3}\right) + 18c_1 - 6y}}{3} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = \left(\mathrm{e}^{-\frac{5\,\mathrm{LambertW}\left(-\frac{243^{4/5}\mathrm{e}^{-\frac{3c_1}{5}}}{405}\right)}{3} - c_1}\right)^{3/5}$$

Will add steps showing solving for IC soon.

The solution

$$\int^y -\frac{3}{\sqrt{18\ln\left(\tau^{5/3}\right) + 18c_1 - 6\tau}} d\tau = x + c_2$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$y = \left(\mathrm{e}^{-\frac{5\,\mathrm{LambertW}\left(-\frac{243^{4/5}\mathrm{e}^{-\frac{3c_1}{5}}}{405}\right)}{3} - c_1}\right)^{3/5}$$

was found not to satisfy the ode or the IC. Hence it is removed.

Summary of solutions found

$$\int^{y} \frac{3}{\sqrt{18 \ln \left(\tau^{5/3}\right) + 18c_1 - 6\tau}} d\tau = x + c_3$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+(1/3)*(_a-5)/_a = 0, _b(_a
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   <- Bernoulli successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

Maple dsolve solution

Solving time : 0.062 (sec)
Leaf size : 59

```
dsolve(3*y(x)*diff(diff(y(x),x),x)+y(x) = 5,
       y(x),singsol=all)
```

$$-3\left(\int^{y} \frac{1}{\sqrt{30 \ln(\_a) + 9c_1 - 6\_a}} d\_a\right) - x - c_2 = 0$$
$$3\left(\int^{y} \frac{1}{\sqrt{30 \ln(\_a) + 9c_1 - 6\_a}} d\_a\right) - x - c_2 = 0$$

Mathematica DSolve solution

Solving time : 0.324 (sec)
Leaf size : 41

```
DSolve[{3*y[x]*D[y[x],{x,2}]+y[x]==5,{}},
      y[x],x,IncludeSingularSolutions->True]
```

$$\text{Solve}\left[\int_1^{y(x)} \frac{1}{\sqrt{c_1 + \frac{2}{3}(5\log(K[1]) - K[1])}} dK[1]^2 = (x + c_2)^2, y(x)\right]$$

2.1.68 problem 68

Internal problem ID [8456]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 68
Date solved : `Thursday, December 12, 2024 at 09:18:12 AM`
CAS classification :
`[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]`

Solve

$$ayy'' + by = c$$

Solved as second order missing x ode

Time used: 2.042 (sec)

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dp}{dy}\frac{dy}{dx} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$ayp(y)\left(\frac{d}{dy}p(y)\right) + by = c$$

Which is now solved as first order ode for $p(y)$.

The ode $p' = -\frac{by-c}{pay}$ is separable as it can be written as

$$\begin{aligned} p' &= -\frac{by-c}{pay} \\ &= f(y)g(p) \end{aligned}$$

Where

$$\begin{aligned} f(y) &= -\frac{by-c}{ay} \\ g(p) &= \frac{1}{p} \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(p)}\,dp &= \int f(y)\,dy \\ \int p\,dp &= \int -\frac{by-c}{ay}\,dy \\ \frac{p^2}{2} &= \frac{c\ln(y)-by}{a} + c_1 \end{aligned}$$

Solving for p gives

$$p = \frac{\sqrt{2}\,\sqrt{a\,(c\ln(y)+c_1a-by)}}{a}$$

$$p = -\frac{\sqrt{2}\,\sqrt{a\,(c\ln(y)+c_1a-by)}}{a}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{\sqrt{2}\,\sqrt{a\,(c\ln(y)+c_1a-by)}}{a}$$

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} \frac{a\sqrt{2}}{2\sqrt{a\,(c\ln(\tau)+c_1a-b\tau)}}d\tau = x + c_2$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\frac{\sqrt{2}\,\sqrt{a\,(c\ln(y)+c_1a-by)}}{a}$$

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} -\frac{a\sqrt{2}}{2\sqrt{a\left(c\ln\left(\tau\right)+c_1a-b\tau\right)}}d\tau = x + c_3$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$\int^{y} \frac{a\sqrt{2}}{2\sqrt{a\left(c\ln\left(\tau\right)+c_1a-b\tau\right)}}d\tau = x + c_2$$

$$\int^{y} -\frac{a\sqrt{2}}{2\sqrt{a\left(c\ln\left(\tau\right)+c_1a-b\tau\right)}}d\tau = x + c_3$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+(_a*b-c)/(_a*a) = 0, _b(_a
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   <- Bernoulli successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

Maple dsolve solution

Solving time : `0.056` (sec)
Leaf size : 68

```
dsolve(a*y(x)*diff(diff(y(x),x),x)+b*y(x) = c,
       y(x),singsol=all)
```

$$a\left(\int^{y}\frac{1}{\sqrt{a\left(2\ln\left(\_a\right)c+c_{1}a-2b\_a\right)}}d\_a\right)-x-c_{2}=0$$
$$-a\left(\int^{y}\frac{1}{\sqrt{a\left(2\ln\left(\_a\right)c+c_{1}a-2b\_a\right)}}d\_a\right)-x-c_{2}=0$$

Mathematica DSolve solution

Solving time : `0.47` (sec)
Leaf size : 43

```
DSolve[{a*y[x]*D[y[x],{x,2}]+b*y[x]==c,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$\text{Solve}\left[\int_{1}^{y(x)}\frac{1}{\sqrt{c_{1}+\frac{2(c\log(K[1])-bK[1])}{a}}}dK[1]^{2}=(x+c_{2})^{2},y(x)\right]$$

2.1.69 problem 69

Internal problem ID [8457]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 69
Date solved : `Thursday, December 12, 2024 at 09:18:15 AM`
CAS classification :
`[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]`

Solve

$$ay^2y'' + by^2 = c$$

Solved as second order missing x ode

Time used: 1.433 (sec)

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dp}{dy}\frac{dy}{dx} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$a\,y^2 p(y)\left(\frac{d}{dy}p(y)\right) + b\,y^2 = c$$

Which is now solved as first order ode for $p(y)$.

The ode $p' = -\frac{b\,y^2-c}{pa\,y^2}$ is separable as it can be written as

$$\begin{aligned} p' &= -\frac{b\,y^2-c}{pa\,y^2} \\ &= f(y)g(p) \end{aligned}$$

Where

$$\begin{aligned} f(y) &= -\frac{b\,y^2-c}{a\,y^2} \\ g(p) &= \frac{1}{p} \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(p)}\,dp &= \int f(y)\,dy \\ \int p\,dp &= \int -\frac{b\,y^2-c}{a\,y^2}\,dy \\ \frac{p^2}{2} &= \frac{-b\,y^2-c}{ay} + c_1 \end{aligned}$$

Solving for p gives

$$p = \frac{\sqrt{2}\,\sqrt{ay\,(c_1ay-b\,y^2-c)}}{ay}$$

$$p = -\frac{\sqrt{2}\,\sqrt{ay\,(c_1ay-b\,y^2-c)}}{ay}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{\sqrt{2}\,\sqrt{ay\,(c_1ay-by^2-c)}}{ay}$$

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^y \frac{a\tau\sqrt{2}}{2\sqrt{a\tau\,(c_1a\tau-b\,\tau^2-c)}}d\tau = x + c_2$$

Singular solutions are found by solving

$$\frac{\sqrt{2}\,\sqrt{ay\,(c_1ay-b\,y^2-c)}}{ay} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = \frac{ac_1 + \sqrt{c_1^2a^2 - 4bc}}{2b}$$
$$y = -\frac{-ac_1 + \sqrt{c_1^2a^2 - 4bc}}{2b}$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\frac{\sqrt{2}\,\sqrt{ay\,(c_1ay - by^2 - c)}}{ay}$$

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^y -\frac{a\tau\sqrt{2}}{2\sqrt{a\tau\,(c_1a\tau - b\,\tau^2 - c)}}d\tau = x + c_3$$

Singular solutions are found by solving

$$-\frac{\sqrt{2}\,\sqrt{ay\,(c_1ay - b\,y^2 - c)}}{ay} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = \frac{ac_1 + \sqrt{c_1^2a^2 - 4bc}}{2b}$$
$$y = -\frac{-ac_1 + \sqrt{c_1^2a^2 - 4bc}}{2b}$$

Will add steps showing solving for IC soon.

The solution

$$y = \frac{ac_1 + \sqrt{c_1^2a^2 - 4bc}}{2b}$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$y = -\frac{-ac_1 + \sqrt{c_1^2a^2 - 4bc}}{2b}$$

was found not to satisfy the ode or the IC. Hence it is removed.

Summary of solutions found

$$\int^{y} \frac{a\tau\sqrt{2}}{2\sqrt{a\tau\left(c_1 a\tau - b\,\tau^2 - c\right)}}d\tau = x + c_2$$

$$\int^{y} -\frac{a\tau\sqrt{2}}{2\sqrt{a\tau\left(c_1 a\tau - b\,\tau^2 - c\right)}}d\tau = x + c_3$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+(_a^2*b-c)/(_a^2*a) = 0, _
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   <- Bernoulli successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

Maple dsolve solution

Solving time : 0.054 (sec)
Leaf size : 76

```
dsolve(a*y(x)^2*diff(diff(y(x),x),x)+b*y(x)^2 = c,
       y(x),singsol=all)
```

$$a\left(\int^{y}\frac{\_a}{\sqrt{\_aa\left(c_1\_aa-2b\_a^2-2c\right)}}d\_a\right)-x-c_2=0$$
$$-a\left(\int^{y}\frac{\_a}{\sqrt{\_aa\left(c_1\_aa-2b\_a^2-2c\right)}}d\_a\right)-x-c_2=0$$

Mathematica DSolve solution

Solving time : 0.835 (sec)
Leaf size : 346

```
DSolve[{a*y[x]^2*D[y[x],{x,2}]+b*y[x]^2==c,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$\text{Solve}\left[-\frac{\left(\sqrt{-16bc+a^2{c_1}^2}-ac_1\right)\left(\sqrt{-16bc+a^2{c_1}^2}+ac_1\right)^2\left(1+\frac{4by(x)}{\sqrt{-16bc+a^2{c_1}^2}-ac_1}\right)\left(1-\frac{4by(x)}{\sqrt{-16bc+a^2{c_1}^2}+ac_1}\right)}{}\right.$$

2.1.70 problem 70

Internal problem ID [8458]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 70
Date solved : `Thursday, December 12, 2024 at 09:18:17 AM`
CAS classification : `[[_2nd_order, _quadrature]]`

Solve

$$
ayy'' + by = 0
$$

Solved as second order missing x ode

Time used: 0.500 (sec)

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$
y' = p
$$

Then

$$
\begin{aligned}
y'' &= \frac{dp}{dx} \\
&= \frac{dp}{dy}\frac{dy}{dx} \\
&= p\frac{dp}{dy}
\end{aligned}
$$

Hence the ode becomes

$$
ayp(y)\left(\frac{d}{dy}p(y)\right) + by = 0
$$

Which is now solved as first order ode for $p(y)$.

Integrating gives

$$\int -\frac{pa}{b}dp = dy$$

$$-\frac{p^2a}{2b} = y + c_1$$

Solving for p gives

$$p = \frac{\sqrt{-2ab\,(y+c_1)}}{a}$$

$$p = -\frac{\sqrt{-2ab\,(y+c_1)}}{a}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{\sqrt{-2ab\,(y+c_1)}}{a}$$

Integrating gives

$$\int \frac{a}{\sqrt{-2ab\,(y+c_1)}}dy = dx$$

$$-\frac{\sqrt{2}\,\sqrt{-ab\,(y+c_1)}}{b} = x + c_2$$

Singular solutions are found by solving

$$\frac{\sqrt{-2ab\,(y+c_1)}}{a} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = -c_1$$

Solving for y gives

$$y = -\frac{c_2^2b + 2c_2bx + b\,x^2 + 2c_1a}{2a}$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\frac{\sqrt{-2ab\,(y+c_1)}}{a}$$

Integrating gives

$$
\begin{aligned}
\int -\frac{a}{\sqrt{-2ab\left(y+c_1\right)}}dy &= dx \\
\frac{\sqrt{2}\,\sqrt{-ab\left(y+c_1\right)}}{b} &= x+c_3
\end{aligned}
$$

Singular solutions are found by solving

$$
-\frac{\sqrt{-2ab\left(y+c_1\right)}}{a}=0
$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$
y=-c_1
$$

Solving for y gives

$$
y=-\frac{c_3^2 b+2c_3 bx+b\,x^2+2c_1 a}{2a}
$$

Will add steps showing solving for IC soon.

The solution

$$
y=-c_1
$$

was found not to satisfy the ode or the IC. Hence it is removed.

Summary of solutions found

$$
\begin{aligned}
y &= -\frac{c_2^2 b+2c_2 bx+b\,x^2+2c_1 a}{2a} \\
y &= -\frac{c_3^2 b+2c_3 bx+b\,x^2+2c_1 a}{2a}
\end{aligned}
$$

Maple step by step solution

Let's solve

$ay(x)\left(\frac{d^2}{dx^2}y(x)\right)+by(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{b}{a}$
- Characteristic polynomial of homogeneous ODE
  $r^2 = 0$
- Use quadratic formula to solve for r
  $r = \frac{0 \pm (\sqrt{0})}{2}$
- Roots of the characteristic polynomial
  $r = 0$
- 1st solution of the homogeneous ODE
  $y_1(x) = 1$
- Repeated root, multiply $y_1(x)$ by x to ensure linear independence
  $y_2(x) = x$
- General solution of the ODE
  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE
  $y(x) = C1 + C2x + y_p(x)$
- □ Find a particular solution $y_p(x)$ of the ODE
  - Use variation of parameters to find y_p here $f(x)$ is the forcing function
    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = -\frac{b}{a}\right]$
  - Wronskian of solutions of the homogeneous equation
    $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$
  - Compute Wronskian
    $W(y_1(x), y_2(x)) = 1$
  - Substitute functions into equation for $y_p(x)$
    $y_p(x) = \frac{b\left(\int x dx - \left(\int 1 dx\right)x\right)}{a}$
  - Compute integrals
    $y_p(x) = -\frac{b x^2}{2a}$
- Substitute particular solution into general solution to ODE
  $y(x) = C1 + C2x - \frac{b x^2}{2a}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 22

```
dsolve(a*y(x)*diff(diff(y(x),x),x)+b*y(x) = 0,
       y(x),singsol=all)
```

$$
\begin{aligned}
y &= 0 \\
y &= -\frac{b\,x^2}{2a} + c_1 x + c_2
\end{aligned}
$$

Mathematica DSolve solution

Solving time : 0.003 (sec)
Leaf size : 28

```
DSolve[{a*y[x]*D[y[x],{x,2}]+b*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
y(x) &\to 0 \\
y(x) &\to -\frac{bx^2}{2a} + c_2 x + c_1
\end{aligned}
$$

2.1.71 problem 71

Internal problem ID [8459]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 71
Date solved : `Thursday, December 12, 2024 at 09:18:18 AM`
CAS classification : `system_of_ODEs`

$$
\begin{aligned}
x' &= 9x + 4y \\
y' &= -6x - y \\
z' &= 6x + 4y + 3z
\end{aligned}
$$

Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} -2\,\mathrm{e}^{3t} + 3\,\mathrm{e}^{5t} & -2\,\mathrm{e}^{3t} + 2\,\mathrm{e}^{5t} & 0 \\ -3\,\mathrm{e}^{5t} + 3\,\mathrm{e}^{3t} & 3\,\mathrm{e}^{3t} - 2\,\mathrm{e}^{5t} & 0 \\ 3\,\mathrm{e}^{5t} - 3\,\mathrm{e}^{3t} & -2\,\mathrm{e}^{3t} + 2\,\mathrm{e}^{5t} & \mathrm{e}^{3t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} -2\,\mathrm{e}^{3t}+3\,\mathrm{e}^{5t} & -2\,\mathrm{e}^{3t}+2\,\mathrm{e}^{5t} & 0 \\ -3\,\mathrm{e}^{5t}+3\,\mathrm{e}^{3t} & 3\,\mathrm{e}^{3t}-2\,\mathrm{e}^{5t} & 0 \\ 3\,\mathrm{e}^{5t}-3\,\mathrm{e}^{3t} & -2\,\mathrm{e}^{3t}+2\,\mathrm{e}^{5t} & \mathrm{e}^{3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} (-2\,\mathrm{e}^{3t}+3\,\mathrm{e}^{5t})\,c_1+(-2\,\mathrm{e}^{3t}+2\,\mathrm{e}^{5t})\,c_2 \\ (-3\,\mathrm{e}^{5t}+3\,\mathrm{e}^{3t})\,c_1+(3\,\mathrm{e}^{3t}-2\,\mathrm{e}^{5t})\,c_2 \\ (3\,\mathrm{e}^{5t}-3\,\mathrm{e}^{3t})\,c_1+(-2\,\mathrm{e}^{3t}+2\,\mathrm{e}^{5t})\,c_2+\mathrm{e}^{3t}c_3 \end{bmatrix} \\
&= \begin{bmatrix} (-2c_1-2c_2)\,\mathrm{e}^{3t}+3\left(c_1+\frac{2c_2}{3}\right)\mathrm{e}^{5t} \\ (3c_1+3c_2)\,\mathrm{e}^{3t}-3\left(c_1+\frac{2c_2}{3}\right)\mathrm{e}^{5t} \\ (-3c_1-2c_2+c_3)\,\mathrm{e}^{3t}+3\left(c_1+\frac{2c_2}{3}\right)\mathrm{e}^{5t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A-\lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 9-\lambda & 4 & 0 \\ -6 & -1-\lambda & 0 \\ 6 & 4 & 3-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 11\lambda^2 + 39\lambda - 45 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 5$$
$$\lambda_2 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
3	1	real eigenvalue
5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 6 & 4 & 0 \\ -6 & -4 & 0 \\ 6 & 4 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 6 & 4 & 0 & 0 \\ -6 & -4 & 0 & 0 \\ 6 & 4 & 0 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{ccc|c} 6 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 6 & 4 & 0 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 6 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 6 & 4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{3}\right\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \\ s \end{bmatrix} = \begin{bmatrix} -\frac{2t}{3} \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} v_1 \\ t \\ s \end{bmatrix} &= \begin{bmatrix} -\frac{2t}{3} \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} v_1 \\ t \\ s \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

Which are normalized to

$$\left(\begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

Considering the eigenvalue $\lambda_2 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{bmatrix} - (5)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 & 0 \\ -6 & -6 & 0 \\ 6 & 4 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 4 & 4 & 0 & 0 \\ -6 & -6 & 0 & 0 \\ 6 & 4 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 4 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 6 & 4 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{3R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c} 4 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{ccc|c} 4 & 4 & 0 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 4 & 4 & 0 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = -t\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ t \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ v_2 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ v_2 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
5	1	1	No	$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$
3	2	2	No	$\begin{bmatrix} 0 & -\frac{2}{3} \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 5 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{5t} \\ &= \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{5t} \end{aligned}$$

eigenvalue 3 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

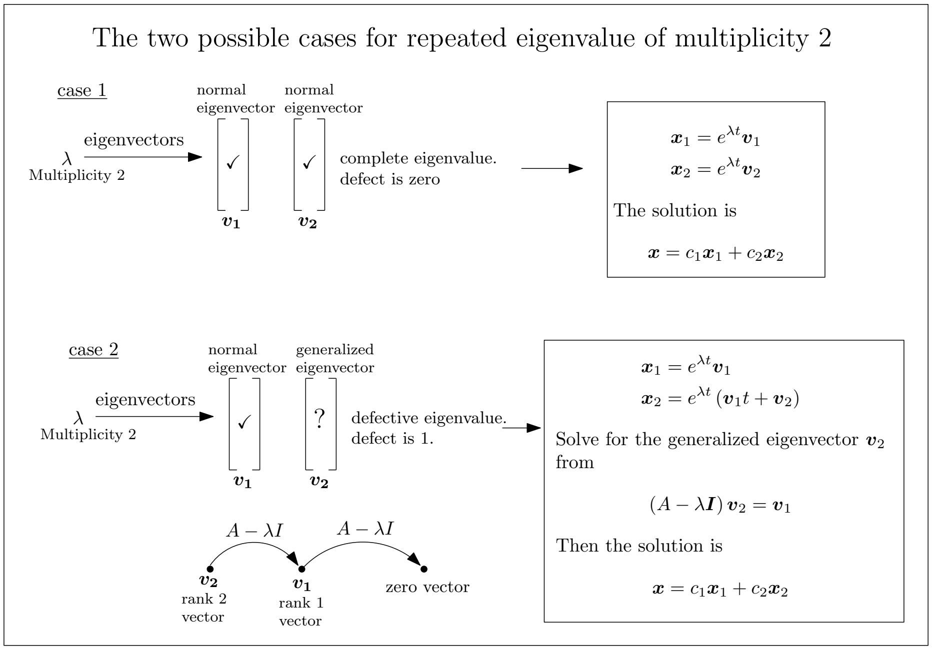


Figure 2.165: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{3t} \\
&= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{3t}
\end{aligned}
$$

$$
\begin{aligned}
\vec{x}_3(t) &= \vec{v}_3 e^{3t} \\
&= \begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix} e^{3t}
\end{aligned}
$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\left[\begin{array}{c} x \\ y \\ z \end{array}\right] = c_1 \left[\begin{array}{c} \mathrm{e}^{5t} \\ -\mathrm{e}^{5t} \\ \mathrm{e}^{5t} \end{array}\right] + c_2 \left[\begin{array}{c} 0 \\ 0 \\ \mathrm{e}^{3t} \end{array}\right] + c_3 \left[\begin{array}{c} -\frac{2\,\mathrm{e}^{3t}}{3} \\ \mathrm{e}^{3t} \\ 0 \end{array}\right]$$

Which becomes

$$\left[\begin{array}{c} x \\ y \\ z \end{array}\right] = \left[\begin{array}{c} c_1\,\mathrm{e}^{5t} - \frac{2c_3\,\mathrm{e}^{3t}}{3} \\ -c_1\,\mathrm{e}^{5t} + c_3\,\mathrm{e}^{3t} \\ c_1\,\mathrm{e}^{5t} + c_2\,\mathrm{e}^{3t} \end{array}\right]$$

Maple step by step solution

Let's solve
$\left[\frac{d}{dt}x(t) = 9x(t) + 4y(t)\,,\ \frac{d}{dt}y(t) = -6x(t) - y(t)\,,\ \frac{d}{dt}z(t) = 6x(t) + 4y(t) + 3z(t)\right]$

- Define vector

$$\vec{x}(t) = \left[\begin{array}{c} x(t) \\ y(t) \\ z(t) \end{array}\right]$$

- Convert system into a vector equation

$$\frac{d}{dt}\vec{x}(t) = \left[\begin{array}{ccc} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{array}\right] \cdot \vec{x}(t) + \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]$$

- System to solve

$$\frac{d}{dt}\vec{x}(t) = \left[\begin{array}{ccc} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{array}\right] \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{array}\right]$$

- Rewrite the system as

$\frac{d}{dt}\vec{x}(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[3, \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]\right], \left[3, \left[\begin{array}{c} -\frac{2}{3} \\ 1 \\ 0 \end{array}\right]\right], \left[5, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[3, \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]\right]$$

- First solution from eigenvalue 3

$$\vec{x}_1(t) = e^{3t} \cdot \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 3$ is the eigenvalue, and

$$\vec{x}_2(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\vec{x}_2(t)$ into the homogeneous system

$$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = \left(e^{\lambda t}A\right) \cdot \left(t\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{x}_2(t)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 3

$$\left(\left[\begin{array}{ccc} 9 & 4 & 0 \\ -6 & -1 & 0 \\ 6 & 4 & 3 \end{array}\right] - 3 \cdot \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]\right) \cdot \vec{p} = \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]$$

- Choice of $\overrightarrow{p}$

  $$\overrightarrow{p} = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]$$

- Second solution from eigenvalue 3

  $$\overrightarrow{x}_2(t) = \mathrm{e}^{3t} \cdot \left(t \cdot \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right] + \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]\right)$$

- Consider eigenpair

  $$\left[5, \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

  $$\overrightarrow{x}_3 = \mathrm{e}^{5t} \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- General solution to the system of ODEs

  $\overrightarrow{x} = \mathit{C1}\,\overrightarrow{x}_1(t) + \mathit{C2}\,\overrightarrow{x}_2(t) + \mathit{C3}\,\overrightarrow{x}_3$

- Substitute solutions into the general solution

  $$\overrightarrow{x} = \mathit{C1}\,\mathrm{e}^{3t} \cdot \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right] + \mathit{C2}\,\mathrm{e}^{3t} \cdot \left(t \cdot \left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right] + \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]\right) + \mathit{C3}\,\mathrm{e}^{5t} \cdot \left[\begin{array}{c} 1 \\ -1 \\ 1 \end{array}\right]$$

- Substitute in vector of dependent variables

  $$\left[\begin{array}{c} x(t) \\ y(t) \\ z(t) \end{array}\right] = \left[\begin{array}{c} \mathit{C3}\,\mathrm{e}^{5t} \\ -\mathit{C3}\,\mathrm{e}^{5t} \\ \mathrm{e}^{3t}(\mathit{C2}t + \mathit{C1}) + \mathit{C3}\,\mathrm{e}^{5t} \end{array}\right]$$

- Solution to the system of ODEs

  $\{x(t) = \mathit{C3}\,\mathrm{e}^{5t}, y(t) = -\mathit{C3}\,\mathrm{e}^{5t}, z(t) = \mathrm{e}^{3t}(\mathit{C2}t + \mathit{C1}) + \mathit{C3}\,\mathrm{e}^{5t}\}$

Maple dsolve solution

Solving time : 0.095 (sec)
Leaf size : 57

```
dsolve([diff(x(t),t) = 9*x(t)+4*y(t), diff(y(t),t) = -6*x(t)-y(t), diff(z(t),t) = 6*x(
       ,{op([x(t), y(t), z(t)])})
```

$$
\begin{aligned}
x(t) &= c_2 \mathrm{e}^{5t} + c_3\,\mathrm{e}^{3t} \\
y(t) &= -c_2 \mathrm{e}^{5t} - \frac{3c_3\,\mathrm{e}^{3t}}{2} \\
z(t) &= c_2 \mathrm{e}^{5t} + c_3\,\mathrm{e}^{3t} + c_1 \mathrm{e}^{3t}
\end{aligned}
$$

Mathematica DSolve solution

Solving time : 0.008 (sec)
Leaf size : 103

```
DSolve[{{D[x[t],t]==9*x[t]+4*y[t],D[y[t],t]==-6*x[t]-y[t],D[z[t],t]==6*x[t]+4*y[t]+3*z[t]},{
       {x[t],y[t],z[t]},t,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
x(t) &\to e^{3t}\left(c_1\left(3e^{2t}-2\right)+2c_2\left(e^{2t}-1\right)\right) \\
y(t) &\to -e^{3t}\left(3c_1\left(e^{2t}-1\right)+c_2\left(2e^{2t}-3\right)\right) \\
z(t) &\to e^{3t}\left(3c_1\left(e^{2t}-1\right)+2c_2\left(e^{2t}-1\right)+c_3\right)
\end{aligned}
$$

2.1.72 problem 72

Internal problem ID [8460]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 72
Date solved : `Thursday, December 12, 2024 at 09:18:18 AM`
CAS classification : `system_of_ODEs`

$$
\begin{aligned}
x' &= x - 3y \\
y' &= 3x + 7y
\end{aligned}
$$

Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{4t}(1-3t) & -3t\,\mathrm{e}^{4t} \\ 3t\,\mathrm{e}^{4t} & \mathrm{e}^{4t}(1+3t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{4t}(1-3t) & -3t\,e^{4t} \\ 3t\,e^{4t} & e^{4t}(1+3t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{4t}(1-3t)\,c_1 - 3t\,e^{4t}c_2 \\ 3t\,e^{4t}c_1 + e^{4t}(1+3t)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} (c_1(1-3t) - 3c_2t)\,e^{4t} \\ e^{4t}(3tc_1 + 3c_2t + c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -3 \\ 3 & 7-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 8\lambda + 16 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 4$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
4	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 4$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} - (4)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3 & -3 & 0 \\ 3 & 3 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -3 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3 & -3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
4	2	1	Yes	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 4 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

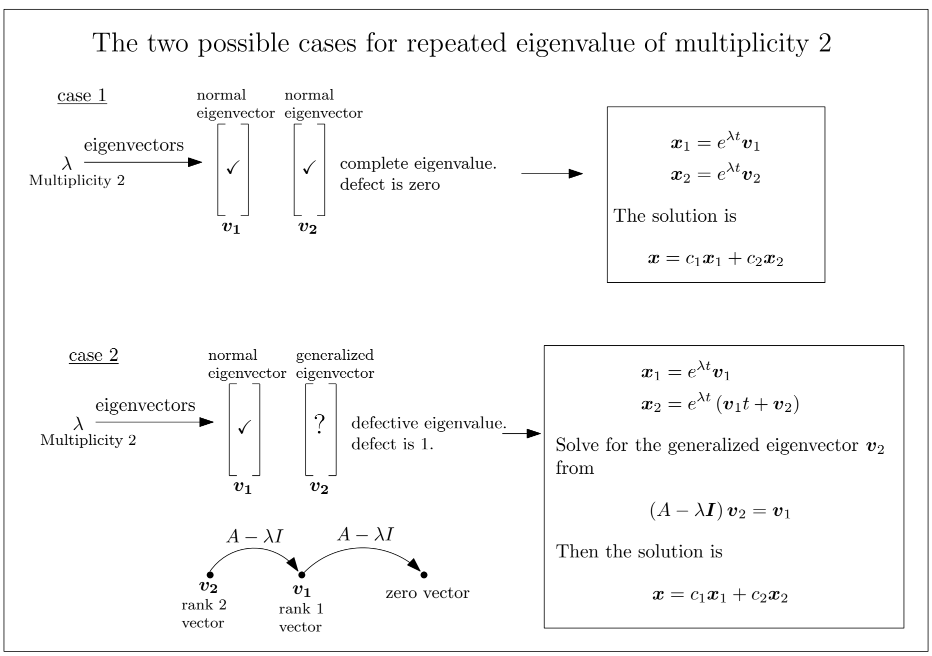


Figure 2.166: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\begin{aligned}\left(\begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} - (4)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} -1 \\ 1 \end{bmatrix}\end{aligned}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 4. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \mathrm{e}^{4t} \\ &= \begin{bmatrix} -\mathrm{e}^{4t} \\ \mathrm{e}^{4t} \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\ &= \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}\right) \mathrm{e}^{4t} \\ &= \begin{bmatrix} -\frac{\mathrm{e}^{4t}(3t+2)}{3} \\ \mathrm{e}^{4t}(t+1) \end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{4t} \\ \mathrm{e}^{4t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{4t}\left(-t - \frac{2}{3}\right) \\ \mathrm{e}^{4t}(t+1) \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e^{4t}\left(-c_1 - c_2 t - \frac{2}{3}c_2\right) \\ e^{4t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

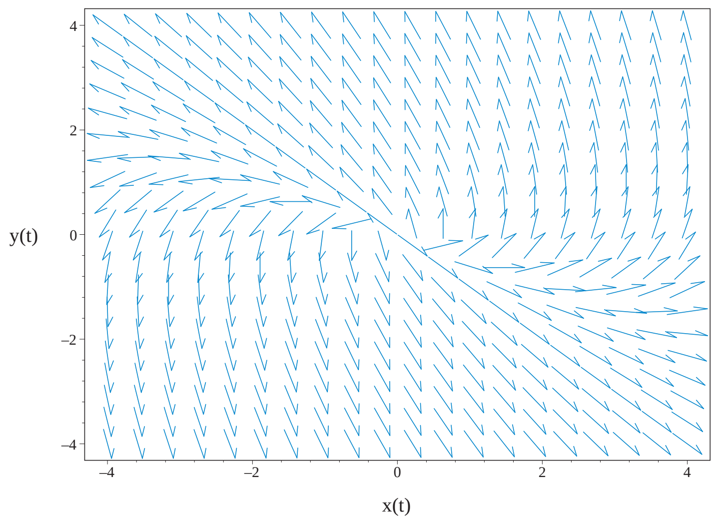


Figure 2.167: Phase plot

Maple step by step solution

Let's solve

$\left[\frac{d}{dt}x(t) = x(t) - 3y(t), \frac{d}{dt}y(t) = 3x(t) + 7y(t)\right]$

- Define vector

  $\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

- Convert system into a vector equation

  $\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

- System to solve

  $\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \cdot \vec{x}(t)$

- Define the coefficient matrix

  $A = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix}$

- Rewrite the system as
  $\frac{d}{dt}\vec{x}(t) = A \cdot \vec{x}(t)$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A
  $\left[\left[4, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right], \left[4, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]\right]$
- Consider eigenpair, with eigenvalue of algebraic multiplicity 2
  $\left[4, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]$
- First solution from eigenvalue 4
  $\vec{x}_1(t) = \mathrm{e}^{4t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$
- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 4$ is the eigenvalue, and
  $\vec{x}_2(t) = \mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right)$
- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\vec{x}_2(t)$ into the homogeneous system
  $\lambda\,\mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \left(\mathrm{e}^{\lambda t}A\right) \cdot \left(t\vec{v} + \vec{p}\right)$
- Use the fact that $\vec{v}$ is an eigenvector of A
  $\lambda\,\mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \mathrm{e}^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$
- Simplify equation
  $\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$
- Make use of the identity matrix I
  $(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$
- Condition $\vec{p}$ must meet for $\vec{x}_2(t)$ to be a solution to the homogeneous system
  $(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$
- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 4
  $\left(\begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} - 4 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$
- Choice of $\vec{p}$
  $\vec{p} = \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}$

- Second solution from eigenvalue 4

$$\vec{x}_2(t) = \mathrm{e}^{4t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}\right)$$

- General solution to the system of ODEs

$$\vec{x} = C1\vec{x}_1(t) + C2\vec{x}_2(t)$$

- Substitute solutions into the general solution

$$\vec{x} = C1\,\mathrm{e}^{4t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C2\,\mathrm{e}^{4t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}\right)$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{4t}\left(-C1 - C2t + \frac{1}{3}C2\right) \\ \mathrm{e}^{4t}(C2t + C1) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x(t) = \mathrm{e}^{4t}\left(-C1 - C2t + \tfrac{1}{3}C2\right), y(t) = \mathrm{e}^{4t}(C2t + C1)\right\}$$

Maple dsolve solution

Solving time : `0.034` (sec)
Leaf size : 32

```
dsolve([diff(x(t),t) = x(t)-3*y(t), diff(y(t),t) = 3*x(t)+7*y(t)]
       ,{op([x(t), y(t)])})
```

$$x(t) = \mathrm{e}^{4t}(c_2t + c_1)$$
$$y(t) = -\frac{\mathrm{e}^{4t}(3c_2t + 3c_1 + c_2)}{3}$$

Mathematica DSolve solution

Solving time : `0.003` (sec)
Leaf size : 46

```
DSolve[{{D[x[t],t]==x[t]-3*y[t],D[y[t],t]==3*x[t]+7*y[t]},{}},
       {x[t],y[t]},t,IncludeSingularSolutions->True]
```

$$x(t) \to -e^{4t}(c_1(3t - 1) + 3c_2t)$$
$$y(t) \to e^{4t}(3(c_1 + c_2)t + c_2)$$

2.1.73 problem 73

Internal problem ID [8461]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 73
Date solved : `Thursday, December 12, 2024 at 09:18:19 AM`
CAS classification : `system_of_ODEs`

$$
\begin{aligned}
x' &= x - 2y \\
y' &= 2x + 5y
\end{aligned}
$$

Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{3t}(1-2t) & -2t\,\mathrm{e}^{3t} \\ 2t\,\mathrm{e}^{3t} & \mathrm{e}^{3t}(1+2t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{3t}(1-2t) & -2t\,e^{3t} \\ 2t\,e^{3t} & e^{3t}(1+2t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{3t}(1-2t)\,c_1 - 2t\,e^{3t}c_2 \\ 2t\,e^{3t}c_1 + e^{3t}(1+2t)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} (c_1(1-2t) - 2c_2t)\,e^{3t} \\ e^{3t}(2tc_1 + 2c_2t + c_2) \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & -2 \\ 2 & 5-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 6\lambda + 9 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
3	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 3$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix} - (3)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -2 & -2 & 0 \\ 2 & 2 & 0 \end{array}\right]$$

$$R_2 = R_2 + R_1 \Longrightarrow \left[\begin{array}{cc|c} -2 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
3	2	1	Yes	$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 3 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

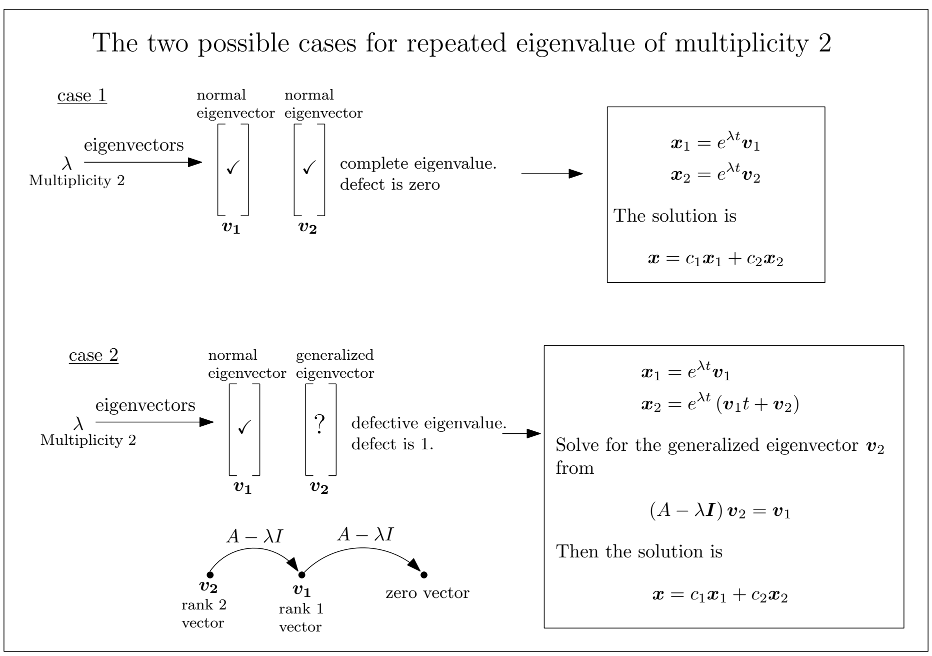


Figure 2.168: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\begin{aligned}\left(\begin{bmatrix}1 & -2\\2 & 5\end{bmatrix} - (3)\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\v_2\end{bmatrix} &= \begin{bmatrix}-1\\1\end{bmatrix}\\ \begin{bmatrix}-2 & -2\\2 & 2\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix} &= \begin{bmatrix}-1\\1\end{bmatrix}\end{aligned}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix}-\frac{1}{2}\\1\end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 3. Therefore the two basis solution associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t}\\ &= \begin{bmatrix}-1\\1\end{bmatrix}\mathrm{e}^{3t}\\ &= \begin{bmatrix}-\mathrm{e}^{3t}\\ \mathrm{e}^{3t}\end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\,e^{\lambda t}\\ &= \left(\begin{bmatrix}-1\\1\end{bmatrix}t + \begin{bmatrix}-\frac{1}{2}\\1\end{bmatrix}\right)\mathrm{e}^{3t}\\ &= \begin{bmatrix}-\frac{\mathrm{e}^{3t}(1+2t)}{2}\\ \mathrm{e}^{3t}(t+1)\end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix}x\\y\end{bmatrix} = c_1\begin{bmatrix}-\mathrm{e}^{3t}\\ \mathrm{e}^{3t}\end{bmatrix} + c_2\begin{bmatrix}\mathrm{e}^{3t}\left(-t-\frac{1}{2}\right)\\ \mathrm{e}^{3t}(t+1)\end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e^{3t}\left(-c_1 - c_2 t - \frac{1}{2}c_2\right) \\ e^{3t}(c_2 t + c_1 + c_2) \end{bmatrix}$$

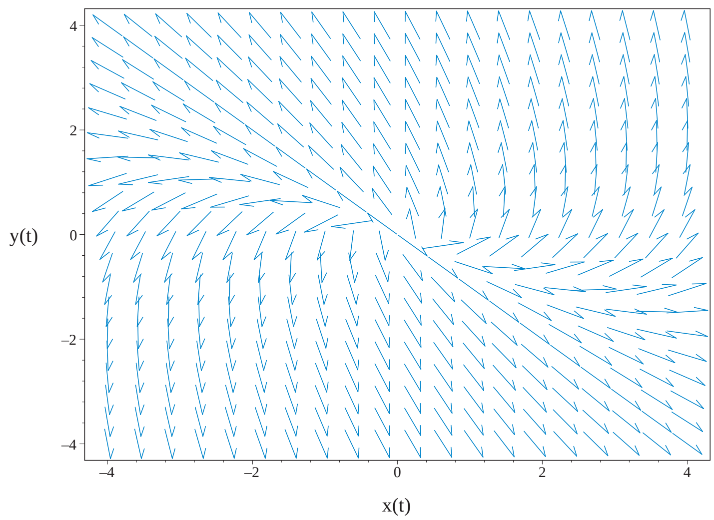


Figure 2.169: Phase plot

Maple step by step solution

Let's solve

$\left[\frac{d}{dt}x(t) = x(t) - 2y(t), \frac{d}{dt}y(t) = 2x(t) + 5y(t)\right]$

- Define vector

$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

- Convert system into a vector equation

$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

- System to solve

$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix} \cdot \vec{x}(t)$

- Define the coefficient matrix

$A = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}$

- Rewrite the system as
  $\frac{d}{dt}\vec{x}(t) = A \cdot \vec{x}(t)$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A
  $$\left[\left[3, \left[\begin{array}{c} -1 \\ 1 \end{array}\right]\right], \left[3, \left[\begin{array}{c} 0 \\ 0 \end{array}\right]\right]\right]$$
- Consider eigenpair, with eigenvalue of algebraic multiplicity 2
  $$\left[3, \left[\begin{array}{c} -1 \\ 1 \end{array}\right]\right]$$
- First solution from eigenvalue 3
  $$\vec{x}_1(t) = e^{3t} \cdot \left[\begin{array}{c} -1 \\ 1 \end{array}\right]$$
- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 3$ is the eigenvalue, and
  $$\vec{x}_2(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$
- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\vec{x}_2(t)$ into the homogeneous system
  $$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = \left(e^{\lambda t}A\right) \cdot \left(t\vec{v} + \vec{p}\right)$$
- Use the fact that $\vec{v}$ is an eigenvector of A
  $$\lambda e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$$
- Simplify equation
  $$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$
- Make use of the identity matrix I
  $$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$
- Condition $\vec{p}$ must meet for $\vec{x}_2(t)$ to be a solution to the homogeneous system
  $$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$
- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 3
  $$\left(\left[\begin{array}{cc} 1 & -2 \\ 2 & 5 \end{array}\right] - 3 \cdot \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right]\right) \cdot \vec{p} = \left[\begin{array}{c} -1 \\ 1 \end{array}\right]$$
- Choice of $\vec{p}$
  $$\vec{p} = \left[\begin{array}{c} \frac{1}{2} \\ 0 \end{array}\right]$$

- Second solution from eigenvalue 3
$$\vec{x}_2(t) = e^{3t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}\right)$$
- General solution to the system of ODEs
$$\vec{x} = C1\vec{x}_1(t) + C2\vec{x}_2(t)$$
- Substitute solutions into the general solution
$$\vec{x} = C1\, e^{3t} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C2\, e^{3t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}\right)$$
- Substitute in vector of dependent variables
$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e^{3t}\left(-C1 - C2t + \frac{1}{2}C2\right) \\ e^{3t}(C2t + C1) \end{bmatrix}$$
- Solution to the system of ODEs
$$\left\{x(t) = e^{3t}\left(-C1 - C2t + \frac{1}{2}C2\right), y(t) = e^{3t}(C2t + C1)\right\}$$

Maple dsolve solution

Solving time : 0.027 (sec)
Leaf size : 32

```
dsolve([diff(x(t),t) = x(t)-2*y(t), diff(y(t),t) = 2*x(t)+5*y(t)]
       ,{op([x(t), y(t)])})
```

$$x(t) = e^{3t}(c_2t + c_1)$$
$$y(t) = -\frac{e^{3t}(2c_2t + 2c_1 + c_2)}{2}$$

Mathematica DSolve solution

Solving time : 0.003 (sec)
Leaf size : 46

```
DSolve[{{D[x[t],t]== x[t]-2*y[t],D[y[t],t] == 2*x[t]+5*y[t]},{}},
       {x[t],y[t]},t,IncludeSingularSolutions->True]
```

$$x(t) \to -e^{3t}(c_1(2t - 1) + 2c_2t)$$
$$y(t) \to e^{3t}(2(c_1 + c_2)t + c_2)$$

2.1.74 problem 74

Internal problem ID [8462]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 74
Date solved : `Thursday, December 12, 2024 at 09:18:20 AM`
CAS classification : `system_of_ODEs`

$$
\begin{aligned}
x' &= 7x + y \\
y' &= -4x + 3y
\end{aligned}
$$

Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} e^{5t}(1+2t) & t\,e^{5t} \\ -4t\,e^{5t} & e^{5t}(1-2t) \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{5t}(1+2t) & t\,e^{5t} \\ -4t\,e^{5t} & e^{5t}(1-2t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{5t}(1+2t)\,c_1 + t\,e^{5t}c_2 \\ -4t\,e^{5t}c_1 + e^{5t}(1-2t)\,c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{5t}(2tc_1 + c_2t + c_1) \\ (c_2(1-2t) - 4tc_1)\,e^{5t} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 7-\lambda & 1 \\ -4 & 3-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$
\lambda^2 - 10\lambda + 25 = 0
$$

The roots of the above are the eigenvalues.

$$
\lambda_1 = 5
$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
5	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 5$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} - (5)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ -4 & -2 & 0 \end{array}\right]$$

$$R_2 = R_2 + 2R_1 \Longrightarrow \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m-k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
5	2	1	Yes	$\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 5 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

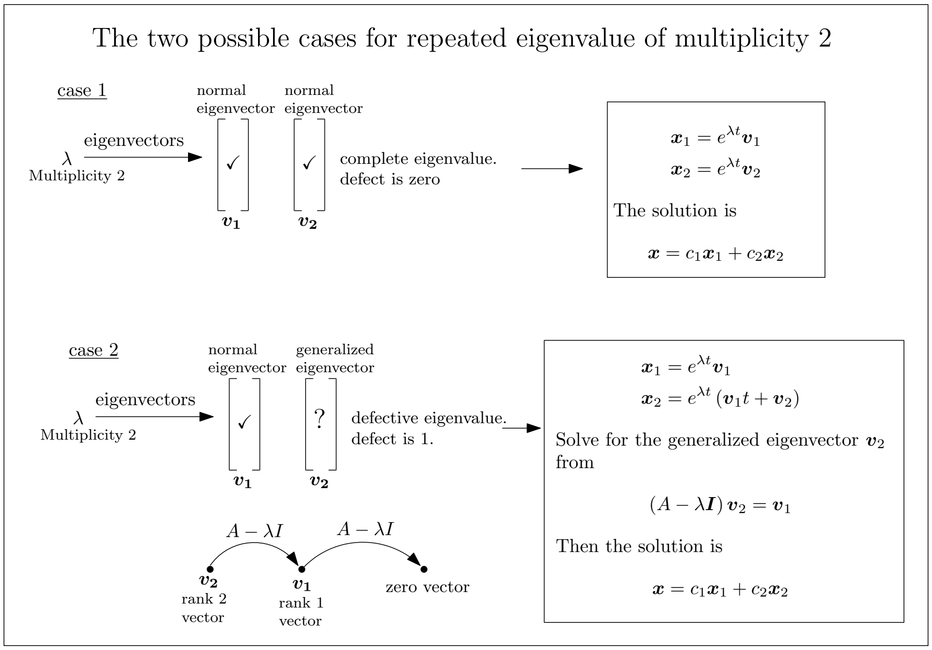


Figure 2.170: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to generate the missing additonal generalized eigevector $\vec{v}_2$ by solving

$$(A - \lambda I)\,\vec{v}_2 = \vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence we need to solve

$$\left(\begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} - (5) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -\frac{5}{2} \end{bmatrix}$$

We have found two generalized eigenvectors for eigenvalue 5. Therefore the two basis solution associated with this eigenvalue are

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\
&= \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \mathrm{e}^{5t} \\
&= \begin{bmatrix} -\frac{\mathrm{e}^{5t}}{2} \\ \mathrm{e}^{5t} \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ -\frac{5}{2} \end{bmatrix} \right) \mathrm{e}^{5t} \\
&= \begin{bmatrix} -\frac{\mathrm{e}^{5t}(t-2)}{2} \\ \frac{\mathrm{e}^{5t}(2t-5)}{2} \end{bmatrix}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} -\frac{\mathrm{e}^{5t}}{2} \\ \mathrm{e}^{5t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{5t}\left(-\frac{t}{2} + 1\right) \\ \mathrm{e}^{5t}\left(t - \frac{5}{2}\right) \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{\mathrm{e}^{5t}((t-2)c_2 + c_1)}{2} \\ \mathrm{e}^{5t}\left(c_1 + c_2 t - \frac{5}{2} c_2\right) \end{bmatrix}
$$

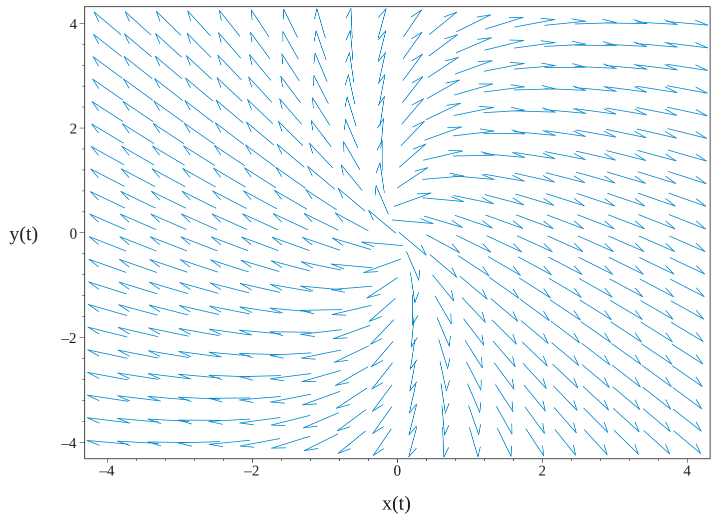


Figure 2.171: Phase plot

Maple step by step solution

Let's solve

$\left[\frac{d}{dt}x(t) = 7x(t) + y(t), \frac{d}{dt}y(t) = -4x(t) + 3y(t)\right]$

- Define vector

$\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

- Convert system into a vector equation

$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

- System to solve

$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} \cdot \vec{x}(t)$

- Define the coefficient matrix

$A = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix}$

- Rewrite the system as

$\frac{d}{dt}\vec{x}(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[5, \left[\begin{array}{c}-\frac{1}{2}\\ 1\end{array}\right]\right], \left[5, \left[\begin{array}{c}0\\ 0\end{array}\right]\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[5, \left[\begin{array}{c}-\frac{1}{2}\\ 1\end{array}\right]\right]$$

- First solution from eigenvalue 5

$$\vec{x}_1(t) = \mathrm{e}^{5t} \cdot \left[\begin{array}{c}-\frac{1}{2}\\ 1\end{array}\right]$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = 5$ is the eigenvalue, and

$$\vec{x}_2(t) = \mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\vec{x}_2(t)$ into the homogeneous system

$$\lambda\,\mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \left(\mathrm{e}^{\lambda t} A\right) \cdot \left(t\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda\,\mathrm{e}^{\lambda t}\left(t\vec{v} + \vec{p}\right) + \mathrm{e}^{\lambda t}\vec{v} = \mathrm{e}^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{x}_2(t)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue 5

$$\left(\left[\begin{array}{cc}7 & 1\\ -4 & 3\end{array}\right] - 5 \cdot \left[\begin{array}{cc}1 & 0\\ 0 & 1\end{array}\right]\right) \cdot \vec{p} = \left[\begin{array}{c}-\frac{1}{2}\\ 1\end{array}\right]$$

- Choice of $\vec{p}$

$$\vec{p} = \left[\begin{array}{c}-\frac{1}{4}\\ 0\end{array}\right]$$

- Second solution from eigenvalue 5

$$\vec{x}_2(t) = \mathrm{e}^{5t} \cdot \left(t \cdot \left[\begin{array}{c}-\frac{1}{2}\\ 1\end{array}\right] + \left[\begin{array}{c}-\frac{1}{4}\\ 0\end{array}\right]\right)$$

- General solution to the system of ODEs
  $\overrightarrow{x} = C1\overrightarrow{x}_1(t) + C2\overrightarrow{x}_2(t)$
- Substitute solutions into the general solution
  $\overrightarrow{x} = C1\,\mathrm{e}^{5t} \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + C2\,\mathrm{e}^{5t} \cdot \left(t \cdot \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{4} \\ 0 \end{bmatrix}\right)$
- Substitute in vector of dependent variables
  $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{\mathrm{e}^{5t}(2C2t+2C1+C2)}{4} \\ \mathrm{e}^{5t}(C2t + C1) \end{bmatrix}$
- Solution to the system of ODEs
  $\left\{x(t) = -\frac{\mathrm{e}^{5t}(2C2t+2C1+C2)}{4}, y(t) = \mathrm{e}^{5t}(C2t + C1)\right\}$

Maple dsolve solution

Solving time : `0.039` (sec)
Leaf size : 34

```
dsolve([diff(x(t),t) = 7*x(t)+y(t), diff(y(t),t) = -4*x(t)+3*y(t)]
      ,{op([x(t), y(t)])})
```

$$x(t) = \mathrm{e}^{5t}(c_2t + c_1)$$
$$y(t) = -\mathrm{e}^{5t}(2c_2t + 2c_1 - c_2)$$

Mathematica DSolve solution

Solving time : `0.003` (sec)
Leaf size : 45

```
DSolve[{{D[x[t],t]== 7*x[t]+y[t],D[y[t],t] == -4*x[t]+3*y[t]},{}},
      {x[t],y[t]},t,IncludeSingularSolutions->True]
```

$$x(t) \to e^{5t}(2c_1t + c_2t + c_1)$$
$$y(t) \to e^{5t}(c_2 - 2(2c_1 + c_2)t)$$

2.1.75 problem 75

Internal problem ID [8463]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 75
Date solved : `Thursday, December 12, 2024 at 09:18:20 AM`
CAS classification : `system_of_ODEs`

$$
\begin{aligned}
x' &= x + y \\
y' &= y \\
z' &= z
\end{aligned}
$$

Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \mathrm{e}^{t} & t\,\mathrm{e}^{t} & 0 \\ 0 & \mathrm{e}^{t} & 0 \\ 0 & 0 & \mathrm{e}^{t} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^t & t\,e^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^t c_1 + t\,e^t c_2 \\ e^t c_2 \\ e^t c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^t (c_2 t + c_1) \\ e^t c_2 \\ e^t c_3 \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}\right) = 0
$$

Since the matrix A is triangular matrix, then the determinant is the product of the elements along the diagonal. Therefore the above becomes

$$(1-\lambda)(1-\lambda)(1-\lambda)=0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
1	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A-\lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - (1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_1, v_3\}$ and the leading variables are $\{v_2\}$. Let $v_1 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_2 = 0\}$

Hence the solution is

$$\begin{bmatrix} t \\ v_2 \\ s \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned}
\begin{bmatrix} t \\ v_2 \\ s \end{bmatrix} &= \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} \\
&= t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} t \\ v_2 \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
1	3	2	Yes	$\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 1 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

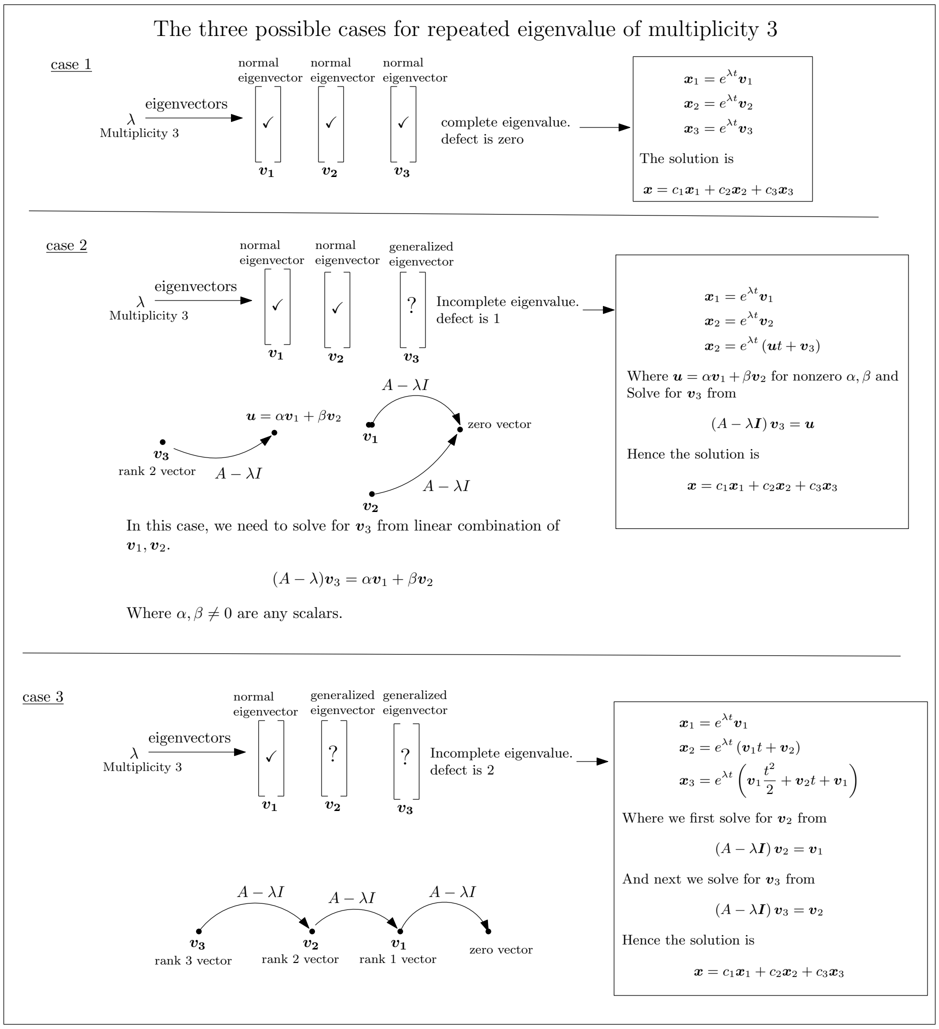


Figure 2.172: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 2, therefore this is defective eigenvalue. The defect is 1. This falls into case 2 shown above. We need to find rank-2 eigenvector $\vec{v}_3$. This eigenvector must therefore satisfy $(A - \lambda I)^2\,\vec{v}_3 = \vec{0}$.

But

$$
\begin{aligned}
(A-\lambda I)^2 &= \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^2 \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

Therefore $\vec{v}_3$ could be any eigenvector vector we want (but not the zero vector). Let

$$
\vec{v}_3 = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}
$$

To determine the actual $\vec{v}_3$ we need now to enforce the condition that $\vec{v}_3$ satisfies

$$
(A-\lambda I)\,\vec{v}_3 = \vec{u} \tag{1}
$$

Where $\vec{u}$ is linear combination of $\vec{v}_1, \vec{v}_2$. Hence

$$
\vec{u} = \alpha \vec{v}_1 + \beta \vec{v}_2
$$

Where α, β are arbitrary constants (not both zero). Eq. (1) becomes

$$
\begin{aligned}
(A-\lambda I) \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} &= \alpha \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} &= \alpha \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} \eta_2 \\ 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} \beta \\ 0 \\ \alpha \end{bmatrix}
\end{aligned}
$$

Expanding the above gives the following equations equations

$$
\begin{aligned}
\eta_2 &= \beta \\
0 &= \alpha
\end{aligned}
$$

solving for α, β from the above gives

$$
\begin{aligned}
\eta_2 &= \beta \\
0 &= \alpha
\end{aligned}
$$

Since α, β are not both zero, then we just need to determine η_i values, not all zero, which satisfy the above equations for α, β not both zero. By inspection we see that the following values satisfy this condition

$$[\eta_2 = -1]$$

Hence we found the missing generalized eigenvector

$$\vec{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

Which implies that

$$\begin{aligned} \alpha &= 0 \\ \beta &= -1 \end{aligned}$$

Therefore

$$\begin{aligned} \vec{u} &= \alpha \vec{v}_1 + \beta \vec{v}_2 \\ &= 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

Therefore the missing generalized eigenvector is now found. We have found three generalized eigenvectors for eigenvalue 1. Therefore the three basis solutions associated with this eigenvalue are

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{\lambda t} \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mathrm{e}^{t} \\ &= \begin{bmatrix} 0 \\ 0 \\ \mathrm{e}^{t} \end{bmatrix} \end{aligned}$$

And

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{\lambda t} \\
&= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mathrm{e}^{t} \\
&= \begin{bmatrix} \mathrm{e}^{t} \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
\vec{x}_3(t) &= (\vec{u}t + \vec{v}_3)\, e^{\lambda t} \\
&= \left(\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right) \mathrm{e}^{t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)
$$

Which is written as

$$
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 0 \\ \mathrm{e}^{t} \end{bmatrix} + c_2 \begin{bmatrix} \mathrm{e}^{t} \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -t\,\mathrm{e}^{t} \\ -\mathrm{e}^{t} \\ 0 \end{bmatrix}
$$

Which becomes

$$
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{t}(-t c_3 + c_2) \\ -c_3\,\mathrm{e}^{t} \\ c_1\,\mathrm{e}^{t} \end{bmatrix}
$$

Maple step by step solution

Maple dsolve solution

Solving time : `0.059` (sec)
Leaf size : 26

```
dsolve([diff(x(t),t) = x(t)+y(t), diff(y(t),t) = y(t), diff(z(t),t) = z(t)]
       ,{op([x(t), y(t), z(t)])})
```

$$
\begin{aligned}
x(t) &= (c_2 t + c_1)\,\mathrm{e}^{t} \\
y(t) &= \mathrm{e}^{t} c_2 \\
z(t) &= c_3\,\mathrm{e}^{t}
\end{aligned}
$$

Mathematica DSolve solution

Solving time : 0.022 (sec)
Leaf size : 62

```
DSolve[{{D[x[t],t]== x[t]+y[t],D[y[t],t] == y[t],D[z[t],t]==z[t]},{}},
       {x[t],y[t],z[t]},t,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
x(t) &\to e^t(c_2t + c_1) \\
y(t) &\to c_2e^t \\
z(t) &\to c_3e^t \\
x(t) &\to e^t(c_2t + c_1) \\
y(t) &\to c_2e^t \\
z(t) &\to 0
\end{aligned}
$$

2.1.76 problem 76

Internal problem ID [8464]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 76
Date solved : `Thursday, December 12, 2024 at 09:18:21 AM`
CAS classification : `system_of_ODEs`

$$
\begin{aligned}
x' &= 2x + y - z \\
y' &= -x + 2z \\
z' &= -x - 2y + 4z
\end{aligned}
$$

Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & 2 \\ -1 & -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \mathrm{e}^{2t} & t\,\mathrm{e}^{2t} & -t\,\mathrm{e}^{2t} \\ -t\,\mathrm{e}^{2t} & \mathrm{e}^{2t}\left(1 - \frac{1}{2}t^2 - 2t\right) & \frac{\mathrm{e}^{2t}t(t+4)}{2} \\ -t\,\mathrm{e}^{2t} & -\frac{\mathrm{e}^{2t}t(t+4)}{2} & \mathrm{e}^{2t}\left(1 + \frac{1}{2}t^2 + 2t\right) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} e^{2t} & t\,e^{2t} & -t\,e^{2t} \\ -t\,e^{2t} & e^{2t}\left(1-\frac{1}{2}t^2-2t\right) & \frac{e^{2t}t(t+4)}{2} \\ -t\,e^{2t} & -\frac{e^{2t}t(t+4)}{2} & e^{2t}\left(1+\frac{1}{2}t^2+2t\right) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}c_1 + t\,e^{2t}c_2 - t\,e^{2t}c_3 \\ -t\,e^{2t}c_1 + e^{2t}\left(1-\frac{1}{2}t^2-2t\right)c_2 + \frac{e^{2t}t(t+4)c_3}{2} \\ -t\,e^{2t}c_1 - \frac{e^{2t}t(t+4)c_2}{2} + e^{2t}\left(1+\frac{1}{2}t^2+2t\right)c_3 \end{bmatrix} \\
&= \begin{bmatrix} e^{2t}((c_2-c_3)\,t+c_1) \\ -\frac{((c_2-c_3)t^2+(2c_1+4c_2-4c_3)t-2c_2)e^{2t}}{2} \\ -\frac{((c_2-c_3)t^2+(2c_1+4c_2-4c_3)t-2c_3)e^{2t}}{2} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & 2 \\ -1 & -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & 2 \\ -1 & -2 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 1 & -1 \\ -1 & -\lambda & 2 \\ -1 & -2 & 4-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 6\lambda^2 + 12\lambda - 8 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & 2 \\ -1 & -2 & 4 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -1 \\ -1 & -2 & 2 \\ -1 & -2 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ -1 & -2 & 2 & 0 \\ -1 & -2 & 2 & 0 \end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 2 gives

$$\left[\begin{array}{ccc|c} -1 & -2 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & -2 & 2 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} -1 & -2 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -1 & -2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = 0, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ v_2 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ v_2 \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
2	3	1	Yes	$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue 2 is real and repated eigenvalue of multiplicity 3.There are three possible cases that can happen. This is illustrated in this diagram

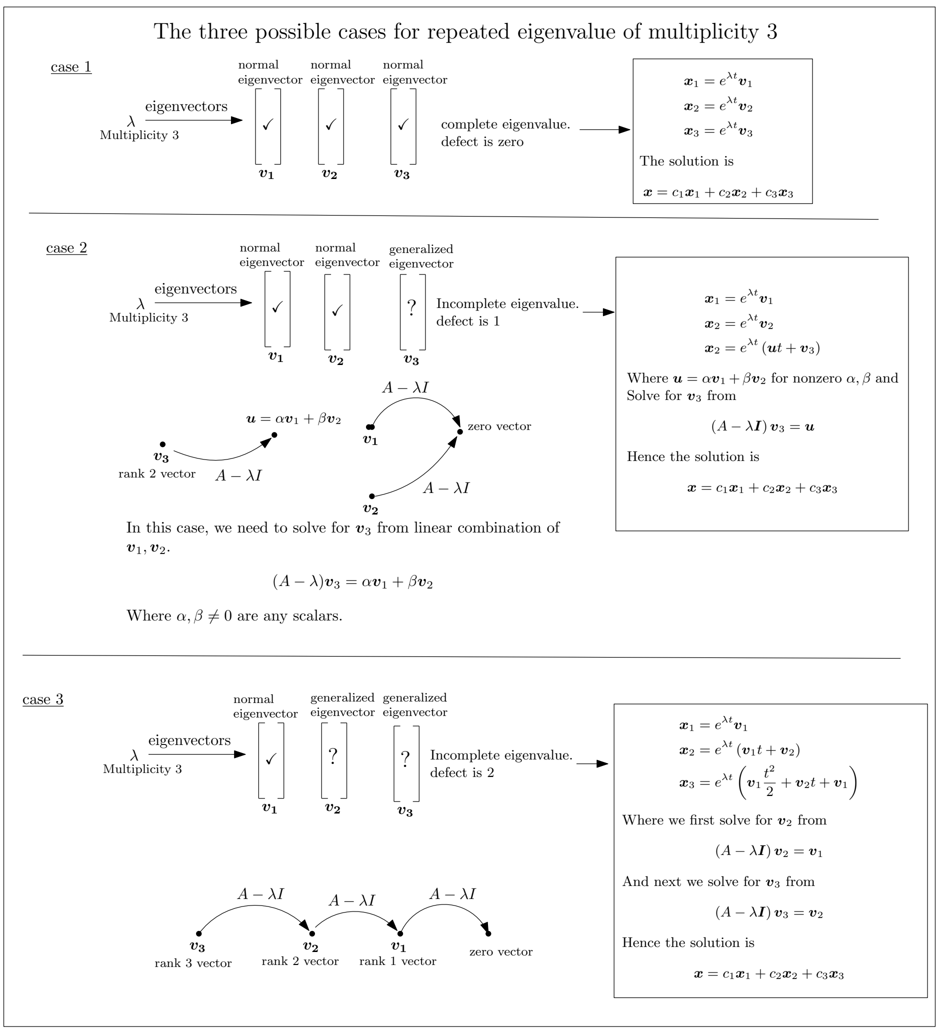


Figure 2.173: Possible case for repeated λ of multiplicity 3

This eigenvalue has algebraic multiplicity of 3, and geometric multiplicity 1, therefore this is defective eigenvalue. The defect is 2. This falls into case 3 shown above. First we find generalized eigenvector $\vec{v}_2$ of rank 2 and then use this to find generalized eigenvector

$\vec{v}_3$ of rank $3.\vec{v}_2$ is found by solving

$$(A-\lambda I)\,\vec{v}_2=\vec{v}_1$$

Where $\vec{v}_1$ is the normal (rank 1) eigenvector found above. Hence

$$\left(\begin{bmatrix}2 & 1 & -1\\ -1 & 0 & 2\\ -1 & -2 & 4\end{bmatrix}-(2)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}0\\ 1\\ 1\end{bmatrix}$$

$$\begin{bmatrix}0 & 1 & -1\\ -1 & -2 & 2\\ -1 & -2 & 2\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}0\\ 1\\ 1\end{bmatrix}$$

Solving for $\vec{v}_2$ gives

$$\vec{v}_2=\begin{bmatrix}-1\\ 1\\ 1\end{bmatrix}$$

Now $\vec{v}_3$ is found by solving

$$(A-\lambda I)\,\vec{v}_3=\vec{v}_2$$

Where $\vec{v}_2$ is the (rank 2) generalized eigenvector found above. Hence

$$\left(\begin{bmatrix}2 & 1 & -1\\ -1 & 0 & 2\\ -1 & -2 & 4\end{bmatrix}-(2)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}-1\\ 1\\ 1\end{bmatrix}$$

$$\begin{bmatrix}0 & 1 & -1\\ -1 & -2 & 2\\ -1 & -2 & 2\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}-1\\ 1\\ 1\end{bmatrix}$$

Solving for $\vec{v}_3$ gives

$$\vec{v}_3=\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$$

We have found three generalized eigenvectors for eigenvalue 2. Therefore the three basis solutions associated with this eigenvalue are

$$\begin{aligned}\vec{x}_1(t)&=\vec{v}_1e^{\lambda t}\\ &=\begin{bmatrix}0\\ 1\\ 1\end{bmatrix}\mathrm{e}^{2t}\\ &=\begin{bmatrix}0\\ \mathrm{e}^{2t}\\ \mathrm{e}^{2t}\end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_2(t) &= e^{\lambda t}(\vec{v}_1 t + \vec{v}_2)\\ &= \mathrm{e}^{2t}\left(\begin{bmatrix}0\\1\\1\end{bmatrix}t + \begin{bmatrix}-1\\1\\1\end{bmatrix}\right)\\ &= \begin{bmatrix}-\mathrm{e}^{2t}\\ \mathrm{e}^{2t}(t+1)\\ \mathrm{e}^{2t}(t+1)\end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\vec{x}_3(t) &= \left(\vec{v}_1\frac{t^2}{2} + \vec{v}_2 t + \vec{v}_3\right)e^{\lambda t}\\ &= \left(\begin{bmatrix}0\\1\\1\end{bmatrix}\frac{t^2}{2} + \begin{bmatrix}-1\\1\\1\end{bmatrix}t + \begin{bmatrix}1\\0\\1\end{bmatrix}\right)\mathrm{e}^{2t}\\ &= \begin{bmatrix}-\mathrm{e}^{2t}(t-1)\\ \frac{\mathrm{e}^{2t}t(t+2)}{2}\\ \frac{\mathrm{e}^{2t}(t^2+2t+2)}{2}\end{bmatrix}\end{aligned}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = c_1\begin{bmatrix}0\\ \mathrm{e}^{2t}\\ \mathrm{e}^{2t}\end{bmatrix} + c_2\begin{bmatrix}-\mathrm{e}^{2t}\\ \mathrm{e}^{2t}(t+1)\\ \mathrm{e}^{2t}(t+1)\end{bmatrix} + c_3\begin{bmatrix}\mathrm{e}^{2t}(-t+1)\\ \mathrm{e}^{2t}\left(\frac{1}{2}t^2+t\right)\\ \mathrm{e}^{2t}\left(\frac{1}{2}t^2+t+1\right)\end{bmatrix}$$

Which becomes

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-((t-1)\,c_3 + c_2)\,\mathrm{e}^{2t}\\ \frac{\mathrm{e}^{2t}(c_3\,t^2+(2c_2+2c_3)t+2c_1+2c_2)}{2}\\ \frac{\mathrm{e}^{2t}((t^2+2t+2)c_3+2c_2t+2c_1+2c_2)}{2}\end{bmatrix}$$

Maple step by step solution

Maple dsolve solution

Solving time : 0.068 (sec)
Leaf size : 58

```
dsolve([diff(x(t),t) = 2*x(t)+y(t)-z(t), diff(y(t),t) = -x(t)+2*z(t), diff(z(t),t) = -
       ,{op([x(t), y(t), z(t)])})
```

$$
\begin{aligned}
x(t) &= -\mathrm{e}^{2t}(2tc_3 - 4c_3 + c_2) \\
y(t) &= \mathrm{e}^{2t}\left(c_3\, t^2 + c_2 t + c_1\right) \\
z(t) &= \mathrm{e}^{2t}\left(c_3\, t^2 + c_2 t + 2c_3 + c_1\right)
\end{aligned}
$$

Mathematica DSolve solution

Solving time : 0.004 (sec)
Leaf size : 107

```
DSolve[{{D[x[t],t]== 2*x[t]+y[t]-z[t],D[y[t],t] == -x[t]+2*z[t],D[z[t],t]==-x[t]-2*y[t]+4*z[
       {x[t],y[t],z[t]},t,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
x(t) &\to e^{2t}((c_2 - c_3)t + c_1) \\
y(t) &\to -\frac{1}{2}e^{2t}\left((c_2 - c_3)t^2 + 2(c_1 + 2c_2 - 2c_3)t - 2c_2\right) \\
z(t) &\to -\frac{1}{2}e^{2t}\left((c_2 - c_3)t^2 + 2(c_1 + 2c_2 - 2c_3)t - 2c_3\right)
\end{aligned}
$$

2.1.77 problem 77

Internal problem ID [8465]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 77
Date solved : `Tuesday, December 17, 2024 at 12:52:55 PM`
CAS classification : `[_quadrature]`

Solve

$$x' = 4Ak\left(\frac{x}{A}\right)^{3/4} - 3kx$$

Solved as first order autonomous ode

Time used: 2.007 (sec)

Integrating gives

$$\int \frac{1}{4Ak\left(\frac{x}{A}\right)^{3/4} - 3kx}dx = dt$$
$$-\frac{4\ln\left(3\left(\frac{x}{A}\right)^{1/4} - 4\right)}{3k} = t + c_1$$

Singular solutions are found by solving

$$4Ak\left(\frac{x}{A}\right)^{3/4} - 3kx = 0$$

for x. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$x = 0$$
$$x = \frac{256A}{81}$$

The following diagram is the phase line diagram. It classifies each of the above equilibrium points as stable or not stable or semi-stable.

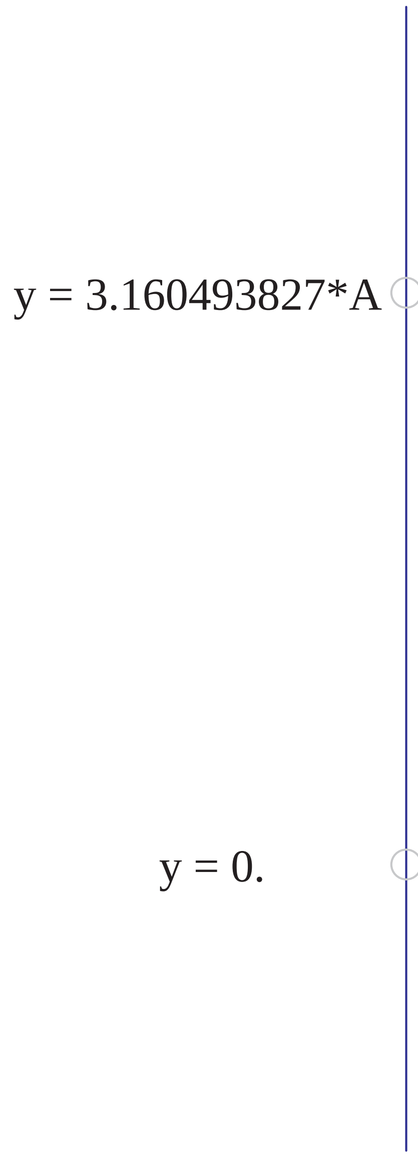


Figure 2.174: Phase line diagram

Solving for x gives

$$
\begin{aligned}
x &= 0 \\
x &= \frac{256A}{81} \\
x &= \frac{\left(\mathrm{e}^{-\frac{3}{4}c_1k-\frac{3}{4}tk}+4\right)^4 A}{81}
\end{aligned}
$$

Summary of solutions found

$$
\begin{aligned}
x &= 0 \\
x &= \frac{256A}{81} \\
x &= \frac{\left(\mathrm{e}^{-\frac{3}{4}c_1 k-\frac{3}{4}tk}+4\right)^4 A}{81}
\end{aligned}
$$

Solved as first order Exact ode

Time used: 1.187 (sec)

To solve an ode of the form

$$
M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}
$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y)=0
$$

Hence

$$
\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}
$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial\phi}{\partial x} &= M \\
\frac{\partial\phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(t,x)\,\mathrm{d}t+N(t,x)\,\mathrm{d}x=0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}x &= \left(4Ak\left(\frac{x}{A}\right)^{3/4} - 3kx\right)\mathrm{d}t \\
\left(-4Ak\left(\frac{x}{A}\right)^{3/4} + 3kx\right)\mathrm{d}t + \mathrm{d}x &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(t,x) &= -4Ak\left(\frac{x}{A}\right)^{3/4} + 3kx \\
N(t,x) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial x} &= \frac{\partial}{\partial x}\left(-4Ak\left(\frac{x}{A}\right)^{3/4} + 3kx\right) \\
&= 3k\left(1 - \frac{1}{\left(\frac{x}{A}\right)^{1/4}}\right)
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial t} &= \frac{\partial}{\partial t}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial t}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial x} - \frac{\partial N}{\partial t}\right) \\
&= 1\left(\left(-\frac{3k}{\left(\frac{x}{A}\right)^{1/4}} + 3k\right) - (0)\right) \\
&= 3k\left(1 - \frac{1}{\left(\frac{x}{A}\right)^{1/4}}\right)
\end{aligned}
$$

Since A depends on x, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial t} - \frac{\partial M}{\partial x}\right) \\
&= -\frac{1}{k\left(4A\left(\frac{x}{A}\right)^{3/4} - 3x\right)}\left((0) - \left(-\frac{3k}{\left(\frac{x}{A}\right)^{1/4}} + 3k\right)\right) \\
&= -\frac{3\left(-1 + \frac{1}{\left(\frac{x}{A}\right)^{1/4}}\right)}{4A\left(\frac{x}{A}\right)^{3/4} - 3x}
\end{aligned}$$

Since B does not depend on t, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}
\mu &= e^{\int B \,\mathrm{d}x} \\
&= e^{\int -\frac{3\left(-1+\frac{1}{\left(\frac{x}{A}\right)^{1/4}}\right)}{4A\left(\frac{x}{A}\right)^{3/4} - 3x} \mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-\frac{3\ln\left(\frac{x}{A}\right)}{4} - \ln\left(3\left(\frac{x}{A}\right)^{1/4} - 4\right)} \\
&= \frac{1}{\left(\frac{x}{A}\right)^{3/4}\left(3\left(\frac{x}{A}\right)^{1/4} - 4\right)}
\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{\left(\frac{x}{A}\right)^{3/4}\left(3\left(\frac{x}{A}\right)^{1/4} - 4\right)}\left(-4Ak\left(\frac{x}{A}\right)^{3/4} + 3kx\right) \\
&= -\frac{4\left(A\left(\frac{x}{A}\right)^{3/4} - \frac{3x}{4}\right)k}{\left(\frac{x}{A}\right)^{3/4}\left(3\left(\frac{x}{A}\right)^{1/4} - 4\right)}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{\left(\frac{x}{A}\right)^{3/4}\left(3\left(\frac{x}{A}\right)^{1/4}-4\right)}(1) \\
&= \frac{1}{\left(\frac{x}{A}\right)^{3/4}\left(3\left(\frac{x}{A}\right)^{1/4}-4\right)}
\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\overline{M}+\overline{N}\frac{\mathrm{d}x}{\mathrm{d}t}=0$$

$$\left(-\frac{4\left(A\left(\frac{x}{A}\right)^{3/4}-\frac{3x}{4}\right)k}{\left(\frac{x}{A}\right)^{3/4}\left(3\left(\frac{x}{A}\right)^{1/4}-4\right)}\right)+\left(\frac{1}{\left(\frac{x}{A}\right)^{3/4}\left(3\left(\frac{x}{A}\right)^{1/4}-4\right)}\right)\frac{\mathrm{d}x}{\mathrm{d}t}=0$$

The following equations are now set up to solve for the function $\phi(t,x)$

$$\frac{\partial\phi}{\partial t}=\overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial x}=\overline{N} \tag{2}$$

Integrating (1) w.r.t. t gives

$$\begin{aligned}
\int\frac{\partial\phi}{\partial t}\,\mathrm{d}t &= \int\overline{M}\,\mathrm{d}t \\
\int\frac{\partial\phi}{\partial t}\,\mathrm{d}t &= \int-\frac{4\left(A\left(\frac{x}{A}\right)^{3/4}-\frac{3x}{4}\right)k}{\left(\frac{x}{A}\right)^{3/4}\left(3\left(\frac{x}{A}\right)^{1/4}-4\right)}\,\mathrm{d}t
\end{aligned}$$

$$\phi=-\frac{4\left(A\left(\frac{x}{A}\right)^{3/4}-\frac{3x}{4}\right)kt}{\left(\frac{x}{A}\right)^{3/4}\left(3\left(\frac{x}{A}\right)^{1/4}-4\right)}+f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both t and x. Taking derivative of equation (3) w.r.t x gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= -\frac{4\left(\frac{3}{4\left(\frac{x}{A}\right)^{1/4}} - \frac{3}{4}\right) kt}{\left(\frac{x}{A}\right)^{3/4}\left(3\left(\frac{x}{A}\right)^{1/4} - 4\right)} + \frac{3\left(A\left(\frac{x}{A}\right)^{3/4} - \frac{3x}{4}\right) kt}{\left(\frac{x}{A}\right)^{7/4}\left(3\left(\frac{x}{A}\right)^{1/4} - 4\right) A} \\
&\quad + \frac{3\left(A\left(\frac{x}{A}\right)^{3/4} - \frac{3x}{4}\right) kt}{\left(\frac{x}{A}\right)^{3/2}\left(3\left(\frac{x}{A}\right)^{1/4} - 4\right)^2 A} + f'(x) \\
&= 0 + f'(x)
\end{aligned} \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial x} = \frac{1}{\left(\frac{x}{A}\right)^{3/4}\left(3\left(\frac{x}{A}\right)^{1/4} - 4\right)}$. Therefore equation (4) becomes

$$
\frac{1}{\left(\frac{x}{A}\right)^{3/4}\left(3\left(\frac{x}{A}\right)^{1/4} - 4\right)} = 0 + f'(x) \tag{5}
$$

Solving equation (5) for $f'(x)$ gives

$$
f'(x) = \frac{1}{\left(\frac{x}{A}\right)^{3/4}\left(3\left(\frac{x}{A}\right)^{1/4} - 4\right)}
$$

Integrating the above w.r.t x gives

$$
\begin{aligned}
\int f'(x)\,\mathrm{d}x &= \int \left(\frac{1}{\left(\frac{x}{A}\right)^{3/4}\left(3\left(\frac{x}{A}\right)^{1/4} - 4\right)}\right) \mathrm{d}x \\
f(x) &= \frac{4A\ln\left(3\left(\frac{x}{A}\right)^{1/4} - 4\right)}{3} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$
\phi = -\frac{4\left(A\left(\frac{x}{A}\right)^{3/4} - \frac{3x}{4}\right) kt}{\left(\frac{x}{A}\right)^{3/4}\left(3\left(\frac{x}{A}\right)^{1/4} - 4\right)} + \frac{4A\ln\left(3\left(\frac{x}{A}\right)^{1/4} - 4\right)}{3} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -\frac{4\left(A\left(\frac{x}{A}\right)^{3/4} - \frac{3x}{4}\right) kt}{\left(\frac{x}{A}\right)^{3/4} \left(3\left(\frac{x}{A}\right)^{1/4} - 4\right)} + \frac{4A\ln\left(3\left(\frac{x}{A}\right)^{1/4} - 4\right)}{3}$$

Solving for x gives

$$x = \frac{\left(e^{-\frac{3(Akt-c_1)}{4A}} + 4\right)^4 A}{81}$$

Summary of solutions found

$$x = \frac{\left(e^{-\frac{3(Akt-c_1)}{4A}} + 4\right)^4 A}{81}$$

Solved using Lie symmetry for first order ode

Time used: 1.545 (sec)

Writing the ode as

$$\begin{aligned}
x' &= 4Ak\left(\frac{x}{A}\right)^{3/4} - 3kx \\
x' &= \omega(t, x)
\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_t + \omega(\eta_x - \xi_t) - \omega^2\xi_x - \omega_t\xi - \omega_x\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = ta_2 + xa_3 + a_1 \tag{1E}$$

$$\eta = tb_2 + xb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 + \left(4Ak\left(\frac{x}{A}\right)^{3/4} - 3kx\right)(b_3 - a_2) \\
&- \left(4Ak\left(\frac{x}{A}\right)^{3/4} - 3kx\right)^2 a_3 - \left(\frac{3k}{\left(\frac{x}{A}\right)^{1/4}} - 3k\right)(tb_2 + xb_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{16A^2\left(\frac{x}{A}\right)^{7/4} k^2 a_3 - 24x^2k^2a_3 + 4xka_2 - kxb_3 + 9\left(\frac{x}{A}\right)^{1/4} k^2x^2a_3 - 3\left(\frac{x}{A}\right)^{1/4} ktb_2 - 3\left(\frac{x}{A}\right)^{1/4} kxa_2 - 3\left(\frac{x}{A}\right)^{1/}}{\left(\frac{x}{A}\right)^{1/4}}
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-16A^2\left(\frac{x}{A}\right)^{7/4} k^2a_3 - 9\left(\frac{x}{A}\right)^{1/4} k^2x^2a_3 + 24x^2k^2a_3 + 3\left(\frac{x}{A}\right)^{1/4} ktb_2 \\
&+3\left(\frac{x}{A}\right)^{1/4} kxa_2 + 3\left(\frac{x}{A}\right)^{1/4} kb_1 - 3ktb_2 - 4xka_2 + kxb_3 + b_2\left(\frac{x}{A}\right)^{1/4} - 3kb_1 = 0
\end{aligned} \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&-16Ax\left(\frac{x}{A}\right)^{3/4} k^2a_3 - 9\left(\frac{x}{A}\right)^{1/4} k^2x^2a_3 + 24x^2k^2a_3 + 3\left(\frac{x}{A}\right)^{1/4} ktb_2 \\
&+3\left(\frac{x}{A}\right)^{1/4} kxa_2 - 3ktb_2 - 4xka_2 + kxb_3 + 3\left(\frac{x}{A}\right)^{1/4} kb_1 - 3kb_1 + b_2\left(\frac{x}{A}\right)^{1/4} = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{t, x\}$ in them.

$$
\left\{t, x, \left(\frac{x}{A}\right)^{1/4}, \left(\frac{x}{A}\right)^{3/4}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{t, x\}$ in them

$$
\left\{t = v_1, x = v_2, \left(\frac{x}{A}\right)^{1/4} = v_3, \left(\frac{x}{A}\right)^{3/4} = v_4\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-16Av_2v_4k^2a_3 - 9v_3k^2v_2^2a_3 + 24v_2^2k^2a_3 + 3v_3kv_2a_2 + 3v_3kv_1b_2 \\
&- 4v_2ka_2 + 3v_3kb_1 - 3kv_1b_2 + kv_2b_3 - 3kb_1 + b_2v_3 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$\begin{aligned} &3v_3kv_1b_2 - 3kv_1b_2 - 9v_3k^2v_2^2a_3 + 24v_2^2k^2a_3 + 3v_3kv_2a_2 \\ &- 16Av_2v_4k^2a_3 + (-4ka_2 + kb_3)\,v_2 + (3kb_1 + b_2)\,v_3 - 3kb_1 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 3ka_2 &= 0 \\ -3kb_1 &= 0 \\ -3kb_2 &= 0 \\ 3kb_2 &= 0 \\ -9k^2a_3 &= 0 \\ 24k^2a_3 &= 0 \\ -16A\,k^2a_3 &= 0 \\ -4ka_2 + kb_3 &= 0 \\ 3kb_1 + b_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= a_1 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= 0 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 1 \\ \eta &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(t, x) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dt}{\xi} = \frac{dx}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial x}\right) S(t, x) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dt \\ &= \int \frac{1}{1} dt \end{aligned}$$

Which results in

$$S = t$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_t + \omega(t, x) S_x}{R_t + \omega(t, x) R_x} \tag{2}$$

Where in the above R_t, R_x, S_t, S_x are all partial derivatives and $\omega(t, x)$ is the right hand side of the original ode given by

$$\omega(t, x) = 4Ak \left(\frac{x}{A}\right)^{3/4} - 3kx$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_t &= 0 \\ R_x &= 1 \\ S_t &= 1 \\ S_x &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{k \left(4A \left(\frac{x}{A}\right)^{3/4} - 3x\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for t, x in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{k\left(4A\left(\frac{R}{A}\right)^{3/4} - 3R\right)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int \frac{1}{k\left(4A\left(\frac{R}{A}\right)^{3/4} - 3R\right)}\,dR$$

$$S(R) = -\frac{4\ln\left(3\left(\frac{R}{A}\right)^{1/4} - 4\right)}{3k} + c_2$$

To complete the solution, we just need to transform the above back to t, x coordinates. This results in

$$t = -\frac{4\ln\left(3\left(\frac{x}{A}\right)^{1/4} - 4\right)}{3k} + c_2$$

Which gives

$$x = \frac{\left(\mathrm{e}^{\frac{3}{4}c_2k - \frac{3}{4}tk} + 4\right)^4 A}{81}$$

Summary of solutions found

$$x = \frac{\left(\mathrm{e}^{\frac{3}{4}c_2k - \frac{3}{4}tk} + 4\right)^4 A}{81}$$

Maple step by step solution

Let's solve

$\frac{d}{dt}x(t) = 4Ak\left(\frac{x(t)}{A}\right)^{3/4} - 3kx(t)$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dt}x(t)$

- Solve for the highest derivative

$$\frac{d}{dt}x(t) = 4Ak\left(\frac{x(t)}{A}\right)^{3/4} - 3kx(t)$$

- Separate variables

$$\frac{\frac{d}{dt}x(t)}{4Ak\left(\frac{x(t)}{A}\right)^{3/4} - 3kx(t)} = 1$$

- Integrate both sides with respect to t

$$\int \frac{\frac{d}{dt}x(t)}{4Ak\left(\frac{x(t)}{A}\right)^{3/4} - 3kx(t)}dt = \int 1dt + C1$$

- Evaluate integral

$$\frac{-\frac{\ln(256A-81x(t))}{3} + \frac{\ln\left(9\sqrt{\frac{x(t)}{A}}+16\right)}{3} - \frac{\ln\left(9\sqrt{\frac{x(t)}{A}}-16\right)}{3} - \frac{2\ln\left(3\left(\frac{x(t)}{A}\right)^{1/4}-4\right)}{3} + \frac{2\ln\left(3\left(\frac{x(t)}{A}\right)^{1/4}+4\right)}{3}}{k} = t + C1$$

- Solve for $x(t)$

$$\left\{x(t) = \frac{-\frac{16\left(-A^3 e^{C1k} e^{kt}\right)^{3/4} e^{3(t+C1)k}}{A^2\left(e^{C1k}\right)^3\left(e^{kt}\right)^3} + \frac{96\, e^{3(t+C1)k}\sqrt{-A^3 e^{C1k} e^{kt}}}{A\left(e^{C1k}\right)^2\left(e^{kt}\right)^2} - \frac{256\, e^{3(t+C1)k}\left(-A^3 e^{C1k} e^{kt}\right)^{1/4}}{e^{C1k} e^{kt}} + 256A\, e^{3(t+C1)k} - 1}{81\, e^{3(t+C1)k}}, x(t) =\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 85

```
dsolve(diff(x(t),t) = 4*A*k*(x(t)/A)^(3/4)-3*k*x(t),
       x(t),singsol=all)
```

$$\frac{\ln\left(9\sqrt{\frac{x(t)}{A}} - 16\right) - \ln\left(9\sqrt{\frac{x(t)}{A}} + 16\right) + 2\ln\left(3\left(\frac{x(t)}{A}\right)^{1/4} - 4\right) - 2\ln\left(3\left(\frac{x(t)}{A}\right)^{1/4} + 4\right) + \ln\left(256A - 81\right.}{3k}$$

Mathematica DSolve solution

Solving time : 0.369 (sec)
Leaf size : 51

```
DSolve[{D[x[t],t]==4*A*k*(x[t]/A)^(3/4)-3*k*x[t],{}},
       x[t],t,IncludeSingularSolutions->True]
```

$$x(t) \to \frac{1}{81} A e^{-3kt} \left(4e^{\frac{3kt}{4}} + e^{\frac{3c_1}{4}}\right)^4$$
$$x(t) \to 0$$
$$x(t) \to \frac{256A}{81}$$

2.1.78 problem 78

Internal problem ID [8466]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 78
Date solved : `Tuesday, December 17, 2024 at 12:53:01 PM`
CAS classification : `[[_homogeneous, 'class A'], _dAlembert]`

Solve

$$\frac{y'y}{1+\frac{\sqrt{1+y'^2}}{2}}=-x$$

Solved as first order dAlembert ode

Time used: 1.300 (sec)

Let $p=y'$ the ode becomes

$$\frac{py}{1+\frac{\sqrt{p^2+1}}{2}}=-x$$

Solving for y from the above results in

$$y=-\frac{x\left(2+\sqrt{p^2+1}\right)}{2p} \tag{1}$$

This has the form

$$y=xf(p)+g(p) \tag{*}$$

Where f,g are functions of $p=y'(x)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f+(xf'+g')\frac{dp}{dx} \\ p-f &= (xf'+g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = \frac{-2 - \sqrt{p^2+1}}{2p}$$
$$g = 0$$

Hence (2) becomes

$$p - \frac{-2 - \sqrt{p^2+1}}{2p} = \left(-\frac{x}{2\sqrt{p^2+1}} + \frac{x}{p^2} + \frac{x\sqrt{p^2+1}}{2p^2}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{-2 - \sqrt{p^2+1}}{2p} = 0$$

Solving the above for p results in

$$p_1 = i$$
$$p_2 = -i$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$y = ix$$
$$y = -ix$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{-2-\sqrt{p(x)^2+1}}{2p(x)}}{-\frac{x}{2\sqrt{p(x)^2+1}} + \frac{x}{p(x)^2} + \frac{x\sqrt{p(x)^2+1}}{2p(x)^2}} \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed. The ode $p'(x) = \frac{\left(2p(x)^2+\sqrt{p(x)^2+1}+2\right)\sqrt{p(x)^2+1}\,p(x)}{x\left(1+2\sqrt{p(x)^2+1}\right)}$ is separable as it can be written as

$$\begin{aligned} p'(x) &= \frac{\left(2p(x)^2 + \sqrt{p(x)^2+1} + 2\right)\sqrt{p(x)^2+1}\,p(x)}{x\left(1+2\sqrt{p(x)^2+1}\right)} \\ &= f(x)g(p) \end{aligned}$$

Where

$$
\begin{aligned}
f(x) &= \frac{1}{x} \\
g(p) &= \frac{\left(2p^2+\sqrt{p^2+1}+2\right)\sqrt{p^2+1}\,p}{1+2\sqrt{p^2+1}}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(p)}\,dp &= \int f(x)\,dx \\
\int \frac{1+2\sqrt{p^2+1}}{\left(2p^2+\sqrt{p^2+1}+2\right)\sqrt{p^2+1}\,p}\,dp &= \int \frac{1}{x}\,dx \\
\ln\left(\frac{p(x)}{\sqrt{p(x)^2+1}}\right) &= \ln(x)+c_1
\end{aligned}
$$

We now need to find the singular solutions, these are found by finding for what values $g(p)$ is zero, since we had to divide by this above. Solving $g(p)=0$ or $\frac{\left(2p^2+\sqrt{p^2+1}+2\right)\sqrt{p^2+1}\,p}{1+2\sqrt{p^2+1}}=0$ for $p(x)$ gives

$$
\begin{aligned}
p(x) &= 0 \\
p(x) &= -i \\
p(x) &= i
\end{aligned}
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
\ln\left(\frac{p(x)}{\sqrt{p(x)^2+1}}\right) &= \ln(x)+c_1 \\
p(x) &= 0 \\
p(x) &= -i \\
p(x) &= i
\end{aligned}
$$

Solving for $p(x)$ gives

$$
\begin{aligned}
p(x) &= 0 \\
p(x) &= -i \\
p(x) &= i
\end{aligned}
$$

$$
p(x) = \frac{e^{c_1} x}{\sqrt{1 - x^2 e^{2c_1}}}
$$
$$
p(x) = -\frac{e^{c_1} x}{\sqrt{1 - x^2 e^{2c_1}}}
$$

Substituing the above solution for p in (2A) gives

$$
\begin{aligned}
y &= -ix \\
y &= ix \\
y &= \frac{\left(-2 - \sqrt{\frac{e^{2c_1} x^2}{1 - x^2 e^{2c_1}} + 1}\right) \sqrt{1 - x^2 e^{2c_1}}\, e^{-c_1}}{2} \\
y &= -\frac{\left(-2 - \sqrt{\frac{e^{2c_1} x^2}{1 - x^2 e^{2c_1}} + 1}\right) \sqrt{1 - x^2 e^{2c_1}}\, e^{-c_1}}{2}
\end{aligned}
$$

<u>Summary of solutions found</u>

$$
\begin{aligned}
y &= -ix \\
y &= ix \\
y &= -\frac{\left(-2 - \sqrt{\frac{e^{2c_1} x^2}{1 - x^2 e^{2c_1}} + 1}\right) \sqrt{1 - x^2 e^{2c_1}}\, e^{-c_1}}{2} \\
y &= \frac{\left(-2 - \sqrt{\frac{e^{2c_1} x^2}{1 - x^2 e^{2c_1}} + 1}\right) \sqrt{1 - x^2 e^{2c_1}}\, e^{-c_1}}{2}
\end{aligned}
$$

Maple step by step solution

Let's solve

$$
\frac{\left(\frac{d}{dx} y(x)\right) y(x)}{1 + \frac{\sqrt{1 + \left(\frac{d}{dx} y(x)\right)^2}}{2}} = -x
$$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx} y(x)$

- Solve for the highest derivative

  $$
  \left[\frac{d}{dx} y(x) = -\frac{\left(4y(x) - \sqrt{4y(x)^2 + 3x^2}\right) x}{4y(x)^2 - x^2}, \frac{d}{dx} y(x) = -\frac{\left(4y(x) + \sqrt{4y(x)^2 + 3x^2}\right) x}{4y(x)^2 - x^2}\right]
  $$

- Solve the equation $\frac{d}{dx}y(x) = -\frac{\left(4y(x)-\sqrt{4y(x)^2+3x^2}\right)x}{4y(x)^2-x^2}$
- Solve the equation $\frac{d}{dx}y(x) = -\frac{\left(4y(x)+\sqrt{4y(x)^2+3x^2}\right)x}{4y(x)^2-x^2}$
- Set of solutions
  $\{workingODE, workingODE\}$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

Maple dsolve solution

Solving time : `2.606` (sec)
Leaf size : 187

```
dsolve(diff(y(x),x)*y(x)/(1+1/2*(1+diff(y(x),x)^2)^(1/2)) = -x,
       y(x),singsol=all)
```

$$y = -\frac{\sqrt{-x^2+c_1}\left(2+\sqrt{\frac{c_1}{-x^2+c_1}}\right)}{2}$$
$$y = \frac{\sqrt{-x^2+c_1}\left(2+\sqrt{\frac{c_1}{-x^2+c_1}}\right)}{2}$$
$$y = -\frac{\sqrt{-9x^2+15c_1-6\sqrt{-3c_1x^2+4c_1^2}}}{3}$$
$$y = \frac{\sqrt{-9x^2+15c_1-6\sqrt{-3c_1x^2+4c_1^2}}}{3}$$
$$y = -\frac{\sqrt{-9x^2+15c_1+6\sqrt{-3c_1x^2+4c_1^2}}}{3}$$
$$y = \frac{\sqrt{-9x^2+15c_1+6\sqrt{-3c_1x^2+4c_1^2}}}{3}$$

Mathematica DSolve solution

Solving time : `2.113` (sec)
Leaf size : 153

```
DSolve[{D[y[x],x]*y[x]/(1+1/2*Sqrt[1+(D[y[x],x])^2])==-x,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{3}\left(e^{c_1}-\sqrt{-9x^2+4e^{2c_1}}\right)$$
$$y(x) \to \frac{1}{3}\left(\sqrt{-9x^2+4e^{2c_1}}+e^{c_1}\right)$$
$$y(x) \to -\sqrt{-x^2+4e^{2c_1}}-e^{c_1}$$
$$y(x) \to \sqrt{-x^2+4e^{2c_1}}-e^{c_1}$$
$$y(x) \to -\sqrt{-x^2}$$
$$y(x) \to \sqrt{-x^2}$$

2.1.79 problem 78

Internal problem ID [8467]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 78
Date solved : `Tuesday, December 17, 2024 at 12:53:02 PM`
CAS classification : `[[_homogeneous, ‘class A‘], _dAlembert]`

Solve

$$\frac{y'y}{1+\frac{\sqrt{1+y'^2}}{2}} = -x$$

With initial conditions

$$y(0) = 3$$

Solved as first order dAlembert ode

Time used: 0.688 (sec)

Let $p = y'$ the ode becomes

$$\frac{py}{1+\frac{\sqrt{p^2+1}}{2}} = -x$$

Solving for y from the above results in

$$y = -\frac{x\left(2+\sqrt{p^2+1}\right)}{2p} \tag{1}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. x gives

$$
\begin{aligned}
p &= f + (xf' + g')\frac{dp}{dx} \\
p - f &= (xf' + g')\frac{dp}{dx} \qquad (2)
\end{aligned}
$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= \frac{-2 - \sqrt{p^2 + 1}}{2p} \\
g &= 0
\end{aligned}
$$

Hence (2) becomes

$$
p - \frac{-2 - \sqrt{p^2 + 1}}{2p} = \left(-\frac{x}{2\sqrt{p^2 + 1}} + \frac{x}{p^2} + \frac{x\sqrt{p^2 + 1}}{2p^2}\right) p'(x) \qquad (2A)
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - \frac{-2 - \sqrt{p^2 + 1}}{2p} = 0
$$

No valid singular solutions found.

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) - \frac{-2 - \sqrt{p(x)^2 + 1}}{2p(x)}}{-\frac{x}{2\sqrt{p(x)^2 + 1}} + \frac{x}{p(x)^2} + \frac{x\sqrt{p(x)^2 + 1}}{2p(x)^2}} \qquad (3)
$$

This ODE is now solved for $p(x)$. No inversion is needed. The ode $p'(x) = \frac{\left(2p(x)^2 + \sqrt{p(x)^2 + 1} + 2\right)\sqrt{p(x)^2 + 1}\, p(x)}{x\left(1 + 2\sqrt{p(x)^2 + 1}\right)}$ is separable as it can be written as

$$
\begin{aligned}
p'(x) &= \frac{\left(2p(x)^2 + \sqrt{p(x)^2 + 1} + 2\right)\sqrt{p(x)^2 + 1}\, p(x)}{x\left(1 + 2\sqrt{p(x)^2 + 1}\right)} \\
&= f(x)g(p)
\end{aligned}
$$

Where

$$
\begin{aligned}
f(x) &= \frac{1}{x} \\
g(p) &= \frac{\left(2p^2 + \sqrt{p^2 + 1} + 2\right)\sqrt{p^2 + 1}\, p}{1 + 2\sqrt{p^2 + 1}}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(p)}\,dp &= \int f(x)\,dx \\
\int \frac{1+2\sqrt{p^2+1}}{\left(2p^2+\sqrt{p^2+1}+2\right)\sqrt{p^2+1}\,p}\,dp &= \int \frac{1}{x}\,dx \\
\ln\left(\frac{p(x)}{\sqrt{p(x)^2+1}}\right) &= \ln(x)+c_1
\end{aligned}
$$

We now need to find the singular solutions, these are found by finding for what values $g(p)$ is zero, since we had to divide by this above. Solving $g(p)=0$ or $\frac{\left(2p^2+\sqrt{p^2+1}+2\right)\sqrt{p^2+1}\,p}{1+2\sqrt{p^2+1}} = 0$ for $p(x)$ gives

$$
\begin{aligned}
p(x) &= 0 \\
p(x) &= -i \\
p(x) &= i
\end{aligned}
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
\ln\left(\frac{p(x)}{\sqrt{p(x)^2+1}}\right) &= \ln(x)+c_1 \\
p(x) &= 0 \\
p(x) &= -i \\
p(x) &= i
\end{aligned}
$$

Solving for $p(x)$ gives

$$
\begin{aligned}
p(x) &= 0 \\
p(x) &= -i \\
p(x) &= i \\
p(x) &= \frac{e^{c_1}x}{\sqrt{1-x^2 e^{2c_1}}} \\
p(x) &= -\frac{e^{c_1}x}{\sqrt{1-x^2 e^{2c_1}}}
\end{aligned}
$$

Substituing the above solution for p in (2A) gives

$$
\begin{aligned}
y &= -ix \\
y &= ix \\
y &= \frac{\left(-2-\sqrt{\frac{\mathrm{e}^{2c_1}x^2}{1-x^2\mathrm{e}^{2c_1}}+1}\right)\sqrt{1-x^2\mathrm{e}^{2c_1}}\,\mathrm{e}^{-c_1}}{2} \\
y &= -\frac{\left(-2-\sqrt{\frac{\mathrm{e}^{2c_1}x^2}{1-x^2\mathrm{e}^{2c_1}}+1}\right)\sqrt{1-x^2\mathrm{e}^{2c_1}}\,\mathrm{e}^{-c_1}}{2}
\end{aligned}
$$

Solving for the constant of integration from initial conditions, the solution becomes

$$
y = -\left(-2-\sqrt{\frac{x^2}{-x^2+4}+1}\right)\sqrt{1-\frac{x^2}{4}}
$$

Solving for the constant of integration from initial conditions, the solution becomes

$$
y = -\left(-2-\sqrt{\frac{x^2}{-x^2+4}+1}\right)\sqrt{1-\frac{x^2}{4}}
$$

The solution

$$
y = -ix
$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$
y = ix
$$

was found not to satisfy the ode or the IC. Hence it is removed.

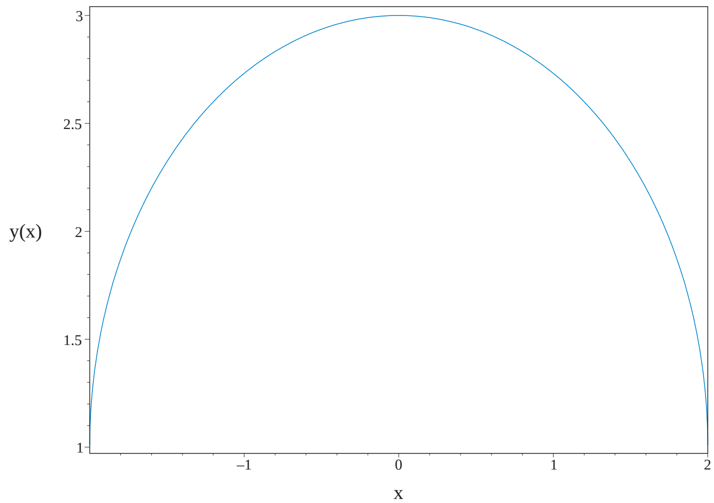


Figure 2.175: Solution plot
$y = -\left(-2-\sqrt{\frac{x^2}{-x^2+4}+1}\right)\sqrt{1-\frac{x^2}{4}}$

Summary of solutions found

$$y = -\left(-2-\sqrt{\frac{x^2}{-x^2+4}+1}\right)\sqrt{1-\frac{x^2}{4}}$$

Maple step by step solution

Let's solve

$$\left[\frac{\left(\frac{d}{dx}y(x)\right)y(x)}{1+\frac{\sqrt{1+\left(\frac{d}{dx}y(x)\right)^2}}{2}} = -x, y(0) = 3\right]$$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $$\left[\frac{d}{dx}y(x) = -\frac{\left(4y(x)-\sqrt{4y(x)^2+3x^2}\right)x}{4y(x)^2-x^2}, \frac{d}{dx}y(x) = -\frac{\left(4y(x)+\sqrt{4y(x)^2+3x^2}\right)x}{4y(x)^2-x^2}\right]$$

□ Solve the equation $\frac{d}{dx}y(x) = -\frac{\left(4y(x)-\sqrt{4y(x)^2+3x^2}\right)x}{4y(x)^2-x^2}$

- ○ Use initial condition $y(0) = 3$

  0

- ○ Solve for 0

$0 = 0$

- Substitute $0 = 0$ into general solution and simplify
  0
- Solution to the IVP
  0

□ Solve the equation $\frac{d}{dx}y(x) = -\frac{\left(4y(x)+\sqrt{4y(x)^2+3x^2}\right)x}{4y(x)^2-x^2}$

- Use initial condition $y(0) = 3$
  0
- Solve for 0
  $0 = 0$
- Substitute $0 = 0$ into general solution and simplify
  0
- Solution to the IVP
  0

• Set of solutions
  $\{0\}$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

Maple dsolve solution

Solving time : 13.325 (sec)
Leaf size : 29

```
dsolve([diff(y(x),x)*y(x)/(1+1/2*(1+diff(y(x),x)^2)^(1/2)) = -x,
       op([y(0) = 3])],y(x),singsol=all)
```

$$y = -3 + \sqrt{-x^2 + 36}$$
$$y = 1 + \sqrt{-x^2 + 4}$$

Mathematica DSolve solution

Solving time : 0.5 (sec)
Leaf size : 35

```
DSolve[{D[y[x],x]*y[x]/(1+1/2*Sqrt[1+(D[y[x],x])^2])==-x,y[0]==3},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \sqrt{4 - x^2} + 1$$
$$y(x) \to \sqrt{36 - x^2} - 3$$

2.1.80 problem 79

Internal problem ID [8468]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 79
Date solved : `Tuesday, December 17, 2024 at 12:53:04 PM`
CAS classification : `[_separable]`

Solve

$$y' = \frac{y\left(1+\frac{a^2 x}{\sqrt{a^2(x^2+1)}}\right)}{\sqrt{a^2\left(x^2+1\right)}}$$

Solved as first order linear ode

Time used: 0.226 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = -\frac{a^2 x + \sqrt{a^2\left(x^2+1\right)}}{a^2\left(x^2+1\right)}$$

$$p(x) = 0$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= \mathrm{e}^{\int -\frac{a^2x+\sqrt{a^2\left(x^2+1\right)}}{a^2\left(x^2+1\right)}dx} \\
&= \mathrm{e}^{-\frac{\ln\left(\frac{a^2x}{\sqrt{a^2}}+\sqrt{a^2x^2+a^2}\right)}{\sqrt{a^2}}-\frac{\ln\left(x^2+1\right)}{2}}
\end{aligned}
$$

The ode becomes

$$
\frac{\mathrm{d}}{\mathrm{d}x}\mu y = 0
$$

$$
\frac{\mathrm{d}}{\mathrm{d}x}\left(y\,\mathrm{e}^{-\frac{\ln\left(\frac{a^2x}{\sqrt{a^2}}+\sqrt{a^2x^2+a^2}\right)}{\sqrt{a^2}}-\frac{\ln\left(x^2+1\right)}{2}}\right) = 0
$$

Integrating gives

$$
\begin{aligned}
y\,\mathrm{e}^{-\frac{\ln\left(\frac{a^2x}{\sqrt{a^2}}+\sqrt{a^2x^2+a^2}\right)}{\sqrt{a^2}}-\frac{\ln\left(x^2+1\right)}{2}} &= \int 0\,dx + c_1 \\
&= c_1
\end{aligned}
$$

Dividing throughout by the integrating factor $\mathrm{e}^{-\frac{\ln\left(\frac{a^2x}{\sqrt{a^2}}+\sqrt{a^2x^2+a^2}\right)}{\sqrt{a^2}}-\frac{\ln\left(x^2+1\right)}{2}}$ gives the final solution

$$
y = \mathrm{e}^{\frac{\ln\left(\frac{a^2x}{\sqrt{a^2}}+\sqrt{a^2x^2+a^2}\right)}{\sqrt{a^2}}+\ln\left(\sqrt{x^2+1}\right)}c_1
$$

<u>Summary of solutions found</u>

$$
y = \mathrm{e}^{\frac{\ln\left(\frac{a^2x}{\sqrt{a^2}}+\sqrt{a^2x^2+a^2}\right)}{\sqrt{a^2}}+\ln\left(\sqrt{x^2+1}\right)}c_1
$$

Solved as first order separable ode

Time used: 0.993 (sec)

The ode $y' = \frac{y\left(a^2x+\sqrt{a^2(x^2+1)}\right)}{a^2(x^2+1)}$ is separable as it can be written as

$$
\begin{aligned}
y' &= \frac{y\left(a^2x+\sqrt{a^2\left(x^2+1\right)}\right)}{a^2\left(x^2+1\right)} \\
&= f(x)g(y)
\end{aligned}
$$

Where

$$
\begin{aligned}
f(x) &= \frac{a^2 x + \sqrt{a^2 \left(x^2+1\right)}}{a^2 \left(x^2+1\right)} \\
g(y) &= y
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(y)}\, dy &= \int f(x)\, dx \\
\int \frac{1}{y}\, dy &= \int \frac{a^2 x + \sqrt{a^2 \left(x^2+1\right)}}{a^2 \left(x^2+1\right)}\, dx \\
\ln(y) &= \frac{\ln\left(\frac{a^2 x}{\sqrt{a^2}} + \sqrt{a^2 x^2 + a^2}\right)}{\sqrt{a^2}} + \ln\left(\sqrt{x^2+1}\right) + c_1
\end{aligned}
$$

We now need to find the singular solutions, these are found by finding for what values $g(y)$ is zero, since we had to divide by this above. Solving $g(y) = 0$ or $y = 0$ for y gives

$$
y = 0
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
\ln(y) &= \frac{\ln\left(\frac{a^2 x}{\sqrt{a^2}} + \sqrt{a^2 x^2 + a^2}\right)}{\sqrt{a^2}} + \ln\left(\sqrt{x^2+1}\right) + c_1 \\
y &= 0
\end{aligned}
$$

Solving for y gives

$$
\begin{aligned}
y &= 0 \\
y &= \mathrm{e}^{\frac{2\ln\left(\sqrt{x^2+1}\right)\sqrt{a^2}+2c_1\sqrt{a^2}+2\ln\left(a^2 x+\sqrt{a^2 x^2+a^2}\,\sqrt{a^2}\right)-\ln\left(a^2\right)}{2\sqrt{a^2}}}
\end{aligned}
$$

<u>Summary of solutions found</u>

$$
\begin{aligned}
y &= 0 \\
y &= \mathrm{e}^{\frac{\ln\left(x^2+1\right)\sqrt{a^2}+2c_1\sqrt{a^2}+2\ln\left(a^2 x+\sqrt{a^2 x^2+a^2}\,\sqrt{a^2}\right)-\ln\left(a^2\right)}{2\sqrt{a^2}}}
\end{aligned}
$$

Solved as first order homogeneous class D2 ode

Time used: 1.287 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$u'(x)\,x + u(x) = \frac{u(x)\,x\left(1 + \frac{a^2 x}{\sqrt{a^2(x^2+1)}}\right)}{\sqrt{a^2\,(x^2+1)}}$$

Which is now solved The ode $u'(x) = \frac{u(x)\left(x\sqrt{a^2(x^2+1)}-a^2\right)}{a^2(x^2+1)x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= \frac{u(x)\left(x\sqrt{a^2\,(x^2+1)} - a^2\right)}{a^2\,(x^2+1)\,x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{x\sqrt{a^2\,(x^2+1)} - a^2}{(x^2+1)\,x\,a^2} \\ g(u) &= u \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\ \int \frac{1}{u}\,du &= \int \frac{x\sqrt{a^2\,(x^2+1)} - a^2}{(x^2+1)\,x\,a^2}\,dx \\ \ln\left(u(x)\right) &= \frac{\ln\left(\frac{a^2 x}{\sqrt{a^2}} + \sqrt{a^2x^2+a^2}\right)}{\sqrt{a^2}} + \ln\left(\frac{\sqrt{x^2+1}}{x}\right) + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} \ln\left(u(x)\right) &= \frac{\ln\left(\frac{a^2 x}{\sqrt{a^2}} + \sqrt{a^2x^2+a^2}\right)}{\sqrt{a^2}} + \ln\left(\frac{\sqrt{x^2+1}}{x}\right) + c_1 \\ u(x) &= 0 \end{aligned}$$

Solving for $u(x)$ gives

$$u(x) = 0$$
$$u(x) = e^{\frac{\ln(x^2+1)\sqrt{a^2}+2\sqrt{a^2}\,\ln\left(\frac{1}{x}\right)+2c_1\sqrt{a^2}+2\ln\left(a^2x+\sqrt{a^2x^2+a^2}\,\sqrt{a^2}\right)-\ln(a^2)}{2\sqrt{a^2}}}$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = e^{\frac{\ln(x^2+1)\sqrt{a^2}+2\sqrt{a^2}\,\ln\left(\frac{1}{x}\right)+2c_1\sqrt{a^2}+2\ln\left(a^2x+\sqrt{a^2x^2+a^2}\,\sqrt{a^2}\right)-\ln(a^2)}{2\sqrt{a^2}}}$ back to y gives

$$y = x\,e^{\frac{\ln(x^2+1)\sqrt{a^2}+2\sqrt{a^2}\,\ln\left(\frac{1}{x}\right)+2c_1\sqrt{a^2}+2\ln\left(a^2x+\sqrt{a^2x^2+a^2}\,\sqrt{a^2}\right)-\ln(a^2)}{2\sqrt{a^2}}}$$

Summary of solutions found

$$y = 0$$
$$y = x\,e^{\frac{\ln(x^2+1)\sqrt{a^2}+2\sqrt{a^2}\,\ln\left(\frac{1}{x}\right)+2c_1\sqrt{a^2}+2\ln\left(a^2x+\sqrt{a^2x^2+a^2}\,\sqrt{a^2}\right)-\ln(a^2)}{2\sqrt{a^2}}}$$

Solved as first order Exact ode

Time used: 2.493 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(\frac{y\left(1 + \frac{a^2 x}{\sqrt{a^2(x^2+1)}}\right)}{\sqrt{a^2\left(x^2+1\right)}}\right)\mathrm{d}x$$

$$\left(-\frac{y\left(1 + \frac{a^2 x}{\sqrt{a^2(x^2+1)}}\right)}{\sqrt{a^2\left(x^2+1\right)}}\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{y\left(1 + \frac{a^2 x}{\sqrt{a^2(x^2+1)}}\right)}{\sqrt{a^2\left(x^2+1\right)}}$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{y\left(1 + \frac{a^2 x}{\sqrt{a^2(x^2+1)}}\right)}{\sqrt{a^2\left(x^2+1\right)}}\right)\\ &= \frac{-a^2 x - \sqrt{a^2\left(x^2+1\right)}}{a^2\left(x^2+1\right)}\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(-\frac{1 + \frac{a^2 x}{\sqrt{a^2(x^2+1)}}}{\sqrt{a^2\left(x^2+1\right)}}\right) - (0)\right) \\
&= \frac{-a^2 x - \sqrt{a^2\left(x^2+1\right)}}{a^2\left(x^2+1\right)}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \,\mathrm{d}x} \\
&= e^{\int \frac{-a^2 x - \sqrt{a^2\left(x^2+1\right)}}{a^2\left(x^2+1\right)} \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{\ln\left(\frac{a^2 x}{\sqrt{a^2}} + \sqrt{a^2 x^2 + a^2}\right)}{\sqrt{a^2}} - \frac{\ln\left(x^2+1\right)}{2}} \\
&= \frac{\left(ax \operatorname{csgn}(a) + \sqrt{a^2\left(x^2+1\right)}\right)^{-\frac{\operatorname{csgn}(a)}{a}}}{\sqrt{x^2+1}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{\left(ax \operatorname{csgn}(a) + \sqrt{a^2\left(x^2+1\right)}\right)^{-\frac{\operatorname{csgn}(a)}{a}}}{\sqrt{x^2+1}} \left(-\frac{y\left(1 + \frac{a^2 x}{\sqrt{a^2(x^2+1)}}\right)}{\sqrt{a^2\left(x^2+1\right)}}\right) \\
&= \frac{\left(-a^2 x - \sqrt{a^2\left(x^2+1\right)}\right) y \left(ax \operatorname{csgn}(a) + \sqrt{a^2\left(x^2+1\right)}\right)^{-\frac{\operatorname{csgn}(a)}{a}}}{a^2\left(x^2+1\right)^{3/2}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{\left(ax\,\mathrm{csgn}\left(a\right)+\sqrt{a^{2}\left(x^{2}+1\right)}\right)^{-\frac{\mathrm{csgn}(a)}{a}}}{\sqrt{x^{2}+1}}(1) \\
&= \frac{\left(ax\,\mathrm{csgn}\left(a\right)+\sqrt{a^{2}\left(x^{2}+1\right)}\right)^{-\frac{\mathrm{csgn}(a)}{a}}}{\sqrt{x^{2}+1}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\left(\frac{\left(-a^{2}x-\sqrt{a^{2}\left(x^{2}+1\right)}\right)y\left(ax\,\mathrm{csgn}\left(a\right)+\sqrt{a^{2}\left(x^{2}+1\right)}\right)^{-\frac{\mathrm{csgn}(a)}{a}}}{a^{2}\left(x^{2}+1\right)^{3/2}}\right)+\left(\frac{\left(ax\,\mathrm{csgn}\left(a\right)+\sqrt{a^{2}\left(x^{2}+1\right)}\right)^{-\frac{\mathrm{cs}}{}}}{\sqrt{x^{2}+1}}\right.
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial\phi}{\partial x}=\overline{M} \tag{1}
$$

$$
\frac{\partial\phi}{\partial y}=\overline{N} \tag{2}
$$

Integrating (2) w.r.t. y gives

$$
\int\frac{\partial\phi}{\partial y}\,\mathrm{d}y=\int\overline{N}\,\mathrm{d}y
$$

$$
\int\frac{\partial\phi}{\partial y}\,\mathrm{d}y=\int\frac{\left(ax\,\mathrm{csgn}\left(a\right)+\sqrt{a^{2}\left(x^{2}+1\right)}\right)^{-\frac{\mathrm{csgn}(a)}{a}}}{\sqrt{x^{2}+1}}\,\mathrm{d}y
$$

$$
\phi=\frac{\left(ax\,\mathrm{csgn}\left(a\right)+\sqrt{a^{2}\left(x^{2}+1\right)}\right)^{-\frac{\mathrm{csgn}(a)}{a}}y}{\sqrt{x^{2}+1}}+f(x) \tag{3}
$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= -\frac{\left(ax \operatorname{csgn}(a)+\sqrt{a^2\left(x^2+1\right)}\right)^{-\frac{\operatorname{csgn}(a)}{a}} \operatorname{csgn}(a)\left(\operatorname{csgn}(a)\, a+\frac{a^2 x}{\sqrt{a^2(x^2+1)}}\right) y}{a\left(ax \operatorname{csgn}(a)+\sqrt{a^2\left(x^2+1\right)}\right)\sqrt{x^2+1}} \\
&\quad -\frac{\left(ax \operatorname{csgn}(a)+\sqrt{a^2\left(x^2+1\right)}\right)^{-\frac{\operatorname{csgn}(a)}{a}} yx}{\left(x^2+1\right)^{3/2}}+f'(x) \\
&= -\frac{y\left(\left(\operatorname{csgn}(a)\, a\, x^2+x^2+1\right)\sqrt{a^2\left(x^2+1\right)}+ax(x^2+1)\left(a+\operatorname{csgn}(a)\right)\right)\left(ax \operatorname{csgn}(a)+\sqrt{a^2\left(x^2+1\right.}\right.}{\left(x^2+1\right)^{3/2}\sqrt{a^2\left(x^2+1\right)}} \\
&\quad +f'(x)
\end{aligned}
\tag{4}
$$

But equation (1) says that $\frac{\partial \phi}{\partial x}=\frac{\left(-a^2 x-\sqrt{a^2(x^2+1)}\right) y\left(ax \operatorname{csgn}(a)+\sqrt{a^2(x^2+1)}\right)^{-\frac{\operatorname{csgn}(a)}{a}}}{a^2(x^2+1)^{3/2}}$. Therefore equation (4) becomes

$$
\begin{aligned}
&\frac{\left(-a^2 x-\sqrt{a^2\left(x^2+1\right)}\right) y\left(ax \operatorname{csgn}(a)+\sqrt{a^2\left(x^2+1\right)}\right)^{-\frac{\operatorname{csgn}(a)}{a}}}{a^2\left(x^2+1\right)^{3/2}} = \\
&-\frac{y\left(\left(\operatorname{csgn}(a)\, a\, x^2+x^2+1\right)\sqrt{a^2\left(x^2+1\right)}+ax(x^2+1)\left(a+\operatorname{csgn}(a)\right)\right)\left(ax \operatorname{csgn}(a)+\sqrt{a^2\left(x^2+1\right)}\right)^{\frac{-a-c}{}}}{\left(x^2+1\right)^{3/2}\sqrt{a^2\left(x^2+1\right)}} \\
&+f'(x)
\end{aligned}
\tag{5}
$$

Solving equation (5) for $f'(x)$ gives

$$
\begin{aligned}
f'(x) &= \frac{y\left(\operatorname{csgn}(a)\sqrt{a^2\left(x^2+1\right)}\left(ax \operatorname{csgn}(a)+\sqrt{a^2\left(x^2+1\right)}\right)^{\frac{-a-\operatorname{csgn}(a)}{a}} a\, x^2+\operatorname{csgn}(a)\left(ax \operatorname{csgn}(a)+\sqrt{a^2\left(x^2\right.}\right.\right.}{} \\
&= \frac{y\left(\left(\left(\operatorname{csgn}(a)\, a\, x^2+x^2+1\right)\sqrt{a^2\left(x^2+1\right)}+ax(x^2+1)\left(a+\operatorname{csgn}(a)\right)\right)\left(ax \operatorname{csgn}(a)+\sqrt{a^2\left(x^2+\right.}\right.\right.}{\left(x^2+1\right)^{3/2}\sqrt{a^2\left(x^2-\right.}}
\end{aligned}
$$

Integrating the above w.r.t x results in

$$
\int f'(x)\,\mathrm{d}x = \int \left(\frac{y\left(\left(\left(\operatorname{csgn}(a)\,a\,x^2+x^2+1\right)\sqrt{a^2\left(x^2+1\right)}+ax(x^2+1)\left(a+\operatorname{csgn}(a)\right)\right)\left(ax\ \operatorname{csgn}(a)+\sqrt{a^2\left(x^2+ \cdots\right.}\right.\right.}{\left(x^2+1\right)^{3/2}\sqrt{a^2\left(x^2+\cdots\right.}} \cdots \right.
$$

$$
f(x) = \int_0^x \frac{y\left(\left(\left(\operatorname{csgn}(a)\,a\,\tau^2+\tau^2+1\right)\sqrt{a^2\left(\tau^2+1\right)}+a\tau(\tau^2+1)\left(a+\operatorname{csgn}(a)\right)\right)\left(a\tau\ \operatorname{csgn}(a)+\sqrt{a^2\left(\cdots\right.}\right.\right.}{\left(\tau^2+1\right)^{3/2}\sqrt{a^2\cdots}} \cdots + c_1
$$

Assuming $0 < a$ then

$$
f(x) = \frac{y\left(\int_0^x \frac{\left(\left(\tau^3+\tau^2\sqrt{\tau^2+1}+\tau\right)a^{\frac{a-1}{a}}+a^{-\frac{1}{a}}\left(\tau^2+1\right)\left(\sqrt{\tau^2+1}+\tau\right)\right)\left(\sqrt{\tau^2+1}+\tau\right)^{\frac{-a-1}{a}}-\left(\sqrt{\tau^2+1}+\tau\right)^{-\frac{1}{a}}\left(a^{-\frac{1}{a}}\tau^2+a^{\frac{a-1}{a}}\sqrt{\tau^2+1}\,\tau+a^{-\cdots}\right.}{\left(\tau^2+1\right)^2}\cdots\right.}{a}\cdots
$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$
\phi = \frac{\left(ax\ \operatorname{csgn}(a)+\sqrt{a^2\left(x^2+1\right)}\right)^{-\frac{\operatorname{csgn}(a)}{a}} y}{\sqrt{x^2+1}} + \frac{y\left(\int_0^x \frac{\left(\left(\tau^3+\tau^2\sqrt{\tau^2+1}+\tau\right)a^{\frac{a-1}{a}}+a^{-\frac{1}{a}}\left(\tau^2+1\right)\left(\sqrt{\tau^2+1}+\tau\right)\right)\left(\sqrt{\tau^2+1}+\tau\right)^{\frac{-a-1}{a}}-\left(\sqrt{\tau^2+1}+\tau\right)^{-\frac{1}{a}}\left(a^{-\frac{1}{a}}\tau^2+a^{\frac{a-1}{a}}\sqrt{\tau^2+1}\,\tau+a^{-\frac{1}{a}}\right)}{\left(\tau^2+1\right)^2} \cdots\right.}{a} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \frac{\left(ax\ \mathrm{csgn}\left(a\right)+\sqrt{a^2\left(x^2+1\right)}\right)^{-\frac{\mathrm{csgn}(a)}{a}} y}{\sqrt{x^2+1}} + \frac{y\left(\int_0^x \frac{\left(\left(\tau^3+\tau^2\sqrt{\tau^2+1}+\tau\right)a^{\frac{a-1}{a}}+a^{-\frac{1}{a}}\left(\tau^2+1\right)\left(\sqrt{\tau^2+1}+\tau\right)\right)\left(\sqrt{\tau^2+1}+\tau\right)^{\frac{-a-1}{a}}-\left(\sqrt{\tau^2+1}+\tau\right)^{-\frac{1}{a}}\left(a^{-\frac{1}{a}}\tau^2+a^{\frac{a-1}{a}}\sqrt{\tau^2+1}\,\tau+a^{-\frac{1}{a}}\right)}{\left(\tau^2+1\right)^2} d\right.}{a}$$

Summary of solutions found

$$\frac{\left(ax\ \mathrm{csgn}\left(a\right)+\sqrt{a^2\left(x^2+1\right)}\right)^{-\frac{\mathrm{csgn}(a)}{a}} y}{\sqrt{x^2+1}} + \frac{y\left(\int_0^x \frac{\left(\left(\tau^3+\tau^2\sqrt{\tau^2+1}+\tau\right)a^{\frac{a-1}{a}}+a^{-\frac{1}{a}}\left(\tau^2+1\right)\left(\sqrt{\tau^2+1}+\tau\right)\right)\left(\sqrt{\tau^2+1}+\tau\right)^{\frac{-a-1}{a}}-\left(\sqrt{\tau^2+1}+\tau\right)^{-\frac{1}{a}}\left(a^{-\frac{1}{a}}\tau^2+a^{\frac{a-1}{a}}\sqrt{\tau^2+1}\,\tau+a^{-\frac{1}{a}}\right)}{\left(\tau^2+1\right)^2} d\tau\right)}{a} = c_1$$

Solved using Lie symmetry for first order ode

Time used: 1.401 (sec)

Writing the ode as

$$\begin{aligned} y' &= \frac{y\left(a^2x+\sqrt{a^2\left(x^2+1\right)}\right)}{a^2\left(x^2+1\right)} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{y\left(a^2x + \sqrt{a^2(x^2+1)}\right)(b_3 - a_2)}{a^2(x^2+1)} - \frac{y^2\left(a^2x + \sqrt{a^2(x^2+1)}\right)^2 a_3}{a^4(x^2+1)^2} \\ & - \left(\frac{y\left(a^2 + \frac{a^2x}{\sqrt{a^2(x^2+1)}}\right)}{a^2(x^2+1)} - \frac{2y\left(a^2x + \sqrt{a^2(x^2+1)}\right)x}{a^2(x^2+1)^2}\right)(xa_2 + ya_3 + a_1) \\ & - \frac{\left(a^2x + \sqrt{a^2(x^2+1)}\right)(xb_2 + yb_3 + b_1)}{a^2(x^2+1)} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{a^4x^5b_2 + a^4x^3y^2a_3 + \sqrt{a^2(x^2+1)}\,a^4x^3b_1 - \sqrt{a^2(x^2+1)}\,a^4x^2ya_1 + a^4x^4b_1 - a^4x^3ya_1 - \sqrt{a^2(x^2+1)}\,a}{}$$

Setting the numerator to zero gives

$$\begin{aligned} & -a^4x^5b_2 - a^4x^3y^2a_3 - \sqrt{a^2(x^2+1)}\,a^4x^3b_1 + \sqrt{a^2(x^2+1)}\,a^4x^2ya_1 \\ & - a^4x^4b_1 + a^4x^3ya_1 + \sqrt{a^2(x^2+1)}\,a^4x^2b_2 - 2\sqrt{a^2(x^2+1)}\,a^4xya_2 \\ & - \sqrt{a^2(x^2+1)}\,a^4y^2a_3 - 2a^4x^3b_2 - a^4x^2ya_2 - a^4x\,y^2a_3 \\ & - \sqrt{a^2(x^2+1)}\,a^4xb_1 - \sqrt{a^2(x^2+1)}\,a^4ya_1 - 2a^4x^2b_1 + a^4xya_1 \\ & - \left(a^2(x^2+1)\right)^{3/2}y^2a_3 + b_2a^4\sqrt{a^2(x^2+1)} - a^4xb_2 - a^4ya_2 - a^4b_1 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} & -a^4x^4ya_2 - a^4x^3y^2a_3 - \sqrt{a^2(x^2+1)}\,a^4x^3b_1 + \sqrt{a^2(x^2+1)}\,a^4x^2ya_1 \\ & - a^4x^3ya_1 - a^4(x^2+1)\,x^3b_2 + a^4(x^2+1)\,x^2ya_2 + \sqrt{a^2(x^2+1)}\,a^4x^2b_2 \\ & - 2\sqrt{a^2(x^2+1)}\,a^4xya_2 - \sqrt{a^2(x^2+1)}\,a^4y^2a_3 - a^4x^2ya_2 - a^4x\,y^2a_3 \\ & - a^4(x^2+1)\,x^2b_1 + 2a^4(x^2+1)\,xya_1 - \sqrt{a^2(x^2+1)}\,a^4xb_1 \\ & - \sqrt{a^2(x^2+1)}\,a^4ya_1 - a^4xya_1 - \left(a^2(x^2+1)\right)^{3/2}y^2a_3 \\ & - a^4(x^2+1)\,xb_2 - a^4(x^2+1)\,ya_2 + b_2a^4\sqrt{a^2(x^2+1)} - a^4(x^2+1)\,b_1 = 0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&-a^2\Big(a^2x^5b_2 + a^2x^3y^2a_3 + \sqrt{a^2\,(x^2+1)}\,a^2x^3b_1 - \sqrt{a^2\,(x^2+1)}\,a^2x^2ya_1 \\
&+ a^2x^4b_1 - a^2x^3ya_1 - \sqrt{a^2\,(x^2+1)}\,a^2x^2b_2 + 2\sqrt{a^2\,(x^2+1)}\,a^2xya_2 \\
&+ a^2\sqrt{a^2\,(x^2+1)}\,y^2a_3 + \sqrt{a^2\,(x^2+1)}\,x^2y^2a_3 + 2a^2x^3b_2 + a^2x^2ya_2 \\
&+ a^2x\,y^2a_3 + \sqrt{a^2\,(x^2+1)}\,a^2xb_1 + \sqrt{a^2\,(x^2+1)}\,a^2ya_1 + 2a^2x^2b_1 - a^2xya_1 \\
&- \sqrt{a^2\,(x^2+1)}\,a^2b_2 + y^2a_3\sqrt{a^2\,(x^2+1)} + a^2xb_2 + a^2ya_2 + a^2b_1\Big) = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{a^2\,(x^2+1)}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x = v_1, y = v_2, \sqrt{a^2\,(x^2+1)} = v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-a^2\big(a^2v_1^3v_2^2a_3 + a^2v_1^5b_2 - a^2v_1^3v_2a_1 - v_3a^2v_1^2v_2a_1 + a^2v_1^4b_1 + v_3a^2v_1^3b_1 + a^2v_1^2v_2a_2 \\
&+ 2v_3a^2v_1v_2a_2 + a^2v_1v_2^2a_3 + a^2v_3v_2^2a_3 + 2a^2v_1^3b_2 - v_3a^2v_1^2b_2 + v_3v_1^2v_2^2a_3 - a^2v_1v_2a_1 \\
&+ v_3a^2v_2a_1 + 2a^2v_1^2b_1 + v_3a^2v_1b_1 + a^2v_2a_2 + a^2v_1b_2 - v_3a^2b_2 + v_2^2a_3v_3 + a^2b_1\big) = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&-a^4b_2v_1^5 - a^4b_1v_1^4 - a^4a_3v_1^3v_2^2 + a^4a_1v_1^3v_2 - a^4b_1v_1^3v_3 - 2a^4b_2v_1^3 \\
&+ a^4a_1v_1^2v_2v_3 - a^4a_2v_1^2v_2 + a^4b_2v_1^2v_3 - a^2a_3v_3v_1^2v_2^2 - 2a^4b_1v_1^2 \\
&- 2a^4a_2v_1v_2v_3 - a^4a_3v_1v_2^2 + a^4a_1v_1v_2 - a^4b_1v_1v_3 - a^4b_2v_1 \\
&- a^2\left(a^2a_3 + a_3\right)v_2^2v_3 - a^4a_1v_2v_3 - a^4a_2v_2 + a^4b_2v_3 - a^4b_1 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a^4 a_1 &= 0 \\
a^4 b_2 &= 0 \\
-a^2 a_3 &= 0 \\
-a^2\left(a^2 a_3 + a_3\right) &= 0 \\
-a^4 a_1 &= 0 \\
-2a^4 a_2 &= 0 \\
-a^4 a_2 &= 0 \\
-a^4 a_3 &= 0 \\
-2a^4 b_1 &= 0 \\
-a^4 b_1 &= 0 \\
-2a^4 b_2 &= 0 \\
-a^4 b_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 0 \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y} dy \end{aligned}$$

Which results in

$$S = \ln(y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y\left(a^2 x + \sqrt{a^2 (x^2+1)}\right)}{a^2 (x^2+1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 0 \\ S_y &= \frac{1}{y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{a^2 x + \sqrt{a^2 (x^2+1)}}{a^2 (x^2+1)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{a^2 R + \sqrt{a^2 (R^2+1)}}{a^2 (R^2+1)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int \frac{a^2 R + \sqrt{a^2 \left(R^2+1\right)}}{a^2 \left(R^2+1\right)}\, dR$$

$$S(R) = \frac{\ln\left(\frac{a^2 R}{\sqrt{a^2}} + \sqrt{R^2 a^2 + a^2}\right)}{\sqrt{a^2}} + \frac{\ln\left(R^2+1\right)}{2} + c_2$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\ln\left(y\right) = \frac{\ln\left(\frac{a^2 x}{\sqrt{a^2}} + \sqrt{a^2 x^2 + a^2}\right)}{\sqrt{a^2}} + \frac{\ln\left(x^2+1\right)}{2} + c_2$$

Which gives

$$y = \mathrm{e}^{\frac{\ln\left(x^2+1\right)\sqrt{a^2}+2c_2\sqrt{a^2}+2\ln\left(a^2 x+\sqrt{a^2x^2+a^2}\,\sqrt{a^2}\right)-\ln\left(a^2\right)}{2\sqrt{a^2}}}$$

Summary of solutions found

$$y = \mathrm{e}^{\frac{\ln\left(x^2+1\right)\sqrt{a^2}+2c_2\sqrt{a^2}+2\ln\left(a^2 x+\sqrt{a^2x^2+a^2}\,\sqrt{a^2}\right)-\ln\left(a^2\right)}{2\sqrt{a^2}}}$$

Maple step by step solution

Let's solve

$$\frac{d}{dx}y(x) = \frac{y(x)\left(1+\frac{a^2 x}{\sqrt{a^2\left(x^2+1\right)}}\right)}{\sqrt{a^2(x^2+1)}}$$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $$\frac{d}{dx}y(x) = \frac{y(x)\left(1+\frac{a^2 x}{\sqrt{a^2\left(x^2+1\right)}}\right)}{\sqrt{a^2(x^2+1)}}$$

- Separate variables

$$\frac{\frac{d}{dx}y(x)}{y(x)} = \frac{1+\frac{a^2x}{\sqrt{a^2(x^2+1)}}}{\sqrt{a^2(x^2+1)}}$$

- Integrate both sides with respect to x

$$\int \frac{\frac{d}{dx}y(x)}{y(x)}dx = \int \frac{1+\frac{a^2x}{\sqrt{a^2(x^2+1)}}}{\sqrt{a^2(x^2+1)}}dx + C1$$

- Evaluate integral

$$\ln(y(x)) = \frac{\ln(x^2+1)}{2} + \frac{\ln\left(\frac{x a^2}{\sqrt{a^2}}+\sqrt{a^2x^2+a^2}\right)}{\sqrt{a^2}} + C1$$

- Solve for $y(x)$

$$y(x) = e^{\frac{\ln(x^2+1)\sqrt{a^2}+2C1\sqrt{a^2}+2\ln\left(x a^2+\sqrt{a^2x^2+a^2}\sqrt{a^2}\right)-\ln(a^2)}{2\sqrt{a^2}}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.018 (sec)
Leaf size : 36

```
dsolve(diff(y(x),x) = y(x)*(1+a^2*x/(a^2*(x^2+1))^(1/2))/(a^2*(x^2+1))^(1/2),
       y(x),singsol=all)
```

$$y = c_1\left(ax \operatorname{csgn}(a) + \sqrt{a^2(x^2+1)}\right)^{\frac{1}{\sqrt{a^2}}}\sqrt{x^2+1}$$

Mathematica DSolve solution

Solving time : 0.301 (sec)
Leaf size : 116

```
DSolve[{D[y[x],x]== y[x]*(1+ a^2*x/Sqrt[a^2*(x^2+1)])/Sqrt[a^2*(x^2+1)],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1\left(a\left(-\sqrt{a^2\left(x^2+1\right)}+\sqrt{a^2}+ax\right)\right)^{-\frac{a+1}{a}}\left(a\left(\sqrt{a^2\left(x^2+1\right)}-\sqrt{a^2}+ax\right)\right)^{\frac{1}{a}-1}\left(\sqrt{a^2}\sqrt{a^2\left(x^2+1\right)}-a^2\left(x^2+1\right)\right)$$

$$y(x) \to 0$$

2.1.81 problem 80

Internal problem ID [8469]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 80
Date solved : `Tuesday, December 17, 2024 at 12:54:59 PM`
CAS classification : `[[_Riccati, _special]]`

Solve

$$y' = x^2 + y^2$$

Solved as first order ode of type reduced Riccati

Time used: 0.091 (sec)

This is reduced Riccati ode of the form

$$y' = a\,x^n + by^2$$

Comparing the given ode to the above shows that

$$\begin{aligned} a &= 1 \\ b &= 1 \\ n &= 2 \end{aligned}$$

Since $n \neq -2$ then the solution of the reduced Riccati ode is given by

$$\begin{aligned} w &= \sqrt{x}\begin{cases} c_1 \operatorname{BesselJ}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{ab}x^k\right) + c_2 \operatorname{BesselY}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{ab}x^k\right) & ab > 0 \\ c_1 \operatorname{BesselI}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{-ab}x^k\right) + c_2 \operatorname{BesselK}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{-ab}x^k\right) & ab < 0 \end{cases} \qquad (1) \\ y &= -\frac{1}{b}\frac{w'}{w} \\ k &= 1 + \frac{n}{2} \end{aligned}$$

Since $ab > 0$ then EQ(1) gives

$$
\begin{aligned}
k &= 2 \\
w &= \sqrt{x}\left(c_1 \operatorname{BesselJ}\left(\frac{1}{4}, \frac{x^2}{2}\right) + c_2 \operatorname{BesselY}\left(\frac{1}{4}, \frac{x^2}{2}\right)\right)
\end{aligned}
$$

Therefore the solution becomes

$$
y = -\frac{1}{b}\frac{w'}{w}
$$

Substituting the value of b, w found above and simplyfing gives

$$
y = \frac{\left(-\operatorname{BesselY}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_2 - \operatorname{BesselJ}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_1\right) x}{c_1 \operatorname{BesselJ}\left(\frac{1}{4}, \frac{x^2}{2}\right) + c_2 \operatorname{BesselY}\left(\frac{1}{4}, \frac{x^2}{2}\right)}
$$

Letting $c_2 = 1$ the above becomes

$$
y = \frac{\left(-\operatorname{BesselY}\left(-\frac{3}{4}, \frac{x^2}{2}\right) - \operatorname{BesselJ}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_1\right) x}{c_1 \operatorname{BesselJ}\left(\frac{1}{4}, \frac{x^2}{2}\right) + \operatorname{BesselY}\left(\frac{1}{4}, \frac{x^2}{2}\right)}
$$

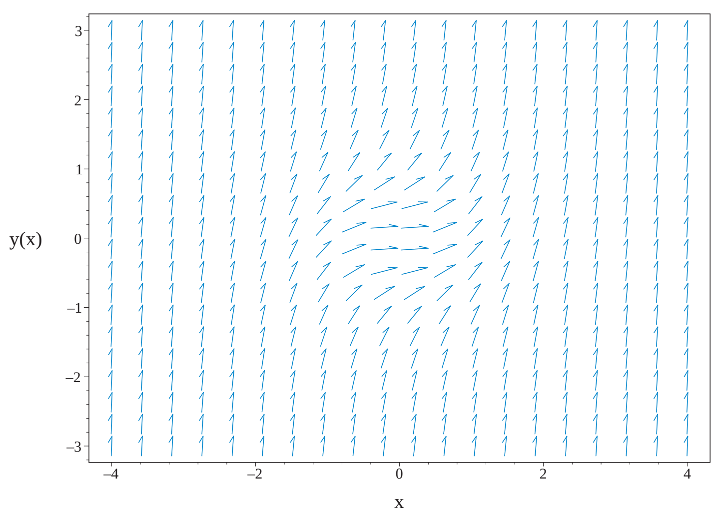


Figure 2.176: Slope field plot
$y' = x^2 + y^2$

Summary of solutions found

$$
y = \frac{\left(-\operatorname{BesselY}\left(-\frac{3}{4}, \frac{x^2}{2}\right) - \operatorname{BesselJ}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_1\right) x}{c_1 \operatorname{BesselJ}\left(\frac{1}{4}, \frac{x^2}{2}\right) + \operatorname{BesselY}\left(\frac{1}{4}, \frac{x^2}{2}\right)}
$$

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) = x^2 + y(x)^2$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = x^2 + y(x)^2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
<- Riccati Special successful`
```

Maple dsolve solution

Solving time : `0.013` (sec)
Leaf size : 43

```
dsolve(diff(y(x),x) = x^2+y(x)^2,
       y(x),singsol=all)
```

$$y = -\frac{x\left(\text{BesselJ}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_1 + \text{BesselY}\left(-\frac{3}{4}, \frac{x^2}{2}\right)\right)}{c_1 \,\text{BesselJ}\left(\frac{1}{4}, \frac{x^2}{2}\right) + \text{BesselY}\left(\frac{1}{4}, \frac{x^2}{2}\right)}$$

Mathematica DSolve solution

Solving time : `0.134` (sec)
Leaf size : 169

```
DSolve[{D[y[x],x]==x^2+y[x]^2,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x^2\left(-2\,\mathrm{BesselJ}\left(-\frac{3}{4},\frac{x^2}{2}\right)+c_1\left(\mathrm{BesselJ}\left(\frac{3}{4},\frac{x^2}{2}\right)-\mathrm{BesselJ}\left(-\frac{5}{4},\frac{x^2}{2}\right)\right)\right)-c_1\,\mathrm{BesselJ}\left(-\frac{1}{4},\frac{x^2}{2}\right)}{2x\left(\mathrm{BesselJ}\left(\frac{1}{4},\frac{x^2}{2}\right)+c_1\,\mathrm{BesselJ}\left(-\frac{1}{4},\frac{x^2}{2}\right)\right)}$$

$$y(x) \to -\frac{x^2\,\mathrm{BesselJ}\left(-\frac{5}{4},\frac{x^2}{2}\right)-x^2\,\mathrm{BesselJ}\left(\frac{3}{4},\frac{x^2}{2}\right)+\mathrm{BesselJ}\left(-\frac{1}{4},\frac{x^2}{2}\right)}{2x\,\mathrm{BesselJ}\left(-\frac{1}{4},\frac{x^2}{2}\right)}$$

2.1.82 problem 81

Internal problem ID [8470]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 81
Date solved : `Tuesday, December 17, 2024 at 12:55:01 PM`
CAS classification : `[_quadrature]`

Solve

$$y' = 2\sqrt{y}$$

With initial conditions

$$y(0) = 0$$

Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= 2\sqrt{y} \end{aligned}$$

The y domain of $f(x, y)$ when $x = 0$ is

$$\{0 \leq y\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(2\sqrt{y}) \\ &= \frac{1}{\sqrt{y}} \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{0 < y\}$$

But the point $y_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. Solution exists but no guarantee that unique solution exists.

Solved as first order autonomous ode

Time used: 0.078 (sec)

Since the ode has the form $y' = f(y)$ and initial conditions (x_0, y_0) are given such that they satisfy the ode itself, then we can write

$$\begin{aligned} 0 &= f(y)|_{y=y_0} \\ 0 &= 0 \end{aligned}$$

And the solution is immediately written as

$$\begin{aligned} y &= y_0 \\ y &= 0 \end{aligned}$$

Singular solutions are found by solving

$$2\sqrt{y} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = 0$$

The following diagram is the phase line diagram. It classifies each of the above equilibrium points as stable or not stable or semi-stable.

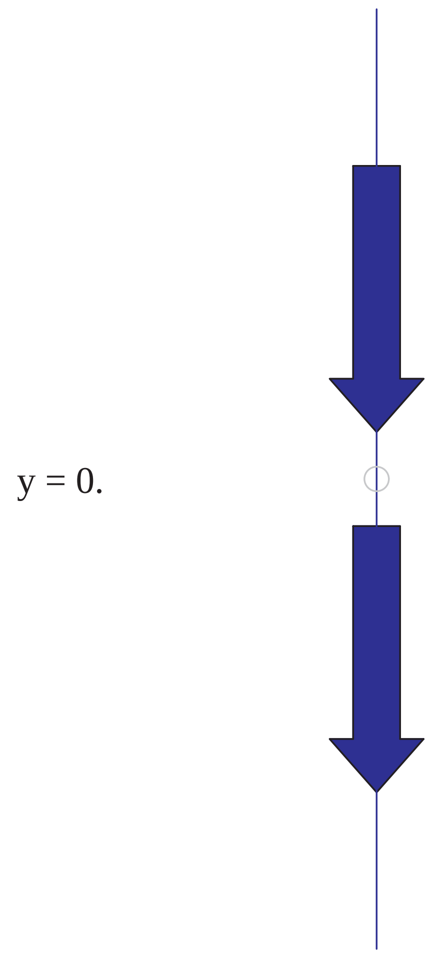
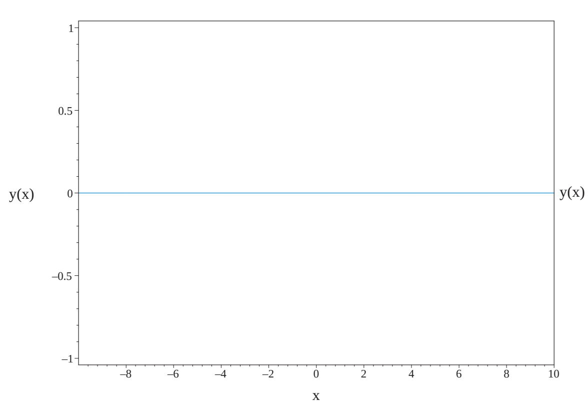
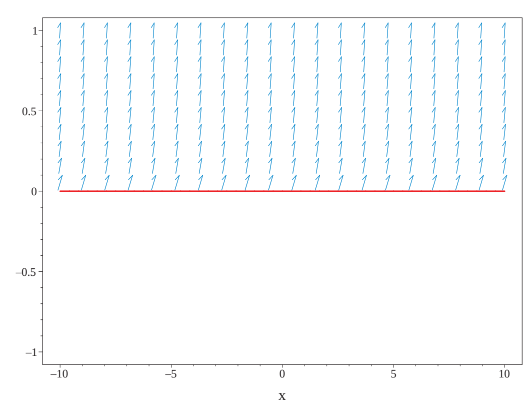


Figure 2.177: Phase line diagram



(a) Solution plot
$y = 0$



(b) Slope field plot
$y' = 2\sqrt{y}$

Summary of solutions found

$$y = 0$$

Solved as first order Bernoulli ode

Time used: 0.115 (sec)

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= 2\sqrt{y} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = (2)\sqrt{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

Comparing this to (1) shows that

$$\begin{aligned} f_0 &= 0 \\ f_1 &= 2 \end{aligned}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_1(x) \tag{3}$$

The next step is use the substitution $v = y^{1-n}$ in equation (3) which generates a new ODE in $v(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= 0 \\ f_1(x) &= 2 \\ n &= \frac{1}{2} \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \sqrt{y}$ gives

$$y' \frac{1}{\sqrt{y}} = 0 + 2 \tag{4}$$

Let

$$\begin{aligned} v &= y^{1-n} \\ &= \sqrt{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$v' = \frac{1}{2\sqrt{y}} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} 2v'(x) &= 2 \\ v' &= 1 \end{aligned} \tag{7}$$

The above now is a linear ODE in $v(x)$ which is now solved.

Since the ode has the form $v'(x) = f(x)$, then we only need to integrate $f(x)$.

$$\begin{aligned} \int dv &= \int 1\, dx \\ v(x) &= x + c_1 \end{aligned}$$

The substitution $v = y^{1-n}$ is now used to convert the above solution back to y which results in

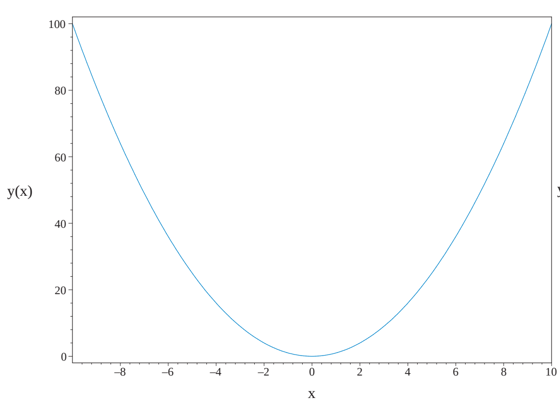
$$\sqrt{y} = x + c_1$$

Solving for the constant of integration from initial conditions, the solution becomes

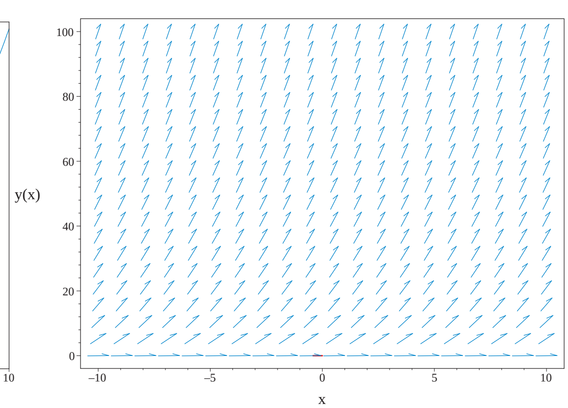
$$\sqrt{y} = x$$

Solving for y gives

$$y = x^2$$



(a) Solution plot
$y = x^2$



(b) Slope field plot
$y' = 2\sqrt{y}$

Summary of solutions found

$$y = x^2$$

Solved as first order Exact ode

Time used: 0.178 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= (2\sqrt{y})\,\mathrm{d}x\\ (-2\sqrt{y})\,\mathrm{d}x + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= -2\sqrt{y} \\
N(x,y) &= 1
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-2\sqrt{y}) \\
&= -\frac{1}{\sqrt{y}}
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(-\frac{1}{\sqrt{y}}\right) - (0)\right) \\
&= -\frac{1}{\sqrt{y}}
\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= -\frac{1}{2\sqrt{y}}\left((0) - \left(-\frac{1}{\sqrt{y}}\right)\right) \\
&= -\frac{1}{2y}
\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int -\frac{1}{2y} \, \mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{\ln(y)}{2}} \\ &= \frac{1}{\sqrt{y}}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{\sqrt{y}}(-2\sqrt{y}) \\ &= -2\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{\sqrt{y}}(1) \\ &= \frac{1}{\sqrt{y}}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (-2) + \left(\frac{1}{\sqrt{y}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -2 \, \mathrm{d}x \\
\phi &= -2x + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\sqrt{y}}$. Therefore equation (4) becomes

$$
\frac{1}{\sqrt{y}} = 0 + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{\sqrt{y}}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(\frac{1}{\sqrt{y}}\right) \mathrm{d}y \\
f(y) &= 2\sqrt{y} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = -2x + 2\sqrt{y} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

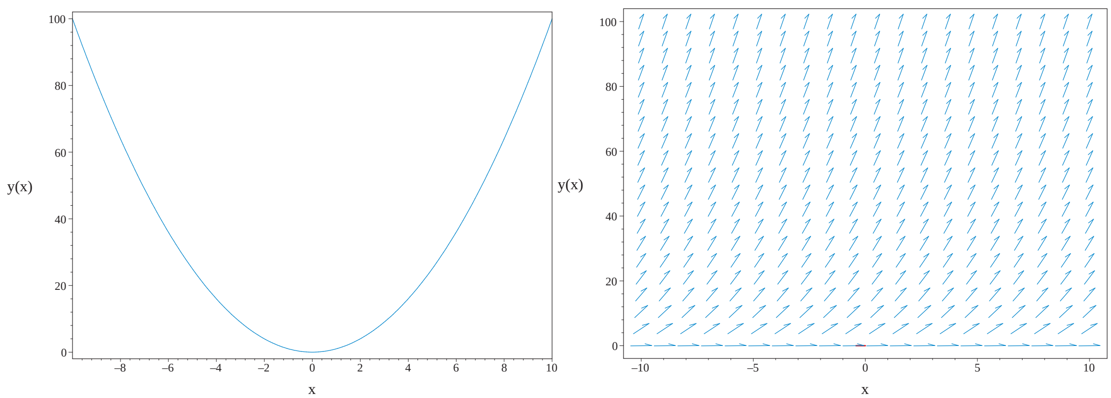
$$
c_1 = -2x + 2\sqrt{y}
$$

Solving for the constant of integration from initial conditions, the solution becomes

$$-2x + 2\sqrt{y} = 0$$

Solving for y gives

$$y = x^2$$



(a) Solution plot
$y = x^2$

(b) Slope field plot
$y' = 2\sqrt{y}$

Summary of solutions found

$$y = x^2$$

Solved using Lie symmetry for first order ode

Time used: 0.346 (sec)

Writing the ode as

$$y' = 2\sqrt{y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + 2\sqrt{y}\,(b_3 - a_2) - 4ya_3 - \frac{xb_2 + yb_3 + b_1}{\sqrt{y}} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{2ya_2 - yb_3 + 4y^{3/2}a_3 - b_2\sqrt{y} + xb_2 + b_1}{\sqrt{y}} = 0$$

Setting the numerator to zero gives

$$-4y^{3/2}a_3 + b_2\sqrt{y} - xb_2 - 2ya_2 + yb_3 - b_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{y}, y^{3/2}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{y} = v_3, y^{3/2} = v_4\right\}$$

The above PDE (6E) now becomes

$$-2v_2a_2 - 4v_4a_3 - v_1b_2 + b_2v_3 + v_2b_3 - b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$-v_1b_2 + (-2a_2 + b_3)\,v_2 + b_2v_3 - 4v_4a_3 - b_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_2 &= 0 \\
-4a_3 &= 0 \\
-b_1 &= 0 \\
-b_2 &= 0 \\
-2a_2 + b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= a_1 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 2a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 1 \\
\eta &= 0
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= 0 - (2\sqrt{y})\,(1) \\
&= -2\sqrt{y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-2\sqrt{y}} dy \end{aligned}$$

Which results in

$$S = -\sqrt{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = 2\sqrt{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 0 \\ S_y &= -\frac{1}{2\sqrt{y}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int -1\, dR$$
$$S(R) = -R + c_2$$

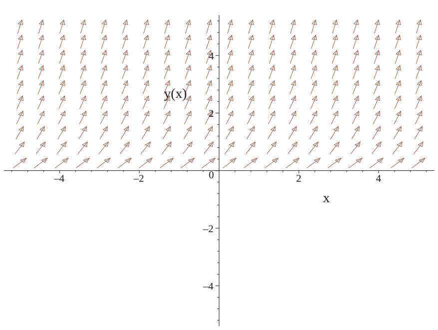
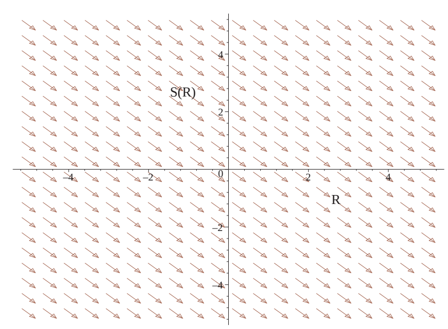
To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$-\sqrt{y} = -x + c_2$$

Which gives

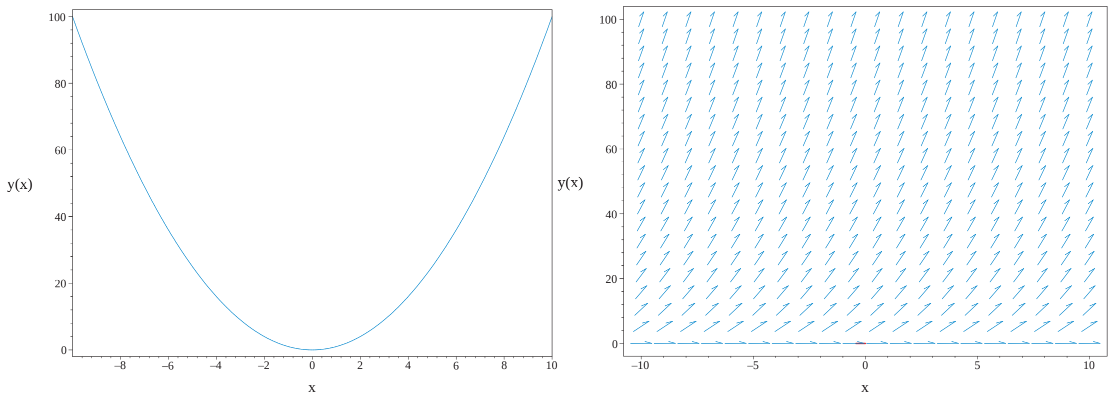
$$y = c_2^2 - 2c_2x + x^2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 2\sqrt{y}$ 	$R = x$ $S = -\sqrt{y}$	$\frac{dS}{dR} = -1$ 

Solving for the constant of integration from initial conditions, the solution becomes

$$y = x^2$$



(a) Solution plot
$y = x^2$

(b) Slope field plot
$y' = 2\sqrt{y}$

Summary of solutions found

$$y = x^2$$

Maple step by step solution

Let's solve

$\left[\frac{d}{dx}y(x) = 2\sqrt{y(x)}, y(0) = 0\right]$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = 2\sqrt{y(x)}$

- Separate variables

  $\frac{\frac{d}{dx}y(x)}{\sqrt{y(x)}} = 2$

- Integrate both sides with respect to x

  $\int \frac{\frac{d}{dx}y(x)}{\sqrt{y(x)}}dx = \int 2dx + C1$

- Evaluate integral

  $2\sqrt{y(x)} = 2x + C1$

- Solve for $y(x)$

  $y(x) = \frac{1}{4}C1^2 + C1x + x^2$

- Use initial condition $y(0) = 0$

  $0 = \frac{C1^2}{4}$

- Solve for $\_C1$
  $C1 = (0, 0)$
- Substitute $\_C1 = (0, 0)$ into general solution and simplify
  $y(x) = x^2$
- Solution to the IVP
  $y(x) = x^2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 5

```
dsolve([diff(y(x),x) = 2*y(x)^(1/2),
       op([y(0) = 0])],y(x),singsol=all)
```

$$y = 0$$

Mathematica DSolve solution

Solving time : 0.003 (sec)
Leaf size : 8

```
DSolve[{D[y[x],x]==2*Sqrt[y[x]],{y[0]==0}},
      y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x^2$$

2.1.83 problem 82

Internal problem ID [8471]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 82
Date solved : `Thursday, December 12, 2024 at 09:20:30 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$z'' + 3z' + 2z = 24\,\mathrm{e}^{-3t} - 24\,\mathrm{e}^{-4t}$$

Solved as second order linear constant coeff ode

Time used: 0.157 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Az''(t) + Bz'(t) + Cz(t) = f(t)$$

Where $A = 1, B = 3, C = 2, f(t) = 24\,\mathrm{e}^{-3t} - 24\,\mathrm{e}^{-4t}$. Let the solution be

$$z = z_h + z_p$$

Where z_h is the solution to the homogeneous ODE $Az''(t) + Bz'(t) + Cz(t) = 0$, and z_p is a particular solution to the non-homogeneous ODE $Az''(t) + Bz'(t) + Cz(t) = f(t)$. z_h is the solution to

$$z'' + 3z' + 2z = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Az''(t) + Bz'(t) + Cz(t) = 0$$

Where in the above $A = 1, B = 3, C = 2$. Let the solution be $z = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{t\lambda} + 3\lambda\,\mathrm{e}^{t\lambda} + 2\,\mathrm{e}^{t\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-3}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{3^2 - (4)(1)(2)}\\ &= -\frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{3}{2} + \frac{1}{2}\\ \lambda_2 &= -\frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}z &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\\ z &= c_1 e^{(-1)t} + c_2 e^{(-2)t}\end{aligned}$$

Or

$$z = c_1 \,\mathrm{e}^{-t} + c_2 \,\mathrm{e}^{-2t}$$

Therefore the homogeneous solution z_h is

$$z_h = c_1 \,\mathrm{e}^{-t} + c_2 \,\mathrm{e}^{-2t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$24 \,\mathrm{e}^{-3t} - 24 \,\mathrm{e}^{-4t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-4t}\}, \{e^{-3t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2t}, e^{-t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$z_p = A_1 e^{-4t} + A_2 e^{-3t}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution z_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1 e^{-4t} + 2A_2 e^{-3t} = 24\, e^{-3t} - 24\, e^{-4t}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -4, A_2 = 12]$$

Substituting the above back in the above trial solution z_p, gives the particular solution

$$z_p = -4\, e^{-4t} + 12\, e^{-3t}$$

Therefore the general solution is

$$\begin{aligned} z &= z_h + z_p \\ &= \left(c_1\, e^{-t} + c_2\, e^{-2t}\right) + \left(-4\, e^{-4t} + 12\, e^{-3t}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$z = -4\, e^{-4t} + 12\, e^{-3t} + c_1\, e^{-t} + c_2\, e^{-2t}$$

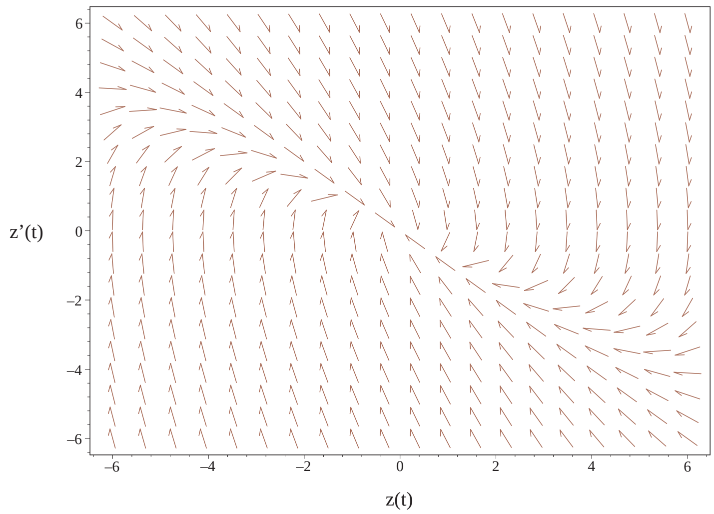


Figure 2.182: Slope field plot
$z'' + 3z' + 2z = 24\,\mathrm{e}^{-3t} - 24\,\mathrm{e}^{-4t}$

Solved as second order ode using Kovacic algorithm

Time used: 0.106 (sec)

Writing the ode as

$$z'' + 3z' + 2z = 0 \tag{1}$$

$$Az'' + Bz' + Cz = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = ze^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = \frac{z}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then z is found using the inverse transformation

$$z = ze^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.87: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-\frac{t}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in z is found from

$$\begin{aligned}
z_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dt} \\
&= z_1 e^{-\frac{3t}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{3t}{2}}\right)
\end{aligned}$$

Which simplifies to

$$z_1 = \mathrm{e}^{-2t}$$

The second solution z_2 to the original ode is found using reduction of order

$$z_2 = z_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{z_1^2}\,dt$$

Substituting gives

$$\begin{aligned}
z_2 &= z_1 \int \frac{e^{\int -\frac{3}{1}\,dt}}{(z_1)^2}\,dt \\
&= z_1 \int \frac{e^{-3t}}{(z_1)^2}\,dt \\
&= z_1 \left(\mathrm{e}^{-3t}\mathrm{e}^{4t}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
z &= c_1 z_1 + c_2 z_2 \\
&= c_1\left(\mathrm{e}^{-2t}\right) + c_2\left(\mathrm{e}^{-2t}\left(\mathrm{e}^{-3t}\mathrm{e}^{4t}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$z = z_h + z_p$$

Where z_h is the solution to the homogeneous ODE $Az''(t) + Bz'(t) + Cz(t) = 0$, and z_p is a particular solution to the nonhomogeneous ODE $Az''(t) + Bz'(t) + Cz(t) = f(t)$. z_h is the solution to

$$z'' + 3z' + 2z = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$z_h = c_1 \,\mathrm{e}^{-2t} + c_2 \,\mathrm{e}^{-t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$24 \,\mathrm{e}^{-3t} - 24 \,\mathrm{e}^{-4t}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-4t}\}, \{\mathrm{e}^{-3t}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-2t}, \mathrm{e}^{-t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$z_p = A_1 \mathrm{e}^{-4t} + A_2 \mathrm{e}^{-3t}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution z_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1 \mathrm{e}^{-4t} + 2A_2 \mathrm{e}^{-3t} = 24 \,\mathrm{e}^{-3t} - 24 \,\mathrm{e}^{-4t}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -4, A_2 = 12]$$

Substituting the above back in the above trial solution z_p, gives the particular solution

$$z_p = -4\,\mathrm{e}^{-4t} + 12\,\mathrm{e}^{-3t}$$

Therefore the general solution is

$$\begin{aligned} z &= z_h + z_p \\ &= \left(c_1\,\mathrm{e}^{-2t} + c_2\,\mathrm{e}^{-t}\right) + \left(-4\,\mathrm{e}^{-4t} + 12\,\mathrm{e}^{-3t}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$z = c_1\,\mathrm{e}^{-2t} + c_2\,\mathrm{e}^{-t} - 4\,\mathrm{e}^{-4t} + 12\,\mathrm{e}^{-3t}$$

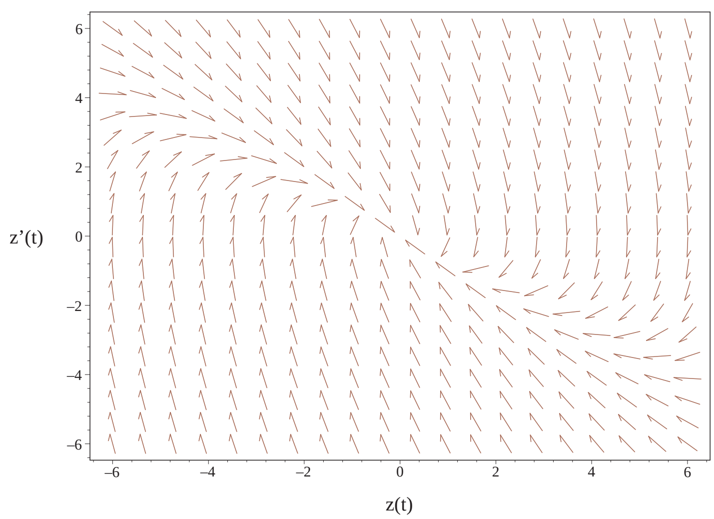


Figure 2.183: Slope field plot
$z'' + 3z' + 2z = 24\,\mathrm{e}^{-3t} - 24\,\mathrm{e}^{-4t}$

Solved as second order ode adjoint method

Time used: 0.540 (sec)

In normal form the ode

$$z'' + 3z' + 2z = 24\,\mathrm{e}^{-3t} - 24\,\mathrm{e}^{-4t} \tag{1}$$

Becomes

$$z'' + p(t)\,z' + q(t)\,z = r(t) \tag{2}$$

Where

$$
\begin{aligned}
p(t) &= 3 \\
q(t) &= 2 \\
r(t) &= 24\,\mathrm{e}^{-3t} - 24\,\mathrm{e}^{-4t}
\end{aligned}
$$

The Lagrange adjoint ode is given by

$$
\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - (3\xi(t))' + (2\xi(t)) &= 0 \\
\xi''(t) - 3\xi'(t) + 2\xi(t) &= 0
\end{aligned}
$$

Which is solved for $\xi(t)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
A\xi''(t) + B\xi'(t) + C\xi(t) = 0
$$

Where in the above $A = 1, B = -3, C = 2$. Let the solution be $\xi = e^{\lambda t}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{t\lambda} - 3\lambda\,\mathrm{e}^{t\lambda} + 2\,\mathrm{e}^{t\lambda} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$
\lambda^2 - 3\lambda + 2 = 0 \tag{2}
$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$
\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}
$$

Substituting $A = 1, B = -3, C = 2$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-3^2 - (4)\,(1)\,(2)} \\
&= \frac{3}{2} \pm \frac{1}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= \frac{3}{2} + \frac{1}{2} \\
\lambda_2 &= \frac{3}{2} - \frac{1}{2}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= 2 \\
\lambda_2 &= 1
\end{aligned}
$$

Since roots are real and distinct, then the solution is

$$
\begin{aligned}
\xi &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\
\xi &= c_1 e^{(2)t} + c_2 e^{(1)t}
\end{aligned}
$$

Or

$$
\xi = c_1\,\mathrm{e}^{2t} + c_2\,\mathrm{e}^{t}
$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$
\begin{aligned}
\xi(t)\,z' - z\xi'(t) + \xi(t)\,p(t)\,z &= \int \xi(t)\,r(t)\,dt \\
z' + z\left(p(t) - \frac{\xi'(t)}{\xi(t)}\right) &= \frac{\int \xi(t)\,r(t)\,dt}{\xi(t)}
\end{aligned}
$$

Or

$$
z' + z\left(3 - \frac{2c_1\,\mathrm{e}^{2t} + c_2\,\mathrm{e}^{t}}{c_1\,\mathrm{e}^{2t} + c_2\,\mathrm{e}^{t}}\right) = \frac{-24c_1\,\mathrm{e}^{-t} - 12c_2\,\mathrm{e}^{-2t} + 12c_1\,\mathrm{e}^{-2t} + 8c_2\,\mathrm{e}^{-3t}}{c_1\,\mathrm{e}^{2t} + c_2\,\mathrm{e}^{t}}
$$

Which is now a first order ode. This is now solved for z. In canonical form a linear first order is

$$
z' + q(t)z = p(t)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(t) &= -\frac{-c_1\,\mathrm{e}^{t} - 2c_2}{c_1\,\mathrm{e}^{t} + c_2} \\
p(t) &= \frac{12\,\mathrm{e}^{-4t}\left(-2c_1\,\mathrm{e}^{2t} + (c_1 - c_2)\,\mathrm{e}^{t} + \frac{2c_2}{3}\right)}{c_1\,\mathrm{e}^{t} + c_2}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\,dt} \\
&= \mathrm{e}^{\int -\frac{-c_1\,\mathrm{e}^{t} - 2c_2}{c_1\,\mathrm{e}^{t} + c_2}dt} \\
&= \frac{\mathrm{e}^{2t}}{c_1\,\mathrm{e}^{t} + c_2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu z) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}t}(\mu z) &= (\mu)\left(\frac{12\,\mathrm{e}^{-4t}\left(-2c_1\,\mathrm{e}^{2t}+(c_1-c_2)\,\mathrm{e}^{t}+\frac{2c_2}{3}\right)}{c_1\,\mathrm{e}^{t}+c_2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{z\,\mathrm{e}^{2t}}{c_1\,\mathrm{e}^{t}+c_2}\right) &= \left(\frac{\mathrm{e}^{2t}}{c_1\,\mathrm{e}^{t}+c_2}\right)\left(\frac{12\,\mathrm{e}^{-4t}\left(-2c_1\,\mathrm{e}^{2t}+(c_1-c_2)\,\mathrm{e}^{t}+\frac{2c_2}{3}\right)}{c_1\,\mathrm{e}^{t}+c_2}\right) \\
\mathrm{d}\left(\frac{z\,\mathrm{e}^{2t}}{c_1\,\mathrm{e}^{t}+c_2}\right) &= \left(\frac{12\,\mathrm{e}^{-4t}\left(-2c_1\,\mathrm{e}^{2t}+(c_1-c_2)\,\mathrm{e}^{t}+\frac{2c_2}{3}\right)\mathrm{e}^{2t}}{\left(c_1\,\mathrm{e}^{t}+c_2\right)^2}\right)\mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{z\,\mathrm{e}^{2t}}{c_1\,\mathrm{e}^{t}+c_2} &= \int \frac{12\,\mathrm{e}^{-4t}\left(-2c_1\,\mathrm{e}^{2t}+(c_1-c_2)\,\mathrm{e}^{t}+\frac{2c_2}{3}\right)\mathrm{e}^{2t}}{\left(c_1\,\mathrm{e}^{t}+c_2\right)^2}\,dt \\
&= -\frac{12c_1}{c_2\left(c_1\,\mathrm{e}^{t}+c_2\right)}-\frac{4c_1^2}{c_2^2\left(c_1\,\mathrm{e}^{t}+c_2\right)}-\frac{4\,\mathrm{e}^{-2t}}{c_2}+\frac{4c_1\,\mathrm{e}^{-t}}{c_2^2}+\frac{12\,\mathrm{e}^{-t}}{c_2}+c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{\mathrm{e}^{2t}}{c_1\,\mathrm{e}^{t}+c_2}$ gives the final solution

$$
z = \left(\mathrm{e}^{3t}c_1c_3+\mathrm{e}^{2t}c_2c_3+12\,\mathrm{e}^{t}-4\right)\mathrm{e}^{-4t}
$$

Hence, the solution found using Lagrange adjoint equation method is

$$
z = \left(\mathrm{e}^{3t}c_1c_3+\mathrm{e}^{2t}c_2c_3+12\,\mathrm{e}^{t}-4\right)\mathrm{e}^{-4t}
$$

The constants can be merged to give

$$
z = \left(\mathrm{e}^{3t}c_1+\mathrm{e}^{2t}c_2+12\,\mathrm{e}^{t}-4\right)\mathrm{e}^{-4t}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
z = \left(\mathrm{e}^{3t}c_1+\mathrm{e}^{2t}c_2+12\,\mathrm{e}^{t}-4\right)\mathrm{e}^{-4t}
$$

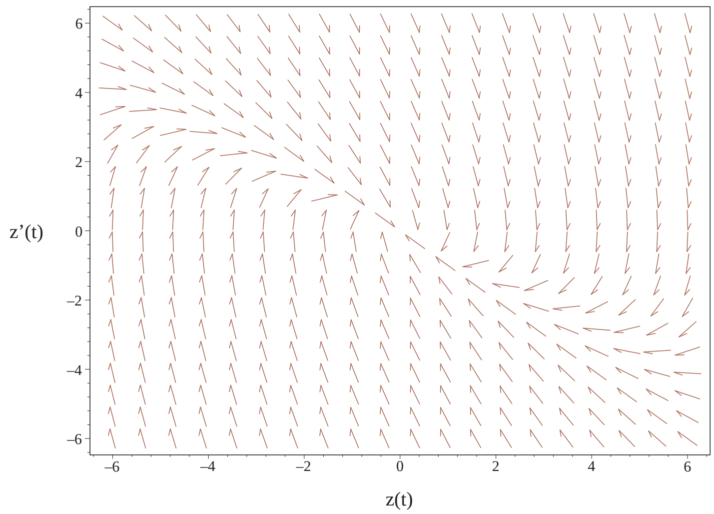


Figure 2.184: Slope field plot
$z'' + 3z' + 2z = 24\,\mathrm{e}^{-3t} - 24\,\mathrm{e}^{-4t}$

Maple step by step solution

Let's solve
$\frac{d^2}{dt^2}z(t) + 3\frac{d}{dt}z(t) + 2z(t) = 24\,\mathrm{e}^{-3t} - 24\,\mathrm{e}^{-4t}$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dt^2}z(t)$
- Characteristic polynomial of homogeneous ODE
  $r^2 + 3r + 2 = 0$
- Factor the characteristic polynomial
  $(r+2)(r+1) = 0$
- Roots of the characteristic polynomial
  $r = (-2, -1)$
- 1st solution of the homogeneous ODE
  $z_1(t) = \mathrm{e}^{-2t}$
- 2nd solution of the homogeneous ODE
  $z_2(t) = \mathrm{e}^{-t}$
- General solution of the ODE
  $z(t) = C1\, z_1(t) + C2\, z_2(t) + z_p(t)$
- Substitute in solutions of the homogeneous ODE
  $z(t) = C1\,\mathrm{e}^{-2t} + C2\,\mathrm{e}^{-t} + z_p(t)$

☐ Find a particular solution $z_p(t)$ of the ODE

- ◦ Use variation of parameters to find z_p here $f(t)$ is the forcing function
  $\left[z_p(t) = -z_1(t)\left(\int \frac{z_2(t)f(t)}{W(z_1(t),z_2(t))}dt\right) + z_2(t)\left(\int \frac{z_1(t)f(t)}{W(z_1(t),z_2(t))}dt\right), f(t) = 24\,\mathrm{e}^{-3t} - 24\,\mathrm{e}^{-4t}\right]$
- ◦ Wronskian of solutions of the homogeneous equation
  $W(z_1(t), z_2(t)) = \begin{bmatrix} \mathrm{e}^{-2t} & \mathrm{e}^{-t} \\ -2\,\mathrm{e}^{-2t} & -\mathrm{e}^{-t} \end{bmatrix}$
- ◦ Compute Wronskian
  $W(z_1(t), z_2(t)) = \mathrm{e}^{-3t}$
- ◦ Substitute functions into equation for $z_p(t)$
  $z_p(t) = -24\,\mathrm{e}^{-2t}\left(\int (\mathrm{e}^{t} - 1)\,\mathrm{e}^{-2t}dt\right) + 24\,\mathrm{e}^{-t}\left(\int (\mathrm{e}^{t} - 1)\,\mathrm{e}^{-3t}dt\right)$
- ◦ Compute integrals
  $z_p(t) = 12\,\mathrm{e}^{-3t} - 4\,\mathrm{e}^{-4t}$
- Substitute particular solution into general solution to ODE
  $z(t) = C1\,\mathrm{e}^{-2t} + C2\,\mathrm{e}^{-t} + 12\,\mathrm{e}^{-3t} - 4\,\mathrm{e}^{-4t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 32

```
dsolve(diff(diff(z(t),t),t)+3*diff(z(t),t)+2*z(t) = 24*exp(-3*t)-24*exp(-4*t),
       z(t),singsol=all)
```

$$z = -\left(4\,\mathrm{e}^{-3t} - 12\,\mathrm{e}^{-2t} + \mathrm{e}^{-t}c_1 - c_2\right)\mathrm{e}^{-t}$$

Mathematica DSolve solution

Solving time : 0.247 (sec)
Leaf size : 34

```
DSolve[{D[z[t],{t,2}]+3*D[z[t],t]+2*z[t]==24*(Exp[-3*t]-Exp[-4*t]),{}},
       z[t],t,IncludeSingularSolutions->True]
```

$$z(t) \to e^{-4t}\left(12e^{t} + c_1 e^{2t} + c_2 e^{3t} - 4\right)$$

2.1.84 problem 83

Internal problem ID [8472]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 83
Date solved : `Tuesday, December 17, 2024 at 12:55:02 PM`
CAS classification : `[_quadrature]`

Solve

$$y' = \sqrt{1 - y^2}$$

Solved as first order autonomous ode

Time used: 0.243 (sec)

Integrating gives

$$\int \frac{1}{\sqrt{-y^2 + 1}} dy = dx$$
$$\arcsin(y) = x + c_1$$

Singular solutions are found by solving

$$\sqrt{-y^2 + 1} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = -1$$
$$y = 1$$

The following diagram is the phase line diagram. It classifies each of the above equilibrium points as stable or not stable or semi-stable.

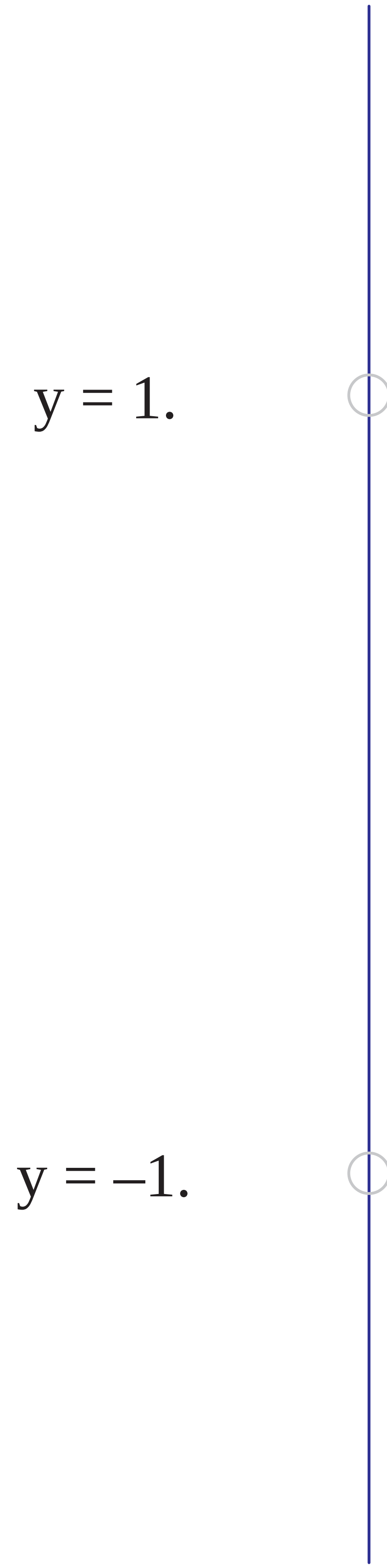


Figure 2.185: Phase line diagram

Solving for y gives

$$y = -1$$
$$y = 1$$
$$y = \sin(x + c_1)$$

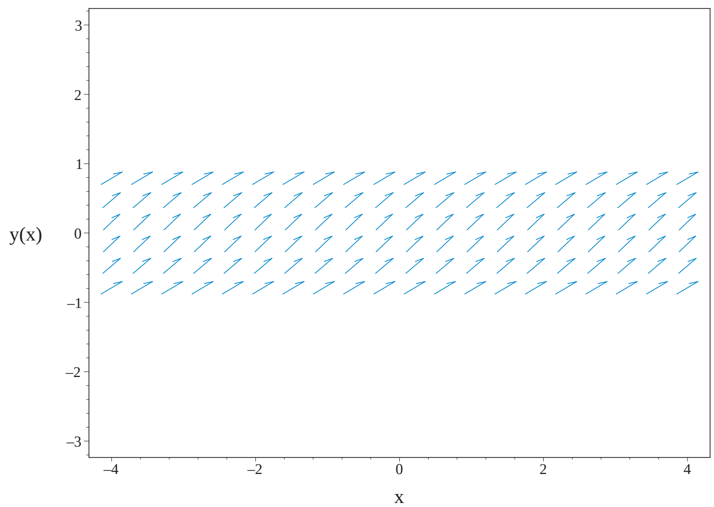


Figure 2.186: Slope field plot
$y' = \sqrt{1 - y^2}$

Summary of solutions found

$$y = -1$$
$$y = 1$$
$$y = \sin(x + c_1)$$

Solved as first order Exact ode

Time used: 38.216 (sec)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y&=\left(\sqrt{-y^2+1}\right)\mathrm{d}x\\ \left(-\sqrt{-y^2+1}\right)\mathrm{d}x+\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\sqrt{-y^2+1}\\ N(x,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\sqrt{-y^2+1}\right)\\ &=\frac{y}{\sqrt{-y^2+1}}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1\left(\left(\frac{y}{\sqrt{-y^2+1}}\right) - (0)\right)\\ &= \frac{y}{\sqrt{-y^2+1}}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= -\frac{1}{\sqrt{-y^2+1}}\left((0) - \left(\frac{y}{\sqrt{-y^2+1}}\right)\right)\\ &= -\frac{y}{y^2-1}\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B \, \mathrm{d}y}\\ &= e^{\int -\frac{y}{y^2-1} \, \mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{\ln(y-1)}{2} - \frac{\ln(y+1)}{2}}\\ &= \frac{1}{\sqrt{y-1}\,\sqrt{y+1}}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{\sqrt{y-1}\,\sqrt{y+1}}\left(-\sqrt{-y^2+1}\right)\\ &= -\frac{\sqrt{-y^2+1}}{\sqrt{y-1}\,\sqrt{y+1}}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{\sqrt{y-1}\,\sqrt{y+1}}(1) \\
&= \frac{1}{\sqrt{y-1}\,\sqrt{y+1}}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\frac{\sqrt{-y^2+1}}{\sqrt{y-1}\,\sqrt{y+1}}\right) + \left(\frac{1}{\sqrt{y-1}\,\sqrt{y+1}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{\sqrt{-y^2+1}}{\sqrt{y-1}\,\sqrt{y+1}}\,\mathrm{d}x \\
\phi &= -\frac{\sqrt{-y^2+1}\,x}{\sqrt{y-1}\,\sqrt{y+1}} + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= \frac{xy}{\sqrt{-y^2+1}\,\sqrt{y-1}\,\sqrt{y+1}} + \frac{\sqrt{-y^2+1}\,x}{2\,(y-1)^{3/2}\,\sqrt{y+1}} + \frac{\sqrt{-y^2+1}\,x}{2\sqrt{y-1}\,(y+1)^{3/2}} + f'(y) \\
&= 0 + f'(y)
\end{aligned} \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\sqrt{y-1}\,\sqrt{y+1}}$. Therefore equation (4) becomes

$$
\frac{1}{\sqrt{y-1}\,\sqrt{y+1}} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{\sqrt{y-1}\,\sqrt{y+1}}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{\sqrt{y-1}\,\sqrt{y+1}}\right) \mathrm{d}y$$

$$f(y) = \frac{\sqrt{(y-1)\,(y+1)}\,\ln\left(y+\sqrt{y^2-1}\right)}{\sqrt{y-1}\,\sqrt{y+1}} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\sqrt{-y^2+1}\,x}{\sqrt{y-1}\,\sqrt{y+1}} + \frac{\sqrt{(y-1)\,(y+1)}\,\ln\left(y+\sqrt{y^2-1}\right)}{\sqrt{y-1}\,\sqrt{y+1}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -\frac{\sqrt{-y^2+1}\,x}{\sqrt{y-1}\,\sqrt{y+1}} + \frac{\sqrt{(y-1)\,(y+1)}\,\ln\left(y+\sqrt{y^2-1}\right)}{\sqrt{y-1}\,\sqrt{y+1}}$$

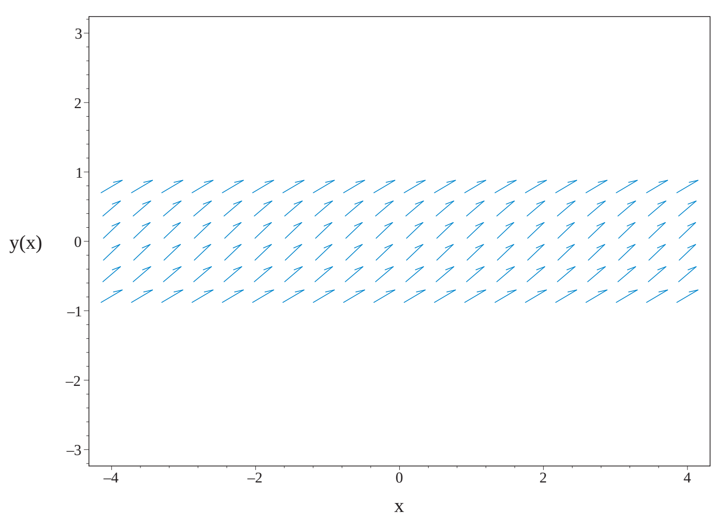


Figure 2.187: Slope field plot
$y' = \sqrt{1-y^2}$

Summary of solutions found

$$-\frac{\sqrt{1-y^2}\,x}{\sqrt{y-1}\,\sqrt{y+1}}+\frac{\sqrt{(y-1)\,(y+1)}\,\ln\left(y+\sqrt{y^2-1}\right)}{\sqrt{y-1}\,\sqrt{y+1}}=c_1$$

Solved using Lie symmetry for first order ode

Time used: 0.400 (sec)

Writing the ode as

$$y' = \sqrt{-y^2+1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2+\sqrt{-y^2+1}\,(b_3-a_2)-\left(-y^2+1\right)a_3+\frac{y(xb_2+yb_3+b_1)}{\sqrt{-y^2+1}}=0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{\sqrt{-y^2+1}\,y^2a_3+xyb_2+y^2a_2-a_3\sqrt{-y^2+1}+b_2\sqrt{-y^2+1}+yb_1-a_2+b_3}{\sqrt{-y^2+1}}=0$$

Setting the numerator to zero gives

$$\sqrt{-y^2+1}\,y^2a_3+xyb_2+y^2a_2-a_3\sqrt{-y^2+1}+b_2\sqrt{-y^2+1}+yb_1-a_2+b_3=0 \tag{6E}$$

Simplifying the above gives

$$\sqrt{-y^2+1}\,y^2a_3-\left(-y^2+1\right)a_2+\left(-y^2+1\right)b_3+xyb_2 \\ +y^2b_3-a_3\sqrt{-y^2+1}+b_2\sqrt{-y^2+1}+yb_1=0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\sqrt{-y^2+1}\,y^2a_3+xyb_2+y^2a_2-a_3\sqrt{-y^2+1}+b_2\sqrt{-y^2+1}+yb_1-a_2+b_3=0$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$\left\{x,y,\sqrt{-y^2+1}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$\left\{x=v_1,y=v_2,\sqrt{-y^2+1}=v_3\right\}$$

The above PDE (6E) now becomes

$$v_3v_2^2a_3+v_2^2a_2+v_1v_2b_2-a_3v_3+v_2b_1+b_2v_3-a_2+b_3=0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1,v_2,v_3\}$$

Equation (7E) now becomes

$$v_1v_2b_2+v_3v_2^2a_3+v_2^2a_2+v_2b_1+(-a_3+b_2)\,v_3+b_3-a_2=0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_2&=0\\ a_3&=0\\ b_1&=0\\ b_2&=0\\ -a_3+b_2&=0\\ b_3-a_2&=0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= a_1 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 1 \\
\eta &= 0
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= 0 - \left(\sqrt{-y^2+1}\right)(1) \\
&= -\sqrt{-y^2+1} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\sqrt{-y^2+1}} dy
\end{aligned}
$$

Which results in

$$S = -\arcsin(y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \sqrt{-y^2 + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 0 \\ S_y &= -\frac{1}{\sqrt{-y^2 + 1}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned} \int dS &= \int -1\, dR \\ S(R) &= -R + c_2 \end{aligned}$$

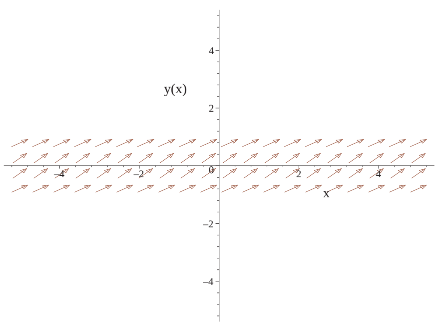
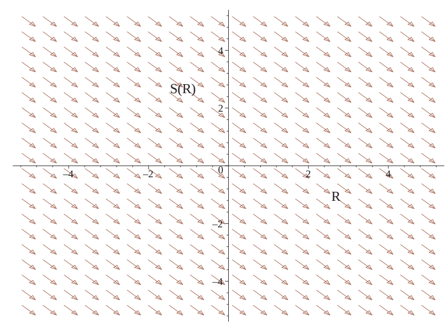
To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$-\arcsin(y) = -x + c_2$$

Which gives

$$y = -\sin(-x + c_2)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \sqrt{-y^2 + 1}$

</td><td>$R = x$
$S = -\arcsin(y)$</td><td>$\frac{dS}{dR} = -1$

</td></tr>
</table>

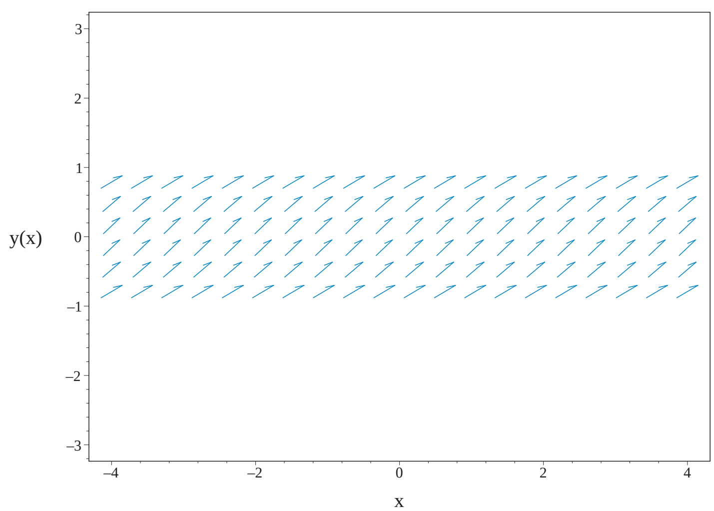


Figure 2.188: Slope field plot
$y' = \sqrt{1 - y^2}$

Summary of solutions found

$$y = -\sin(-x + c_2)$$

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) = \sqrt{1 - y(x)^2}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = \sqrt{1 - y(x)^2}$

- Separate variables

  $\frac{\frac{d}{dx}y(x)}{\sqrt{1-y(x)^2}} = 1$

- Integrate both sides with respect to x

  $\int \frac{\frac{d}{dx}y(x)}{\sqrt{1-y(x)^2}}dx = \int 1dx + C1$

- Evaluate integral

  $\arcsin(y(x)) = x + C1$

- Solve for $y(x)$

  $y(x) = \sin(x + C1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 8

```
dsolve(diff(y(x),x) = (1-y(x)^2)^(1/2),
       y(x),singsol=all)
```

$$y = \sin(x + c_1)$$

Mathematica DSolve solution

Solving time : 3.464 (sec)
Leaf size : 54

```
DSolve[{D[y[x],x]==Sqrt[1-y[x]^2],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
y(x) &\to -\frac{\tan(x + c_1)}{\sqrt{\sec^2(x + c_1)}} \\
y(x) &\to \frac{\tan(x + c_1)}{\sqrt{\sec^2(x + c_1)}} \\
y(x) &\to -1 \\
y(x) &\to 1
\end{aligned}
$$

2.1.85 problem 84

Internal problem ID [8473]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 84
Date solved : `Tuesday, December 17, 2024 at 12:55:42 PM`
CAS classification : `[_Riccati]`

Solve

$$y' = x^2 + y^2 - 1$$

Solved as first order ode of type Riccati

Time used: 0.442 (sec)

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= x^2 + y^2 - 1 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^2 + y^2 - 1$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x^2 - 1$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= x^2 - 1 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \left(x^2 - 1\right) u(x) = 0$$

Unable to solve. Will ask Maple to solve this ode now.

Solution obtained is

$$u(x) = \frac{c_1 \,\mathrm{WhittakerM}\left(\frac{i}{4}, \frac{1}{4}, i x^2\right) + c_2 \,\mathrm{WhittakerW}\left(\frac{i}{4}, \frac{1}{4}, i x^2\right)}{\sqrt{x}}$$

Taking derivative gives

$$u'(x) = \frac{2 i c_1 \left(\left(\frac{1}{2} - \frac{1}{4x^2}\right) \mathrm{WhittakerM}\left(\frac{i}{4}, \frac{1}{4}, i x^2\right) + \frac{\left(\frac{1}{4} - \frac{3i}{4}\right) \mathrm{WhittakerM}\left(1 + \frac{i}{4}, \frac{1}{4}, i x^2\right)}{x^2}\right) x + 2 i c_2 \left(\left(\frac{1}{2} - \frac{1}{4x^2}\right) \mathrm{WhittakerW}\left(\right.\right.}{\sqrt{x}} - \frac{c_1 \,\mathrm{WhittakerM}\left(\frac{i}{4}, \frac{1}{4}, i x^2\right) + c_2 \,\mathrm{WhittakerW}\left(\frac{i}{4}, \frac{1}{4}, i x^2\right)}{2 x^{3/2}}$$

Doing change of constants, the solution becomes

$$y = -\frac{\left(\frac{2 i c_5 \left(\left(\frac{1}{2} - \frac{1}{4x^2}\right) \mathrm{WhittakerM}\left(\frac{i}{4}, \frac{1}{4}, i x^2\right) + \frac{\left(\frac{1}{4} - \frac{3i}{4}\right) \mathrm{WhittakerM}\left(1 + \frac{i}{4}, \frac{1}{4}, i x^2\right)}{x^2}\right) x + 2 i \left(\left(\frac{1}{2} - \frac{1}{4x^2}\right) \mathrm{WhittakerW}\left(\frac{i}{4}, \frac{1}{4}, i x^2\right) + \frac{i \,\mathrm{WhittakerW}\left(1 + \frac{i}{4}\right.}{x^2}\right.}{\sqrt{x}}\right.}{c_5 \,\mathrm{WhittakerM}\left(\frac{i}{4}, \frac{1}{4}, i x^2\right) + \mathrm{WhittakerW}\left(\frac{i}{4}, \frac{1}{4}, ix\right.}$$

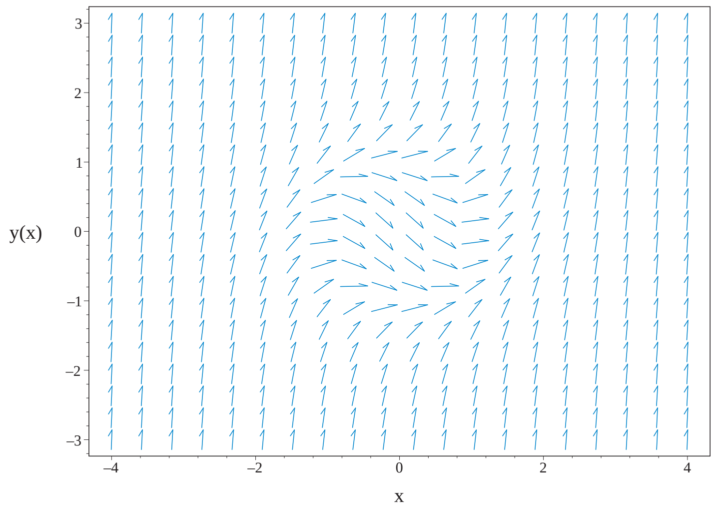


Figure 2.189: Slope field plot
$y' = x^2 + y^2 - 1$

Summary of solutions found

$$y = -\frac{\left(\frac{2ic_5\left(\left(\frac{1}{2}-\frac{1}{4x^2}\right)\mathrm{WhittakerM}\left(\frac{i}{4},\frac{1}{4},ix^2\right)+\frac{\left(\frac{1}{4}-\frac{3i}{4}\right)\mathrm{WhittakerM}\left(1+\frac{i}{4},\frac{1}{4},ix^2\right)}{x^2}\right)x+2i\left(\left(\frac{1}{2}-\frac{1}{4x^2}\right)\mathrm{WhittakerW}\left(\frac{i}{4},\frac{1}{4},ix^2\right)+\frac{i\,\mathrm{WhittakerW}\left(1+\frac{i}{4}\right.}{x^2}\right.}{\sqrt{x}}\right.}{c_5\,\mathrm{WhittakerM}\left(\frac{i}{4},\frac{1}{4},ix^2\right)+\mathrm{WhittakerW}\left(\frac{i}{4},\frac{1}{4},ix\right.}$$

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) = x^2 + y(x)^2 - 1$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = x^2 + y(x)^2 - 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (-x^2+1)*y(x), y(x)`
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      checking if the LODE has constant coefficients
      checking if the LODE is of Euler type
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying a Liouvillian solution using Kovacics algorithm
      <- No Liouvillian solutions exists
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Whittaker
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
            <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
         <- Whittaker successful
      <- special function solution successful
   <- Riccati to 2nd Order successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 85

```
dsolve(diff(y(x),x) = x^2+y(x)^2-1,
       y(x),singsol=all)
```

$$y = \frac{(-3-i)\,\mathrm{WhittakerM}\left(1+\frac{i}{4}, \frac{1}{4}, i x^2\right) + 4\,\mathrm{WhittakerW}\left(1+\frac{i}{4}, \frac{1}{4}, i x^2\right) c_1 + \left(-2 i x^2 + i + 1\right) \mathrm{WhittakerM}\,(}{2x\left(c_1\,\mathrm{WhittakerW}\left(\frac{i}{4}, \frac{1}{4}, i x^2\right) + \mathrm{WhittakerM}\left(\frac{i}{4}, \frac{1}{4}, i\right.\right.}$$

Mathematica DSolve solution

Solving time : 0.225 (sec)
Leaf size : 153

```
DSolve[{D[y[x],x]==x^2+y[x]^2-1,{}},
      y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{i\big(x\,\mathrm{ParabolicCylinderD}\left(-\frac{1}{2}-\frac{i}{2}, (-1+i)x\right) + (1+i)\,\mathrm{ParabolicCylinderD}\left(\frac{1}{2}-\frac{i}{2}, (-1+i)x\right) - c_1 x\,\mathrm{P}}{\mathrm{ParabolicCylinderD}\left(-\frac{1}{2}-\frac{i}{2}, (-1+i)x\right) + c_1\,\mathrm{Par}}$$

$$y(x) \to \frac{(1+i)\,\mathrm{ParabolicCylinderD}\left(\frac{1}{2}+\frac{i}{2}, (1+i)x\right)}{\mathrm{ParabolicCylinderD}\left(-\frac{1}{2}+\frac{i}{2}, (1+i)x\right)} - i x$$

2.1.86 problem 85

Internal problem ID [8474]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 85
Date solved : `Tuesday, December 17, 2024 at 12:55:44 PM`
CAS classification : `[_Bernoulli]`

Solve

$$y' = 2y(x\sqrt{y} - 1)$$

With initial conditions

$$y(0) = 1$$

Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= 2y(x\sqrt{y} - 1) \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(2y(x\sqrt{y} - 1)) \\ &= 3x\sqrt{y} - 2 \end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

Solved as first order Bernoulli ode

Time used: 0.232 (sec)

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= 2y(x\sqrt{y} - 1) \end{aligned}$$

This is a Bernoulli ODE.

$$y' = (-2)\, y + (2x)\, y^{3/2} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

Comparing this to (1) shows that

$$\begin{aligned} f_0 &= -2 \\ f_1 &= 2x \end{aligned}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $v = y^{1-n}$ in equation (3) which generates a new ODE in $v(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -2 \\ f_1(x) &= 2x \\ n &= \frac{3}{2} \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^{3/2}$ gives

$$y' \frac{1}{y^{3/2}} = -\frac{2}{\sqrt{y}} + 2x \tag{4}$$

Let

$$\begin{aligned} v &= y^{1-n} \\ &= \frac{1}{\sqrt{y}} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$v' = -\frac{1}{2y^{3/2}} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -2v'(x) &= -2v(x) + 2x \\ v' &= v - x \end{aligned} \tag{7}$$

The above now is a linear ODE in $v(x)$ which is now solved.

In canonical form a linear first order is

$$v'(x) + q(x)v(x) = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= -1 \\ p(x) &= -x \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q \, dx} \\ &= \mathrm{e}^{\int (-1) dx} \\ &= \mathrm{e}^{-x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu v) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu v) &= (\mu)(-x) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(v\,\mathrm{e}^{-x}\right) &= \left(\mathrm{e}^{-x}\right)(-x) \\ \mathrm{d}\left(v\,\mathrm{e}^{-x}\right) &= \left(-x\,\mathrm{e}^{-x}\right)\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$
\begin{aligned}
v\,\mathrm{e}^{-x} &= \int -x\,\mathrm{e}^{-x}\,dx \\
&= (x+1)\,\mathrm{e}^{-x} + c_1
\end{aligned}
$$

Dividing throughout by the integrating factor e^{-x} gives the final solution

$$
v(x) = c_1\,\mathrm{e}^{x} + x + 1
$$

The substitution $v = y^{1-n}$ is now used to convert the above solution back to y which results in

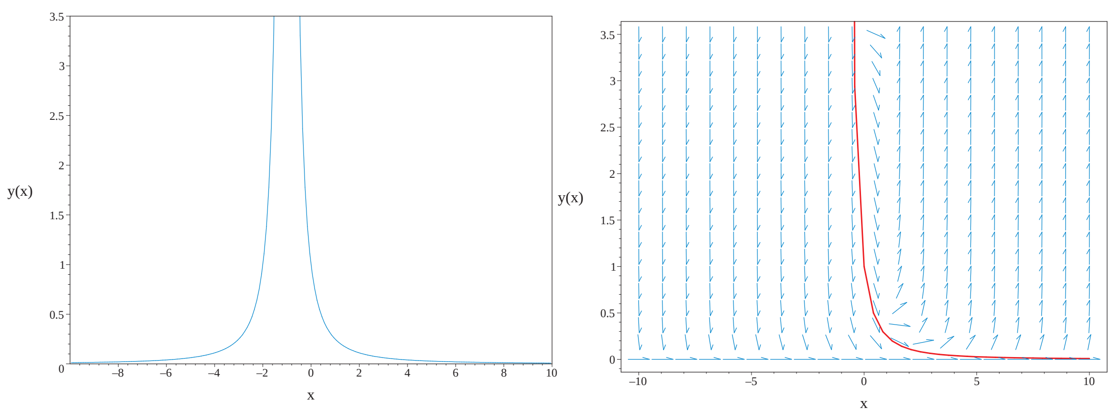
$$
\frac{1}{\sqrt{y}} = c_1\,\mathrm{e}^{x} + x + 1
$$

Solving for the constant of integration from initial conditions, the solution becomes

$$
\frac{1}{\sqrt{y}} = x + 1
$$

Solving for y gives

$$
y = \frac{1}{(x+1)^2}
$$



(a) Solution plot
$y = \frac{1}{(x+1)^2}$

(b) Slope field plot
$y' = 2y\left(x\sqrt{y} - 1\right)$

<u>Summary of solutions found</u>

$$
y = \frac{1}{(x+1)^2}
$$

Maple step by step solution

Let's solve

$\left[\frac{d}{dx}y(x) = 2y(x)\left(x\sqrt{y(x)}-1\right), y(0) = 1\right]$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = 2y(x)\left(x\sqrt{y(x)}-1\right)$

- Use initial condition $y(0) = 1$

  0

- Solve for 0

  $0 = 0$

- Substitute $0 = 0$ into general solution and simplify

  0

- Solution to the IVP

  0

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

Maple dsolve solution

Solving time : 0.062 (sec)
Leaf size : 9

```
dsolve([diff(y(x),x) = 2*y(x)*(x*y(x)^(1/2)-1),
       op([y(0) = 1])],y(x),singsol=all)
```

$$y = \frac{1}{(x+1)^2}$$

Mathematica DSolve solution

Solving time : 0.684 (sec)
Leaf size : 20

```
DSolve[{D[y[x],x]==2*y[x]*(x*Sqrt[y[x]-1]),{y[0]==1}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to 1$$
$$y(x) \to \sec^2\left(\frac{x^2}{2}\right)$$

2.1.87 problem 86

Internal problem ID [8475]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 86
Date solved : `Thursday, December 12, 2024 at 09:21:15 AM`
CAS classification :
`[[_2nd_order, _exact, _nonlinear], [_2nd_order, _with_linear_symmetries], [_2nd_order,`

Solve

$$y'' = \frac{1}{y} - \frac{xy'}{y^2}$$

Solved as second order nonlinear exact ode

Time used: 34.198 (sec)

An exact non-linear second order ode has the form

$$a_2(x, y, y')\, y'' + a_1(x, y, y')\, y' + a_0(x, y, y') = 0$$

Where the following conditions are satisfied

$$\frac{\partial a_2}{\partial y} = \frac{\partial a_1}{\partial y'}$$
$$\frac{\partial a_2}{\partial x} = \frac{\partial a_0}{\partial y'}$$
$$\frac{\partial a_1}{\partial x} = \frac{\partial a_0}{\partial y}$$

Looking at the the ode given we see that

$$a_2 = 1$$
$$a_1 = \frac{x}{y^2}$$
$$a_0 = -\frac{1}{y}$$

Applying the conditions to the above shows this is a nonlinear exact second order ode. Therefore it can be reduced to first order ode given by

$$
\begin{aligned}
\int a_2\, dy' + \int a_1\, dy + \int a_0\, dx &= c_1 \\
\int 1\, dy' + \int \frac{x}{y^2}\, dy + \int -\frac{1}{y}\, dx &= c_1
\end{aligned}
$$

Which results in

$$
y' - \frac{x}{y} = c_1
$$

Which is now solved.

In canonical form, the ODE is

$$
\begin{aligned}
y' &= F(x, y) \\
&= \frac{c_1 y + x}{y} \qquad (1)
\end{aligned}
$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x, y)$ and $N(x, y)$ are both homogeneous functions and of the same order. Recall that a function $f(x, y)$ is homogeneous of order n if

$$
f(t^n x, t^n y) = t^n f(x, y)
$$

In this case, it can be seen that both $M = c_1 y + x$ and $N = y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$
\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} x + u
$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}x} x + u &= c_1 + \frac{1}{u} \\
\frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{c_1 + \frac{1}{u(x)} - u(x)}{x}
\end{aligned}
$$

Or

$$
u'(x) - \frac{c_1 + \frac{1}{u(x)} - u(x)}{x} = 0
$$

Or

$$u'(x)\,u(x)\,x+u(x)^2-c_1u(x)-1=0$$

Which is now solved as separable in $u(x)$.

The ode $u'(x)=-\frac{u(x)^2-c_1u(x)-1}{u(x)x}$ is separable as it can be written as

$$\begin{aligned}u'(x)&=-\frac{u(x)^2-c_1u(x)-1}{u\left(x\right)x}\\&=f(x)g(u)\end{aligned}$$

Where

$$\begin{aligned}f(x)&=-\frac{1}{x}\\g(u)&=\frac{-c_1u+u^2-1}{u}\end{aligned}$$

Integrating gives

$$\begin{aligned}\int\frac{1}{g(u)}\,du&=\int f(x)\,dx\\\int\frac{u}{-c_1u+u^2-1}\,du&=\int-\frac{1}{x}\,dx\\\frac{\ln\left(u(x)^2-c_1u(x)-1\right)}{2}+\frac{c_1\,\operatorname{arctanh}\left(\frac{-2u(x)+c_1}{\sqrt{c_1^2+4}}\right)}{\sqrt{c_1^2+4}}&=\ln\left(\frac{1}{x}\right)+c_2\end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u)=0$ or $\frac{-c_1u+u^2-1}{u}=0$ for $u(x)$ gives

$$\begin{aligned}u(x)&=\frac{c_1}{2}-\frac{\sqrt{c_1^2+4}}{2}\\u(x)&=\frac{c_1}{2}+\frac{\sqrt{c_1^2+4}}{2}\end{aligned}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\frac{\ln\left(u(x)^2 - c_1 u(x) - 1\right)}{2} + \frac{c_1 \operatorname{arctanh}\left(\frac{-2u(x)+c_1}{\sqrt{c_1^2+4}}\right)}{\sqrt{c_1^2+4}} = \ln\left(\frac{1}{x}\right) + c_2$$
$$u(x) = \frac{c_1}{2} - \frac{\sqrt{c_1^2+4}}{2}$$
$$u(x) = \frac{c_1}{2} + \frac{\sqrt{c_1^2+4}}{2}$$

Converting $\frac{\ln\left(u(x)^2 - c_1 u(x) - 1\right)}{2} + \frac{c_1 \operatorname{arctanh}\left(\frac{-2u(x)+c_1}{\sqrt{c_1^2+4}}\right)}{\sqrt{c_1^2+4}} = \ln\left(\frac{1}{x}\right) + c_2$ back to y gives

$$\frac{\ln\left(\frac{-c_1 y x + y^2 - x^2}{x^2}\right)}{2} + \frac{c_1 \operatorname{arctanh}\left(\frac{c_1 x - 2y}{x\sqrt{c_1^2+4}}\right)}{\sqrt{c_1^2+4}} = \ln\left(\frac{1}{x}\right) + c_2$$

Converting $u(x) = \frac{c_1}{2} - \frac{\sqrt{c_1^2+4}}{2}$ back to y gives

$$y = x\left(\frac{c_1}{2} - \frac{\sqrt{c_1^2+4}}{2}\right)$$

Converting $u(x) = \frac{c_1}{2} + \frac{\sqrt{c_1^2+4}}{2}$ back to y gives

$$y = x\left(\frac{c_1}{2} + \frac{\sqrt{c_1^2+4}}{2}\right)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$\frac{\ln\left(\frac{-c_1 y x + y^2 - x^2}{x^2}\right)}{2} + \frac{c_1 \operatorname{arctanh}\left(\frac{c_1 x - 2y}{x\sqrt{c_1^2+4}}\right)}{\sqrt{c_1^2+4}} = \ln\left(\frac{1}{x}\right) + c_2$$
$$y = x\left(\frac{c_1}{2} - \frac{\sqrt{c_1^2+4}}{2}\right)$$
$$y = x\left(\frac{c_1}{2} + \frac{\sqrt{c_1^2+4}}{2}\right)$$

Solved as second order integrable as is ode (ABC method)

Time used: 34.203 (sec)

Writing the ode as

$$y'' - \frac{1}{y} + \frac{xy'}{y^2} = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int \left(y'' - \frac{1}{y} + \frac{xy'}{y^2}\right) dx = 0$$
$$y' - \frac{x}{y} = c_1$$

Which is now solved for y. In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{c_1 y + x}{y} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = c_1 y + x$ and $N = y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\frac{\mathrm{d}u}{\mathrm{d}x} x + u = c_1 + \frac{1}{u}$$
$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{c_1 + \frac{1}{u(x)} - u(x)}{x}$$

Or

$$u'(x) - \frac{c_1 + \frac{1}{u(x)} - u(x)}{x} = 0$$

Or

$$u'(x)\, u(x)\, x + u(x)^2 - c_1 u(x) - 1 = 0$$

Which is now solved as separable in $u(x)$.

The ode $u'(x) = -\frac{u(x)^2 - c_1 u(x) - 1}{u(x)x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)^2 - c_1 u(x) - 1}{u(x)\, x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(u) &= \frac{-c_1 u + u^2 - 1}{u} \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\, du &= \int f(x)\, dx \\ \int \frac{u}{-c_1 u + u^2 - 1}\, du &= \int -\frac{1}{x}\, dx \\ \frac{\ln\left(u(x)^2 - c_1 u(x) - 1\right)}{2} + \frac{c_1 \operatorname{arctanh}\left(\frac{-2u(x)+c_1}{\sqrt{c_1^2+4}}\right)}{\sqrt{c_1^2+4}} &= \ln\left(\frac{1}{x}\right) + c_2 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $\frac{-c_1 u + u^2 - 1}{u} = 0$ for $u(x)$ gives

$$\begin{aligned} u(x) &= \frac{c_1}{2} - \frac{\sqrt{c_1^2+4}}{2} \\ u(x) &= \frac{c_1}{2} + \frac{\sqrt{c_1^2+4}}{2} \end{aligned}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} \frac{\ln\left(u(x)^2 - c_1 u(x) - 1\right)}{2} + \frac{c_1 \operatorname{arctanh}\left(\frac{-2u(x)+c_1}{\sqrt{c_1^2+4}}\right)}{\sqrt{c_1^2+4}} &= \ln\left(\frac{1}{x}\right) + c_2 \\ u(x) &= \frac{c_1}{2} - \frac{\sqrt{c_1^2+4}}{2} \\ u(x) &= \frac{c_1}{2} + \frac{\sqrt{c_1^2+4}}{2} \end{aligned}$$

Converting $\frac{\ln\left(u(x)^2-c_1u(x)-1\right)}{2}+\frac{c_1\,\operatorname{arctanh}\left(\frac{-2u(x)+c_1}{\sqrt{c_1^2+4}}\right)}{\sqrt{c_1^2+4}}=\ln\left(\frac{1}{x}\right)+c_2$ back to y gives

$$\frac{\ln\left(\frac{-c_1yx+y^2-x^2}{x^2}\right)}{2}+\frac{c_1\,\operatorname{arctanh}\left(\frac{c_1x-2y}{x\sqrt{c_1^2+4}}\right)}{\sqrt{c_1^2+4}}=\ln\left(\frac{1}{x}\right)+c_2$$

Converting $u(x)=\frac{c_1}{2}-\frac{\sqrt{c_1^2+4}}{2}$ back to y gives

$$y=x\left(\frac{c_1}{2}-\frac{\sqrt{c_1^2+4}}{2}\right)$$

Converting $u(x)=\frac{c_1}{2}+\frac{\sqrt{c_1^2+4}}{2}$ back to y gives

$$y=x\left(\frac{c_1}{2}+\frac{\sqrt{c_1^2+4}}{2}\right)$$

Will add steps showing solving for IC soon.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
-> trying 2nd order, dynamical_symmetries, fully reducible to Abel through one integra
   --- trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods fo
   -> trying 2nd order, dynamical_symmetries, fully reducible to Abel through one inte
trying 2nd order, integrating factors of the form mu(x,y)/(y)^n, only the singular cas
trying symmetries linear in x and y(x)
trying differential order: 2; exact nonlinear
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(_b(_a)*c__1-_a)/_b(_a), _b(_a)`
```

```
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   trying separable
   trying inverse linear
   trying homogeneous types:
   trying homogeneous D
   <- homogeneous successful
<- differential order: 2; exact nonlinear successful`
```

Maple dsolve solution

Solving time : 0.049 (sec)
Leaf size : 56

```
dsolve(diff(diff(y(x),x),x) = 1/y(x)-x/y(x)^2*diff(y(x),x),
       y(x),singsol=all)
```

$$y = \operatorname{RootOf}\left(\_Z^2 - e^{\operatorname{RootOf}\left(\left(4\,e^{\_Z}\cosh\left(\frac{\sqrt{c_1^2+4}\,(2c_2+\_Z+2\ln(x))}{2c_1}\right)^2+c_1^2+4\right)x^2\right)} - 1 + \_Zc_1\right)x$$

Mathematica DSolve solution

Solving time : 0.212 (sec)
Leaf size : 77

```
DSolve[{D[y[x],{x,2}]==1/y[x]-x/y[x]^2*D[y[x],x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$\text{Solve}\left[\frac{1}{2}\log\left(-\frac{y(x)^2}{x^2}-\frac{c_1 y(x)}{x}+1\right)-\frac{c_1 \arctan\left(\frac{\frac{2y(x)}{x}+c_1}{\sqrt{-4-c_1{}^2}}\right)}{\sqrt{-4-c_1{}^2}}=-\log(x)+c_2,y(x)\right]$$

2.1.88 problem 87

Internal problem ID [8476]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 87
Date solved : `Thursday, December 12, 2024 at 09:22:24 AM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$y'' + y' + y = 0$$

With initial conditions

$$y(0) = 0$$

Solved as second order linear constant coeff ode

Time used: 0.142 (sec)

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} + \lambda \,\mathrm{e}^{x\lambda} + \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(1)} \\ &= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \mathrm{e}^{-\frac{x}{2}} c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)$$

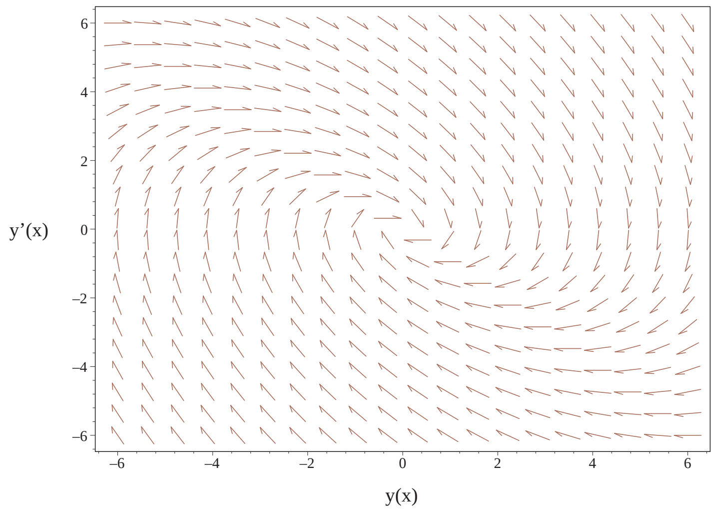


Figure 2.191: Slope field plot
$y'' + y' + y = 0$

Solved as second order ode using Kovacic algorithm

Time used: 0.161 (sec)

Writing the ode as

$$y'' + y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.92: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)\right) + c_2\left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)\left(\frac{2\sqrt{3}\, \tan\left(\frac{\sqrt{3}\, x}{2}\right)}{3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = \frac{2 c_2\, \mathrm{e}^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3}
$$

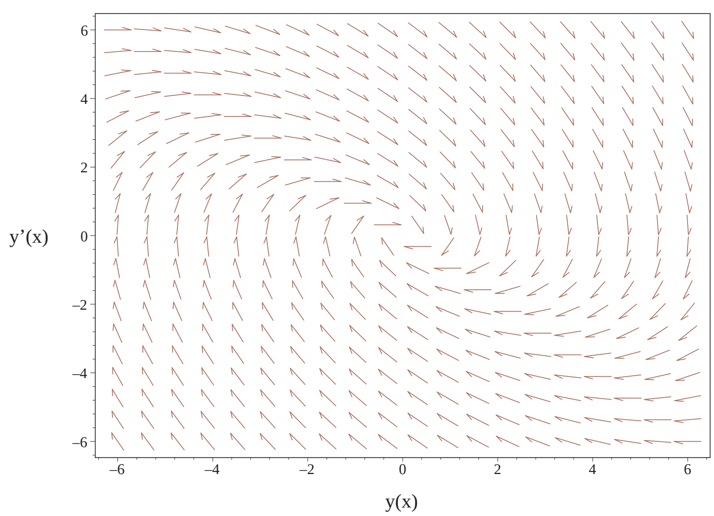


Figure 2.192: Slope field plot
$y'' + y' + y = 0$

Solved as second order ode adjoint method

Time used: 1.602 (sec)

In normal form the ode

$$
y'' + y' + y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= 1 \\
q(x) &= 1 \\
r(x) &= 0
\end{aligned}
$$

The Lagrange adjoint ode is given by

$$
\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - (\xi(x))' + (\xi(x)) &= 0 \\
\xi''(x) - \xi'(x) + \xi(x) &= 0
\end{aligned}
$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
A\xi''(x) + B\xi'(x) + C\xi(x) = 0
$$

Where in the above $A = 1, B = -1, C = 1$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$
\lambda^2 e^{x\lambda} - \lambda\, e^{x\lambda} + e^{x\lambda} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$
\lambda^2 - \lambda + 1 = 0 \tag{2}
$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$
\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}
$$

Substituting $A = 1, B = -1, C = 1$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2 - (4)\,(1)\,(1)} \\
&= \frac{1}{2} \pm \frac{i\sqrt{3}}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= \frac{1}{2} + \frac{i\sqrt{3}}{2} \\
\lambda_2 &= \frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}
$$

Which simplifies to

$$\lambda_1 = \frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = \frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = \frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$\xi = e^{\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y = \int \xi(x)\,r(x)\,dx$$
$$y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) = \frac{\int \xi(x)\,r(x)\,dx}{\xi\,(x)}$$

Or

$$y' + y\left(1 - \frac{\left(\frac{e^{\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{2} + e^{\frac{x}{2}}\left(-\frac{c_1\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{c_2\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right)\right)e^{-\frac{x}{2}}}{c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)}\right) = 0$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= -\frac{\left(c_2\sqrt{3}-c_1\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3}+c_2\right)}{2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)} \\
p(x) &= 0
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= \mathrm{e}^{\int -\frac{\left(c_2\sqrt{3}-c_1\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3}+c_2\right)}{2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)}dx} \\
&= \frac{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{\tan\left(\frac{\sqrt{3}\,x}{2}\right)c_2+c_1}
\end{aligned}
$$

The ode becomes

$$
\frac{\mathrm{d}}{\mathrm{d}x}\mu y = 0
$$

$$
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{\tan\left(\frac{\sqrt{3}\,x}{2}\right)c_2+c_1}\right) = 0
$$

Integrating gives

$$
\begin{aligned}
\frac{y\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{\tan\left(\frac{\sqrt{3}\,x}{2}\right)c_2+c_1} &= \int 0\,dx + c_3 \\
&= c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{\tan\left(\frac{\sqrt{3}\,x}{2}\right)c_2+c_1}$ gives the final solution

$$
y = \frac{\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)c_2+c_1\right)\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}c_3}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}
$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right) c_2 + c_1\right) \mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}} c_3}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

The constants can be merged to give

$$y = \frac{\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right) c_2 + c_1\right) \mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{\tan\left(\frac{\sqrt{3}\,x}{2}\right) c_2\, \mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

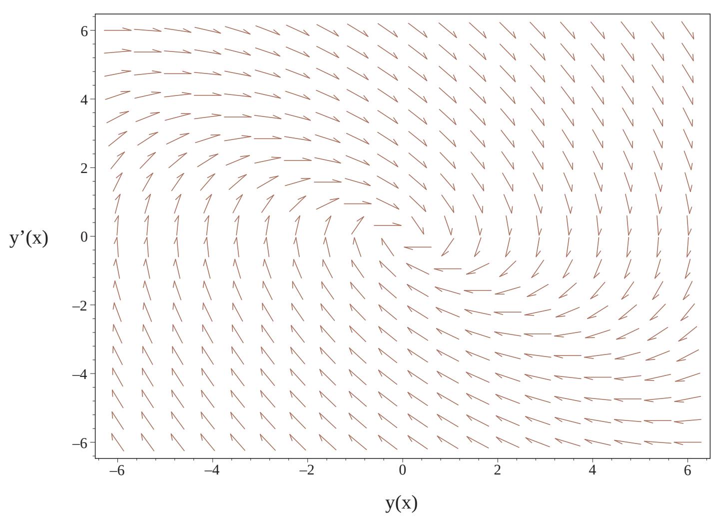


Figure 2.193: Slope field plot
$y'' + y' + y = 0$

Maple step by step solution

Let's solve

$\left[\frac{d^2}{dx^2}y(x)+\frac{d}{dx}y(x)+y(x)=0, y(0)=0\right]$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Characteristic polynomial of ODE

  $r^2+r+1=0$
- Use quadratic formula to solve for r

  $r=\frac{(-1)\pm(\sqrt{-3})}{2}$
- Roots of the characteristic polynomial

  $r=\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}, -\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)$
- 1st solution of the ODE

  $y_1(x)=\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)$
- 2nd solution of the ODE

  $y_2(x)=\mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)$
- General solution of the ODE

  $y(x)=C1y_1(x)+C2y_2(x)$
- Substitute in solutions

  $y(x)=C1\,\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)+C2\,\mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

Maple dsolve solution

Solving time : 0.020 (sec)
Leaf size : 17

```
dsolve([diff(diff(y(x),x),x)+diff(y(x),x)+y(x) = 0,
       op([y(0) = 0])],y(x),singsol=all)
```

$$y = c_1 \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right)$$

Mathematica DSolve solution

Solving time : 0.023 (sec)
Leaf size : 26

```
DSolve[{D[y[x],{x,2}]+D[y[x],x]+y[x]==0,{y[0]==0}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{-x/2} \sin\left(\frac{\sqrt{3}x}{2}\right)$$

2.1.89 problem 88

Internal problem ID [8477]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 88
Date solved : `Thursday, December 12, 2024 at 09:22:26 AM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$y'' + y' + y = 0$$

With initial conditions

$$y'(0) = 0$$

Solved as second order linear constant coeff ode

Time used: 0.132 (sec)

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + \lambda e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(1)} \\ &= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \mathrm{e}^{-\frac{x}{2}}\left(c_2\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

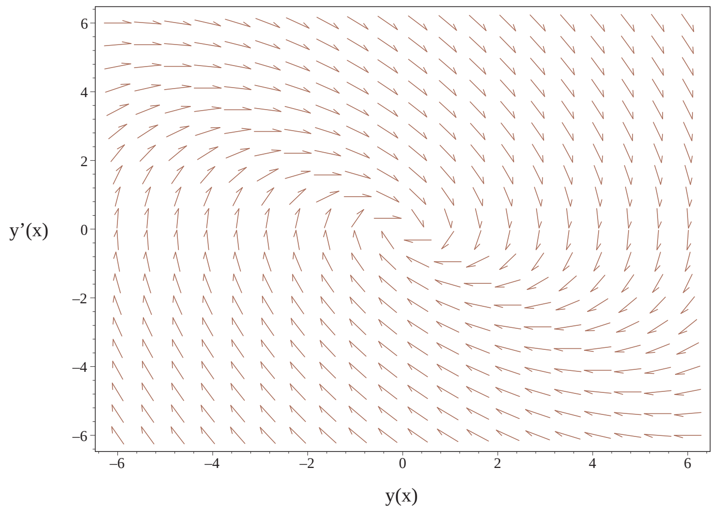


Figure 2.194: Slope field plot
$y'' + y' + y = 0$

Solved as second order ode using Kovacic algorithm

Time used: 0.166 (sec)

Writing the ode as

$$y'' + y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.94: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-\frac{x}{2}} \cos \left(\frac{\sqrt{3}\, x}{2} \right) \right) + c_2 \left(e^{-\frac{x}{2}} \cos \left(\frac{\sqrt{3}\, x}{2} \right) \left(\frac{2\sqrt{3} \tan \left(\frac{\sqrt{3}\, x}{2} \right)}{3} \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = 2c_2 e^{-\frac{x}{2}} \cos \left(\frac{\sqrt{3}\, x}{2} \right) + \frac{2c_2 e^{-\frac{x}{2}} \sqrt{3} \sin \left(\frac{\sqrt{3}\, x}{2} \right)}{3}
$$

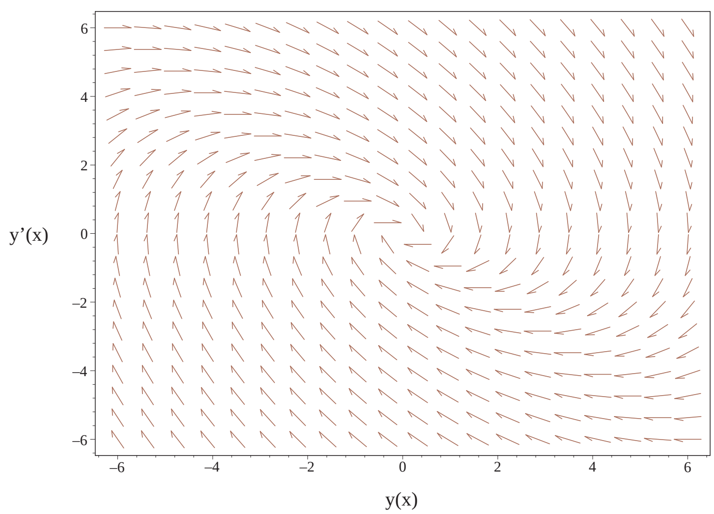


Figure 2.195: Slope field plot
$y'' + y' + y = 0$

Solved as second order ode adjoint method

Time used: 1.603 (sec)

In normal form the ode

$$
y'' + y' + y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= 1 \\
q(x) &= 1 \\
r(x) &= 0
\end{aligned}
$$

The Lagrange adjoint ode is given by

$$
\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - (\xi(x))' + (\xi(x)) &= 0 \\
\xi''(x) - \xi'(x) + \xi(x) &= 0
\end{aligned}
$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
A\xi''(x) + B\xi'(x) + C\xi(x) = 0
$$

Where in the above $A = 1, B = -1, C = 1$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{x\lambda} - \lambda\, \mathrm{e}^{x\lambda} + \mathrm{e}^{x\lambda} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$
\lambda^2 - \lambda + 1 = 0 \tag{2}
$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$
\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}
$$

Substituting $A = 1, B = -1, C = 1$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2 - (4)\,(1)\,(1)} \\
&= \frac{1}{2} \pm \frac{i\sqrt{3}}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= \frac{1}{2} + \frac{i\sqrt{3}}{2} \\
\lambda_2 &= \frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}
$$

Which simplifies to

$$\lambda_1 = \frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = \frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = \frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$\xi = e^{\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y = \int \xi(x)\,r(x)\,dx$$
$$y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) = \frac{\int \xi(x)\,r(x)\,dx}{\xi\,(x)}$$

Or

$$y' + y\left(1 - \frac{\left(\frac{e^{\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right)+c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{2} + e^{\frac{x}{2}}\left(-\frac{c_1\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{c_2\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right)\right)e^{-\frac{x}{2}}}{c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)}\right) = 0$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= -\frac{\left(c_2\sqrt{3}-c_1\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3}+c_2\right)}{2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)} \\
p(x) &= 0
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= \mathrm{e}^{\int -\frac{\left(c_2\sqrt{3}-c_1\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(c_1\sqrt{3}+c_2\right)}{2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)}dx} \\
&= \frac{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{\tan\left(\frac{\sqrt{3}\,x}{2}\right)c_2+c_1}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{\tan\left(\frac{\sqrt{3}\,x}{2}\right)c_2+c_1}\right) &= 0
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{\tan\left(\frac{\sqrt{3}\,x}{2}\right)c_2+c_1} &= \int 0\,dx + c_3 \\
&= c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,\mathrm{e}^{\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{\tan\left(\frac{\sqrt{3}\,x}{2}\right)c_2+c_1}$ gives the final solution

$$
y = \frac{\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)c_2+c_1\right)\mathrm{e}^{-\frac{\sqrt{3}\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}c_3}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}
$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right) c_2 + c_1\right) e^{-\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}} c_3}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

The constants can be merged to give

$$y = \frac{\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right) c_2 + c_1\right) e^{-\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right) c_2 + c_2\sqrt{3}\right) e^{-\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

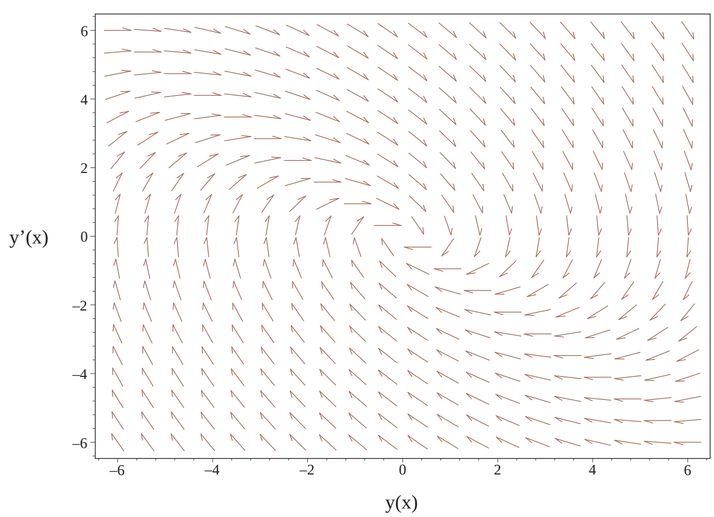


Figure 2.196: Slope field plot
$y'' + y' + y = 0$

Maple step by step solution

Let's solve

$$\left[\frac{d^2}{dx^2}y(x)+\frac{d}{dx}y(x)+y(x)=0, \left(\frac{d}{dx}y(x)\right)\bigg|_{\{x=0\}}=0\right]$$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2}y(x)$
- Characteristic polynomial of ODE
$r^2+r+1=0$
- Use quadratic formula to solve for r
$r=\frac{(-1)\pm(\sqrt{-3})}{2}$
- Roots of the characteristic polynomial
$r=\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}, -\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)$
- 1st solution of the ODE
$y_1(x)=\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)$
- 2nd solution of the ODE
$y_2(x)=\mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)$
- General solution of the ODE
$y(x)=C1y_1(x)+C2y_2(x)$
- Substitute in solutions
$y(x)=C1\,\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)+C2\,\mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

Maple dsolve solution

Solving time : `0.033` (sec)
Leaf size : 29

```
dsolve([diff(diff(y(x),x),x)+diff(y(x),x)+y(x) = 0,
       op([D(y)(0) = 0])],y(x),singsol=all)
```

$$y = \mathrm{e}^{-\frac{x}{2}} c_1 \left(\sqrt{3}\, \cos\left(\frac{\sqrt{3}\, x}{2}\right)+\sin\left(\frac{\sqrt{3}\, x}{2}\right)\right)$$

Mathematica DSolve solution

Solving time : `0.021` (sec)
Leaf size : 44

```
DSolve[{D[y[x],{x,2}]+D[y[x],x]+y[x]==0,{Derivative[1][y][0] ==0}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{-x/2} \left(\sin\left(\frac{\sqrt{3}x}{2}\right)+\sqrt{3}\cos\left(\frac{\sqrt{3}x}{2}\right)\right)$$

2.1.90 problem 88

Internal problem ID [8478]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 88
Date solved : `Thursday, December 12, 2024 at 09:22:29 AM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$y'' + y' + y = 0$$

With initial conditions

$$\begin{aligned} y'(0) &= 0 \\ y(0) &= 1 \end{aligned}$$

Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + y' + y = 0$$

The domain of $p(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solved as second order linear constant coeff ode

Time used: 0.169 (sec)

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + \lambda e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(1)} \\ &= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

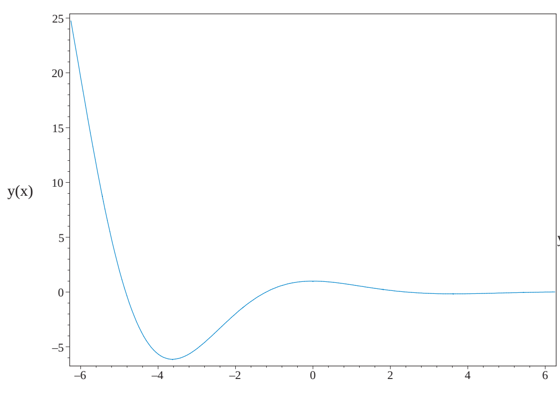
Which becomes

$$y = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

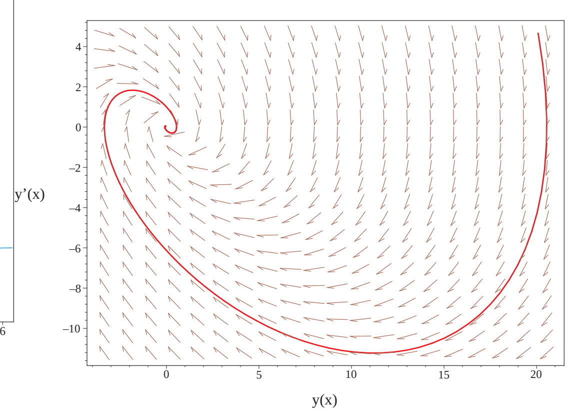
Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \mathrm{e}^{-\frac{x}{2}} \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) + \frac{\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)$$



(a) Solution plot
$y = \mathrm{e}^{-\frac{x}{2}} \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) + \frac{\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)$



(b) Slope field plot
$y'' + y' + y = 0$

Solved as second order ode using Kovacic algorithm

Time used: 0.232 (sec)

Writing the ode as

$$y'' + y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.96: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{3}\, x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)
\end{aligned}
$$

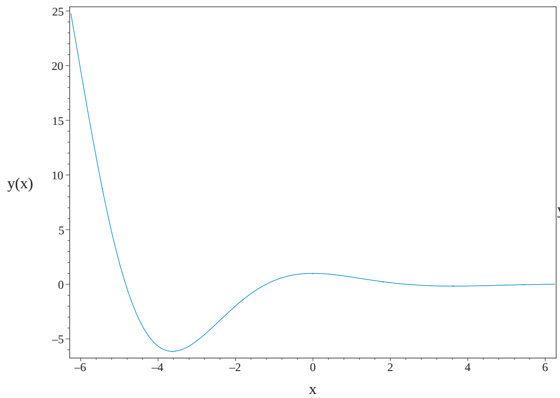
Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right) + c_2\left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)\right)
\end{aligned}
$$

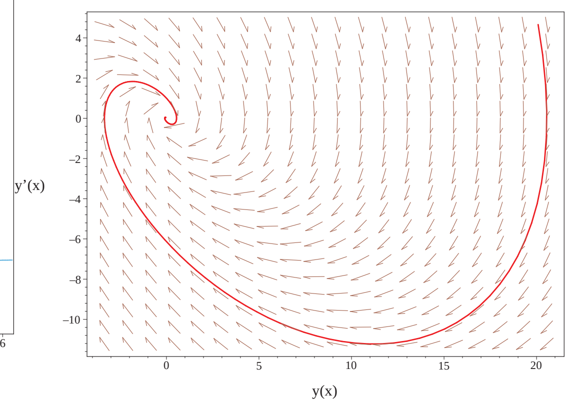
Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{\mathrm{e}^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3}$$



(a) Solution plot
$y = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{\mathrm{e}^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3}$



(b) Slope field plot
$y'' + y' + y = 0$

Solved as second order ode adjoint method

Time used: 1.622 (sec)

In normal form the ode

$$y'' + y' + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 1 \\ r(x) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (\xi(x))' + (\xi(x)) &= 0 \\ \xi''(x) - \xi'(x) + \xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = -1, C = 1$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} - \lambda \,\mathrm{e}^{x\lambda} + \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = 1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2 - (4)\,(1)\,(1)} \\
&= \frac{1}{2} \pm \frac{i\sqrt{3}}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= \frac{1}{2} + \frac{i\sqrt{3}}{2} \\
\lambda_2 &= \frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}$$

Which simplifies to

$$\lambda_1 = \frac{1}{2} + \frac{i\sqrt{3}}{2}$$

$$\lambda_2 = \frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = \frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$\xi = e^{\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}
\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y &= \int \xi(x)\,r(x)\,dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi\,(x)}
\end{aligned}$$

Or

$$y' + y\left(1 - \frac{\left(\frac{\mathrm{e}^{\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right)+c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{2} + \mathrm{e}^{\frac{x}{2}}\left(-\frac{c_1\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{c_2\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2}\right)\right)\mathrm{e}^{-\frac{x}{2}}}{c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)}\right) = 0$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned}
q(x) &= -\frac{(c_2\sqrt{3} - c_1)\cos\left(\frac{\sqrt{3}\,x}{2}\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)(c_1\sqrt{3} + c_2)}{2c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + 2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)} \\
p(x) &= 0
\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= \mathrm{e}^{\int -\frac{(c_2\sqrt{3}-c_1)\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\sin\left(\frac{\sqrt{3}\,x}{2}\right)(c_1\sqrt{3}+c_2)}{2c_1\cos\left(\frac{\sqrt{3}\,x}{2}\right)+2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)}dx} \\
&= \frac{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2 \mathrm{e}^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}}{\tan\left(\frac{\sqrt{3}\,x}{2}\right)c_2 + c_1}
\end{aligned}$$

The ode becomes

$$\frac{d}{dx}\mu y = 0$$

$$\frac{d}{dx}\left(\frac{y\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,e^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{\tan\left(\frac{\sqrt{3}\,x}{2}\right)c_2 + c_1}\right) = 0$$

Integrating gives

$$\begin{aligned}
\frac{y\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,e^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{\tan\left(\frac{\sqrt{3}\,x}{2}\right)c_2 + c_1} &= \int 0\,dx + c_3 \\
&= c_3
\end{aligned}$$

Dividing throughout by the integrating factor $\frac{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}\,e^{\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{\tan\left(\frac{\sqrt{3}\,x}{2}\right)c_2 + c_1}$ gives the final solution

$$y = \frac{\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)c_2 + c_1\right)e^{-\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}c_3}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)c_2 + c_1\right)e^{-\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}c_3}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

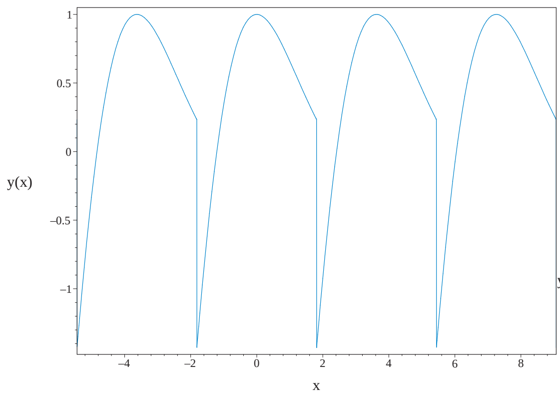
The constants can be merged to give

$$y = \frac{\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)c_2 + c_1\right)e^{-\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^2}}$$

Will add steps showing solving for IC soon.

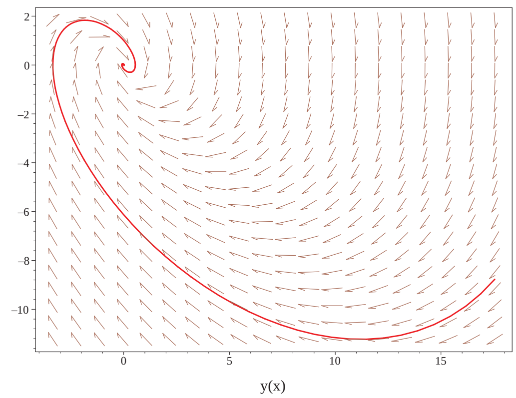
Summary of solutions found

$$y = \frac{\left(\frac{\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}+1\right)\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^{2}}}$$



(a) Solution plot

$$y = \frac{\left(\frac{\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}+1\right)\mathrm{e}^{-\frac{\sqrt{3}\,\arctan\left(\tan\left(\frac{\sqrt{3}\,x}{2}\right)\right)}{3}}}{\sqrt{\sec\left(\frac{\sqrt{3}\,x}{2}\right)^{2}}}$$



(b) Slope field plot
$y'' + y' + y = 0$

Maple step by step solution

Let's solve

$\left[\frac{d^2}{dx^2}y(x) + \frac{d}{dx}y(x) + y(x) = 0, \left(\frac{d}{dx}y(x)\right)\Big|_{\{x=0\}} = 0, y(0) = 1\right]$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Characteristic polynomial of ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, -\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)$

- 1st solution of the ODE

$$y_1(x) = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)$$

- 2nd solution of the ODE

$$y_2(x) = \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right)$$

- General solution of the ODE

$$y(x) = C1 y_1(x) + C2 y_2(x)$$

- Substitute in solutions

$$y(x) = C1\, \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + C2\, \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right)$$

☐ Check validity of solution $y(x) = \_C1 \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \_C2 \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right)$

- Use initial condition $y(0) = 1$

$$1 = \_C1$$

- Compute derivative of the solution

$$\frac{d}{dx} y(x) = -\frac{\_C1 \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)}{2} - \frac{\_C1 \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) \sqrt{3}}{2} - \frac{\_C2 \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{2} + \frac{\_C2 \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) \sqrt{3}}{2}$$

- Use the initial condition $\left(\frac{d}{dx} y(x)\right)\Big|_{\{x=0\}} = 0$

$$0 = -\frac{\_C1}{2} + \frac{\sqrt{3}\, \_C2}{2}$$

- Solve for $\_C1$ and $\_C2$

$$\left\{\_C1 = 1, \_C2 = \frac{\sqrt{3}}{3}\right\}$$

- Substitute constant values into general solution and simplify

$$y(x) = \frac{\left(\sin\left(\frac{\sqrt{3}\, x}{2}\right) \sqrt{3} + 3 \cos\left(\frac{\sqrt{3}\, x}{2}\right)\right) \mathrm{e}^{-\frac{x}{2}}}{3}$$

- Solution to the IVP

$$y(x) = \frac{\left(\sin\left(\frac{\sqrt{3}\, x}{2}\right) \sqrt{3} + 3 \cos\left(\frac{\sqrt{3}\, x}{2}\right)\right) \mathrm{e}^{-\frac{x}{2}}}{3}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

Maple dsolve solution

Solving time : 0.039 (sec)
Leaf size : 31

```
dsolve([diff(diff(y(x),x),x)+diff(y(x),x)+y(x) = 0,
       op([D(y)(0) = 0, y(0) = 1])],y(x),singsol=all)
```

$$y = \frac{e^{-\frac{x}{2}} \left(\sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)+3 \cos\left(\frac{\sqrt{3}\, x}{2}\right)\right)}{3}$$

Mathematica DSolve solution

Solving time : 0.023 (sec)
Leaf size : 47

```
DSolve[{D[y[x],{x,2}]+D[y[x],x]+y[x]==0,{Derivative[1][y][0] ==0,y[0]==1}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{3} e^{-x/2} \left(\sqrt{3} \sin\left(\frac{\sqrt{3} x}{2}\right)+3 \cos\left(\frac{\sqrt{3} x}{2}\right)\right)$$

2.1.91 problem 89

Internal problem ID [8479]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 89
Date solved : Wednesday, December 18, 2024 at 01:53:08 AM
CAS classification :
[[_2nd_order, _exact, _nonlinear], [_2nd_order, _reducible, _mu_x_y1], [_2nd_order, _r

Solve

$$y'' - y'y = 2x$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
-> trying 2nd order, dynamical_symmetries, fully reducible to Abel through one integra
   --- trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods for
   -> trying 2nd order, dynamical_symmetries, fully reducible to Abel through one inte
trying 2nd order, integrating factors of the form mu(x,y)/(y)^n, only the singular cas
trying symmetries linear in x and y(x)
trying differential order: 2; exact nonlinear
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = (1/2)*_b(_a)^2+_a^2-c__1, _b(_a)`
   Methods for first order ODEs:
   --- Trying classification methods ---
```

```
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   trying separable
   trying inverse linear
   trying homogeneous types:
   trying Chini
   differential order: 1; looking for linear symmetries
   trying exact
   Looking for potential symmetries
   trying Riccati
   trying Riccati Special
   trying Riccati sub-methods:
      trying Riccati to 2nd Order
      -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (-(1/2)*x^2+(1/2)*c__
         Methods for second order ODEs:
         --- Trying classification methods ---
         trying a quadrature
         checking if the LODE has constant coefficients
         checking if the LODE is of Euler type
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying a Liouvillian solution using Kovacics algorithm
         <- No Liouvillian solutions exists
         -> Trying a solution in terms of special functions:
            -> Bessel
            -> elliptic
            -> Legendre
            -> Whittaker
               -> hyper3: Equivalence to 1F1 under a power @ Moebius
               <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
            <- Whittaker successful
         <- special function solution successful
      <- Riccati to 2nd Order successful
<- differential order: 2; exact nonlinear successful`
```

Maple dsolve solution

Solving time : `0.093` (sec)
Leaf size : 147

```
dsolve(diff(diff(y(x),x),x)-y(x)*diff(y(x),x) = 2*x,
       y(x),singsol=all)
```

$$y = \frac{-\operatorname{WhittakerM}\left(\frac{ic_1\sqrt{2}}{8}+1, \frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right)(6+ic_1\sqrt{2}) + 8c_2 \operatorname{WhittakerW}\left(\frac{ic_1\sqrt{2}}{8}+1, \frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right) + 2(1-i(x^2}{2x\left(c_2 \operatorname{WhittakerW}\left(\frac{ic_1\sqrt{2}}{8}, \frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right) + \operatorname{Whittak}\right.}$$

Mathematica DSolve solution

Solving time : `45.186` (sec)
Leaf size : 318

```
DSolve[{D[y[x],{x,2}]+D[y[x],x]*y[x]==2*x,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{\sqrt[4]{2}\left(\sqrt[4]{2}x \operatorname{ParabolicCylinderD}\left(\frac{1}{4}\left(-\sqrt{2}c_1-2\right), i\sqrt[4]{2}x\right) + 2i \operatorname{ParabolicCylinderD}\left(\frac{1}{4}\left(2-\sqrt{2}c_1\right), i\sqrt[4]{2}\right.\right.}{\operatorname{ParabolicCylinderD}\left(\frac{1}{4}\left(-\sqrt{2}c_1-2\right), i\sqrt[4]{2}x\right.}$$

$$y(x) \to \sqrt{2}x - \frac{2\sqrt[4]{2} \operatorname{ParabolicCylinderD}\left(\frac{1}{4}\left(\sqrt{2}c_1+2\right), \sqrt[4]{2}x\right)}{\operatorname{ParabolicCylinderD}\left(\frac{1}{4}\left(\sqrt{2}c_1-2\right), \sqrt[4]{2}x\right)}$$

$$y(x) \to \sqrt{2}x - \frac{2\sqrt[4]{2} \operatorname{ParabolicCylinderD}\left(\frac{1}{4}\left(\sqrt{2}c_1+2\right), \sqrt[4]{2}x\right)}{\operatorname{ParabolicCylinderD}\left(\frac{1}{4}\left(\sqrt{2}c_1-2\right), \sqrt[4]{2}x\right)}$$

2.1.92 problem 90

Internal problem ID [8480]
Book : Own collection of miscellaneous problems
Section : section 1.0
Problem number : 90
Date solved : `Tuesday, December 17, 2024 at 12:55:46 PM`
CAS classification : `[_Riccati]`

Solve

$$y' - y^2 - x - x^2 = 0$$

Solved as first order ode of type Riccati

Time used: 0.494 (sec)

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= x^2 + y^2 + x \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^2 + y^2 + x$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x^2 + x$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= x^2 + x \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + (x^2 + x) u(x) = 0$$

Unable to solve. Will ask Maple to solve this ode now.

Solution obtained is

$$\begin{aligned} u(x) = 2\Bigg(& c_2\left(x+\frac{1}{2}\right) \mathrm{hypergeom}\left(\left[\frac{3}{4}-\frac{i}{16}\right], \left[\frac{3}{2}\right], \frac{i(2x+1)^2}{4}\right) \\ & + \frac{\mathrm{hypergeom}\left(\left[\frac{1}{4}-\frac{i}{16}\right], \left[\frac{1}{2}\right], \frac{i(2x+1)^2}{4}\right) c_1}{2}\Bigg) \mathrm{e}^{-\frac{ix(x+1)}{2}} \end{aligned}$$

Taking derivative gives

$$\begin{aligned} u'(x) = 2\Bigg(& \mathrm{hypergeom}\left(\left[\frac{3}{4}-\frac{i}{16}\right], \left[\frac{3}{2}\right], \frac{i(2x+1)^2}{4}\right) c_2 \\ & + \left(\frac{1}{24}+\frac{i}{2}\right) c_2\left(x+\frac{1}{2}\right) \mathrm{hypergeom}\left(\left[\frac{7}{4}-\frac{i}{16}\right], \left[\frac{5}{2}\right], \frac{i(2x+1)^2}{4}\right)(2x+1) \\ & + \left(\frac{1}{16}+\frac{i}{4}\right) \mathrm{hypergeom}\left(\left[\frac{5}{4}-\frac{i}{16}\right], \left[\frac{3}{2}\right], \frac{i(2x+1)^2}{4}\right)(2x+1) c_1\Bigg) \mathrm{e}^{-\frac{ix(x+1)}{2}} \\ & + 2\Bigg(c_2\left(x+\frac{1}{2}\right) \mathrm{hypergeom}\left(\left[\frac{3}{4}-\frac{i}{16}\right], \left[\frac{3}{2}\right], \frac{i(2x+1)^2}{4}\right) \\ & + \frac{\mathrm{hypergeom}\left(\left[\frac{1}{4}-\frac{i}{16}\right], \left[\frac{1}{2}\right], \frac{i(2x+1)^2}{4}\right) c_1}{2}\Bigg)\left(-\frac{i(x+1)}{2}-\frac{ix}{2}\right) \mathrm{e}^{-\frac{ix(x+1)}{2}} \end{aligned}$$

Doing change of constants, the solution becomes

$$y = -\frac{\left(2\left(\operatorname{hypergeom}\left(\left[\frac{3}{4}-\frac{i}{16}\right],\left[\frac{3}{2}\right],\frac{i(2x+1)^2}{4}\right)+\left(\frac{1}{24}+\frac{i}{2}\right)\left(x+\frac{1}{2}\right)\operatorname{hypergeom}\left(\left[\frac{7}{4}-\frac{i}{16}\right],\left[\frac{5}{2}\right],\frac{i(2x+1)^2}{4}\right)(2x\right.\right.}{}$$

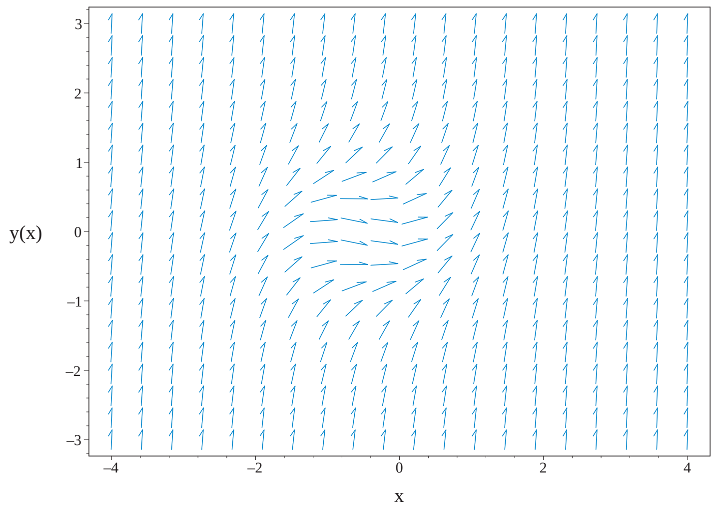


Figure 2.200: Slope field plot
$y' - y^2 - x - x^2 = 0$

Summary of solutions found

$$y = -\frac{\left(2\left(\operatorname{hypergeom}\left(\left[\frac{3}{4}-\frac{i}{16}\right],\left[\frac{3}{2}\right],\frac{i(2x+1)^2}{4}\right)+\left(\frac{1}{24}+\frac{i}{2}\right)\left(x+\frac{1}{2}\right)\operatorname{hypergeom}\left(\left[\frac{7}{4}-\frac{i}{16}\right],\left[\frac{5}{2}\right],\frac{i(2x+1)^2}{4}\right)(2x\right.\right.}{}$$

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) - y(x)^2 - x - x^2 = 0$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = y(x)^2 + x + x^2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = (-x^2-x)*y(x), y(x)`
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      checking if the LODE has constant coefficients
      checking if the LODE is of Euler type
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Trying a Liouvillian solution using Kovacics algorithm
      <- No Liouvillian solutions exists
      -> Trying a solution in terms of special functions:
```

```
         -> Bessel
         -> elliptic
         -> Legendre
         -> Whittaker
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
            <- hyper3 successful: indirect Equivalence to 0F1 under \`\`^ @ Moebius\`\
         <- hypergeometric successful
      <- special function solution successful
   <- Riccati to 2nd Order successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 128

```
dsolve(diff(y(x),x)-y(x)^2-x-x^2 = 0,
       y(x),singsol=all)
```

$$y = \frac{2c_1\left(ix^2+ix-1+\frac{1}{4}i\right)\operatorname{hypergeom}\left(\left[\frac{3}{4}-\frac{i}{16}\right],\left[\frac{3}{2}\right],\frac{i(2x+1)^2}{4}\right)+2\left(\left(-\frac{1}{12}-i\right)c_1\left(x+\frac{1}{2}\right)\operatorname{hypergeom}\left(\left[\frac{7}{4}-\right.\right.\right.}{(2x+1)\,c_1\operatorname{hypergeom}\left(\left[\frac{3}{4}-\frac{i}{16}\right],\left[\frac{3}{2}\right],\frac{i(}{}\right.}$$

Mathematica DSolve solution

Solving time : 0.282 (sec)
Leaf size : 298

```
DSolve[{D[y[x],x]-y[x]^2-x-x^2==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{i\left((2x+1)\,\text{ParabolicCylinderD}\left(-\frac{1}{2}-\frac{i}{8},\left(-\frac{1}{2}+\frac{i}{2}\right)(2x+1)\right)-c_1(2x+1)\,\text{ParabolicCylinderD}\left(-\frac{1}{2}+\right.\right.}{2\left(\text{ParabolicCylinderD}\left(-\frac{1}{2}-\frac{i}{8},\left(-\right.\right.\right.}$$

$$y(x) \to \frac{(1+i)\,\text{ParabolicCylinderD}\left(\frac{1}{2}+\frac{i}{8},(1+i)x+\left(\frac{1}{2}+\frac{i}{2}\right)\right)}{\text{ParabolicCylinderD}\left(-\frac{1}{2}+\frac{i}{8},(1+i)x+\left(\frac{1}{2}+\frac{i}{2}\right)\right)}-\frac{1}{2}i(2x+1)$$

$$y(x) \to \frac{(1+i)\,\text{ParabolicCylinderD}\left(\frac{1}{2}+\frac{i}{8},(1+i)x+\left(\frac{1}{2}+\frac{i}{2}\right)\right)}{\text{ParabolicCylinderD}\left(-\frac{1}{2}+\frac{i}{8},(1+i)x+\left(\frac{1}{2}+\frac{i}{2}\right)\right)}-\frac{1}{2}i(2x+1)$$

2.2 section 2.0

2.2.1 problem 1

Internal problem ID [8481]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 1
Date solved : `Thursday, December 12, 2024 at 09:22:42 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy - x = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.488 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^2 + 4x - 2}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^2 + 4x - 2 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.99: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 2\\ &= -2\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{1} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i\\ &= \frac{x}{2} + 1 \qquad (10)\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 4x - 2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (2 + x) \, \mathrm{e}^{\int \left(-1 - \frac{x}{2}\right) dx} \\ &= (2 + x) \, \mathrm{e}^{-x - \frac{1}{4}x^2} \\ &= (2 + x) \, \mathrm{e}^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-x}{1} \, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2 + x) \, \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-2 + \frac{(2+x)^2}{2}}}{2 + x} - \frac{i\sqrt{\pi} \, \mathrm{e}^{-2} \sqrt{2} \, \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' - xy' - xy = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1(2+x)\,\mathrm{e}^{-x} - \frac{c_2\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= (2+x)\,\mathrm{e}^{-x} \\
y_2 &= -\frac{\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} (2+x)\,\mathrm{e}^{-x} & -\frac{\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2} \\ \frac{d}{dx}\left((2+x)\,\mathrm{e}^{-x}\right) & \frac{d}{dx}\left(-\frac{\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}\right) \end{vmatrix}
$$

Which gives

$$
W = \begin{vmatrix} (2+x)\,\mathrm{e}^{-x} & -\frac{\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2} \\ \mathrm{e}^{-x}-(2+x)\,\mathrm{e}^{-x} & -\frac{\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2\,\mathrm{e}^{-2}(2+x)\mathrm{e}^{\frac{(2+x)^2}{2}}+2(2+x)\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2} + \frac{\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}}{}\right.\right.}{2} \end{vmatrix}
$$

Therefore

$$
\begin{aligned}
W = {} & ((2 \\
& +x)\,\mathrm{e}^{-x})\left(-\frac{\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2\,\mathrm{e}^{-2}(2+x)\,\mathrm{e}^{\frac{(2+x)^2}{2}}+2(2+x)\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}\right. \\
& \left.+\frac{\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}\right) \\
& -\left(-\frac{\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}\right)\left(\mathrm{e}^{-x}-(2+x)\,\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
W = {} & \mathrm{e}^{\frac{(2+x)^2}{2}}\mathrm{e}^{-2}\mathrm{e}^{-2x}x^2+4\,\mathrm{e}^{\frac{(2+x)^2}{2}}\mathrm{e}^{-2}\mathrm{e}^{-2x}x-\mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-2x}x^2 \\
& +4\,\mathrm{e}^{\frac{(2+x)^2}{2}}\mathrm{e}^{-2}\mathrm{e}^{-2x}-4\,\mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-2x}x-3\,\mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-2x}
\end{aligned}
$$

Which simplifies to

$$W = \mathrm{e}^{\frac{x^2}{2}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}x}{2}}{\mathrm{e}^{\frac{x^2}{2}}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{\mathrm{e}^{-\frac{x(2+x)}{2}}x\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}dx$$

Hence

$$\begin{aligned}u_1 = &-\mathrm{e}^{-\frac{x(2+x)}{2}}\mathrm{e}^{\frac{x(4+x)}{2}} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)\mathrm{e}^{-\frac{x(2+x)}{2}}}{2} - \mathrm{e}^{-2}\mathrm{e}^{-\frac{x(2+x)}{2}}\mathrm{e}^{\frac{(2+x)^2}{2}}x \\ &+ \mathrm{e}^{-\frac{x(2+x)}{2}}x\,\mathrm{e}^{\frac{x(4+x)}{2}} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)x\,\mathrm{e}^{-\frac{x(2+x)}{2}}}{2}\end{aligned}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(2+x)\,\mathrm{e}^{-x}x}{\mathrm{e}^{\frac{x^2}{2}}}\,dx$$

Which simplifies to

$$u_2 = \int x(2+x)\,\mathrm{e}^{-\frac{x(2+x)}{2}}dx$$

Hence

$$u_2 = -(x+1)\,\mathrm{e}^{-\frac{x(2+x)}{2}}$$

Therefore the particular solution, from equation (1) is

$$
\begin{aligned}
y_p(x) = {} & \left(-\mathrm{e}^{-\frac{x(2+x)}{2}} \mathrm{e}^{\frac{x(4+x)}{2}} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)\mathrm{e}^{-\frac{x(2+x)}{2}}}{2} - \mathrm{e}^{-2}\mathrm{e}^{-\frac{x(2+x)}{2}}\mathrm{e}^{\frac{(2+x)^2}{2}}x \right. \\
& \left. + \mathrm{e}^{-\frac{x(2+x)}{2}} x\,\mathrm{e}^{\frac{x(4+x)}{2}} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) x\,\mathrm{e}^{-\frac{x(2+x)}{2}}}{2}\right)(2+x)\,\mathrm{e}^{-x} \\
& + \frac{\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,(2+x)\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}(x+1)\,\mathrm{e}^{-\frac{x(2+x)}{2}}}{2}
\end{aligned}
$$

Which simplifies to

$$
y_p(x) = -1
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1(2+x)\,\mathrm{e}^{-x} - \frac{c_2\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,(2+x)\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}\right) + (-1)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = c_1(2+x)\,\mathrm{e}^{-x} - \frac{c_2\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,(2+x)\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2} - 1
$$

Solved as second order ode adjoint method

Time used: 2.339 (sec)

In normal form the ode

$$
y'' - xy' - xy - x = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}
$$

Where

$$\begin{aligned} p(x) &= -x \\ q(x) &= -x \\ r(x) &= x \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (-x\xi(x))' + (-x\xi(x)) &= 0 \\ \xi''(x) + x\xi'(x) - (x-1)\,\xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. Writing the ode as

$$\xi'' + x\xi' + (1-x)\,\xi = 0 \tag{1}$$
$$A\xi'' + B\xi' + C\xi = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 1-x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = \xi e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^2 + 4x - 2 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then ξ is found using the inverse transformation

$$
\xi = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.100: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 2 \\
&= -2
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2} + 1 \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2+4x-2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{1}{4}x^2 + x - \frac{1}{2}
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2} + 1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (2+x)\, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\
&= (2+x)\, \mathrm{e}^{-x-\frac{1}{4}x^2} \\
&= (2+x)\, \mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}
$$

The first solution to the original ode in ξ is found from

$$
\begin{aligned}
\xi_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{1} \, dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
\xi_1 = (2+x)\, \mathrm{e}^{-\frac{x(2+x)}{2}}
$$

The second solution ξ_2 to the original ode is found using reduction of order

$$
\xi_2 = \xi_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{\xi_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
\xi_2 &= \xi_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(\xi_1)^2} \, dx \\
&= \xi_1 \int \frac{e^{-\frac{x^2}{2}}}{(\xi_1)^2} \, dx \\
&= \xi_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\, \mathrm{e}^{-2} \sqrt{2}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
\xi &= c_1\xi_1 + c_2\xi_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-\frac{x(2+x)}{2}}\right) \\
&\quad + c_2\left((2+x)\,\mathrm{e}^{-\frac{x(2+x)}{2}}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$
\begin{aligned}
\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y &= \int \xi(x)\,r(x)\,dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi\,(x)}
\end{aligned}
$$

Or

$$
y' + y\left(-x - \frac{c_1\,\mathrm{e}^{-\frac{x(2+x)}{2}} + c_1(2+x)\,(-1-x)\,\mathrm{e}^{-\frac{x(2+x)}{2}} - \frac{c_2(-1-x)\mathrm{e}^{-\frac{x(2+x)}{2}}\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right.}{2}}{c_1\,(2+x)\,\mathrm{e}^{-\frac{x(2+x)}{2}} - \frac{c_2\,\mathrm{e}^{-\frac{x(2+x)}{2}}\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,(2+x)\,\mathrm{er}\right.}{2}}\right.
$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$
y' + q(x)y = p(x)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= -\frac{-i\sqrt{2}\,c_2(x+1)\,\mathrm{e}^{-2}\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 + 2c_1(x+1)}{i\sqrt{2}\,c_2\,\mathrm{e}^{-2}\,(2+x)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 - 2c_1\,(2+x)} \\
p(x) &= \frac{-i\sqrt{2}\,c_2(x+1)\,\mathrm{e}^{-2}\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 + 2c_1(x+1)}{i\sqrt{2}\,c_2\,\mathrm{e}^{-2}\,(2+x)\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 - 2c_1\,(2+x)}
\end{aligned}
$$

The integrating factor μ is

$$
\mu = \mathrm{e}^{\int -\frac{-i\sqrt{2}\,c_2(x+1)\mathrm{e}^{-2}\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2+2c_1(x+1)}{i\sqrt{2}\,c_2\,\mathrm{e}^{-2}(2+x)\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2-2c_1(2+x)}dx}
$$

Therefore the solution is

$$y = \left(\int \frac{\left(-i\sqrt{2}\, c_2(x+1)\,\mathrm{e}^{-2}\sqrt{\pi}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2+2c_1(x+1)\right)\mathrm{e}^{\int -\frac{-i\sqrt{2}\, c_2(x+1)\mathrm{e}^{-2}\sqrt{\pi}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{i\sqrt{2}\, c_2\, \mathrm{e}^{-2}(2+x)\sqrt{\pi}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}}}{i\sqrt{2}\, c_2\, \mathrm{e}^{-2}\,(2+x)\,\sqrt{\pi}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2-2c_1\,(2+x)}\right.$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \left(\int \frac{\left(-i\sqrt{2}\, c_2(x+1)\,\mathrm{e}^{-2}\sqrt{\pi}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2+2c_1(x+1)\right)\mathrm{e}^{\int -\frac{-i\sqrt{2}\, c_2(x+1)\mathrm{e}^{-2}\sqrt{\pi}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2}{i\sqrt{2}\, c_2\, \mathrm{e}^{-2}(2+x)\sqrt{\pi}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2}}}{i\sqrt{2}\, c_2\, \mathrm{e}^{-2}\,(2+x)\,\sqrt{\pi}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2-2c_1\,(2+x)} \right.$$
$$\left. + c_3\right)\mathrm{e}^{-\int -\frac{-i\sqrt{2}\, c_2(x+1)\mathrm{e}^{-2}\sqrt{\pi}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2+2c_1(x+1)}{i\sqrt{2}\, c_2\, \mathrm{e}^{-2}(2+x)\sqrt{\pi}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2-2c_1(2+x)}dx}$$

The constants can be merged to give

$$y = \left(\int \frac{\left(-i\sqrt{2}\, c_2(x+1)\,\mathrm{e}^{-2}\sqrt{\pi}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2+2c_1(x+1)\right)\mathrm{e}^{\int -\frac{-i\sqrt{2}\, c_2(x+1)\mathrm{e}^{-2}\sqrt{\pi}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2}{i\sqrt{2}\, c_2\, \mathrm{e}^{-2}(2+x)\sqrt{\pi}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2}}}{i\sqrt{2}\, c_2\, \mathrm{e}^{-2}\,(2+x)\,\sqrt{\pi}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2-2c_1\,(2+x)} \right.$$
$$\left. + 1\right)\mathrm{e}^{-\int -\frac{-i\sqrt{2}\, c_2(x+1)\mathrm{e}^{-2}\sqrt{\pi}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2+2c_1(x+1)}{i\sqrt{2}\, c_2\, \mathrm{e}^{-2}(2+x)\sqrt{\pi}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2-2c_1(2+x)}dx}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = \left(\int \frac{\left(-i\sqrt{2}\, c_2 (x+1)\, \mathrm{e}^{-2}\sqrt{\pi}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2 + 2c_1(x+1)\right) \mathrm{e}^{\int -\frac{-i\sqrt{2}\, c_2 (x+1)\mathrm{e}^{-2}\sqrt{\pi}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2 + 2c_1(x+1)}{i\sqrt{2}\, c_2\, \mathrm{e}^{-2}(2+x)\sqrt{\pi}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2 - 2c_1(2+x)} dx}}{i\sqrt{2}\, c_2\, \mathrm{e}^{-2}\,(2+x)\sqrt{\pi}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2 - 2c_1\,(2+x)} dx + 1\right) \mathrm{e}^{-\int -\frac{-i\sqrt{2}\, c_2 (x+1)\mathrm{e}^{-2}\sqrt{\pi}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2 + 2c_1(x+1)}{i\sqrt{2}\, c_2\, \mathrm{e}^{-2}(2+x)\sqrt{\pi}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2 - 2c_1(2+x)} dx}
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : `0.011` (sec)
Leaf size : 52

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x)-x = 0,
       y(x),singsol=all)
```

$$y = ic_1\sqrt{\pi}\,(x+2)\,\sqrt{2}\,\mathrm{e}^{-2-x}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)+2\,\mathrm{e}^{\frac{x(x+2)}{2}}c_1-1+\mathrm{e}^{-x}(x+2)\,c_2$$

Mathematica DSolve solution

Solving time : `3.196` (sec)
Leaf size : 216

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]-x==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$\begin{aligned} y(x) \to \frac{1}{2}e^{-\frac{1}{2}(x+2)^2}\Bigg(2\sqrt{2}e^{\frac{x^2}{2}+x+2}(x+2)\int_1^x\Bigg(\frac{e^{K[1]}K[1]}{\sqrt{2}} \\ -\frac{1}{2}e^{-\frac{1}{2}K[1]^2-K[1]-2}\sqrt{\pi}\mathrm{erfi}\left(\frac{\sqrt{(K[1]+2)^2}}{\sqrt{2}}\right)K[1]\sqrt{(K[1]+2)^2}\Bigg)dK[1] \\ -\sqrt{2\pi}\sqrt{(x+2)^2}\left(c_2e^{\frac{x^2}{2}+x+2}+x+1\right)\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) \\ +2e^{\frac{x^2}{2}+x+2}\left(e^x(x+1)+\sqrt{2}c_1(x+2)+c_2e^{\frac{1}{2}(x+2)^2}\right)\Bigg) \end{aligned}$$

2.2.2 problem 2

Internal problem ID [8482]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 2
Date solved : `Thursday, December 12, 2024 at 09:22:46 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy - 2x = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.475 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.101: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 4x - 2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (2 + x)\, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\ &= (2 + x)\, \mathrm{e}^{-x-\frac{1}{4}x^2} \\ &= (2 + x)\, \mathrm{e}^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1} \, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2 + x)\, \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2 + x} - \frac{i\sqrt{\pi}\, \mathrm{e}^{-2}\sqrt{2}\, \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' - xy' - xy = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1(2+x)\,\mathrm{e}^{-x} - \frac{c_2\left(i\sqrt{\pi}\,\sqrt{2}\,(2+x)\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= (2+x)\,\mathrm{e}^{-x} \\
y_2 &= -\frac{\left(i\sqrt{\pi}\,\sqrt{2}\,(2+x)\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (2+x)\,\mathrm{e}^{-x} & -\frac{\left(i\sqrt{\pi}\,\sqrt{2}\,(2+x)\mathrm{e}^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2} \\ \frac{d}{dx}\left((2+x)\,\mathrm{e}^{-x}\right) & \frac{d}{dx}\left(-\frac{\left(i\sqrt{\pi}\,\sqrt{2}\,(2+x)\mathrm{e}^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (2+x)\,\mathrm{e}^{-x} & -\frac{\left(i\sqrt{\pi}\,\sqrt{2}\,(2+x)\mathrm{e}^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2} \\ \mathrm{e}^{-x}-(2+x)\,\mathrm{e}^{-x} & -\frac{\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2(2+x)\mathrm{e}^{-2}\mathrm{e}^{\frac{(2+x)^2}{2}}+2(2+x)\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}+\frac{\left(i\sqrt{\pi}\,\sqrt{2}\,(2+x)\mathrm{e}^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2} \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & \left((2+x)\,\mathrm{e}^{-x}\right)\left(-\frac{\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2(2+x)\,\mathrm{e}^{-2}\mathrm{e}^{\frac{(2+x)^2}{2}}+2(2+x)\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}\right. \\ & \left.+\frac{\left(i\sqrt{\pi}\,\sqrt{2}\,(2+x)\,\mathrm{e}^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}\right) \\ & -\left(-\frac{\left(i\sqrt{\pi}\,\sqrt{2}\,(2+x)\,\mathrm{e}^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}\right)\left(\mathrm{e}^{-x}-(2+x)\,\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$\begin{aligned} W = {} & \mathrm{e}^{\frac{(2+x)^2}{2}}\mathrm{e}^{-2}\mathrm{e}^{-2x}x^2+4\,\mathrm{e}^{\frac{(2+x)^2}{2}}\mathrm{e}^{-2}\mathrm{e}^{-2x}x-\mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-2x}x^2 \\ & +4\,\mathrm{e}^{\frac{(2+x)^2}{2}}\mathrm{e}^{-2}\mathrm{e}^{-2x}-4\,\mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-2x}x-3\,\mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-2x} \end{aligned}$$

Which simplifies to

$$W = \mathrm{e}^{\frac{x^2}{2}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\left(i\sqrt{\pi}\,\sqrt{2}\,(2+x)\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}x}{\mathrm{e}^{\frac{x^2}{2}}}\,dx$$

Which simplifies to

$$u_1 = -\int -x\,\mathrm{e}^{-\frac{x(2+x)}{2}}\left(i\sqrt{\pi}\,\sqrt{2}\,(2+x)\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)dx$$

Hence

$$\begin{aligned} u_1 = {} & -i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)\mathrm{e}^{-\frac{x(2+x)}{2}} - 2\,\mathrm{e}^{-\frac{x(2+x)}{2}}\mathrm{e}^{\frac{x(4+x)}{2}} - 2\,\mathrm{e}^{-2}\mathrm{e}^{-\frac{x(2+x)}{2}}\mathrm{e}^{\frac{(2+x)^2}{2}}x \\ & + 2x\,\mathrm{e}^{-\frac{x(2+x)}{2}}\mathrm{e}^{\frac{x(4+x)}{2}} - i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)x\,\mathrm{e}^{-\frac{x(2+x)}{2}} \end{aligned}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2(2+x)\,\mathrm{e}^{-x}x}{\mathrm{e}^{\frac{x^2}{2}}}\,dx$$

Which simplifies to

$$u_2 = \int 2x(2+x)\,\mathrm{e}^{-\frac{x(2+x)}{2}}\,dx$$

Hence

$$u_2 = -2(x+1)\,\mathrm{e}^{-\frac{x(2+x)}{2}}$$

Therefore the particular solution, from equation (1) is

$$
\begin{aligned}
y_p(x) = \Bigg(&-i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)\mathrm{e}^{-\frac{x(2+x)}{2}} - 2\,\mathrm{e}^{-\frac{x(2+x)}{2}}\mathrm{e}^{\frac{x(4+x)}{2}} \\
&- 2\,\mathrm{e}^{-2}\mathrm{e}^{-\frac{x(2+x)}{2}}\mathrm{e}^{\frac{(2+x)^2}{2}}x + 2x\,\mathrm{e}^{-\frac{x(2+x)}{2}}\mathrm{e}^{\frac{x(4+x)}{2}} \\
&- i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)x\,\mathrm{e}^{-\frac{x(2+x)}{2}}\Bigg)(2+x)\,\mathrm{e}^{-x} \\
&+ \left(i\sqrt{\pi}\,\sqrt{2}\,(2+x)\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}(x+1)\,\mathrm{e}^{-\frac{x(2+x)}{2}}
\end{aligned}
$$

Which simplifies to

$$
y_p(x) = -2
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1(2+x)\,\mathrm{e}^{-x} - \frac{c_2\left(i\sqrt{\pi}\,\sqrt{2}\,(2+x)\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}\right) + (-2)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = c_1(2+x)\,\mathrm{e}^{-x} - \frac{c_2\left(i\sqrt{\pi}\,\sqrt{2}\,(2+x)\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2} - 2
$$

Solved as second order ode adjoint method

Time used: 2.351 (sec)

In normal form the ode

$$
y'' - xy' - xy - 2x = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}
$$

Where

$$\begin{aligned} p(x) &= -x \\ q(x) &= -x \\ r(x) &= 2x \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (-x\xi(x))' + (-x\xi(x)) &= 0 \\ \xi''(x) + x\xi'(x) - (x-1)\,\xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. Writing the ode as

$$\xi'' + x\xi' + (1-x)\,\xi = 0 \tag{1}$$
$$A\xi'' + B\xi' + C\xi = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= x \\ C &= 1-x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = \xi e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = x^2 + 4x - 2$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then ξ is found using the inverse transformation

$$\xi = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.102: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2+4x-2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$
\begin{aligned}
d &= \alpha_\infty^- \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= (-)[\sqrt{r}]_\infty \\
&= 0 + (-)\left(\frac{x}{2}+1\right) \\
&= -1 - \frac{x}{2} \\
&= -1 - \frac{x}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\
-2 + a_0 &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 2\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = 2 + x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (2+x)\, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\
&= (2+x)\, \mathrm{e}^{-x-\frac{1}{4}x^2} \\
&= (2+x)\, \mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}
$$

The first solution to the original ode in ξ is found from

$$
\begin{aligned}
\xi_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{1} \, dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
\xi_1 = (2+x)\, \mathrm{e}^{-\frac{x(2+x)}{2}}
$$

The second solution ξ_2 to the original ode is found using reduction of order

$$
\xi_2 = \xi_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{\xi_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
\xi_2 &= \xi_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(\xi_1)^2} \, dx \\
&= \xi_1 \int \frac{e^{-\frac{x^2}{2}}}{(\xi_1)^2} \, dx \\
&= \xi_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\, \mathrm{e}^{-2}\sqrt{2}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
\xi &= c_1\xi_1 + c_2\xi_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-\frac{x(2+x)}{2}}\right) \\
&\quad + c_2\left((2+x)\,\mathrm{e}^{-\frac{x(2+x)}{2}}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$
\begin{aligned}
\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y &= \int \xi(x)\,r(x)\,dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi(x)}
\end{aligned}
$$

Or

$$
y' + y\left(-x - \frac{c_1\,\mathrm{e}^{-\frac{x(2+x)}{2}} + c_1(2+x)\,(-1-x)\,\mathrm{e}^{-\frac{x(2+x)}{2}} - \frac{c_2(-1-x)\mathrm{e}^{-\frac{x(2+x)}{2}}\left(i\sqrt{\pi}\,\sqrt{2}\,(2+x)\mathrm{e}^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right.}{2}}{c_1\,(2+x)\,\mathrm{e}^{-\frac{x(2+x)}{2}} - \frac{c_2\,\mathrm{e}^{-\frac{x(2+x)}{2}}\left(i\sqrt{\pi}\,\sqrt{2}\,(2+x)\mathrm{e}^{-2}\,\mathrm{er}\right.}{2}}\right.
$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$
y' + q(x)y = p(x)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= -\frac{-i\sqrt{2}\,\sqrt{\pi}\,(x+1)\,c_2\,\mathrm{e}^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 + 2c_1(x+1)}{i\sqrt{2}\,(2+x)\,\sqrt{\pi}\,c_2\,\mathrm{e}^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 - 2c_1\,(2+x)} \\
p(x) &= \frac{-2i\sqrt{2}\,\sqrt{\pi}\,(x+1)\,c_2\,\mathrm{e}^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 4\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 + 4c_1(x+1)}{i\sqrt{2}\,(2+x)\,\sqrt{\pi}\,c_2\,\mathrm{e}^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 - 2c_1\,(2+x)}
\end{aligned}
$$

The integrating factor μ is

$$
\mu = \mathrm{e}^{\int -\frac{-i\sqrt{2}\,\sqrt{\pi}\,(x+1)c_2\,\mathrm{e}^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2+2c_1(x+1)}{i\sqrt{2}\,(2+x)\sqrt{\pi}\,c_2\,\mathrm{e}^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2-2c_1(2+x)}dx}
$$

Therefore the solution is

$$y = \left(\int \frac{\left(-2i\sqrt{2}\,\sqrt{\pi}\,(x+1)\,c_2\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-4\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2+4c_1(x+1)\right)\mathrm{e}^{\int -\frac{-i\sqrt{2}\,\sqrt{\pi}\,(x+1)c_2\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2+2c_1(x+1)}{i\sqrt{2}\,(2+x)\sqrt{\pi}\,c_2\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2-2c_1(2+x)}dx}}{i\sqrt{2}\,(2+x)\,\sqrt{\pi}\,c_2\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2-2c_1\,(2+x)}dx\right)$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \left(\int \frac{\left(-2i\sqrt{2}\,\sqrt{\pi}\,(x+1)\,c_2\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-4\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2+4c_1(x+1)\right)\mathrm{e}^{\int -\frac{-i\sqrt{2}\,\sqrt{\pi}\,(x+1)c_2\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2+2c_1(x+1)}{i\sqrt{2}\,(2+x)\sqrt{\pi}\,c_2\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2-2c_1(2+x)}dx}}{i\sqrt{2}\,(2+x)\,\sqrt{\pi}\,c_2\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2-2c_1\,(2+x)}dx + c_3\right)\mathrm{e}^{-\int -\frac{-i\sqrt{2}\,\sqrt{\pi}\,(x+1)c_2\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2+2c_1(x+1)}{i\sqrt{2}\,(2+x)\sqrt{\pi}\,c_2\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2-2c_1(2+x)}dx}$$

The constants can be merged to give

$$y = \left(\int \frac{\left(-2i\sqrt{2}\,\sqrt{\pi}\,(x+1)\,c_2\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-4\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2+4c_1(x+1)\right)\mathrm{e}^{\int -\frac{-i\sqrt{2}\,\sqrt{\pi}\,(x+1)c_2\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2+2c_1(x+1)}{i\sqrt{2}\,(2+x)\sqrt{\pi}\,c_2\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2-2c_1(2+x)}dx}}{i\sqrt{2}\,(2+x)\,\sqrt{\pi}\,c_2\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2-2c_1\,(2+x)}dx + 1\right)\mathrm{e}^{-\int -\frac{-i\sqrt{2}\,\sqrt{\pi}\,(x+1)c_2\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2+2c_1(x+1)}{i\sqrt{2}\,(2+x)\sqrt{\pi}\,c_2\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2-2c_1(2+x)}dx}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = \left(\int \frac{\left(-2i\sqrt{2}\,\sqrt{\pi}\,(x+1)\,c_2\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-4\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2+4c_1(x+1)\right)\mathrm{e}^{\int -\frac{-i\sqrt{2}\,\sqrt{\pi}\,(x+1)c_2\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2+2c_1(x+1)}{i\sqrt{2}\,(2+x)\sqrt{\pi}\,c_2\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2-2c_1(2+x)}dx}}{i\sqrt{2}\,(2+x)\,\sqrt{\pi}\,c_2\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2-2c_1\,(2+x)}dx + 1\right)\mathrm{e}^{-\int -\frac{-i\sqrt{2}\,\sqrt{\pi}\,(x+1)c_2\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2+2c_1(x+1)}{i\sqrt{2}\,(2+x)\sqrt{\pi}\,c_2\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2-2c_1(2+x)}dx}
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : `0.013` (sec)
Leaf size : 52

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x)-2*x = 0,
       y(x),singsol=all)
```

$$y = ic_1\sqrt{\pi}\,(x+2)\,\sqrt{2}\,\mathrm{e}^{-2-x}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)+2\,\mathrm{e}^{\frac{x(x+2)}{2}}c_1-2+\mathrm{e}^{-x}(x+2)\,c_2$$

Mathematica DSolve solution

Solving time : `1.258` (sec)
Leaf size : 217

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]-2*x==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$\begin{aligned} y(x) \to \frac{1}{2}e^{-\frac{1}{2}(x+2)^2}\Bigg(2\sqrt{2}e^{\frac{x^2}{2}+x+2}(x+2)\int_1^x\Bigg(\sqrt{2}e^{K[1]}K[1] \\ - e^{-\frac{1}{2}K[1]^2-K[1]-2}\sqrt{\pi}\mathrm{erfi}\left(\frac{\sqrt{(K[1]+2)^2}}{\sqrt{2}}\right)K[1]\sqrt{(K[1]+2)^2}\Bigg)dK[1] \\ - \sqrt{2\pi}\sqrt{(x+2)^2}\left(c_2e^{\frac{x^2}{2}+x+2}+2x+2\right)\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) \\ + 2e^{\frac{x^2}{2}+x+2}\left(2e^x(x+1)+\sqrt{2}c_1(x+2)+c_2e^{\frac{1}{2}(x+2)^2}\right)\Bigg) \end{aligned}$$

2.2.3 problem 3

Internal problem ID [8483]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 3
Date solved : Thursday, December 12, 2024 at 09:22:49 AM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$y'' - xy' - xy - 3x = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.483 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.103: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 4x - 2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1-\frac{x}{2} \\ &= -1-\frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega \, dx} \\
&= (2 + x)\,\mathrm{e}^{\int \left(-1-\frac{x}{2}\right)dx} \\
&= (2 + x)\,\mathrm{e}^{-x-\frac{1}{4}x^2} \\
&= (2 + x)\,\mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = (2 + x)\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2 + x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - xy' - xy = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1(2+x)\,\mathrm{e}^{-x} - \frac{c_2\left(i\mathrm{e}^{-2}(2+x)\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= (2+x)\,\mathrm{e}^{-x} \\
y_2 &= -\frac{\left(i\mathrm{e}^{-2}(2+x)\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (2+x)\,\mathrm{e}^{-x} & -\frac{\left(i\mathrm{e}^{-2}(2+x)\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2} \\ \frac{d}{dx}\left((2+x)\,\mathrm{e}^{-x}\right) & \frac{d}{dx}\left(-\frac{\left(i\mathrm{e}^{-2}(2+x)\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (2+x)\,\mathrm{e}^{-x} & -\frac{\left(i\mathrm{e}^{-2}(2+x)\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2} \\ \mathrm{e}^{-x}-(2+x)\,\mathrm{e}^{-x} & -\frac{\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2\,\mathrm{e}^{-2}(2+x)\mathrm{e}^{\frac{(2+x)^2}{2}}+2(2+x)\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}+\frac{\left(i\mathrm{e}^{-2}(2+x)\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\right.\right.}{2} \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & \left((2+x)\,\mathrm{e}^{-x}\right)\left(-\frac{\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2\,\mathrm{e}^{-2}(2+x)\,\mathrm{e}^{\frac{(2+x)^2}{2}}+2(2+x)\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}\right. \\ & \left.+\frac{\left(i\mathrm{e}^{-2}(2+x)\,\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}\right) \\ & -\left(-\frac{\left(i\mathrm{e}^{-2}(2+x)\,\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}\right)\left(\mathrm{e}^{-x}-(2+x)\,\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$\begin{aligned} W = {} & \mathrm{e}^{\frac{(2+x)^2}{2}}\mathrm{e}^{-2}\mathrm{e}^{-2x}x^2+4\,\mathrm{e}^{\frac{(2+x)^2}{2}}\mathrm{e}^{-2}\mathrm{e}^{-2x}x-\mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-2x}x^2 \\ & +4\,\mathrm{e}^{\frac{(2+x)^2}{2}}\mathrm{e}^{-2}\mathrm{e}^{-2x}-4\,\mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-2x}x-3\,\mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-2x} \end{aligned}$$

Which simplifies to

$$W = \mathrm{e}^{\frac{x^2}{2}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{3\left(i\mathrm{e}^{-2}(2+x)\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}x}{2}}{\mathrm{e}^{\frac{x^2}{2}}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{3x\,\mathrm{e}^{-\frac{x(2+x)}{2}}\left(i\mathrm{e}^{-2}(2+x)\,\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}dx$$

Hence

$$\begin{aligned} u_1 = &-3\,\mathrm{e}^{-\frac{x(2+x)}{2}}\mathrm{e}^{\frac{x(4+x)}{2}} - \frac{3i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)\mathrm{e}^{-\frac{x(2+x)}{2}}}{2} - 3\,\mathrm{e}^{-2}\mathrm{e}^{-\frac{x(2+x)}{2}}\mathrm{e}^{\frac{(2+x)^2}{2}}x \\ &+ 3x\,\mathrm{e}^{-\frac{x(2+x)}{2}}\mathrm{e}^{\frac{x(4+x)}{2}} - \frac{3i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)x\,\mathrm{e}^{-\frac{x(2+x)}{2}}}{2} \end{aligned}$$

And Eq. (3) becomes

$$u_2 = \int \frac{3(2+x)\,\mathrm{e}^{-x}x}{\mathrm{e}^{\frac{x^2}{2}}}\,dx$$

Which simplifies to

$$u_2 = \int 3x(2+x)\,\mathrm{e}^{-\frac{x(2+x)}{2}}dx$$

Hence

$$u_2 = -3(x+1)\,\mathrm{e}^{-\frac{x(2+x)}{2}}$$

Therefore the particular solution, from equation (1) is

$$
\begin{aligned}
y_p(x) = & \left(-3\,\mathrm{e}^{-\frac{x(2+x)}{2}}\,\mathrm{e}^{\frac{x(4+x)}{2}} - \frac{3i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)\mathrm{e}^{-\frac{x(2+x)}{2}}}{2} - 3\,\mathrm{e}^{-2}\mathrm{e}^{-\frac{x(2+x)}{2}}\,\mathrm{e}^{\frac{(2+x)^2}{2}}\,x \right. \\
& \left. + 3x\,\mathrm{e}^{-\frac{x(2+x)}{2}}\,\mathrm{e}^{\frac{x(4+x)}{2}} - \frac{3i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)x\,\mathrm{e}^{-\frac{x(2+x)}{2}}}{2}\right)(2+x)\,\mathrm{e}^{-x} \\
& + \frac{3\left(i\mathrm{e}^{-2}(2+x)\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}(x+1)\,\mathrm{e}^{-\frac{x(2+x)}{2}}}{2}
\end{aligned}
$$

Which simplifies to

$$
y_p(x) = -3
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1(2+x)\,\mathrm{e}^{-x} - \frac{c_2\left(i\mathrm{e}^{-2}(2+x)\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}\right) + (-3)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = c_1(2+x)\,\mathrm{e}^{-x} - \frac{c_2\left(i\mathrm{e}^{-2}(2+x)\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2} - 3
$$

Solved as second order ode adjoint method

Time used: 2.349 (sec)

In normal form the ode

$$
y'' - xy' - xy - 3x = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -x \\
q(x) &= -x \\
r(x) &= 3x
\end{aligned}
$$

The Lagrange adjoint ode is given by

$$
\begin{aligned}
\xi^{''} - (\xi\, p)' + \xi q &= 0 \\
\xi^{''} - (-x\xi(x))' + (-x\xi(x)) &= 0 \\
\xi''(x) + x\xi'(x) - (x-1)\,\xi(x) &= 0
\end{aligned}
$$

Which is solved for $\xi(x)$. Writing the ode as

$$
\xi'' + x\xi' + (1-x)\,\xi = 0 \tag{1}
$$

$$
A\xi'' + B\xi' + C\xi = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= x \\
C &= 1-x
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = \xi e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^2 + 4x - 2}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^2 + 4x - 2 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then ξ is found using the inverse transformation

$$
\xi = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.104: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 2 \\
&= -2
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2} + 1 \qquad (10)
\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2+4x-2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$
\begin{aligned}
d &= \alpha_\infty^- \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= (-)[\sqrt{r}]_\infty \\
&= 0 + (-)\left(\frac{x}{2} + 1\right) \\
&= -1 - \frac{x}{2} \\
&= -1 - \frac{x}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\
-2 + a_0 &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 2\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = 2 + x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (2+x)\,\mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\
&= (2+x)\,\mathrm{e}^{-x-\frac{1}{4}x^2} \\
&= (2+x)\,\mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}
$$

The first solution to the original ode in ξ is found from

$$
\begin{aligned}
\xi_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{1} \, dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
\xi_1 = (2+x)\,\mathrm{e}^{-\frac{x(2+x)}{2}}
$$

The second solution ξ_2 to the original ode is found using reduction of order

$$
\xi_2 = \xi_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{\xi_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
\xi_2 &= \xi_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(\xi_1)^2} \, dx \\
&= \xi_1 \int \frac{e^{-\frac{x^2}{2}}}{(\xi_1)^2} \, dx \\
&= \xi_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
\xi &= c_1\xi_1 + c_2\xi_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-\frac{x(2+x)}{2}}\right) \\
&\quad + c_2\left((2+x)\,\mathrm{e}^{-\frac{x(2+x)}{2}}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$
\begin{aligned}
\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y &= \int \xi(x)\,r(x)\,dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi\,(x)}
\end{aligned}
$$

Or

$$
y' + y\left(-x - \frac{c_1\,\mathrm{e}^{-\frac{x(2+x)}{2}} + c_1(2+x)\,(-1-x)\,\mathrm{e}^{-\frac{x(2+x)}{2}} - \frac{c_2\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 2\,\mathrm{e}^{-2}(2+x)\mathrm{e}^{\frac{(2+x)^2}{2}} + 2(2+x)\mathrm{e}^{\frac{x(4}{}}}{2}}{c_1\,(2+x)\,\mathrm{e}^{-\frac{x(2+x)}{2}} - \frac{c_2\left(i\mathrm{e}^{-2}(2+x)\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+}{2}\right.\right.}{2}}\right.
$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$
y' + q(x)y = p(x)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= -\frac{-ic_2(x+1)\,\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 + 2c_1(x+1)}{ic_2\sqrt{\pi}\,\,(2+x)\,\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 - 2c_1\,(2+x)} \\
p(x) &= \frac{-3ic_2(x+1)\,\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 6\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 + 6c_1(x+1)}{ic_2\sqrt{\pi}\,\,(2+x)\,\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 - 2c_1\,(2+x)}
\end{aligned}
$$

The integrating factor μ is

$$
\mu = \mathrm{e}^{\int -\frac{-ic_2(x+1)\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 + 2c_1(x+1)}{ic_2\sqrt{\pi}\,(2+x)\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 - 2c_1(2+x)}dx}
$$

Therefore the solution is

$$y = \left(\int \frac{\left(-3ic_2(x+1)\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 6\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2 + 6c_1(x+1)\right)\mathrm{e}^{\int -\frac{-ic_2(x+1)\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) \cdots}{ic_2\sqrt{\pi}\,(2+x)\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) \cdots}}}{ic_2\sqrt{\pi}\,(2+x)\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2 - 2c_1\,(2+x)} \cdots \right.$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \left(\int \frac{\left(-3ic_2(x+1)\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 6\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2 + 6c_1(x+1)\right)\mathrm{e}^{\int -\frac{-ic_2(x+1)\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - \cdots}{ic_2\sqrt{\pi}\,(2+x)\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + \cdots}}}{ic_2\sqrt{\pi}\,(2+x)\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2 - 2c_1\,(2+x)} \cdots + c_3 \right) \mathrm{e}^{-\int -\frac{-ic_2(x+1)\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2 + 2c_1(x+1)}{ic_2\sqrt{\pi}\,(2+x)\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2 - 2c_1(2+x)}dx}$$

The constants can be merged to give

$$y = \left(\int \frac{\left(-3ic_2(x+1)\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 6\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2 + 6c_1(x+1)\right)\mathrm{e}^{\int -\frac{-ic_2(x+1)\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - \cdots}{ic_2\sqrt{\pi}\,(2+x)\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + \cdots}}}{ic_2\sqrt{\pi}\,(2+x)\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2 - 2c_1\,(2+x)} \cdots + 1 \right) \mathrm{e}^{-\int -\frac{-ic_2(x+1)\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2 + 2c_1(x+1)}{ic_2\sqrt{\pi}\,(2+x)\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2 - 2c_1(2+x)}dx}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = \left(\int \frac{\left(-3ic_2(x+1)\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-6\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2+6c_1(x+1)\right)\mathrm{e}^{\int -\frac{-ic_2(x+1)\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-\cdots}{ic_2\sqrt{\pi}\,(2+x)\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+\cdots}}}{ic_2\sqrt{\pi}\,(2+x)\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2-2c_1\,(2+x)\cdots}\cdots + 1\right)\mathrm{e}^{-\int -\frac{-ic_2(x+1)\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2+2c_1(x+1)}{ic_2\sqrt{\pi}\,(2+x)\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2-2c_1(2+x)}dx}
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 52

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x)-3*x = 0,
       y(x),singsol=all)
```

$$y = ic_1\sqrt{\pi}\,(x+2)\,\sqrt{2}\,\mathrm{e}^{-2-x}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)+2\,\mathrm{e}^{\frac{x(x+2)}{2}}c_1-3+\mathrm{e}^{-x}(x+2)\,c_2$$

Mathematica DSolve solution

Solving time : 1.532 (sec)
Leaf size : 220

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]-3*x==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$\begin{aligned} y(x) \to \frac{1}{2}e^{-\frac{1}{2}(x+2)^2}\Bigg(2\sqrt{2}e^{\frac{x^2}{2}+x+2}(x+2)\int_1^x\Bigg(\frac{3e^{K[1]}K[1]}{\sqrt{2}} \\ -\frac{3}{2}e^{-\frac{1}{2}K[1]^2-K[1]-2}\sqrt{\pi}\mathrm{erfi}\left(\frac{\sqrt{(K[1]+2)^2}}{\sqrt{2}}\right)K[1]\sqrt{(K[1]+2)^2}\Bigg)dK[1] \\ -\sqrt{2\pi}\sqrt{(x+2)^2}\left(c_2e^{\frac{x^2}{2}+x+2}+3x+3\right)\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) \\ +2e^{\frac{x^2}{2}+x+2}\left(3e^x(x+1)+\sqrt{2}c_1(x+2)+c_2e^{\frac{1}{2}(x+2)^2}\right)\Bigg) \end{aligned}$$

2.2.4 problem 4

Internal problem ID [8484]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 4
Date solved : `Thursday, December 12, 2024 at 09:22:52 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy - x^2 - x = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.965 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.105: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x - 2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (2 + x) \, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\ &= (2 + x) \, \mathrm{e}^{-x-\frac{1}{4}x^2} \\ &= (2 + x) \, \mathrm{e}^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-x}{1} \, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2 + x) \, \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2 + x} - \frac{i\sqrt{\pi}\, \mathrm{e}^{-2} \sqrt{2}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - xy' - xy = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1(2+x)\,\mathrm{e}^{-x} - \frac{c_2\,\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}(2+x)\,\sqrt{2}\,\sqrt{\pi}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= (2+x)\,\mathrm{e}^{-x} \\
y_2 &= -\frac{\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}(2+x)\,\sqrt{2}\,\sqrt{\pi}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (2+x)\,\mathrm{e}^{-x} & -\frac{\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}(2+x)\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} \\ \frac{d}{dx}\left((2+x)\,\mathrm{e}^{-x}\right) & \frac{d}{dx}\left(-\frac{\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}(2+x)\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (2+x)\,\mathrm{e}^{-x} & -\frac{\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}(2+x)\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} \\ \mathrm{e}^{-x}-(2+x)\,\mathrm{e}^{-x} & \frac{\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}(2+x)\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} - \frac{\mathrm{e}^{-x}\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2\,\mathrm{e}^{-2}(2+x)\mathrm{e}^{\frac{(2+x)^2}{2}} \cdots\right.}{2} \end{vmatrix}$$

Therefore

$$W = \left((2+x)\,\mathrm{e}^{-x}\right)\left(\frac{\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}(2+x)\,\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} - \frac{\mathrm{e}^{-x}\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2\,\mathrm{e}^{-2}(2+x)\,\mathrm{e}^{\frac{(2+x)^2}{2}}+2(2+x)\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}\right) - \left(-\frac{\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}(2+x)\,\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}\right)\left(\mathrm{e}^{-x}-(2+x)\,\mathrm{e}^{-x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{\frac{(2+x)^2}{2}}\mathrm{e}^{-2}\mathrm{e}^{-2x}x^2 + 4\,\mathrm{e}^{\frac{(2+x)^2}{2}}\mathrm{e}^{-2}\mathrm{e}^{-2x}x - \mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-2x}x^2 + 4\,\mathrm{e}^{\frac{(2+x)^2}{2}}\mathrm{e}^{-2}\mathrm{e}^{-2x} - 4\,\mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-2x}x - 3\,\mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-2x}$$

Which simplifies to

$$W = e^{\frac{x^2}{2}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{e^{-x}\left(i e^{-2}(2+x)\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,e^{\frac{x(4+x)}{2}}\right)(x^2+x)}{2}}{e^{\frac{x^2}{2}}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{e^{-\frac{x(2+x)}{2}}\left(i e^{-2}(2+x)\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,e^{\frac{x(4+x)}{2}}\right)(1+x)\,x}{2}dx$$

Hence

$$u_1 = -\int_0^x -\frac{e^{-\frac{\alpha(2+\alpha)}{2}}\left(i e^{-2}(2+\alpha)\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+\alpha)}{2}\right)+2\,e^{\frac{\alpha(4+\alpha)}{2}}\right)(1+\alpha)\,\alpha}{2}d\alpha$$

And Eq. (3) becomes

$$u_2 = \int \frac{(2+x)\,e^{-x}(x^2+x)}{e^{\frac{x^2}{2}}}\,dx$$

Which simplifies to

$$u_2 = \int (2+x)\,x(1+x)\,e^{-\frac{x(2+x)}{2}}dx$$

Hence

$$u_2 = -\left(x^2+2x+2\right)e^{-\frac{x(2+x)}{2}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\int_0^x -\frac{e^{-\frac{\alpha(2+\alpha)}{2}}\left(i e^{-2}(2+\alpha)\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+\alpha)}{2}\right)+2\,e^{\frac{\alpha(4+\alpha)}{2}}\right)(1+\alpha)\,\alpha}{2}d\alpha(2+x)\,e^{-x} + \frac{e^{-x}\left(i e^{-2}(2+x)\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,e^{\frac{x(4+x)}{2}}\right)(x^2+2x+2)\,e^{-\frac{x(2+x)}{2}}}{2}$$

Which simplifies to

$$
\begin{aligned}
y_p(x) &= \frac{\mathrm{e}^{-x}(2+x)\int_0^x \mathrm{e}^{-\frac{\alpha(2+\alpha)}{2}}\left(i\mathrm{e}^{-2}(2+\alpha)\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+\alpha)}{2}\right)+2\,\mathrm{e}^{\frac{\alpha(4+\alpha)}{2}}\right)(1+\alpha)\,\alpha d\alpha}{2} \\
&+ \frac{\left(2+i\sqrt{2}\,\sqrt{\pi}\,(2+x)\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)\mathrm{e}^{-\frac{(2+x)^2}{2}}\right)(x^2+2x+2)}{2}
\end{aligned}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1(2+x)\,\mathrm{e}^{-x} - \frac{c_2\,\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}(2+x)\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}\right) \\
&+ \left(\frac{\mathrm{e}^{-x}(2+x)\int_0^x \mathrm{e}^{-\frac{\alpha(2+\alpha)}{2}}\left(i\mathrm{e}^{-2}(2+\alpha)\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+\alpha)}{2}\right)+2\,\mathrm{e}^{\frac{\alpha(4+\alpha)}{2}}\right)(1+\alpha)\,\alpha d\alpha}{2}\right. \\
&\left. + \frac{\left(2+i\sqrt{2}\,\sqrt{\pi}\,(2+x)\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)\mathrm{e}^{-\frac{(2+x)^2}{2}}\right)(x^2+2x+2)}{2}\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
\begin{aligned}
y &= c_1(2+x)\,\mathrm{e}^{-x} - \frac{c_2\,\mathrm{e}^{-x}\left(i\mathrm{e}^{-2}(2+x)\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} \\
&+ \frac{\mathrm{e}^{-x}(2+x)\int_0^x \mathrm{e}^{-\frac{\alpha(2+\alpha)}{2}}\left(i\mathrm{e}^{-2}(2+\alpha)\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+\alpha)}{2}\right)+2\,\mathrm{e}^{\frac{\alpha(4+\alpha)}{2}}\right)(1+\alpha)\,\alpha d\alpha}{2} \\
&+ \frac{\left(2+i\sqrt{2}\,\sqrt{\pi}\,(2+x)\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)\mathrm{e}^{-\frac{(2+x)^2}{2}}\right)(x^2+2x+2)}{2}
\end{aligned}
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.012 (sec)
Leaf size : 54

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x)-x^2-x = 0,
       y(x),singsol=all)
```

$$y = ic_1\sqrt{\pi}\,(x+2)\,\sqrt{2}\,\mathrm{e}^{-2-x}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)+2\,\mathrm{e}^{\frac{x(x+2)}{2}}c_1+\mathrm{e}^{-x}(x+2)\,c_2-x$$

Mathematica DSolve solution

Solving time : `1.764` (sec)
Leaf size : 84

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]-x^2-x==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) - 2e^x x + 2\sqrt{2}c_1(x+2) + 2c_2 e^{\frac{1}{2}(x+2)^2}\right)$$

2.2.5 problem 5

Internal problem ID [8485]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 5
Date solved : `Thursday, December 12, 2024 at 09:22:54 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' - xy' - xy - x^3 + 2 = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 1.027 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.106: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 4x - 2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\,p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (2+x)\, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\ &= (2+x)\, \mathrm{e}^{-x-\frac{1}{4}x^2} \\ &= (2+x)\, \mathrm{e}^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2+x)\, \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\, \mathrm{e}^{-2}\sqrt{2}\, \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' - xy' - xy = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1(2+x)\,\mathrm{e}^{-x} - \frac{c_2\left(i\sqrt{\pi}\,(2+x)\,\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= (2+x)\,\mathrm{e}^{-x} \\
y_2 &= -\frac{\left(i\sqrt{\pi}\,(2+x)\,\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (2+x)\,e^{-x} & -\frac{\left(i\sqrt{\pi}\,(2+x)e^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,e^{\frac{x(4+x)}{2}}\right)e^{-x}}{2} \\ \frac{d}{dx}\left((2+x)\,e^{-x}\right) & \frac{d}{dx}\left(-\frac{\left(i\sqrt{\pi}\,(2+x)e^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,e^{\frac{x(4+x)}{2}}\right)e^{-x}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (2+x)\,e^{-x} & -\frac{\left(i\sqrt{\pi}\,(2+x)e^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,e^{\frac{x(4+x)}{2}}\right)e^{-x}}{2} \\ e^{-x}-(2+x)\,e^{-x} & -\frac{\left(i\sqrt{\pi}\,e^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2(2+x)e^{-2}e^{\frac{(2+x)^2}{2}}+2(2+x)e^{\frac{x(4+x)}{2}}\right)e^{-x}}{2} + \frac{\left(i\sqrt{\pi}\,(2+x)e^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\right.\right.}{2} \end{vmatrix}$$

Therefore

$$\begin{aligned} W = ((2 \\ +x)\,e^{-x}) & \left(-\frac{\left(i\sqrt{\pi}\,e^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2(2+x)\,e^{-2}e^{\frac{(2+x)^2}{2}}+2(2+x)\,e^{\frac{x(4+x)}{2}}\right)e^{-x}}{2}\right. \\ & \left.+\frac{\left(i\sqrt{\pi}\,(2+x)\,e^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,e^{\frac{x(4+x)}{2}}\right)e^{-x}}{2}\right) \\ & -\left(-\frac{\left(i\sqrt{\pi}\,(2+x)\,e^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,e^{\frac{x(4+x)}{2}}\right)e^{-x}}{2}\right)\left(e^{-x}-(2+x)\,e^{-x}\right) \end{aligned}$$

Which simplifies to

$$\begin{aligned} W = {} & e^{\frac{(2+x)^2}{2}}e^{-2}e^{-2x}x^2 - e^{\frac{x(4+x)}{2}}e^{-2x}x^2 + 4\,e^{\frac{(2+x)^2}{2}}e^{-2}e^{-2x}x \\ & - 4\,e^{\frac{x(4+x)}{2}}e^{-2x}x + 4\,e^{\frac{(2+x)^2}{2}}e^{-2}e^{-2x} - 3\,e^{\frac{x(4+x)}{2}}e^{-2x} \end{aligned}$$

Which simplifies to

$$W = \mathrm{e}^{\frac{x^2}{2}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{\left(i\sqrt{\pi}\,(2+x)\mathrm{e}^{-2}\sqrt{2}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}\,(x^3-2)}{2}}{\mathrm{e}^{\frac{x^2}{2}}}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{\left(i\sqrt{\pi}\,(2+x)\,\mathrm{e}^{-2}\sqrt{2}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)(x^3-2)\,\mathrm{e}^{-\frac{x(2+x)}{2}}}{2} dx$$

Hence

$$u_1 = -\int_0^x -\frac{\left(i\sqrt{\pi}\,(2+\alpha)\,\mathrm{e}^{-2}\sqrt{2}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+\alpha)}{2}\right)+2\,\mathrm{e}^{\frac{\alpha(4+\alpha)}{2}}\right)(\alpha^3-2)\,\mathrm{e}^{-\frac{\alpha(2+\alpha)}{2}}}{2} d\alpha$$

And Eq. (3) becomes

$$u_2 = \int \frac{(2+x)\,\mathrm{e}^{-x}(x^3-2)}{\mathrm{e}^{\frac{x^2}{2}}}\, dx$$

Which simplifies to

$$u_2 = \int (2+x)\left(x^3-2\right)\mathrm{e}^{-\frac{x(2+x)}{2}} dx$$

Hence

$$u_2 = -\left(x^3+x^2+2x-2\right)\mathrm{e}^{-\frac{x(2+x)}{2}}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned}y_p(x) = -\int_0^x -\frac{\left(i\sqrt{\pi}\,(2+\alpha)\,\mathrm{e}^{-2}\sqrt{2}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+\alpha)}{2}\right)+2\,\mathrm{e}^{\frac{\alpha(4+\alpha)}{2}}\right)(\alpha^3-2)\,\mathrm{e}^{-\frac{\alpha(2+\alpha)}{2}}}{2} d\alpha(2 \\ + x)\,\mathrm{e}^{-x} \\ + \frac{\left(i\sqrt{\pi}\,(2+x)\,\mathrm{e}^{-2}\sqrt{2}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}(x^3+x^2+2x-2)\,\mathrm{e}^{-\frac{x(2+x)}{2}}}{2}\end{aligned}$$

Which simplifies to

$$
\begin{aligned}
&y_p(x) \\
&= \frac{\mathrm{e}^{-x}(2+x)\int_0^x \left(i\sqrt{\pi}\,(2+\alpha)\,\mathrm{e}^{-2}\sqrt{2}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+\alpha)}{2}\right)+2\,\mathrm{e}^{\frac{\alpha(4+\alpha)}{2}}\right)(\alpha^3-2)\,\mathrm{e}^{-\frac{\alpha(2+\alpha)}{2}}d\alpha}{2} \\
&\quad + \frac{\left(2+i(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{-\frac{(2+x)^2}{2}}\right)(x^3+x^2+2x-2)}{2}
\end{aligned}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1(2+x)\,\mathrm{e}^{-x} - \frac{c_2\left(i\sqrt{\pi}\,(2+x)\,\mathrm{e}^{-2}\sqrt{2}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}\right) \\
&\quad + \left(\frac{\mathrm{e}^{-x}(2+x)\int_0^x \left(i\sqrt{\pi}\,(2+\alpha)\,\mathrm{e}^{-2}\sqrt{2}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+\alpha)}{2}\right)+2\,\mathrm{e}^{\frac{\alpha(4+\alpha)}{2}}\right)(\alpha^3-2)\,\mathrm{e}^{-\frac{\alpha(2+\alpha)}{2}}d\alpha}{2}\right. \\
&\quad \left. + \frac{\left(2+i(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{-\frac{(2+x)^2}{2}}\right)(x^3+x^2+2x-2)}{2}\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

<u>Summary of solutions found</u>

$$
\begin{aligned}
y &= c_1(2+x)\,\mathrm{e}^{-x} - \frac{c_2\left(i\sqrt{\pi}\,(2+x)\,\mathrm{e}^{-2}\sqrt{2}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2} \\
&\quad + \frac{\mathrm{e}^{-x}(2+x)\int_0^x \left(i\sqrt{\pi}\,(2+\alpha)\,\mathrm{e}^{-2}\sqrt{2}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+\alpha)}{2}\right)+2\,\mathrm{e}^{\frac{\alpha(4+\alpha)}{2}}\right)(\alpha^3-2)\,\mathrm{e}^{-\frac{\alpha(2+\alpha)}{2}}d\alpha}{2} \\
&\quad + \frac{\left(2+i(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{-\frac{(2+x)^2}{2}}\right)(x^3+x^2+2x-2)}{2}
\end{aligned}
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.012 (sec)
Leaf size : 60

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x)-x^3+2 = 0,
       y(x),singsol=all)
```

$$y = ic_1\sqrt{\pi}\,(x+2)\,\sqrt{2}\,\mathrm{e}^{-2-x}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)+2\,\mathrm{e}^{\frac{x(x+2)}{2}}c_1+\mathrm{e}^{-x}(x+2)\,c_2-x^2+2x-2$$

Mathematica DSolve solution

Solving time : 3.133 (sec)
Leaf size : 91

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]-x^3+2==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) - 2e^x\left(x^2-2x+2\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

2.2.6 problem 6

Internal problem ID [8486]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 6
Date solved : `Thursday, December 12, 2024 at 09:22:56 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' - xy' - xy - x^4 - 6 = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 1.053 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^2 + 4x - 2}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^2 + 4x - 2 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.107: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x - 2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) = 0 \\ -2 + a_0 = 0 \end{gathered}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (2 + x)\, e^{\int \left(-1-\frac{x}{2}\right) dx} \\ &= (2 + x)\, e^{-x-\frac{1}{4}x^2} \\ &= (2 + x)\, e^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1} \, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(e^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2 + x)\, e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{e^{-2+\frac{(2+x)^2}{2}}}{2 + x} - \frac{i\sqrt{\pi}\, e^{-2}\sqrt{2}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
y'' - xy' - xy = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1(2+x)\,\mathrm{e}^{-x} - \frac{c_2\,\mathrm{e}^{-x}\left(i(2+x)\,\mathrm{e}^{-2}\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= (2+x)\,\mathrm{e}^{-x} \\
y_2 &= -\frac{\mathrm{e}^{-x}\left(i(2+x)\,\mathrm{e}^{-2}\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (2+x)\,\mathrm{e}^{-x} & -\frac{\mathrm{e}^{-x}\left(i(2+x)\mathrm{e}^{-2}\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} \\ \frac{d}{dx}\left((2+x)\,\mathrm{e}^{-x}\right) & \frac{d}{dx}\left(-\frac{\mathrm{e}^{-x}\left(i(2+x)\mathrm{e}^{-2}\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (2+x)\,\mathrm{e}^{-x} & -\frac{\mathrm{e}^{-x}\left(i(2+x)\mathrm{e}^{-2}\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} \\ \mathrm{e}^{-x}-(2+x)\,\mathrm{e}^{-x} & \frac{\mathrm{e}^{-x}\left(i(2+x)\mathrm{e}^{-2}\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} - \frac{\mathrm{e}^{-x}\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2(2+x)\mathrm{e}^{-2}\mathrm{e}^{\frac{(2+x}{2}}\right.}{2} \end{vmatrix}$$

Therefore

$$\begin{aligned} W = \left((2+x)\,\mathrm{e}^{-x}\right) & \left(\frac{\mathrm{e}^{-x}\left(i(2+x)\,\mathrm{e}^{-2}\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}\right. \\ & \left.-\frac{\mathrm{e}^{-x}\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2(2+x)\,\mathrm{e}^{-2}\mathrm{e}^{\frac{(2+x)^2}{2}}+2(2+x)\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}\right) \\ & -\left(-\frac{\mathrm{e}^{-x}\left(i(2+x)\,\mathrm{e}^{-2}\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}\right)\left(\mathrm{e}^{-x}-(2+x)\,\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$\begin{aligned} W = {} & \mathrm{e}^{\frac{(2+x)^2}{2}}\mathrm{e}^{-2}\mathrm{e}^{-2x}x^2+4\,\mathrm{e}^{\frac{(2+x)^2}{2}}\mathrm{e}^{-2}\mathrm{e}^{-2x}x-\mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-2x}x^2 \\ & +4\,\mathrm{e}^{\frac{(2+x)^2}{2}}\mathrm{e}^{-2}\mathrm{e}^{-2x}-4\,\mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-2x}x-3\,\mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-2x} \end{aligned}$$

Which simplifies to

$$W = e^{\frac{x^2}{2}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{e^{-x}\left(i(2+x)e^{-2}\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,e^{\frac{x(4+x)}{2}}\right)(x^4+6)}{2}}{e^{\frac{x^2}{2}}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{e^{-\frac{x(2+x)}{2}}(x^4+6)\left(i(2+x)\,e^{-2}\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,e^{\frac{x(4+x)}{2}}\right)}{2}dx$$

Hence

$$u_1 = -\int_0^x -\frac{e^{-\frac{\alpha(2+\alpha)}{2}}(\alpha^4+6)\left(i(2+\alpha)\,e^{-2}\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+\alpha)}{2}\right)+2\,e^{\frac{\alpha(4+\alpha)}{2}}\right)}{2}d\alpha$$

And Eq. (3) becomes

$$u_2 = \int \frac{(2+x)\,e^{-x}(x^4+6)}{e^{\frac{x^2}{2}}}\,dx$$

Which simplifies to

$$u_2 = \int (2+x)\left(x^4+6\right)e^{-\frac{x(2+x)}{2}}dx$$

Hence

$$u_2 = -\left(x^4+x^3+3x^2+12\right)e^{-\frac{x(2+x)}{2}}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned}
y_p(x) &= -\int_0^x -\frac{e^{-\frac{\alpha(2+\alpha)}{2}}(\alpha^4+6)\left(i(2+\alpha)\,e^{-2}\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+\alpha)}{2}\right)+2\,e^{\frac{\alpha(4+\alpha)}{2}}\right)}{2}d\alpha(2+x)\,e^{-x} \\
&\quad + \frac{e^{-x}\left(i(2+x)\,e^{-2}\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,e^{\frac{x(4+x)}{2}}\right)(x^4+x^3+3x^2+12)\,e^{-\frac{x(2+x)}{2}}}{2}
\end{aligned}$$

Which simplifies to

$$
\begin{aligned}
y_p(x) &= \frac{\mathrm{e}^{-x}(2+x)\int_0^x \mathrm{e}^{-\frac{\alpha(2+\alpha)}{2}}(\alpha^4+6)\left(i(2+\alpha)\,\mathrm{e}^{-2}\sqrt{\pi}\,\sqrt{2}\;\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+\alpha)}{2}\right)+2\,\mathrm{e}^{\frac{\alpha(4+\alpha)}{2}}\right)d\alpha}{2} \\
&\quad+\frac{\left(2+i(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{-\frac{(2+x)^2}{2}}\right)(x^4+x^3+3x^2+12)}{2}
\end{aligned}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1(2+x)\,\mathrm{e}^{-x} - \frac{c_2\,\mathrm{e}^{-x}\left(i(2+x)\,\mathrm{e}^{-2}\sqrt{\pi}\,\sqrt{2}\;\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}\right) \\
&\quad+\left(\frac{\mathrm{e}^{-x}(2+x)\int_0^x \mathrm{e}^{-\frac{\alpha(2+\alpha)}{2}}(\alpha^4+6)\left(i(2+\alpha)\,\mathrm{e}^{-2}\sqrt{\pi}\,\sqrt{2}\;\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+\alpha)}{2}\right)+2\,\mathrm{e}^{\frac{\alpha(4+\alpha)}{2}}\right)d\alpha}{2}\right. \\
&\quad\left.+\frac{\left(2+i(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{-\frac{(2+x)^2}{2}}\right)(x^4+x^3+3x^2+12)}{2}\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

<u>Summary of solutions found</u>

$$
\begin{aligned}
y &= c_1(2+x)\,\mathrm{e}^{-x} - \frac{c_2\,\mathrm{e}^{-x}\left(i(2+x)\,\mathrm{e}^{-2}\sqrt{\pi}\,\sqrt{2}\;\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} \\
&\quad+\frac{\mathrm{e}^{-x}(2+x)\int_0^x \mathrm{e}^{-\frac{\alpha(2+\alpha)}{2}}(\alpha^4+6)\left(i(2+\alpha)\,\mathrm{e}^{-2}\sqrt{\pi}\,\sqrt{2}\;\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+\alpha)}{2}\right)+2\,\mathrm{e}^{\frac{\alpha(4+\alpha)}{2}}\right)d\alpha}{2} \\
&\quad+\frac{\left(2+i(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{-\frac{(2+x)^2}{2}}\right)(x^4+x^3+3x^2+12)}{2}
\end{aligned}
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.012 (sec)
Leaf size : 64

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x)-x^4-6 = 0,
       y(x),singsol=all)
```

$$y = ic_1\sqrt{\pi}\,(x+2)\,\sqrt{2}\,\mathrm{e}^{-2-x}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)+2\,\mathrm{e}^{\frac{x(x+2)}{2}}c_1+\mathrm{e}^{-x}(x+2)\,c_2-x^3+3x^2-6x$$

Mathematica DSolve solution

Solving time : 4.38 (sec)
Leaf size : 92

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]-x^4-6==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) - 2e^x x\left(x^2-3x+6\right) + 2\sqrt{2}c_1(x+2) + 2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

2.2.7 problem 7

Internal problem ID [8487]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 7
Date solved : `Thursday, December 12, 2024 at 09:22:58 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' - xy' - xy - x^5 + 24 = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.998 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.108: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 2\\ &= -2\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{1} a_i x^i \qquad (8)\end{aligned}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \qquad (9)$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i\\ &= \frac{x}{2} + 1 \qquad (10)\end{aligned}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x - 2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= (2+x)\, e^{\int \left(-1-\frac{x}{2}\right) dx} \\ &= (2+x)\, e^{-x-\frac{1}{4}x^2} \\ &= (2+x)\, e^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(e^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2+x)\, e^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{e^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\, e^{-2}\sqrt{2}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
y'' - xy' - xy = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1(2+x)\,\mathrm{e}^{-x} - \frac{c_2\left(i\mathrm{e}^{-2}(2+x)\,\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= (2+x)\,\mathrm{e}^{-x} \\
y_2 &= -\frac{\left(i\mathrm{e}^{-2}(2+x)\,\sqrt{\pi}\,\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (2+x)\,\mathrm{e}^{-x} & -\frac{\left(i\mathrm{e}^{-2}(2+x)\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2} \\ \frac{d}{dx}\left((2+x)\,\mathrm{e}^{-x}\right) & \frac{d}{dx}\left(-\frac{\left(i\mathrm{e}^{-2}(2+x)\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (2+x)\,\mathrm{e}^{-x} & -\frac{\left(i\mathrm{e}^{-2}(2+x)\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2} \\ \mathrm{e}^{-x}-(2+x)\,\mathrm{e}^{-x} & -\frac{\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2\,\mathrm{e}^{-2}(2+x)\mathrm{e}^{\frac{(2+x)^2}{2}}+2(2+x)\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2} + \frac{\left(i\mathrm{e}^{-2}(2+x)\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\right.}{2} \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & ((2 \\ & +x)\,\mathrm{e}^{-x})\left(-\frac{\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2\,\mathrm{e}^{-2}(2+x)\,\mathrm{e}^{\frac{(2+x)^2}{2}}+2(2+x)\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}\right. \\ & \left.+\frac{\left(i\mathrm{e}^{-2}(2+x)\,\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}\right) \\ & -\left(-\frac{\left(i\mathrm{e}^{-2}(2+x)\,\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}\right)\left(\mathrm{e}^{-x}-(2+x)\,\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$\begin{aligned} W = {} & \mathrm{e}^{\frac{(2+x)^2}{2}}\mathrm{e}^{-2}\mathrm{e}^{-2x}x^2+4\,\mathrm{e}^{\frac{(2+x)^2}{2}}\mathrm{e}^{-2}\mathrm{e}^{-2x}x-\mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-2x}x^2 \\ & +4\,\mathrm{e}^{\frac{(2+x)^2}{2}}\mathrm{e}^{-2}\mathrm{e}^{-2x}-4\,\mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-2x}x-3\,\mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-2x} \end{aligned}$$

Which simplifies to

$$W = e^{\frac{x^2}{2}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{\left(i e^{-2(2+x)}\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,e^{\frac{x(4+x)}{2}}\right)e^{-x}\left(x^5-24\right)}{2}}{e^{\frac{x^2}{2}}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{\left(i e^{-2}(2+x)\,\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,e^{\frac{x(4+x)}{2}}\right)e^{-\frac{x(2+x)}{2}}\left(x^5-24\right)}{2}dx$$

Hence

$$u_1 = -\int_0^x -\frac{\left(i e^{-2}(2+\alpha)\,\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+\alpha)}{2}\right)+2\,e^{\frac{\alpha(4+\alpha)}{2}}\right)e^{-\frac{\alpha(2+\alpha)}{2}}\left(\alpha^5-24\right)}{2}d\alpha$$

And Eq. (3) becomes

$$u_2 = \int \frac{(2+x)\,e^{-x}\left(x^5-24\right)}{e^{\frac{x^2}{2}}}\,dx$$

Which simplifies to

$$u_2 = \int (2+x)\,e^{-\frac{x(2+x)}{2}}\left(x^5-24\right)dx$$

Hence

$$u_2 = -\left(x^5+x^4+4x^3+12x-36\right)e^{-\frac{x(2+x)}{2}}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = &-\int_0^x \\ &-\frac{\left(i e^{-2}(2+\alpha)\,\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+\alpha)}{2}\right)+2\,e^{\frac{\alpha(4+\alpha)}{2}}\right)e^{-\frac{\alpha(2+\alpha)}{2}}\left(\alpha^5-24\right)}{2}d\alpha(2+x)\,e^{-x} \\ &+\frac{\left(i e^{-2}(2+x)\,\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,e^{\frac{x(4+x)}{2}}\right)e^{-x}\left(x^5+x^4+4x^3+12x-36\right)e^{-\frac{x(2+x)}{2}}}{2} \end{aligned}$$

Which simplifies to

$$
\begin{aligned}
y_p(x) &\\
&= \frac{\mathrm{e}^{-x}(2+x)\int_0^x \left(i\mathrm{e}^{-2}(2+\alpha)\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+\alpha)}{2}\right)+2\,\mathrm{e}^{\frac{\alpha(4+\alpha)}{2}}\right)\mathrm{e}^{-\frac{\alpha(2+\alpha)}{2}}(\alpha^5-24)\,d\alpha}{2} \\
&\quad+\frac{(x^5+x^4+4x^3+12x-36)\left(2+i\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)\sqrt{\pi}\,\sqrt{2}\,(2+x)\,\mathrm{e}^{-\frac{(2+x)^2}{2}}\right)}{2}
\end{aligned}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1(2+x)\,\mathrm{e}^{-x}-\frac{c_2\left(i\mathrm{e}^{-2}(2+x)\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}\right) \\
&\quad+\left(\frac{\mathrm{e}^{-x}(2+x)\int_0^x \left(i\mathrm{e}^{-2}(2+\alpha)\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+\alpha)}{2}\right)+2\,\mathrm{e}^{\frac{\alpha(4+\alpha)}{2}}\right)\mathrm{e}^{-\frac{\alpha(2+\alpha)}{2}}(\alpha^5-24)\,d\alpha}{2}\right. \\
&\quad\left.+\frac{(x^5+x^4+4x^3+12x-36)\left(2+i\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)\sqrt{\pi}\,\sqrt{2}\,(2+x)\,\mathrm{e}^{-\frac{(2+x)^2}{2}}\right)}{2}\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

<u>Summary of solutions found</u>

$$
\begin{aligned}
y &= c_1(2+x)\,\mathrm{e}^{-x}-\frac{c_2\left(i\mathrm{e}^{-2}(2+x)\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2} \\
&\quad+\frac{\mathrm{e}^{-x}(2+x)\int_0^x \left(i\mathrm{e}^{-2}(2+\alpha)\sqrt{\pi}\,\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+\alpha)}{2}\right)+2\,\mathrm{e}^{\frac{\alpha(4+\alpha)}{2}}\right)\mathrm{e}^{-\frac{\alpha(2+\alpha)}{2}}(\alpha^5-24)\,d\alpha}{2} \\
&\quad+\frac{(x^5+x^4+4x^3+12x-36)\left(2+i\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)\sqrt{\pi}\,\sqrt{2}\,(2+x)\,\mathrm{e}^{-\frac{(2+x)^2}{2}}\right)}{2}
\end{aligned}
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.011 (sec)
Leaf size : 70

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x)-x^5+24 = 0,
       y(x),singsol=all)
```

$$
\begin{aligned}
y = & i c_1 \sqrt{\pi}\,(x+2)\sqrt{2}\,\mathrm{e}^{-2-x}\operatorname{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)+2\,\mathrm{e}^{\frac{x(x+2)}{2}} c_1 \\
& +\mathrm{e}^{-x}(x+2)\,c_2-x^4+4x^3-12x^2+12x+12
\end{aligned}
$$

Mathematica DSolve solution

Solving time : `2.655` (sec)
Leaf size : 102

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]-x^5+24==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^{-x}\left(-\sqrt{2\pi}c_2\sqrt{(x+2)^2}\operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right)\right.$$
$$\left.+e^x\left(-2x^4+8x^3-24x^2+24x+24\right)+2\sqrt{2}c_1(x+2)+2c_2e^{\frac{1}{2}(x+2)^2}\right)$$

2.2.8 problem 8

Internal problem ID [8488]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 8
Date solved : `Thursday, December 12, 2024 at 09:23:00 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy - x = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.484 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.109: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1\end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x - 2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= (2 + x) \, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\ &= (2 + x) \, \mathrm{e}^{-x-\frac{1}{4}x^2} \\ &= (2 + x) \, \mathrm{e}^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{1} \, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2 + x) \, \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2} \, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2 + x} - \frac{i\sqrt{\pi} \, \mathrm{e}^{-2}\sqrt{2} \, \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' - xy' - xy = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1(2+x)\,\mathrm{e}^{-x} - \frac{c_2\left(i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{-2}(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= (2+x)\,\mathrm{e}^{-x} \\
y_2 &= -\frac{\left(i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{-2}(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (2+x)\,\mathrm{e}^{-x} & -\frac{\left(i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{-2}(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2} \\ \frac{d}{dx}\left((2+x)\,\mathrm{e}^{-x}\right) & \frac{d}{dx}\left(-\frac{\left(i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{-2}(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (2+x)\,\mathrm{e}^{-x} & -\frac{\left(i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{-2}(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2} \\ \mathrm{e}^{-x}-(2+x)\,\mathrm{e}^{-x} & -\frac{\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2\,\mathrm{e}^{-2}(2+x)\mathrm{e}^{\frac{(2+x)^2}{2}}+2(2+x)\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}+\frac{\left(i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{-2}(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\right.}{2} \end{vmatrix}$$

Therefore

$$\begin{aligned} W = ((2 &\\ +x)\,\mathrm{e}^{-x}) &\left(-\frac{\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2\,\mathrm{e}^{-2}(2+x)\,\mathrm{e}^{\frac{(2+x)^2}{2}}+2(2+x)\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}\right. \\ &\left.+\frac{\left(i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{-2}(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}\right) \\ &-\left(-\frac{\left(i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{-2}(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}\right)\left(\mathrm{e}^{-x}-(2+x)\,\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$\begin{aligned} W = {} & \mathrm{e}^{\frac{(2+x)^2}{2}}\mathrm{e}^{-2}\mathrm{e}^{-2x}x^2+4\,\mathrm{e}^{\frac{(2+x)^2}{2}}\mathrm{e}^{-2}\mathrm{e}^{-2x}x-\mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-2x}x^2 \\ & +4\,\mathrm{e}^{\frac{(2+x)^2}{2}}\mathrm{e}^{-2}\mathrm{e}^{-2x}-4\,\mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-2x}x-3\,\mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-2x} \end{aligned}$$

Which simplifies to

$$W = \mathrm{e}^{\frac{x^2}{2}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{\left(i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{-2}(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}x}{2}}{\mathrm{e}^{\frac{x^2}{2}}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{x\left(i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{-2}(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-\frac{x(2+x)}{2}}}{2}dx$$

Hence

$$\begin{aligned}u_1 = &-\mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-\frac{x(2+x)}{2}} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)\mathrm{e}^{-\frac{x(2+x)}{2}}}{2} - \mathrm{e}^{-2}\mathrm{e}^{-\frac{x(2+x)}{2}}x\,\mathrm{e}^{\frac{(2+x)^2}{2}} \\ &+ x\,\mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-\frac{x(2+x)}{2}} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)x\,\mathrm{e}^{-\frac{x(2+x)}{2}}}{2}\end{aligned}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(2+x)\,\mathrm{e}^{-x}x}{\mathrm{e}^{\frac{x^2}{2}}}\,dx$$

Which simplifies to

$$u_2 = \int x(2+x)\,\mathrm{e}^{-\frac{x(2+x)}{2}}dx$$

Hence

$$u_2 = -(x+1)\,\mathrm{e}^{-\frac{x(2+x)}{2}}$$

Therefore the particular solution, from equation (1) is

$$
\begin{aligned}
y_p(x) = \Bigg(& -\mathrm{e}^{\frac{x(4+x)}{2}} \mathrm{e}^{-\frac{x(2+x)}{2}} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)\mathrm{e}^{-\frac{x(2+x)}{2}}}{2} - \mathrm{e}^{-2}\mathrm{e}^{-\frac{x(2+x)}{2}}\,x\,\mathrm{e}^{\frac{(2+x)^2}{2}} \\
& + x\,\mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-\frac{x(2+x)}{2}} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)x\,\mathrm{e}^{-\frac{x(2+x)}{2}}}{2}\Bigg)(2+x)\,\mathrm{e}^{-x} \\
& + \frac{\left(i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{-2}(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}(x+1)\,\mathrm{e}^{-\frac{x(2+x)}{2}}}{2}
\end{aligned}
$$

Which simplifies to

$$
y_p(x) = -1
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1(2+x)\,\mathrm{e}^{-x} - \frac{c_2\left(i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{-2}(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}\right) + (-1)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = c_1(2+x)\,\mathrm{e}^{-x} - \frac{c_2\left(i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{-2}(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2} - 1
$$

Solved as second order ode adjoint method

Time used: 2.335 (sec)

In normal form the ode

$$
y'' - xy' - xy - x = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}
$$

Where

$$\begin{aligned}p(x) &= -x\\ q(x) &= -x\\ r(x) &= x\end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned}\xi'' - (\xi\, p)' + \xi q &= 0\\ \xi'' - (-x\xi(x))' + (-x\xi(x)) &= 0\\ \xi''(x) + x\xi'(x) - (x-1)\,\xi(x) &= 0\end{aligned}$$

Which is solved for $\xi(x)$. Writing the ode as

$$\xi'' + x\xi' + (1-x)\,\xi = 0 \tag{1}$$

$$A\xi'' + B\xi' + C\xi = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= 1\\ B &= x\\ C &= 1-x\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = \xi e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r &= \frac{s}{t}\\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= x^2 + 4x - 2 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then ξ is found using the inverse transformation

$$
\xi = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.110: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 2 \\
&= -2
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2+4x-2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2} + 1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{gathered} (0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) = 0 \\ -2 + a_0 = 0 \end{gathered}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= (2+x)\, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\
&= (2+x)\, \mathrm{e}^{-x-\frac{1}{4}x^2} \\
&= (2+x)\, \mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}$$

The first solution to the original ode in ξ is found from

$$\begin{aligned}
\xi_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{1} \, dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$\xi_1 = (2+x)\, \mathrm{e}^{-\frac{x(2+x)}{2}}$$

The second solution ξ_2 to the original ode is found using reduction of order

$$\xi_2 = \xi_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{\xi_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
\xi_2 &= \xi_1 \int \frac{e^{\int -\frac{x}{1} \, dx}}{(\xi_1)^2} \, dx \\
&= \xi_1 \int \frac{e^{-\frac{x^2}{2}}}{(\xi_1)^2} \, dx \\
&= \xi_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\, \mathrm{e}^{-2}\sqrt{2}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
\xi &= c_1\xi_1 + c_2\xi_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-\frac{x(2+x)}{2}}\right) \\
&\quad + c_2\left((2+x)\,\mathrm{e}^{-\frac{x(2+x)}{2}}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$
\begin{aligned}
\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y &= \int \xi(x)\,r(x)\,dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi\,(x)}
\end{aligned}
$$

Or

$$
y' + y\left(-x - \frac{c_1\,\mathrm{e}^{-\frac{x(2+x)}{2}} + c_1(2+x)\,(-1-x)\,\mathrm{e}^{-\frac{x(2+x)}{2}} - \frac{c_2\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 2\,\mathrm{e}^{-2}(2+x)\mathrm{e}^{\frac{(2+x)^2}{2}} + 2(2+x)\mathrm{e}^{\frac{x(4}{}}\cdots}{2}}{c_1\,(2+x)\,\mathrm{e}^{-\frac{x(2+x)}{2}} - \frac{c_2\left(i\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{-2}(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+\cdots}{2}\right.\cdots}{2}}\cdots\right.
$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$
y' + q(x)y = p(x)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= -\frac{-i(x+1)\,\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}c_2\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 + 2c_1(x+1)}{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}c_2\,(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 - 2c_1\,(2+x)} \\
p(x) &= \frac{-i(x+1)\,\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}c_2\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 + 2c_1(x+1)}{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}c_2\,(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 - 2c_1\,(2+x)}
\end{aligned}
$$

The integrating factor μ is

$$
\mu = \mathrm{e}^{\int -\frac{-i(x+1)\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}c_2\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 + 2c_1(x+1)}{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}c_2(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 - 2c_1(2+x)}dx}
$$

Therefore the solution is

$$y = \left(\int \frac{\left(-i(x+1)\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}c_2\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 + 2c_1(x+1)\right)\mathrm{e}^{\int -\frac{-i(x+1)\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}c_2\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 + 2c_1(x+1)}{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}c_2(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 - 2c_1(2+x)}dx}}{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}c_2\,(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 - 2c_1\,(2+x)} \cdots \right.$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \left(\int \frac{\left(-i(x+1)\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}c_2\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 + 2c_1(x+1)\right)\mathrm{e}^{\int -\frac{-i(x+1)\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}c_2\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 + 2c_1(x+1)}{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}c_2(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 - 2c_1(2+x)}dx}}{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}c_2\,(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 - 2c_1\,(2+x)} \cdots + c_3 \right) \mathrm{e}^{-\int -\frac{-i(x+1)\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}c_2\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 + 2c_1(x+1)}{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}c_2(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 - 2c_1(2+x)}dx}$$

The constants can be merged to give

$$y = \left(\int \frac{\left(-i(x+1)\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}c_2\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 + 2c_1(x+1)\right)\mathrm{e}^{\int -\frac{-i(x+1)\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}c_2\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 + 2c_1(x+1)}{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}c_2(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 - 2c_1(2+x)}dx}}{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}c_2\,(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 - 2c_1\,(2+x)} \cdots + 1 \right) \mathrm{e}^{-\int -\frac{-i(x+1)\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}c_2\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 + 2c_1(x+1)}{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}c_2(2+x)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 - 2c_1(2+x)}dx}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
\begin{aligned}
y &= \left(\int \frac{\left(-i(x+1)\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}c_2\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2+2c_1(x+1)\right)\mathrm{e}^{\int -\frac{-i(x+1)\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}c_2\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2+2c_1(x+1)}{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}c_2(2+x)\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2-2c_1(2+x)}dx}}{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}c_2\,(2+x)\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2-2c_1\,(2+x)}dx\right. \\
&\left.\quad +1\right)\mathrm{e}^{-\int -\frac{-i(x+1)\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}c_2\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2+2c_1(x+1)}{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}c_2(2+x)\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2-2c_1(2+x)}dx}
\end{aligned}
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.012 (sec)
Leaf size : 52

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x)-x = 0,
       y(x),singsol=all)
```

$$y = i c_1 \sqrt{\pi}\,(x+2)\,\sqrt{2}\,\mathrm{e}^{-2-x}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)+2\,\mathrm{e}^{\frac{x(x+2)}{2}} c_1 - 1 + \mathrm{e}^{-x}(x+2)\,c_2$$

Mathematica DSolve solution

Solving time : 0.628 (sec)
Leaf size : 216

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]-x==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$\begin{aligned} y(x) \rightarrow \frac{1}{2} e^{-\frac{1}{2}(x+2)^2}\Bigg(2\sqrt{2} e^{\frac{x^2}{2}+x+2}(x+2)\int_1^x\Bigg(\frac{e^{K[1]}K[1]}{\sqrt{2}} \\ -\frac{1}{2} e^{-\frac{1}{2}K[1]^2-K[1]-2}\sqrt{\pi}\,\mathrm{erfi}\left(\frac{\sqrt{(K[1]+2)^2}}{\sqrt{2}}\right)K[1]\sqrt{(K[1]+2)^2}\Bigg)dK[1] \\ -\sqrt{2\pi}\sqrt{(x+2)^2}\left(c_2 e^{\frac{x^2}{2}+x+2}+x+1\right)\mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) \\ +2e^{\frac{x^2}{2}+x+2}\left(e^x(x+1)+\sqrt{2}c_1(x+2)+c_2 e^{\frac{1}{2}(x+2)^2}\right)\Bigg) \end{aligned}$$

2.2.9 problem 9

Internal problem ID [8489]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 9
Date solved : `Thursday, December 12, 2024 at 09:23:04 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy - x^2 = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.887 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.111: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 2 \\
&= -2
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

<u>Attempting to find a solution using case $n = 1$.</u>

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i
\end{aligned} \tag{8}
$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = \frac{1}{2}
$$

From Eq. (9) the sum up to $v = 1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= \frac{x}{2} + 1
\end{aligned} \tag{10}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 4x - 2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) = 0$$
$$-2 + a_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= (2 + x)\, \mathrm{e}^{\int \left(-1 - \frac{x}{2}\right) dx} \\ &= (2 + x)\, \mathrm{e}^{-x - \frac{1}{4}x^2} \\ &= (2 + x)\, \mathrm{e}^{-\frac{x(4+x)}{4}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-x}{1}\, dx} \\ &= z_1 e^{\frac{x^2}{4}} \\ &= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = (2 + x)\, \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\, dx \\ &= y_1 \left(-\frac{\mathrm{e}^{-2 + \frac{(2+x)^2}{2}}}{2 + x} - \frac{i\sqrt{\pi}\, \mathrm{e}^{-2} \sqrt{2}\, \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - xy' - xy = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1(2+x)\,\mathrm{e}^{-x} - \frac{c_2\,\mathrm{e}^{-x}\left(i\sqrt{2}\,(2+x)\,\sqrt{\pi}\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= (2+x)\,\mathrm{e}^{-x} \\
y_2 &= -\frac{\mathrm{e}^{-x}\left(i\sqrt{2}\,(2+x)\,\sqrt{\pi}\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (2+x)\,\mathrm{e}^{-x} & -\frac{\mathrm{e}^{-x}\left(i\sqrt{2}\,(2+x)\sqrt{\pi}\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} \\ \frac{d}{dx}\left((2+x)\,\mathrm{e}^{-x}\right) & \frac{d}{dx}\left(-\frac{\mathrm{e}^{-x}\left(i\sqrt{2}\,(2+x)\sqrt{\pi}\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (2+x)\,\mathrm{e}^{-x} & -\frac{\mathrm{e}^{-x}\left(i\sqrt{2}\,(2+x)\sqrt{\pi}\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} \\ \mathrm{e}^{-x}-(2+x)\,\mathrm{e}^{-x} & \frac{\mathrm{e}^{-x}\left(i\sqrt{2}\,(2+x)\sqrt{\pi}\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} - \frac{\mathrm{e}^{-x}\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2(2+x)\mathrm{e}^{-2}\mathrm{e}^{\frac{(2+x}{2}}\right.}{2}\end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & \left((2+x)\,\mathrm{e}^{-x}\right)\left(\frac{\mathrm{e}^{-x}\left(i\sqrt{2}\,(2+x)\,\sqrt{\pi}\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}\right. \\ & \left. - \frac{\mathrm{e}^{-x}\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2(2+x)\,\mathrm{e}^{-2}\mathrm{e}^{\frac{(2+x)^2}{2}}+2(2+x)\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}\right) \\ & - \left(-\frac{\mathrm{e}^{-x}\left(i\sqrt{2}\,(2+x)\,\sqrt{\pi}\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}\right)\left(\mathrm{e}^{-x}-(2+x)\,\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$\begin{aligned} W = {} & \mathrm{e}^{\frac{(2+x)^2}{2}}\mathrm{e}^{-2}\mathrm{e}^{-2x}x^2+4\,\mathrm{e}^{\frac{(2+x)^2}{2}}\mathrm{e}^{-2}\mathrm{e}^{-2x}x-\mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-2x}x^2 \\ & +4\,\mathrm{e}^{\frac{(2+x)^2}{2}}\mathrm{e}^{-2}\mathrm{e}^{-2x}-4\,\mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-2x}x-3\,\mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-2x} \end{aligned}$$

Which simplifies to

$$W = \mathrm{e}^{\frac{x^2}{2}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{\mathrm{e}^{-x}\left(i\sqrt{2}\,(2+x)\sqrt{\pi}\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)x^2}{2}}{\mathrm{e}^{\frac{x^2}{2}}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{\mathrm{e}^{-\frac{x(2+x)}{2}}\left(i\sqrt{2}\,(2+x)\sqrt{\pi}\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)x^2}{2}dx$$

Hence

$$u_1 = -\int_0^x -\frac{\mathrm{e}^{-\frac{\alpha(2+\alpha)}{2}}\left(i\sqrt{2}\,(2+\alpha)\sqrt{\pi}\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+\alpha)}{2}\right)+2\,\mathrm{e}^{\frac{\alpha(4+\alpha)}{2}}\right)\alpha^2}{2}d\alpha$$

And Eq. (3) becomes

$$u_2 = \int \frac{(2+x)\,\mathrm{e}^{-x}x^2}{\mathrm{e}^{\frac{x^2}{2}}}\,dx$$

Which simplifies to

$$u_2 = \int x^2(2+x)\,\mathrm{e}^{-\frac{x(2+x)}{2}}\,dx$$

Hence

$$u_2 = -\left(x^2+x+1\right)\mathrm{e}^{-\frac{x(2+x)}{2}}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned}y_p(x) = &-\int_0^x -\frac{\mathrm{e}^{-\frac{\alpha(2+\alpha)}{2}}\left(i\sqrt{2}\,(2+\alpha)\sqrt{\pi}\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+\alpha)}{2}\right)+2\,\mathrm{e}^{\frac{\alpha(4+\alpha)}{2}}\right)\alpha^2}{2}d\alpha(2+x)\,\mathrm{e}^{-x} \\ &+\frac{\mathrm{e}^{-x}\left(i\sqrt{2}\,(2+x)\sqrt{\pi}\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)(x^2+x+1)\,\mathrm{e}^{-\frac{x(2+x)}{2}}}{2}\end{aligned}$$

Which simplifies to

$$y_p(x) = \frac{e^{-x}(2+x)\int_0^x e^{-\frac{\alpha(2+\alpha)}{2}}\left(i\sqrt{2}\,(2+\alpha)\sqrt{\pi}\,e^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+\alpha)}{2}\right)+2\,e^{\frac{\alpha(4+\alpha)}{2}}\right)\alpha^2 d\alpha}{2} + \frac{(x^2+x+1)\left(2+i\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)\sqrt{\pi}\,(2+x)\sqrt{2}\,e^{-\frac{(2+x)^2}{2}}\right)}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1(2+x)\,e^{-x} - \frac{c_2\,e^{-x}\left(i\sqrt{2}\,(2+x)\sqrt{\pi}\,e^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,e^{\frac{x(4+x)}{2}}\right)}{2}\right) \\ &\quad + \left(\frac{e^{-x}(2+x)\int_0^x e^{-\frac{\alpha(2+\alpha)}{2}}\left(i\sqrt{2}\,(2+\alpha)\sqrt{\pi}\,e^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+\alpha)}{2}\right)+2\,e^{\frac{\alpha(4+\alpha)}{2}}\right)\alpha^2 d\alpha}{2} \right. \\ &\qquad \left. + \frac{(x^2+x+1)\left(2+i\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)\sqrt{\pi}\,(2+x)\sqrt{2}\,e^{-\frac{(2+x)^2}{2}}\right)}{2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1(2+x)\,e^{-x} - \frac{c_2\,e^{-x}\left(i\sqrt{2}\,(2+x)\sqrt{\pi}\,e^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,e^{\frac{x(4+x)}{2}}\right)}{2} + \frac{e^{-x}(2+x)\int_0^x e^{-\frac{\alpha(2+\alpha)}{2}}\left(i\sqrt{2}\,(2+\alpha)\sqrt{\pi}\,e^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+\alpha)}{2}\right)+2\,e^{\frac{\alpha(4+\alpha)}{2}}\right)\alpha^2 d\alpha}{2} + \frac{(x^2+x+1)\left(2+i\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)\sqrt{\pi}\,(2+x)\sqrt{2}\,e^{-\frac{(2+x)^2}{2}}\right)}{2}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.013 (sec)
Leaf size : 55

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x)-x^2 = 0,
       y(x),singsol=all)
```

$$y = ic_1\sqrt{\pi}\,(x+2)\,\sqrt{2}\,\mathrm{e}^{-2-x}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)+2\,\mathrm{e}^{\frac{x(x+2)}{2}}c_1+\mathrm{e}^{-x}(x+2)\,c_2-x+1$$

Mathematica DSolve solution

Solving time : `3.118` (sec)
Leaf size : 226

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]-x^2==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
y(x) \to \frac{1}{2}e^{-\frac{1}{2}(x+2)^2}\Bigg(2\sqrt{2}e^{\frac{x^2}{2}+x+2}(x+2)\int_1^x\Bigg(&\frac{e^{K[1]}K[1]^2}{\sqrt{2}} \\
&- \frac{1}{2}e^{-\frac{1}{2}K[1]^2-K[1]-2}\sqrt{\pi}\operatorname{erfi}\left(\frac{\sqrt{(K[1]+2)^2}}{\sqrt{2}}\right)K[1]^2\sqrt{(K[1]+2)^2}\Bigg)dK[1] \\
&- \sqrt{2\pi}\sqrt{(x+2)^2}\left(x^2+c_2e^{\frac{x^2}{2}+x+2}+x+1\right)\operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right) \\
&+ 2e^{\frac{x^2}{2}+x+2}\left(e^x\left(x^2+x+1\right)+\sqrt{2}c_1(x+2)+c_2e^{\frac{1}{2}(x+2)^2}\right)\Bigg)
\end{aligned}
$$

2.2.10 problem 10

Internal problem ID [8490]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 10
Date solved : `Thursday, December 12, 2024 at 09:23:06 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' - xy' - xy - x^3 = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 1.004 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.112: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2 + 4x - 2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\ -2 + a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega\, dx} \\
&= (2+x)\, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\
&= (2+x)\, \mathrm{e}^{-x-\frac{1}{4}x^2} \\
&= (2+x)\, \mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\, dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = (2+x)\, \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\, dx \\
&= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\, \mathrm{e}^{-2}\sqrt{2}\, \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2} \right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
y'' - xy' - xy = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1(2+x)\,\mathrm{e}^{-x} - \frac{c_2\left(i(2+x)\,\mathrm{e}^{-2}\sqrt{2}\,\sqrt{\pi}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= (2+x)\,\mathrm{e}^{-x} \\
y_2 &= -\frac{\left(i(2+x)\,\mathrm{e}^{-2}\sqrt{2}\,\sqrt{\pi}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (2+x)\,\mathrm{e}^{-x} & -\frac{\left(i(2+x)\mathrm{e}^{-2}\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2} \\ \frac{d}{dx}\left((2+x)\,\mathrm{e}^{-x}\right) & \frac{d}{dx}\left(-\frac{\left(i(2+x)\mathrm{e}^{-2}\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (2+x)\,\mathrm{e}^{-x} & -\frac{\left(i(2+x)\mathrm{e}^{-2}\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2} \\ \mathrm{e}^{-x}-(2+x)\,\mathrm{e}^{-x} & -\frac{\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2(2+x)\mathrm{e}^{-2}\mathrm{e}^{\frac{(2+x)^2}{2}}+2(2+x)\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}+\frac{\left(i(2+x)\mathrm{e}^{-2}\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\right.}{2}\right.}{2}\end{vmatrix}$$

Therefore

$$\begin{aligned}W = {} & ((2 \\ & +x)\,\mathrm{e}^{-x})\left(-\frac{\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2(2+x)\,\mathrm{e}^{-2}\mathrm{e}^{\frac{(2+x)^2}{2}}+2(2+x)\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}\right. \\ & \left.+\frac{\left(i(2+x)\,\mathrm{e}^{-2}\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}\right) \\ & -\left(-\frac{\left(i(2+x)\,\mathrm{e}^{-2}\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}\right)\left(\mathrm{e}^{-x}-(2+x)\,\mathrm{e}^{-x}\right)\end{aligned}$$

Which simplifies to

$$\begin{aligned}W = {} & \mathrm{e}^{\frac{(2+x)^2}{2}}\mathrm{e}^{-2}\mathrm{e}^{-2x}x^2+4\,\mathrm{e}^{\frac{(2+x)^2}{2}}\mathrm{e}^{-2}\mathrm{e}^{-2x}x-\mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-2x}x^2 \\ & +4\,\mathrm{e}^{\frac{(2+x)^2}{2}}\mathrm{e}^{-2}\mathrm{e}^{-2x}-4\,\mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-2x}x-3\,\mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-2x}\end{aligned}$$

Which simplifies to

$$W = \mathrm{e}^{\frac{x^2}{2}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{\left(i(2+x)\mathrm{e}^{-2}\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}x^3}{2}}{\mathrm{e}^{\frac{x^2}{2}}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{\left(i(2+x)\,\mathrm{e}^{-2}\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)x^3\mathrm{e}^{-\frac{x(2+x)}{2}}}{2}dx$$

Hence

$$u_1 = -\int_0^x -\frac{\left(i(2+\alpha)\,\mathrm{e}^{-2}\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+\alpha)}{2}\right)+2\,\mathrm{e}^{\frac{\alpha(4+\alpha)}{2}}\right)\alpha^3\mathrm{e}^{-\frac{\alpha(2+\alpha)}{2}}}{2}d\alpha$$

And Eq. (3) becomes

$$u_2 = \int \frac{(2+x)\,\mathrm{e}^{-x}x^3}{\mathrm{e}^{\frac{x^2}{2}}}\,dx$$

Which simplifies to

$$u_2 = \int x^3(2+x)\,\mathrm{e}^{-\frac{x(2+x)}{2}}dx$$

Hence

$$u_2 = -x^3\mathrm{e}^{-x-\frac{1}{2}x^2} - x^2\mathrm{e}^{-x-\frac{1}{2}x^2} - 2x\,\mathrm{e}^{-x-\frac{1}{2}x^2} + \sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}}\sqrt{2}\,\operatorname{erf}\left(\frac{\sqrt{2}\,x}{2}+\frac{\sqrt{2}}{2}\right)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\int_0^x -\frac{\left(i(2+\alpha)\,\mathrm{e}^{-2}\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+\alpha)}{2}\right)+2\,\mathrm{e}^{\frac{\alpha(4+\alpha)}{2}}\right)\alpha^3\mathrm{e}^{-\frac{\alpha(2+\alpha)}{2}}}{2}d\alpha(2+x)\,\mathrm{e}^{-x}$$
$$-\frac{\left(i(2+x)\,\mathrm{e}^{-2}\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}\left(-x^3\mathrm{e}^{-x-\frac{1}{2}x^2} - x^2\mathrm{e}^{-x-\frac{1}{2}x^2} - 2x\,\mathrm{e}^{-x-\frac{1}{2}x^2} + \sqrt{\pi}\,\mathrm{e}^{\frac{1}{2}}\right.}{2}$$

Which simplifies to

$$
\begin{aligned}
y_p(x) = {} & \frac{\mathrm{e}^{-x}(2+x)\int_0^x \left(i(2+\alpha)\,\mathrm{e}^{-2}\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+\alpha)}{2}\right)+2\,\mathrm{e}^{\frac{\alpha(4+\alpha)}{2}}\right)\alpha^3\mathrm{e}^{-\frac{\alpha(2+\alpha)}{2}}d\alpha}{2} \\
& + \frac{i(2+x)\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)x\sqrt{\pi}\,\sqrt{2}\,(x^2+x+2)\,\mathrm{e}^{-\frac{(2+x)^2}{2}}}{2} \\
& - \sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{\frac{(x+1)^2}{2}}\operatorname{erf}\left(\frac{\sqrt{2}\,(x+1)}{2}\right) \\
& - i\pi(2+x)\,\mathrm{e}^{-\frac{3}{2}-x}\operatorname{erf}\left(\frac{\sqrt{2}\,(x+1)}{2}\right)\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+x^3+x^2+2x
\end{aligned}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1(2+x)\,\mathrm{e}^{-x} - \frac{c_2\left(i(2+x)\,\mathrm{e}^{-2}\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2}\right) \\
&\quad + \left(\frac{\mathrm{e}^{-x}(2+x)\int_0^x \left(i(2+\alpha)\,\mathrm{e}^{-2}\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+\alpha)}{2}\right)+2\,\mathrm{e}^{\frac{\alpha(4+\alpha)}{2}}\right)\alpha^3\mathrm{e}^{-\frac{\alpha(2+\alpha)}{2}}d\alpha}{2}\right. \\
&\quad + \frac{i(2+x)\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)x\sqrt{\pi}\,\sqrt{2}\,(x^2+x+2)\,\mathrm{e}^{-\frac{(2+x)^2}{2}}}{2} \\
&\quad - \sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{\frac{(x+1)^2}{2}}\operatorname{erf}\left(\frac{\sqrt{2}\,(x+1)}{2}\right) \\
&\quad \left. - i\pi(2+x)\,\mathrm{e}^{-\frac{3}{2}-x}\operatorname{erf}\left(\frac{\sqrt{2}\,(x+1)}{2}\right)\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+x^3+x^2+2x\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
\begin{aligned}
y &= c_1(2+x)\,\mathrm{e}^{-x} - \frac{c_2\left(i(2+x)\,\mathrm{e}^{-2}\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)\mathrm{e}^{-x}}{2} \\
&\quad + \frac{\mathrm{e}^{-x}(2+x)\int_0^x \left(i(2+\alpha)\,\mathrm{e}^{-2}\sqrt{2}\,\sqrt{\pi}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+\alpha)}{2}\right)+2\,\mathrm{e}^{\frac{\alpha(4+\alpha)}{2}}\right)\alpha^3\mathrm{e}^{-\frac{\alpha(2+\alpha)}{2}}d\alpha}{2} \\
&\quad + \frac{i(2+x)\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)x\sqrt{\pi}\,\sqrt{2}\,(x^2+x+2)\,\mathrm{e}^{-\frac{(2+x)^2}{2}}}{2} \\
&\quad - \sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{\frac{(x+1)^2}{2}}\operatorname{erf}\left(\frac{\sqrt{2}\,(x+1)}{2}\right) \\
&\quad - i\pi(2+x)\,\mathrm{e}^{-\frac{3}{2}-x}\operatorname{erf}\left(\frac{\sqrt{2}\,(x+1)}{2}\right)\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+x^3+x^2+2x
\end{aligned}
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.016 (sec)
Leaf size : 194

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x)-x^3 = 0,
       y(x),singsol=all)
```

$$
\begin{aligned}
y = &\frac{\mathrm{e}^{-x}(x+2)\left(\int \mathrm{e}^{-\frac{x(x+2)}{2}} x^3\left(i\sqrt{\pi}\,\sqrt{2}\,(x+2)\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)+2\,\mathrm{e}^{\frac{x(x+4)}{2}}\right)dx\right)}{2} \\
&+\frac{i\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right)(x^2+x+2)\,x(x+2)\,\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{-\frac{(x+2)^2}{2}}}{2} \\
&-\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{\frac{(x+1)^2}{2}}\,\mathrm{erf}\left(\frac{\sqrt{2}\,(x+1)}{2}\right) \\
&-i\left(-\sqrt{2}\,\sqrt{\pi}\,\mathrm{e}^{-2-x}c_1+\pi\,\mathrm{e}^{-\frac{3}{2}-x}\,\mathrm{erf}\left(\frac{\sqrt{2}\,(x+1)}{2}\right)\right)(x+2)\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) \\
&+2\,\mathrm{e}^{\frac{x(x+2)}{2}}c_1+\mathrm{e}^{-x}(x+2)\,c_2+x^3+x^2+2x
\end{aligned}
$$

Mathematica DSolve solution

Solving time : `4.2` (sec)
Leaf size : 453

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]-x^3==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
y(x) \to{}& \frac{1}{2}e^{-\frac{1}{2}(x+2)^2}\left(2\sqrt{2}e^{\frac{x^2}{2}+x+2}(x+2)\int_1^x\left(\frac{e^{K[1]}K[1]^3}{\sqrt{2}}\right.\right.\\
&\left.-\frac{1}{2}e^{-\frac{1}{2}K[1]^2-K[1]-2}\sqrt{\pi}\operatorname{erfi}\left(\frac{\sqrt{(K[1]+2)^2}}{\sqrt{2}}\right)K[1]^3\sqrt{(K[1]+2)^2}\right)dK[1]\\
&-2\operatorname{erf}\left(\frac{x+1}{\sqrt{2}}\right)\left(\sqrt{2\pi}e^{x^2+3x+\frac{5}{2}}-\pi e^{\frac{1}{2}(x+1)^2}\sqrt{(x+2)^2}\operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right)\right)\\
&-\sqrt{2\pi}\sqrt{(x+2)^2}x^3\operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right)-\sqrt{2\pi}\sqrt{(x+2)^2}x^2\operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right)\\
&-\sqrt{2\pi}c_2e^{\frac{x^2}{2}+x+2}\sqrt{(x+2)^2}\operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right)\\
&-2\sqrt{2\pi}\sqrt{(x+2)^2}x\operatorname{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}}\right)+2e^{\frac{1}{2}(x+2)^2}x^3+2e^{\frac{1}{2}(x+2)^2}x^2\\
&\left.+2\sqrt{2}c_1e^{\frac{x^2}{2}+x+2}x+4\sqrt{2}c_1e^{\frac{x^2}{2}+x+2}+2c_2e^{x^2+3x+4}+4e^{\frac{1}{2}(x+2)^2}x\right)
\end{aligned}
$$

2.2.11 problem 11

Internal problem ID [8491]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 11
Date solved : `Wednesday, December 18, 2024 at 01:53:14 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - axy' - bxy - cx = 0$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
```

```
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.036 (sec)
Leaf size : 86

```
dsolve(diff(diff(y(x),x),x)-a*x*diff(y(x),x)-b*x*y(x)-c*x = 0,
       y(x),singsol=all)
```

$$y = \frac{e^{-\frac{bx}{a}} \operatorname{KummerU}\left(-\frac{b^2}{2a^3}, \frac{1}{2}, \frac{(a^2x+2b)^2}{2a^3}\right) c_1 b + e^{-\frac{bx}{a}} \operatorname{KummerM}\left(-\frac{b^2}{2a^3}, \frac{1}{2}, \frac{(a^2x+2b)^2}{2a^3}\right) c_2 b - c}{b}$$

Mathematica DSolve solution

Solving time : 5.112 (sec)
Leaf size : 565

```
DSolve[{D[y[x],{x,2}]-a*x*D[y[x],x]-b*x*y[x]-c*x==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-\frac{bx}{a}} \left(\operatorname{HermiteH}\left(\frac{b^2}{a^3}, \frac{xa^2+2b}{\sqrt{2}a^{3/2}}\right) \int_1^x \frac{a^4 c e^{\frac{bK[}{a}}}{b^2 \left(\sqrt{2} \operatorname{HermiteH}\left(\frac{b^2}{a^3} - 1, \frac{K[1]a^2+2b}{\sqrt{2}a^{3/2}}\right) \operatorname{Hypergeometric1F1}\left(-\frac{b^2}{2a^3}, \right.\right.}\right.$$

2.2.12 problem 12

Internal problem ID [8492]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 12
Date solved : `Wednesday, December 18, 2024 at 01:53:15 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - axy' - bxy - c\,x^2 = 0$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
```

```
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.036 (sec)
Leaf size : 95

```
dsolve(diff(diff(y(x),x),x)-a*x*diff(y(x),x)-b*x*y(x)-c*x^2 = 0,
       y(x),singsol=all)
```

$$y = \frac{e^{-\frac{bx}{a}} \operatorname{KummerM}\left(-\frac{b^2}{2a^3}, \frac{1}{2}, \frac{(a^2x+2b)^2}{2a^3}\right) c_2 b^2 + e^{-\frac{bx}{a}} \operatorname{KummerU}\left(-\frac{b^2}{2a^3}, \frac{1}{2}, \frac{(a^2x+2b)^2}{2a^3}\right) c_1 b^2 + c(-bx+a)}{b^2}$$

Mathematica DSolve solution

Solving time : 2.905 (sec)
Leaf size : 569

```
DSolve[{D[y[x],{x,2}]-a*x*D[y[x],x]-b*x*y[x]-c*x^2==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-\frac{bx}{a}} \left(\operatorname{HermiteH}\left(\frac{b^2}{a^3}, \frac{xa^2+2b}{\sqrt{2}a^{3/2}}\right) \int_1^x \frac{a^4 c e^{\frac{bK[1}{a}}}{b^2 \left(\sqrt{2} \operatorname{HermiteH}\left(\frac{b^2}{a^3}-1, \frac{K[1]a^2+2b}{\sqrt{2}a^{3/2}}\right) \operatorname{Hypergeometric1F1}\left(-\frac{b^2}{2a^3},\right.\right.}\right.$$

2.2.13 problem 13

Internal problem ID [8493]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 13
Date solved : Wednesday, December 18, 2024 at 01:53:16 AM
CAS classification : [[_2nd_order, _linear, _nonhomogeneous]]

Solve

$$y'' - axy' - bxy - c\,x^3 = 0$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
```

```
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.037 (sec)
Leaf size : 517

```
dsolve(diff(diff(y(x),x),x)-a*x*diff(y(x),x)-b*x*y(x)-c*x^3 = 0,
       y(x),singsol=all)
```

$$y = \frac{e^{-\frac{bx}{a}}\left(2\,\text{KummerU}\left(-\frac{b^2}{2a^3}, \frac{1}{2}, \frac{(a^2x+2b)^2}{2a^3}\right)\left(\int -\frac{(a^2x+2b)x^3\,\text{KummerM}\left(-\right.}{\text{KummerM}\left(-\frac{b^2}{2a^3}, \frac{1}{2}, \frac{(a^2x+2b)^2}{2a^3}\right)(a^3-b^2)\,\text{KummerU}\left(\frac{2a^3-b^2}{2a^3}, \frac{1}{2}, \frac{(a^2x+2b)^2}{2a^3}\right)}\right.\right.}{}$$

Mathematica DSolve solution

Solving time : 2.885 (sec)
Leaf size : 569

```
DSolve[{D[y[x],{x,2}]-a*x*D[y[x],x]-b*x*y[x]-c*x^3==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-\frac{bx}{a}}\left(\text{HermiteH}\left(\frac{b^2}{a^3}, \frac{xa^2+2b}{\sqrt{2}a^{3/2}}\right)\int_1^x \frac{a^4ce^{\frac{bK[1}{a}}}{b^2\left(\sqrt{2}\,\text{HermiteH}\left(\frac{b^2}{a^3}-1, \frac{K[1]a^2+2b}{\sqrt{2}a^{3/2}}\right)\text{Hypergeometric1F1}\left(-\frac{b^2}{2a^3},\right.\right.}\right.$$

2.2.14 problem 14

Internal problem ID [8494]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 14
Date solved : `Thursday, December 12, 2024 at 09:23:10 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - y' - xy - x = 0$$

Solved as second order Airy ode

Time used: 0.060 (sec)

This is Airy ODE. It has the general form

$$ay'' + by' + cxy = F(x)$$

Where in this case

$$\begin{aligned} a &= 1 \\ b &= -1 \\ c &= -1 \\ F &= x \end{aligned}$$

Therefore the solution to the homogeneous Airy ODE becomes

$$y = c_1 \,\mathrm{e}^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x - \frac{1}{4}\right)(-1)^{1/3}\right) + c_2 \,\mathrm{e}^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x - \frac{1}{4}\right)(-1)^{1/3}\right)$$

Since this is inhomogeneous Airy ODE, then we need to find the particular solution. The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right), e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_2 - x(A_2 x + A_1) - x = 0$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) + c_2 e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)\right) + (-1) \\ &= c_1 e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) + c_2 e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) - 1 \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) + c_2 e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) - 1$$

Solved as second order ode adjoint method

Time used: 1.047 (sec)

In normal form the ode

$$y'' - y' - xy - x = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= -1 \\ q(x) &= -x \\ r(x) &= x \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (-\xi(x))' + (-x\xi(x)) &= 0 \\ \xi''(x) + \xi'(x) - x\xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is Airy ODE. It has the general form

$$a\xi'' + b\xi' + cx\xi = F(x)$$

Where in this case

$$\begin{aligned} a &= 1 \\ b &= 1 \\ c &= -1 \\ F &= 0 \end{aligned}$$

Therefore the solution to the homogeneous Airy ODE becomes

$$\xi = c_3\, \mathrm{e}^{-\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x - \frac{1}{4}\right)(-1)^{1/3}\right) + c_4\, \mathrm{e}^{-\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x - \frac{1}{4}\right)(-1)^{1/3}\right)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y = \int \xi(x)\,r(x)\,dx$$
$$y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) = \frac{\int \xi(x)\,r(x)\,dx}{\xi(x)}$$

Or

$$y' + y\left(-1 - \frac{-\frac{c_3\,\mathrm{e}^{-\frac{x}{2}}\operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)}{2} - c_3\,\mathrm{e}^{-\frac{x}{2}}(-1)^{1/3}\operatorname{AiryAi}\left(1, \left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) - \frac{c_4\,\mathrm{e}^{-\frac{x}{2}}\operatorname{AiryBi}\left((-\right.}{2}}{c_3\,\mathrm{e}^{-\frac{x}{2}}\operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) + c_4\,\mathrm{e}^{-\frac{x}{2}}\operatorname{AiryBi}\left(\right.}\right.$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = -\frac{\left(-ic_3\sqrt{3} - c_3\right)\operatorname{AiryAi}\left(1, -\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right) + \left(-i\sqrt{3}\,c_4 - c_4\right)\operatorname{AiryBi}\left(1, -\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right) + c_3\,\mathrm{A}}{2c_3\operatorname{AiryAi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right) + 2c_4\operatorname{AiryBi}\left(-\frac{\left(x+\frac{1}{4}\right.}{}\right.}$$

$$p(x) = \frac{\left(-ic_3\sqrt{3} - c_3\right)\operatorname{AiryAi}\left(1, -\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right) + \left(-i\sqrt{3}\,c_4 - c_4\right)\operatorname{AiryBi}\left(1, -\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right) + c_3\,\mathrm{Air}}{2c_3\operatorname{AiryAi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right) + 2c_4\operatorname{AiryBi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1\right.}{2}\right.}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= \mathrm{e}^{\int -\frac{\left(-ic_3\sqrt{3}-c_3\right)\operatorname{AiryAi}\left(1,-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)+\left(-i\sqrt{3}\,c_4-c_4\right)\operatorname{AiryBi}\left(1,-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)+c_3\operatorname{AiryAi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)+c_4\operatorname{AiryBi}\left(-\frac{\left(x+\frac{1}{4}\right.}{}\right.}{2c_3\operatorname{AiryAi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)+2c_4\operatorname{AiryBi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)}} \\
&= \frac{\mathrm{e}^{-\frac{x}{2}}}{c_3\operatorname{AiryAi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right) + c_4\operatorname{AiryBi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)}
\end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu p$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{\left(-ic_3\sqrt{3}-c_3\right)\mathrm{AiryAi}\left(1, -\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right) + \left(-i\sqrt{3}\,c_4 - c_4\right)\mathrm{AiryBi}\left(1, -\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right) + c_3\,\mathrm{Air}}{2c_3\,\mathrm{AiryAi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right) + 2c_4\,\mathrm{AiryBi}\left(-\frac{\left(x+\frac{1}{4}\right)(}{2}\right.}\right.$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y\,\mathrm{e}^{-\frac{x}{2}}}{c_3\,\mathrm{AiryAi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right) + c_4\,\mathrm{AiryBi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)}\right) = \left(\frac{\mathrm{e}^{-\frac{x}{2}}}{c_3\,\mathrm{AiryAi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right) + c_4\,\mathrm{AiryBi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)}\right)\left(\frac{\left(-ic_3\sqrt{3}-c_3\right)\mathrm{AiryAi}\left(1, -\frac{\left(x+\frac{1}{4}\right)\left(1+i\right.}{2}\right.}{}\right.$$

$$\mathrm{d}\left(\frac{y\,\mathrm{e}^{-\frac{x}{2}}}{c_3\,\mathrm{AiryAi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right) + c_4\,\mathrm{AiryBi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)}\right) = \left(\frac{\left(\left(-ic_3\sqrt{3}-c_3\right)\mathrm{AiryAi}\left(1, -\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right) + \left(-i\sqrt{3}\,c_4 - c_4\right)\mathrm{AiryBi}\left(1, -\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right) + c_3\,\mathrm{Airy}\right.}{\left(2c_3\,\mathrm{AiryAi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right) + 2c_4\,\mathrm{AiryBi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)\right)\left(c_3\,\mathrm{AiryAi}\left(-\frac{\left(x+\frac{1}{4}\right)}{}\right.\right.}\right.$$

Integrating gives

$$\frac{y\,\mathrm{e}^{-\frac{x}{2}}}{c_3\,\mathrm{AiryAi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right) + c_4\,\mathrm{AiryBi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)} = \int \frac{\left(\left(-ic_3\sqrt{3}-c_3\right)\mathrm{AiryAi}\left(1, -\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{}\right.}{2}\right.\right.}{\left(2c_3\,\mathrm{AiryAi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+\right.}{2}\right.\right.}$$

$$= -\frac{\mathrm{e}^{-\frac{x}{2}}}{c_3\,\mathrm{AiryAi}\left(-\frac{(4x+1)\left(1+i\sqrt{3}\right)}{8}\right) + c_4\,\mathrm{AiryBi}\left(\right.}$$

Dividing throughout by the integrating factor $\frac{e^{-\frac{x}{2}}}{c_3 \operatorname{AiryAi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)+c_4 \operatorname{AiryBi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)}$ gives the final solution

$$y = \operatorname{AiryAi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right) e^{\frac{x}{2}} c_3 c_5 + \operatorname{AiryBi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right) e^{\frac{x}{2}} c_4 c_5 - 1$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \operatorname{AiryAi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right) e^{\frac{x}{2}} c_3 c_5 + \operatorname{AiryBi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right) e^{\frac{x}{2}} c_4 c_5 - 1$$

The constants can be merged to give

$$y = \operatorname{AiryAi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right) e^{\frac{x}{2}} c_3 + \operatorname{AiryBi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right) e^{\frac{x}{2}} c_4 - 1$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \operatorname{AiryAi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right) e^{\frac{x}{2}} c_3 + \operatorname{AiryBi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right) e^{\frac{x}{2}} c_4 - 1$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
```

```
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.020 (sec)
Leaf size : 26

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)-x*y(x)-x = 0,
       y(x),singsol=all)
```

$$y = e^{\frac{x}{2}} \operatorname{AiryAi}\left(x+\frac{1}{4}\right) c_2 + e^{\frac{x}{2}} \operatorname{AiryBi}\left(x+\frac{1}{4}\right) c_1 - 1$$

Mathematica DSolve solution

Solving time : 12.951 (sec)
Leaf size : 99

```
DSolve[{D[y[x],{x,2}]-D[y[x],x]-x*y[x]-x==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$\begin{aligned} y(x) \to e^{x/2}\Bigg(&\operatorname{AiryAi}\left(x+\frac{1}{4}\right)\int_1^x -e^{-\frac{K[1]}{2}}\pi \operatorname{AiryBi}\left(K[1]+\frac{1}{4}\right)K[1]dK[1] \\ &+ \operatorname{AiryBi}\left(x+\frac{1}{4}\right)\int_1^x e^{-\frac{K[2]}{2}}\pi \operatorname{AiryAi}\left(K[2]+\frac{1}{4}\right)K[2]dK[2] \\ &+ c_1 \operatorname{AiryAi}\left(x+\frac{1}{4}\right) + c_2 \operatorname{AiryBi}\left(x+\frac{1}{4}\right)\Bigg) \end{aligned}$$

2.2.15 problem 15

Internal problem ID [8495]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 15
Date solved : `Thursday, December 12, 2024 at 09:23:12 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' - y' - xy - x^2 = 0$$

Solved as second order Airy ode

Time used: 1.015 (sec)

This is Airy ODE. It has the general form

$$ay'' + by' + cxy = F(x)$$

Where in this case

$$\begin{aligned} a &= 1 \\ b &= -1 \\ c &= -1 \\ F &= x^2 \end{aligned}$$

Therefore the solution to the homogeneous Airy ODE becomes

$$y = c_1 \,\mathrm{e}^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x - \frac{1}{4}\right) (-1)^{1/3}\right) + c_2 \,\mathrm{e}^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x - \frac{1}{4}\right) (-1)^{1/3}\right)$$

Since this is inhomogeneous Airy ODE, then we need to find the particular solution. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will

be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)$$
$$y_2 = e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) & e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) \\ \frac{d}{dx}\left(e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)\right) & \frac{d}{dx}\left(e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) & e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(- \right.\right. \\ \frac{e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)}{2} - e^{\frac{x}{2}}(-1)^{1/3} \operatorname{AiryAi}\left(1, \left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) & \frac{e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)}{2} - e^{\frac{x}{2}}(-1 \end{vmatrix}$$

Therefore

$$W = \left(e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)\right)\left(\frac{e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)}{2}\right.$$
$$\left. - e^{\frac{x}{2}}(-1)^{1/3} \operatorname{AiryBi}\left(1, \left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)\right) - \left(e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)\right)\left(\frac{e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x \right.\right.}{2}\right.$$

Which simplifies to

$$W = -\mathrm{e}^{x}\,\mathrm{AiryAi}\left(-\frac{(4x+1)\,(-1)^{1/3}}{4}\right)(-1)^{1/3}\,\mathrm{AiryBi}\left(1, -\frac{(4x+1)\,(-1)^{1/3}}{4}\right) + \mathrm{e}^{x}\,\mathrm{AiryBi}\left(-\frac{(4x+1)\,(-1}{4}\right.$$

Which simplifies to

$$W = -\frac{\mathrm{e}^{x}\left(1+i\sqrt{3}\right)}{2\pi}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\mathrm{e}^{\frac{x}{2}}\,\mathrm{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)x^2}{-\frac{\mathrm{e}^{x}\left(1+i\sqrt{3}\right)}{2\pi}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{2\pi\,x^2\mathrm{e}^{-\frac{x}{2}}\,\mathrm{AiryBi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)}{1+i\sqrt{3}}dx$$

Hence

$$u_1 = -\int_0^x -\frac{2\pi\,\alpha^2\mathrm{e}^{-\frac{\alpha}{2}}\,\mathrm{AiryBi}\left(-\frac{\left(\alpha+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)}{1+i\sqrt{3}}d\alpha$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{\frac{x}{2}}\,\mathrm{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)x^2}{-\frac{\mathrm{e}^{x}\left(1+i\sqrt{3}\right)}{2\pi}}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{2\pi\,x^2\mathrm{e}^{-\frac{x}{2}}\,\mathrm{AiryAi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)}{1+i\sqrt{3}}dx$$

Hence

$$u_2 = \int_0^x -\frac{2\pi\,\alpha^2 \mathrm{e}^{-\frac{\alpha}{2}}\,\mathrm{AiryAi}\left(-\frac{(\alpha+\frac{1}{4})(1+i\sqrt{3})}{2}\right)}{1+i\sqrt{3}} d\alpha$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\int_0^x -\frac{2\pi\,\alpha^2 \mathrm{e}^{-\frac{\alpha}{2}}\,\mathrm{AiryBi}\left(-\frac{(\alpha+\frac{1}{4})(1+i\sqrt{3})}{2}\right)}{1+i\sqrt{3}} d\alpha \mathrm{e}^{\frac{x}{2}}\,\mathrm{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) + \mathrm{e}^{\frac{x}{2}}\,\mathrm{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)\int_0^x -\frac{2\pi\,\alpha^2 \mathrm{e}^{-\frac{\alpha}{2}}\,\mathrm{AiryAi}\left(-\frac{(\alpha+\frac{1}{4})(1+i\sqrt{3})}{2}\right)}{1+i\sqrt{3}} d\alpha$$

Which simplifies to

$$y_p(x) = \frac{2\,\mathrm{e}^{\frac{x}{2}}\pi\left(\int_0^x \alpha^2 \mathrm{e}^{-\frac{\alpha}{2}}\,\mathrm{AiryBi}\left(-\frac{(\alpha+\frac{1}{4})(1+i\sqrt{3})}{2}\right) d\alpha\,\mathrm{AiryAi}\left(-\frac{(x+\frac{1}{4})(1+i\sqrt{3})}{2}\right) - \mathrm{AiryBi}\left(-\frac{(x+\frac{1}{4})(1+i\sqrt{3})}{2}\right)\int_0^x}{1+i\sqrt{3}}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\,\mathrm{e}^{\frac{x}{2}}\,\mathrm{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) + c_2\,\mathrm{e}^{\frac{x}{2}}\,\mathrm{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)\right) \\ &\quad + \left(\frac{2\,\mathrm{e}^{\frac{x}{2}}\pi\left(\int_0^x \alpha^2 \mathrm{e}^{-\frac{\alpha}{2}}\,\mathrm{AiryBi}\left(-\frac{(\alpha+\frac{1}{4})(1+i\sqrt{3})}{2}\right) d\alpha\,\mathrm{AiryAi}\left(-\frac{(x+\frac{1}{4})(1+i\sqrt{3})}{2}\right) - \mathrm{AiryBi}\left(-\frac{(x+\frac{1}{4})(1+i\sqrt{3})}{2}\right.}{1+i\sqrt{3}}\right. \end{aligned}$$

$$\begin{aligned} &= c_1\,\mathrm{e}^{\frac{x}{2}}\,\mathrm{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) + c_2\,\mathrm{e}^{\frac{x}{2}}\,\mathrm{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) \\ &\quad + \frac{2\,\mathrm{e}^{\frac{x}{2}}\pi\left(\int_0^x \alpha^2 \mathrm{e}^{-\frac{\alpha}{2}}\,\mathrm{AiryBi}\left(-\frac{(\alpha+\frac{1}{4})(1+i\sqrt{3})}{2}\right) d\alpha\,\mathrm{AiryAi}\left(-\frac{(x+\frac{1}{4})(1+i\sqrt{3})}{2}\right) - \mathrm{AiryBi}\left(-\frac{(x+\frac{1}{4})(1+i\sqrt{3})}{2}\right)\right.}{1+i\sqrt{3}} \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \,\mathrm{e}^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)+c_2 \,\mathrm{e}^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) + \frac{2\,\mathrm{e}^{\frac{x}{2}}\pi\left(\int_0^x \alpha^2 \mathrm{e}^{-\frac{\alpha}{2}} \operatorname{AiryBi}\left(-\frac{\left(\alpha+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)d\alpha \operatorname{AiryAi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)-\operatorname{AiryBi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)\int}{1+i\sqrt{3}}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : `0.023` (sec)
Leaf size : 63

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)-x*y(x)-x^2 = 0,
       y(x),singsol=all)
```

$$y = e^{\frac{x}{2}} \left(\operatorname{AiryBi}\left(x+\frac{1}{4}\right) \pi \left(\int x^{2} \operatorname{AiryAi}\left(x+\frac{1}{4}\right) e^{-\frac{x}{2}}d x \right) - \operatorname{AiryAi}\left(x+\frac{1}{4}\right) \pi \left(\int x^{2} \operatorname{AiryBi}\left(x+\frac{1}{4}\right) e^{-\frac{x}{2}}d x \right) + c_{1} \operatorname{AiryBi}\left(x+\frac{1}{4}\right) + c_{2} \operatorname{AiryAi}\left(x+\frac{1}{4}\right)\right)$$

Mathematica DSolve solution

Solving time : `9.261` (sec)
Leaf size : 103

```
DSolve[{D[y[x],{x,2}]-D[y[x],x]-x*y[x]-x^2==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to e^{x/2}\left(\operatorname{AiryAi}\left(x+\frac{1}{4}\right)\int_1^x -e^{-\frac{K[1]}{2}}\pi \operatorname{AiryBi}\left(K[1]+\frac{1}{4}\right)K[1]^2dK[1] + \operatorname{AiryBi}\left(x+\frac{1}{4}\right)\int_1^x e^{-\frac{K[2]}{2}}\pi \operatorname{AiryAi}\left(K[2]+\frac{1}{4}\right)K[2]^2dK[2] + c_1 \operatorname{AiryAi}\left(x+\frac{1}{4}\right) + c_2 \operatorname{AiryBi}\left(x+\frac{1}{4}\right)\right)$$

2.2.16 problem 16

Internal problem ID [8496]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 16
Date solved : `Thursday, December 12, 2024 at 09:23:13 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - y' - xy - x^2 - 1 = 0$$

Solved as second order Airy ode

Time used: 0.065 (sec)

This is Airy ODE. It has the general form

$$ay'' + by' + cxy = F(x)$$

Where in this case

$$\begin{aligned} a &= 1 \\ b &= -1 \\ c &= -1 \\ F &= x^2 + 1 \end{aligned}$$

Therefore the solution to the homogeneous Airy ODE becomes

$$y = c_1 \,\mathrm{e}^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x - \frac{1}{4}\right)(-1)^{1/3}\right) + c_2 \,\mathrm{e}^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x - \frac{1}{4}\right)(-1)^{1/3}\right)$$

Since this is inhomogeneous Airy ODE, then we need to find the particular solution. The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right), e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_3 - 2xA_3 - A_2 - x(A_3x^2 + A_2x + A_1) - x^2 - 1 = 0$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = -1, A_3 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) + c_2 e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)\right) + (-x) \\
&= c_1 e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) + c_2 e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) - x
\end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) + c_2 e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) - x$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : `0.021` (sec)
Leaf size : 28

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)-x*y(x)-x^2-1 = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{\frac{x}{2}} \operatorname{AiryAi}\left(x+\frac{1}{4}\right) c_2 + \mathrm{e}^{\frac{x}{2}} \operatorname{AiryBi}\left(x+\frac{1}{4}\right) c_1 - x$$

Mathematica DSolve solution

Solving time : `4.326` (sec)
Leaf size : 107

```
DSolve[{D[y[x],{x,2}]-D[y[x],x]-x*y[x]-x^2-1==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
y(x) \to e^{x/2}\Bigg(&\operatorname{AiryAi}\left(x+\frac{1}{4}\right)\int_1^x -e^{-\frac{K[1]}{2}}\pi \operatorname{AiryBi}\left(K[1]+\frac{1}{4}\right)\left(K[1]^2+1\right)dK[1] \\
&+\operatorname{AiryBi}\left(x+\frac{1}{4}\right)\int_1^x e^{-\frac{K[2]}{2}}\pi \operatorname{AiryAi}\left(K[2]+\frac{1}{4}\right)\left(K[2]^2+1\right)dK[2] \\
&+c_1\operatorname{AiryAi}\left(x+\frac{1}{4}\right)+c_2\operatorname{AiryBi}\left(x+\frac{1}{4}\right)\Bigg)
\end{aligned}
$$

2.2.17 problem 16

Internal problem ID [8497]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 16
Date solved : `Thursday, December 12, 2024 at 09:23:14 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - y' - xy - x^2 - 1 = 0$$

Solved as second order Airy ode

Time used: 0.065 (sec)

This is Airy ODE. It has the general form

$$ay'' + by' + cxy = F(x)$$

Where in this case

$$\begin{aligned} a &= 1 \\ b &= -1 \\ c &= -1 \\ F &= x^2 + 1 \end{aligned}$$

Therefore the solution to the homogeneous Airy ODE becomes

$$y = c_1 \,\mathrm{e}^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x - \frac{1}{4}\right)(-1)^{1/3}\right) + c_2 \,\mathrm{e}^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x - \frac{1}{4}\right)(-1)^{1/3}\right)$$

Since this is inhomogeneous Airy ODE, then we need to find the particular solution. The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right), e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_3 - 2xA_3 - A_2 - x(A_3x^2 + A_2x + A_1) - x^2 - 1 = 0$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = -1, A_3 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) + c_2 e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)\right) + (-x) \\
&= c_1 e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) + c_2 e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) - x
\end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) + c_2 e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) - x$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.020 (sec)
Leaf size : 28

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)-x*y(x)-x^2-1 = 0,
       y(x),singsol=all)
```

$$y = e^{\frac{x}{2}} \operatorname{AiryAi}\left(x+\frac{1}{4}\right) c_2 + e^{\frac{x}{2}} \operatorname{AiryBi}\left(x+\frac{1}{4}\right) c_1 - x$$

Mathematica DSolve solution

Solving time : `1.224` (sec)
Leaf size : 107

```
DSolve[{D[y[x],{x,2}]-D[y[x],x]-x*y[x]-x^2-1==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{x/2}\left(\operatorname{AiryAi}\left(x+\frac{1}{4}\right)\int_1^x -e^{-\frac{K[1]}{2}}\pi \operatorname{AiryBi}\left(K[1]+\frac{1}{4}\right)\left(K[1]^2+1\right)dK[1] + \operatorname{AiryBi}\left(x+\frac{1}{4}\right)\int_1^x e^{-\frac{K[2]}{2}}\pi \operatorname{AiryAi}\left(K[2]+\frac{1}{4}\right)\left(K[2]^2+1\right)dK[2] + c_1 \operatorname{AiryAi}\left(x+\frac{1}{4}\right) + c_2 \operatorname{AiryBi}\left(x+\frac{1}{4}\right)\right)$$

2.2.18 problem 17

Internal problem ID [8498]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 17
Date solved : `Thursday, December 12, 2024 at 09:23:15 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - 2y' - xy - x^2 - 2 = 0$$

Solved as second order Airy ode

Time used: 0.068 (sec)

This is Airy ODE. It has the general form

$$ay'' + by' + cxy = F(x)$$

Where in this case

$$\begin{aligned} a &= 1 \\ b &= -2 \\ c &= -1 \\ F &= x^2 + 2 \end{aligned}$$

Therefore the solution to the homogeneous Airy ODE becomes

$$y = c_1 \,\mathrm{e}^x \operatorname{AiryAi}\left((-x-1)(-1)^{1/3}\right) + c_2 \,\mathrm{e}^x \operatorname{AiryBi}\left((-x-1)(-1)^{1/3}\right)$$

Since this is inhomogeneous Airy ODE, then we need to find the particular solution. The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^x \operatorname{AiryAi}\left((-x-1)(-1)^{1/3}\right), e^x \operatorname{AiryBi}\left((-x-1)(-1)^{1/3}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_3 - 4xA_3 - 2A_2 - x\left(A_3x^2 + A_2x + A_1\right) - x^2 - 2 = 0$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = -1, A_3 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^x \operatorname{AiryAi}\left((-x-1)(-1)^{1/3}\right) + c_2 e^x \operatorname{AiryBi}\left((-x-1)(-1)^{1/3}\right)\right) + (-x) \\
&= c_1 e^x \operatorname{AiryAi}\left((-x-1)(-1)^{1/3}\right) + c_2 e^x \operatorname{AiryBi}\left((-x-1)(-1)^{1/3}\right) - x
\end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 e^x \operatorname{AiryAi}\left((-x-1)(-1)^{1/3}\right) + c_2 e^x \operatorname{AiryBi}\left((-x-1)(-1)^{1/3}\right) - x$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.020 (sec)
Leaf size : 24

```
dsolve(diff(diff(y(x),x),x)-2*diff(y(x),x)-x*y(x)-x^2-2 = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{x} \operatorname{AiryAi}\left(x+1\right) c_{2}+\mathrm{e}^{x} \operatorname{AiryBi}\left(x+1\right) c_{1}-x$$

Mathematica DSolve solution

Solving time : `5.473` (sec)
Leaf size : 87

```
DSolve[{D[y[x],{x,2}]-2*D[y[x],x]-x*y[x]-x^2-2==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
y(x) \to e^x \Bigg(& \operatorname{AiryAi}(x+1) \int_1^x -e^{-K[1]}\pi \operatorname{AiryBi}(K[1]+1) \left(K[1]^2+2\right) dK[1] \\
& + \operatorname{AiryBi}(x+1) \int_1^x e^{-K[2]}\pi \operatorname{AiryAi}(K[2]+1) \left(K[2]^2+2\right) dK[2] \\
& + c_1 \operatorname{AiryAi}(x+1) + c_2 \operatorname{AiryBi}(x+1) \Bigg)
\end{aligned}
$$

2.2.19 problem 18

Internal problem ID [8499]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 18
Date solved : `Thursday, December 12, 2024 at 09:23:16 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - 4y' - xy - x^2 - 4 = 0$$

Solved as second order Airy ode

Time used: 0.065 (sec)

This is Airy ODE. It has the general form

$$ay'' + by' + cxy = F(x)$$

Where in this case

$$\begin{aligned} a &= 1 \\ b &= -4 \\ c &= -1 \\ F &= x^2 + 4 \end{aligned}$$

Therefore the solution to the homogeneous Airy ODE becomes

$$y = c_1 \,\mathrm{e}^{2x} \operatorname{AiryAi}\left((-x-4)\,(-1)^{1/3}\right) + c_2 \,\mathrm{e}^{2x} \operatorname{AiryBi}\left((-x-4)\,(-1)^{1/3}\right)$$

Since this is inhomogeneous Airy ODE, then we need to find the particular solution. The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2 + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{2x} \operatorname{AiryAi}\left((-x-4)(-1)^{1/3}\right), e^{2x} \operatorname{AiryBi}\left((-x-4)(-1)^{1/3}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_3 - 8xA_3 - 4A_2 - x(A_3x^2 + A_2x + A_1) - x^2 - 4 = 0$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = -1, A_3 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{2x} \operatorname{AiryAi}\left((-x-4)(-1)^{1/3}\right) + c_2 e^{2x} \operatorname{AiryBi}\left((-x-4)(-1)^{1/3}\right)\right) + (-x) \\
&= c_1 e^{2x} \operatorname{AiryAi}\left((-x-4)(-1)^{1/3}\right) + c_2 e^{2x} \operatorname{AiryBi}\left((-x-4)(-1)^{1/3}\right) - x
\end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 e^{2x} \operatorname{AiryAi}\left((-x-4)(-1)^{1/3}\right) + c_2 e^{2x} \operatorname{AiryBi}\left((-x-4)(-1)^{1/3}\right) - x$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.021 (sec)
Leaf size : 28

```
dsolve(diff(diff(y(x),x),x)-4*diff(y(x),x)-x*y(x)-x^2-4 = 0,
       y(x),singsol=all)
```

$$y = e^{2x} \operatorname{AiryAi}(x+4) c_2 + e^{2x} \operatorname{AiryBi}(x+4) c_1 - x$$

Mathematica DSolve solution

Solving time : `5.75` (sec)
Leaf size : 89

```
DSolve[{D[y[x],{x,2}]-4*D[y[x],x]-x*y[x]-x^2-4==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{2x}\left(\operatorname{AiryAi}(x+4)\int_1^x -e^{-2K[1]}\pi \operatorname{AiryBi}(K[1]+4)\left(K[1]^2+4\right)dK[1] + \operatorname{AiryBi}(x+4)\int_1^x e^{-2K[2]}\pi \operatorname{AiryAi}(K[2]+4)\left(K[2]^2+4\right)dK[2] + c_1 \operatorname{AiryAi}(x+4) + c_2 \operatorname{AiryBi}(x+4)\right)$$

2.2.20 problem 19

Internal problem ID [8500]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 19
Date solved : `Thursday, December 12, 2024 at 09:23:16 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' - y' - xy - x^3 + 1 = 0$$

Solved as second order Airy ode

Time used: 1.057 (sec)

This is Airy ODE. It has the general form

$$ay'' + by' + cxy = F(x)$$

Where in this case

$$\begin{aligned} a &= 1 \\ b &= -1 \\ c &= -1 \\ F &= x^3 - 1 \end{aligned}$$

Therefore the solution to the homogeneous Airy ODE becomes

$$y = c_1 \, \mathrm{e}^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x - \frac{1}{4}\right) (-1)^{1/3}\right) + c_2 \, \mathrm{e}^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x - \frac{1}{4}\right) (-1)^{1/3}\right)$$

Since this is inhomogeneous Airy ODE, then we need to find the particular solution. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will

be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x - \frac{1}{4}\right)(-1)^{1/3}\right)$$
$$y_2 = e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x - \frac{1}{4}\right)(-1)^{1/3}\right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x - \frac{1}{4}\right)(-1)^{1/3}\right) & e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x - \frac{1}{4}\right)(-1)^{1/3}\right) \\ \frac{d}{dx}\left(e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x - \frac{1}{4}\right)(-1)^{1/3}\right)\right) & \frac{d}{dx}\left(e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x - \frac{1}{4}\right)(-1)^{1/3}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x - \frac{1}{4}\right)(-1)^{1/3}\right) & e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-\right.\right. \\ \frac{e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)}{2} - e^{\frac{x}{2}}(-1)^{1/3} \operatorname{AiryAi}\left(1, \left(-x - \frac{1}{4}\right)(-1)^{1/3}\right) & \frac{e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)}{2} - e^{\frac{x}{2}}(-1 \end{vmatrix}$$

Therefore

$$W = \left(e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x - \frac{1}{4}\right)(-1)^{1/3}\right)\right)\left(\frac{e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x - \frac{1}{4}\right)(-1)^{1/3}\right)}{2}\right.$$
$$\left. - e^{\frac{x}{2}}(-1)^{1/3} \operatorname{AiryBi}\left(1, \left(-x - \frac{1}{4}\right)(-1)^{1/3}\right)\right) - \left(e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x - \frac{1}{4}\right)(-1)^{1/3}\right)\right)\left(\frac{e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x\right.\right.}{2}\right.$$

Which simplifies to

$$W = -\mathrm{e}^{x} \operatorname{AiryAi}\left(-\frac{(4x+1)(-1)^{1/3}}{4}\right)(-1)^{1/3} \operatorname{AiryBi}\left(1, -\frac{(4x+1)(-1)^{1/3}}{4}\right) + \mathrm{e}^{x} \operatorname{AiryBi}\left(-\frac{(4x+1)(-1}{4}\right.$$

Which simplifies to

$$W = -\frac{\mathrm{e}^{x}\left(1+i\sqrt{3}\right)}{2\pi}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\mathrm{e}^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)(x^3-1)}{-\frac{\mathrm{e}^{x}\left(1+i\sqrt{3}\right)}{2\pi}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{(-2x^3+2)\,\pi\,\mathrm{e}^{-\frac{x}{2}} \operatorname{AiryBi}\left(-\frac{\left(1+i\sqrt{3}\right)\left(x+\frac{1}{4}\right)}{2}\right)}{1+i\sqrt{3}} dx$$

Hence

$$u_1 = -\int_0^x \frac{(-2\alpha^3+2)\,\pi\,\mathrm{e}^{-\frac{\alpha}{2}} \operatorname{AiryBi}\left(-\frac{\left(1+i\sqrt{3}\right)\left(\alpha+\frac{1}{4}\right)}{2}\right)}{1+i\sqrt{3}} d\alpha$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)(x^3-1)}{-\frac{\mathrm{e}^{x}\left(1+i\sqrt{3}\right)}{2\pi}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{(-2x^3+2)\,\pi\,\mathrm{e}^{-\frac{x}{2}} \operatorname{AiryAi}\left(-\frac{\left(1+i\sqrt{3}\right)\left(x+\frac{1}{4}\right)}{2}\right)}{1+i\sqrt{3}} dx$$

Hence

$$u_2 = \int_0^x \frac{(-2\alpha^3+2)\,\pi\,\mathrm{e}^{-\frac{\alpha}{2}}\,\mathrm{AiryAi}\left(-\frac{\left(1+i\sqrt{3}\right)\left(\alpha+\frac{1}{4}\right)}{2}\right)}{1+i\sqrt{3}}d\alpha$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned}
y_p(x) &\\
&= -\int_0^x \frac{(-2\alpha^3+2)\,\pi\,\mathrm{e}^{-\frac{\alpha}{2}}\,\mathrm{AiryBi}\left(-\frac{\left(1+i\sqrt{3}\right)\left(\alpha+\frac{1}{4}\right)}{2}\right)}{1+i\sqrt{3}}d\alpha\mathrm{e}^{\frac{x}{2}}\,\mathrm{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)\\
&\quad+\mathrm{e}^{\frac{x}{2}}\,\mathrm{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)\int_0^x \frac{(-2\alpha^3+2)\,\pi\,\mathrm{e}^{-\frac{\alpha}{2}}\,\mathrm{AiryAi}\left(-\frac{\left(1+i\sqrt{3}\right)\left(\alpha+\frac{1}{4}\right)}{2}\right)}{1+i\sqrt{3}}d\alpha
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
y_p(x) &\\
&= \frac{2\,\mathrm{e}^{\frac{x}{2}}\pi\left(\mathrm{AiryAi}\left(-\frac{\left(1+i\sqrt{3}\right)\left(x+\frac{1}{4}\right)}{2}\right)\int_0^x (\alpha^3-1)\,\mathrm{e}^{-\frac{\alpha}{2}}\,\mathrm{AiryBi}\left(-\frac{\left(1+i\sqrt{3}\right)\left(\alpha+\frac{1}{4}\right)}{2}\right)d\alpha-\int_0^x (\alpha^3-1)\,\mathrm{e}^{-\frac{\alpha}{2}}\,\mathrm{AiryA}\right.}{1+i\sqrt{3}}
\end{aligned}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p\\
&= \left(c_1\,\mathrm{e}^{\frac{x}{2}}\,\mathrm{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)+c_2\,\mathrm{e}^{\frac{x}{2}}\,\mathrm{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)\right)\\
&\quad+\left(\frac{2\,\mathrm{e}^{\frac{x}{2}}\pi\left(\mathrm{AiryAi}\left(-\frac{\left(1+i\sqrt{3}\right)\left(x+\frac{1}{4}\right)}{2}\right)\int_0^x (\alpha^3-1)\,\mathrm{e}^{-\frac{\alpha}{2}}\,\mathrm{AiryBi}\left(-\frac{\left(1+i\sqrt{3}\right)\left(\alpha+\frac{1}{4}\right)}{2}\right)d\alpha-\int_0^x (\alpha^3-1)\,\mathrm{e}^{-\frac{\alpha}{2}}\,\mathrm{A}\right.}{1+i\sqrt{3}}\right.\\
&= c_1\,\mathrm{e}^{\frac{x}{2}}\,\mathrm{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)+c_2\,\mathrm{e}^{\frac{x}{2}}\,\mathrm{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)\\
&\quad+\frac{2\,\mathrm{e}^{\frac{x}{2}}\pi\left(\mathrm{AiryAi}\left(-\frac{\left(1+i\sqrt{3}\right)\left(x+\frac{1}{4}\right)}{2}\right)\int_0^x (\alpha^3-1)\,\mathrm{e}^{-\frac{\alpha}{2}}\,\mathrm{AiryBi}\left(-\frac{\left(1+i\sqrt{3}\right)\left(\alpha+\frac{1}{4}\right)}{2}\right)d\alpha-\int_0^x (\alpha^3-1)\,\mathrm{e}^{-\frac{\alpha}{2}}\,\mathrm{Air}\right.}{1+i\sqrt{3}}
\end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \,\mathrm{e}^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) + c_2 \,\mathrm{e}^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) + \frac{2\,\mathrm{e}^{\frac{x}{2}}\pi \left(\operatorname{AiryAi}\left(-\frac{\left(1+i\sqrt{3}\right)\left(x+\frac{1}{4}\right)}{2}\right) \int_0^x \left(\alpha^3-1\right)\mathrm{e}^{-\frac{\alpha}{2}} \operatorname{AiryBi}\left(-\frac{\left(1+i\sqrt{3}\right)\left(\alpha+\frac{1}{4}\right)}{2}\right) d\alpha - \int_0^x \left(\alpha^3-1\right)\mathrm{e}^{-\frac{\alpha}{2}} \operatorname{Airy}\right.}{1+i\sqrt{3}}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.025 (sec)
Leaf size : 67

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)-x*y(x)-x^3+1 = 0,
       y(x),singsol=all)
```

$$y = e^{\frac{x}{2}}\left(\operatorname{AiryBi}\left(x+\frac{1}{4}\right)\pi\left(\int \left(x^3-1\right)\operatorname{AiryAi}\left(x+\frac{1}{4}\right)e^{-\frac{x}{2}}dx\right) - \operatorname{AiryAi}\left(x+\frac{1}{4}\right)\pi\left(\int \left(x^3-1\right)\operatorname{AiryBi}\left(x+\frac{1}{4}\right)e^{-\frac{x}{2}}dx\right) + c_1\operatorname{AiryBi}\left(x+\frac{1}{4}\right) + c_2\operatorname{AiryAi}\left(x+\frac{1}{4}\right)\right)$$

Mathematica DSolve solution

Solving time : 3.812 (sec)
Leaf size : 107

```
DSolve[{D[y[x],{x,2}]-D[y[x],x]-x*y[x]-x^3+1==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{x/2}\left(\operatorname{AiryAi}\left(x+\frac{1}{4}\right)\int_1^x -e^{-\frac{K[1]}{2}}\pi\operatorname{AiryBi}\left(K[1]+\frac{1}{4}\right)\left(K[1]^3-1\right)dK[1] + \operatorname{AiryBi}\left(x+\frac{1}{4}\right)\int_1^x e^{-\frac{K[2]}{2}}\pi\operatorname{AiryAi}\left(K[2]+\frac{1}{4}\right)\left(K[2]^3-1\right)dK[2] + c_1\operatorname{AiryAi}\left(x+\frac{1}{4}\right) + c_2\operatorname{AiryBi}\left(x+\frac{1}{4}\right)\right)$$

2.2.21 problem 20

Internal problem ID [8501]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 20
Date solved : `Thursday, December 12, 2024 at 09:23:18 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' - 2y' - xy - x^3 - x^2 = 0$$

Solved as second order Airy ode

Time used: 0.069 (sec)

This is Airy ODE. It has the general form

$$ay'' + by' + cxy = F(x)$$

Where in this case

$$\begin{aligned} a &= 1 \\ b &= -2 \\ c &= -1 \\ F &= x^2(x+1) \end{aligned}$$

Therefore the solution to the homogeneous Airy ODE becomes

$$y = c_1 \,\mathrm{e}^{x} \operatorname{AiryAi}\left((-x-1)\,(-1)^{1/3}\right) + c_2 \,\mathrm{e}^{x} \operatorname{AiryBi}\left((-x-1)\,(-1)^{1/3}\right)$$

Since this is inhomogeneous Airy ODE, then we need to find the particular solution. The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3 + x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^x \operatorname{AiryAi}\left((-x-1)(-1)^{1/3}\right), e^x \operatorname{AiryBi}\left((-x-1)(-1)^{1/3}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_4x^3 + A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6xA_4 + 2A_3 - 6x^2A_4 - 4xA_3 - 2A_2 - x(A_4x^3 + A_3x^2 + A_2x + A_1) - x^3 - x^2 = 0$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 4, A_2 = -1, A_3 = -1, A_4 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x^2 - x + 4$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^x \operatorname{AiryAi}\left((-x-1)(-1)^{1/3}\right) + c_2 e^x \operatorname{AiryBi}\left((-x-1)(-1)^{1/3}\right)\right) + (-x^2 - x + 4) \\
&= c_1 e^x \operatorname{AiryAi}\left((-x-1)(-1)^{1/3}\right) + c_2 e^x \operatorname{AiryBi}\left((-x-1)(-1)^{1/3}\right) - x^2 - x + 4
\end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 e^x \operatorname{AiryAi}\left((-x-1)(-1)^{1/3}\right) + c_2 e^x \operatorname{AiryBi}\left((-x-1)(-1)^{1/3}\right) - x^2 - x + 4$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.020 (sec)
Leaf size : 30

```
dsolve(diff(diff(y(x),x),x)-2*diff(y(x),x)-x*y(x)-x^3-x^2 = 0,
       y(x),singsol=all)
```

$$y = e^{x} \operatorname{AiryAi}\left(x+1\right) c_{2}+e^{x} \operatorname{AiryBi}\left(x+1\right) c_{1}-x^{2}-x+4$$

Mathematica DSolve solution

Solving time : 7.976 (sec)
Leaf size : 91

```
DSolve[{D[y[x],{x,2}]-2*D[y[x],x]-x*y[x]-x^3-x^2==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
y(x) \to e^x \bigg(&\operatorname{AiryAi}(x+1) \int_1^x -e^{-K[1]}\pi \operatorname{AiryBi}(K[1]+1)K[1]^2(K[1]+1)dK[1] \\
&+ \operatorname{AiryBi}(x+1) \int_1^x e^{-K[2]}\pi \operatorname{AiryAi}(K[2]+1)K[2]^2(K[2]+1)dK[2] \\
&+ c_1 \operatorname{AiryAi}(x+1) + c_2 \operatorname{AiryBi}(x+1) \bigg)
\end{aligned}
$$

2.2.22 problem 21

Internal problem ID [8502]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 21
Date solved : `Thursday, December 12, 2024 at 09:23:19 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' - y' - xy - x^3 + 2 = 0$$

Solved as second order Airy ode

Time used: 0.074 (sec)

This is Airy ODE. It has the general form

$$ay'' + by' + cxy = F(x)$$

Where in this case

$$\begin{aligned} a &= 1 \\ b &= -1 \\ c &= -1 \\ F &= x^3 - 2 \end{aligned}$$

Therefore the solution to the homogeneous Airy ODE becomes

$$y = c_1 \,\mathrm{e}^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x - \frac{1}{4}\right)(-1)^{1/3}\right) + c_2 \,\mathrm{e}^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x - \frac{1}{4}\right)(-1)^{1/3}\right)$$

Since this is inhomogeneous Airy ODE, then we need to find the particular solution. The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3 + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right), e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_4 x^3 + A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6xA_4 + 2A_3 - 3x^2 A_4 - 2xA_3 - A_2 - x\left(A_4 x^3 + A_3 x^2 + A_2 x + A_1\right) - x^3 + 2 = 0$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = 0, A_3 = -1, A_4 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x^2 + 2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) + c_2 e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)\right) + \left(-x^2+2\right) \\ &= c_1 e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) + c_2 e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) - x^2 + 2 \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) + c_2 e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) - x^2 + 2$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.021 (sec)
Leaf size : 31

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)-x*y(x)-x^3+2 = 0,
       y(x),singsol=all)
```

$$y = e^{\frac{x}{2}} \operatorname{AiryAi}\left(x+\frac{1}{4}\right) c_2 + e^{\frac{x}{2}} \operatorname{AiryBi}\left(x+\frac{1}{4}\right) c_1 - x^2 + 2$$

Mathematica DSolve solution

Solving time : `3.756` (sec)
Leaf size : 107

```
DSolve[{D[y[x],{x,2}]-D[y[x],x]-x*y[x]-x^3+2==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
y(x) \to e^{x/2}\bigg(&\operatorname{AiryAi}\left(x+\frac{1}{4}\right)\int_1^x -e^{-\frac{K[1]}{2}}\pi \operatorname{AiryBi}\left(K[1]+\frac{1}{4}\right)\left(K[1]^3-2\right)dK[1] \\
&+ \operatorname{AiryBi}\left(x+\frac{1}{4}\right)\int_1^x e^{-\frac{K[2]}{2}}\pi \operatorname{AiryAi}\left(K[2]+\frac{1}{4}\right)\left(K[2]^3-2\right)dK[2] \\
&+ c_1 \operatorname{AiryAi}\left(x+\frac{1}{4}\right) + c_2 \operatorname{AiryBi}\left(x+\frac{1}{4}\right)\bigg)
\end{aligned}
$$

2.2.23 problem 22

Internal problem ID [8503]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 22
Date solved : `Thursday, December 12, 2024 at 09:23:19 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' - 2y' - xy - x^3 + 2 = 0$$

Solved as second order Airy ode

Time used: 0.067 (sec)

This is Airy ODE. It has the general form

$$ay'' + by' + cxy = F(x)$$

Where in this case

$$\begin{aligned} a &= 1 \\ b &= -2 \\ c &= -1 \\ F &= x^3 - 2 \end{aligned}$$

Therefore the solution to the homogeneous Airy ODE becomes

$$y = c_1 \,\mathrm{e}^{x} \operatorname{AiryAi}\left((-x-1)\,(-1)^{1/3}\right) + c_2 \,\mathrm{e}^{x} \operatorname{AiryBi}\left((-x-1)\,(-1)^{1/3}\right)$$

Since this is inhomogeneous Airy ODE, then we need to find the particular solution. The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3 + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^x \operatorname{AiryAi}\left((-x-1)(-1)^{1/3}\right), e^x \operatorname{AiryBi}\left((-x-1)(-1)^{1/3}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_4x^3 + A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6xA_4 + 2A_3 - 6x^2A_4 - 4xA_3 - 2A_2 - x(A_4x^3 + A_3x^2 + A_2x + A_1) - x^3 + 2 = 0$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 4, A_2 = 0, A_3 = -1, A_4 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x^2 + 4$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^x \operatorname{AiryAi}\left((-x-1)(-1)^{1/3}\right) + c_2 e^x \operatorname{AiryBi}\left((-x-1)(-1)^{1/3}\right)\right) + (-x^2 + 4) \\
&= c_1 e^x \operatorname{AiryAi}\left((-x-1)(-1)^{1/3}\right) + c_2 e^x \operatorname{AiryBi}\left((-x-1)(-1)^{1/3}\right) - x^2 + 4
\end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 e^x \operatorname{AiryAi}\left((-x-1)(-1)^{1/3}\right) + c_2 e^x \operatorname{AiryBi}\left((-x-1)(-1)^{1/3}\right) - x^2 + 4$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.021 (sec)
Leaf size : 27

```
dsolve(diff(diff(y(x),x),x)-2*diff(y(x),x)-x*y(x)-x^3+2 = 0,
       y(x),singsol=all)
```

$$y = e^{x} \operatorname{AiryAi}\left(x+1\right) c_{2}+e^{x} \operatorname{AiryBi}\left(x+1\right) c_{1}-x^{2}+4$$

Mathematica DSolve solution

Solving time : 2.465 (sec)
Leaf size : 87

```
DSolve[{D[y[x],{x,2}]-2*D[y[x],x]-x*y[x]-x^3+2==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
y(x) \to e^x \Bigg(& \operatorname{AiryAi}(x+1) \int_1^x -e^{-K[1]} \pi \operatorname{AiryBi}(K[1]+1) \left(K[1]^3-2\right) dK[1] \\
& + \operatorname{AiryBi}(x+1) \int_1^x e^{-K[2]} \pi \operatorname{AiryAi}(K[2]+1) \left(K[2]^3-2\right) dK[2] \\
& + c_1 \operatorname{AiryAi}(x+1) + c_2 \operatorname{AiryBi}(x+1) \Bigg)
\end{aligned}
$$

2.2.24 problem 23

Internal problem ID [8504]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 23
Date solved : `Thursday, December 12, 2024 at 09:23:20 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' - 4y' - xy - x^3 + 2 = 0$$

Solved as second order Airy ode

Time used: 0.069 (sec)

This is Airy ODE. It has the general form

$$ay'' + by' + cxy = F(x)$$

Where in this case

$$\begin{aligned} a &= 1 \\ b &= -4 \\ c &= -1 \\ F &= x^3 - 2 \end{aligned}$$

Therefore the solution to the homogeneous Airy ODE becomes

$$y = c_1 \, \mathrm{e}^{2x} \operatorname{AiryAi}\left((-x-4)(-1)^{1/3}\right) + c_2 \, \mathrm{e}^{2x} \operatorname{AiryBi}\left((-x-4)(-1)^{1/3}\right)$$

Since this is inhomogeneous Airy ODE, then we need to find the particular solution. The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3 + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{2x} \operatorname{AiryAi}\left((-x-4)(-1)^{1/3}\right), e^{2x} \operatorname{AiryBi}\left((-x-4)(-1)^{1/3}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_4x^3 + A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6xA_4 + 2A_3 - 12x^2A_4 - 8xA_3 - 4A_2 - x\left(A_4x^3 + A_3x^2 + A_2x + A_1\right) - x^3 + 2 = 0$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 8, A_2 = 0, A_3 = -1, A_4 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x^2 + 8$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} \operatorname{AiryAi}\left((-x-4)(-1)^{1/3}\right) + c_2 e^{2x} \operatorname{AiryBi}\left((-x-4)(-1)^{1/3}\right)\right) + \left(-x^2 + 8\right) \\ &= c_1 e^{2x} \operatorname{AiryAi}\left((-x-4)(-1)^{1/3}\right) + c_2 e^{2x} \operatorname{AiryBi}\left((-x-4)(-1)^{1/3}\right) - x^2 + 8 \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 e^{2x} \operatorname{AiryAi}\left((-x-4)(-1)^{1/3}\right) + c_2 e^{2x} \operatorname{AiryBi}\left((-x-4)(-1)^{1/3}\right) - x^2 + 8$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.021 (sec)
Leaf size : 31

```
dsolve(diff(diff(y(x),x),x)-4*diff(y(x),x)-x*y(x)-x^3+2 = 0,
       y(x),singsol=all)
```

$$y = e^{2x} \operatorname{AiryAi}(x+4)\, c_2 + e^{2x} \operatorname{AiryBi}(x+4)\, c_1 - x^2 + 8$$

Mathematica DSolve solution

Solving time : 2.634 (sec)
Leaf size : 89

```
DSolve[{D[y[x],{x,2}]-4*D[y[x],x]-x*y[x]-x^3+2==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{2x}\left(\operatorname{AiryAi}(x+4)\int_1^x -e^{-2K[1]}\pi \operatorname{AiryBi}(K[1]+4)\left(K[1]^3-2\right)dK[1] + \operatorname{AiryBi}(x+4)\int_1^x e^{-2K[2]}\pi \operatorname{AiryAi}(K[2]+4)\left(K[2]^3-2\right)dK[2] + c_1 \operatorname{AiryAi}(x+4) + c_2 \operatorname{AiryBi}(x+4)\right)$$

2.2.25 problem 24

Internal problem ID [8505]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 24
Date solved : `Thursday, December 12, 2024 at 09:23:21 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' - 6y' - xy - x^3 + 2 = 0$$

Solved as second order Airy ode

Time used: 0.072 (sec)

This is Airy ODE. It has the general form

$$ay'' + by' + cxy = F(x)$$

Where in this case

$$\begin{aligned}a &= 1\\ b &= -6\\ c &= -1\\ F &= x^3 - 2\end{aligned}$$

Therefore the solution to the homogeneous Airy ODE becomes

$$y = c_1 \,\mathrm{e}^{3x} \operatorname{AiryAi}\left((-x-9)(-1)^{1/3}\right) + c_2 \,\mathrm{e}^{3x} \operatorname{AiryBi}\left((-x-9)(-1)^{1/3}\right)$$

Since this is inhomogeneous Airy ODE, then we need to find the particular solution. The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3 + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{3x} \operatorname{AiryAi}\left((-x-9)(-1)^{1/3}\right), e^{3x} \operatorname{AiryBi}\left((-x-9)(-1)^{1/3}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_4x^3 + A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6xA_4 + 2A_3 - 18x^2A_4 - 12xA_3 - 6A_2 - x\left(A_4x^3 + A_3x^2 + A_2x + A_1\right) - x^3 + 2 = 0$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 12, A_2 = 0, A_3 = -1, A_4 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x^2 + 12$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{3x} \operatorname{AiryAi}\left((-x-9)(-1)^{1/3}\right) + c_2 e^{3x} \operatorname{AiryBi}\left((-x-9)(-1)^{1/3}\right)\right) + \left(-x^2 + 12\right) \\ &= c_1 e^{3x} \operatorname{AiryAi}\left((-x-9)(-1)^{1/3}\right) + c_2 e^{3x} \operatorname{AiryBi}\left((-x-9)(-1)^{1/3}\right) - x^2 + 12 \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 e^{3x} \operatorname{AiryAi}\left((-x-9)(-1)^{1/3}\right) + c_2 e^{3x} \operatorname{AiryBi}\left((-x-9)(-1)^{1/3}\right) - x^2 + 12$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.021 (sec)
Leaf size : 31

```
dsolve(diff(diff(y(x),x),x)-6*diff(y(x),x)-x*y(x)-x^3+2 = 0,
       y(x),singsol=all)
```

$$y = e^{3x} \operatorname{AiryAi}(9 + x) c_2 + e^{3x} \operatorname{AiryBi}(9 + x) c_1 - x^2 + 12$$

Mathematica DSolve solution

Solving time : `6.195` (sec)
Leaf size : 89

```
DSolve[{D[y[x],{x,2}]-6*D[y[x],x]-x*y[x]-x^3+2==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
y(x) \to e^{3x}\Bigg(&\text{AiryAi}(x+9)\int_1^x -e^{-3K[1]}\pi\,\text{AiryBi}(K[1]+9)\left(K[1]^3-2\right)dK[1] \\
&+\text{AiryBi}(x+9)\int_1^x e^{-3K[2]}\pi\,\text{AiryAi}(K[2]+9)\left(K[2]^3-2\right)dK[2] \\
&+c_1\,\text{AiryAi}(x+9)+c_2\,\text{AiryBi}(x+9)\Bigg)
\end{aligned}
$$

2.2.26 problem 25

Internal problem ID [8506]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 25
Date solved : `Thursday, December 12, 2024 at 09:23:22 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' - 8y' - xy - x^3 + 2 = 0$$

Solved as second order Airy ode

Time used: 0.071 (sec)

This is Airy ODE. It has the general form

$$ay'' + by' + cxy = F(x)$$

Where in this case

$$\begin{aligned}
a &= 1 \\
b &= -8 \\
c &= -1 \\
F &= x^3 - 2
\end{aligned}$$

Therefore the solution to the homogeneous Airy ODE becomes

$$y = c_1\, \mathrm{e}^{4x} \operatorname{AiryAi}\left((-x-16)\,(-1)^{1/3}\right) + c_2\, \mathrm{e}^{4x} \operatorname{AiryBi}\left((-x-16)\,(-1)^{1/3}\right)$$

Since this is inhomogeneous Airy ODE, then we need to find the particular solution. The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3 + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ \mathrm{e}^{4x} \operatorname{AiryAi}\left((-x-16)\,(-1)^{1/3} \right), \mathrm{e}^{4x} \operatorname{AiryBi}\left((-x-16)\,(-1)^{1/3} \right) \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_4x^3 + A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6xA_4 + 2A_3 - 24x^2A_4 - 16xA_3 - 8A_2 - x\left(A_4x^3 + A_3x^2 + A_2x + A_1\right) - x^3 + 2 = 0$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 16, A_2 = 0, A_3 = -1, A_4 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x^2 + 16$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1\,\mathrm{e}^{4x} \operatorname{AiryAi}\left((-x-16)\,(-1)^{1/3} \right) + c_2\,\mathrm{e}^{4x} \operatorname{AiryBi}\left((-x-16)\,(-1)^{1/3} \right)\right) + \left(-x^2+16\right) \\
&= c_1\,\mathrm{e}^{4x} \operatorname{AiryAi}\left((-x-16)\,(-1)^{1/3} \right) + c_2\,\mathrm{e}^{4x} \operatorname{AiryBi}\left((-x-16)\,(-1)^{1/3} \right) - x^2 + 16
\end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1\,\mathrm{e}^{4x} \operatorname{AiryAi}\left((-x-16)\,(-1)^{1/3} \right) + c_2\,\mathrm{e}^{4x} \operatorname{AiryBi}\left((-x-16)\,(-1)^{1/3} \right) - x^2 + 16$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.020 (sec)
Leaf size : 31

```
dsolve(diff(diff(y(x),x),x)-8*diff(y(x),x)-x*y(x)-x^3+2 = 0,
       y(x),singsol=all)
```

$$y = e^{4x} \operatorname{AiryAi}(16 + x) c_2 + e^{4x} \operatorname{AiryBi}(16 + x) c_1 - x^2 + 16$$

Mathematica DSolve solution

Solving time : 6.306 (sec)
Leaf size : 89

```
DSolve[{D[y[x],{x,2}]-8*D[y[x],x]-x*y[x]-x^3+2==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{4x} \left(\text{AiryAi}(x+16) \int_1^x -e^{-4K[1]} \pi \, \text{AiryBi}(K[1]+16) \left(K[1]^3-2\right) dK[1] + \text{AiryBi}(x+16) \int_1^x e^{-4K[2]} \pi \, \text{AiryAi}(K[2]+16) \left(K[2]^3-2\right) dK[2] + c_1 \, \text{AiryAi}(x+16) + c_2 \, \text{AiryBi}(x+16) \right)$$

2.2.27 problem 26

Internal problem ID [8507]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 26
Date solved : `Thursday, December 12, 2024 at 09:23:22 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' - y' - xy - x^4 + 3 = 0$$

Solved as second order Airy ode

Time used: 0.078 (sec)

This is Airy ODE. It has the general form

$$ay'' + by' + cxy = F(x)$$

Where in this case

$$\begin{aligned}a &= 1\\ b &= -1\\ c &= -1\\ F &= x^4 - 3\end{aligned}$$

Therefore the solution to the homogeneous Airy ODE becomes

$$y = c_1 \,\mathrm{e}^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x - \frac{1}{4}\right)(-1)^{1/3}\right) + c_2 \,\mathrm{e}^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x - \frac{1}{4}\right)(-1)^{1/3}\right)$$

Since this is inhomogeneous Airy ODE, then we need to find the particular solution. The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^4 + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2, x^3, x^4\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right), e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_5x^4 + A_4x^3 + A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} &12x^2A_5 + 6xA_4 + 2A_3 - 4x^3A_5 - 3x^2A_4 - 2xA_3 - A_2 \\ &- x(A_5x^4 + A_4x^3 + A_3x^2 + A_2x + A_1) - x^4 + 3 = 0 \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -6, A_2 = 3, A_3 = 0, A_4 = -1, A_5 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x^3 + 3x - 6$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) \right. \\ &\qquad \left. + c_2 e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)\right) + (-x^3 + 3x - 6) \\ &= c_1 e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) + c_2 e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) - x^3 + 3x - 6 \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 e^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) + c_2 e^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) - x^3 + 3x - 6$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.022 (sec)
Leaf size : 34

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)-x*y(x)-x^4+3 = 0,
       y(x),singsol=all)
```

$$y = e^{\frac{x}{2}} \operatorname{AiryAi}\left(x+\frac{1}{4}\right) c_2 + e^{\frac{x}{2}} \operatorname{AiryBi}\left(x+\frac{1}{4}\right) c_1 - x^3 + 3x - 6$$

Mathematica DSolve solution

Solving time : 3.797 (sec)
Leaf size : 107

```
DSolve[{D[y[x],{x,2}]-D[y[x],x]-x*y[x]-x^4+3==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{x/2}\left(\text{AiryAi}\left(x+\frac{1}{4}\right)\int_1^x -e^{-\frac{K[1]}{2}}\pi\,\text{AiryBi}\left(K[1]+\frac{1}{4}\right)\left(K[1]^4-3\right)dK[1]\right.$$
$$+\text{AiryBi}\left(x+\frac{1}{4}\right)\int_1^x e^{-\frac{K[2]}{2}}\pi\,\text{AiryAi}\left(K[2]+\frac{1}{4}\right)\left(K[2]^4-3\right)dK[2]$$
$$\left.+c_1\,\text{AiryAi}\left(x+\frac{1}{4}\right)+c_2\,\text{AiryBi}\left(x+\frac{1}{4}\right)\right)$$

2.2.28 problem 27

Internal problem ID [8508]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 27
Date solved : `Thursday, December 12, 2024 at 09:23:23 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' - y' - xy - x^3 = 0$$

Solved as second order Airy ode

Time used: 1.040 (sec)

This is Airy ODE. It has the general form

$$ay'' + by' + cxy = F(x)$$

Where in this case

$$\begin{aligned} a &= 1 \\ b &= -1 \\ c &= -1 \\ F &= x^3 \end{aligned}$$

Therefore the solution to the homogeneous Airy ODE becomes

$$y = c_1 \, \mathrm{e}^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x - \frac{1}{4}\right)(-1)^{1/3}\right) + c_2 \, \mathrm{e}^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x - \frac{1}{4}\right)(-1)^{1/3}\right)$$

Since this is inhomogeneous Airy ODE, then we need to find the particular solution. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will

be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)$$
$$y_2 = \mathrm{e}^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) & \mathrm{e}^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) \\ \frac{d}{dx}\left(\mathrm{e}^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)\right) & \frac{d}{dx}\left(\mathrm{e}^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) & \mathrm{e}^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(- \right.\right. \\ \frac{\mathrm{e}^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)}{2} - \mathrm{e}^{\frac{x}{2}}(-1)^{1/3} \operatorname{AiryAi}\left(1, \left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) & \frac{\mathrm{e}^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)}{2} - \mathrm{e}^{\frac{x}{2}}(-1 \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{e}^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)\right)\left(\frac{\mathrm{e}^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)}{2}\right.$$
$$\left. -\mathrm{e}^{\frac{x}{2}}(-1)^{1/3} \operatorname{AiryBi}\left(1, \left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)\right) - \left(\mathrm{e}^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)\right)\left(\frac{\mathrm{e}^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x\right.\right.}{2}\right.$$

Which simplifies to

$$W = -\mathrm{e}^{x} \operatorname{AiryAi}\left(-\frac{(4x+1)(-1)^{1/3}}{4}\right)(-1)^{1/3} \operatorname{AiryBi}\left(1, -\frac{(4x+1)(-1)^{1/3}}{4}\right) + \mathrm{e}^{x} \operatorname{AiryBi}\left(-\frac{(4x+1)(-1}{4}\right.$$

Which simplifies to

$$W = -\frac{\mathrm{e}^{x}\left(1+i\sqrt{3}\right)}{2\pi}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\mathrm{e}^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) x^3}{-\frac{\mathrm{e}^{x}\left(1+i\sqrt{3}\right)}{2\pi}}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{2\pi\, x^3 \mathrm{e}^{-\frac{x}{2}} \operatorname{AiryBi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)}{1+i\sqrt{3}} dx$$

Hence

$$u_1 = -\int_0^x -\frac{2\pi\, \alpha^3 \mathrm{e}^{-\frac{\alpha}{2}} \operatorname{AiryBi}\left(-\frac{\left(\alpha+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)}{1+i\sqrt{3}} d\alpha$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) x^3}{-\frac{\mathrm{e}^{x}\left(1+i\sqrt{3}\right)}{2\pi}}\, dx$$

Which simplifies to

$$u_2 = \int -\frac{2\pi\, x^3 \mathrm{e}^{-\frac{x}{2}} \operatorname{AiryAi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)}{1+i\sqrt{3}} dx$$

Hence

$$u_2 = \int_0^x -\frac{2\pi\,\alpha^3 \mathrm{e}^{-\frac{\alpha}{2}}\operatorname{AiryAi}\left(-\frac{\left(\alpha+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)}{1+i\sqrt{3}}d\alpha$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = &-\int_0^x -\frac{2\pi\,\alpha^3 \mathrm{e}^{-\frac{\alpha}{2}}\operatorname{AiryBi}\left(-\frac{\left(\alpha+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)}{1+i\sqrt{3}}d\alpha \mathrm{e}^{\frac{x}{2}}\operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) \\ &+ \mathrm{e}^{\frac{x}{2}}\operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)\int_0^x -\frac{2\pi\,\alpha^3 \mathrm{e}^{-\frac{\alpha}{2}}\operatorname{AiryAi}\left(-\frac{\left(\alpha+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)}{1+i\sqrt{3}}d\alpha \end{aligned}$$

Which simplifies to

$$\begin{aligned} &y_p(x) \\ &= \frac{2\,\mathrm{e}^{\frac{x}{2}}\pi\left(\operatorname{AiryAi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)\int_0^x \alpha^3 \mathrm{e}^{-\frac{\alpha}{2}}\operatorname{AiryBi}\left(-\frac{\left(\alpha+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)d\alpha - \operatorname{AiryBi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)\int_0^x \right.}{1+i\sqrt{3}} \end{aligned}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\,\mathrm{e}^{\frac{x}{2}}\operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) + c_2\,\mathrm{e}^{\frac{x}{2}}\operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right)\right) \\ &\quad + \left(\frac{2\,\mathrm{e}^{\frac{x}{2}}\pi\left(\operatorname{AiryAi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)\int_0^x \alpha^3 \mathrm{e}^{-\frac{\alpha}{2}}\operatorname{AiryBi}\left(-\frac{\left(\alpha+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)d\alpha - \operatorname{AiryBi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right.\right.}{1+i\sqrt{3}}\right. \\ &= c_1\,\mathrm{e}^{\frac{x}{2}}\operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) + c_2\,\mathrm{e}^{\frac{x}{2}}\operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) \\ &\quad + \frac{2\,\mathrm{e}^{\frac{x}{2}}\pi\left(\operatorname{AiryAi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)\int_0^x \alpha^3 \mathrm{e}^{-\frac{\alpha}{2}}\operatorname{AiryBi}\left(-\frac{\left(\alpha+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)d\alpha - \operatorname{AiryBi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)\right.}{1+i\sqrt{3}} \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \,\mathrm{e}^{\frac{x}{2}} \operatorname{AiryAi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) + c_2 \,\mathrm{e}^{\frac{x}{2}} \operatorname{AiryBi}\left(\left(-x-\frac{1}{4}\right)(-1)^{1/3}\right) + \frac{2\,\mathrm{e}^{\frac{x}{2}}\pi\left(\operatorname{AiryAi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right)\int_0^x \alpha^3 \mathrm{e}^{-\frac{\alpha}{2}} \operatorname{AiryBi}\left(-\frac{\left(\alpha+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right) d\alpha - \operatorname{AiryBi}\left(-\frac{\left(x+\frac{1}{4}\right)\left(1+i\sqrt{3}\right)}{2}\right) \int}{1+i\sqrt{3}}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : `0.024` (sec)
Leaf size : 63

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)-x*y(x)-x^3 = 0,
       y(x),singsol=all)
```

$$
\begin{aligned}
y = \mathrm{e}^{\frac{x}{2}} \Bigg(& \mathrm{AiryBi}\left(x+\frac{1}{4}\right)\pi\left(\int x^{3}\,\mathrm{AiryAi}\left(x+\frac{1}{4}\right)\mathrm{e}^{-\frac{x}{2}}dx\right) \\
& - \mathrm{AiryAi}\left(x+\frac{1}{4}\right)\pi\left(\int x^{3}\,\mathrm{AiryBi}\left(x+\frac{1}{4}\right)\mathrm{e}^{-\frac{x}{2}}dx\right)+c_{1}\,\mathrm{AiryBi}\left(x+\frac{1}{4}\right) \\
& + c_{2}\,\mathrm{AiryAi}\left(x+\frac{1}{4}\right)\Bigg)
\end{aligned}
$$

Mathematica DSolve solution

Solving time : `9.893` (sec)
Leaf size : 103

```
DSolve[{D[y[x],{x,2}]-D[y[x],x]-x*y[x]-x^3==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
y(x)\to e^{x/2}\Bigg(& \mathrm{AiryAi}\left(x+\frac{1}{4}\right)\int_{1}^{x}-e^{-\frac{K[1]}{2}}\pi\,\mathrm{AiryBi}\left(K[1]+\frac{1}{4}\right)K[1]^{3}dK[1] \\
& + \mathrm{AiryBi}\left(x+\frac{1}{4}\right)\int_{1}^{x}e^{-\frac{K[2]}{2}}\pi\,\mathrm{AiryAi}\left(K[2]+\frac{1}{4}\right)K[2]^{3}dK[2] \\
& + c_{1}\,\mathrm{AiryAi}\left(x+\frac{1}{4}\right)+c_{2}\,\mathrm{AiryBi}\left(x+\frac{1}{4}\right)\Bigg)
\end{aligned}
$$

2.2.29 problem 28

Internal problem ID [8509]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 28
Date solved : `Thursday, December 12, 2024 at 09:23:25 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' - xy - x^3 + 2 = 0$$

Solved as second order Airy ode

Time used: 0.061 (sec)

This is Airy ODE. It has the general form

$$ay'' + by' + cxy = F(x)$$

Where in this case

$$\begin{aligned} a &= 1 \\ b &= 0 \\ c &= -1 \\ F &= x^3 - 2 \end{aligned}$$

Therefore the solution to the homogeneous Airy ODE becomes

$$y = c_1 \operatorname{AiryAi}\left(-x(-1)^{1/3}\right) + c_2 \operatorname{AiryBi}\left(-x(-1)^{1/3}\right)$$

Since this is inhomogeneous Airy ODE, then we need to find the particular solution. The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^3 + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2, x^3\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\text{AiryAi}\left(-x(-1)^{1/3}\right), \text{AiryBi}\left(-x(-1)^{1/3}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_4x^3 + A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6xA_4 + 2A_3 - x(A_4x^3 + A_3x^2 + A_2x + A_1) - x^3 + 2 = 0$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 0, A_3 = -1, A_4 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x^2$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \text{AiryAi}\left(-x(-1)^{1/3}\right) + c_2 \text{AiryBi}\left(-x(-1)^{1/3}\right)\right) + (-x^2) \\
&= c_1 \text{AiryAi}\left(-x(-1)^{1/3}\right) + c_2 \text{AiryBi}\left(-x(-1)^{1/3}\right) - x^2
\end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \text{AiryAi}\left(-x(-1)^{1/3}\right) + c_2 \text{AiryBi}\left(-x(-1)^{1/3}\right) - x^2$$

Solved as second order ode adjoint method

Time used: 1.019 (sec)

In normal form the ode

$$y'' - xy - x^3 + 2 = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= -x \\ r(x) &= x^3 - 2 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (0)' + (-x\xi(x)) &= 0 \\ \xi''(x) - x\xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is Airy ODE. It has the general form

$$a\xi'' + b\xi' + cx\xi = F(x)$$

Where in this case

$$\begin{aligned} a &= 1 \\ b &= 0 \\ c &= -1 \\ F &= 0 \end{aligned}$$

Therefore the solution to the homogeneous Airy ODE becomes

$$\xi = c_5 \operatorname{AiryAi}\left(-x(-1)^{1/3}\right) + c_6 \operatorname{AiryBi}\left(-x(-1)^{1/3}\right)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$
\begin{aligned}
\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)}
\end{aligned}
$$

Or

$$
y' - \frac{y\left(-c_5(-1)^{1/3}\operatorname{AiryAi}\left(1, -x(-1)^{1/3}\right) - c_6(-1)^{1/3}\operatorname{AiryBi}\left(1, -x(-1)^{1/3}\right)\right)}{c_5\operatorname{AiryAi}\left(-x\,(-1)^{1/3}\right) + c_6\operatorname{AiryBi}\left(-x\,(-1)^{1/3}\right)} = \frac{-2c_5 x\operatorname{AiryAi}\left(-x(-\right.}{}
$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$
y' + q(x)y = p(x)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= \frac{\left(1+i\sqrt{3}\right)\left(\operatorname{AiryAi}\left(1, -\frac{x\left(1+i\sqrt{3}\right)}{2}\right)c_5 + \operatorname{AiryBi}\left(1, -\frac{x\left(1+i\sqrt{3}\right)}{2}\right)c_6\right)}{2c_5\operatorname{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + 2c_6\operatorname{AiryBi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)} \\
p(x) &= -\frac{x\left(xc_5\left(1+i\sqrt{3}\right)\operatorname{AiryAi}\left(1, -\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + xc_6\left(1+i\sqrt{3}\right)\operatorname{AiryBi}\left(1, -\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + 4c_5\operatorname{AiryAi}\left(\right.\right.}{2c_5\operatorname{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + 2c_6\operatorname{AiryBi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= \mathrm{e}^{\int \frac{\left(1+i\sqrt{3}\right)\left(\operatorname{AiryAi}\left(1, -\frac{x\left(1+i\sqrt{3}\right)}{2}\right)c_5 + \operatorname{AiryBi}\left(1, -\frac{x\left(1+i\sqrt{3}\right)}{2}\right)c_6\right)}{2c_5\operatorname{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + 2c_6\operatorname{AiryBi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)}dx} \\
&= \frac{1}{c_5\operatorname{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + c_6\operatorname{AiryBi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)}
\end{aligned}
$$

The ode becomes

$$\frac{d}{dx}(\mu y) = \mu p$$

$$\frac{d}{dx}(\mu y) = (\mu)\left(-\frac{x\left(xc_5(1+i\sqrt{3})\operatorname{AiryAi}\left(1, -\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + xc_6(1+i\sqrt{3})\operatorname{AiryBi}\left(1, -\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + 4c_5\operatorname{AiryAi}\left(\right.\right.}{2c_5\operatorname{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + 2c_6\operatorname{AiryBi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)}\right.$$

$$\frac{d}{dx}\left(\frac{y}{c_5\operatorname{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + c_6\operatorname{AiryBi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)}\right) = \left(\frac{1}{c_5\operatorname{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + c_6\operatorname{AiryBi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)}\right)\left(-\frac{x\left(xc_5(1+i\sqrt{3})\operatorname{AiryAi}\left(1, -\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + \right.}{2}\right.$$

$$d\left(\frac{y}{c_5\operatorname{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + c_6\operatorname{AiryBi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)}\right) = \left(-\frac{x\left(xc_5(1+i\sqrt{3})\operatorname{AiryAi}\left(1, -\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + xc_6(1+i\sqrt{3})\operatorname{AiryBi}\left(1, -\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + 4c_5\operatorname{AiryAi}\left(-\right.\right.}{\left(2c_5\operatorname{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + 2c_6\operatorname{AiryBi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)\right)\left(c_5\operatorname{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + \right.}\right.$$

Integrating gives

$$\begin{aligned}\frac{y}{c_5\operatorname{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + c_6\operatorname{AiryBi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)} &= \int -\frac{x\left(xc_5(1+i\sqrt{3})\operatorname{AiryAi}\left(1, -\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + xc_6\right.}{\left(2c_5\operatorname{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + 2c_6\right.} \\ &= -\frac{x^2}{c_5\operatorname{AiryAi}\left(\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)x\right) + c_6\operatorname{AiryBi}\left(\left(-\frac{1}{2} - \right.\right.}\end{aligned}$$

Dividing throughout by the integrating factor $\frac{1}{c_5\operatorname{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)+c_6\operatorname{AiryBi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)}$

gives the final solution

$$y = \mathrm{AiryAi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) c_5 c_7 + \mathrm{AiryBi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) c_6 c_7 - x^2$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \mathrm{AiryAi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) c_5 c_7 + \mathrm{AiryBi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) c_6 c_7 - x^2$$

The constants can be merged to give

$$y = c_5 \,\mathrm{AiryAi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) + c_6 \,\mathrm{AiryBi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) - x^2$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_5 \,\mathrm{AiryAi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) + c_6 \,\mathrm{AiryBi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) - x^2$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
```

```
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 18

```
dsolve(diff(diff(y(x),x),x)-x*y(x)-x^3+2 = 0,
       y(x),singsol=all)
```

$$y = \mathrm{AiryAi}\left(x\right) c_2 + \mathrm{AiryBi}\left(x\right) c_1 - x^2$$

Mathematica DSolve solution

Solving time : 0.456 (sec)
Leaf size : 290

```
DSolve[{D[y[x],{x,2}]-x*y[x]-x^3+2==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{6\sqrt[3]{3}\pi x\,\mathrm{Gamma}\left(\frac{1}{3}\right)\mathrm{Gamma}\left(\frac{5}{3}\right)\mathrm{Gamma}\left(\frac{7}{3}\right)\mathrm{Gamma}\left(\frac{8}{3}\right)\left(\sqrt{3}\,\mathrm{AiryAi}(x) - \mathrm{AiryBi}(x)\right)\,{}_1F_2\left(\frac{1}{3};\frac{2}{3},\frac{4}{3};\frac{x^3}{9}\right)}{}$$

2.2.30 problem 29

Internal problem ID [8510]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 29
Date solved : `Thursday, December 12, 2024 at 09:23:28 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' - xy - x^6 + 64 = 0$$

Solved as second order Airy ode

Time used: 0.646 (sec)

This is Airy ODE. It has the general form

$$ay'' + by' + cxy = F(x)$$

Where in this case

$$\begin{aligned} a &= 1 \\ b &= 0 \\ c &= -1 \\ F &= x^6 - 64 \end{aligned}$$

Therefore the solution to the homogeneous Airy ODE becomes

$$y = c_1 \operatorname{AiryAi}\left(-x(-1)^{1/3}\right) + c_2 \operatorname{AiryBi}\left(-x(-1)^{1/3}\right)$$

Since this is inhomogeneous Airy ODE, then we need to find the particular solution. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will

be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \text{AiryAi}\left(-x(-1)^{1/3}\right)$$
$$y_2 = \text{AiryBi}\left(-x(-1)^{1/3}\right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \text{AiryAi}\left(-x(-1)^{1/3}\right) & \text{AiryBi}\left(-x(-1)^{1/3}\right) \\ \frac{d}{dx}\left(\text{AiryAi}\left(-x(-1)^{1/3}\right)\right) & \frac{d}{dx}\left(\text{AiryBi}\left(-x(-1)^{1/3}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \text{AiryAi}\left(-x(-1)^{1/3}\right) & \text{AiryBi}\left(-x(-1)^{1/3}\right) \\ -(-1)^{1/3}\,\text{AiryAi}\left(1, -x(-1)^{1/3}\right) & -(-1)^{1/3}\,\text{AiryBi}\left(1, -x(-1)^{1/3}\right) \end{vmatrix}$$

Therefore

$$W = \left(\text{AiryAi}\left(-x(-1)^{1/3}\right)\right)\left(-(-1)^{1/3}\,\text{AiryBi}\left(1, -x(-1)^{1/3}\right)\right) - \left(\text{AiryBi}\left(-x(-1)^{1/3}\right)\right)\left(-(-1)^{1/3}\,\text{AiryA}\right.$$

Which simplifies to

$$W = -\operatorname{AiryAi}\left(-x(-1)^{1/3}\right)(-1)^{1/3}\operatorname{AiryBi}\left(1,-x(-1)^{1/3}\right)+\operatorname{AiryBi}\left(-x(-1)^{1/3}\right)(-1)^{1/3}\operatorname{AiryAi}\left(1,-x(-\right.$$

Which simplifies to

$$W = \frac{-1-i\sqrt{3}}{2\pi}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\operatorname{AiryBi}\left(-x(-1)^{1/3}\right)(x^6-64)}{\frac{-1-i\sqrt{3}}{2\pi}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{(-2x^6+128)\,\pi \operatorname{AiryBi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)}{1+i\sqrt{3}}dx$$

Hence

$$u_1 = -\int_0^x \frac{(-2\alpha^6+128)\,\pi \operatorname{AiryBi}\left(-\frac{\alpha\left(1+i\sqrt{3}\right)}{2}\right)}{1+i\sqrt{3}}d\alpha$$

And Eq. (3) becomes

$$u_2 = \int \frac{\operatorname{AiryAi}\left(-x(-1)^{1/3}\right)(x^6-64)}{\frac{-1-i\sqrt{3}}{2\pi}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{(-2x^6+128)\,\pi \operatorname{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)}{1+i\sqrt{3}}dx$$

Hence

$$u_2 = \int_0^x \frac{(-2\alpha^6+128)\,\pi \operatorname{AiryAi}\left(-\frac{\alpha\left(1+i\sqrt{3}\right)}{2}\right)}{1+i\sqrt{3}}d\alpha$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\int_0^x \frac{(-2\alpha^6+128)\,\pi\,\mathrm{AiryBi}\left(-\frac{\alpha\left(1+i\sqrt{3}\right)}{2}\right)}{1+i\sqrt{3}}d\alpha\,\mathrm{AiryAi}\left(-x(-1)^{1/3}\right) + \mathrm{AiryBi}\left(-x(-1)^{1/3}\right)\int_0^x \frac{(-2\alpha^6+128)\,\pi\,\mathrm{AiryAi}\left(-\frac{\alpha\left(1+i\sqrt{3}\right)}{2}\right)}{1+i\sqrt{3}}d\alpha$$

Which simplifies to

$$y_p(x) = \frac{2\pi\left(\int_0^x (\alpha^6-64)\,\mathrm{AiryBi}\left(-\frac{\alpha\left(1+i\sqrt{3}\right)}{2}\right)d\alpha\,\mathrm{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right) - \mathrm{AiryBi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)\int_0^x (\alpha^6-64)\,\mathrm{Ai}\right.}{1+i\sqrt{3}}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\,\mathrm{AiryAi}\left(-x(-1)^{1/3}\right) + c_2\,\mathrm{AiryBi}\left(-x(-1)^{1/3}\right)\right) \\ &\quad + \left(\frac{2\pi\left(\int_0^x (\alpha^6-64)\,\mathrm{AiryBi}\left(-\frac{\alpha\left(1+i\sqrt{3}\right)}{2}\right)d\alpha\,\mathrm{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right) - \mathrm{AiryBi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)\int_0^x (\alpha^6 -\right.}{1+i\sqrt{3}}\right. \\ &= c_1\,\mathrm{AiryAi}\left(-x(-1)^{1/3}\right) + c_2\,\mathrm{AiryBi}\left(-x(-1)^{1/3}\right) \\ &\quad + \frac{2\pi\left(\int_0^x (\alpha^6-64)\,\mathrm{AiryBi}\left(-\frac{\alpha\left(1+i\sqrt{3}\right)}{2}\right)d\alpha\,\mathrm{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right) - \mathrm{AiryBi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)\int_0^x (\alpha^6-64\right.}{1+i\sqrt{3}} \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1\,\mathrm{AiryAi}\left(-x(-1)^{1/3}\right) + c_2\,\mathrm{AiryBi}\left(-x(-1)^{1/3}\right) + \frac{2\pi\left(\int_0^x (\alpha^6-64)\,\mathrm{AiryBi}\left(-\frac{\alpha\left(1+i\sqrt{3}\right)}{2}\right)d\alpha\,\mathrm{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right) - \mathrm{AiryBi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)\int_0^x (\alpha^6-64)\right.}{1+i\sqrt{3}}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : `0.014` (sec)
Leaf size : 149

```
dsolve(diff(diff(y(x),x),x)-x*y(x)-x^6+64 = 0,
       y(x),singsol=all)
```

$$y = \frac{16x^7\pi\left(3^{1/3}\operatorname{AiryBi}(x) - \operatorname{AiryAi}(x)\,3^{5/6}\right)\operatorname{hypergeom}\left(\left[\frac{7}{3}\right],\left[\frac{2}{3},\frac{10}{3}\right],\frac{x^3}{9}\right) - 21x^8\Gamma\left(\frac{2}{3}\right)^2\left(3^{2/3}\operatorname{AiryAi}(x) + \right.}{}$$

Mathematica DSolve solution

Solving time : 0.47 (sec)
Leaf size : 256

```
DSolve[{D[y[x],{x,2}]-x*y[x]-x^6+64==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$y(x)$

$$\rightarrow \frac{192\sqrt[3]{3}\pi x\,\text{Gamma}\left(\frac{1}{3}\right)\left(\sqrt{3}\,\text{AiryAi}(x)-\text{AiryBi}(x)\right)\,{}_1F_2\left(\frac{1}{3};\frac{2}{3},\frac{4}{3};\frac{x^3}{9}\right)-\frac{\sqrt[6]{3}\pi x^8\,\text{Gamma}\left(\frac{2}{3}\right)\text{Gamma}\left(\frac{8}{3}\right)\left(3\,\text{AiryAi}\right.}{\text{Gamma}\left(\right.}}{}$$

2.2.31 problem 30

Internal problem ID [8511]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 30
Date solved : `Thursday, December 12, 2024 at 09:23:36 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy - x = 0$$

Solved as second order Airy ode

Time used: 0.052 (sec)

This is Airy ODE. It has the general form

$$ay'' + by' + cxy = F(x)$$

Where in this case

$$\begin{aligned} a &= 1 \\ b &= 0 \\ c &= -1 \\ F &= x \end{aligned}$$

Therefore the solution to the homogeneous Airy ODE becomes

$$y = c_1 \operatorname{AiryAi}\left(-x(-1)^{1/3}\right) + c_2 \operatorname{AiryBi}\left(-x(-1)^{1/3}\right)$$

Since this is inhomogeneous Airy ODE, then we need to find the particular solution. The particular solution is now found using the method of undetermined coefficients.

Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\operatorname{AiryAi}\left(-x(-1)^{1/3}\right), \operatorname{AiryBi}\left(-x(-1)^{1/3}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-x(A_2 x + A_1) - x = 0$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \operatorname{AiryAi}\left(-x(-1)^{1/3}\right) + c_2 \operatorname{AiryBi}\left(-x(-1)^{1/3}\right)\right) + (-1) \\ &= c_1 \operatorname{AiryAi}\left(-x(-1)^{1/3}\right) + c_2 \operatorname{AiryBi}\left(-x(-1)^{1/3}\right) - 1 \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \operatorname{AiryAi}\left(-x(-1)^{1/3}\right) + c_2 \operatorname{AiryBi}\left(-x(-1)^{1/3}\right) - 1$$

Solved as second order Bessel ode

Time used: 0.611 (sec)

Writing the ode as

$$x^2y'' - x^3y = x^3 \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}\left(n, \beta\, x^{\gamma}\right) + c_2 \operatorname{BesselY}\left(n, \beta\, x^{\gamma}\right)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}
\alpha &= \frac{1}{2} \\
\beta &= \frac{2i}{3} \\
n &= \frac{1}{3} \\
\gamma &= \frac{3}{2}
\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = c_1\sqrt{x}\, \operatorname{BesselJ}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right) + c_2\sqrt{x}\, \operatorname{BesselY}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1\sqrt{x}\, \operatorname{BesselJ}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right) + c_2\sqrt{x}\, \operatorname{BesselY}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right)$$

$$y_2 = \sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right) & \sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right) \\ \frac{d}{dx}\left(\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right)\right) & \frac{d}{dx}\left(\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right) & \sqrt{x}\,\mathrm{BesselY} \\ \frac{\mathrm{BesselJ}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right)}{2\sqrt{x}} + ix\left(-\mathrm{BesselJ}\left(\frac{4}{3}, \frac{2ix^{3/2}}{3}\right) - \frac{i\,\mathrm{BesselJ}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right)}{2x^{3/2}}\right) & \frac{\mathrm{BesselY}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right)}{2\sqrt{x}} + ix\left(-\mathrm{BesselY}\right. \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right)\right)\left(\frac{\mathrm{BesselY}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right)}{2\sqrt{x}}\right.$$
$$\left.+ix\left(-\mathrm{BesselY}\left(\frac{4}{3}, \frac{2ix^{3/2}}{3}\right) - \frac{i\,\mathrm{BesselY}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right)}{2x^{3/2}}\right)\right) - \left(\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right)\right)\left(\frac{\mathrm{BesselJ}\left(\frac{1}{3}, \right.}{2\sqrt{x}}\right.$$

Which simplifies to

$$W = -ix^{3/2}\left(\mathrm{BesselJ}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right)\mathrm{BesselY}\left(\frac{4}{3}, \frac{2ix^{3/2}}{3}\right) - \mathrm{BesselY}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right)\mathrm{BesselJ}\left(\frac{4}{3}, \frac{2ix^{3/2}}{3}\right)\right)$$

Which simplifies to

$$W = \frac{3}{\pi}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^{7/2}\,\mathrm{BesselY}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right)}{\frac{3x^2}{\pi}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{x^{3/2}\,\mathrm{BesselY}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right)\pi}{3}\,dx$$

Hence

$$u_1 = -\int_0^x \frac{\alpha^{3/2}\,\mathrm{BesselY}\left(\frac{1}{3}, \frac{2i\alpha^{3/2}}{3}\right)\pi}{3}\,d\alpha$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^{7/2}\,\mathrm{BesselJ}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right)}{\frac{3x^2}{\pi}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{x^{3/2} \operatorname{BesselJ}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right) \pi}{3} dx$$

Hence

$$u_2 = \int_0^x \frac{\alpha^{3/2} \operatorname{BesselJ}\left(\frac{1}{3}, \frac{2i\alpha^{3/2}}{3}\right) \pi}{3} d\alpha$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\int_0^x \frac{\alpha^{3/2} \operatorname{BesselY}\left(\frac{1}{3}, \frac{2i\alpha^{3/2}}{3}\right) \pi}{3} d\alpha \sqrt{x} \operatorname{BesselJ}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right) + \sqrt{x} \operatorname{BesselY}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right) \int_0^x \frac{\alpha^{3/2} \operatorname{BesselJ}}{}$$

Which simplifies to

$$y_p(x) = -\frac{\pi\sqrt{x}\left(\int_0^x \alpha^{3/2} \operatorname{BesselY}\left(\frac{1}{3}, \frac{2i\alpha^{3/2}}{3}\right) d\alpha \operatorname{BesselJ}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right) - \operatorname{BesselY}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right) \int_0^x \alpha^{3/2} \operatorname{BesselJ}\left(\frac{1}{3}, \frac{2i\alpha^{3/2}}{3}\right.\right.}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\sqrt{x} \operatorname{BesselJ}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right) + c_2\sqrt{x} \operatorname{BesselY}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right)\right) \\ &\quad + \left(-\frac{\pi\sqrt{x}\left(\int_0^x \alpha^{3/2} \operatorname{BesselY}\left(\frac{1}{3}, \frac{2i\alpha^{3/2}}{3}\right) d\alpha \operatorname{BesselJ}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right) - \operatorname{BesselY}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right) \int_0^x \alpha^{3/2} \operatorname{BesselJ}\left(\frac{1}{3}\right.\right.}{3}\right. \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1\sqrt{x} \operatorname{BesselJ}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right) + c_2\sqrt{x} \operatorname{BesselY}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right) - \frac{\pi\sqrt{x}\left(\int_0^x \alpha^{3/2} \operatorname{BesselY}\left(\frac{1}{3}, \frac{2i\alpha^{3/2}}{3}\right) d\alpha \operatorname{BesselJ}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right) - \operatorname{BesselY}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right) \int_0^x \alpha^{3/2} \operatorname{BesselJ}\left(\frac{1}{3}, \frac{2i\alpha^{3/2}}{3}\right.\right.}{3}$$

Solved as second order ode adjoint method

Time used: 0.782 (sec)

In normal form the ode

$$y'' - xy - x = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= -x \\ r(x) &= x \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (0)' + (-x\xi(x)) &= 0 \\ \xi''(x) - x\xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is Airy ODE. It has the general form

$$a\xi'' + b\xi' + cx\xi = F(x)$$

Where in this case

$$\begin{aligned} a &= 1 \\ b &= 0 \\ c &= -1 \\ F &= 0 \end{aligned}$$

Therefore the solution to the homogeneous Airy ODE becomes

$$\xi = c_3 \operatorname{AiryAi}\left(-x(-1)^{1/3}\right) + c_4 \operatorname{AiryBi}\left(-x(-1)^{1/3}\right)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$
\begin{aligned}
\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi\left(x\right)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi\left(x\right)}
\end{aligned}
$$

Or

$$
y' - \frac{y\left(-c_3(-1)^{1/3} \operatorname{AiryAi}\left(1, -x(-1)^{1/3}\right) - c_4(-1)^{1/3} \operatorname{AiryBi}\left(1, -x(-1)^{1/3}\right)\right)}{c_3 \operatorname{AiryAi}\left(-x\left(-1\right)^{1/3}\right) + c_4 \operatorname{AiryBi}\left(-x\left(-1\right)^{1/3}\right)} = -\frac{\frac{c_3 \operatorname{AiryAi}\left(1, -x(-1)^{1/3}\right)\sqrt{}}{2}}{c}
$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$
y' + q(x)y = p(x)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= \frac{(1+i\sqrt{3})\left(c_3 \operatorname{AiryAi}\left(1, -\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + c_4 \operatorname{AiryBi}\left(1, -\frac{x\left(1+i\sqrt{3}\right)}{2}\right)\right)}{2c_3 \operatorname{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + 2c_4 \operatorname{AiryBi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)} \\
p(x) &= -\frac{(1+i\sqrt{3})\left(c_3 \operatorname{AiryAi}\left(1, -\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + c_4 \operatorname{AiryBi}\left(1, -\frac{x\left(1+i\sqrt{3}\right)}{2}\right)\right)}{2c_3 \operatorname{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + 2c_4 \operatorname{AiryBi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\, dx} \\
&= \mathrm{e}^{\int \frac{(1+i\sqrt{3})\left(c_3 \operatorname{AiryAi}\left(1, -\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + c_4 \operatorname{AiryBi}\left(1, -\frac{x\left(1+i\sqrt{3}\right)}{2}\right)\right)}{2c_3 \operatorname{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + 2c_4 \operatorname{AiryBi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)} dx} \\
&= \frac{1}{c_3 \operatorname{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + c_4 \operatorname{AiryBi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)}
\end{aligned}
$$

The ode becomes

$$
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu p
$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(-\frac{(1+i\sqrt{3})\left(c_3 \operatorname{AiryAi}\left(1, -\frac{x(1+i\sqrt{3})}{2}\right) + c_4 \operatorname{AiryBi}\left(1, -\frac{x(1+i\sqrt{3})}{2}\right)\right)}{2c_3 \operatorname{AiryAi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) + 2c_4 \operatorname{AiryBi}\left(-\frac{x(1+i\sqrt{3})}{2}\right)}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{c_3 \operatorname{AiryAi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) + c_4 \operatorname{AiryBi}\left(-\frac{x(1+i\sqrt{3})}{2}\right)}\right)$$
$$= \left(\frac{1}{c_3 \operatorname{AiryAi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) + c_4 \operatorname{AiryBi}\left(-\frac{x(1+i\sqrt{3})}{2}\right)}\right)\left(-\frac{(1+i\sqrt{3})\left(c_3 \operatorname{AiryAi}\left(1, -\frac{x(1+i\sqrt{3})}{2}\right) + c_4 \right.}{2c_3 \operatorname{AiryAi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) + 2c_4 \operatorname{Airy}}\right.$$

$$\mathrm{d}\left(\frac{y}{c_3 \operatorname{AiryAi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) + c_4 \operatorname{AiryBi}\left(-\frac{x(1+i\sqrt{3})}{2}\right)}\right)$$
$$= \left(-\frac{(1+i\sqrt{3})\left(c_3 \operatorname{AiryAi}\left(1, -\frac{x(1+i\sqrt{3})}{2}\right) + c_4 \operatorname{AiryBi}\left(1, -\frac{x(1+i\sqrt{3})}{2}\right)\right)}{\left(2c_3 \operatorname{AiryAi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) + 2c_4 \operatorname{AiryBi}\left(-\frac{x(1+i\sqrt{3})}{2}\right)\right)\left(c_3 \operatorname{AiryAi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) + c_4 \operatorname{AiryBi}\left(-\frac{x}{}\right.\right.}\right.$$

Integrating gives

$$\frac{y}{c_3 \operatorname{AiryAi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) + c_4 \operatorname{AiryBi}\left(-\frac{x(1+i\sqrt{3})}{2}\right)} = \int -\frac{(1+i\sqrt{3})\left(c_3 \operatorname{AiryAi}\left(1, \right.\right.}{\left(2c_3 \operatorname{AiryAi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) + 2c_4 \operatorname{AiryBi}\left(-\frac{x(1}{}\right.\right.}$$
$$= \frac{1+i\sqrt{3}}{2\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\left(c_3 \operatorname{AiryAi}\left(\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)x\right) + c_4 \operatorname{Airy}\right.}$$

Dividing throughout by the integrating factor $\frac{1}{c_3 \operatorname{AiryAi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) + c_4 \operatorname{AiryBi}\left(-\frac{x(1+i\sqrt{3})}{2}\right)}$ gives the final solution

$$y = \operatorname{AiryBi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) c_4 c_5 + \operatorname{AiryAi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) c_3 c_5 - 1$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \text{AiryBi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) c_4 c_5 + \text{AiryAi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) c_3 c_5 - 1$$

The constants can be merged to give

$$y = c_4 \,\text{AiryBi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) + c_3 \,\text{AiryAi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) - 1$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_4 \,\text{AiryBi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) + c_3 \,\text{AiryAi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) - 1$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 14

```
dsolve(diff(diff(y(x),x),x)-x*y(x)-x = 0,
       y(x),singsol=all)
```

$$y = \operatorname{AiryAi}(x) c_2 + \operatorname{AiryBi}(x) c_1 - 1$$

Mathematica DSolve solution

Solving time : 0.036 (sec)
Leaf size : 28

```
DSolve[{D[y[x],{x,2}]-x*y[x]-x==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$\begin{aligned} y(x) \to & \pi \operatorname{AiryAiPrime}(x) \operatorname{AiryBi}(x) + c_2 \operatorname{AiryBi}(x) \\ & + \operatorname{AiryAi}(x)(-\pi \operatorname{AiryBiPrime}(x) + c_1) \end{aligned}$$

2.2.32 problem 31

Internal problem ID [8512]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 31
Date solved : `Thursday, December 12, 2024 at 09:23:38 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy - x^2 = 0$$

Solved as second order Airy ode

Time used: 0.057 (sec)

This is Airy ODE. It has the general form

$$ay'' + by' + cxy = F(x)$$

Where in this case

$$\begin{aligned} a &= 1 \\ b &= 0 \\ c &= -1 \\ F &= x^2 \end{aligned}$$

Therefore the solution to the homogeneous Airy ODE becomes

$$y = c_1 \operatorname{AiryAi}\left(-x(-1)^{1/3}\right) + c_2 \operatorname{AiryBi}\left(-x(-1)^{1/3}\right)$$

Since this is inhomogeneous Airy ODE, then we need to find the particular solution. The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\text{AiryAi}\left(-x(-1)^{1/3}\right), \text{AiryBi}\left(-x(-1)^{1/3}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_3 - x(A_3x^2 + A_2x + A_1) - x^2 = 0$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = -1, A_3 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \text{AiryAi}\left(-x(-1)^{1/3}\right) + c_2 \text{AiryBi}\left(-x(-1)^{1/3}\right)\right) + (-x) \\ &= c_1 \text{AiryAi}\left(-x(-1)^{1/3}\right) + c_2 \text{AiryBi}\left(-x(-1)^{1/3}\right) - x \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \text{AiryAi}\left(-x(-1)^{1/3}\right) + c_2 \text{AiryBi}\left(-x(-1)^{1/3}\right) - x$$

Solved as second order ode adjoint method

Time used: 0.775 (sec)

In normal form the ode

$$y'' - xy - x^2 = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= -x \\ r(x) &= x^2 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (0)' + (-x\xi(x)) &= 0 \\ \xi''(x) - x\xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is Airy ODE. It has the general form

$$a\xi'' + b\xi' + cx\xi = F(x)$$

Where in this case

$$\begin{aligned} a &= 1 \\ b &= 0 \\ c &= -1 \\ F &= 0 \end{aligned}$$

Therefore the solution to the homogeneous Airy ODE becomes

$$\xi = c_5 \operatorname{AiryAi}\left(-x(-1)^{1/3}\right) + c_6 \operatorname{AiryBi}\left(-x(-1)^{1/3}\right)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$
\begin{aligned}
\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)}
\end{aligned}
$$

Or

$$
y' - \frac{y\left(-c_5(-1)^{1/3}\operatorname{AiryAi}\left(1, -x(-1)^{1/3}\right) - c_6(-1)^{1/3}\operatorname{AiryBi}\left(1, -x(-1)^{1/3}\right)\right)}{c_5\operatorname{AiryAi}\left(-x(-1)^{1/3}\right) + c_6\operatorname{AiryBi}\left(-x(-1)^{1/3}\right)} = \frac{-c_5\operatorname{AiryAi}\left(-x(-1)^{1}\right.}{}
$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$
y' + q(x)y = p(x)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= \frac{(1+i\sqrt{3})\left(\operatorname{AiryBi}\left(1, -\frac{x(1+i\sqrt{3})}{2}\right)c_6 + \operatorname{AiryAi}\left(1, -\frac{x(1+i\sqrt{3})}{2}\right)c_5\right)}{2c_5\operatorname{AiryAi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) + 2c_6\operatorname{AiryBi}\left(-\frac{x(1+i\sqrt{3})}{2}\right)} \\
p(x) &= \frac{-xc_5(1+i\sqrt{3})\operatorname{AiryAi}\left(1, -\frac{x(1+i\sqrt{3})}{2}\right) - c_6x(1+i\sqrt{3})\operatorname{AiryBi}\left(1, -\frac{x(1+i\sqrt{3})}{2}\right) - 2c_5\operatorname{AiryAi}\left(-\right.}{2c_5\operatorname{AiryAi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) + 2c_6\operatorname{AiryBi}\left(-\frac{x(1+i\sqrt{3})}{2}\right)}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\, dx} \\
&= e^{\int \frac{(1+i\sqrt{3})\left(\operatorname{AiryBi}\left(1, -\frac{x(1+i\sqrt{3})}{2}\right)c_6 + \operatorname{AiryAi}\left(1, -\frac{x(1+i\sqrt{3})}{2}\right)c_5\right)}{2c_5\operatorname{AiryAi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) + 2c_6\operatorname{AiryBi}\left(-\frac{x(1+i\sqrt{3})}{2}\right)}dx} \\
&= \frac{1}{c_5\operatorname{AiryAi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) + c_6\operatorname{AiryBi}\left(-\frac{x(1+i\sqrt{3})}{2}\right)}
\end{aligned}
$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu p$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{-xc_5(1+i\sqrt{3})\,\mathrm{AiryAi}\left(1, -\frac{x\left(1+i\sqrt{3}\right)}{2}\right) - c_6 x(1+i\sqrt{3})\,\mathrm{AiryBi}\left(1, -\frac{x\left(1+i\sqrt{3}\right)}{2}\right) - 2c_5\,\mathrm{AiryAi}\left(-\right.}{2c_5\,\mathrm{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + 2c_6\,\mathrm{AiryBi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)}\right.$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{c_5\,\mathrm{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + c_6\,\mathrm{AiryBi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)}\right) = \left(\frac{1}{c_5\,\mathrm{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + c_6\,\mathrm{AiryBi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)}\right)\left(\frac{-xc_5(1+i\sqrt{3})\,\mathrm{AiryAi}\left(1, -\frac{x\left(1+i\sqrt{3}\right)}{2}\right) - c_6 x}{2c_5\,A}\right.$$

$$\mathrm{d}\left(\frac{y}{c_5\,\mathrm{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + c_6\,\mathrm{AiryBi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)}\right) = \left(\frac{-xc_5(1+i\sqrt{3})\,\mathrm{AiryAi}\left(1, -\frac{x\left(1+i\sqrt{3}\right)}{2}\right) - c_6 x(1+i\sqrt{3})\,\mathrm{AiryBi}\left(1, -\frac{x\left(1+i\sqrt{3}\right)}{2}\right) - 2c_5\,\mathrm{AiryAi}\left(-\frac{x(1+}{}\right.}{\left(2c_5\,\mathrm{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + 2c_6\,\mathrm{AiryBi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)\right)\left(c_5\,\mathrm{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + c_6\,\mathrm{A}\right.}\right.$$

Integrating gives

$$\begin{aligned}\frac{y}{c_5\,\mathrm{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + c_6\,\mathrm{AiryBi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)} &= \int \frac{-xc_5(1+i\sqrt{3})\,\mathrm{AiryAi}\left(1, -\frac{x\left(1+i\sqrt{3}\right)}{2}\right) - c_6 x(1+}{\left(2c_5\,\mathrm{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right) + 2c_6\,\mathrm{Air}\right.} \\ &= -\frac{x}{c_5\,\mathrm{AiryAi}\left(\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)x\right) + c_6\,\mathrm{AiryBi}\left(\left(-\frac{1}{2} - \frac{i\sqrt{}}{}\right.\right.}\end{aligned}$$

Dividing throughout by the integrating factor $\frac{1}{c_5\,\mathrm{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)+c_6\,\mathrm{AiryBi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)}$

gives the final solution

$$y = \text{AiryAi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) c_5 c_7 + \text{AiryBi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) c_6 c_7 - x$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \text{AiryAi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) c_5 c_7 + \text{AiryBi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) c_6 c_7 - x$$

The constants can be merged to give

$$y = c_5 \,\text{AiryAi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) + c_6 \,\text{AiryBi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) - x$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_5 \,\text{AiryAi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) + c_6 \,\text{AiryBi}\left(-\frac{x(1+i\sqrt{3})}{2}\right) - x$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
```

```
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 16

```
dsolve(diff(diff(y(x),x),x)-x*y(x)-x^2 = 0,
       y(x),singsol=all)
```

$$y = \operatorname{AiryAi}(x)\, c_2 + \operatorname{AiryBi}(x)\, c_1 - x$$

Mathematica DSolve solution

Solving time : 0.042 (sec)
Leaf size : 30

```
DSolve[{D[y[x],{x,2}]-x*y[x]-x^2==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \pi x \operatorname{AiryAiPrime}(x) \operatorname{AiryBi}(x) + c_2 \operatorname{AiryBi}(x) + \operatorname{AiryAi}(x)(-\pi x \operatorname{AiryBiPrime}(x) + c_1)$$

2.2.33 problem 32

Internal problem ID [8513]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 32
Date solved : `Thursday, December 12, 2024 at 09:23:44 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' - xy - x^3 = 0$$

Solved as second order Airy ode

Time used: 0.635 (sec)

This is Airy ODE. It has the general form

$$ay'' + by' + cxy = F(x)$$

Where in this case

$$\begin{aligned} a &= 1 \\ b &= 0 \\ c &= -1 \\ F &= x^3 \end{aligned}$$

Therefore the solution to the homogeneous Airy ODE becomes

$$y = c_1 \operatorname{AiryAi}\left(-x(-1)^{1/3}\right) + c_2 \operatorname{AiryBi}\left(-x(-1)^{1/3}\right)$$

Since this is inhomogeneous Airy ODE, then we need to find the particular solution. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will

be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{AiryAi}\left(-x(-1)^{1/3}\right)$$
$$y_2 = \mathrm{AiryBi}\left(-x(-1)^{1/3}\right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{AiryAi}\left(-x(-1)^{1/3}\right) & \mathrm{AiryBi}\left(-x(-1)^{1/3}\right) \\ \frac{d}{dx}\left(\mathrm{AiryAi}\left(-x(-1)^{1/3}\right)\right) & \frac{d}{dx}\left(\mathrm{AiryBi}\left(-x(-1)^{1/3}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{AiryAi}\left(-x(-1)^{1/3}\right) & \mathrm{AiryBi}\left(-x(-1)^{1/3}\right) \\ -(-1)^{1/3}\,\mathrm{AiryAi}\left(1, -x(-1)^{1/3}\right) & -(-1)^{1/3}\,\mathrm{AiryBi}\left(1, -x(-1)^{1/3}\right) \end{vmatrix}$$

Therefore

$$W = \left(\mathrm{AiryAi}\left(-x(-1)^{1/3}\right)\right)\left(-(-1)^{1/3}\,\mathrm{AiryBi}\left(1, -x(-1)^{1/3}\right)\right) - \left(\mathrm{AiryBi}\left(-x(-1)^{1/3}\right)\right)\left(-(-1)^{1/3}\,\mathrm{AiryA}\right.$$

Which simplifies to

$$W = -\operatorname{AiryAi}\left(-x(-1)^{1/3}\right)(-1)^{1/3}\operatorname{AiryBi}\left(1,-x(-1)^{1/3}\right)+\operatorname{AiryBi}\left(-x(-1)^{1/3}\right)(-1)^{1/3}\operatorname{AiryAi}\left(1,-x(-\right.$$

Which simplifies to

$$W = \frac{-1-i\sqrt{3}}{2\pi}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\operatorname{AiryBi}\left(-x(-1)^{1/3}\right)x^3}{\frac{-1-i\sqrt{3}}{2\pi}}dx$$

Which simplifies to

$$u_1 = -\int -\frac{2\operatorname{AiryBi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)x^3\pi}{1+i\sqrt{3}}dx$$

Hence

$$u_1 = -\int_0^x -\frac{2\operatorname{AiryBi}\left(-\frac{\alpha\left(1+i\sqrt{3}\right)}{2}\right)\alpha^3\pi}{1+i\sqrt{3}}d\alpha$$

And Eq. (3) becomes

$$u_2 = \int \frac{\operatorname{AiryAi}\left(-x(-1)^{1/3}\right)x^3}{\frac{-1-i\sqrt{3}}{2\pi}}dx$$

Which simplifies to

$$u_2 = \int -\frac{2\operatorname{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)x^3\pi}{1+i\sqrt{3}}dx$$

Hence

$$u_2 = \int_0^x -\frac{2\operatorname{AiryAi}\left(-\frac{\alpha\left(1+i\sqrt{3}\right)}{2}\right)\alpha^3\pi}{1+i\sqrt{3}}d\alpha$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\int_0^x -\frac{2\,\text{AiryBi}\left(-\frac{\alpha\left(1+i\sqrt{3}\right)}{2}\right)\alpha^3\pi}{1+i\sqrt{3}}d\alpha\,\text{AiryAi}\left(-x(-1)^{1/3}\right) + \text{AiryBi}\left(-x(-1)^{1/3}\right)\int_0^x -\frac{2\,\text{AiryAi}\left(-\frac{\alpha\left(1+i\sqrt{3}\right)}{2}\right)\alpha^3\pi}{1+i\sqrt{3}}d\alpha$$

Which simplifies to

$$y_p(x) = \frac{2\pi\left(\text{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)\int_0^x \text{AiryBi}\left(-\frac{\alpha\left(1+i\sqrt{3}\right)}{2}\right)\alpha^3 d\alpha - \text{AiryBi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)\int_0^x \text{AiryAi}\left(-\frac{\alpha\left(1+i\sqrt{3}\right)}{2}\right.\right.}{1+i\sqrt{3}}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\,\text{AiryAi}\left(-x(-1)^{1/3}\right) + c_2\,\text{AiryBi}\left(-x(-1)^{1/3}\right)\right) \\ &\quad + \left(\frac{2\pi\left(\text{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)\int_0^x \text{AiryBi}\left(-\frac{\alpha\left(1+i\sqrt{3}\right)}{2}\right)\alpha^3 d\alpha - \text{AiryBi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)\int_0^x \text{AiryAi}\left(-\frac{\alpha\left(1\right.}{}\right.\right.}{1+i\sqrt{3}}\right. \\ &= c_1\,\text{AiryAi}\left(-x(-1)^{1/3}\right) + c_2\,\text{AiryBi}\left(-x(-1)^{1/3}\right) \\ &\quad + \frac{2\pi\left(\text{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)\int_0^x \text{AiryBi}\left(-\frac{\alpha\left(1+i\sqrt{3}\right)}{2}\right)\alpha^3 d\alpha - \text{AiryBi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)\int_0^x \text{AiryAi}\left(-\frac{\alpha\left(1+i\right.}{2}\right.\right.}{1+i\sqrt{3}} \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1\,\text{AiryAi}\left(-x(-1)^{1/3}\right) + c_2\,\text{AiryBi}\left(-x(-1)^{1/3}\right) + \frac{2\pi\left(\text{AiryAi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)\int_0^x \text{AiryBi}\left(-\frac{\alpha\left(1+i\sqrt{3}\right)}{2}\right)\alpha^3 d\alpha - \text{AiryBi}\left(-\frac{x\left(1+i\sqrt{3}\right)}{2}\right)\int_0^x \text{AiryAi}\left(-\frac{\alpha\left(1+i\sqrt{}\right.}{2}\right.\right.}{1+i\sqrt{3}}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.015 (sec)
Leaf size : 87

```
dsolve(diff(diff(y(x),x),x)-x*y(x)-x^3 = 0,
       y(x),singsol=all)
```

$$y = \frac{5x^4\pi\left(3^{1/3}\,\mathrm{AiryBi}\left(x\right)-\mathrm{AiryAi}\left(x\right)3^{5/6}\right)\mathrm{hypergeom}\left(\left[\frac{4}{3}\right],\left[\frac{2}{3},\frac{7}{3}\right],\frac{x^3}{9}\right)-6\Gamma\left(\frac{2}{3}\right)\left(x^5\,\mathrm{hypergeom}\left(\left[\frac{5}{3}\right],\left[\frac{4}{3},\right.\right.\right.}{60\Gamma\left(\frac{2}{3}\right)}$$

Mathematica DSolve solution

Solving time : 0.095 (sec)
Leaf size : 137

```
DSolve[{D[y[x],{x,2}]-x*y[x]-x^3==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
y(x) \to & -\frac{\pi x^5 \operatorname{Gamma}\left(\frac{5}{3}\right)\left(3 \operatorname{AiryAi}(x)+\sqrt{3} \operatorname{AiryBi}(x)\right) {}_1F_2\left(\frac{5}{3}; \frac{4}{3}, \frac{8}{3}; \frac{x^3}{9}\right)}{9\, 3^{5/6} \operatorname{Gamma}\left(\frac{4}{3}\right) \operatorname{Gamma}\left(\frac{8}{3}\right)} \\
& + \frac{\pi x^4 \operatorname{Gamma}\left(\frac{4}{3}\right)\left(\operatorname{AiryBi}(x)-\sqrt{3} \operatorname{AiryAi}(x)\right) {}_1F_2\left(\frac{4}{3}; \frac{2}{3}, \frac{7}{3}; \frac{x^3}{9}\right)}{3\, 3^{2/3} \operatorname{Gamma}\left(\frac{2}{3}\right) \operatorname{Gamma}\left(\frac{7}{3}\right)} \\
& + c_1 \operatorname{AiryAi}(x)+c_2 \operatorname{AiryBi}(x)
\end{aligned}
$$

2.2.34 problem 33

Internal problem ID [8514]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 33
Date solved : `Thursday, December 12, 2024 at 09:23:49 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' - xy - x^6 - x^3 + 42 = 0$$

Solved as second order Airy ode

Time used: 0.071 (sec)

This is Airy ODE. It has the general form

$$ay'' + by' + cxy = F(x)$$

Where in this case

$$\begin{aligned} a &= 1 \\ b &= 0 \\ c &= -1 \\ F &= x^6 + x^3 - 42 \end{aligned}$$

Therefore the solution to the homogeneous Airy ODE becomes

$$y = c_1 \operatorname{AiryAi}\left(-x(-1)^{1/3}\right) + c_2 \operatorname{AiryBi}\left(-x(-1)^{1/3}\right)$$

Since this is inhomogeneous Airy ODE, then we need to find the particular solution. The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^6 + x^3 + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2, x^3, x^4, x^5, x^6\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\text{AiryAi}\left(-x(-1)^{1/3}\right), \text{AiryBi}\left(-x(-1)^{1/3}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_7x^6 + A_6x^5 + A_5x^4 + A_4x^3 + A_3x^2 + A_2x + A_1$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6, A_7\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&30x^4A_7 + 20x^3A_6 + 12x^2A_5 + 6xA_4 + 2A_3\\&\quad - x\left(A_7x^6 + A_6x^5 + A_5x^4 + A_4x^3 + A_3x^2 + A_2x + A_1\right) - x^6 - x^3 + 42 = 0\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 0, A_3 = -21, A_4 = 0, A_5 = 0, A_6 = -1, A_7 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x^5 - 21x^2$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\&= \left(c_1 \text{AiryAi}\left(-x(-1)^{1/3}\right) + c_2 \text{AiryBi}\left(-x(-1)^{1/3}\right)\right) + \left(-x^5 - 21x^2\right)\\&= c_1 \text{AiryAi}\left(-x(-1)^{1/3}\right) + c_2 \text{AiryBi}\left(-x(-1)^{1/3}\right) - x^5 - 21x^2\end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \text{AiryAi}\left(-x(-1)^{1/3}\right) + c_2 \text{AiryBi}\left(-x(-1)^{1/3}\right) - x^5 - 21x^2$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 23

```
dsolve(diff(diff(y(x),x),x)-x*y(x)-x^6-x^3+42 = 0,
       y(x),singsol=all)
```

$$y = \operatorname{AiryAi}(x) c_2 + \operatorname{AiryBi}(x) c_1 - x^5 - 21x^2$$

Mathematica DSolve solution

Solving time : `1.093` (sec)
Leaf size : 367

```
DSolve[{D[y[x],{x,2}]-x*y[x]-x^6-x^3+42==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$y(x) \rightarrow$

$$-\frac{-126\sqrt[3]{3}\pi x\,\mathrm{Gamma}\left(\frac{1}{3}\right)\left(\sqrt{3}\,\mathrm{AiryAi}(x)-\mathrm{AiryBi}(x)\right)\ {}_1F_2\left(\frac{1}{3};\frac{2}{3},\frac{4}{3};\frac{x^3}{9}\right)+\frac{\sqrt[6]{3}\pi x^8\,\mathrm{Gamma}\left(\frac{2}{3}\right)\mathrm{Gamma}\left(\frac{8}{3}\right)\left(3\,\mathrm{Ai}\right.}{\mathrm{Gam}}}{}$$

2.2.35 problem 34

Internal problem ID [8515]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 34
Date solved : `Thursday, December 12, 2024 at 09:23:57 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - x^2 y - x^2 = 0$$

Solved as second order Bessel ode

Time used: 0.611 (sec)

Writing the ode as

$$x^2 y'' - x^4 y = x^4 \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2 y'' + x y' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha)\, x y' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha (c_1 \,\mathrm{BesselJ}\left(n, \beta\, x^\gamma\right) + c_2 \,\mathrm{BesselY}\left(n, \beta\, x^\gamma\right)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$
\begin{aligned}
\alpha &= \frac{1}{2} \\
\beta &= \frac{i}{2} \\
n &= \frac{1}{4} \\
\gamma &= 2
\end{aligned}
$$

Substituting all the above into (4) gives the solution as

$$
y = c_1\sqrt{x}\ \text{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) + c_2\sqrt{x}\ \text{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)
$$

Therefore the homogeneous solution y_h is

$$
y_h = c_1\sqrt{x}\ \text{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) + c_2\sqrt{x}\ \text{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \sqrt{x}\ \text{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) \\
y_2 &= \sqrt{x}\ \text{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right) & \sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{ix^2}{2}\right) \\ \frac{d}{dx}\left(\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right)\right) & \frac{d}{dx}\left(\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{ix^2}{2}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right) & \sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{ix^2}{2}\right) \\ \frac{\mathrm{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right)}{2\sqrt{x}} + ix^{3/2}\left(-\mathrm{BesselJ}\left(\frac{5}{4}, \frac{ix^2}{2}\right) - \frac{i\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right)}{2x^2}\right) & \frac{\mathrm{BesselY}\left(\frac{1}{4}, \frac{ix^2}{2}\right)}{2\sqrt{x}} + ix^{3/2}\left(-\mathrm{BesselY}\left(\frac{5}{4}, \frac{ix^2}{2}\right)\right. \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & \left(\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right)\right)\left(\frac{\mathrm{BesselY}\left(\frac{1}{4}, \frac{ix^2}{2}\right)}{2\sqrt{x}}\right. \\ & \left. + ix^{3/2}\left(-\mathrm{BesselY}\left(\frac{5}{4}, \frac{ix^2}{2}\right) - \frac{i\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{ix^2}{2}\right)}{2x^2}\right)\right) \\ & - \left(\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{ix^2}{2}\right)\right)\left(\frac{\mathrm{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right)}{2\sqrt{x}}\right. \\ & \left. + ix^{3/2}\left(-\mathrm{BesselJ}\left(\frac{5}{4}, \frac{ix^2}{2}\right) - \frac{i\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right)}{2x^2}\right)\right) \end{aligned}$$

Which simplifies to

$$W = -ix^2\left(\mathrm{BesselY}\left(\frac{5}{4}, \frac{ix^2}{2}\right)\mathrm{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right) - \mathrm{BesselJ}\left(\frac{5}{4}, \frac{ix^2}{2}\right)\mathrm{BesselY}\left(\frac{1}{4}, \frac{ix^2}{2}\right)\right)$$

Which simplifies to

$$W = \frac{4}{\pi}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^{9/2}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{ix^2}{2}\right)}{\frac{4x^2}{\pi}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{x^{5/2}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)\pi}{4} dx$$

Hence

$$u_1 = -\int_0^x \frac{\alpha^{5/2}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i \alpha^2}{2}\right)\pi}{4} d\alpha$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^{9/2}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right)}{\frac{4x^2}{\pi}} dx$$

Which simplifies to

$$u_2 = \int \frac{x^{5/2}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right)\pi}{4} dx$$

Hence

$$u_2 = \int_0^x \frac{\alpha^{5/2}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i \alpha^2}{2}\right)\pi}{4} d\alpha$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = &-\int_0^x \frac{\alpha^{5/2}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i \alpha^2}{2}\right)\pi}{4} d\alpha \sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) \\ &+ \sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right) \int_0^x \frac{\alpha^{5/2}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i \alpha^2}{2}\right)\pi}{4} d\alpha \end{aligned}$$

Which simplifies to

$$y_p(x) = -\frac{\pi\sqrt{x}\left(\int_0^x \alpha^{5/2}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i\alpha^2}{2}\right) d\alpha\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) - \mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)\int_0^x \alpha^{5/2}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i\alpha^2}{2}\right) d\alpha\right)}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)\right) \\
&\quad + \left(-\frac{\pi\sqrt{x}\left(\int_0^x \alpha^{5/2}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i\alpha^2}{2}\right) d\alpha\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) - \mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)\int_0^x \alpha^{5/2}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i\alpha^2}{2}\right) d\right.}{4}\right.
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
\begin{aligned}
y &= c_1\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right) \\
&\quad - \frac{\pi\sqrt{x}\left(\int_0^x \alpha^{5/2}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i\alpha^2}{2}\right) d\alpha\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) - \mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)\int_0^x \alpha^{5/2}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i\alpha^2}{2}\right) d\alpha\right)}{4}
\end{aligned}
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : `0.017` (sec)
Leaf size : 30

```
dsolve(diff(diff(y(x),x),x)-x^2*y(x)-x^2 = 0,
       y(x),singsol=all)
```

$$y = \sqrt{x}\,\mathrm{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) c_2 + \sqrt{x}\,\mathrm{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right) c_1 - 1$$

Mathematica DSolve solution

Solving time : `5.651` (sec)
Leaf size : 213

```
DSolve[{D[y[x],{x,2}]-x^2*y[x]-x^2==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \mathrm{ParabolicCylinderD}\left(-\frac{1}{2}, \sqrt{2}x\right)\left(\int_1^x \frac{K[1]^2\,\mathrm{ParabolicCyl}}{\sqrt{2}\left(\mathrm{HermiteH}\left(-\frac{1}{2}, K[1]\right)\left(i\,\mathrm{HermiteH}\left(\frac{1}{2}, iK[1]\right) + 2\,\mathrm{HermiteH}\right.\right.} + c_1\right)$$
$$+\mathrm{ParabolicCylinderD}\left(-\frac{1}{2}, i\sqrt{2}x\right)\left(\int_1^x \frac{K[2]^2\,\mathrm{Parabolic}}{\sqrt{2}\left(\mathrm{HermiteH}\left(-\frac{1}{2}, iK[2]\right)\mathrm{HermiteH}\left(\frac{1}{2}, K[2]\right) + \mathrm{HermiteH}\right.} + c_2\right)$$

2.2.36 problem 35

Internal problem ID [8516]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 35
Date solved : `Friday, December 13, 2024 at 05:03:59 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - x^2 y - x^3 = 0$$

Solved as second order Bessel ode

Time used: 15.276 (sec)

Writing the ode as

$$x^2 y'' - x^4 y = x^5 \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2 y'' + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha) xy' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha (c_1 \operatorname{BesselJ}(n, \beta x^\gamma) + c_2 \operatorname{BesselY}(n, \beta x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= \frac{1}{2}\\ \beta &= \frac{i}{2}\\ n &= \frac{1}{4}\\ \gamma &= 2\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = c_1\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= \sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right)\\ y_2 &= \sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) & \sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right) \\ \frac{d}{dx}\left(\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right)\right) & \frac{d}{dx}\left(\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) & \sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right) \\ \frac{\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right)}{2\sqrt{x}} + i x^{3/2}\left(-\mathrm{BesselJ}\left(\frac{5}{4}, \frac{i x^2}{2}\right) - \frac{i\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right)}{2x^2}\right) & \frac{\mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)}{2\sqrt{x}} + i x^{3/2}\left(-\mathrm{BesselY}\left(\frac{5}{4}, \frac{i x^2}{2}\right)\right. \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & \left(\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right)\right)\left(\frac{\mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)}{2\sqrt{x}}\right. \\ & \left. + i x^{3/2}\left(-\mathrm{BesselY}\left(\frac{5}{4}, \frac{i x^2}{2}\right) - \frac{i\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)}{2x^2}\right)\right) \\ & - \left(\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)\right)\left(\frac{\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right)}{2\sqrt{x}}\right. \\ & \left. + i x^{3/2}\left(-\mathrm{BesselJ}\left(\frac{5}{4}, \frac{i x^2}{2}\right) - \frac{i\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right)}{2x^2}\right)\right) \end{aligned}$$

Which simplifies to

$$W = -i x^2\left(\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right)\mathrm{BesselY}\left(\frac{5}{4}, \frac{i x^2}{2}\right) - \mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)\mathrm{BesselJ}\left(\frac{5}{4}, \frac{i x^2}{2}\right)\right)$$

Which simplifies to

$$W = \frac{4}{\pi}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^{11/2}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)}{\frac{4x^2}{\pi}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{x^{7/2}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)\pi}{4} dx$$

Hence

$$u_1 = -\int_0^x \frac{\alpha^{7/2}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i \alpha^2}{2}\right)\pi}{4} d\alpha$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^{11/2}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right)}{\frac{4x^2}{\pi}} dx$$

Which simplifies to

$$u_2 = \int \frac{x^{7/2}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right)\pi}{4} dx$$

Hence

$$u_2 = \frac{(-1)^{1/8}\,\pi\, x^3\,\mathrm{BesselI}\left(\frac{5}{4}, \frac{x^2}{2}\right)}{4\,(x^2)^{1/4}}$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = &-\int_0^x \frac{\alpha^{7/2}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i\alpha^2}{2}\right)\pi}{4} d\alpha \sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) \\ &+ \frac{x^{7/2}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)(-1)^{1/8}\,\pi\,\mathrm{BesselI}\left(\frac{5}{4}, \frac{x^2}{2}\right)}{4\,(x^2)^{1/4}} \end{aligned}$$

Which simplifies to

$$y_p(x) = \frac{\left(-\int_0^x \alpha^{7/2}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i\alpha^2}{2}\right) d\alpha\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right)(x^2)^{1/4} + x\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)(-1)^{1/8}\left(\mathrm{BesselI}\left(-\frac{3}{4}, \frac{x^2}{2}\right)\right.\right.}{4\,(x^2)^{1/4}}$$

Therefore the general solution is

$$y = y_h + y_p$$
$$= \left(c_1\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{ix^2}{2}\right)\right)$$
$$+ \left(\frac{\left(-\int_0^x \alpha^{7/2}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i\alpha^2}{2}\right) d\alpha\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right)(x^2)^{1/4} + x\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{ix^2}{2}\right)(-1)^{1/8}\left(\mathrm{BesselI}\left(-\frac{3}{4}\right.\right.\right.}{4\,(x^2)^{1/4}}\right.$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{ix^2}{2}\right)$$
$$+ \frac{\left(-\int_0^x \alpha^{7/2}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i\alpha^2}{2}\right) d\alpha\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right)(x^2)^{1/4} + x\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{ix^2}{2}\right)(-1)^{1/8}\left(\mathrm{BesselI}\left(-\frac{3}{4}, \frac{x^2}{2}\right.\right.\right.}{4\,(x^2)^{1/4}}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.016 (sec)
Leaf size : 32

```
dsolve(diff(diff(y(x),x),x)-x^2*y(x)-x^3 = 0,
       y(x),singsol=all)
```

$$y = \sqrt{x}\,\mathrm{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) c_2 + \sqrt{x}\,\mathrm{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right) c_1 - x$$

Mathematica DSolve solution

Solving time : 5.149 (sec)
Leaf size : 213

```
DSolve[{D[y[x],{x,2}]-x^2*y[x]-x^3==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \rightarrow \mathrm{ParabolicCylinderD}\left(-\frac{1}{2}, \sqrt{2}x\right)\left(\int_1^x \frac{K[1]^3\,\mathrm{ParabolicCyl}}{\sqrt{2}\left(\mathrm{HermiteH}\left(-\frac{1}{2}, K[1]\right)\left(i\,\mathrm{HermiteH}\left(\frac{1}{2}, iK[1]\right) + 2\,\mathrm{HermiteH}\right.\right.} + c_1\right)$$
$$+\mathrm{ParabolicCylinderD}\left(-\frac{1}{2}, i\sqrt{2}x\right)\left(\int_1^x \frac{K[2]^3\,\mathrm{Parabolic}}{\sqrt{2}\left(\mathrm{HermiteH}\left(-\frac{1}{2}, iK[2]\right)\mathrm{HermiteH}\left(\frac{1}{2}, K[2]\right) + \mathrm{HermiteH}\right.} + c_2\right)$$

2.2.37 problem 36

Internal problem ID [8517]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 36
Date solved : `Thursday, December 12, 2024 at 09:24:13 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' - x^2 y - x^4 = 0$$

Solved as second order Bessel ode

Time used: 0.615 (sec)

Writing the ode as

$$x^2 y'' - x^4 y = x^6 \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2 y'' + y'x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha) x y' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha} (c_1 \operatorname{BesselJ}(n, \beta x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}
\alpha &= \frac{1}{2} \\
\beta &= \frac{i}{2} \\
n &= \frac{1}{4} \\
\gamma &= 2
\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = c_1\sqrt{x}\,\text{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) + c_2\sqrt{x}\,\text{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1\sqrt{x}\,\text{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) + c_2\sqrt{x}\,\text{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= \sqrt{x}\,\text{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) \\
y_2 &= \sqrt{x}\,\text{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right) & \sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{ix^2}{2}\right) \\ \frac{d}{dx}\left(\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right)\right) & \frac{d}{dx}\left(\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{ix^2}{2}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right) & \sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{ix^2}{2} \right. \\ \frac{\mathrm{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right)}{2\sqrt{x}} + ix^{3/2}\left(-\mathrm{BesselJ}\left(\frac{5}{4}, \frac{ix^2}{2}\right) - \frac{i\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right)}{2x^2}\right) & \frac{\mathrm{BesselY}\left(\frac{1}{4}, \frac{ix^2}{2}\right)}{2\sqrt{x}} + ix^{3/2}\left(-\mathrm{BesselY}\left(\frac{5}{4}, \frac{ix^2}{2}\right.\right. \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & \left(\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right)\right)\left(\frac{\mathrm{BesselY}\left(\frac{1}{4}, \frac{ix^2}{2}\right)}{2\sqrt{x}}\right. \\ & \left. + ix^{3/2}\left(-\mathrm{BesselY}\left(\frac{5}{4}, \frac{ix^2}{2}\right) - \frac{i\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{ix^2}{2}\right)}{2x^2}\right)\right) \\ & - \left(\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{ix^2}{2}\right)\right)\left(\frac{\mathrm{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right)}{2\sqrt{x}}\right. \\ & \left. + ix^{3/2}\left(-\mathrm{BesselJ}\left(\frac{5}{4}, \frac{ix^2}{2}\right) - \frac{i\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right)}{2x^2}\right)\right) \end{aligned}$$

Which simplifies to

$$W = -ix^2\left(\mathrm{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right)\mathrm{BesselY}\left(\frac{5}{4}, \frac{ix^2}{2}\right) - \mathrm{BesselY}\left(\frac{1}{4}, \frac{ix^2}{2}\right)\mathrm{BesselJ}\left(\frac{5}{4}, \frac{ix^2}{2}\right)\right)$$

Which simplifies to

$$W = \frac{4}{\pi}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^{13/2}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{ix^2}{2}\right)}{\frac{4x^2}{\pi}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{x^{9/2} \operatorname{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right) \pi}{4} dx$$

Hence

$$u_1 = -\int_0^x \frac{\alpha^{9/2} \operatorname{BesselY}\left(\frac{1}{4}, \frac{i \alpha^2}{2}\right) \pi}{4} d\alpha$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^{13/2} \operatorname{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right)}{\frac{4x^2}{\pi}} dx$$

Which simplifies to

$$u_2 = \int \frac{x^{9/2} \operatorname{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) \pi}{4} dx$$

Hence

$$u_2 = \int_0^x \frac{\alpha^{9/2} \operatorname{BesselJ}\left(\frac{1}{4}, \frac{i \alpha^2}{2}\right) \pi}{4} d\alpha$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = &-\int_0^x \frac{\alpha^{9/2} \operatorname{BesselY}\left(\frac{1}{4}, \frac{i \alpha^2}{2}\right) \pi}{4} d\alpha \sqrt{x} \operatorname{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) \\ &+ \sqrt{x} \operatorname{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right) \int_0^x \frac{\alpha^{9/2} \operatorname{BesselJ}\left(\frac{1}{4}, \frac{i \alpha^2}{2}\right) \pi}{4} d\alpha \end{aligned}$$

Which simplifies to

$$y_p(x) = -\frac{\pi \sqrt{x} \left(\int_0^x \alpha^{9/2} \operatorname{BesselY}\left(\frac{1}{4}, \frac{i \alpha^2}{2}\right) d\alpha \operatorname{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) - \operatorname{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right) \int_0^x \alpha^{9/2} \operatorname{BesselJ}\left(\frac{1}{4}, \frac{i \alpha^2}{2}\right) d\alpha\right)}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)\right) \\
&\quad + \left(-\frac{\pi\sqrt{x}\left(\int_0^x \alpha^{9/2}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i\alpha^2}{2}\right) d\alpha\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) - \mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)\int_0^x \alpha^{9/2}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i\alpha^2}{2}\right) d\right.}{4}\right.
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
\begin{aligned}
y &= c_1\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right) \\
&\quad - \frac{\pi\sqrt{x}\left(\int_0^x \alpha^{9/2}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i\alpha^2}{2}\right) d\alpha\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) - \mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)\int_0^x \alpha^{9/2}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i\alpha^2}{2}\right) d\alpha\right)}{4}
\end{aligned}
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.022 (sec)
Leaf size : 124

```
dsolve(diff(diff(y(x),x),x)-x^2*y(x)-x^4 = 0,
       y(x),singsol=all)
```

$$y = -\frac{\sqrt{x}\left(-\frac{6\pi^2 x^5 \operatorname{hypergeom}\left(\left[\frac{5}{4}\right],\left[\frac{3}{4},\frac{5}{2}\right],\frac{x^4}{16}\right) \operatorname{BesselI}\left(\frac{1}{4},\frac{x^2}{2}\right) \operatorname{csgn}(x)}{5} + \Gamma\left(\frac{3}{4}\right)\left(2x^6\Gamma\left(\frac{3}{4}\right) \operatorname{BesselK}\left(\frac{1}{4},\frac{x^2}{2}\right) \operatorname{hypergeom}\left(\left[\frac{3}{2}\right]\right.\right.\right.}{}$$

Mathematica DSolve solution

Solving time : 5.191 (sec)
Leaf size : 213

```
DSolve[{D[y[x],{x,2}]-x^2*y[x]-x^4==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \operatorname{ParabolicCylinderD}\left(-\frac{1}{2},\sqrt{2}x\right)\left(\int_1^x \frac{K[1]^4 \operatorname{ParabolicCyl}}{\sqrt{2}\left(\operatorname{HermiteH}\left(-\frac{1}{2},K[1]\right)\left(i\operatorname{HermiteH}\left(\frac{1}{2},iK[1]\right)+2\operatorname{HermiteH}\right.\right.} + c_1\right)$$
$$+\operatorname{ParabolicCylinderD}\left(-\frac{1}{2},i\sqrt{2}x\right)\left(\int_1^x \frac{K[2]^4 \operatorname{Parabolic}}{\sqrt{2}\left(\operatorname{HermiteH}\left(-\frac{1}{2},iK[2]\right)\operatorname{HermiteH}\left(\frac{1}{2},K[2]\right)+\operatorname{HermiteH}\right.} + c_2\right)$$

2.2.38 problem 37

Internal problem ID [8518]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 37
Date solved : `Thursday, December 12, 2024 at 09:24:17 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' - x^2 y - x^4 + 2 = 0$$

Solved as second order Bessel ode

Time used: 0.659 (sec)

Writing the ode as

$$x^2 y'' - x^4 y = x^2 \left(x^4 - 2\right) \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2 y'' + y' x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha) x y' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha (c_1 \operatorname{BesselJ}(n, \beta x^\gamma) + c_2 \operatorname{BesselY}(n, \beta x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$
\begin{aligned}
\alpha &= \frac{1}{2} \\
\beta &= \frac{i}{2} \\
n &= \frac{1}{4} \\
\gamma &= 2
\end{aligned}
$$

Substituting all the above into (4) gives the solution as

$$
y = c_1 \sqrt{x}\, \mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) + c_2 \sqrt{x}\, \mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)
$$

Therefore the homogeneous solution y_h is

$$
y_h = c_1 \sqrt{x}\, \mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) + c_2 \sqrt{x}\, \mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \sqrt{x}\, \mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) \\
y_2 &= \sqrt{x}\, \mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x}\,\text{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right) & \sqrt{x}\,\text{BesselY}\left(\frac{1}{4}, \frac{ix^2}{2}\right) \\ \frac{d}{dx}\left(\sqrt{x}\,\text{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right)\right) & \frac{d}{dx}\left(\sqrt{x}\,\text{BesselY}\left(\frac{1}{4}, \frac{ix^2}{2}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x}\,\text{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right) & \sqrt{x}\,\text{BesselY}\left(\frac{1}{4}, \frac{ix^2}{2}\right) \\ \frac{\text{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right)}{2\sqrt{x}} + ix^{3/2}\left(-\text{BesselJ}\left(\frac{5}{4}, \frac{ix^2}{2}\right) - \frac{i\,\text{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right)}{2x^2}\right) & \frac{\text{BesselY}\left(\frac{1}{4}, \frac{ix^2}{2}\right)}{2\sqrt{x}} + ix^{3/2}\left(-\text{BesselY}\left(\frac{5}{4}, \frac{ix^2}{2}\right.\right. \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & \left(\sqrt{x}\,\text{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right)\right)\left(\frac{\text{BesselY}\left(\frac{1}{4}, \frac{ix^2}{2}\right)}{2\sqrt{x}}\right. \\ & \left. + ix^{3/2}\left(-\text{BesselY}\left(\frac{5}{4}, \frac{ix^2}{2}\right) - \frac{i\,\text{BesselY}\left(\frac{1}{4}, \frac{ix^2}{2}\right)}{2x^2}\right)\right) \\ & - \left(\sqrt{x}\,\text{BesselY}\left(\frac{1}{4}, \frac{ix^2}{2}\right)\right)\left(\frac{\text{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right)}{2\sqrt{x}}\right. \\ & \left. + ix^{3/2}\left(-\text{BesselJ}\left(\frac{5}{4}, \frac{ix^2}{2}\right) - \frac{i\,\text{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right)}{2x^2}\right)\right) \end{aligned}$$

Which simplifies to

$$W = -ix^2\left(\text{BesselJ}\left(\frac{1}{4}, \frac{ix^2}{2}\right)\text{BesselY}\left(\frac{5}{4}, \frac{ix^2}{2}\right) - \text{BesselY}\left(\frac{1}{4}, \frac{ix^2}{2}\right)\text{BesselJ}\left(\frac{5}{4}, \frac{ix^2}{2}\right)\right)$$

Which simplifies to

$$W = \frac{4}{\pi}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^{5/2}\,\text{BesselY}\left(\frac{1}{4}, \frac{ix^2}{2}\right)(x^4 - 2)}{\frac{4x^2}{\pi}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\sqrt{x}\, \text{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right) \left(x^4-2\right) \pi}{4} d x$$

Hence

$$u_1 = -\int_0^x \frac{\sqrt{\alpha}\, \text{BesselY}\left(\frac{1}{4}, \frac{i \alpha^2}{2}\right) \left(\alpha^4-2\right) \pi}{4} d \alpha$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^{5/2}\, \text{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) \left(x^4-2\right)}{\frac{4 x^2}{\pi}} \, d x$$

Which simplifies to

$$u_2 = \int \frac{\sqrt{x}\, \text{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) \left(x^4-2\right) \pi}{4} d x$$

Hence

$$u_2 = \int_0^x \frac{\sqrt{\alpha}\, \text{BesselJ}\left(\frac{1}{4}, \frac{i \alpha^2}{2}\right) \left(\alpha^4-2\right) \pi}{4} d \alpha$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = &-\int_0^x \frac{\sqrt{\alpha}\, \text{BesselY}\left(\frac{1}{4}, \frac{i \alpha^2}{2}\right) \left(\alpha^4-2\right) \pi}{4} d \alpha \sqrt{x}\, \text{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) \\ &+ \sqrt{x}\, \text{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right) \int_0^x \frac{\sqrt{\alpha}\, \text{BesselJ}\left(\frac{1}{4}, \frac{i \alpha^2}{2}\right) \left(\alpha^4-2\right) \pi}{4} d \alpha \end{aligned}$$

Which simplifies to

$$y_p(x) =$$
$$-\frac{\sqrt{x}\, \pi \left(\int_0^x \sqrt{\alpha}\, \text{BesselY}\left(\frac{1}{4}, \frac{i \alpha^2}{2}\right) \left(\alpha^4-2\right) d \alpha\, \text{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) - \text{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right) \int_0^x \sqrt{\alpha}\, \text{BesselJ}\left(\frac{1}{4}, \frac{i \alpha^2}{2}\right. \right.}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)\right) \\
&\quad + \left(-\frac{\sqrt{x}\,\pi\left(\int_0^x \sqrt{\alpha}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i\alpha^2}{2}\right)(\alpha^4-2)\,d\alpha\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) - \mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)\int_0^x \sqrt{\alpha}\,\mathrm{BesselJ}\left(\right.\right.}{4}\right.
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
\begin{aligned}
y &= c_1\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right) \\
&\quad - \frac{\sqrt{x}\,\pi\left(\int_0^x \sqrt{\alpha}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i\alpha^2}{2}\right)(\alpha^4-2)\,d\alpha\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i x^2}{2}\right) - \mathrm{BesselY}\left(\frac{1}{4}, \frac{i x^2}{2}\right)\int_0^x \sqrt{\alpha}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i\alpha^2}{2}\right.\right.}{4}
\end{aligned}
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : `0.016` (sec)
Leaf size : 34

```
dsolve(diff(diff(y(x),x),x)-x^2*y(x)-x^4+2 = 0,
       y(x),singsol=all)
```

$$y = \sqrt{x}\, \mathrm{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) c_2 + \sqrt{x}\, \mathrm{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right) c_1 - x^2$$

Mathematica DSolve solution

Solving time : `6.382` (sec)
Leaf size : 217

```
DSolve[{D[y[x],{x,2}]-x^2*y[x]-x^4+2==0,{}},
      y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \mathrm{ParabolicCylinderD}\left(-\frac{1}{2}, \sqrt{2}x\right)\left(\int_1^x -\frac{(K[1]^4-2)\,\mathrm{ParabolicCylinderD}\left(-\frac{1}{2}, i\sqrt{2}K[1]\right)}{\sqrt{2}\left(\mathrm{HermiteH}\left(-\frac{1}{2}, iK[1]\right)\mathrm{HermiteH}\left(\frac{1}{2}, K[1]\right) + \mathrm{HermiteH}\left(-\frac{1}{2}, K[1]\right)\left(-i\,\mathrm{HermiteH}\left(\frac{1}{2}, iK[1]\right) - 2\right.\right.} + c_1\right)$$

$$+\mathrm{ParabolicCylinderD}\left(-\frac{1}{2}, i\sqrt{2}x\right)\left(\int_1^x \frac{(K[2]^4-2)\,\mathrm{Parabo}}{\sqrt{2}\left(\mathrm{HermiteH}\left(-\frac{1}{2}, iK[2]\right)\mathrm{HermiteH}\left(\frac{1}{2}, K[2]\right) + \mathrm{HermiteH}\right.} + c_2\right)$$

2.2.39 problem 38

Internal problem ID [8519]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 38
Date solved : `Thursday, December 12, 2024 at 09:24:26 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' - 2x^2 y - x^4 + 1 = 0$$

Solved as second order Bessel ode

Time used: 0.688 (sec)

Writing the ode as

$$x^2 y'' - 2x^4 y = x^2 (x^4 - 1) \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2 y'' + y' x + (-n^2 + x^2)\, y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha)\, x y' + (\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2)\, y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha (c_1 \operatorname{BesselJ}(n, \beta x^\gamma) + c_2 \operatorname{BesselY}(n, \beta x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$
\begin{aligned}
\alpha &= \frac{1}{2} \\
\beta &= \frac{i\sqrt{2}}{2} \\
n &= \frac{1}{4} \\
\gamma &= 2
\end{aligned}
$$

Substituting all the above into (4) gives the solution as

$$
y = c_1\sqrt{x}\, \operatorname{BesselJ}\left(\frac{1}{4}, \frac{i\sqrt{2}\, x^2}{2}\right) + c_2\sqrt{x}\, \operatorname{BesselY}\left(\frac{1}{4}, \frac{i\sqrt{2}\, x^2}{2}\right)
$$

Therefore the homogeneous solution y_h is

$$
y_h = c_1\sqrt{x}\, \operatorname{BesselJ}\left(\frac{1}{4}, \frac{i\sqrt{2}\, x^2}{2}\right) + c_2\sqrt{x}\, \operatorname{BesselY}\left(\frac{1}{4}, \frac{i\sqrt{2}\, x^2}{2}\right)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \sqrt{x}\, \operatorname{BesselJ}\left(\frac{1}{4}, \frac{i\sqrt{2}\, x^2}{2}\right) \\
y_2 &= \sqrt{x}\, \operatorname{BesselY}\left(\frac{1}{4}, \frac{i\sqrt{2}\, x^2}{2}\right)
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right) & \sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right) \\ \frac{d}{dx}\left(\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right)\right) & \frac{d}{dx}\left(\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right) & \sqrt{} \\ \frac{\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right)}{2\sqrt{x}} + i x^{3/2}\left(-\mathrm{BesselJ}\left(\frac{5}{4}, \frac{i\sqrt{2}\,x^2}{2}\right) - \frac{i\sqrt{2}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right)}{4x^2}\right)\sqrt{2} & \frac{\mathrm{BesselY}\left(\frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right)}{2\sqrt{x}} + i x^{3/2}\left(- \right. \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & \left(\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right)\right)\left(\frac{\mathrm{BesselY}\left(\frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right)}{2\sqrt{x}}\right. \\ & \left. + i x^{3/2}\left(-\mathrm{BesselY}\left(\frac{5}{4}, \frac{i\sqrt{2}\,x^2}{2}\right) - \frac{i\sqrt{2}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right)}{4x^2}\right)\sqrt{2}\right) \\ & - \left(\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right)\right)\left(\frac{\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right)}{2\sqrt{x}}\right. \\ & \left. + i x^{3/2}\left(-\mathrm{BesselJ}\left(\frac{5}{4}, \frac{i\sqrt{2}\,x^2}{2}\right) - \frac{i\sqrt{2}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right)}{4x^2}\right)\sqrt{2}\right) \end{aligned}$$

Which simplifies to

$$\begin{aligned} W = -i x^2 \sqrt{2}\,\Bigg(& \mathrm{BesselJ}\left(\frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right)\mathrm{BesselY}\left(\frac{5}{4}, \frac{i\sqrt{2}\,x^2}{2}\right) \\ & - \mathrm{BesselY}\left(\frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right)\mathrm{BesselJ}\left(\frac{5}{4}, \frac{i\sqrt{2}\,x^2}{2}\right)\Bigg) \end{aligned}$$

Which simplifies to

$$W = \frac{4}{\pi}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^{5/2}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right)(x^4-1)}{\frac{4x^2}{\pi}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right)(x^4-1)\,\pi}{4}\,dx$$

Hence

$$u_1 = -\int_0^x \frac{\sqrt{\alpha}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i\sqrt{2}\,\alpha^2}{2}\right)(\alpha^4-1)\,\pi}{4}\,d\alpha$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^{5/2}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right)(x^4-1)}{\frac{4x^2}{\pi}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right)(x^4-1)\,\pi}{4}\,dx$$

Hence

$$u_2 = \int_0^x \frac{\sqrt{\alpha}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i\sqrt{2}\,\alpha^2}{2}\right)(\alpha^4-1)\,\pi}{4}\,d\alpha$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = &-\int_0^x \frac{\sqrt{\alpha}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i\sqrt{2}\,\alpha^2}{2}\right)(\alpha^4-1)\,\pi}{4}\,d\alpha\,\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right) \\ &+ \sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right)\int_0^x \frac{\sqrt{\alpha}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i\sqrt{2}\,\alpha^2}{2}\right)(\alpha^4-1)\,\pi}{4}\,d\alpha \end{aligned}$$

Which simplifies to

$$
y_p(x) = -\frac{\sqrt{x}\,\pi\left(\int_0^x \sqrt{\alpha}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i\sqrt{2}\,\alpha^2}{2}\right)(\alpha^4-1)\,d\alpha\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right) - \mathrm{BesselY}\left(\frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right)\int_0^x \sqrt{\alpha}\,\mathrm{Bessel}\right.}{4}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right)\right) \\
&\quad + \left(-\frac{\sqrt{x}\,\pi\left(\int_0^x \sqrt{\alpha}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i\sqrt{2}\,\alpha^2}{2}\right)(\alpha^4-1)\,d\alpha\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right) - \mathrm{BesselY}\left(\frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right)\int_0^x \sqrt{\alpha}\,\mathrm{B}\right.}{4}\right.
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
\begin{aligned}
y &= c_1\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right) \\
&\quad - \frac{\sqrt{x}\,\pi\left(\int_0^x \sqrt{\alpha}\,\mathrm{BesselY}\left(\frac{1}{4}, \frac{i\sqrt{2}\,\alpha^2}{2}\right)(\alpha^4-1)\,d\alpha\,\mathrm{BesselJ}\left(\frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right) - \mathrm{BesselY}\left(\frac{1}{4}, \frac{i\sqrt{2}\,x^2}{2}\right)\int_0^x \sqrt{\alpha}\,\mathrm{Bessel}\right.}{4}
\end{aligned}
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
```

```
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.016 (sec)
Leaf size : 40

```
dsolve(diff(diff(y(x),x),x)-2*x^2*y(x)-x^4+1 = 0,
       y(x),singsol=all)
```

$$y = \sqrt{x}\,\mathrm{BesselI}\left(\frac{1}{4}, \frac{\sqrt{2}\,x^2}{2}\right) c_2 + \sqrt{x}\,\mathrm{BesselK}\left(\frac{1}{4}, \frac{\sqrt{2}\,x^2}{2}\right) c_1 - \frac{x^2}{2}$$

Mathematica DSolve solution

Solving time : 4.134 (sec)
Leaf size : 288

```
DSolve[{D[y[x],{x,2}]-2*x^2*y[x]-x^4+1==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$y(x)$

$$\to \mathrm{ParabolicCylinderD}\left(-\frac{1}{2}, 2^{3/4}x\right)\left(\left(\int_1^x \frac{}{i2^{3/4}\,\mathrm{HermiteH}\left(-\frac{1}{2}, \sqrt[4]{2}K[1]\right)\mathrm{HermiteH}\left(\frac{1}{2}, i\sqrt[4]{2}K[1]\right) + \mathrm{Para}\right.\right.$$

2.2.40 problem 39

Internal problem ID [8520]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 39
Date solved : `Thursday, December 12, 2024 at 09:24:36 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - x^3 y - x^3 = 0$$

Solved as second order Bessel ode

Time used: 0.619 (sec)

Writing the ode as

$$x^2 y'' - x^5 y = x^5 \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2 y'' + y' x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha)\, x y' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha} (c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\alpha = \frac{1}{2}$$
$$\beta = \frac{2i}{5}$$
$$n = \frac{1}{5}$$
$$\gamma = \frac{5}{2}$$

Substituting all the above into (4) gives the solution as

$$y = c_1\sqrt{x}\ \mathrm{BesselJ}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right) + c_2\sqrt{x}\ \mathrm{BesselY}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1\sqrt{x}\ \mathrm{BesselJ}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right) + c_2\sqrt{x}\ \mathrm{BesselY}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sqrt{x}\ \mathrm{BesselJ}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)$$
$$y_2 = \sqrt{x}\ \mathrm{BesselY}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right) & \sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right) \\ \frac{d}{dx}\left(\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)\right) & \frac{d}{dx}\left(\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right) & \sqrt{x}\,\mathrm{BesselY} \\ \frac{\mathrm{BesselJ}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)}{2\sqrt{x}} + ix^2\left(-\mathrm{BesselJ}\left(\frac{6}{5}, \frac{2ix^{5/2}}{5}\right) - \frac{i\,\mathrm{BesselJ}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)}{2x^{5/2}}\right) & \frac{\mathrm{BesselY}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)}{2\sqrt{x}} + ix^2\left(-\mathrm{BesselY}\right.\end{vmatrix}$$

Therefore

$$W = \left(\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)\right)\left(\frac{\mathrm{BesselY}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)}{2\sqrt{x}}\right.$$
$$\left. + ix^2\left(-\mathrm{BesselY}\left(\frac{6}{5}, \frac{2ix^{5/2}}{5}\right) - \frac{i\,\mathrm{BesselY}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)}{2x^{5/2}}\right)\right) - \left(\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)\right)\left(\frac{\mathrm{BesselJ}\left(\frac{1}{5},\right.}{2\sqrt{x}}\right.$$

Which simplifies to

$$W = -ix^{5/2}\left(\mathrm{BesselJ}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)\mathrm{BesselY}\left(\frac{6}{5}, \frac{2ix^{5/2}}{5}\right) - \mathrm{BesselY}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)\mathrm{BesselJ}\left(\frac{6}{5}, \frac{2ix^{5/2}}{5}\right)\right)$$

Which simplifies to

$$W = \frac{5}{\pi}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^{11/2}\,\mathrm{BesselY}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)}{\frac{5x^2}{\pi}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{x^{7/2}\,\text{BesselY}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)\pi}{5}dx$$

Hence

$$u_1 = -\int_0^x \frac{\alpha^{7/2}\,\text{BesselY}\left(\frac{1}{5}, \frac{2i\alpha^{5/2}}{5}\right)\pi}{5}d\alpha$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^{11/2}\,\text{BesselJ}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)}{\frac{5x^2}{\pi}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{x^{7/2}\,\text{BesselJ}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)\pi}{5}dx$$

Hence

$$u_2 = \int_0^x \frac{\alpha^{7/2}\,\text{BesselJ}\left(\frac{1}{5}, \frac{2i\alpha^{5/2}}{5}\right)\pi}{5}d\alpha$$

Therefore the particular solution, from equation (1) is

$$y_p(x) =$$
$$-\int_0^x \frac{\alpha^{7/2}\,\text{BesselY}\left(\frac{1}{5}, \frac{2i\alpha^{5/2}}{5}\right)\pi}{5}d\alpha\sqrt{x}\,\text{BesselJ}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)+\sqrt{x}\,\text{BesselY}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)\int_0^x \frac{\alpha^{7/2}\,\text{BesselJ}}{}$$

Which simplifies to

$$y_p(x)$$
$$= \frac{\pi\sqrt{x}\left(-\int_0^x \alpha^{7/2}\,\text{BesselY}\left(\frac{1}{5}, \frac{2i\alpha^{5/2}}{5}\right)d\alpha\,\text{BesselJ}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)+\text{BesselY}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)\int_0^x \alpha^{7/2}\,\text{BesselJ}\left(\frac{1}{5}, \frac{2i\alpha^{5/}}{5}\right.\right.}{5}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{5}, \frac{2i x^{5/2}}{5}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{5}, \frac{2i x^{5/2}}{5}\right)\right) \\
&\quad + \left(\frac{\pi\sqrt{x}\left(-\int_0^x \alpha^{7/2}\,\mathrm{BesselY}\left(\frac{1}{5}, \frac{2i\alpha^{5/2}}{5}\right) d\alpha\,\mathrm{BesselJ}\left(\frac{1}{5}, \frac{2i x^{5/2}}{5}\right) + \mathrm{BesselY}\left(\frac{1}{5}, \frac{2i x^{5/2}}{5}\right)\int_0^x \alpha^{7/2}\,\mathrm{BesselJ}\left(\frac{1}{5}\right.\right.}{5}\right.
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
\begin{aligned}
y &= c_1\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{5}, \frac{2i x^{5/2}}{5}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{5}, \frac{2i x^{5/2}}{5}\right) \\
&\quad + \frac{\pi\sqrt{x}\left(-\int_0^x \alpha^{7/2}\,\mathrm{BesselY}\left(\frac{1}{5}, \frac{2i\alpha^{5/2}}{5}\right) d\alpha\,\mathrm{BesselJ}\left(\frac{1}{5}, \frac{2i x^{5/2}}{5}\right) + \mathrm{BesselY}\left(\frac{1}{5}, \frac{2i x^{5/2}}{5}\right)\int_0^x \alpha^{7/2}\,\mathrm{BesselJ}\left(\frac{1}{5}, \frac{2i}{}\right.\right.}{5}
\end{aligned}
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.016 (sec)
Leaf size : 30

```
dsolve(diff(diff(y(x),x),x)-y(x)*x^3-x^3 = 0,
       y(x),singsol=all)
```

$$y = \sqrt{x}\, \text{BesselI}\left(\frac{1}{5}, \frac{2x^{5/2}}{5}\right) c_2 + \sqrt{x}\, \text{BesselK}\left(\frac{1}{5}, \frac{2x^{5/2}}{5}\right) c_1 - 1$$

Mathematica DSolve solution

Solving time : 0.244 (sec)
Leaf size : 217

```
DSolve[{D[y[x],{x,2}]-x^3*y[x]-x^3==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt[5]{-1}\, \text{Gamma}\left(\frac{4}{5}\right) \left(5^{4/5} x^5\, \text{Gamma}\left(\frac{6}{5}\right) \text{Hypergeometric0F1Regularized}\left(\frac{9}{5}, \frac{x^5}{25}\right) \text{BesselI}\left(\frac{1}{5}, \frac{2x^{5/2}}{5}\right) + 5\ 5\right.}{25\sqrt[5]{x^{5/2}}\text{Root}\left[25\right.} + \frac{c_1\sqrt{x}\, \text{Gamma}\left(\frac{4}{5}\right) \text{BesselI}\left(-\frac{1}{5}, \frac{2x^{5/2}}{5}\right)}{\sqrt[5]{5}} + \sqrt[5]{-\frac{1}{5}} c_2\sqrt{x}\, \text{Gamma}\left(\frac{6}{5}\right) \text{BesselI}\left(\frac{1}{5}, \frac{2x^{5/2}}{5}\right)$$

2.2.41 problem 40

Internal problem ID [8521]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 40
Date solved : `Friday, December 13, 2024 at 05:05:02 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - x^3 y - x^4 = 0$$

Solved as second order Bessel ode

Time used: 65.896 (sec)

Writing the ode as

$$x^2 y'' - x^5 y = x^6 \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2 y'' + y' x + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha)\, x y' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha} (c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= \frac{1}{2}\\ \beta &= \frac{2i}{5}\\ n &= \frac{1}{5}\\ \gamma &= \frac{5}{2}\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = c_1\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{5}, \frac{2i x^{5/2}}{5}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{5}, \frac{2i x^{5/2}}{5}\right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{5}, \frac{2i x^{5/2}}{5}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{5}, \frac{2i x^{5/2}}{5}\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= \sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{5}, \frac{2i x^{5/2}}{5}\right)\\ y_2 &= \sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{5}, \frac{2i x^{5/2}}{5}\right)\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right) & \sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right) \\ \frac{d}{dx}\left(\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)\right) & \frac{d}{dx}\left(\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right) & \sqrt{x}\,\mathrm{BesselY} \\ \frac{\mathrm{BesselJ}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)}{2\sqrt{x}} + ix^2\left(-\mathrm{BesselJ}\left(\frac{6}{5}, \frac{2ix^{5/2}}{5}\right) - \frac{i\,\mathrm{BesselJ}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)}{2x^{5/2}}\right) & \frac{\mathrm{BesselY}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)}{2\sqrt{x}} + ix^2\left(-\mathrm{BesselY}\right.
\end{vmatrix}$$

Therefore

$$W = \left(\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)\right)\left(\frac{\mathrm{BesselY}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)}{2\sqrt{x}}\right.$$
$$\left. + ix^2\left(-\mathrm{BesselY}\left(\frac{6}{5}, \frac{2ix^{5/2}}{5}\right) - \frac{i\,\mathrm{BesselY}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)}{2x^{5/2}}\right)\right) - \left(\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)\right)\left(\frac{\mathrm{BesselJ}\left(\frac{1}{5},\right.}{2\sqrt{x}}\right.$$

Which simplifies to

$$W = -ix^{5/2}\left(\mathrm{BesselJ}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)\mathrm{BesselY}\left(\frac{6}{5}, \frac{2ix^{5/2}}{5}\right) - \mathrm{BesselY}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)\mathrm{BesselJ}\left(\frac{6}{5}, \frac{2ix^{5/2}}{5}\right)\right)$$

Which simplifies to

$$W = \frac{5}{\pi}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^{13/2}\,\mathrm{BesselY}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)}{\frac{5x^2}{\pi}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{x^{9/2}\,\text{BesselY}\left(\frac{1}{5}, \frac{2i x^{5/2}}{5}\right)\pi}{5} dx$$

Hence

$$u_1 = -\int_0^x \frac{\alpha^{9/2}\,\text{BesselY}\left(\frac{1}{5}, \frac{2i\alpha^{5/2}}{5}\right)\pi}{5} d\alpha$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^{13/2}\,\text{BesselJ}\left(\frac{1}{5}, \frac{2i x^{5/2}}{5}\right)}{\frac{5x^2}{\pi}} dx$$

Which simplifies to

$$u_2 = \int \frac{x^{9/2}\,\text{BesselJ}\left(\frac{1}{5}, \frac{2i x^{5/2}}{5}\right)\pi}{5} dx$$

Hence

$$u_2 = \frac{(-1)^{1/10}\,\pi\, x^{7/2}\,\text{BesselI}\left(\frac{6}{5}, \frac{2x^{5/2}}{5}\right)}{5\left(x^{5/2}\right)^{1/5}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\int_0^x \frac{\alpha^{9/2}\,\text{BesselY}\left(\frac{1}{5}, \frac{2i\alpha^{5/2}}{5}\right)\pi}{5} d\alpha\sqrt{x}\,\text{BesselJ}\left(\frac{1}{5}, \frac{2i x^{5/2}}{5}\right) + \frac{x^4\,\text{BesselY}\left(\frac{1}{5}, \frac{2i x^{5/2}}{5}\right)(-1)^{1/10}\,\pi\,\text{BesselI}\left(\right.}{5\left(x^{5/2}\right)^{1/5}}$$

Which simplifies to

$$y_p(x) = \frac{\left(-\int_0^x \alpha^{9/2}\,\text{BesselY}\left(\frac{1}{5}, \frac{2i\alpha^{5/2}}{5}\right) d\alpha\sqrt{x}\,\text{BesselJ}\left(\frac{1}{5}, \frac{2i x^{5/2}}{5}\right)\left(x^{5/2}\right)^{1/5} + \text{BesselY}\left(\frac{1}{5}, \frac{2i x^{5/2}}{5}\right)(-1)^{1/10}\left(x^4\,\text{Be}\right.\right.}{5\left(x^{5/2}\right)^{1/5}}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)\right) \\
&\quad + \left(\frac{\left(-\int_0^x \alpha^{9/2}\,\mathrm{BesselY}\left(\frac{1}{5}, \frac{2i\alpha^{5/2}}{5}\right)d\alpha\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)\left(x^{5/2}\right)^{1/5} + \mathrm{BesselY}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)(-1)^{1/10}\left(\right.\right.}{5\left(x^{5/2}\right)^{1/5}}\right.
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
\begin{aligned}
y &= c_1\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right) + c_2\sqrt{x}\,\mathrm{BesselY}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right) \\
&\quad + \frac{\left(-\int_0^x \alpha^{9/2}\,\mathrm{BesselY}\left(\frac{1}{5}, \frac{2i\alpha^{5/2}}{5}\right)d\alpha\sqrt{x}\,\mathrm{BesselJ}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)\left(x^{5/2}\right)^{1/5} + \mathrm{BesselY}\left(\frac{1}{5}, \frac{2ix^{5/2}}{5}\right)(-1)^{1/10}\left(x^4\right.\right.}{5\left(x^{5/2}\right)^{1/5}}
\end{aligned}
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
```

```
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.015 (sec)
Leaf size : 32

```
dsolve(diff(diff(y(x),x),x)-y(x)*x^3-x^4 = 0,
       y(x),singsol=all)
```

$$y = \sqrt{x}\, \text{BesselI}\left(\frac{1}{5}, \frac{2x^{5/2}}{5}\right) c_2 + \sqrt{x}\, \text{BesselK}\left(\frac{1}{5}, \frac{2x^{5/2}}{5}\right) c_1 - x$$

Mathematica DSolve solution

Solving time : 0.191 (sec)
Leaf size : 219

```
DSolve[{D[y[x],{x,2}]-x^3*y[x]-x^4==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt[5]{-1}\,\text{Gamma}\left(\frac{6}{5}\right)\left(-5^{2/5}\sqrt[5]{x^{5/2}}x^{15/2}\,\text{Gamma}\left(\frac{4}{5}\right)\text{Hypergeometric0F1Regularized}\left(\frac{11}{5}, \frac{x^5}{25}\right)\text{BesselI}\left(-\frac{1}{5}\right.\right.}{25x^{3/2}\text{Root}\,[}$$
$$+ \frac{c_1\sqrt{x}\,\text{Gamma}\left(\frac{4}{5}\right)\text{BesselI}\left(-\frac{1}{5}, \frac{2x^{5/2}}{5}\right)}{\sqrt[5]{5}}$$
$$+ \sqrt[5]{-\frac{1}{5}}c_2\sqrt{x}\,\text{Gamma}\left(\frac{6}{5}\right)\text{BesselI}\left(\frac{1}{5}, \frac{2x^{5/2}}{5}\right)$$

2.2.42 problem 41

Internal problem ID [8522]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 41
Date solved : `Wednesday, December 18, 2024 at 01:53:17 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - x^2y' - x^2y - x^2 = 0$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
```

```
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   trying a solution in terms of MeijerG functions
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
   <- Heun successful: received ODE is equivalent to the  HeunT  ODE, case  c = 0
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 2.695 (sec)
Leaf size : 55

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x^2-x^2*y(x)-x^2 = 0,
       y(x),singsol=all)
```

$$y = \operatorname{HeunT}\left(3^{2/3}, 3, 2\,3^{1/3}, \frac{3^{2/3}x}{3}\right) e^{-x} c_2 + \operatorname{HeunT}\left(3^{2/3}, -3, 2\,3^{1/3}, -\frac{3^{2/3}x}{3}\right) e^{\frac{x\left(x^2+3\right)}{3}} c_1 - 1$$

Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{D[y[x],{x,2}]-x^2*D[y[x],x]-x^2*y[x]-x^2==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

Not solved

2.2.43 problem 42

Internal problem ID [8523]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 42
Date solved : `Thursday, December 12, 2024 at 09:25:37 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - x^3y' - x^3y - x^3 = 0$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
```

```
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   trying a solution in terms of MeijerG functions
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      trying 2nd order, integrating factor of the form mu(x,y)
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Kummer
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         -> Mathieu
            -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying to convert to an ODE of Bessel type
      trying to convert to an ODE of Bessel type
      -> trying reduction of order to Riccati
         trying Riccati sub-methods:
            -> trying a symmetry pattern of the form [F(x)*G(y), 0]
            -> trying a symmetry pattern of the form [0, F(x)*G(y)]
            -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3`[0, y+1]
```

Maple dsolve solution

Solving time : 0.296 (sec)
Leaf size : maple_leaf_size

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x^3-y(x)*x^3-x^3 = 0,
       y(x),singsol=all)
```

No solution found

Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{D[y[x],{x,2}]-x^3*D[y[x],x]-x^3*y[x]-x^3==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

Not solved

2.2.44 problem 43

Internal problem ID [8524]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 43
Date solved : `Thursday, December 12, 2024 at 09:25:38 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - xy' - xy - x = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.493 (sec)

Writing the ode as

$$y'' - xy' - xy = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -x \\ C &= -x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 4x - 2}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.113: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

<u>Attempting to find a solution using case $n = 1$.</u>

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2 + 4x - 2}{4} \\ &= Q + \frac{R}{4} \\ &= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\ &= \frac{1}{4}x^2 + x - \frac{1}{2} \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$\begin{aligned} b &= \left(-\frac{1}{2}\right) - (1) \\ &= -\frac{3}{2} \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{1}{4}x^2 + x - \frac{1}{2}$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$\begin{aligned} d &= \alpha_\infty^- \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= (-)[\sqrt{r}]_\infty \\ &= 0 + (-)\left(\frac{x}{2}+1\right) \\ &= -1 - \frac{x}{2} \\ &= -1 - \frac{x}{2} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-1-\frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1-\frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) = 0$$

$$-2 + a_0 = 0$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 2\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = 2 + x$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \,dx} \\
&= (2 + x)\,\mathrm{e}^{\int \left(-1-\frac{x}{2}\right)dx} \\
&= (2 + x)\,\mathrm{e}^{-x-\frac{1}{4}x^2} \\
&= (2 + x)\,\mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-x}{1}\,dx} \\
&= z_1 e^{\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = (2 + x)\,\mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{x^2}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2 + x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-x}\right) + c_2\left((2+x)\,\mathrm{e}^{-x}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' - xy' - xy = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1(2+x)\,\mathrm{e}^{-x} - \frac{c_2\,\mathrm{e}^{-x}\left(i\sqrt{\pi}\,\sqrt{2}\,(2+x)\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= (2+x)\,\mathrm{e}^{-x} \\
y_2 &= -\frac{\mathrm{e}^{-x}\left(i\sqrt{\pi}\,\sqrt{2}\,(2+x)\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (2+x)\,\mathrm{e}^{-x} & -\frac{\mathrm{e}^{-x}\left(i\sqrt{\pi}\,\sqrt{2}\,(2+x)\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} \\ \frac{d}{dx}\left((2+x)\,\mathrm{e}^{-x}\right) & \frac{d}{dx}\left(-\frac{\mathrm{e}^{-x}\left(i\sqrt{\pi}\,\sqrt{2}\,(2+x)\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (2+x)\,\mathrm{e}^{-x} & -\frac{\mathrm{e}^{-x}\left(i\sqrt{\pi}\,\sqrt{2}\,(2+x)\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} \\ \mathrm{e}^{-x}-(2+x)\,\mathrm{e}^{-x} & \frac{\mathrm{e}^{-x}\left(i\sqrt{\pi}\,\sqrt{2}\,(2+x)\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} - \frac{\mathrm{e}^{-x}\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2(2+x)\mathrm{e}^{-2}\mathrm{e}^{\frac{(2+x}{2}}\right.}{2} \end{vmatrix}$$

Therefore

$$W = \left((2+x)\,\mathrm{e}^{-x}\right)\left(\frac{\mathrm{e}^{-x}\left(i\sqrt{\pi}\,\sqrt{2}\,(2+x)\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} - \frac{\mathrm{e}^{-x}\left(i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2(2+x)\,\mathrm{e}^{-2}\mathrm{e}^{\frac{(2+x)^2}{2}}+2(2+x)\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}\right) - \left(-\frac{\mathrm{e}^{-x}\left(i\sqrt{\pi}\,\sqrt{2}\,(2+x)\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}\right)\left(\mathrm{e}^{-x}-(2+x)\,\mathrm{e}^{-x}\right)$$

Which simplifies to

$$W = \mathrm{e}^{\frac{(2+x)^2}{2}}\mathrm{e}^{-2}\mathrm{e}^{-2x}x^2 + 4\,\mathrm{e}^{\frac{(2+x)^2}{2}}\mathrm{e}^{-2}\mathrm{e}^{-2x}x - \mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-2x}x^2 + 4\,\mathrm{e}^{\frac{(2+x)^2}{2}}\mathrm{e}^{-2}\mathrm{e}^{-2x} - 4\,\mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-2x}x - 3\,\mathrm{e}^{\frac{x(4+x)}{2}}\mathrm{e}^{-2x}$$

Which simplifies to

$$W = \mathrm{e}^{\frac{x^2}{2}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{\mathrm{e}^{-x}\left(i\sqrt{\pi}\,\sqrt{2}\,(2+x)\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)x}{2}}{\mathrm{e}^{\frac{x^2}{2}}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{\mathrm{e}^{-\frac{x(2+x)}{2}}\left(i\sqrt{\pi}\,\sqrt{2}\,(2+x)\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)x}{2}dx$$

Hence

$$\begin{aligned}u_1 = &-\mathrm{e}^{-2}\mathrm{e}^{-\frac{x(2+x)}{2}}\mathrm{e}^{\frac{(2+x)^2}{2}}x+\mathrm{e}^{-\frac{x(2+x)}{2}}\mathrm{e}^{\frac{x(4+x)}{2}}x-\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)x\,\mathrm{e}^{-\frac{x(2+x)}{2}}}{2}\\ &-\mathrm{e}^{-\frac{x(2+x)}{2}}\mathrm{e}^{\frac{x(4+x)}{2}}-\frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)\mathrm{e}^{-\frac{x(2+x)}{2}}}{2}\end{aligned}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(2+x)\,\mathrm{e}^{-x}x}{\mathrm{e}^{\frac{x^2}{2}}}\,dx$$

Which simplifies to

$$u_2 = \int x(2+x)\,\mathrm{e}^{-\frac{x(2+x)}{2}}dx$$

Hence

$$u_2 = -(x+1)\,\mathrm{e}^{-\frac{x(2+x)}{2}}$$

Therefore the particular solution, from equation (1) is

$$
\begin{aligned}
y_p(x) = {} & \left(-\mathrm{e}^{-2} \mathrm{e}^{-\frac{x(2+x)}{2}} \mathrm{e}^{\frac{(2+x)^2}{2}} x + \mathrm{e}^{-\frac{x(2+x)}{2}} \mathrm{e}^{\frac{x(4+x)}{2}} x - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) x\,\mathrm{e}^{-\frac{x(2+x)}{2}}}{2} \right. \\
& \left. - \mathrm{e}^{-\frac{x(2+x)}{2}} \mathrm{e}^{\frac{x(4+x)}{2}} - \frac{i\sqrt{\pi}\,\sqrt{2}\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) \mathrm{e}^{-\frac{x(2+x)}{2}}}{2}\right) (2+x)\,\mathrm{e}^{-x} \\
& + \frac{\mathrm{e}^{-x}\left(i\sqrt{\pi}\,\sqrt{2}\,(2+x)\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)(x+1)\,\mathrm{e}^{-\frac{x(2+x)}{2}}}{2}
\end{aligned}
$$

Which simplifies to

$$
y_p(x) = -1
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1(2+x)\,\mathrm{e}^{-x} - \frac{c_2\,\mathrm{e}^{-x}\left(i\sqrt{\pi}\,\sqrt{2}\,(2+x)\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2}\right) + (-1)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = c_1(2+x)\,\mathrm{e}^{-x} - \frac{c_2\,\mathrm{e}^{-x}\left(i\sqrt{\pi}\,\sqrt{2}\,(2+x)\,\mathrm{e}^{-2}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right)}{2} - 1
$$

Solved as second order ode adjoint method

Time used: 2.331 (sec)

In normal form the ode

$$
y'' - xy' - xy - x = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= -x \\
q(x) &= -x \\
r(x) &= x
\end{aligned}
$$

The Lagrange adjoint ode is given by

$$
\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - (-x\xi(x))' + (-x\xi(x)) &= 0 \\
\xi''(x) + x\xi'(x) - (x-1)\,\xi(x) &= 0
\end{aligned}
$$

Which is solved for $\xi(x)$. Writing the ode as

$$
\xi'' + x\xi' + (1-x)\,\xi = 0 \tag{1}
$$

$$
A\xi'' + B\xi' + C\xi = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= x \\
C &= 1-x
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = \xi e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{x^2 + 4x - 2}{4} \tag{6}
$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 4x - 2 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then ξ is found using the inverse transformation

$$\xi = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.114: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \le i \le v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx \frac{x}{2} + 1 - \frac{3}{2x} + \frac{3}{x^2} - \frac{33}{4x^3} + \frac{51}{2x^4} - \frac{339}{4x^5} + \frac{591}{2x^6} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = \frac{1}{2}$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= \frac{x}{2} + 1 \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = \frac{1}{4}x^2 + x + 1$$

This shows that the coefficient of 1 in the above is 1. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{x^2+4x-2}{4} \\
&= Q + \frac{R}{4} \\
&= \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right) + (0) \\
&= \frac{1}{4}x^2 + x - \frac{1}{2}
\end{aligned}
$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is $-\frac{1}{2}$. Now b can be found.

$$
\begin{aligned}
b &= \left(-\frac{1}{2}\right) - (1) \\
&= -\frac{3}{2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \frac{x}{2} + 1 \\
\alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= -2 \\
\alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{-\frac{3}{2}}{\frac{1}{2}} - 1\right) &&= 1
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{1}{4}x^2 + x - \frac{1}{2}
$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	$\frac{x}{2}+1$	-2	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = 1$, and since there are no poles then

$$
\begin{aligned}
d &= \alpha_\infty^- \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= (-)[\sqrt{r}]_\infty \\
&= 0 + (-)\left(\frac{x}{2} + 1\right) \\
&= -1 - \frac{x}{2} \\
&= -1 - \frac{x}{2}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-1 - \frac{x}{2}\right)(1) + \left(\left(-\frac{1}{2}\right) + \left(-1 - \frac{x}{2}\right)^2 - \left(\frac{1}{4}x^2 + x - \frac{1}{2}\right)\right) &= 0 \\
-2 + a_0 &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 2\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = 2 + x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\, dx} \\
&= (2+x)\, \mathrm{e}^{\int \left(-1-\frac{x}{2}\right) dx} \\
&= (2+x)\, \mathrm{e}^{-x-\frac{1}{4}x^2} \\
&= (2+x)\, \mathrm{e}^{-\frac{x(4+x)}{4}}
\end{aligned}$$

The first solution to the original ode in ξ is found from

$$\begin{aligned}
\xi_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{1}\, dx} \\
&= z_1 e^{-\frac{x^2}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{4}}\right)
\end{aligned}$$

Which simplifies to

$$\xi_1 = (2+x)\, \mathrm{e}^{-\frac{x(2+x)}{2}}$$

The second solution ξ_2 to the original ode is found using reduction of order

$$\xi_2 = \xi_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{\xi_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
\xi_2 &= \xi_1 \int \frac{e^{\int -\frac{x}{1}\, dx}}{(\xi_1)^2}\, dx \\
&= \xi_1 \int \frac{e^{-\frac{x^2}{2}}}{(\xi_1)^2}\, dx \\
&= \xi_1 \left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\, \mathrm{e}^{-2}\sqrt{2}\ \mathrm{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
\xi &= c_1\xi_1 + c_2\xi_2 \\
&= c_1\left((2+x)\,\mathrm{e}^{-\frac{x(2+x)}{2}}\right) \\
&\quad + c_2\left((2+x)\,\mathrm{e}^{-\frac{x(2+x)}{2}}\left(-\frac{\mathrm{e}^{-2+\frac{(2+x)^2}{2}}}{2+x} - \frac{i\sqrt{\pi}\,\mathrm{e}^{-2}\sqrt{2}\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$
\begin{aligned}
\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y &= \int \xi(x)\,r(x)\,dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi\,(x)}
\end{aligned}
$$

Or

$$
y' + y\left(-x - \frac{c_1\,\mathrm{e}^{-\frac{x(2+x)}{2}} + c_1(2+x)\,(-1-x)\,\mathrm{e}^{-\frac{x(2+x)}{2}} - \frac{c_2(-1-x)\mathrm{e}^{-\frac{x(2+x)}{2}}\left(i\sqrt{\pi}\,\sqrt{2}\,(2+x)\mathrm{e}^{-2}\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}\right.}{2}}{c_1\,(2+x)\,\mathrm{e}^{-\frac{x(2+x)}{2}} - \frac{c_2\,\mathrm{e}^{-\frac{x(2+x)}{2}}\left(i\sqrt{\pi}\,\sqrt{2}\,(2+x)\mathrm{e}^{-2}\,\mathrm{er}\right.}{2}}\right.
$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$
y' + q(x)y = p(x)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= -\frac{-i\mathrm{e}^{-2}(x+1)\,\sqrt{2}\,\sqrt{\pi}\,c_2\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 + 2c_1(x+1)}{i\mathrm{e}^{-2}\sqrt{2}\,\sqrt{\pi}\,\,(2+x)\,c_2\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 - 2c_1\,(2+x)} \\
p(x) &= \frac{-i\mathrm{e}^{-2}(x+1)\,\sqrt{2}\,\sqrt{\pi}\,c_2\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 + 2c_1(x+1)}{i\mathrm{e}^{-2}\sqrt{2}\,\sqrt{\pi}\,\,(2+x)\,c_2\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2 - 2c_1\,(2+x)}
\end{aligned}
$$

The integrating factor μ is

$$
\mu = \mathrm{e}^{\int -\frac{-i\mathrm{e}^{-2}(x+1)\sqrt{2}\,\sqrt{\pi}\,c_2\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)-2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2+2c_1(x+1)}{i\mathrm{e}^{-2}\sqrt{2}\,\sqrt{\pi}\,(2+x)c_2\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2\,\mathrm{e}^{\frac{x(4+x)}{2}}c_2-2c_1(2+x)}dx}
$$

Therefore the solution is

$$y = \left(\int \frac{\left(-i\mathrm{e}^{-2}(x+1)\sqrt{2}\,\sqrt{\pi}\,c_2\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2 + 2c_1(x+1)\right) \mathrm{e}^{\int -\frac{-i\mathrm{e}^{-2}(x+1)\sqrt{2}\,\sqrt{\pi}\,c_2\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}{i\mathrm{e}^{-2}\sqrt{2}\,\sqrt{\pi}\,(2+x)c_2\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)}}}{i\mathrm{e}^{-2}\sqrt{2}\,\sqrt{\pi}\,(2+x)\,c_2\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2 - 2c_1\,(2+x)} \right.$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \left(\int \frac{\left(-i\mathrm{e}^{-2}(x+1)\sqrt{2}\,\sqrt{\pi}\,c_2\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2 + 2c_1(x+1)\right) \mathrm{e}^{\int -\frac{-i\mathrm{e}^{-2}(x+1)\sqrt{2}\,\sqrt{\pi}\,c_2\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) -}{i\mathrm{e}^{-2}\sqrt{2}\,\sqrt{\pi}\,(2+x)c_2\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2}}}{i\mathrm{e}^{-2}\sqrt{2}\,\sqrt{\pi}\,(2+x)\,c_2\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2 - 2c_1\,(2+x)} \right.$$

$$\left. + c_3 \right) \mathrm{e}^{-\int -\frac{-i\mathrm{e}^{-2}(x+1)\sqrt{2}\,\sqrt{\pi}\,c_2\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2 + 2c_1(x+1)}{i\mathrm{e}^{-2}\sqrt{2}\,\sqrt{\pi}\,(2+x)c_2\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2 - 2c_1(2+x)} dx}$$

The constants can be merged to give

$$y = \left(\int \frac{\left(-i\mathrm{e}^{-2}(x+1)\sqrt{2}\,\sqrt{\pi}\,c_2\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2 + 2c_1(x+1)\right) \mathrm{e}^{\int -\frac{-i\mathrm{e}^{-2}(x+1)\sqrt{2}\,\sqrt{\pi}\,c_2\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) -}{i\mathrm{e}^{-2}\sqrt{2}\,\sqrt{\pi}\,(2+x)c_2\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right)+2}}}{i\mathrm{e}^{-2}\sqrt{2}\,\sqrt{\pi}\,(2+x)\,c_2\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2 - 2c_1\,(2+x)} \right.$$

$$\left. + 1 \right) \mathrm{e}^{-\int -\frac{-i\mathrm{e}^{-2}(x+1)\sqrt{2}\,\sqrt{\pi}\,c_2\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2 + 2c_1(x+1)}{i\mathrm{e}^{-2}\sqrt{2}\,\sqrt{\pi}\,(2+x)c_2\,\operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2 - 2c_1(2+x)} dx}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = \left(\int \frac{\left(-i \mathrm{e}^{-2}(x+1)\sqrt{2}\sqrt{\pi}\, c_2 \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2 + 2c_1(x+1)\right) \mathrm{e}^{\int -\frac{-i \mathrm{e}^{-2}(x+1)\sqrt{2}\sqrt{\pi}\, c_2 \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - \cdots}{i \mathrm{e}^{-2}\sqrt{2}\sqrt{\pi}\,(2+x) c_2 \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2 \cdots}}}{i \mathrm{e}^{-2}\sqrt{2}\sqrt{\pi}\,(2+x)\, c_2 \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2 - 2c_1\,(2+x)} \cdots + 1 \right) \mathrm{e}^{-\int -\frac{-i \mathrm{e}^{-2}(x+1)\sqrt{2}\sqrt{\pi}\, c_2 \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) - 2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2 + 2c_1(x+1)}{i \mathrm{e}^{-2}\sqrt{2}\sqrt{\pi}\,(2+x) c_2 \operatorname{erf}\left(\frac{i\sqrt{2}\,(2+x)}{2}\right) + 2\,\mathrm{e}^{\frac{x(4+x)}{2}} c_2 - 2c_1(2+x)} dx}
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
      Group is reducible, not completely reducible
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 52

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x-x*y(x)-x = 0,
       y(x),singsol=all)
```

$$y = i c_1 \sqrt{\pi}\,(x+2)\,\sqrt{2}\,\mathrm{e}^{-2-x}\,\mathrm{erf}\left(\frac{i\sqrt{2}\,(x+2)}{2}\right) + 2\,\mathrm{e}^{\frac{x(x+2)}{2}} c_1 - 1 + \mathrm{e}^{-x}(x+2)\,c_2$$

Mathematica DSolve solution

Solving time : 0.726 (sec)
Leaf size : 216

```
DSolve[{D[y[x],{x,2}]-x*D[y[x],x]-x*y[x]-x==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$\begin{aligned} y(x) \rightarrow \frac{1}{2} e^{-\frac{1}{2}(x+2)^2} \Bigg(& 2\sqrt{2} e^{\frac{x^2}{2}+x+2}(x+2) \int_1^x \left(\frac{e^{K[1]}K[1]}{\sqrt{2}} \right. \\ & \left. - \frac{1}{2} e^{-\frac{1}{2}K[1]^2-K[1]-2} \sqrt{\pi}\,\mathrm{erfi}\left(\frac{\sqrt{(K[1]+2)^2}}{\sqrt{2}} \right) K[1] \sqrt{(K[1]+2)^2} \right) dK[1] \\ & - \sqrt{2\pi} \sqrt{(x+2)^2} \left(c_2 e^{\frac{x^2}{2}+x+2} + x + 1 \right) \mathrm{erfi}\left(\frac{\sqrt{(x+2)^2}}{\sqrt{2}} \right) \\ & + 2 e^{\frac{x^2}{2}+x+2} \left(e^x (x+1) + \sqrt{2} c_1 (x+2) + c_2 e^{\frac{1}{2}(x+2)^2} \right) \Bigg) \end{aligned}$$

2.2.45 problem 44

Internal problem ID [8525]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 44
Date solved : `Wednesday, December 18, 2024 at 01:53:18 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - x^2 y' - xy - x^2 = 0$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.018 (sec)
Leaf size : 44

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x^2-x*y(x)-x^2 = 0,
       y(x),singsol=all)
```

$$y = e^{\frac{x^3}{6}}\sqrt{x}\,\mathrm{BesselI}\left(\frac{1}{6}, \frac{x^3}{6}\right) c_2 + e^{\frac{x^3}{6}}\sqrt{x}\,\mathrm{BesselK}\left(\frac{1}{6}, \frac{x^3}{6}\right) c_1 - \frac{x}{2}$$

Mathematica DSolve solution

Solving time : 0.319 (sec)
Leaf size : 224

```
DSolve[{D[y[x],{x,2}]-x^2*D[y[x],x]-x*y[x]-x^2==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \rightarrow -\frac{e^{\frac{x^3}{6}}\left(12(x^3)^{5/6}\,\mathrm{Gamma}\left(\frac{1}{6}\right)\mathrm{Gamma}\left(\frac{7}{6}\right)\mathrm{BesselI}\left(\frac{1}{6}, \frac{x^3}{6}\right) {}_1F_1\left(-\frac{2}{3}; -\frac{1}{3}; -\frac{x^3}{3}\right) + \sqrt[3]{2}3^{2/3}\sqrt[6]{x^3}x^6\,\mathrm{Gamma}\left(\right.\right.}{}$$

2.2.46 problem 45

Internal problem ID [8526]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 45
Date solved : Wednesday, December 18, 2024 at 01:53:19 AM
CAS classification : [[_2nd_order, _with_linear_symmetries]]

Solve

$$y'' - x^2y' - x^2y - x^3 - x^2 = 0$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
```

```
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   trying a solution in terms of MeijerG functions
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
   <- Heun successful: received ODE is equivalent to the  HeunT  ODE, case  c = 0
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.247 (sec)
Leaf size : 57

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x^2-x^2*y(x)-x^3-x^2 = 0,
       y(x),singsol=all)
```

$$y = \operatorname{HeunT}\left(3^{2/3}, 3, 2\,3^{1/3}, \frac{3^{2/3}x}{3}\right) \mathrm{e}^{-x} c_2 + \operatorname{HeunT}\left(3^{2/3}, -3, 2\,3^{1/3}, -\frac{3^{2/3}x}{3}\right) \mathrm{e}^{\frac{x\left(x^2+3\right)}{3}} c_1 - x$$

Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{D[y[x],{x,2}]-x^2*D[y[x],x]-x^2*y[x]-x^3-x^2==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

Not solved

2.2.47 problem 46

Internal problem ID [8527]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 46
Date solved : `Wednesday, December 18, 2024 at 01:53:21 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - x^2 y' - x^3 y - x^4 - x^2 = 0$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
```

```
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   trying a solution in terms of MeijerG functions
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
   <- Heun successful: received ODE is equivalent to the  HeunT  ODE, case  c = 0
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.303 (sec)
Leaf size : 74

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x^2-y(x)*x^3-x^4-x^2 = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-\frac{x(x-2)}{2}} \operatorname{HeunT}\left(2\, 3^{2/3}, -3, -3\, 3^{1/3}, \frac{3^{2/3}(x+1)}{3}\right) c_2 + \mathrm{e}^{\frac{1}{3}x^3+\frac{1}{2}x^2-x} \operatorname{HeunT}\left(2\, 3^{2/3}, 3, -3\, 3^{1/3}, -\frac{3^{2/3}(x+1)}{3}\right) c_1 - x$$

Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{D[y[x],{x,2}]-x^2*D[y[x],x]-x^3*y[x]-x^4-x^2==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

Not solved

2.2.48 problem 47

Internal problem ID [8528]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 47
Date solved : `Friday, December 13, 2024 at 05:10:06 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - \frac{y'}{x} - xy - x^2 - \frac{1}{x} = 0$$

Solved as second order Bessel ode

Time used: 6.226 (sec)

Writing the ode as

$$x^2y'' - xy' - x^3y = x^2\left(x^2 + \frac{1}{x}\right) \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2y'' + xy' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\, xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}
\alpha &= 1 \\
\beta &= \frac{2i}{3} \\
n &= \frac{2}{3} \\
\gamma &= \frac{3}{2}
\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = c_1 x \operatorname{BesselJ}\left(\frac{2}{3}, \frac{2i x^{3/2}}{3}\right) + c_2 x \operatorname{BesselY}\left(\frac{2}{3}, \frac{2i x^{3/2}}{3}\right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 x \operatorname{BesselJ}\left(\frac{2}{3}, \frac{2i x^{3/2}}{3}\right) + c_2 x \operatorname{BesselY}\left(\frac{2}{3}, \frac{2i x^{3/2}}{3}\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= x \operatorname{BesselJ}\left(\frac{2}{3}, \frac{2i x^{3/2}}{3}\right) \\
y_2 &= x \operatorname{BesselY}\left(\frac{2}{3}, \frac{2i x^{3/2}}{3}\right)
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x \operatorname{BesselJ}\left(\frac{2}{3}, \frac{2ix^{3/2}}{3}\right) & x \operatorname{BesselY}\left(\frac{2}{3}, \frac{2ix^{3/2}}{3}\right) \\ \frac{d}{dx}\left(x \operatorname{BesselJ}\left(\frac{2}{3}, \frac{2ix^{3/2}}{3}\right)\right) & \frac{d}{dx}\left(x \operatorname{BesselY}\left(\frac{2}{3}, \frac{2ix^{3/2}}{3}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x \operatorname{BesselJ}\left(\frac{2}{3}, \frac{2ix^{3/2}}{3}\right) & x \operatorname{B} \\ \operatorname{BesselJ}\left(\frac{2}{3}, \frac{2ix^{3/2}}{3}\right) + ix^{3/2}\left(\operatorname{BesselJ}\left(-\frac{1}{3}, \frac{2ix^{3/2}}{3}\right) + \frac{i \operatorname{BesselJ}\left(\frac{2}{3}, \frac{2ix^{3/2}}{3}\right)}{x^{3/2}}\right) & \operatorname{BesselY}\left(\frac{2}{3}, \frac{2ix^{3/2}}{3}\right) + ix^{3/2}\Big(\end{vmatrix}$$

Therefore

$$W = \left(x \operatorname{BesselJ}\left(\frac{2}{3}, \frac{2ix^{3/2}}{3}\right)\right)\left(\operatorname{BesselY}\left(\frac{2}{3}, \frac{2ix^{3/2}}{3}\right) + ix^{3/2}\left(\operatorname{BesselY}\left(-\frac{1}{3}, \frac{2ix^{3/2}}{3}\right) + \frac{i \operatorname{BesselY}\left(\frac{2}{3}, \frac{2ix^{3/2}}{3}\right)}{x^{3/2}}\right.\right.$$

Which simplifies to

$$W = ix^{5/2}\left(\operatorname{BesselJ}\left(\frac{2}{3}, \frac{2ix^{3/2}}{3}\right) \operatorname{BesselY}\left(-\frac{1}{3}, \frac{2ix^{3/2}}{3}\right) - \operatorname{BesselY}\left(\frac{2}{3}, \frac{2ix^{3/2}}{3}\right) \operatorname{BesselJ}\left(-\frac{1}{3}, \frac{2ix^{3/2}}{3}\right)\right)$$

Which simplifies to

$$W = \frac{3x}{\pi}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^3 \operatorname{BesselY}\left(\frac{2}{3}, \frac{2ix^{3/2}}{3}\right)\left(x^2 + \frac{1}{x}\right)}{\frac{3x^3}{\pi}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\operatorname{BesselY}\left(\frac{2}{3}, \frac{2ix^{3/2}}{3}\right)(x^3 + 1)\,\pi}{3x} dx$$

Hence

$$u_1 = i\pi \Bigg(-\frac{i \Bigg(\frac{2ix^{3/2}\,\mathrm{BesselY}\left(\frac{2}{3}, \frac{2ix^{3/2}}{3}\right)\pi\left(\frac{\sqrt{3}\ \mathrm{AngerJ}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right)}{3} - \mathrm{WeberE}\left(\frac{1}{3}, \frac{2ix^{3/2}}{3}\right)\right)}{9} - \frac{2ix^{3/2}\,\mathrm{BesselY}\left(-\frac{1}{3}, \frac{2ix^{3/2}}{3}\right)\mathrm{LommelS1}\Big(1, \frac{2}{3}, \cdots}{3} \cdots}{2}$$

$$+ \frac{2i \Bigg(-\frac{8ix^{3/2}\,\mathrm{BesselY}\left(\frac{2}{3}, \frac{2ix^{3/2}}{3}\right)\mathrm{LommelS1}\left(-2, \frac{1}{3}, \frac{2ix^{3/2}}{3}\right)}{9} + \frac{ix^{3/2}\,\mathrm{BesselY}\left(-\frac{1}{3}, \frac{2ix^{3/2}}{3}\right)\pi\Big(\frac{\sqrt{3}\ \mathrm{AngerJ}\left(\frac{2}{3}, \frac{2ix^{3/2}}{3}\right)}{3} + \mathrm{WeberE}\Big(\frac{2}{3}, \frac{2ix \cdots}{}}{2} \cdots}{9}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^3\,\mathrm{BesselJ}\left(\frac{2}{3}, \frac{2ix^{3/2}}{3}\right)\left(x^2 + \frac{1}{x}\right)}{\frac{3x^3}{\pi}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\mathrm{BesselJ}\left(\frac{2}{3}, \frac{2ix^{3/2}}{3}\right)\left(x^3 + 1\right)\pi}{3x}\,dx$$

Hence

$$u_2 = -i\pi\left(-\frac{i\left(\frac{2ix^{3/2}\,\mathrm{BesselJ}\left(\frac{2}{3},\frac{2ix^{3/2}}{3}\right)\pi\left(\frac{\sqrt{3}\,\mathrm{AngerJ}\left(\frac{1}{3},\frac{2ix^{3/2}}{3}\right)}{3}-\mathrm{WeberE}\left(\frac{1}{3},\frac{2ix^{3/2}}{3}\right)\right)}{9}-\frac{2ix^{3/2}\,\mathrm{BesselJ}\left(-\frac{1}{3},\frac{2ix^{3/2}}{3}\right)\mathrm{LommelS1}\left(1,\right.}{3}\right.}{2}\right.$$

$$+\frac{2i\left(-\frac{8ix^{3/2}\,\mathrm{BesselJ}\left(\frac{2}{3},\frac{2ix^{3/2}}{3}\right)\mathrm{LommelS1}\left(-2,\frac{1}{3},\frac{2ix^{3/2}}{3}\right)}{9}+\frac{ix^{3/2}\,\mathrm{BesselJ}\left(-\frac{1}{3},\frac{2ix^{3/2}}{3}\right)\pi\left(\frac{\sqrt{3}\,\mathrm{AngerJ}\left(\frac{2}{3},\frac{2ix^{3/2}}{3}\right)}{3}+\mathrm{WeberE}\left(\frac{2}{3},\right.\right.}{2}\right.}{9}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = i\pi\left(-\frac{i\left(\frac{2ix^{3/2}\,\mathrm{BesselY}\left(\frac{2}{3},\frac{2ix^{3/2}}{3}\right)\pi\left(\frac{\sqrt{3}\,\mathrm{AngerJ}\left(\frac{1}{3},\frac{2ix^{3/2}}{3}\right)}{3}-\mathrm{WeberE}\left(\frac{1}{3},\frac{2ix^{3/2}}{3}\right)\right)}{9}-\frac{2ix^{3/2}\,\mathrm{BesselY}\left(-\frac{1}{3},\frac{2ix^{3/2}}{3}\right)\mathrm{LommelS1}\left(1,\frac{2}{3},\right.}{3}\right.}{2}\right.$$

$$+\frac{2i\left(-\frac{8ix^{3/2}\,\mathrm{BesselY}\left(\frac{2}{3},\frac{2ix^{3/2}}{3}\right)\mathrm{LommelS1}\left(-2,\frac{1}{3},\frac{2ix^{3/2}}{3}\right)}{9}+\frac{ix^{3/2}\,\mathrm{BesselY}\left(-\frac{1}{3},\frac{2ix^{3/2}}{3}\right)\pi\left(\frac{\sqrt{3}\,\mathrm{AngerJ}\left(\frac{2}{3},\frac{2ix^{3/2}}{3}\right)}{3}+\mathrm{WeberE}\left(\frac{2}{3},\right.\right.}{2}\right.}{9}$$

Which simplifies to

$$y_p(x) = -\frac{\left(9\,\text{LommelS1}\left(1, \frac{2}{3}, \frac{2i x^{3/2}}{3}\right) + \pi\left(\sqrt{3}\,\text{AngerJ}\left(\frac{2}{3}, \frac{2i x^{3/2}}{3}\right) + 3\,\text{WeberE}\left(\frac{2}{3}, \frac{2i x^{3/2}}{3}\right)\right)\right) x}{9}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x\,\text{BesselJ}\left(\frac{2}{3}, \frac{2i x^{3/2}}{3}\right) + c_2 x\,\text{BesselY}\left(\frac{2}{3}, \frac{2i x^{3/2}}{3}\right)\right) \\ &\quad + \left(-\frac{\left(9\,\text{LommelS1}\left(1, \frac{2}{3}, \frac{2i x^{3/2}}{3}\right) + \pi\left(\sqrt{3}\,\text{AngerJ}\left(\frac{2}{3}, \frac{2i x^{3/2}}{3}\right) + 3\,\text{WeberE}\left(\frac{2}{3}, \frac{2i x^{3/2}}{3}\right)\right)\right) x}{9}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$\begin{aligned} y &= c_1 x\,\text{BesselJ}\left(\frac{2}{3}, \frac{2i x^{3/2}}{3}\right) + c_2 x\,\text{BesselY}\left(\frac{2}{3}, \frac{2i x^{3/2}}{3}\right) \\ &\quad - \frac{\left(9\,\text{LommelS1}\left(1, \frac{2}{3}, \frac{2i x^{3/2}}{3}\right) + \pi\left(\sqrt{3}\,\text{AngerJ}\left(\frac{2}{3}, \frac{2i x^{3/2}}{3}\right) + 3\,\text{WeberE}\left(\frac{2}{3}, \frac{2i x^{3/2}}{3}\right)\right)\right) x}{9} \end{aligned}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
```

```
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 26

```
dsolve(diff(diff(y(x),x),x)-1/x*diff(y(x),x)-x*y(x)-x^2-1/x = 0,
       y(x),singsol=all)
```

$$y = x\left(-1+\operatorname{BesselI}\left(\frac{2}{3}, \frac{2x^{3/2}}{3}\right)c_2+\operatorname{BesselK}\left(\frac{2}{3}, \frac{2x^{3/2}}{3}\right)c_1\right)$$

Mathematica DSolve solution

Solving time : 0.452 (sec)
Leaf size : 253

```
DSolve[{D[y[x],{x,2}]-1/x*D[y[x],x]-x*y[x]-x^2-1/x==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\frac{3\sqrt[6]{3}\pi\,\operatorname{Gamma}\left(-\frac{1}{3}\right)\left(3\operatorname{AiryAiPrime}(x)+\sqrt{3}\operatorname{AiryBiPrime}(x)\right){}_1F_2\left(-\frac{1}{3};\frac{1}{3},\frac{2}{3};\frac{x^3}{9}\right)}{x\operatorname{Gamma}\left(\frac{2}{3}\right)}+\frac{\sqrt[3]{3}\pi x\operatorname{Gamma}\left(\frac{1}{3}\right)^2\left(\sqrt{3}\operatorname{AiryAiPrime}(x)-\operatorname{AiryBiPrime}(x)\right)}{\operatorname{Gamma}\left(\frac{4}{3}\right)}}{}$$

2.2.49 problem 48

Internal problem ID [8529]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 48
Date solved : `Thursday, December 12, 2024 at 09:26:23 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - \frac{y'}{x} - x^2 y - x^3 - \frac{1}{x} = 0$$

Solved as second order ode using change of variable on x method 2

Time used: 0.419 (sec)

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - \frac{y'}{x} - x^2 y = 0$$

In normal form the ode

$$y'' - \frac{y'}{x} - x^2 y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= -x^2 \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\, \tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int e^{-\int p(x)dx} dx \\ &= \int e^{-\int -\frac{1}{x}dx} dx \\ &= \int e^{\ln(x)}\, dx \\ &= \int x dx \\ &= \frac{x^2}{2} \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{-x^2}{x^2} \\ &= -1 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) - y(\tau) = 0$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\tau\lambda} - \mathrm{e}^{\tau\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\lambda_1 = +1$$
$$\lambda_2 = -1$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y(\tau) = c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau}$$
$$y(\tau) = c_1 e^{(1)\tau} + c_2 e^{(-1)\tau}$$

Or

$$y(\tau) = c_1 \,\mathrm{e}^{\tau} + c_2 \,\mathrm{e}^{-\tau}$$

Will add steps showing solving for IC soon.

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \,\mathrm{e}^{\frac{x^2}{2}} + c_2 \,\mathrm{e}^{-\frac{x^2}{2}}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \,\mathrm{e}^{\frac{x^2}{2}} + c_2 \,\mathrm{e}^{-\frac{x^2}{2}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-\frac{x^2}{2}}$$

$$y_2 = \mathrm{e}^{\frac{x^2}{2}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-\frac{x^2}{2}} & \mathrm{e}^{\frac{x^2}{2}} \\ \frac{d}{dx}\left(\mathrm{e}^{-\frac{x^2}{2}}\right) & \frac{d}{dx}\left(\mathrm{e}^{\frac{x^2}{2}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-\frac{x^2}{2}} & e^{\frac{x^2}{2}} \\ -x\,e^{-\frac{x^2}{2}} & x\,e^{\frac{x^2}{2}} \end{vmatrix}$$

Therefore

$$W = \left(e^{-\frac{x^2}{2}}\right)\left(x\,e^{\frac{x^2}{2}}\right) - \left(e^{\frac{x^2}{2}}\right)\left(-x\,e^{-\frac{x^2}{2}}\right)$$

Which simplifies to

$$W = 2\,e^{-\frac{x^2}{2}} x\,e^{\frac{x^2}{2}}$$

Which simplifies to

$$W = 2x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{\frac{x^2}{2}}\left(x^3 + \frac{1}{x}\right)}{2x}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{e^{\frac{x^2}{2}}\left(x^4 + 1\right)}{2x^2} dx$$

Hence

$$u_1 = -\frac{(x^2 - 1)\,e^{\frac{x^2}{2}}}{2x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-\frac{x^2}{2}}\left(x^3 + \frac{1}{x}\right)}{2x}\,dx$$

Which simplifies to

$$u_2 = \int \frac{e^{-\frac{x^2}{2}}\left(x^4 + 1\right)}{2x^2} dx$$

Hence

$$u_2 = -\frac{(x^2 + 1)\,e^{-\frac{x^2}{2}}}{2x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{(x^2-1)\,e^{\frac{x^2}{2}}\,e^{-\frac{x^2}{2}}}{2x} - \frac{e^{\frac{x^2}{2}}\,(x^2+1)\,e^{-\frac{x^2}{2}}}{2x}$$

Which simplifies to

$$y_p(x) = -x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\,e^{\frac{x^2}{2}} + c_2\,e^{-\frac{x^2}{2}}\right) + (-x) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -x + c_1\,e^{\frac{x^2}{2}} + c_2\,e^{-\frac{x^2}{2}}$$

Solved as second order ode using change of variable on x method 1

Time used: 1.223 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -\frac{1}{x}, C = -x^2, f(x) = x^3 + \frac{1}{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$y'' - \frac{y'}{x} - x^2 y = 0$$

In normal form the ode

$$y'' - \frac{y'}{x} - x^2 y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= -x^2 \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{-x^2}}{c} \\ \tau'' &= -\frac{x}{c\sqrt{-x^2}} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{x}{c\sqrt{-x^2}} - \frac{1}{x}\frac{\sqrt{-x^2}}{c}}{\left(\frac{\sqrt{-x^2}}{c}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\ &= \frac{\int \sqrt{-x^2}dx}{c} \\ &= \frac{x\sqrt{-x^2}}{2c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos\left(\frac{x\sqrt{-x^2}}{2}\right) + c_2 \sin\left(\frac{x\sqrt{-x^2}}{2}\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= \cos\left(\frac{x\sqrt{-x^2}}{2}\right) \\ y_2 &= \sin\left(\frac{x\sqrt{-x^2}}{2}\right)\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos\left(\frac{x\sqrt{-x^2}}{2}\right) & \sin\left(\frac{x\sqrt{-x^2}}{2}\right) \\ \frac{d}{dx}\left(\cos\left(\frac{x\sqrt{-x^2}}{2}\right)\right) & \frac{d}{dx}\left(\sin\left(\frac{x\sqrt{-x^2}}{2}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos\left(\frac{x\sqrt{-x^2}}{2}\right) & \sin\left(\frac{x\sqrt{-x^2}}{2}\right) \\ -\left(\frac{\sqrt{-x^2}}{2} - \frac{x^2}{2\sqrt{-x^2}}\right)\sin\left(\frac{x\sqrt{-x^2}}{2}\right) & \left(\frac{\sqrt{-x^2}}{2} - \frac{x^2}{2\sqrt{-x^2}}\right)\cos\left(\frac{x\sqrt{-x^2}}{2}\right) \end{vmatrix}$$

Therefore

$$\begin{aligned} W = &\left(\cos\left(\frac{x\sqrt{-x^2}}{2}\right)\right)\left(\left(\frac{\sqrt{-x^2}}{2} - \frac{x^2}{2\sqrt{-x^2}}\right)\cos\left(\frac{x\sqrt{-x^2}}{2}\right)\right) \\ &- \left(\sin\left(\frac{x\sqrt{-x^2}}{2}\right)\right)\left(-\left(\frac{\sqrt{-x^2}}{2} - \frac{x^2}{2\sqrt{-x^2}}\right)\sin\left(\frac{x\sqrt{-x^2}}{2}\right)\right) \end{aligned}$$

Which simplifies to

$$W = -\frac{x^2\left(\cos\left(\frac{x\sqrt{-x^2}}{2}\right)^2 + \sin\left(\frac{x\sqrt{-x^2}}{2}\right)^2\right)}{\sqrt{-x^2}}$$

Which simplifies to

$$W = -\frac{x^2}{\sqrt{-x^2}}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin\left(\frac{x\sqrt{-x^2}}{2}\right)\left(x^3 + \frac{1}{x}\right)}{-\frac{x^2}{\sqrt{-x^2}}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{(-x^4 - 1)\sin\left(\frac{x\sqrt{-x^2}}{2}\right)\sqrt{-x^2}}{x^3}dx$$

Hence

$$u_1 = \frac{2\sqrt{\pi}\left(\frac{x\left(\frac{\sqrt{-x^2}}{x}\right)^{5/2} \mathrm{e}^{-\frac{x^2}{2}}}{4\sqrt{\pi}} + \frac{x\left(\frac{\sqrt{-x^2}}{x}\right)^{5/2} \mathrm{e}^{\frac{x^2}{2}}}{4\sqrt{\pi}} - \frac{\left(\frac{\sqrt{-x^2}}{x}\right)^{5/2}\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right)}{8} - \frac{\left(\frac{\sqrt{-x^2}}{x}\right)^{5/2}\sqrt{2}\,\mathrm{erfi}\left(\frac{\sqrt{2}\,x}{2}\right)}{8}\right)}{\sqrt{\frac{\sqrt{-x^2}}{x}}} - \frac{\sqrt{\pi}\left(\frac{4\sqrt{\frac{\sqrt{-x^2}}{x}}\,\mathrm{e}^{-\frac{x^2}{2}}}{\sqrt{\pi}\,x} - \frac{4\sqrt{\frac{\sqrt{-x^2}}{x}}\,\mathrm{e}^{\frac{x^2}{2}}}{\sqrt{\pi}\,x} + 2\sqrt{\frac{\sqrt{-x^2}}{x}}\,\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right) + 2\sqrt{\frac{\sqrt{-x^2}}{x}}\,\sqrt{2}\,\mathrm{erfi}\left(\frac{\sqrt{2}\,x}{2}\right)\right)}{8\sqrt{\frac{\sqrt{-x^2}}{x}}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos\left(\frac{x\sqrt{-x^2}}{2}\right)\left(x^3 + \frac{1}{x}\right)}{-\frac{x^2}{\sqrt{-x^2}}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{(-x^4 - 1)\cos\left(\frac{x\sqrt{-x^2}}{2}\right)\sqrt{-x^2}}{x^3}\,dx$$

Hence

$$u_2 = \frac{2\sqrt{\pi}\,\sqrt{-x^2}\,(-1)^{1/4}\left(\frac{x(-1)^{3/4}\mathrm{e}^{\frac{x^2}{2}}}{4\sqrt{\pi}} - \frac{x(-1)^{3/4}\mathrm{e}^{-\frac{x^2}{2}}}{4\sqrt{\pi}} + \frac{(-1)^{3/4}\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right)}{8} - \frac{(-1)^{3/4}\sqrt{2}\,\mathrm{erfi}\left(\frac{\sqrt{2}\,x}{2}\right)}{8}\right)}{x} - \frac{\sqrt{\pi}\,\sqrt{-x^2}\,(-1)^{1/4}\left(\frac{4(-1)^{3/4}\mathrm{e}^{\frac{x^2}{2}}}{\sqrt{\pi}\,x} + \frac{4(-1)^{3/4}\mathrm{e}^{-\frac{x^2}{2}}}{\sqrt{\pi}\,x} + 2(-1)^{3/4}\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right) - 2(-1)^{3/4}\sqrt{2}\,\mathrm{erfi}\left(\frac{\sqrt{2}\,x}{2}\right)\right)}{8x}$$

Therefore the particular solution, from equation (1) is

$$
\begin{aligned}
y_p(x) \\
&= \left(\frac{2\sqrt{\pi}\left(\frac{x\left(\frac{\sqrt{-x^2}}{x}\right)^{5/2} e^{-\frac{x^2}{2}}}{4\sqrt{\pi}} + \frac{x\left(\frac{\sqrt{-x^2}}{x}\right)^{5/2} e^{\frac{x^2}{2}}}{4\sqrt{\pi}} - \frac{\left(\frac{\sqrt{-x^2}}{x}\right)^{5/2}\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right)}{8} - \frac{\left(\frac{\sqrt{-x^2}}{x}\right)^{5/2}\sqrt{2}\,\mathrm{erfi}\left(\frac{\sqrt{2}\,x}{2}\right)}{8}\right)}{\sqrt{\frac{\sqrt{-x^2}}{x}}}\right. \\
&\quad \left. - \frac{\sqrt{\pi}\left(\frac{4\sqrt{\frac{\sqrt{-x^2}}{x}}\, e^{-\frac{x^2}{2}}}{\sqrt{\pi}\, x} - \frac{4\sqrt{\frac{\sqrt{-x^2}}{x}}\, e^{\frac{x^2}{2}}}{\sqrt{\pi}\, x} + 2\sqrt{\frac{\sqrt{-x^2}}{x}}\sqrt{2}\,\mathrm{erf}\left(\frac{\sqrt{2}\,x}{2}\right) + 2\sqrt{\frac{\sqrt{-x^2}}{x}}\sqrt{2}\,\mathrm{erfi}\left(\frac{\sqrt{2}\,x}{2}\right)\right)}{8\sqrt{\frac{\sqrt{-x^2}}{x}}}\right)\cos\left(\frac{x\sqrt{-x^2}}{2}\right)
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
y_p(x) \\
&= \frac{-\left(\left((x^2+1)\,e^{-\frac{x^2}{2}} + e^{\frac{x^2}{2}}(x^2-1)\right)x\cos\left(\frac{x\sqrt{-x^2}}{2}\right) + \sqrt{-x^2}\,\sin\left(\frac{x\sqrt{-x^2}}{2}\right)\left(e^{-\frac{x^2}{2}}x^2 - e^{\frac{x^2}{2}}x^2 + e^{-\frac{x^2}{2}} + e^{\frac{x^2}{2}}\right)\right)}{2x^2}
\end{aligned}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1\cos\left(\frac{x\sqrt{-x^2}}{2}\right) + c_2\sin\left(\frac{x\sqrt{-x^2}}{2}\right)\right) \\
&\quad + \left(\frac{-\left(\left((x^2+1)\,e^{-\frac{x^2}{2}} + e^{\frac{x^2}{2}}(x^2-1)\right)x\cos\left(\frac{x\sqrt{-x^2}}{2}\right) + \sqrt{-x^2}\,\sin\left(\frac{x\sqrt{-x^2}}{2}\right)\left(e^{-\frac{x^2}{2}}x^2 - e^{\frac{x^2}{2}}x^2 + e^{-\frac{x^2}{2}} + e^{\frac{x^2}{2}}\right)\right)}{2x^2}\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
\begin{aligned}
y &= c_1\cos\left(\frac{x\sqrt{-x^2}}{2}\right) + c_2\sin\left(\frac{x\sqrt{-x^2}}{2}\right) \\
&\quad + \frac{-\left(\left((x^2+1)\,e^{-\frac{x^2}{2}} + e^{\frac{x^2}{2}}(x^2-1)\right)x\cos\left(\frac{x\sqrt{-x^2}}{2}\right) + \sqrt{-x^2}\,\sin\left(\frac{x\sqrt{-x^2}}{2}\right)\left(e^{-\frac{x^2}{2}}x^2 - e^{\frac{x^2}{2}}x^2 + e^{-\frac{x^2}{2}} + e^{\frac{x^2}{2}}\right)\right)}{2x^2}
\end{aligned}
$$

Solved as second order Bessel ode

Time used: 0.289 (sec)

Writing the ode as

$$x^2y'' - y'x - x^4y = x^2\left(x^3 + \frac{1}{x}\right) \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2y'' + y'x + \left(-n^2 + x^2\right)y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2y'' + (1 - 2\alpha)\,xy' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha(c_1\,\text{BesselJ}\,(n, \beta\,x^\gamma) + c_2\,\text{BesselY}\,(n, \beta\,x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned}\alpha &= 1\\ \beta &= \frac{i}{2}\\ n &= \frac{1}{2}\\ \gamma &= 2\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = \frac{2ic_1x\sinh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{ix^2}} - \frac{2c_2x\cosh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{ix^2}}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{2ic_1x\sinh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{ix^2}} - \frac{2c_2x\cosh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{ix^2}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{2ix \sinh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{i x^2}}$$

$$y_2 = -\frac{2x \cosh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{i x^2}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{2ix \sinh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{i x^2}} & -\frac{2x \cosh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{i x^2}} \\ \frac{d}{dx}\left(\frac{2ix \sinh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{i x^2}}\right) & \frac{d}{dx}\left(-\frac{2x \cosh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{i x^2}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{2ix \sinh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{i x^2}} & -\frac{2x \cosh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{i x^2}} \\ \frac{2i \sinh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{i x^2}} + \frac{2x^2 \sinh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,(i x^2)^{3/2}} + \frac{2ix^2 \cosh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{i x^2}} & -\frac{2 \cosh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{i x^2}} + \frac{2ix^2 \cosh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,(i x^2)^{3/2}} - \frac{2x^2 \sinh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{i x^2}} \end{vmatrix}$$

Therefore

$$W = \left(\frac{2ix\sinh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{ix^2}}\right)\left(-\frac{2\cosh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{ix^2}} + \frac{2ix^2\cosh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\left(ix^2\right)^{3/2}} - \frac{2x^2\sinh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{ix^2}}\right) - \left(-\frac{2x\cosh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{ix^2}}\right)\left(\frac{2i\sinh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{ix^2}} + \frac{2x^2\sinh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\left(ix^2\right)^{3/2}} + \frac{2ix^2\cosh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{ix^2}}\right)$$

Which simplifies to

$$W = -\frac{4x\left(\sinh\left(\frac{x^2}{2}\right)^2 - \cosh\left(\frac{x^2}{2}\right)^2\right)}{\pi}$$

Which simplifies to

$$W = \frac{4x}{\pi}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{2x^3\cosh\left(\frac{x^2}{2}\right)\left(x^3+\frac{1}{x}\right)}{\sqrt{\pi}\,\sqrt{ix^2}}}{\frac{4x^3}{\pi}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{\sqrt{\pi}\,\cosh\left(\frac{x^2}{2}\right)\left(x^4+1\right)}{2\sqrt{ix^2}\,x}dx$$

Hence

$$u_1 = \frac{\sqrt{\pi}\,(x^2-1)\,\mathrm{e}^{\frac{x^2}{2}}}{4\sqrt{ix^2}} - \frac{\sqrt{\pi}\,(x^2+1)\,\mathrm{e}^{-\frac{x^2}{2}}}{4\sqrt{ix^2}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{2ix^3\sinh\left(\frac{x^2}{2}\right)\left(x^3+\frac{1}{x}\right)}{\sqrt{\pi}\,\sqrt{ix^2}}}{\frac{4x^3}{\pi}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{i\sqrt{\pi}\,\sinh\left(\frac{x^2}{2}\right)(x^4+1)}{2\sqrt{i x^2}\,x}dx$$

Hence

$$u_2 = \frac{i\sqrt{\pi}\,(x^2-1)\,\mathrm{e}^{\frac{x^2}{2}}}{4\sqrt{i x^2}} + \frac{i\sqrt{\pi}\,(x^2+1)\,\mathrm{e}^{-\frac{x^2}{2}}}{4\sqrt{i x^2}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{2i\left(\frac{\sqrt{\pi}\,(x^2-1)\mathrm{e}^{\frac{x^2}{2}}}{4\sqrt{i x^2}} - \frac{\sqrt{\pi}\,(x^2+1)\mathrm{e}^{-\frac{x^2}{2}}}{4\sqrt{i x^2}}\right)x\sinh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{i x^2}} - \frac{2x\cosh\left(\frac{x^2}{2}\right)\left(\frac{i\sqrt{\pi}\,(x^2-1)\mathrm{e}^{\frac{x^2}{2}}}{4\sqrt{i x^2}} + \frac{i\sqrt{\pi}\,(x^2+1)\mathrm{e}^{-\frac{x^2}{2}}}{4\sqrt{i x^2}}\right)}{\sqrt{\pi}\,\sqrt{i x^2}}$$

Which simplifies to

$$y_p(x) = -x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{2ic_1x\sinh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{i x^2}} - \frac{2c_2x\cosh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{i x^2}}\right) + (-x) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{2ic_1x\sinh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{i x^2}} - \frac{2c_2x\cosh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{i x^2}} - x$$

Solved as second order ode using Kovacic algorithm

Time used: 0.332 (sec)

Writing the ode as

$$y'' - \frac{y'}{x} - x^2 y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -\frac{1}{x} \\ C &= -x^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4x^4 + 3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4x^4 + 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{4x^4 + 3}{4x^2} \right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.115: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 4 \\ &= -2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Therefore

$$L = [1, 2]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2} + x^2$$

For the pole at $x=0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b=\frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^{+} &= \frac{1}{2}+\sqrt{1+4b}=\frac{3}{2}\\ \alpha_c^{-} &= \frac{1}{2}-\sqrt{1+4b}=-\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $O_r(\infty)=-2$ then

$$v=\frac{-O_r(\infty)}{2}=\frac{2}{2}=1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0\leq i\leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i\\ &= \sum_{i=0}^{1} a_i x^i \end{aligned}\tag{8}$$

Let a be the coefficient of $x^v=x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r}\approx x+\frac{3}{8x^3}-\frac{9}{128x^7}+\frac{27}{1024x^{11}}-\frac{405}{32768x^{15}}+\frac{1701}{262144x^{19}}-\frac{15309}{4194304x^{23}}+\frac{72171}{33554432x^{27}}+\dots\tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a=1$$

From Eq. (9) the sum up to $v=1$ gives

$$\begin{aligned}[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i\\ &= x\end{aligned}\tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1}=x^0=1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2=x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v=0$ or not. Since $v=1$ which is not zero, then starting $r=\frac{s}{t}$, we do long division and write this in the form

$$r=Q+\frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{4x^4+3}{4x^2} \\
&= Q + \frac{R}{4x^2} \\
&= (x^2) + \left(\frac{3}{4x^2}\right) \\
&= \frac{3}{4x^2} + x^2
\end{aligned}
$$

We see that the coefficient of the term x in the quotient is 0. Now b can be found.

$$
\begin{aligned}
b &= (0) - (0) \\
&= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= x \\
\alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{0}{1} - 1\right) &&= -\frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{0}{1} - 1\right) &&= -\frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{4x^4+3}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	x	$-\frac{1}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)\,(x) \\
&= -\frac{1}{2x} - x \\
&= -\frac{1}{2x} - x
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{2x} - x\right)(0) + \left(\left(\frac{1}{2x^2} - 1\right) + \left(-\frac{1}{2x} - x\right)^2 - \left(\frac{4x^4+3}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\,dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2x} - x\right)dx} \\
&= \frac{\mathrm{e}^{-\frac{x^2}{2}}}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-\frac{1}{x}}{1}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x^2}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-\frac{1}{x}}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{\mathrm{e}^{x^2}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2\left(\mathrm{e}^{-\frac{x^2}{2}}\left(\frac{\mathrm{e}^{x^2}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - \frac{y'}{x} - x^2 y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \,\mathrm{e}^{-\frac{x^2}{2}} + \frac{c_2 \,\mathrm{e}^{\frac{x^2}{2}}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-\frac{x^2}{2}}$$

$$y_2 = \frac{\mathrm{e}^{\frac{x^2}{2}}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-\frac{x^2}{2}} & \frac{\mathrm{e}^{\frac{x^2}{2}}}{2} \\ \frac{d}{dx}\left(\mathrm{e}^{-\frac{x^2}{2}}\right) & \frac{d}{dx}\left(\frac{\mathrm{e}^{\frac{x^2}{2}}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-\frac{x^2}{2}} & \frac{e^{\frac{x^2}{2}}}{2} \\ -x\,e^{-\frac{x^2}{2}} & \frac{x\,e^{\frac{x^2}{2}}}{2} \end{vmatrix}$$

Therefore

$$W = \left(e^{-\frac{x^2}{2}}\right)\left(\frac{x\,e^{\frac{x^2}{2}}}{2}\right) - \left(\frac{e^{\frac{x^2}{2}}}{2}\right)\left(-x\,e^{-\frac{x^2}{2}}\right)$$

Which simplifies to

$$W = e^{-\frac{x^2}{2}} x\,e^{\frac{x^2}{2}}$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{\frac{x^2}{2}}\left(x^3+\frac{1}{x}\right)}{2}}{x}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{e^{\frac{x^2}{2}}\left(x^4+1\right)}{2x^2} dx$$

Hence

$$u_1 = -\frac{\left(x^2-1\right) e^{\frac{x^2}{2}}}{2x}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-\frac{x^2}{2}}\left(x^3+\frac{1}{x}\right)}{x}\,dx$$

Which simplifies to

$$u_2 = \int \frac{e^{-\frac{x^2}{2}}\left(x^4+1\right)}{x^2} dx$$

Hence

$$u_2 = -\frac{(x^2+1)\,\mathrm{e}^{-\frac{x^2}{2}}}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{(x^2-1)\,\mathrm{e}^{\frac{x^2}{2}}\,\mathrm{e}^{-\frac{x^2}{2}}}{2x} - \frac{\mathrm{e}^{\frac{x^2}{2}}\,(x^2+1)\,\mathrm{e}^{-\frac{x^2}{2}}}{2x}$$

Which simplifies to

$$y_p(x) = -x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\,\mathrm{e}^{-\frac{x^2}{2}} + \frac{c_2\,\mathrm{e}^{\frac{x^2}{2}}}{2}\right) + (-x) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1\,\mathrm{e}^{-\frac{x^2}{2}} + \frac{c_2\,\mathrm{e}^{\frac{x^2}{2}}}{2} - x$$

Solved as second order ode adjoint method

Time used: 2.400 (sec)

In normal form the ode

$$y'' - \frac{y'}{x} - x^2y - x^3 - \frac{1}{x} = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= -x^2 \\ r(x) &= x^3 + \frac{1}{x} \end{aligned}$$

The Lagrange adjoint ode is given by

$$
\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - \left(-\frac{\xi(x)}{x}\right)' + \left(-x^2\xi(x)\right) &= 0 \\
\xi''(x) + \frac{\xi'(x)}{x} - \frac{(x^4+1)\,\xi(x)}{x^2} &= 0
\end{aligned}
$$

Which is solved for $\xi(x)$. Writing the ode as

$$
\xi''x^2 + \xi'x + \left(-x^4 - 1\right)\xi = 0 \tag{1}
$$

Bessel ode has the form

$$
\xi''x^2 + \xi'x + \left(-n^2 + x^2\right)\xi = 0 \tag{2}
$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$
\xi''x^2 + (1-2\alpha)\,x\xi' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)\xi = 0 \tag{3}
$$

With the standard solution

$$
\xi = x^{\alpha}(c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}
$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$
\begin{aligned}
\alpha &= 0 \\
\beta &= \frac{i}{2} \\
n &= -\frac{1}{2} \\
\gamma &= 2
\end{aligned}
$$

Substituting all the above into (4) gives the solution as

$$
\xi = \frac{2c_1 \cosh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{i x^2}} + \frac{2ic_2 \sinh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{i x^2}}
$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$
\begin{aligned}
\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y &= \int \xi(x)\,r(x)\,dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi(x)}
\end{aligned}
$$

Or

$$
y'+y\left(-\frac{1}{x}-\frac{-\frac{2ic_1\cosh\left(\frac{x^2}{2}\right)x}{\sqrt{\pi}\,(ix^2)^{3/2}}+\frac{2c_1x\sinh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{ix^2}}+\frac{2c_2\sinh\left(\frac{x^2}{2}\right)x}{\sqrt{\pi}\,(ix^2)^{3/2}}+\frac{2ic_2x\cosh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{ix^2}}}{\frac{2c_1\cosh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{ix^2}}+\frac{2ic_2\sinh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{ix^2}}}\right)=\frac{\frac{(ix^2c_2+c_1x^2-ic_2-c_1)\,\mathrm{e}^{\frac{x^2}{2}}}{\sqrt{\pi}\,\sqrt{ix^2}}-(-ix^2 \cdots}{\frac{2c_1\cosh\left(\frac{x^2}{2}\right)}{\sqrt{\pi}\,\sqrt{ix^2}}+2ic_2 \cdots}
$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$
y'+q(x)y=p(x)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x)&=-\frac{x\left(\sinh\left(\frac{x^2}{2}\right)c_1+i\cosh\left(\frac{x^2}{2}\right)c_2\right)}{ic_2\sinh\left(\frac{x^2}{2}\right)+c_1\cosh\left(\frac{x^2}{2}\right)}\\
p(x)&=\frac{-(ic_1+c_2)\,(x^2+1)\,\mathrm{e}^{-\frac{x^2}{2}}+\mathrm{e}^{\frac{x^2}{2}}(x-1)\,(x+1)\,(ic_1-c_2)}{2ic_1\cosh\left(\frac{x^2}{2}\right)-2c_2\sinh\left(\frac{x^2}{2}\right)}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu&=e^{\int q\,dx}\\
&=\mathrm{e}^{\int-\frac{x\left(\sinh\left(\frac{x^2}{2}\right)c_1+i\cosh\left(\frac{x^2}{2}\right)c_2\right)}{ic_2\sinh\left(\frac{x^2}{2}\right)+c_1\cosh\left(\frac{x^2}{2}\right)}dx}\\
&=\frac{1}{c_2\sinh\left(\frac{x^2}{2}\right)-ic_1\cosh\left(\frac{x^2}{2}\right)}
\end{aligned}
$$

The ode becomes

$$
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y)=\mu p
$$

$$
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y)=(\mu)\left(\frac{-(ic_1+c_2)\,(x^2+1)\,\mathrm{e}^{-\frac{x^2}{2}}+\mathrm{e}^{\frac{x^2}{2}}(x-1)\,(x+1)\,(ic_1-c_2)}{2ic_1\cosh\left(\frac{x^2}{2}\right)-2c_2\sinh\left(\frac{x^2}{2}\right)}\right)
$$

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{c_2\sinh\left(\frac{x^2}{2}\right)-ic_1\cosh\left(\frac{x^2}{2}\right)}\right)\\
&=\left(\frac{1}{c_2\sinh\left(\frac{x^2}{2}\right)-ic_1\cosh\left(\frac{x^2}{2}\right)}\right)\left(\frac{-(ic_1+c_2)\,(x^2+1)\,\mathrm{e}^{-\frac{x^2}{2}}+\mathrm{e}^{\frac{x^2}{2}}(x-1)\,(x+1)\,(ic_1-c_2)}{2ic_1\cosh\left(\frac{x^2}{2}\right)-2c_2\sinh\left(\frac{x^2}{2}\right)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{d}\left(\frac{y}{c_2\sinh\left(\frac{x^2}{2}\right)-ic_1\cosh\left(\frac{x^2}{2}\right)}\right)\\
&=\left(\frac{-(ic_1+c_2)\,(x^2+1)\,\mathrm{e}^{-\frac{x^2}{2}}+\mathrm{e}^{\frac{x^2}{2}}(x-1)\,(x+1)\,(ic_1-c_2)}{\left(2ic_1\cosh\left(\frac{x^2}{2}\right)-2c_2\sinh\left(\frac{x^2}{2}\right)\right)\left(c_2\sinh\left(\frac{x^2}{2}\right)-ic_1\cosh\left(\frac{x^2}{2}\right)\right)}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{c_2 \sinh\left(\frac{x^2}{2}\right) - i c_1 \cosh\left(\frac{x^2}{2}\right)} &= \int \frac{-(i c_1 + c_2)\left(x^2+1\right) \mathrm{e}^{-\frac{x^2}{2}} + \mathrm{e}^{\frac{x^2}{2}} (x-1)(x+1)(i c_1 - c_2)}{\left(2 i c_1 \cosh\left(\frac{x^2}{2}\right) - 2 c_2 \sinh\left(\frac{x^2}{2}\right)\right)\left(c_2 \sinh\left(\frac{x^2}{2}\right) - i c_1 \cosh\left(\frac{x^2}{2}\right)\right)} \, dx \\
&= -\frac{2 i x \,\mathrm{e}^{-\frac{x^2}{2}}}{i c_2 + c_1 - i c_2 \,\mathrm{e}^{-x^2} + \mathrm{e}^{-x^2} c_1} + c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{1}{c_2 \sinh\left(\frac{x^2}{2}\right) - i c_1 \cosh\left(\frac{x^2}{2}\right)}$ gives the final solution

$$
y = \frac{\left(c_2 \sinh\left(\frac{x^2}{2}\right) - i c_1 \cosh\left(\frac{x^2}{2}\right)\right)\left(2x\,\mathrm{e}^{-\frac{x^2}{2}} + c_3\left((i c_1 + c_2)\,\mathrm{e}^{-x^2} + i c_1 - c_2\right)\right)}{(i c_1 + c_2)\,\mathrm{e}^{-x^2} + i c_1 - c_2}
$$

Hence, the solution found using Lagrange adjoint equation method is

$$
y = \frac{\left(c_2 \sinh\left(\frac{x^2}{2}\right) - i c_1 \cosh\left(\frac{x^2}{2}\right)\right)\left(2x\,\mathrm{e}^{-\frac{x^2}{2}} + c_3\left((i c_1 + c_2)\,\mathrm{e}^{-x^2} + i c_1 - c_2\right)\right)}{(i c_1 + c_2)\,\mathrm{e}^{-x^2} + i c_1 - c_2}
$$

The constants can be merged to give

$$
y = \frac{\left(c_2 \sinh\left(\frac{x^2}{2}\right) - i c_1 \cosh\left(\frac{x^2}{2}\right)\right)\left(2x\,\mathrm{e}^{-\frac{x^2}{2}} + (i c_1 + c_2)\,\mathrm{e}^{-x^2} + i c_1 - c_2\right)}{(i c_1 + c_2)\,\mathrm{e}^{-x^2} + i c_1 - c_2}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = \frac{\left(c_2 \sinh\left(\frac{x^2}{2}\right) - i c_1 \cosh\left(\frac{x^2}{2}\right)\right)\left(2x\,\mathrm{e}^{-\frac{x^2}{2}} + (i c_1 + c_2)\,\mathrm{e}^{-x^2} + i c_1 - c_2\right)}{(i c_1 + c_2)\,\mathrm{e}^{-x^2} + i c_1 - c_2}
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
```

```
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 24

```
dsolve(diff(diff(y(x),x),x)-1/x*diff(y(x),x)-x^2*y(x)-x^3-1/x = 0,
       y(x),singsol=all)
```

$$y = \sinh\left(\frac{x^2}{2}\right) c_2 + \cosh\left(\frac{x^2}{2}\right) c_1 - x$$

Mathematica DSolve solution

Solving time : 0.086 (sec)
Leaf size : 34

```
DSolve[{D[y[x],{x,2}]-1/x*D[y[x],x]-x^2*y[x]-x^3-1/x==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 \cosh\left(\frac{x^2}{2}\right) + ic_2 \sinh\left(\frac{x^2}{2}\right) - x$$

2.2.50 problem 49

Internal problem ID [8530]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 49
Date solved : `Friday, December 13, 2024 at 05:13:20 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - \frac{y'}{x} - x^3 y - x^4 - \frac{1}{x} = 0$$

Solved as second order Bessel ode

Time used: 64.734 (sec)

Writing the ode as

$$x^2 y'' - x y' - x^5 y = x^2 \left(x^4 + \frac{1}{x}\right) \tag{1}$$

Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE and y_p is a particular solution to the non-homogeneous ODE. Bessel ode has the form

$$x^2 y'' + x y' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha) x y' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^\alpha (c_1 \operatorname{BesselJ}(n, \beta x^\gamma) + c_2 \operatorname{BesselY}(n, \beta x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned} \alpha &= 1 \\ \beta &= \frac{2i}{5} \\ n &= \frac{2}{5} \\ \gamma &= \frac{5}{2} \end{aligned}$$

Substituting all the above into (4) gives the solution as

$$y = c_1 x \operatorname{BesselJ}\left(\frac{2}{5}, \frac{2i x^{5/2}}{5}\right) + c_2 x \operatorname{BesselY}\left(\frac{2}{5}, \frac{2i x^{5/2}}{5}\right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 x \operatorname{BesselJ}\left(\frac{2}{5}, \frac{2i x^{5/2}}{5}\right) + c_2 x \operatorname{BesselY}\left(\frac{2}{5}, \frac{2i x^{5/2}}{5}\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x \operatorname{BesselJ}\left(\frac{2}{5}, \frac{2i x^{5/2}}{5}\right) \\ y_2 &= x \operatorname{BesselY}\left(\frac{2}{5}, \frac{2i x^{5/2}}{5}\right) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x \operatorname{BesselJ}\left(\frac{2}{5}, \frac{2ix^{5/2}}{5}\right) & x \operatorname{BesselY}\left(\frac{2}{5}, \frac{2ix^{5/2}}{5}\right) \\ \frac{d}{dx}\left(x \operatorname{BesselJ}\left(\frac{2}{5}, \frac{2ix^{5/2}}{5}\right)\right) & \frac{d}{dx}\left(x \operatorname{BesselY}\left(\frac{2}{5}, \frac{2ix^{5/2}}{5}\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x \operatorname{BesselJ}\left(\frac{2}{5}, \frac{2ix^{5/2}}{5}\right) & x \operatorname{B} \\ \operatorname{BesselJ}\left(\frac{2}{5}, \frac{2ix^{5/2}}{5}\right) + ix^{5/2}\left(-\operatorname{BesselJ}\left(\frac{7}{5}, \frac{2ix^{5/2}}{5}\right) - \frac{i \operatorname{BesselJ}\left(\frac{2}{5}, \frac{2ix^{5/2}}{5}\right)}{x^{5/2}}\right) & \operatorname{BesselY}\left(\frac{2}{5}, \frac{2ix^{5/2}}{5}\right) + ix^{5/2}\Bigg(\end{vmatrix}$$

Therefore

$$W = \left(x \operatorname{BesselJ}\left(\frac{2}{5}, \frac{2ix^{5/2}}{5}\right)\right)\left(\operatorname{BesselY}\left(\frac{2}{5}, \frac{2ix^{5/2}}{5}\right) + ix^{5/2}\left(-\operatorname{BesselY}\left(\frac{7}{5}, \frac{2ix^{5/2}}{5}\right) - \frac{i \operatorname{BesselY}\left(\frac{2}{5}, \frac{2ix^{5/2}}{5}\right)}{x^{5/2}}\right.\right.$$

Which simplifies to

$$W = -ix^{7/2}\left(\operatorname{BesselJ}\left(\frac{2}{5}, \frac{2ix^{5/2}}{5}\right)\operatorname{BesselY}\left(\frac{7}{5}, \frac{2ix^{5/2}}{5}\right) - \operatorname{BesselY}\left(\frac{2}{5}, \frac{2ix^{5/2}}{5}\right)\operatorname{BesselJ}\left(\frac{7}{5}, \frac{2ix^{5/2}}{5}\right)\right)$$

Which simplifies to

$$W = \frac{5x}{\pi}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^3 \operatorname{BesselY}\left(\frac{2}{5}, \frac{2ix^{5/2}}{5}\right)\left(x^4 + \frac{1}{x}\right)}{\frac{5x^3}{\pi}}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\operatorname{BesselY}\left(\frac{2}{5}, \frac{2ix^{5/2}}{5}\right)\left(x^5 + 1\right)\pi}{5x} dx$$

Hence

$$u_1 = i\pi\left(-\frac{i\left(\frac{2ix^{5/2}\,\mathrm{BesselY}\left(\frac{2}{5},\frac{2ix^{5/2}}{5}\right)\pi\left(\tan\left(\frac{3\pi}{10}\right)\mathrm{AngerJ}\left(\frac{3}{5},\frac{2ix^{5/2}}{5}\right)-\mathrm{WeberE}\left(\frac{3}{5},\frac{2ix^{5/2}}{5}\right)\right)}{25}-\frac{2ix^{5/2}\,\mathrm{BesselY}\left(-\frac{3}{5},\frac{2ix^{5/2}}{5}\right)\mathrm{LommelS1}\left(\right.}{5}\right.}{2}\right.$$
$$\left.+\frac{2i\left(-\frac{16ix^{5/2}\,\mathrm{BesselY}\left(\frac{2}{5},\frac{2ix^{5/2}}{5}\right)\mathrm{LommelS1}\left(-2,\frac{3}{5},\frac{2ix^{5/2}}{5}\right)}{25}+\frac{ix^{5/2}\,\mathrm{BesselY}\left(-\frac{3}{5},\frac{2ix^{5/2}}{5}\right)\pi\left(\cot\left(\frac{\pi}{5}\right)\mathrm{AngerJ}\left(\frac{2}{5},\frac{2ix^{5/2}}{5}\right)+\mathrm{WeberE}\left(\right.\right.}{2}\right.}{25}\right.$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^3\,\mathrm{BesselJ}\left(\frac{2}{5},\frac{2ix^{5/2}}{5}\right)\left(x^4+\frac{1}{x}\right)}{\frac{5x^3}{\pi}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\mathrm{BesselJ}\left(\frac{2}{5},\frac{2ix^{5/2}}{5}\right)\left(x^5+1\right)\pi}{5x}dx$$

Hence

$$u_2 = -i\pi\left(-\frac{i\left(\frac{2ix^{5/2}\,\mathrm{BesselJ}\left(\frac{2}{5},\frac{2ix^{5/2}}{5}\right)\pi\left(\tan\left(\frac{3\pi}{10}\right)\mathrm{AngerJ}\left(\frac{3}{5},\frac{2ix^{5/2}}{5}\right)-\mathrm{WeberE}\left(\frac{3}{5},\frac{2ix^{5/2}}{5}\right)\right)}{25}-\frac{2ix^{5/2}\,\mathrm{BesselJ}\left(-\frac{3}{5},\frac{2ix^{5/2}}{5}\right)\mathrm{LommelS}}{5}\right.}{2}\right.$$
$$\left.+\frac{2i\left(-\frac{16ix^{5/2}\,\mathrm{BesselJ}\left(\frac{2}{5},\frac{2ix^{5/2}}{5}\right)\mathrm{LommelS1}\left(-2,\frac{3}{5},\frac{2ix^{5/2}}{5}\right)}{25}+\frac{ix^{5/2}\,\mathrm{BesselJ}\left(-\frac{3}{5},\frac{2ix^{5/2}}{5}\right)\pi\left(\cot\left(\frac{\pi}{5}\right)\mathrm{AngerJ}\left(\frac{2}{5},\frac{2ix^{5/2}}{5}\right)+\mathrm{WeberE}\left(\right.\right.}{2}\right.}{25}\right.$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = i\pi\left(-\frac{i\left(\frac{2ix^{5/2}\,\mathrm{BesselY}\left(\frac{2}{5},\frac{2ix^{5/2}}{5}\right)\pi\left(\tan\left(\frac{3\pi}{10}\right)\mathrm{AngerJ}\left(\frac{3}{5},\frac{2ix^{5/2}}{5}\right)-\mathrm{WeberE}\left(\frac{3}{5},\frac{2ix^{5/2}}{5}\right)\right)}{25}-\frac{2ix^{5/2}\,\mathrm{BesselY}\left(-\frac{3}{5},\frac{2ix^{5/2}}{5}\right)\mathrm{LommelS1}\left(\right.}{5}\right.}{2}\right.$$
$$\left.+\frac{2i\left(-\frac{16ix^{5/2}\,\mathrm{BesselY}\left(\frac{2}{5},\frac{2ix^{5/2}}{5}\right)\mathrm{LommelS1}\left(-2,\frac{3}{5},\frac{2ix^{5/2}}{5}\right)}{25}+\frac{ix^{5/2}\,\mathrm{BesselY}\left(-\frac{3}{5},\frac{2ix^{5/2}}{5}\right)\pi\left(\cot\left(\frac{\pi}{5}\right)\mathrm{AngerJ}\left(\frac{2}{5},\frac{2ix^{5/2}}{5}\right)+\mathrm{WeberE}\left(\right.\right.}{2}\right.}{25}\right.$$

Which simplifies to

$$y_p(x) = -\frac{x\left(5\,\mathrm{LommelS1}\left(1,\frac{2}{5},\frac{2ix^{5/2}}{5}\right)+\pi\left(\cot\left(\frac{\pi}{5}\right)\mathrm{AngerJ}\left(\frac{2}{5},\frac{2ix^{5/2}}{5}\right)+\mathrm{WeberE}\left(\frac{2}{5},\frac{2ix^{5/2}}{5}\right)\right)\right)}{5}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 x\,\mathrm{BesselJ}\left(\frac{2}{5},\frac{2ix^{5/2}}{5}\right)+c_2 x\,\mathrm{BesselY}\left(\frac{2}{5},\frac{2ix^{5/2}}{5}\right)\right) \\
&\quad+\left(-\frac{x\left(5\,\mathrm{LommelS1}\left(1,\frac{2}{5},\frac{2ix^{5/2}}{5}\right)+\pi\left(\cot\left(\frac{\pi}{5}\right)\mathrm{AngerJ}\left(\frac{2}{5},\frac{2ix^{5/2}}{5}\right)+\mathrm{WeberE}\left(\frac{2}{5},\frac{2ix^{5/2}}{5}\right)\right)\right)}{5}\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$\begin{aligned}
y &= c_1 x\,\mathrm{BesselJ}\left(\frac{2}{5},\frac{2ix^{5/2}}{5}\right)+c_2 x\,\mathrm{BesselY}\left(\frac{2}{5},\frac{2ix^{5/2}}{5}\right) \\
&\quad-\frac{x\left(5\,\mathrm{LommelS1}\left(1,\frac{2}{5},\frac{2ix^{5/2}}{5}\right)+\pi\left(\cot\left(\frac{\pi}{5}\right)\mathrm{AngerJ}\left(\frac{2}{5},\frac{2ix^{5/2}}{5}\right)+\mathrm{WeberE}\left(\frac{2}{5},\frac{2ix^{5/2}}{5}\right)\right)\right)}{5}
\end{aligned}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 26

```
dsolve(diff(diff(y(x),x),x)-1/x*diff(y(x),x)-y(x)*x^3-x^4-1/x = 0,
       y(x),singsol=all)
```

$$y = x\left(-1 + \operatorname{BesselI}\left(\frac{2}{5}, \frac{2x^{5/2}}{5}\right) c_2 + \operatorname{BesselK}\left(\frac{2}{5}, \frac{2x^{5/2}}{5}\right) c_1\right)$$

Mathematica DSolve solution

Solving time : 0.334 (sec)
Leaf size : 316

```
DSolve[{D[y[x],{x,2}]-1/x*D[y[x],x]-x^3*y[x]-x^4-1/x==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\frac{5(x^{5/2})^{13/5}\,\text{Gamma}\left(\frac{4}{5}\right)\text{Gamma}\left(\frac{7}{5}\right)\text{BesselI}\left(\frac{2}{5},\frac{2x^{5/2}}{5}\right){}_1F_2\left(\frac{4}{5};\frac{3}{5},\frac{9}{5};\frac{x^5}{25}\right)}{\text{Gamma}\left(\frac{9}{5}\right)} - \frac{\sqrt[5]{5}(x^{5/2})^{7/5}\,\text{Gamma}\left(\frac{1}{5}\right)\text{Gamma}\left(\frac{3}{5}\right)\text{BesselI}\left(-\frac{2}{5},\frac{2x^{5/2}}{5}\right){}_1F}{\text{Gamma}\left(\frac{6}{5}\right)}}{}$$

2.2.51 problem 50

Internal problem ID [8531]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 50
Date solved : `Thursday, December 12, 2024 at 09:27:04 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - x^3y' - xy - x^3 - x^2 = 0$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
```

```
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   trying a solution in terms of MeijerG functions
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      trying 2nd order, integrating factor of the form mu(x,y)
      -> Trying a solution in terms of special functions:
         -> Bessel
         -> elliptic
         -> Legendre
         -> Kummer
            -> hyper3: Equivalence to 1F1 under a power @ Moebius
         -> hypergeometric
            -> heuristic approach
            -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
         -> Mathieu
            -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying to convert to an ODE of Bessel type
      trying to convert to an ODE of Bessel type
      -> trying reduction of order to Riccati
         trying Riccati sub-methods:
            -> trying a symmetry pattern of the form [F(x)*G(y), 0]
            -> trying a symmetry pattern of the form [0, F(x)*G(y)]
            -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3`[0, x+y]
```

Maple dsolve solution

Solving time : 0.283 (sec)
Leaf size : maple_leaf_size

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x^3-x*y(x)-x^3-x^2 = 0,
       y(x),singsol=all)
```

No solution found

Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{D[y[x],{x,2}]-x^3*D[y[x],x]-x*y[x]-x^3-x^2==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

Not solved

2.2.52 problem 51

Internal problem ID [8532]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 51
Date solved : `Wednesday, December 18, 2024 at 01:53:22 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - x^3 y' - x^2 y - x^3 = 0$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
```

```
        <- hyper3 successful: received ODE is equivalent to the 1F1 ODE
      <- Kummer successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : `0.216` (sec)
Leaf size : 28

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x^3-x^2*y(x)-x^3 = 0,
       y(x),singsol=all)
```

$$
y = \left(\operatorname{KummerU}\left(\frac{1}{2}, \frac{5}{4}, \frac{x^{4}}{4}\right) c_{1} + \operatorname{KummerM}\left(\frac{1}{2}, \frac{5}{4}, \frac{x^{4}}{4}\right) c_{2} - \frac{1}{2}\right) x
$$

Mathematica DSolve solution

Solving time : `1.16` (sec)
Leaf size : 337

```
DSolve[{D[y[x],{x,2}]-x^3*D[y[x],x]-x^2*y[x]-x^3==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)$$

$$
\rightarrow \text{Hypergeometric1F1}\left(\frac{1}{4}, \frac{3}{4}, \frac{x^4}{4}\right) \int_1^x \frac{}{5\,\text{Hypergeometric1F1}\left(\frac{1}{2}, \frac{5}{4}, \frac{K[1]^4}{4}\right) \text{Hypergeometric1F1}\left(\frac{5}{4}, \frac{7}{4}, \frac{K[1]^4}{4}\right.}
$$

$$
+ \frac{\sqrt[4]{-1}x\,\text{Hypergeometric1F1}\left(\frac{1}{2}, \frac{5}{4}, \frac{x^4}{4}\right) \int_1^x \frac{(15-15i)\,\text{H}}{3\,\text{Hypergeometric1F1}\left(\frac{1}{4}, \frac{3}{4}, \frac{K[2]^4}{4}\right)\left(2\,\text{Hypergeometric1F1}\left(\frac{3}{2}, \frac{9}{4}, \frac{K[2]^4}{4}\right) K[2]^4 + 5\,\text{H}\right.}}{\sqrt{2}}
$$

$$
+ c_1\,\text{Hypergeometric1F1}\left(\frac{1}{4}, \frac{3}{4}, \frac{x^4}{4}\right) + \left(\frac{1}{2} + \frac{i}{2}\right) c_2 x\,\text{Hypergeometric1F1}\left(\frac{1}{2}, \frac{5}{4}, \frac{x^4}{4}\right)
$$

2.2.53 problem 52

Internal problem ID [8533]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 52
Date solved : `Thursday, December 12, 2024 at 09:27:06 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' - x^3 y' - x^3 y - x^4 - x^3 = 0$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
```

```
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
   trying a solution in terms of MeijerG functions
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      trying 2nd order, integrating factor of the form mu(x,y)
      trying to convert to an ODE of Bessel type
      -> trying reduction of order to Riccati
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3`[0, x+y]
```

Maple dsolve solution

Solving time : 0.290 (sec)
Leaf size : maple_leaf_size

```
dsolve(diff(diff(y(x),x),x)-diff(y(x),x)*x^3-y(x)*x^3-x^4-x^3 = 0,
       y(x),singsol=all)
```

No solution found

Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{D[y[x],{x,2}]-x^3*D[y[x],x]-x^3*y[x]-x^4-x^3==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

Not solved

2.2.54 problem 50

Internal problem ID [8534]
Book : Own collection of miscellaneous problems
Section : section 2.0
Problem number : 50
Date solved : `Wednesday, December 18, 2024 at 01:53:23 AM`
CAS classification : `[[_3rd_order, _with_linear_symmetries]]`

Solve

$$y''' - x^3 y' - x^2 y - x^3 = 0$$

Does not support this form of ODE for higher order. Terminating.

Maple step by step solution

Let's solve
$\frac{d^3}{dx^3}y(x) - \left(\frac{d}{dx}y(x)\right)x^3 - x^2 y(x) - x^3 = 0$

- Highest derivative means the order of the ODE is 3
$\frac{d^3}{dx^3}y(x)$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE is of Euler type
trying Louvillian solutions for 3rd order ODEs, imprimitive case
-> pFq: Equivalence to the 3F2 or one of its 3 confluent cases under a power @ Moebius
<- pFq successful: received ODE is equivalent to the  1F2  ODE, case  c = 0 `
```

Maple dsolve solution

Solving time : 0.015 (sec)
Leaf size : 51

```
dsolve(diff(diff(diff(y(x),x),x),x)-diff(y(x),x)*x^3-x^2*y(x)-x^3 = 0,
       y(x),singsol=all)
```

$$y = -\frac{x}{2} + c_1 \,\mathrm{hypergeom}\left(\left[\frac{1}{5}\right], \left[\frac{3}{5}, \frac{4}{5}\right], \frac{x^5}{25}\right) + c_2 x \,\mathrm{hypergeom}\left(\left[\frac{2}{5}\right], \left[\frac{4}{5}, \frac{6}{5}\right], \frac{x^5}{25}\right) + c_3\, x^2 \,\mathrm{hypergeom}\left(\left[\frac{3}{5}\right], \left[\frac{6}{5}, \frac{7}{5}\right], \frac{x^5}{25}\right)$$

Mathematica DSolve solution

Solving time : 11.57 (sec)
Leaf size : 2548

```
DSolve[{D[y[x],{x,3}]-x^3*D[y[x],x]-x^2*y[x]-x^3==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

Too large to display

2.3 section 3.0

2.3.1 problem 1

Internal problem ID [8535]
Book : Own collection of miscellaneous problems
Section : section 3.0
Problem number : 1
Date solved : `Thursday, December 12, 2024 at 09:27:08 AM`
CAS classification : `[[_2nd_order, _missing_x]]`

Solve

$$y'' + cy' + ky = 0$$

Solved as second order linear constant coeff ode

Time used: 0.104 (sec)

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = c, C = k$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} + c\lambda\, \mathrm{e}^{x\lambda} + k\, \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$c\lambda + \lambda^2 + k = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = c, C = k$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-c}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{c^2 - (4)(1)(k)} \\ &= -\frac{c}{2} \pm \frac{\sqrt{c^2 - 4k}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{c}{2} + \frac{\sqrt{c^2 - 4k}}{2} \\ \lambda_2 &= -\frac{c}{2} - \frac{\sqrt{c^2 - 4k}}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -\frac{c}{2} + \frac{\sqrt{c^2 - 4k}}{2} \\ \lambda_2 &= -\frac{c}{2} - \frac{\sqrt{c^2 - 4k}}{2}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{\left(-\frac{c}{2} + \frac{\sqrt{c^2-4k}}{2}\right)x} + c_2 e^{\left(-\frac{c}{2} - \frac{\sqrt{c^2-4k}}{2}\right)x}\end{aligned}$$

Or

$$y = c_1 \,\mathrm{e}^{x\left(-\frac{c}{2} + \frac{\sqrt{c^2-4k}}{2}\right)} + c_2 \,\mathrm{e}^{x\left(-\frac{c}{2} - \frac{\sqrt{c^2-4k}}{2}\right)}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \,\mathrm{e}^{x\left(-\frac{c}{2} + \frac{\sqrt{c^2-4k}}{2}\right)} + c_2 \,\mathrm{e}^{x\left(-\frac{c}{2} - \frac{\sqrt{c^2-4k}}{2}\right)}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.140 (sec)

Writing the ode as

$$y'' + cy' + ky = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= c \\ C &= k \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{c^2 - 4k}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= c^2 - 4k \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{c^2}{4} - k\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.117: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{c^2}{4} - k$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{\frac{\sqrt{c^2-4k}\, x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{c}{1}\,dx} \\
&= z_1 e^{-\frac{cx}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{cx}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{\frac{x\left(-c+\sqrt{c^2-4k}\right)}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{c}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-cx}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{\mathrm{e}^{-cx}\mathrm{e}^{-x\left(-c+\sqrt{c^2-4k}\right)}}{\sqrt{c^2-4k}}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\frac{x\left(-c+\sqrt{c^2-4k}\right)}{2}}\right) + c_2 \left(\mathrm{e}^{\frac{x\left(-c+\sqrt{c^2-4k}\right)}{2}} \left(-\frac{\mathrm{e}^{-cx}\mathrm{e}^{-x\left(-c+\sqrt{c^2-4k}\right)}}{\sqrt{c^2-4k}}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = c_1\,\mathrm{e}^{\frac{x\left(-c+\sqrt{c^2-4k}\right)}{2}} - \frac{c_2\,\mathrm{e}^{-\frac{x\left(c+\sqrt{c^2-4k}\right)}{2}}}{\sqrt{c^2-4k}}
$$

Solved as second order ode adjoint method

Time used: 0.780 (sec)

In normal form the ode

$$y'' + cy' + ky = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= c \\ q(x) &= k \\ r(x) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (c\xi(x))' + (k\xi(x)) &= 0 \\ \xi''(x) - c\xi'(x) + k\xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = -c, C = k$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} - c\lambda\, \mathrm{e}^{x\lambda} + k\, \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$-c\lambda + \lambda^2 + k = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -c, C = k$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{c}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-c^2 - (4)\,(1)\,(k)} \\ &= \frac{c}{2} \pm \frac{\sqrt{c^2 - 4k}}{2} \end{aligned}$$

Hence

$$\lambda_1 = \frac{c}{2} + \frac{\sqrt{c^2 - 4k}}{2}$$
$$\lambda_2 = \frac{c}{2} - \frac{\sqrt{c^2 - 4k}}{2}$$

Which simplifies to

$$\lambda_1 = \frac{c}{2} + \frac{\sqrt{c^2 - 4k}}{2}$$
$$\lambda_2 = \frac{c}{2} - \frac{\sqrt{c^2 - 4k}}{2}$$

Since roots are real and distinct, then the solution is

$$\xi = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$\xi = c_1 e^{\left(\frac{c}{2}+\frac{\sqrt{c^2-4k}}{2}\right)x} + c_2 e^{\left(\frac{c}{2}-\frac{\sqrt{c^2-4k}}{2}\right)x}$$

Or

$$\xi = c_1 \,\mathrm{e}^{x\left(\frac{c}{2}+\frac{\sqrt{c^2-4k}}{2}\right)} + c_2 \,\mathrm{e}^{x\left(\frac{c}{2}-\frac{\sqrt{c^2-4k}}{2}\right)}$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y = \int \xi(x)\, r(x)\, dx$$
$$y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) = \frac{\int \xi(x)\, r(x)\, dx}{\xi\,(x)}$$

Or

$$y' + y\left(c - \frac{c_1\left(\frac{c}{2}+\frac{\sqrt{c^2-4k}}{2}\right)\mathrm{e}^{x\left(\frac{c}{2}+\frac{\sqrt{c^2-4k}}{2}\right)} + c_2\left(\frac{c}{2}-\frac{\sqrt{c^2-4k}}{2}\right)\mathrm{e}^{x\left(\frac{c}{2}-\frac{\sqrt{c^2-4k}}{2}\right)}}{c_1\,\mathrm{e}^{x\left(\frac{c}{2}+\frac{\sqrt{c^2-4k}}{2}\right)} + c_2\,\mathrm{e}^{x\left(\frac{c}{2}-\frac{\sqrt{c^2-4k}}{2}\right)}}\right) = 0$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= -\frac{-c_2\left(c+\sqrt{c^2-4k}\right) \mathrm{e}^{-\frac{x\left(-c+\sqrt{c^2-4k}\right)}{2}}-c_1 \,\mathrm{e}^{\frac{x\left(c+\sqrt{c^2-4k}\right)}{2}}\left(c-\sqrt{c^2-4k}\right)}{2 c_1 \,\mathrm{e}^{\frac{x\left(c+\sqrt{c^2-4k}\right)}{2}}+2 c_2 \,\mathrm{e}^{-\frac{x\left(-c+\sqrt{c^2-4k}\right)}{2}}} \\
p(x) &= 0
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= \mathrm{e}^{\int -\frac{-c_2\left(c+\sqrt{c^2-4k}\right) \mathrm{e}^{-\frac{x\left(-c+\sqrt{c^2-4k}\right)}{2}}-c_1 \,\mathrm{e}^{\frac{x\left(c+\sqrt{c^2-4k}\right)}{2}}\left(c-\sqrt{c^2-4k}\right)}{2 c_1 \,\mathrm{e}^{\frac{x\left(c+\sqrt{c^2-4k}\right)}{2}}+2 c_2 \,\mathrm{e}^{-\frac{x\left(-c+\sqrt{c^2-4k}\right)}{2}}}dx} \\
&= \frac{\mathrm{e}^{-\frac{x\left(-c+\sqrt{c^2-4k}\right)}{2}}}{c_2 \,\mathrm{e}^{-\sqrt{c^2-4k}\,x}+c_1}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y\,\mathrm{e}^{-\frac{x\left(-c+\sqrt{c^2-4k}\right)}{2}}}{c_2 \,\mathrm{e}^{-\sqrt{c^2-4k}\,x}+c_1}\right) &= 0
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y\,\mathrm{e}^{-\frac{x\left(-c+\sqrt{c^2-4k}\right)}{2}}}{c_2 \,\mathrm{e}^{-\sqrt{c^2-4k}\,x}+c_1} &= \int 0\,dx + c_3 \\
&= c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{\mathrm{e}^{-\frac{x\left(-c+\sqrt{c^2-4k}\right)}{2}}}{c_2 \,\mathrm{e}^{-\sqrt{c^2-4k}\,x}+c_1}$ gives the final solution

$$
y = \left(c_2 \,\mathrm{e}^{-\sqrt{c^2-4k}\,x}+c_1\right) \mathrm{e}^{\frac{x\left(-c+\sqrt{c^2-4k}\right)}{2}} c_3
$$

Hence, the solution found using Lagrange adjoint equation method is

$$
y = \left(c_2 \,\mathrm{e}^{-\sqrt{c^2-4k}\,x}+c_1\right) \mathrm{e}^{\frac{x\left(-c+\sqrt{c^2-4k}\right)}{2}} c_3
$$

The constants can be merged to give

$$
y = \left(c_2 \,\mathrm{e}^{-\sqrt{c^2-4k}\,x}+c_1\right) \mathrm{e}^{\frac{x\left(-c+\sqrt{c^2-4k}\right)}{2}}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \left(c_2 \, \mathrm{e}^{-\sqrt{c^2-4k}\, x} + c_1\right) \mathrm{e}^{\frac{x\left(-c+\sqrt{c^2-4k}\right)}{2}}$$

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + c\left(\frac{d}{dx}y(x)\right) + ky(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Characteristic polynomial of ODE

  $cr + r^2 + k = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-c)\pm\left(\sqrt{c^2-4k}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{c}{2} - \frac{\sqrt{c^2-4k}}{2}, -\frac{c}{2} + \frac{\sqrt{c^2-4k}}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{\left(-\frac{c}{2}-\frac{\sqrt{c^2-4k}}{2}\right)x}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{\left(-\frac{c}{2}+\frac{\sqrt{c^2-4k}}{2}\right)x}$

- General solution of the ODE

  $y(x) = C1y_1(x) + C2y_2(x)$

- Substitute in solutions

  $y(x) = C1\,\mathrm{e}^{\left(-\frac{c}{2}-\frac{\sqrt{c^2-4k}}{2}\right)x} + C2\,\mathrm{e}^{\left(-\frac{c}{2}+\frac{\sqrt{c^2-4k}}{2}\right)x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 41

```
dsolve(diff(diff(y(x),x),x)+c*diff(y(x),x)+y(x)*k = 0,
       y(x),singsol=all)
```

$$y = c_1 e^{\frac{\left(-c+\sqrt{c^2-4k}\right)x}{2}} + c_2 e^{-\frac{\left(c+\sqrt{c^2-4k}\right)x}{2}}$$

Mathematica DSolve solution

Solving time : 9.909 (sec)
Leaf size : 2548

```
DSolve[{D[y[x],{x,3}]-x^3*D[y[x],x]-x^2*y[x]-x^3==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

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2.3.2 problem 2

Internal problem ID [8536]
Book : Own collection of miscellaneous problems
Section : section 3.0
Problem number : 2
Date solved : `Tuesday, December 17, 2024 at 12:55:51 PM`
CAS classification : `[_quadrature]`

Solve

$$w' = -\frac{1}{2} - \frac{\sqrt{1-12w}}{2}$$

With initial conditions

$$w(1) = -1$$

Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} w' &= f(z,w) \\ &= -\frac{1}{2} - \frac{\sqrt{1-12w}}{2} \end{aligned}$$

The w domain of $f(z,w)$ when $z=1$ is

$$\left\{w \leq \frac{1}{12}\right\}$$

And the point $w_0 = -1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial w} &= \frac{\partial}{\partial w}\left(-\frac{1}{2} - \frac{\sqrt{1-12w}}{2}\right) \\ &= \frac{3}{\sqrt{1-12w}} \end{aligned}$$

The w domain of $\frac{\partial f}{\partial w}$ when $z = 1$ is

$$\left\{w < \frac{1}{12}\right\}$$

And the point $w_0 = -1$ is inside this domain. Therefore solution exists and is unique.

Solved as first order autonomous ode

Time used: 1.931 (sec)

Integrating gives

$$\int \frac{1}{-\frac{1}{2} - \frac{\sqrt{1-12w}}{2}} dw = dz$$
$$\frac{\sqrt{1-12w}}{3} - \frac{\ln\left(\sqrt{1-12w}+1\right)}{3} = z + c_1$$

Solving for the constant of integration from initial conditions, the solution becomes

$$\frac{\sqrt{1-12w}}{3} - \frac{\ln\left(\sqrt{1-12w}+1\right)}{3} = z - 1 + \frac{\sqrt{13}}{3} - \frac{\ln\left(\sqrt{13}+1\right)}{3}$$

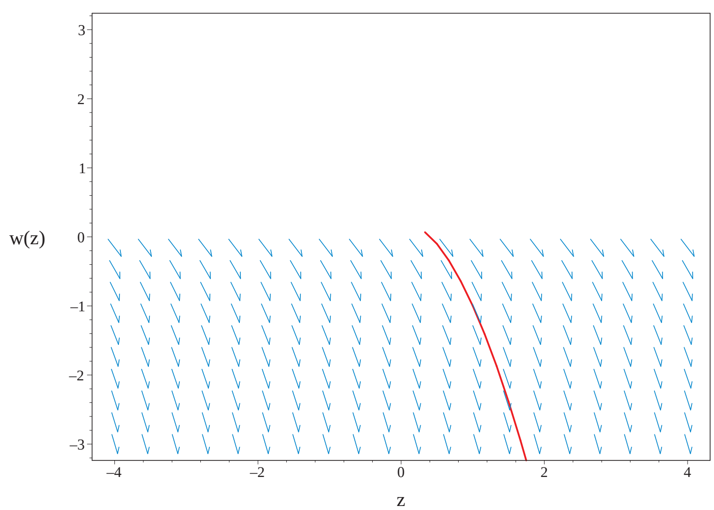


Figure 2.201: Slope field plot
$w' = -\frac{1}{2} - \frac{\sqrt{1-12w}}{2}$

Summary of solutions found

$$\frac{\sqrt{1-12w}}{3} - \frac{\ln\left(\sqrt{1-12w}+1\right)}{3} = z - 1 + \frac{\sqrt{13}}{3} - \frac{\ln\left(\sqrt{13}+1\right)}{3}$$

Solved as first order Exact ode

Time used: 1.204 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(z,w)\,\mathrm{d}z+N(z,w)\,\mathrm{d}w=0 \tag{1A}$$

Therefore

$$\mathrm{d}w=\left(-\frac{1}{2}-\frac{\sqrt{1-12w}}{2}\right)\mathrm{d}z$$
$$\left(\frac{1}{2}+\frac{\sqrt{1-12w}}{2}\right)\mathrm{d}z+\mathrm{d}w=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(z,w) &= \frac{1}{2}+\frac{\sqrt{1-12w}}{2}\\
N(z,w) &= 1
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial w}=\frac{\partial N}{\partial z}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial w} &= \frac{\partial}{\partial w}\left(\frac{1}{2}+\frac{\sqrt{1-12w}}{2}\right)\\
&= -\frac{3}{\sqrt{1-12w}}
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial z} &= \frac{\partial}{\partial z}(1)\\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial w}\neq\frac{\partial N}{\partial z}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial w}-\frac{\partial N}{\partial z}\right)\\
&= 1\left(\left(-\frac{3}{\sqrt{1-12w}}\right)-(0)\right)\\
&= -\frac{3}{\sqrt{1-12w}}
\end{aligned}$$

Since A depends on w, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial z}-\frac{\partial M}{\partial w}\right)\\
&= \frac{2}{\sqrt{1-12w}+1}\left((0)-\left(-\frac{3}{\sqrt{1-12w}}\right)\right)\\
&= \frac{6}{\left(\sqrt{1-12w}+1\right)\sqrt{1-12w}}
\end{aligned}$$

Since B does not depend on z, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}w} \\
&= e^{\int \frac{6}{(\sqrt{1-12w}+1)\sqrt{1-12w}} \, \mathrm{d}w}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\ln\left(\sqrt{1-12w}+1\right)} \\
&= \frac{1}{\sqrt{1-12w}+1}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{\sqrt{1-12w}+1}\left(\frac{1}{2}+\frac{\sqrt{1-12w}}{2}\right) \\
&= \frac{1}{2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{\sqrt{1-12w}+1}(1) \\
&= \frac{1}{\sqrt{1-12w}+1}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M}+\overline{N}\frac{\mathrm{d}w}{\mathrm{d}z} &= 0 \\
\left(\frac{1}{2}\right)+\left(\frac{1}{\sqrt{1-12w}+1}\right)\frac{\mathrm{d}w}{\mathrm{d}z} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(z,w)$

$$
\frac{\partial \phi}{\partial z} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial w} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. z gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial z} \, \mathrm{d}z &= \int \overline{M} \, \mathrm{d}z \\
\int \frac{\partial \phi}{\partial z} \, \mathrm{d}z &= \int \frac{1}{2} \, \mathrm{d}z \\
\phi &= \frac{z}{2} + f(w) \qquad (3)
\end{aligned}
$$

Where $f(w)$ is used for the constant of integration since ϕ is a function of both z and w. Taking derivative of equation (3) w.r.t w gives

$$
\frac{\partial \phi}{\partial w} = 0 + f'(w) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial w} = \frac{1}{\sqrt{1-12w}+1}$. Therefore equation (4) becomes

$$
\frac{1}{\sqrt{1-12w}+1} = 0 + f'(w) \tag{5}
$$

Solving equation (5) for $f'(w)$ gives

$$
f'(w) = \frac{1}{\sqrt{1-12w}+1}
$$

Integrating the above w.r.t w gives

$$
\begin{aligned}
\int f'(w) \, \mathrm{d}w &= \int \left(\frac{1}{\sqrt{1-12w}+1} \right) \mathrm{d}w \\
f(w) &= -\frac{\sqrt{1-12w}}{6} + \frac{\ln\left(\sqrt{1-12w}+1\right)}{6} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(w)$ into equation (3) gives ϕ

$$
\phi = \frac{z}{2} - \frac{\sqrt{1-12w}}{6} + \frac{\ln\left(\sqrt{1-12w}+1\right)}{6} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \frac{z}{2} - \frac{\sqrt{1-12w}}{6} + \frac{\ln\left(\sqrt{1-12w}+1\right)}{6}$$

Solving for the constant of integration from initial conditions, the solution becomes

$$\frac{z}{2} - \frac{\sqrt{1-12w}}{6} + \frac{\ln\left(\sqrt{1-12w}+1\right)}{6} = \frac{1}{2} - \frac{\sqrt{13}}{6} + \frac{\ln\left(\sqrt{13}+1\right)}{6}$$

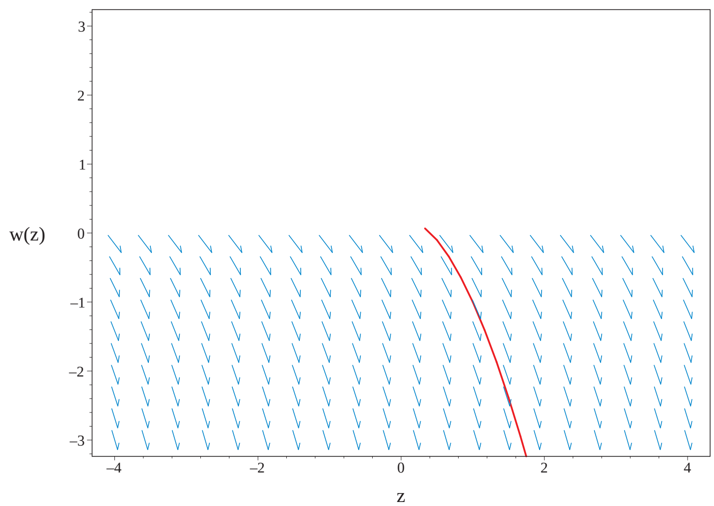


Figure 2.202: Slope field plot
$w' = -\frac{1}{2} - \frac{\sqrt{1-12w}}{2}$

Summary of solutions found

$$\frac{z}{2} - \frac{\sqrt{1-12w}}{6} + \frac{\ln\left(\sqrt{1-12w}+1\right)}{6} = \frac{1}{2} - \frac{\sqrt{13}}{6} + \frac{\ln\left(\sqrt{13}+1\right)}{6}$$

Solved using Lie symmetry for first order ode

Time used: 5.788 (sec)

Writing the ode as

$$w' = -\frac{1}{2} - \frac{\sqrt{1-12w}}{2}$$
$$w' = \omega(z, w)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_z + \omega(\eta_w - \xi_z) - \omega^2 \xi_w - \omega_z \xi - \omega_w \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = w a_3 + z a_2 + a_1 \tag{1E}$$

$$\eta = w b_3 + z b_2 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \left(-\frac{1}{2} - \frac{\sqrt{1-12w}}{2}\right)(b_3 - a_2) - \left(-\frac{1}{2} - \frac{\sqrt{1-12w}}{2}\right)^2 a_3 - \frac{3(w b_3 + z b_2 + b_1)}{\sqrt{1-12w}} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{(1-12w)^{3/2} a_3 - 2\sqrt{1-12w}\, a_2 + a_3\sqrt{1-12w} - 4b_2\sqrt{1-12w} + 2\sqrt{1-12w}\, b_3 + 24a_2 w - 24w a_3 - 12}{4\sqrt{1-12w}}$$

Setting the numerator to zero gives

$$\begin{aligned} &-(1-12w)^{3/2} a_3 + 2\sqrt{1-12w}\, a_2 - a_3\sqrt{1-12w} + 4b_2\sqrt{1-12w} \\ &\quad - 2\sqrt{1-12w}\, b_3 - 24a_2 w + 24w a_3 + 12w b_3 - 12z b_2 + 2a_2 - 2a_3 - 12b_1 - 2b_3 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &-(1-12w)^{3/2} a_3 + 2(1-12w)\, a_2 - 2(1-12w)\, a_3 - 2(1-12w)\, b_3 + 2\sqrt{1-12w}\, a_2 \\ &\quad - a_3\sqrt{1-12w} + 4b_2\sqrt{1-12w} - 2\sqrt{1-12w}\, b_3 - 12w b_3 - 12z b_2 - 12b_1 = 0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned} &12a_3\sqrt{1-12w}\, w - 24a_2 w + 24w a_3 + 12w b_3 + 2\sqrt{1-12w}\, a_2 - 2a_3\sqrt{1-12w} \\ &\quad + 4b_2\sqrt{1-12w} - 2\sqrt{1-12w}\, b_3 - 12z b_2 + 2a_2 - 2a_3 - 12b_1 - 2b_3 = 0 \end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{w, z\}$ in them.

$$\{w, z, \sqrt{1 - 12w}\}$$

The following substitution is now made to be able to collect on all terms with $\{w, z\}$ in them

$$\{w = v_1, z = v_2, \sqrt{1 - 12w} = v_3\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &12a_3v_3v_1 - 24a_2v_1 + 2v_3a_2 + 24v_1a_3 - 2a_3v_3 - 12v_2b_2 \\ &+ 4b_2v_3 + 12v_1b_3 - 2v_3b_3 + 2a_2 - 2a_3 - 12b_1 - 2b_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned} &12a_3v_3v_1 + (-24a_2 + 24a_3 + 12b_3)\, v_1 - 12v_2b_2 \\ &+ (2a_2 - 2a_3 + 4b_2 - 2b_3)\, v_3 + 2a_2 - 2a_3 - 12b_1 - 2b_3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 12a_3 &= 0 \\ -12b_2 &= 0 \\ -24a_2 + 24a_3 + 12b_3 &= 0 \\ 2a_2 - 2a_3 - 12b_1 - 2b_3 &= 0 \\ 2a_2 - 2a_3 + 4b_2 - 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= a_1 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= 0 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = 1$$
$$\eta = 0$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(z,w)\,\xi \\ &= 0 - \left(-\frac{1}{2} - \frac{\sqrt{1-12w}}{2}\right)(1) \\ &= \frac{1}{2} + \frac{\sqrt{1-12w}}{2} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(z,w) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dz}{\xi} = \frac{dw}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial z} + \eta\frac{\partial}{\partial w}\right)S(z,w) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = z$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{2} + \frac{\sqrt{1-12w}}{2}} dy\end{aligned}$$

Which results in

$$S = -\frac{\sqrt{1-12w}}{3} - \frac{\ln\left(\sqrt{1-12w}-1\right)}{6} + \frac{\ln\left(\sqrt{1-12w}+1\right)}{6} + \frac{\ln(w)}{6}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_z + \omega(z,w)S_w}{R_z + \omega(z,w)R_w} \tag{2}$$

Where in the above R_z, R_w, S_z, S_w are all partial derivatives and $\omega(z,w)$ is the right hand side of the original ode given by

$$\omega(z,w) = -\frac{1}{2} - \frac{\sqrt{1-12w}}{2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_z &= 1 \\ R_w &= 0 \\ S_z &= 0 \\ S_w &= \frac{2}{\sqrt{1-12w}+1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for z, w in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

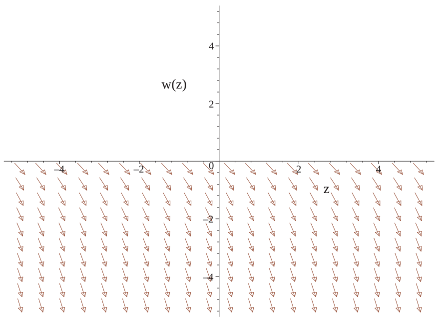
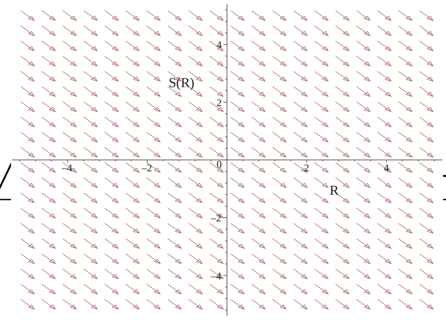
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned} \int dS &= \int -1\, dR \\ S(R) &= -R + c_2 \end{aligned}$$

To complete the solution, we just need to transform the above back to z, w coordinates. This results in

$$-\frac{\sqrt{1-12w}}{3} - \frac{\ln\left(\sqrt{1-12w}-1\right)}{6} + \frac{\ln\left(\sqrt{1-12w}+1\right)}{6} + \frac{\ln(w)}{6} = -z + c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in z, w coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dw}{dz} = -\frac{1}{2} - \frac{\sqrt{1-12w}}{2}$ 	$R = z$ $S = -\frac{\sqrt{1-12w}}{3} - \frac{\ln\left(\sqrt{}\right.}{}$ $\ldots - 1) + \mathrm{l}$	$\frac{dS}{dR} = -1$ 

Solving for the constant of integration from initial conditions, the solution becomes

$$-\frac{\sqrt{1-12w}}{3} - \frac{\ln\left(\sqrt{1-12w}-1\right)}{6} + \frac{\ln\left(\sqrt{1-12w}+1\right)}{6} + \frac{\ln\left(w\right)}{6} = -z + 1 - \frac{\sqrt{13}}{3} - \frac{\ln\left(-1+\sqrt{13}\right)}{6} +$$

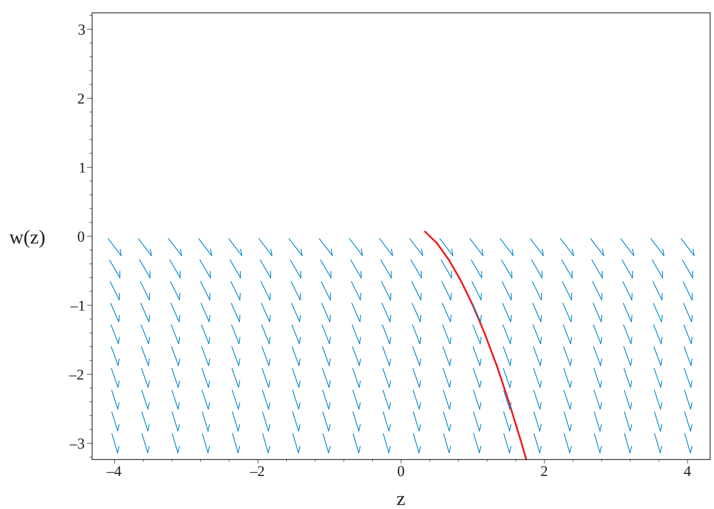


Figure 2.203: Slope field plot
$w' = -\frac{1}{2} - \frac{\sqrt{1-12w}}{2}$

Summary of solutions found

$$
\begin{aligned}
&-\frac{\sqrt{1-12w}}{3}-\frac{\ln\left(\sqrt{1-12w}-1\right)}{6}+\frac{\ln\left(\sqrt{1-12w}+1\right)}{6}+\frac{\ln\left(w\right)}{6} \\
&= -z+1-\frac{\sqrt{13}}{3}-\frac{\ln\left(-1+\sqrt{13}\right)}{6}+\frac{\ln\left(\sqrt{13}+1\right)}{6}+\frac{i\pi}{6}
\end{aligned}
$$

Maple step by step solution

Let's solve

$\left[\frac{d}{dz}w(z)=-\frac{1}{2}-\frac{\sqrt{1-12w(z)}}{2}, w(1)=-1\right]$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dz}w(z)$

- Solve for the highest derivative

  $\frac{d}{dz}w(z)=-\frac{1}{2}-\frac{\sqrt{1-12w(z)}}{2}$

- Separate variables

  $\frac{\frac{d}{dz}w(z)}{-\frac{1}{2}-\frac{\sqrt{1-12w(z)}}{2}}=1$

- Integrate both sides with respect to z

  $\int \frac{\frac{d}{dz}w(z)}{-\frac{1}{2}-\frac{\sqrt{1-12w(z)}}{2}}dz=\int 1dz+C1$

- Evaluate integral

  $-\frac{\ln(w(z))}{6}+\frac{\sqrt{1-12w(z)}}{3}+\frac{\ln\left(-1+\sqrt{1-12w(z)}\right)}{6}-\frac{\ln\left(1+\sqrt{1-12w(z)}\right)}{6}=z+C1$

- Use initial condition $w(1)=-1$

  $-\frac{I\pi}{6}+\frac{\sqrt{13}}{3}+\frac{\ln\left(-1+\sqrt{13}\right)}{6}-\frac{\ln\left(1+\sqrt{13}\right)}{6}=C1+1$

- Solve for $\_C1$

  $C1=-1-\frac{I\pi}{6}+\frac{\sqrt{13}}{3}+\frac{\ln\left(-1+\sqrt{13}\right)}{6}-\frac{\ln\left(1+\sqrt{13}\right)}{6}$

- Substitute $\_C1=-1-\frac{I\pi}{6}+\frac{\sqrt{13}}{3}+\frac{\ln\left(-1+\sqrt{13}\right)}{6}-\frac{\ln\left(1+\sqrt{13}\right)}{6}$ into general solution and simplify

  $-\frac{\ln(w(z))}{6}+\frac{\sqrt{1-12w(z)}}{3}+\frac{\ln\left(-1+\sqrt{1-12w(z)}\right)}{6}-\frac{\ln\left(1+\sqrt{1-12w(z)}\right)}{6}=z-1-\frac{I\pi}{6}+\frac{\sqrt{13}}{3}+\frac{\ln\left(-1+\sqrt{13}\right)}{6}-\ln(1$

- Solution to the IVP

  $-\frac{\ln(w(z))}{6}+\frac{\sqrt{1-12w(z)}}{3}+\frac{\ln\left(-1+\sqrt{1-12w(z)}\right)}{6}-\frac{\ln\left(1+\sqrt{1-12w(z)}\right)}{6}=z-1-\frac{I\pi}{6}+\frac{\sqrt{13}}{3}+\frac{\ln\left(-1+\sqrt{13}\right)}{6}-\ln(1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

Maple dsolve solution

Solving time : 1.513 (sec)
Leaf size : 65

```
dsolve([diff(w(z),z) = -1/2-1/2*(1-12*w(z))^(1/2),
       op([w(1) = -1])],w(z),singsol=all)
```

$$w(z) = \mathrm{RootOf}\left(-i\pi + 2\sqrt{13} - \ln\left(1+\sqrt{13}\right) + \ln\left(-1+\sqrt{13}\right) - 2\sqrt{1-12\_Z} + \ln\left(\_Z\right) - \ln\left(-1+\sqrt{1-12\_Z}\right) + \ln\left(1+\sqrt{1-12\_Z}\right) + 6z - 6\right)$$

Mathematica DSolve solution

Solving time : 11.607 (sec)
Leaf size : 105

```
DSolve[{D[w[z],z] == -1/2 - Sqrt[1/4 - 3*w[z]],{w[1] == -1}},
      w[z],z,IncludeSingularSolutions->True]
```

$$w(z) \to -\frac{1}{12}W\left(\left(\sqrt{13}-1\right)e^{-3z+\sqrt{13}+2}\right)\left(W\left(\left(\sqrt{13}-1\right)e^{-3z+\sqrt{13}+2}\right)+2\right)$$
$$w(z) \to -\frac{1}{12}W\left(\left(\sqrt{13}-1\right)e^{-3z+\sqrt{13}+2}\right)\left(W\left(\left(\sqrt{13}-1\right)e^{-3z+\sqrt{13}+2}\right)+2\right)$$

2.3.3 problem 3

Internal problem ID [8537]
Book : Own collection of miscellaneous problems
Section : section 3.0
Problem number : 3
Date solved : `Thursday, December 12, 2024 at 09:27:11 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' + y = \sin(x)$$

With initial conditions

$$y(0) = 1$$

Solved as second order linear constant coeff ode

Time used: 0.170 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A=1, B=0, C=1$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda}+\mathrm{e}^{x\lambda}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2+1=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=0, C=1$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{0}{(2)\,(1)}\pm\frac{1}{(2)\,(1)}\sqrt{0^2-(4)\,(1)\,(1)}\\&=\pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1&=+i\\\lambda_2&=-i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1&=i\\\lambda_2&=-i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2}=\alpha\pm i\beta$$

Where $\alpha=0$ and $\beta=1$. Therefore the final solution, when using Euler relation, can be written as

$$y=e^{\alpha x}(c_1\cos(\beta x)+c_2\sin(\beta x))$$

Which becomes

$$y=e^0(c_1\cos{(x)}+c_2\sin{(x)})$$

Or

$$y=c_1\cos{(x)}+c_2\sin{(x)}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\cos(x)\, x, \sin(x)\, x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \cos(x)\, x + A_2 \sin(x)\, x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)\, x}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\frac{\cos(x)\, x}{2}\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = -\frac{\cos(x)\, x}{2} + \cos(x) + c_2 \sin(x)
$$

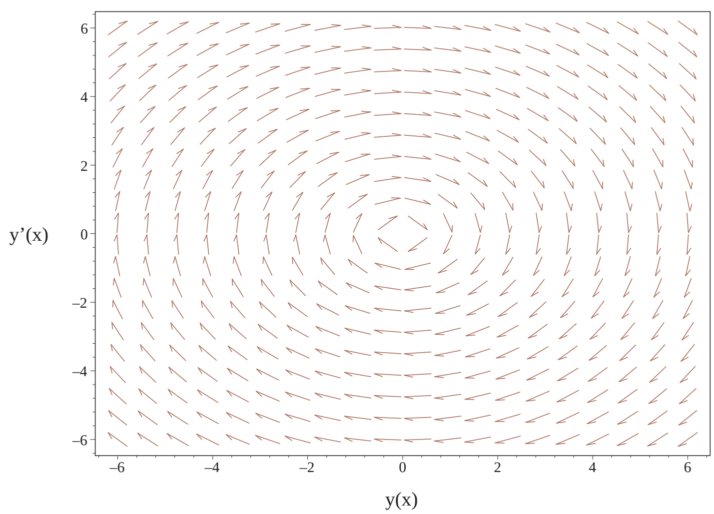


Figure 2.204: Slope field plot
$y'' + y = \sin(x)$

Solved as second order ode using Kovacic algorithm

Time used: 0.139 (sec)

Writing the ode as

$$
y'' + y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= 1
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.120: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)\,(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)(\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \cos(x) + c_2 \sin(x)
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\sin(x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(x), \sin(x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\cos(x), \sin(x)\}
$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{\cos(x)\, x, \sin(x)\, x\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$
y_p = A_1 \cos(x)\, x + A_2 \sin(x)\, x
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)\, x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\frac{\cos(x)\, x}{2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{\cos(x)\, x}{2} + \cos(x) + c_2 \sin(x)$$

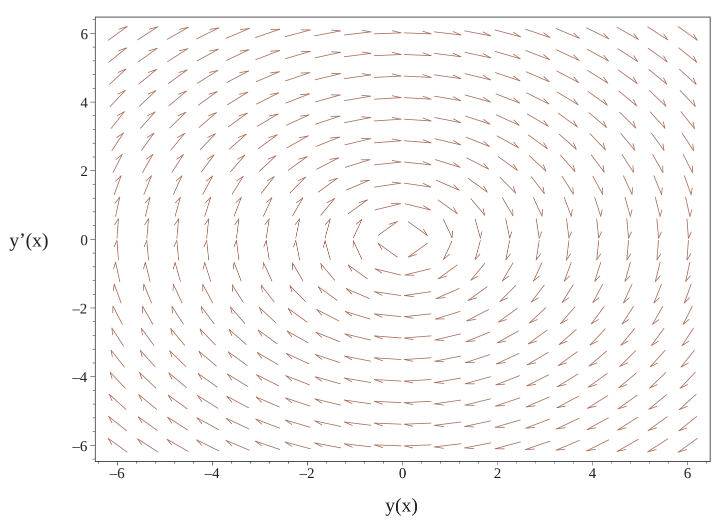


Figure 2.205: Slope field plot
$y'' + y = \sin(x)$

Solved as second order ode adjoint method

Time used: 2.171 (sec)

In normal form the ode

$$y'' + y = \sin(x) \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 1 \\ r(x) &= \sin(x) \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\,p)' + \xi q &= 0 \\ \xi'' - (0)' + (\xi(x)) &= 0 \\ \xi''(x) + \xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)} \\ &= \pm i \end{aligned}$$

Hence

$$\lambda_1 = +i$$
$$\lambda_2 = -i$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$\xi = e^{0}(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$\xi = c_1 \cos(x) + c_2 \sin(x)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}
\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)}
\end{aligned}$$

Or

$$y' - \frac{y(-c_1 \sin(x) + c_2 \cos(x))}{c_1 \cos(x) + c_2 \sin(x)} = \frac{-\frac{\cos(x)^2 c_1}{2} + c_2\left(-\frac{\cos(x)\sin(x)}{2} + \frac{x}{2}\right)}{c_1 \cos(x) + c_2 \sin(x)}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= -\frac{-c_1 \sin(x) + c_2 \cos(x)}{c_1 \cos(x) + c_2 \sin(x)} \\
p(x) &= \frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{2 c_2 \sin(x) + 2 c_1 \cos(x)}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q \, dx} \\
&= \mathrm{e}^{\int -\frac{-c_1 \sin(x) + c_2 \cos(x)}{c_1 \cos(x) + c_2 \sin(x)} dx} \\
&= \frac{1}{c_1 \cos(x) + c_2 \sin(x)}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu p \\
&\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu) \left(\frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{2 c_2 \sin(x) + 2 c_1 \cos(x)} \right) \\
&\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{y}{c_1 \cos(x) + c_2 \sin(x)} \right) \\
&\quad = \left(\frac{1}{c_1 \cos(x) + c_2 \sin(x)} \right) \left(\frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{2 c_2 \sin(x) + 2 c_1 \cos(x)} \right) \\
&\mathrm{d}\left(\frac{y}{c_1 \cos(x) + c_2 \sin(x)} \right) = \left(\frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{(2 c_2 \sin(x) + 2 c_1 \cos(x)) (c_1 \cos(x) + c_2 \sin(x))} \right) \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{c_1 \cos(x) + c_2 \sin(x)} &= \int \frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{(2 c_2 \sin(x) + 2 c_1 \cos(x)) (c_1 \cos(x) + c_2 \sin(x))} \, dx \\
&= \frac{\frac{x}{2} + \frac{x \tan\left(\frac{x}{2}\right)^2}{2} - \frac{x \tan\left(\frac{x}{2}\right)^4}{2} - \frac{x \tan\left(\frac{x}{2}\right)^6}{2}}{\left(1 + \tan\left(\frac{x}{2}\right)^2\right)^2 \left(c_1 \tan\left(\frac{x}{2}\right)^2 - 2 c_2 \tan\left(\frac{x}{2}\right) - c_1\right)} + c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{1}{c_1 \cos(x) + c_2 \sin(x)}$ gives the final solution

$$
y = \frac{(2 c_1 c_3 - x) \cos(x)}{2} + c_2 c_3 \sin(x)
$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{(2c_1c_3 - x)\cos(x)}{2} + c_2c_3\sin(x)$$

The constants can be merged to give

$$y = \frac{(2c_1 - x)\cos(x)}{2} + c_2\sin(x)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{(2 - x)\cos(x)}{2} + c_2\sin(x)$$

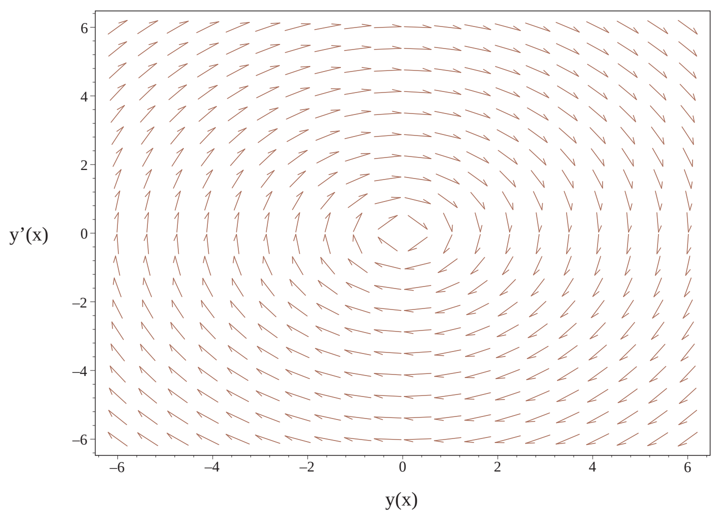


Figure 2.206: Slope field plot
$y'' + y = \sin(x)$

Maple step by step solution

Let's solve
$\left[\frac{d^2}{dx^2}y(x) + y(x) = \sin(x), y(0) = 1\right]$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2}y(x)$

- Characteristic polynomial of homogeneous ODE
  $r^2 + 1 = 0$
- Use quadratic formula to solve for r
  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial
  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the homogeneous ODE
  $y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE
  $y_2(x) = \sin(x)$
- General solution of the ODE
  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE
  $y(x) = C1 \cos(x) + C2 \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE
  - Use variation of parameters to find y_p here $f(x)$ is the forcing function
    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(x)\right]$
  - Wronskian of solutions of the homogeneous equation
    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$
  - Compute Wronskian
    $W(y_1(x), y_2(x)) = 1$
  - Substitute functions into equation for $y_p(x)$
    $y_p(x) = -\cos(x) \left(\int \sin(x)^2 dx\right) + \frac{\sin(x) \left(\int \sin(2x) dx\right)}{2}$
  - Compute integrals
    $y_p(x) = \frac{\sin(x)}{4} - \frac{x \cos(x)}{2}$

- Substitute particular solution into general solution to ODE
  $y(x) = C1 \cos(x) + C2 \sin(x) + \frac{\sin(x)}{4} - \frac{x \cos(x)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.023 (sec)
Leaf size : 21

```
dsolve([diff(diff(y(x),x),x)+y(x) = sin(x),
       op([y(0) = 1])],y(x),singsol=all)
```

$$y = \frac{\sin (x) (2c_2 + 1)}{2} - \frac{\cos (x) (x - 2)}{2}$$

Mathematica DSolve solution

Solving time : 0.023 (sec)
Leaf size : 20

```
DSolve[{D[y[x],{x,2}]+y[x]==Sin[x],{y[0] == 1}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{1}{2} x \cos(x) + \cos(x) + c_2 \sin(x)$$

2.3.4 problem 4

Internal problem ID [8538]
Book : Own collection of miscellaneous problems
Section : section 3.0
Problem number : 4
Date solved : `Thursday, December 12, 2024 at 09:27:14 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' + y = \sin(x)$$

With initial conditions

$$y'(0) = 1$$

Solved as second order linear constant coeff ode

Time used: 0.160 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} + \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\cos(x)\, x, \sin(x)\, x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \cos(x)\, x + A_2 \sin(x)\, x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)\, x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\frac{\cos(x)\, x}{2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{\cos(x)\, x}{2} + c_1 \cos(x) + \frac{3 \sin(x)}{2}$$

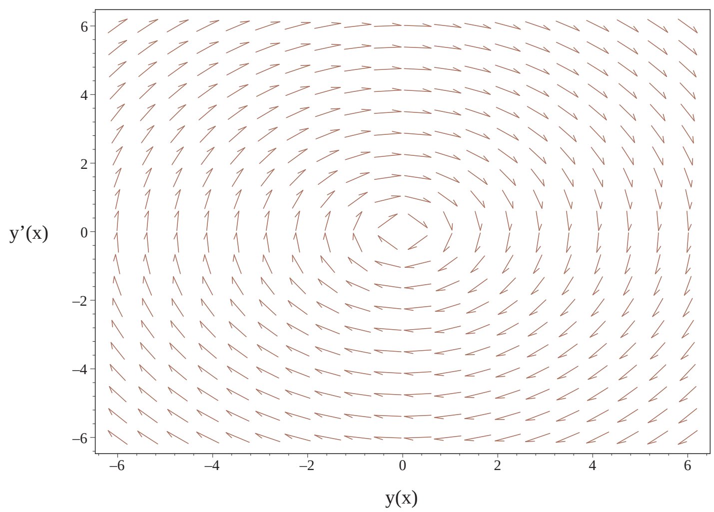


Figure 2.207: Slope field plot
$y'' + y = \sin(x)$

Solved as second order ode using Kovacic algorithm

Time used: 0.188 (sec)

Writing the ode as

$$y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.122: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)(\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \cos(x) + c_2 \sin(x)
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\sin(x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(x), \sin(x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\cos(x), \sin(x)\}
$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{\cos(x)\, x, \sin(x)\, x\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$
y_p = A_1 \cos(x)\, x + A_2 \sin(x)\, x
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)\,x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\frac{\cos(x)\,x}{2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{\cos(x)\,x}{2} + c_1 \cos(x) + \frac{3\sin(x)}{2}$$

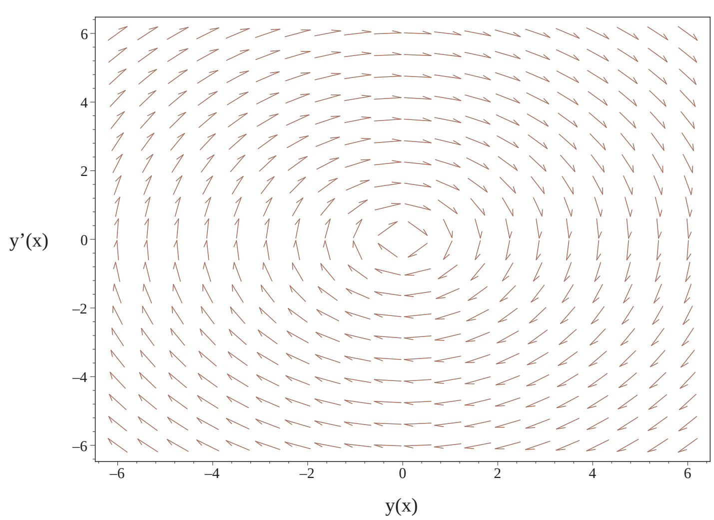


Figure 2.208: Slope field plot
$y'' + y = \sin(x)$

Solved as second order ode adjoint method

Time used: 2.258 (sec)

In normal form the ode

$$y'' + y = \sin(x) \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 1 \\ r(x) &= \sin(x) \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\,p)' + \xi q &= 0 \\ \xi'' - (0)' + (\xi(x)) &= 0 \\ \xi''(x) + \xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)} \\ &= \pm i \end{aligned}$$

Hence

$$
\begin{aligned}
\lambda_1 &= +i \\
\lambda_2 &= -i
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= i \\
\lambda_2 &= -i
\end{aligned}
$$

Since roots are complex conjugate of each others, then let the roots be

$$
\lambda_{1,2} = \alpha \pm i\beta
$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$
\xi = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))
$$

Which becomes

$$
\xi = e^{0}(c_1 \cos(x) + c_2 \sin(x))
$$

Or

$$
\xi = c_1 \cos(x) + c_2 \sin(x)
$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$
\begin{aligned}
\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)}
\end{aligned}
$$

Or

$$
y' - \frac{y(-c_1 \sin(x) + c_2 \cos(x))}{c_1 \cos(x) + c_2 \sin(x)} = \frac{-\frac{\cos(x)^2 c_1}{2} + c_2\left(-\frac{\cos(x)\sin(x)}{2} + \frac{x}{2}\right)}{c_1 \cos(x) + c_2 \sin(x)}
$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$
y' + q(x)y = p(x)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= -\frac{-c_1 \sin(x) + c_2 \cos(x)}{c_1 \cos(x) + c_2 \sin(x)} \\
p(x) &= \frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{2c_2 \sin(x) + 2c_1 \cos(x)}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q \, dx} \\
&= e^{\int -\frac{-c_1 \sin(x) + c_2 \cos(x)}{c_1 \cos(x) + c_2 \sin(x)} dx} \\
&= \frac{1}{c_1 \cos(x) + c_2 \sin(x)}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu p \\
&\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu) \left(\frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{2c_2 \sin(x) + 2c_1 \cos(x)} \right) \\
&\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{c_1 \cos(x) + c_2 \sin(x)} \right) \\
&\quad = \left(\frac{1}{c_1 \cos(x) + c_2 \sin(x)} \right) \left(\frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{2c_2 \sin(x) + 2c_1 \cos(x)} \right) \\
&\mathrm{d}\left(\frac{y}{c_1 \cos(x) + c_2 \sin(x)} \right) = \left(\frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{(2c_2 \sin(x) + 2c_1 \cos(x)) (c_1 \cos(x) + c_2 \sin(x))} \right) \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{c_1 \cos(x) + c_2 \sin(x)} &= \int \frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{(2c_2 \sin(x) + 2c_1 \cos(x)) (c_1 \cos(x) + c_2 \sin(x))} \, dx \\
&= \frac{\frac{x}{2} + \frac{x \tan\left(\frac{x}{2}\right)^2}{2} - \frac{x \tan\left(\frac{x}{2}\right)^4}{2} - \frac{x \tan\left(\frac{x}{2}\right)^6}{2}}{\left(1 + \tan\left(\frac{x}{2}\right)^2\right)^2 \left(c_1 \tan\left(\frac{x}{2}\right)^2 - 2c_2 \tan\left(\frac{x}{2}\right) - c_1\right)} + c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{1}{c_1 \cos(x) + c_2 \sin(x)}$ gives the final solution

$$
y = \frac{(2c_1 c_3 - x) \cos(x)}{2} + c_2 c_3 \sin(x)
$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{(2c_1c_3 - x)\cos(x)}{2} + c_2c_3\sin(x)$$

The constants can be merged to give

$$y = \frac{(2c_1 - x)\cos(x)}{2} + c_2\sin(x)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{(2c_1 - x)\cos(x)}{2} + \frac{3\sin(x)}{2}$$

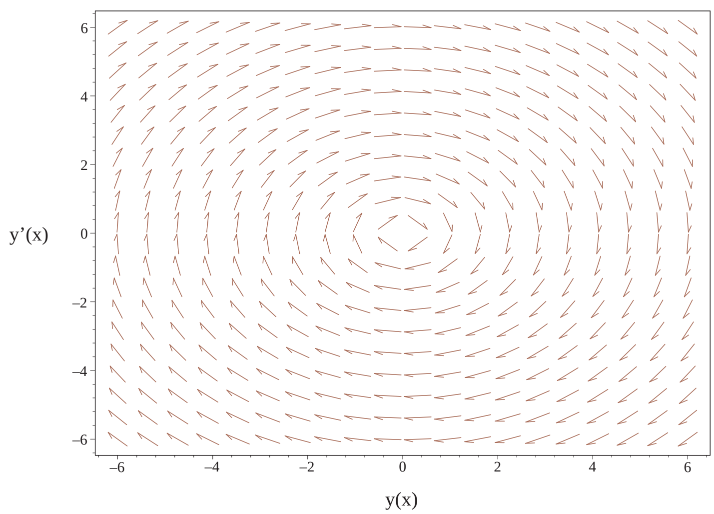


Figure 2.209: Slope field plot
$y'' + y = \sin(x)$

Maple step by step solution

Let's solve

$\left[\frac{d^2}{dx^2}y(x) + y(x) = \sin(x), \left(\frac{d}{dx}y(x)\right)\Big|_{\{x=0\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Characteristic polynomial of homogeneous ODE
  $r^2 + 1 = 0$
- Use quadratic formula to solve for r
  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial
  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the homogeneous ODE
  $y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE
  $y_2(x) = \sin(x)$
- General solution of the ODE
  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE
  $y(x) = C1 \cos(x) + C2 \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function
    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(x)\right]$
  - Wronskian of solutions of the homogeneous equation
    $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{array}\right]$
  - Compute Wronskian
    $W(y_1(x), y_2(x)) = 1$
  - Substitute functions into equation for $y_p(x)$
    $y_p(x) = -\cos(x) \left(\int \sin(x)^2 \, dx\right) + \frac{\sin(x) \left(\int \sin(2x) dx\right)}{2}$
  - Compute integrals
    $y_p(x) = \frac{\sin(x)}{4} - \frac{x \cos(x)}{2}$

- Substitute particular solution into general solution to ODE
  $y(x) = C1 \cos(x) + C2 \sin(x) + \frac{\sin(x)}{4} - \frac{x \cos(x)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.024 (sec)
Leaf size : 20

```
dsolve([diff(diff(y(x),x),x)+y(x) = sin(x),
       op([D(y)(0) = 1])],y(x),singsol=all)
```

$$y = \frac{(2c_1 - x)\cos(x)}{2} + \frac{3\sin(x)}{2}$$

Mathematica DSolve solution

Solving time : 0.022 (sec)
Leaf size : 23

```
DSolve[{D[y[x],{x,2}]+y[x]==Sin[x],{Derivative[1][y][0] == 1}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{3\sin(x)}{2} + \left(-\frac{x}{2} + c_1\right)\cos(x)$$

2.3.5 problem 5

Internal problem ID [8539]
Book : Own collection of miscellaneous problems
Section : section 3.0
Problem number : 5
Date solved : `Thursday, December 12, 2024 at 09:27:18 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' + y = \sin(x)$$

With initial conditions

$$y'(0) = 1$$
$$y(0) = 0$$

Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$p(x) = 0$$
$$q(x) = 1$$
$$F = \sin(x)$$

Hence the ode is

$$y'' + y = \sin(x)$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = \sin(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

Solved as second order linear constant coeff ode

Time used: 0.167 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} + \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\cos(x)\, x, \sin(x)\, x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \cos(x)\, x + A_2 \sin(x)\, x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)\, x}{2}$$

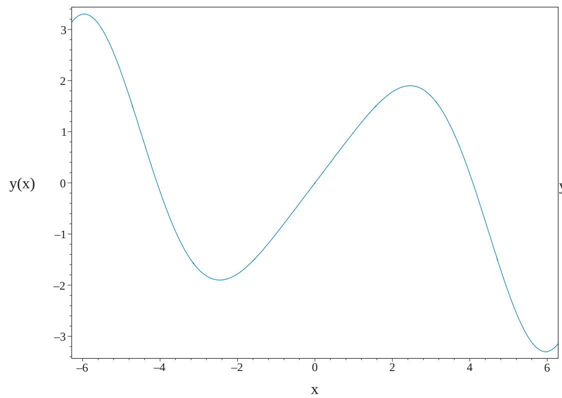
Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\frac{\cos(x)\, x}{2}\right) \end{aligned}$$

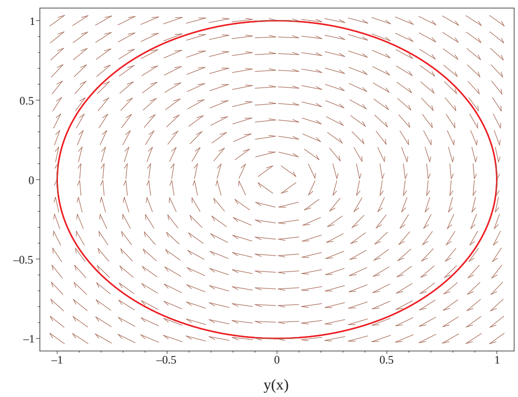
Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{\cos(x)\, x}{2} + \frac{3\sin(x)}{2}$$



(a) Solution plot
$y = -\frac{\cos(x)x}{2} + \frac{3\sin(x)}{2}$



(b) Slope field plot
$y'' + y = \sin(x)$

Solved as second order ode using Kovacic algorithm

Time used: 0.189 (sec)

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.124: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\cos(x)\, x, \sin(x)\, x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \cos(x)\, x + A_2 \sin(x)\, x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)\, x}{2}$$

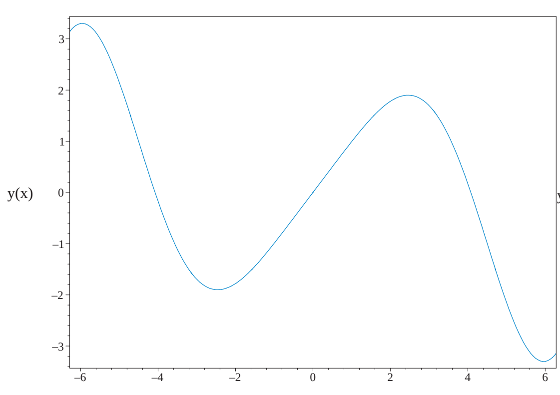
Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\frac{\cos(x)\, x}{2}\right) \end{aligned}$$

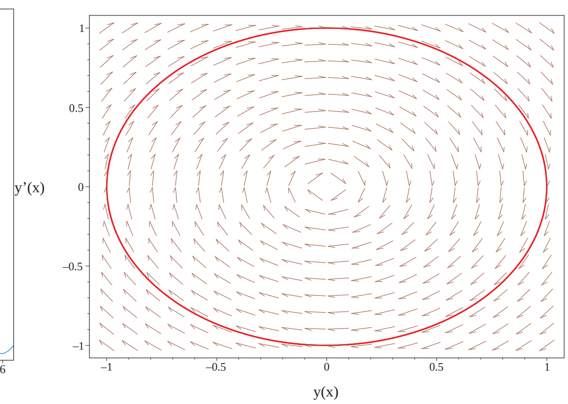
Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{\cos(x)\, x}{2} + \frac{3\sin(x)}{2}$$



(a) Solution plot
$y = -\frac{\cos(x)x}{2} + \frac{3\sin(x)}{2}$



(b) Slope field plot
$y'' + y = \sin(x)$

Solved as second order ode adjoint method

Time used: 2.217 (sec)

In normal form the ode

$$y'' + y = \sin(x) \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 1 \\ r(x) &= \sin(x) \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (0)' + (\xi(x)) &= 0 \\ \xi''(x) + \xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} + \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)} \\ &= \pm i \end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$\xi = e^{0}(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$\xi = c_1 \cos(x) + c_2 \sin(x)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx\\ y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)}\end{aligned}$$

Or

$$y' - \frac{y(-c_1 \sin(x) + c_2 \cos(x))}{c_1 \cos(x) + c_2 \sin(x)} = \frac{-\frac{\cos(x)^2 c_1}{2} + c_2\left(-\frac{\cos(x)\sin(x)}{2} + \frac{x}{2}\right)}{c_1 \cos(x) + c_2 \sin(x)}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= -\frac{-c_1 \sin(x) + c_2 \cos(x)}{c_1 \cos(x) + c_2 \sin(x)} \\
p(x) &= \frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{2c_2 \sin(x) + 2c_1 \cos(x)}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q \, dx} \\
&= e^{\int -\frac{-c_1 \sin(x) + c_2 \cos(x)}{c_1 \cos(x) + c_2 \sin(x)} dx} \\
&= \frac{1}{c_1 \cos(x) + c_2 \sin(x)}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu p \\
&\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu) \left(\frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{2c_2 \sin(x) + 2c_1 \cos(x)} \right) \\
&\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{c_1 \cos(x) + c_2 \sin(x)} \right) \\
&\quad = \left(\frac{1}{c_1 \cos(x) + c_2 \sin(x)} \right) \left(\frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{2c_2 \sin(x) + 2c_1 \cos(x)} \right) \\
&\mathrm{d}\left(\frac{y}{c_1 \cos(x) + c_2 \sin(x)} \right) = \left(\frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{(2c_2 \sin(x) + 2c_1 \cos(x)) (c_1 \cos(x) + c_2 \sin(x))} \right) \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{c_1 \cos(x) + c_2 \sin(x)} &= \int \frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{(2c_2 \sin(x) + 2c_1 \cos(x)) (c_1 \cos(x) + c_2 \sin(x))} \, dx \\
&= \frac{\frac{x}{2} + \frac{x \tan\left(\frac{x}{2}\right)^2}{2} - \frac{x \tan\left(\frac{x}{2}\right)^4}{2} - \frac{x \tan\left(\frac{x}{2}\right)^6}{2}}{\left(1 + \tan\left(\frac{x}{2}\right)^2\right)^2 \left(c_1 \tan\left(\frac{x}{2}\right)^2 - 2c_2 \tan\left(\frac{x}{2}\right) - c_1\right)} + c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{1}{c_1 \cos(x) + c_2 \sin(x)}$ gives the final solution

$$
y = \frac{(2c_1 c_3 - x) \cos(x)}{2} + c_2 c_3 \sin(x)
$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{(2c_1c_3 - x)\cos(x)}{2} + c_2c_3\sin(x)$$

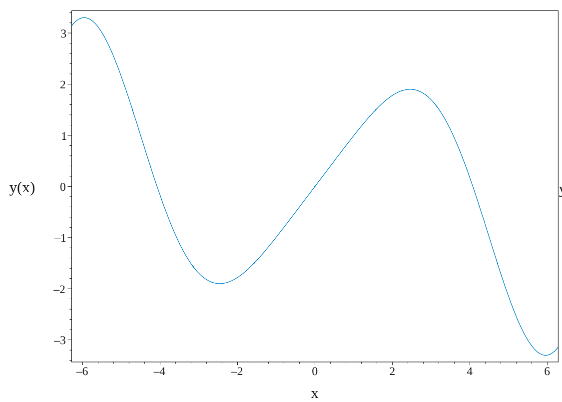
The constants can be merged to give

$$y = \frac{(2c_1 - x)\cos(x)}{2} + c_2\sin(x)$$

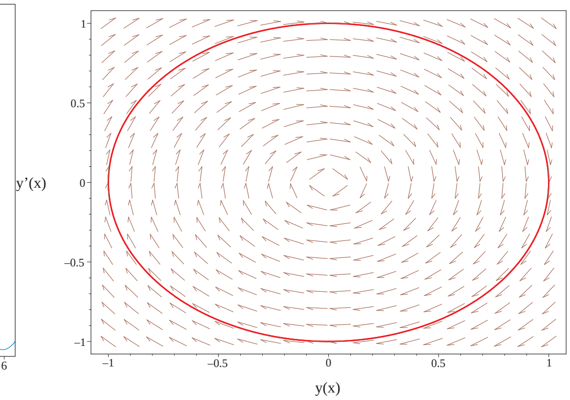
Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{\cos(x)\,x}{2} + \frac{3\sin(x)}{2}$$



(a) Solution plot
$y = -\frac{\cos(x)x}{2} + \frac{3\sin(x)}{2}$



(b) Slope field plot
$y'' + y = \sin(x)$

Maple step by step solution

Let's solve
$\left[\frac{d^2}{dx^2}y(x) + y(x) = \sin(x), \left(\frac{d}{dx}y(x)\right)\Big|_{\{x=0\}} = 1, y(0) = 0\right]$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Characteristic polynomial of homogeneous ODE
  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$
- General solution of the ODE

  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y(x) = C1 \cos(x) + C2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(x)\right]$
- ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$
- ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$
- ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\cos(x)\left(\int \sin(x)^2 dx\right) + \frac{\sin(x)\left(\int \sin(2x) dx\right)}{2}$
- ◦ Compute integrals

  $y_p(x) = \frac{\sin(x)}{4} - \frac{x \cos(x)}{2}$
- Substitute particular solution into general solution to ODE

  $y(x) = C1 \cos(x) + C2 \sin(x) + \frac{\sin(x)}{4} - \frac{x \cos(x)}{2}$

☐ Check validity of solution $y(x) = \_C1 \cos(x) + \_C2 \sin(x) + \frac{\sin(x)}{4} - \frac{x \cos(x)}{2}$

- ◦ Use initial condition $y(0) = 0$

  $0 = \_C1$
- ◦ Compute derivative of the solution

  $\frac{d}{dx} y(x) = -\_C1 \sin(x) + \_C2 \cos(x) - \frac{\cos(x)}{4} + \frac{\sin(x) x}{2}$
- ◦ Use the initial condition $\left(\frac{d}{dx} y(x)\right)\Big|_{\{x=0\}} = 1$

$1 = -\frac{1}{4} + \_C2$

- Solve for $\_C1$ and $\_C2$

  $\{\_C1 = 0, \_C2 = \frac{5}{4}\}$

- Substitute constant values into general solution and simplify

  $y(x) = \frac{3\sin(x)}{2} - \frac{x\cos(x)}{2}$

- Solution to the IVP

  $y(x) = \frac{3\sin(x)}{2} - \frac{x\cos(x)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.019 (sec)
Leaf size : 14

```
dsolve([diff(diff(y(x),x),x)+y(x) = sin(x),
       op([D(y)(0) = 1, y(0) = 0])],y(x),singsol=all)
```

$$y = \frac{3\sin(x)}{2} - \frac{\cos(x)\,x}{2}$$

Mathematica DSolve solution

Solving time : 0.02 (sec)
Leaf size : 19

```
DSolve[{D[y[x],{x,2}]+y[x]==Sin[x],{Derivative[1][y][0] == 1,y[0]==0}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}(3\sin(x) - x\cos(x))$$

2.3.6 problem 6

Internal problem ID [8540]
Book : Own collection of miscellaneous problems
Section : section 3.0
Problem number : 6
Date solved : `Thursday, December 12, 2024 at 09:27:21 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' + y = \sin(x)$$

With initial conditions

$$y(1) = 0$$

Solved as second order linear constant coeff ode

Time used: 0.218 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} + \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\cos(x) x, \sin(x) x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \cos(x) x + A_2 \sin(x) x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x) x}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\frac{\cos(x)\, x}{2}\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = -\frac{\cos(x)\, x}{2} + \left(-c_2 \tan(1) + \frac{1}{2}\right) \cos(x) + c_2 \sin(x)
$$

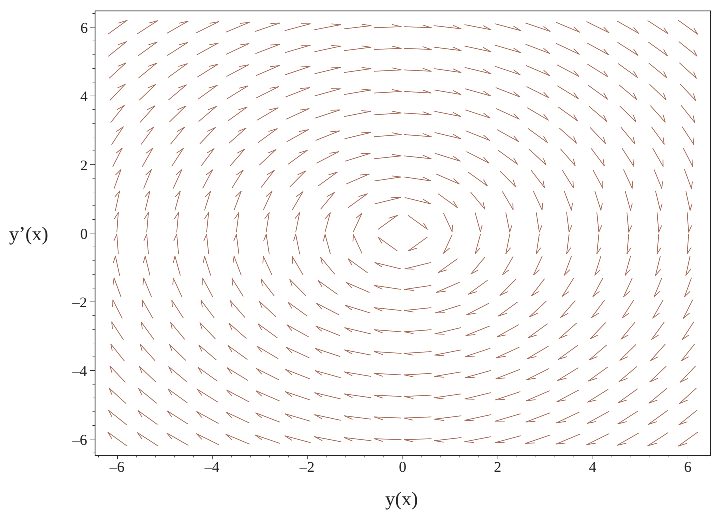


Figure 2.213: Slope field plot
$y'' + y = \sin(x)$

Solved as second order ode using Kovacic algorithm

Time used: 0.148 (sec)

Writing the ode as

$$
y'' + y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= 1
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.126: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2} \, dx \\ &= \cos(x) (\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)(\tan(x)))\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\cos(x)\, x, \sin(x)\, x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \cos(x)\, x + A_2 \sin(x)\, x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)\, x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\frac{\cos(x)\, x}{2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{\cos(x)\, x}{2} + \left(-c_2 \tan(1) + \frac{1}{2}\right) \cos(x) + c_2 \sin(x)$$

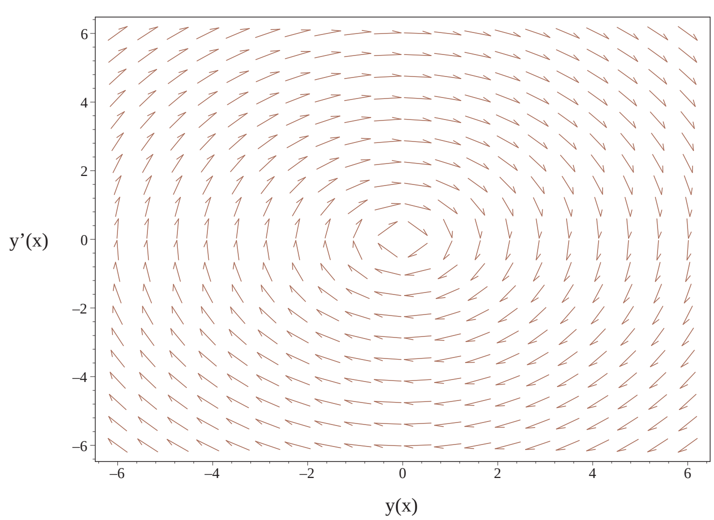


Figure 2.214: Slope field plot
$y'' + y = \sin(x)$

Solved as second order ode adjoint method

Time used: 2.181 (sec)

In normal form the ode

$$y'' + y = \sin(x) \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 1 \\ r(x) &= \sin(x) \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (0)' + (\xi(x)) &= 0 \\ \xi''(x) + \xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)} \\ &= \pm i \end{aligned}$$

Hence

$$\lambda_1 = +i$$
$$\lambda_2 = -i$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$\xi = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$\xi = c_1 \cos(x) + c_2 \sin(x)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}
\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)}
\end{aligned}$$

Or

$$y' - \frac{y(-c_1 \sin(x) + c_2 \cos(x))}{c_1 \cos(x) + c_2 \sin(x)} = \frac{-\frac{\cos(x)^2 c_1}{2} + c_2\left(-\frac{\cos(x)\sin(x)}{2} + \frac{x}{2}\right)}{c_1 \cos(x) + c_2 \sin(x)}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= -\frac{-c_1 \sin(x) + c_2 \cos(x)}{c_1 \cos(x) + c_2 \sin(x)} \\
p(x) &= \frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{2c_2 \sin(x) + 2c_1 \cos(x)}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q \, dx} \\
&= e^{\int -\frac{-c_1 \sin(x)+c_2 \cos(x)}{c_1 \cos(x)+c_2 \sin(x)} dx} \\
&= \frac{1}{c_1 \cos(x) + c_2 \sin(x)}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu) \left(\frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{2c_2 \sin(x) + 2c_1 \cos(x)} \right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{c_1 \cos(x) + c_2 \sin(x)} \right) \\
&= \left(\frac{1}{c_1 \cos(x) + c_2 \sin(x)} \right) \left(\frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{2c_2 \sin(x) + 2c_1 \cos(x)} \right) \\
\mathrm{d}\left(\frac{y}{c_1 \cos(x) + c_2 \sin(x)} \right) &= \left(\frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{(2c_2 \sin(x) + 2c_1 \cos(x)) (c_1 \cos(x) + c_2 \sin(x))} \right) \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{c_1 \cos(x) + c_2 \sin(x)} &= \int \frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{(2c_2 \sin(x) + 2c_1 \cos(x)) (c_1 \cos(x) + c_2 \sin(x))} \, dx \\
&= \frac{\frac{x}{2} + \frac{x \tan(\frac{x}{2})^2}{2} - \frac{x \tan(\frac{x}{2})^4}{2} - \frac{x \tan(\frac{x}{2})^6}{2}}{\left(1 + \tan\left(\frac{x}{2}\right)^2\right)^2 \left(c_1 \tan\left(\frac{x}{2}\right)^2 - 2c_2 \tan\left(\frac{x}{2}\right) - c_1\right)} + c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{1}{c_1 \cos(x)+c_2 \sin(x)}$ gives the final solution

$$
y = \frac{(2c_1 c_3 - x) \cos(x)}{2} + c_2 c_3 \sin(x)
$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{(2c_1c_3 - x)\cos(x)}{2} + c_2c_3\sin(x)$$

The constants can be merged to give

$$y = \frac{(2c_1 - x)\cos(x)}{2} + c_2\sin(x)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{(-2c_2\tan(1) + 1 - x)\cos(x)}{2} + c_2\sin(x)$$

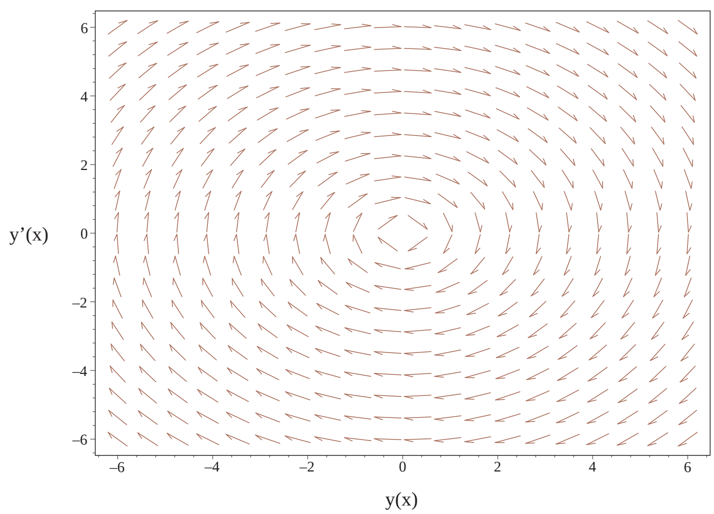


Figure 2.215: Slope field plot
$y'' + y = \sin(x)$

Maple step by step solution

Let's solve
$\left[\frac{d^2}{dx^2}y(x) + y(x) = \sin(x), y(1) = 0\right]$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2}y(x)$

- Characteristic polynomial of homogeneous ODE
  $r^2 + 1 = 0$
- Use quadratic formula to solve for r
  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial
  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the homogeneous ODE
  $y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE
  $y_2(x) = \sin(x)$
- General solution of the ODE
  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE
  $y(x) = C1 \cos(x) + C2 \sin(x) + y_p(x)$
- □ Find a particular solution $y_p(x)$ of the ODE
  - Use variation of parameters to find y_p here $f(x)$ is the forcing function
    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(x)\right]$
  - Wronskian of solutions of the homogeneous equation
    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$
  - Compute Wronskian
    $W(y_1(x), y_2(x)) = 1$
  - Substitute functions into equation for $y_p(x)$
    $y_p(x) = -\cos(x) \left(\int \sin(x)^2 dx\right) + \frac{\sin(x) \left(\int \sin(2x) dx\right)}{2}$
  - Compute integrals
    $y_p(x) = \frac{\sin(x)}{4} - \frac{x \cos(x)}{2}$
- Substitute particular solution into general solution to ODE
  $y(x) = C1 \cos(x) + C2 \sin(x) + \frac{\sin(x)}{4} - \frac{x \cos(x)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.116 (sec)
Leaf size : 31

```
dsolve([diff(diff(y(x),x),x)+y(x) = sin(x),
       op([y(1) = 0])],y(x),singsol=all)
```

$$y = \frac{\left(\left(-2c_2 - 1\right)\tan\left(1\right) - x + 1\right)\cos\left(x\right)}{2} + \frac{\sin\left(x\right)\left(2c_2 + 1\right)}{2}$$

Mathematica DSolve solution

Solving time : 0.02 (sec)
Leaf size : 18

```
DSolve[{D[y[x],{x,2}]+y[x]==Sin[x],{y[0] == 0}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{1}{2}x\cos(x) + c_2\sin(x)$$

2.3.7 problem 7

Internal problem ID [8541]
Book : Own collection of miscellaneous problems
Section : section 3.0
Problem number : 7
Date solved : `Thursday, December 12, 2024 at 09:27:24 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' + y = \sin(x)$$

With initial conditions

$$y'(1) = 0$$

Solved as second order linear constant coeff ode

Time used: 0.197 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\cos(x)\, x, \sin(x)\, x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \cos(x)\, x + A_2 \sin(x)\, x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)\, x}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\frac{\cos(x)\, x}{2}\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = -\frac{\cos(x)\, x}{2} + \left(c_2 \cot(1) + \frac{1}{2} - \frac{\cot(1)}{2}\right) \cos(x) + c_2 \sin(x)
$$

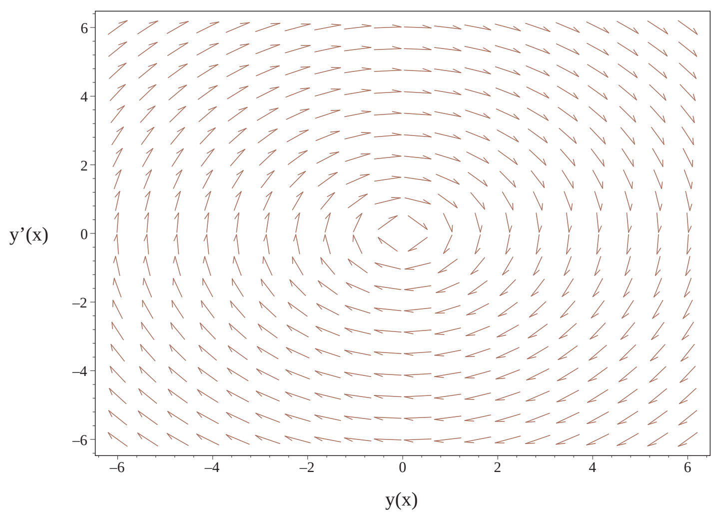


Figure 2.216: Slope field plot
$y'' + y = \sin(x)$

Solved as second order ode using Kovacic algorithm

Time used: 0.198 (sec)

Writing the ode as

$$
y'' + y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= 1
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.128: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)(\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\cos(x)\, x, \sin(x)\, x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \cos(x)\, x + A_2 \sin(x)\, x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)\, x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\frac{\cos(x)\, x}{2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{\cos(x)\, x}{2} + \left(c_2 \cot(1) + \frac{1}{2} - \frac{\cot(1)}{2}\right) \cos(x) + c_2 \sin(x)$$

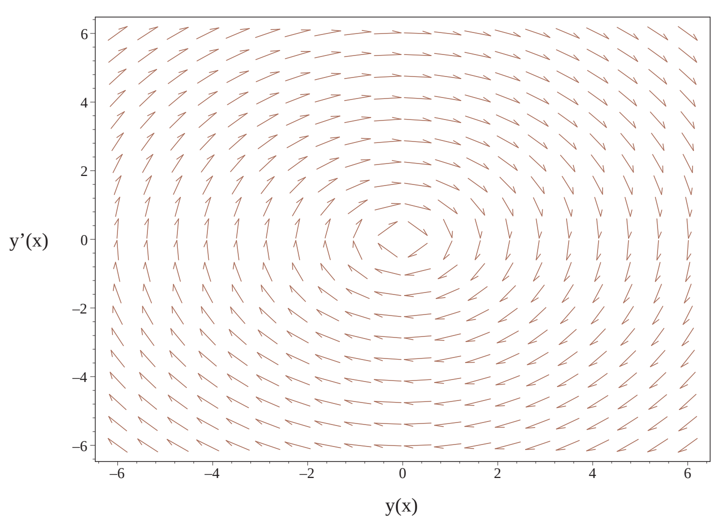


Figure 2.217: Slope field plot
$y'' + y = \sin(x)$

Solved as second order ode adjoint method

Time used: 2.250 (sec)

In normal form the ode

$$y'' + y = \sin(x) \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 1 \\ r(x) &= \sin(x) \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (0)' + (\xi(x)) &= 0 \\ \xi''(x) + \xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} + \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)} \\ &= \pm i \end{aligned}$$

Hence

$$\lambda_1 = +i$$
$$\lambda_2 = -i$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$\xi = e^{0}(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$\xi = c_1 \cos(x) + c_2 \sin(x)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}
\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)}
\end{aligned}$$

Or

$$y' - \frac{y(-c_1 \sin(x) + c_2 \cos(x))}{c_1 \cos(x) + c_2 \sin(x)} = \frac{-\frac{\cos(x)^2 c_1}{2} + c_2\left(-\frac{\cos(x)\sin(x)}{2} + \frac{x}{2}\right)}{c_1 \cos(x) + c_2 \sin(x)}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = -\frac{-c_1 \sin(x) + c_2 \cos(x)}{c_1 \cos(x) + c_2 \sin(x)}$$
$$p(x) = \frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{2c_2 \sin(x) + 2c_1 \cos(x)}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dx} \\ &= e^{\int -\frac{-c_1 \sin(x)+c_2 \cos(x)}{c_1 \cos(x)+c_2 \sin(x)}dx} \\ &= \frac{1}{c_1 \cos(x) + c_2 \sin(x)}\end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu p$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{2c_2 \sin(x) + 2c_1 \cos(x)}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{c_1 \cos(x) + c_2 \sin(x)}\right) = \left(\frac{1}{c_1 \cos(x) + c_2 \sin(x)}\right)\left(\frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{2c_2 \sin(x) + 2c_1 \cos(x)}\right)$$
$$\mathrm{d}\left(\frac{y}{c_1 \cos(x) + c_2 \sin(x)}\right) = \left(\frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{(2c_2 \sin(x) + 2c_1 \cos(x))(c_1 \cos(x) + c_2 \sin(x))}\right)\mathrm{d}x$$

Integrating gives

$$\begin{aligned}\frac{y}{c_1 \cos(x) + c_2 \sin(x)} &= \int \frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{(2c_2 \sin(x) + 2c_1 \cos(x))(c_1 \cos(x) + c_2 \sin(x))}\,dx \\ &= \frac{\frac{x}{2} + \frac{x \tan(\frac{x}{2})^2}{2} - \frac{x \tan(\frac{x}{2})^4}{2} - \frac{x \tan(\frac{x}{2})^6}{2}}{\left(1+\tan\left(\frac{x}{2}\right)^2\right)^2\left(c_1 \tan\left(\frac{x}{2}\right)^2 - 2c_2 \tan\left(\frac{x}{2}\right) - c_1\right)} + c_3\end{aligned}$$

Dividing throughout by the integrating factor $\frac{1}{c_1 \cos(x)+c_2 \sin(x)}$ gives the final solution

$$y = \frac{(2c_1c_3 - x)\cos(x)}{2} + c_2c_3 \sin(x)$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{(2c_1c_3 - x)\cos(x)}{2} + c_2c_3\sin(x)$$

The constants can be merged to give

$$y = \frac{(2c_1 - x)\cos(x)}{2} + c_2\sin(x)$$

Will add steps showing solving for IC soon.

<u>Summary of solutions found</u>

$$y = \frac{(2c_2\cot(1) + 1 - \cot(1) - x)\cos(x)}{2} + c_2\sin(x)$$

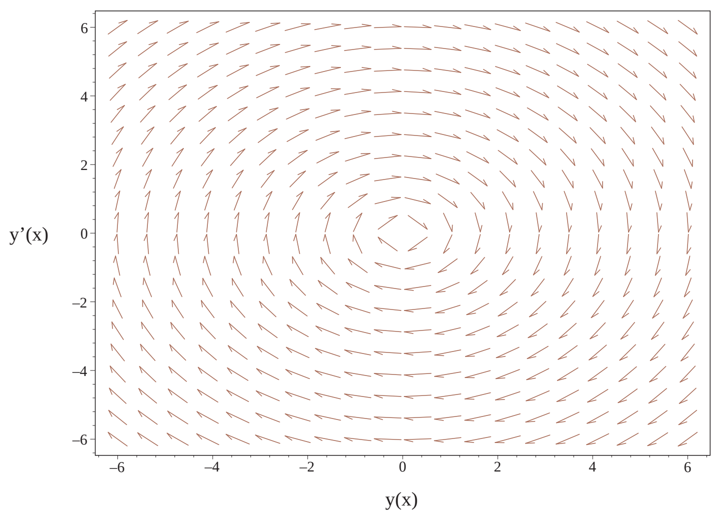


Figure 2.218: Slope field plot
$y'' + y = \sin(x)$

Maple step by step solution

Let's solve

$$\left[\frac{d^2}{dx^2}y(x) + y(x) = \sin(x), \left(\frac{d}{dx}y(x)\right)\bigg|_{\{x=1\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Characteristic polynomial of homogeneous ODE
  $r^2 + 1 = 0$
- Use quadratic formula to solve for r
  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial
  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the homogeneous ODE
  $y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE
  $y_2(x) = \sin(x)$
- General solution of the ODE
  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE
  $y(x) = C1 \cos(x) + C2 \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function
  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(x)\right]$
- Wronskian of solutions of the homogeneous equation
  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$
- Compute Wronskian
  $W(y_1(x), y_2(x)) = 1$
- Substitute functions into equation for $y_p(x)$
  $y_p(x) = -\cos(x)\left(\int \sin(x)^2 dx\right) + \frac{\sin(x)\left(\int \sin(2x) dx\right)}{2}$
- Compute integrals
  $y_p(x) = \frac{\sin(x)}{4} - \frac{x \cos(x)}{2}$

- Substitute particular solution into general solution to ODE
  $y(x) = C1 \cos(x) + C2 \sin(x) + \frac{\sin(x)}{4} - \frac{x \cos(x)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.080 (sec)
Leaf size : 28

```
dsolve([diff(diff(y(x),x),x)+y(x) = sin(x),
       op([D(y)(1) = 0])],y(x),singsol=all)
```

$$y = \frac{(2\cot(1)\, c_2 - x + 1)\cos(x)}{2} + \frac{\sin(x)\,(2c_2 + 1)}{2}$$

Mathematica DSolve solution

Solving time : 0.022 (sec)
Leaf size : 35

```
DSolve[{D[y[x],{x,2}]+y[x]==Sin[x],{Derivative[1][y][1] == 0}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}((1 - \tan(1) + 2c_1 \tan(1))\sin(x) - (x - 2c_1)\cos(x))$$

2.3.8 problem 8

Internal problem ID [8542]
Book : Own collection of miscellaneous problems
Section : section 3.0
Problem number : 8
Date solved : `Thursday, December 12, 2024 at 09:27:28 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' + y = \sin(x)$$

With initial conditions

$$y'(1) = 0$$
$$y(0) = 0$$

Solved as second order linear constant coeff ode

Time used: 0.214 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} + \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\cos(x)\, x, \sin(x)\, x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \cos(x)\, x + A_2 \sin(x)\, x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)\, x}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\frac{\cos(x)\, x}{2}\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = -\frac{\cos(x)\, x}{2} + \left(-\frac{\tan(1)}{2} + \frac{1}{2}\right)\sin(x)
$$

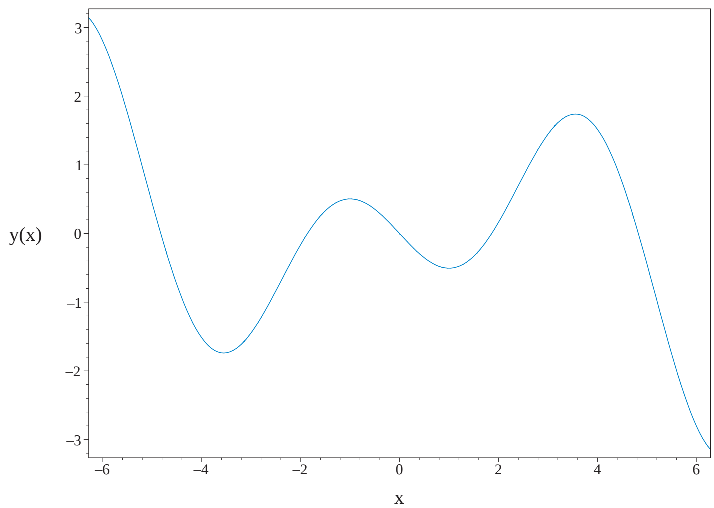


Figure 2.219: Solution plot
$y = -\frac{\cos(x)x}{2} + \left(-\frac{\tan(1)}{2} + \frac{1}{2}\right)\sin(x)$

Solved as second order ode using Kovacic algorithm

Time used: 0.122 (sec)

Writing the ode as

$$
y'' + y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= 1
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.130: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\
&= \cos(x)(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\cos(x)\, x, \sin(x)\, x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \cos(x)\, x + A_2 \sin(x)\, x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)\, x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\frac{\cos(x)\, x}{2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{\cos(x)\, x}{2} + \left(-\frac{\tan(1)}{2} + \frac{1}{2}\right)\sin(x)$$

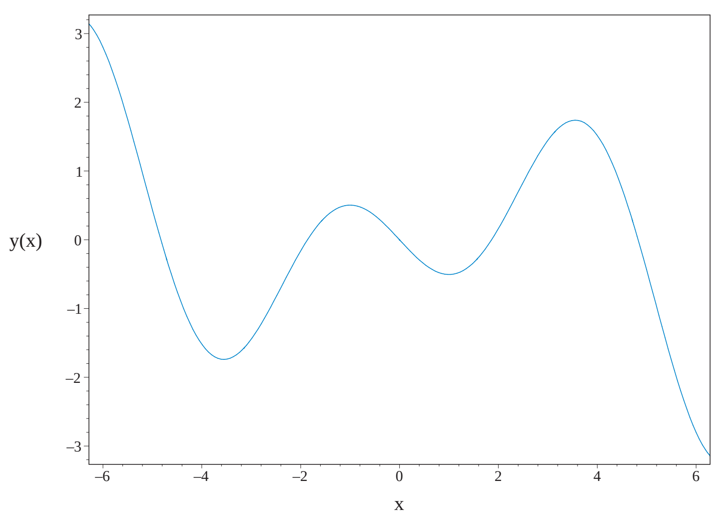


Figure 2.220: Solution plot
$y = -\frac{\cos(x)x}{2} + \left(-\frac{\tan(1)}{2} + \frac{1}{2}\right)\sin(x)$

Solved as second order ode adjoint method

Time used: 2.253 (sec)

In normal form the ode

$$y'' + y = \sin(x) \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 1 \\ r(x) &= \sin(x) \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (0)' + (\xi(x)) &= 0 \\ \xi''(x) + \xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)} \\ &= \pm i \end{aligned}$$

Hence

$$\lambda_1 = +i$$
$$\lambda_2 = -i$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$\xi = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$\xi = c_1 \cos(x) + c_2 \sin(x)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}
\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)}
\end{aligned}$$

Or

$$y' - \frac{y(-c_1 \sin(x) + c_2 \cos(x))}{c_1 \cos(x) + c_2 \sin(x)} = \frac{-\frac{\cos(x)^2 c_1}{2} + c_2\left(-\frac{\cos(x)\sin(x)}{2} + \frac{x}{2}\right)}{c_1 \cos(x) + c_2 \sin(x)}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= -\frac{-c_1 \sin(x) + c_2 \cos(x)}{c_1 \cos(x) + c_2 \sin(x)} \\
p(x) &= \frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{2c_2 \sin(x) + 2c_1 \cos(x)}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q \, dx} \\
&= e^{\int -\frac{-c_1 \sin(x) + c_2 \cos(x)}{c_1 \cos(x) + c_2 \sin(x)} dx} \\
&= \frac{1}{c_1 \cos(x) + c_2 \sin(x)}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu p \\
&\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{2c_2 \sin(x) + 2c_1 \cos(x)}\right) \\
&\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{c_1 \cos(x) + c_2 \sin(x)}\right) \\
&\quad = \left(\frac{1}{c_1 \cos(x) + c_2 \sin(x)}\right)\left(\frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{2c_2 \sin(x) + 2c_1 \cos(x)}\right) \\
&\mathrm{d}\left(\frac{y}{c_1 \cos(x) + c_2 \sin(x)}\right) = \left(\frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{(2c_2 \sin(x) + 2c_1 \cos(x))(c_1 \cos(x) + c_2 \sin(x))}\right) \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{c_1 \cos(x) + c_2 \sin(x)} &= \int \frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{(2c_2 \sin(x) + 2c_1 \cos(x))(c_1 \cos(x) + c_2 \sin(x))} \, dx \\
&= \frac{\frac{x}{2} + \frac{x \tan\left(\frac{x}{2}\right)^2}{2} - \frac{x \tan\left(\frac{x}{2}\right)^4}{2} - \frac{x \tan\left(\frac{x}{2}\right)^6}{2}}{\left(1 + \tan\left(\frac{x}{2}\right)^2\right)^2 \left(c_1 \tan\left(\frac{x}{2}\right)^2 - 2c_2 \tan\left(\frac{x}{2}\right) - c_1\right)} + c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{1}{c_1 \cos(x) + c_2 \sin(x)}$ gives the final solution

$$
y = \frac{(2c_1 c_3 - x) \cos(x)}{2} + c_2 c_3 \sin(x)
$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{(2c_1c_3 - x)\cos(x)}{2} + c_2c_3\sin(x)$$

The constants can be merged to give

$$y = \frac{(2c_1 - x)\cos(x)}{2} + c_2\sin(x)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{\cos(x)\, x}{2} + \left(-\frac{\tan(1)}{2} + \frac{1}{2}\right)\sin(x)$$

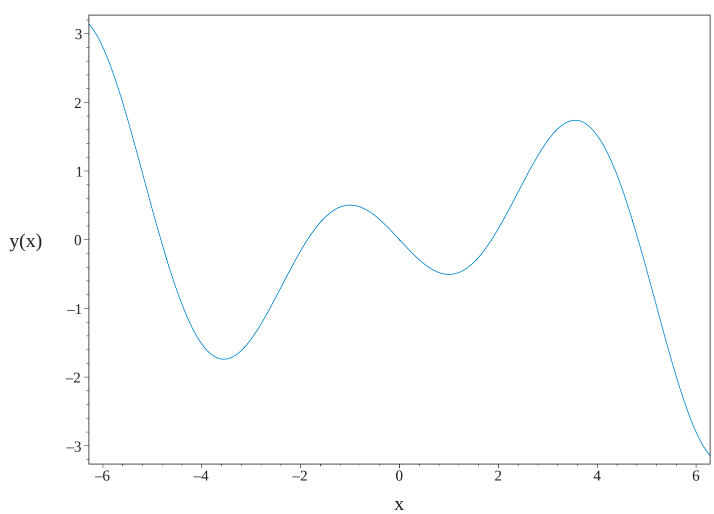


Figure 2.221: Solution plot
$y = -\frac{\cos(x)x}{2} + \left(-\frac{\tan(1)}{2} + \frac{1}{2}\right)\sin(x)$

Maple step by step solution

Let's solve

$$\left[\frac{d^2}{dx^2}y(x) + y(x) = \sin(x), \left(\frac{d}{dx}y(x)\right)\bigg|_{\{x=1\}} = 0, y(0) = 0\right]$$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Characteristic polynomial of homogeneous ODE
  $r^2+1=0$
- Use quadratic formula to solve for r
  $r=\frac{0\pm(\sqrt{-4})}{2}$
- Roots of the characteristic polynomial
  $r=(-\mathrm{I},\mathrm{I})$
- 1st solution of the homogeneous ODE
  $y_1(x)=\cos(x)$
- 2nd solution of the homogeneous ODE
  $y_2(x)=\sin(x)$
- General solution of the ODE
  $y(x)=C1y_1(x)+C2y_2(x)+y_p(x)$
- Substitute in solutions of the homogeneous ODE
  $y(x)=C1\cos(x)+C2\sin(x)+y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE
  - Use variation of parameters to find y_p here $f(x)$ is the forcing function
    $\left[y_p(x)=-y_1(x)\left(\int\frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int\frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right),f(x)=\sin(x)\right]$
  - Wronskian of solutions of the homogeneous equation
    $W(y_1(x),y_2(x))=\begin{bmatrix}\cos(x)&\sin(x)\\-\sin(x)&\cos(x)\end{bmatrix}$
  - Compute Wronskian
    $W(y_1(x),y_2(x))=1$
  - Substitute functions into equation for $y_p(x)$
    $y_p(x)=-\cos(x)\left(\int\sin(x)^2\,dx\right)+\frac{\sin(x)\left(\int\sin(2x)dx\right)}{2}$
  - Compute integrals
    $y_p(x)=\frac{\sin(x)}{4}-\frac{x\cos(x)}{2}$
- Substitute particular solution into general solution to ODE
  $y(x)=C1\cos(x)+C2\sin(x)+\frac{\sin(x)}{4}-\frac{x\cos(x)}{2}$

□ Check validity of solution $y(x)=\_C1\cos(x)+\_C2\sin(x)+\frac{\sin(x)}{4}-\frac{x\cos(x)}{2}$
  - Use initial condition $y(0)=0$
    $0=\_C1$
  - Compute derivative of the solution

$$\frac{d}{dx}y(x) = -\_C1\sin(x) + \_C2\cos(x) - \frac{\cos(x)}{4} + \frac{\sin(x)x}{2}$$

- Use the initial condition $\left(\frac{d}{dx}y(x)\right)\Big|_{\{x=1\}} = 0$

  $$0 = -\_C1\sin(1) + \_C2\cos(1) - \frac{\cos(1)}{4} + \frac{\sin(1)}{2}$$

- Solve for $\_C1$ and $\_C2$

  $$\left\{\_C1 = 0, \_C2 = -\frac{-\cos(1)+2\sin(1)}{4\cos(1)}\right\}$$

- Substitute constant values into general solution and simplify

  $$y(x) = \frac{(-\tan(1)+1)\sin(x)}{2} - \frac{x\cos(x)}{2}$$

- Solution to the IVP

  $$y(x) = \frac{(-\tan(1)+1)\sin(x)}{2} - \frac{x\cos(x)}{2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.048 (sec)
Leaf size : 20

```
dsolve([diff(diff(y(x),x),x)+y(x) = sin(x),
       op([D(y)(1) = 0, y(0) = 0])],y(x),singsol=all)
```

$$y = \frac{(-\tan(1)+1)\sin(x)}{2} - \frac{\cos(x)x}{2}$$

Mathematica DSolve solution

Solving time : 0.022 (sec)
Leaf size : 23

```
DSolve[{D[y[x],{x,2}]+y[x]==Sin[x],{Derivative[1][y][1] == 0,y[0]==0}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}(\sin(x) - x\cos(x) - \tan(1)\sin(x))$$

2.3.9 problem 9

Internal problem ID [8543]
Book : Own collection of miscellaneous problems
Section : section 3.0
Problem number : 9
Date solved : `Thursday, December 12, 2024 at 09:27:31 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' + y = \sin(x)$$

With initial conditions

$$y'(1) = 0$$
$$y(2) = 0$$

Solved as second order linear constant coeff ode

Time used: 0.282 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\cos(x)\, x, \sin(x)\, x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \cos(x)\, x + A_2 \sin(x)\, x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)\, x}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\frac{\cos(x)\, x}{2}\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = -\frac{\cos(x)\, x}{2} + \left(-\frac{\sin(2)}{2} + \frac{\cos(2)}{2} + \frac{1}{2}\right)\cos(x) + \frac{\cos(2)\,(\sin(1) + \cos(1))\sec(1)\sin(x)}{2}
$$

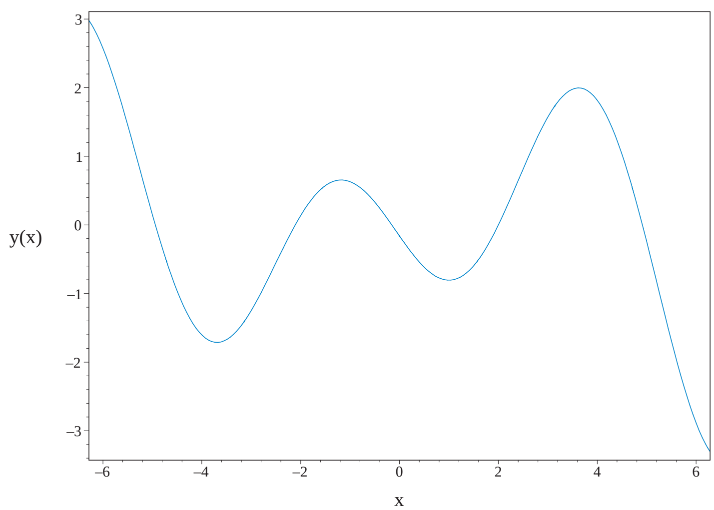


Figure 2.222: Solution plot
$y = -\frac{\cos(x)x}{2} + \left(-\frac{\sin(2)}{2} + \frac{\cos(2)}{2} + \frac{1}{2}\right)\cos(x) + \frac{\cos(2)(\sin(1)+\cos(1))\sec(1)\sin(x)}{2}$

Solved as second order ode using Kovacic algorithm

Time used: 0.115 (sec)

Writing the ode as

$$
y'' + y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.132: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\ &= \cos(x)\,(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\cos(x)\,x, \sin(x)\,x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \cos(x)\,x + A_2 \sin(x)\,x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)\,x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\frac{\cos(x)\,x}{2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{\cos(x)\,x}{2} + \left(-\frac{\sin(2)}{2} + \frac{\cos(2)}{2} + \frac{1}{2}\right)\cos(x) + \frac{\cos(2)\left(\sin(1) + \cos(1)\right)\sec(1)\sin(x)}{2}$$

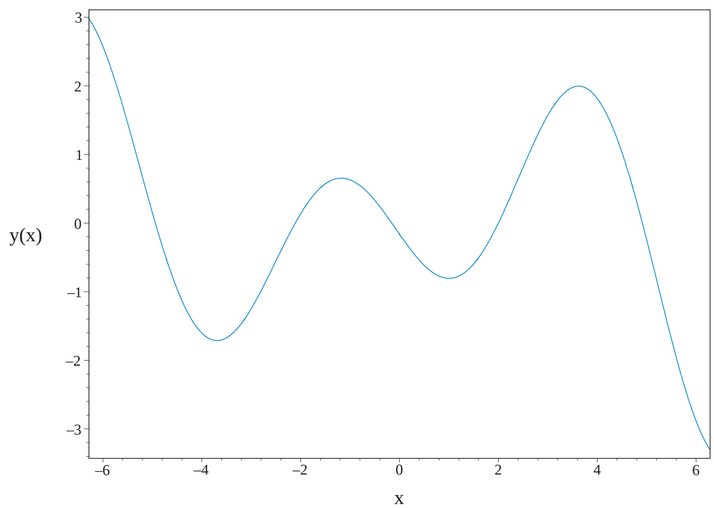


Figure 2.223: Solution plot
$y = -\frac{\cos(x)x}{2} + \left(-\frac{\sin(2)}{2} + \frac{\cos(2)}{2} + \frac{1}{2}\right)\cos(x) + \frac{\cos(2)(\sin(1)+\cos(1))\sec(1)\sin(x)}{2}$

Solved as second order ode adjoint method

Time used: 2.306 (sec)

In normal form the ode

$$y'' + y = \sin(x) \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}$$

Where

$$\begin{aligned}
p(x) &= 0\\
q(x) &= 1\\
r(x) &= \sin(x)
\end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned}\xi'' - (\xi\, p)' + \xi q &= 0\\ \xi'' - (0)' + (\xi(x)) &= 0\\ \xi''(x) + \xi(x) &= 0\end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} + \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$\xi = e^{0}(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$\xi = c_1 \cos(x) + c_2 \sin(x)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}
\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)}
\end{aligned}$$

Or

$$y' - \frac{y(-c_1 \sin(x) + c_2 \cos(x))}{c_1 \cos(x) + c_2 \sin(x)} = \frac{-\frac{\cos(x)^2 c_1}{2} + c_2\left(-\frac{\cos(x)\sin(x)}{2} + \frac{x}{2}\right)}{c_1 \cos(x) + c_2 \sin(x)}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned}
q(x) &= -\frac{-c_1 \sin(x) + c_2 \cos(x)}{c_1 \cos(x) + c_2 \sin(x)} \\
p(x) &= \frac{-c_2 \cos(x)\sin(x) - \cos(x)^2 c_1 + c_2 x}{2c_2 \sin(x) + 2c_1 \cos(x)}
\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int q\, dx} \\
&= e^{\int -\frac{-c_1 \sin(x) + c_2 \cos(x)}{c_1 \cos(x) + c_2 \sin(x)} dx} \\
&= \frac{1}{c_1 \cos(x) + c_2 \sin(x)}
\end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu p$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(\frac{-c_2 \cos(x)\sin(x) - \cos(x)^2 c_1 + c_2 x}{2c_2 \sin(x) + 2c_1 \cos(x)}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{c_1 \cos(x) + c_2 \sin(x)}\right) = \left(\frac{1}{c_1 \cos(x) + c_2 \sin(x)}\right)\left(\frac{-c_2 \cos(x)\sin(x) - \cos(x)^2 c_1 + c_2 x}{2c_2 \sin(x) + 2c_1 \cos(x)}\right)$$

$$\mathrm{d}\left(\frac{y}{c_1 \cos(x) + c_2 \sin(x)}\right) = \left(\frac{-c_2 \cos(x)\sin(x) - \cos(x)^2 c_1 + c_2 x}{(2c_2 \sin(x) + 2c_1 \cos(x))(c_1 \cos(x) + c_2 \sin(x))}\right)\mathrm{d}x$$

Integrating gives

$$\begin{aligned}\frac{y}{c_1 \cos(x) + c_2 \sin(x)} &= \int \frac{-c_2 \cos(x)\sin(x) - \cos(x)^2 c_1 + c_2 x}{(2c_2 \sin(x) + 2c_1 \cos(x))(c_1 \cos(x) + c_2 \sin(x))}\,dx \\ &= \frac{\frac{x}{2} + \frac{x\tan\left(\frac{x}{2}\right)^2}{2} - \frac{x\tan\left(\frac{x}{2}\right)^4}{2} - \frac{x\tan\left(\frac{x}{2}\right)^6}{2}}{\left(1+\tan\left(\frac{x}{2}\right)^2\right)^2\left(c_1\tan\left(\frac{x}{2}\right)^2 - 2c_2\tan\left(\frac{x}{2}\right) - c_1\right)} + c_3\end{aligned}$$

Dividing throughout by the integrating factor $\frac{1}{c_1 \cos(x) + c_2 \sin(x)}$ gives the final solution

$$y = \frac{(2c_1c_3 - x)\cos(x)}{2} + c_2c_3 \sin(x)$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{(2c_1c_3 - x)\cos(x)}{2} + c_2c_3 \sin(x)$$

The constants can be merged to give

$$y = \frac{(2c_1 - x)\cos(x)}{2} + c_2 \sin(x)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{(-\sin(2) + \cos(2) + 1 - x)\cos(x)}{2} + \frac{\cos(2)(\sin(1) + \cos(1))\sec(1)\sin(x)}{2}$$

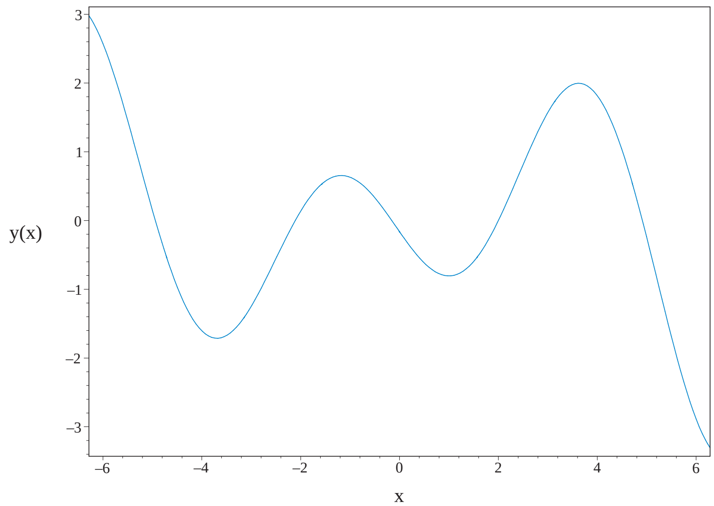


Figure 2.224: Solution plot
$y = \frac{(-\sin(2)+\cos(2)+1-x)\cos(x)}{2} + \frac{\cos(2)(\sin(1)+\cos(1))\sec(1)\sin(x)}{2}$

Maple step by step solution

Let's solve

$\left[\frac{d^2}{dx^2}y(x) + y(x) = \sin(x), \left(\frac{d}{dx}y(x)\right)\bigg|_{\{x=1\}} = 0, y(2) = 0\right]$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$
- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE

$y_2(x) = \sin(x)$

- General solution of the ODE
  $y(x) = C1y_1(x) + C2y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE
  $y(x) = C1\cos(x) + C2\sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function
  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \sin(x)\right]$
- ◦ Wronskian of solutions of the homogeneous equation
  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$
- ◦ Compute Wronskian
  $W(y_1(x), y_2(x)) = 1$
- ◦ Substitute functions into equation for $y_p(x)$
  $y_p(x) = -\cos(x)\left(\int \sin(x)^2 dx\right) + \frac{\sin(x)\left(\int \sin(2x)dx\right)}{2}$
- ◦ Compute integrals
  $y_p(x) = \frac{\sin(x)}{4} - \frac{x\cos(x)}{2}$

- Substitute particular solution into general solution to ODE
  $y(x) = C1\cos(x) + C2\sin(x) + \frac{\sin(x)}{4} - \frac{x\cos(x)}{2}$

□ Check validity of solution $y(x) = \_C1\cos(x) + \_C2\sin(x) + \frac{\sin(x)}{4} - \frac{x\cos(x)}{2}$

- ◦ Use initial condition $y(2) = 0$
  $0 = \_C1\cos(2) + \_C2\sin(2) + \frac{\sin(2)}{4} - \cos(2)$
- ◦ Compute derivative of the solution
  $\frac{d}{dx}y(x) = -\_C1\sin(x) + \_C2\cos(x) - \frac{\cos(x)}{4} + \frac{\sin(x)x}{2}$
- ◦ Use the initial condition $\left(\frac{d}{dx}y(x)\right)\Big|_{\{x=1\}} = 0$
  $0 = -\_C1\sin(1) + \_C2\cos(1) - \frac{\cos(1)}{4} + \frac{\sin(1)}{2}$
- ◦ Solve for $\_C1$ and $\_C2$
  $\left\{\_C1 = \frac{\sin(2)\sin(1)+2\cos(1)\cos(2)-\cos(1)\sin(2)}{2(\sin(2)\sin(1)+\cos(1)\cos(2))}, \_C2 = \frac{2\sin(1)\cos(2)-\sin(2)\sin(1)+\cos(1)\cos(2)}{4(\sin(2)\sin(1)+\cos(1)\cos(2))}\right\}$
- ◦ Substitute constant values into general solution and simplify
  $y(x) = \frac{(-x+\cos(2)-\sin(2)+1)\cos(x)}{2} + \frac{(\sin(2)-\tan(1)+\cos(2))\sin(x)}{2}$

- Solution to the IVP

$$y(x) = \frac{(-x+\cos(2)-\sin(2)+1)\cos(x)}{2} + \frac{(\sin(2)-\tan(1)+\cos(2))\sin(x)}{2}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.233 (sec)
Leaf size : 33

```
dsolve([diff(diff(y(x),x),x)+y(x) = sin(x),
       op([D(y)(1) = 0, y(2) = 0])],y(x),singsol=all)
```

$$y = \frac{(-x+\cos(2)-\sin(2)+1)\cos(x)}{2} + \frac{\sin(x)(\sin(2)-\tan(1)+\cos(2))}{2}$$

Mathematica DSolve solution

Solving time : 0.026 (sec)
Leaf size : 39

```
DSolve[{D[y[x],{x,2}]+y[x]==Sin[x],{Derivative[1][y][1] == 0,y[2]==0}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{4}(\sec(1)\sin(x)(-\sin(1)+\sin(3)+\cos(1)+\cos(3)) - 2\cos(x)(x-1+\sin(2)-\cos(2)))$$

2.3.10 problem 10

Internal problem ID [8544]
Book : Own collection of miscellaneous problems
Section : section 3.0
Problem number : 10
Date solved : `Thursday, December 12, 2024 at 09:27:35 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' + y = \sin(x)$$

With initial conditions

$$y'(1) = 0$$
$$y(0) = 0$$

Solved as second order linear constant coeff ode

Time used: 0.213 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} + \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\cos(x)\,x, \sin(x)\,x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 \cos(x)\,x + A_2 \sin(x)\,x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)\,x}{2}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\frac{\cos(x)\, x}{2}\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = -\frac{\cos(x)\, x}{2} + \left(\frac{1}{2} - \frac{\tan(1)}{2}\right)\sin(x)
$$

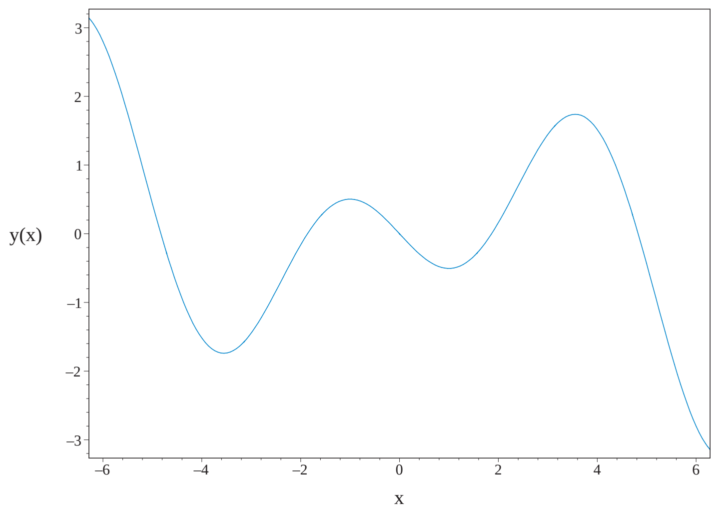


Figure 2.225: Solution plot
$y = -\frac{\cos(x)x}{2} + \left(\frac{1}{2} - \frac{\tan(1)}{2}\right)\sin(x)$

Solved as second order ode using Kovacic algorithm

Time used: 0.187 (sec)

Writing the ode as

$$
y'' + y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= 1
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.134: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1 \\ &= \cos(x)\end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)\,(\tan(x))\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)(\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \cos(x) + c_2 \sin(x)
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\sin(x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(x), \sin(x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\cos(x), \sin(x)\}
$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{\cos(x)\, x, \sin(x)\, x\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$
y_p = A_1 \cos(x)\, x + A_2 \sin(x)\, x
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(x)\, x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(-\frac{\cos(x)\, x}{2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{\cos(x)\, x}{2} + \left(\frac{1}{2} - \frac{\tan(1)}{2}\right) \sin(x)$$

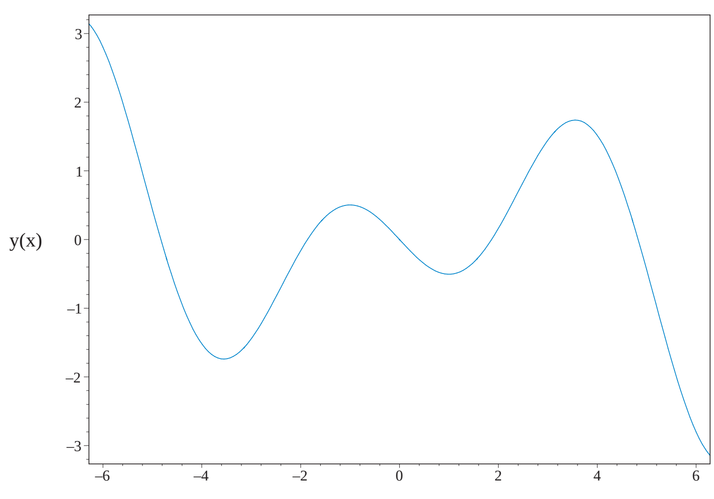


Figure 2.226: Solution plot
$y = -\frac{\cos(x)x}{2} + \left(\frac{1}{2} - \frac{\tan(1)}{2}\right) \sin(x)$

Solved as second order ode adjoint method

Time used: 2.188 (sec)

In normal form the ode

$$y'' + y = \sin(x) \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 1 \\ r(x) &= \sin(x) \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi^{''} - (\xi\, p)' + \xi q &= 0 \\ \xi^{''} - (0)' + (\xi(x)) &= 0 \\ \xi''(x) + \xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} + \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(1)} \\ &= \pm i \end{aligned}$$

Hence

$$\lambda_1 = +i$$
$$\lambda_2 = -i$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$\xi = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$\xi = e^{0}(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$\xi = c_1 \cos(x) + c_2 \sin(x)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y = \int \xi(x)\, r(x)\, dx$$
$$y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) = \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)}$$

Or

$$y' - \frac{y(-c_1 \sin(x) + c_2 \cos(x))}{c_1 \cos(x) + c_2 \sin(x)} = \frac{c_2\left(-\frac{\cos(x)\sin(x)}{2} + \frac{x}{2}\right) - \frac{\cos(x)^2 c_1}{2}}{c_1 \cos(x) + c_2 \sin(x)}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= -\frac{-c_1 \sin(x) + c_2 \cos(x)}{c_1 \cos(x) + c_2 \sin(x)} \\
p(x) &= \frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{2c_2 \sin(x) + 2c_1 \cos(x)}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q \, dx} \\
&= e^{\int -\frac{-c_1 \sin(x) + c_2 \cos(x)}{c_1 \cos(x) + c_2 \sin(x)} dx} \\
&= \frac{1}{c_1 \cos(x) + c_2 \sin(x)}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = \mu p \\
&\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu) \left(\frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{2c_2 \sin(x) + 2c_1 \cos(x)} \right) \\
&\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{y}{c_1 \cos(x) + c_2 \sin(x)} \right) \\
&\quad = \left(\frac{1}{c_1 \cos(x) + c_2 \sin(x)} \right) \left(\frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{2c_2 \sin(x) + 2c_1 \cos(x)} \right) \\
&\mathrm{d}\left(\frac{y}{c_1 \cos(x) + c_2 \sin(x)} \right) = \left(\frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{(2c_2 \sin(x) + 2c_1 \cos(x)) (c_1 \cos(x) + c_2 \sin(x))} \right) \mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{c_1 \cos(x) + c_2 \sin(x)} &= \int \frac{-c_2 \cos(x) \sin(x) - \cos(x)^2 c_1 + c_2 x}{(2c_2 \sin(x) + 2c_1 \cos(x)) (c_1 \cos(x) + c_2 \sin(x))} \, dx \\
&= \frac{\frac{x}{2} + \frac{x \tan\left(\frac{x}{2}\right)^2}{2} - \frac{x \tan\left(\frac{x}{2}\right)^4}{2} - \frac{x \tan\left(\frac{x}{2}\right)^6}{2}}{\left(1 + \tan\left(\frac{x}{2}\right)^2\right)^2 \left(c_1 \tan\left(\frac{x}{2}\right)^2 - 2c_2 \tan\left(\frac{x}{2}\right) - c_1\right)} + c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{1}{c_1 \cos(x) + c_2 \sin(x)}$ gives the final solution

$$
y = \frac{(2c_1 c_3 - x) \cos(x)}{2} + c_2 c_3 \sin(x)
$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{(2c_1c_3 - x)\cos(x)}{2} + c_2c_3\sin(x)$$

The constants can be merged to give

$$y = \frac{(2c_1 - x)\cos(x)}{2} + c_2\sin(x)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{\cos(x)\,x}{2} + \left(\frac{1}{2} - \frac{\tan(1)}{2}\right)\sin(x)$$

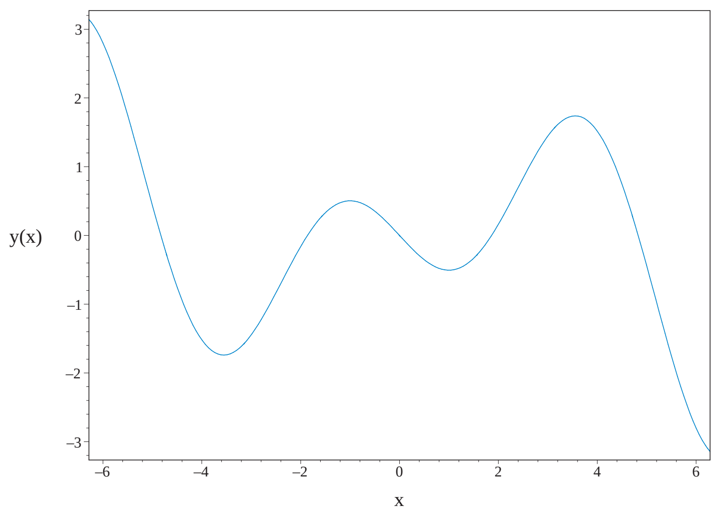


Figure 2.227: Solution plot
$y = -\frac{\cos(x)x}{2} + \left(\frac{1}{2} - \frac{\tan(1)}{2}\right)\sin(x)$

Maple step by step solution

Let's solve
$\left[\frac{d^2}{dx^2}y(x) + y(x) = \sin(x), \left(\frac{d}{dx}y(x)\right)\Big|_{\{x=1\}} = 0, y(0) = 0\right]$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Characteristic polynomial of homogeneous ODE
  $r^2 + 1 = 0$
- Use quadratic formula to solve for r
  $r = \frac{0\pm(\sqrt{-4})}{2}$
- Roots of the characteristic polynomial
  $r = (-\mathrm{I}, \mathrm{I})$
- 1st solution of the homogeneous ODE
  $y_1(x) = \cos(x)$
- 2nd solution of the homogeneous ODE
  $y_2(x) = \sin(x)$
- General solution of the ODE
  $y(x) = C1y_1(x) + C2y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE
  $y(x) = C1\cos(x) + C2\sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE
  - Use variation of parameters to find y_p here $f(x)$ is the forcing function
    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \sin(x)\right]$
  - Wronskian of solutions of the homogeneous equation
    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$
  - Compute Wronskian
    $W(y_1(x), y_2(x)) = 1$
  - Substitute functions into equation for $y_p(x)$
    $y_p(x) = -\cos(x)\left(\int \sin(x)^2 dx\right) + \frac{\sin(x)\left(\int \sin(2x)dx\right)}{2}$
  - Compute integrals
    $y_p(x) = \frac{\sin(x)}{4} - \frac{x\cos(x)}{2}$
- Substitute particular solution into general solution to ODE
  $y(x) = C1\cos(x) + C2\sin(x) + \frac{\sin(x)}{4} - \frac{x\cos(x)}{2}$

□ Check validity of solution $y(x) = \_C1\cos(x) + \_C2\sin(x) + \frac{\sin(x)}{4} - \frac{x\cos(x)}{2}$
  - Use initial condition $y(0) = 0$
    $0 = \_C1$
  - Compute derivative of the solution

$\frac{d}{dx}y(x) = -\_C1 \sin(x) + \_C2 \cos(x) - \frac{\cos(x)}{4} + \frac{\sin(x)x}{2}$

- Use the initial condition $\left(\frac{d}{dx}y(x)\right)\Big|_{\{x=1\}} = 0$

  $0 = -\_C1 \sin(1) + \_C2 \cos(1) - \frac{\cos(1)}{4} + \frac{\sin(1)}{2}$

- Solve for $\_C1$ and $\_C2$

  $\left\{\_C1 = 0, \_C2 = -\frac{-\cos(1)+2\sin(1)}{4\cos(1)}\right\}$

- Substitute constant values into general solution and simplify

  $y(x) = \frac{(-\tan(1)+1)\sin(x)}{2} - \frac{x\cos(x)}{2}$

- Solution to the IVP

  $y(x) = \frac{(-\tan(1)+1)\sin(x)}{2} - \frac{x\cos(x)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.011 (sec)
Leaf size : 20

```
dsolve([diff(diff(y(x),x),x)+y(x) = sin(x),
       op([D(y)(1) = 0, y(0) = 0])],y(x),singsol=all)
```

$$y = \frac{(-\tan(1)+1)\sin(x)}{2} - \frac{\cos(x)x}{2}$$

Mathematica DSolve solution

Solving time : 0.023 (sec)
Leaf size : 23

```
DSolve[{D[y[x],{x,2}]+y[x]==Sin[x],{Derivative[1][y][1] == 0,y[0]==0}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}(\sin(x) - x\cos(x) - \tan(1)\sin(x))$$

2.3.11 problem 11

Internal problem ID [8545]
Book : Own collection of miscellaneous problems
Section : section 3.0
Problem number : 11
Date solved : `Thursday, December 12, 2024 at 09:27:38 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' + y' + y = \sin(x)$$

With initial conditions

$$y'(1) = 0$$
$$y(2) = 0$$

Solved as second order linear constant coeff ode

Time used: 0.799 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 1, f(x) = \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + \lambda e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(1)} \\ &= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right), \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \sin(x) + A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\cos(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\mathrm{e}^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right)\right) + (-\cos(x)) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
\begin{aligned}
y = &-\cos(x) \\
&+ e^{-\frac{x}{2}} \left(\frac{e^{\frac{1}{2}} \left(\cos\left(\frac{\sqrt{3}}{2}\right) e^{\frac{1}{2}} \cos(2) \sqrt{3} - \sin\left(\frac{\sqrt{3}}{2}\right) e^{\frac{1}{2}} \cos(2) + 2 \sin(1) \sin\left(\sqrt{3}\right) \right) \cos\left(\frac{\sqrt{3}\, x}{2}\right)}{\sqrt{3}\, \cos\left(\frac{\sqrt{3}}{2}\right) + \sin\left(\frac{\sqrt{3}}{2}\right)} \right. \\
&\left. + \frac{e^{\frac{1}{2}} \left(\cos(2) \sin\left(\frac{\sqrt{3}}{2}\right) \sqrt{3}\, e^{\frac{1}{2}} + \cos(2) \cos\left(\frac{\sqrt{3}}{2}\right) e^{\frac{1}{2}} - 2 \cos\left(\sqrt{3}\right) \sin(1) \right) \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{\sqrt{3}\, \cos\left(\frac{\sqrt{3}}{2}\right) + \sin\left(\frac{\sqrt{3}}{2}\right)} \right)
\end{aligned}
$$

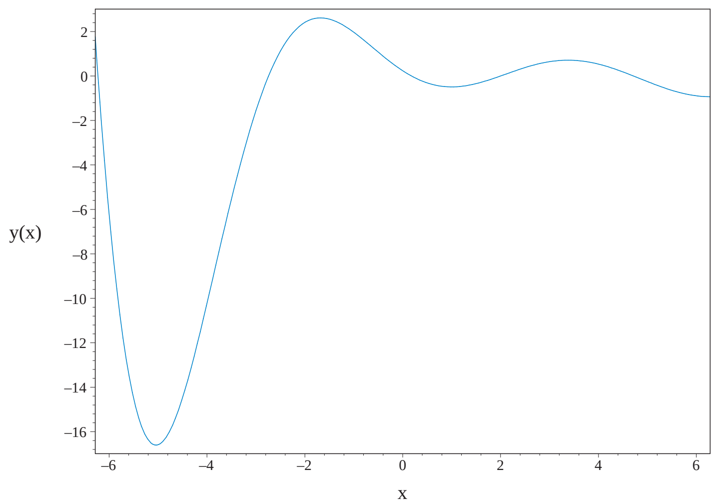


Figure 2.228: Solution plot

$$
y = -\cos(x) + e^{-\frac{x}{2}} \left(\frac{e^{\frac{1}{2}} \left(\cos\left(\frac{\sqrt{3}}{2}\right) e^{\frac{1}{2}} \cos(2) \sqrt{3} - \sin\left(\frac{\sqrt{3}}{2}\right) e^{\frac{1}{2}} \cos(2) + 2 \sin(1) \sin\left(\sqrt{3}\right) \right) \cos\left(\frac{\sqrt{3}\, x}{2}\right)}{\sqrt{3}\, \cos\left(\frac{\sqrt{3}}{2}\right) + \sin\left(\frac{\sqrt{3}}{2}\right)} + \frac{e^{\frac{1}{2}} \left(\cos(2) \sin\left(\frac{\sqrt{3}}{2}\right) \sqrt{3}\, e^{\frac{1}{2}} + \cos(2) \cdots \right.}{\sqrt{3}\, \cos(\cdots}
$$

Solved as second order ode using Kovacic algorithm

Time used: 0.750 (sec)

Writing the ode as

$$
y'' + y' + y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 1 \\
C &= 1
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.136: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{-x}}{\left(y_1\right)^2}\, dx \\
&= y_1 \left(\frac{2\sqrt{3}\, \tan\left(\frac{\sqrt{3}\, x}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) \left(\frac{2\sqrt{3}\, \tan\left(\frac{\sqrt{3}\, x}{2}\right)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1\, \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2c_2\, \mathrm{e}^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\sin(x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(x), \sin(x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right), \frac{2\, e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \sin(x) + A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\cos(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\, e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2c_2\, e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3}\right) + (-\cos(x)) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$\begin{aligned} y &= \frac{e^{\frac{1}{2}} \left(-\cos(2) \sin\left(\frac{\sqrt{3}}{2}\right) \sqrt{3}\, e^{\frac{1}{2}} + 3\cos(2) \cos\left(\frac{\sqrt{3}}{2}\right) e^{\frac{1}{2}} + 2\sin(1) \sqrt{3}\, \sin\left(\sqrt{3}\right)\right) e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)}{\sqrt{3}\, \sin\left(\frac{\sqrt{3}}{2}\right) + 3\cos\left(\frac{\sqrt{3}}{2}\right)} \\ &\quad + \frac{2\, e^{\frac{1}{2}} \left(\cos(2) \sin\left(\frac{\sqrt{3}}{2}\right) \sqrt{3}\, e^{\frac{1}{2}} + \cos(2) \cos\left(\frac{\sqrt{3}}{2}\right) e^{\frac{1}{2}} - 2\cos\left(\sqrt{3}\right) \sin(1)\right) e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{2\sqrt{3}\, \sin\left(\frac{\sqrt{3}}{2}\right) + 6\cos\left(\frac{\sqrt{3}}{2}\right)} \\ &\quad - \cos(x) \end{aligned}$$

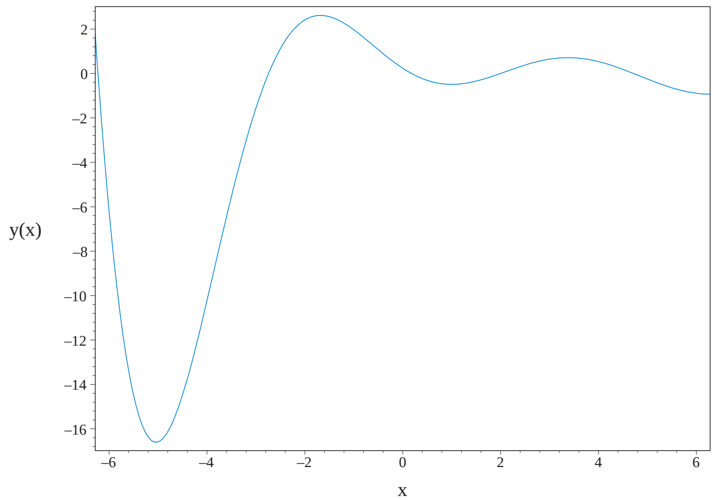


Figure 2.229: Solution plot

$$y = \frac{e^{\frac{1}{2}}\left(-\cos(2)\sin\left(\frac{\sqrt{3}}{2}\right)\sqrt{3}\,e^{\frac{1}{2}}+3\cos(2)\cos\left(\frac{\sqrt{3}}{2}\right)e^{\frac{1}{2}}+2\sin(1)\sqrt{3}\,\sin\left(\sqrt{3}\right)\right)e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{\sqrt{3}\,\sin\left(\frac{\sqrt{3}}{2}\right)+3\cos\left(\frac{\sqrt{3}}{2}\right)} + \frac{2\,e^{\frac{1}{2}}\left(\cos(2)\sin\left(\frac{\sqrt{3}}{2}\right)\sqrt{3}\,e^{\frac{1}{2}}+\cos(2)\cos\left(\frac{\sqrt{3}}{2}\right)e^{\frac{1}{2}}-2\cos\left(\sqrt{3}\right)\sin(1)\right)e^{-\frac{x}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2\sqrt{3}\,\sin\left(\frac{\sqrt{3}}{2}\right)+6\cos\left(\frac{\sqrt{3}}{2}\right)} - \cos(x)$$

Maple step by step solution

Let's solve

$$\left[\frac{d^2}{dx^2}y(x)+\frac{d}{dx}y(x)+y(x)=\sin(x),\left(\frac{d}{dx}y(x)\right)\bigg|_{\{x=1\}}=0,y(2)=0\right]$$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Characteristic polynomial of homogeneous ODE

  $r^2+r+1=0$

- Use quadratic formula to solve for r

  $r=\frac{(-1)\pm(\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r=\left(-\frac{1}{2}-\frac{I\sqrt{3}}{2},-\frac{1}{2}+\frac{I\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x)=e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x)=e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)$

- General solution of the ODE
  $y(x) = C1y_1(x) + C2y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE
  $y(x) = C1\,e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) + C2\,e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function
  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \sin(x)\right]$
- ○ Wronskian of solutions of the homogeneous equation
  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) & e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right) \\ -\frac{e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} & -\frac{e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \end{bmatrix}$
- ○ Compute Wronskian
  $W(y_1(x), y_2(x)) = \frac{\sqrt{3}\,e^{-x}}{2}$
- ○ Substitute functions into equation for $y_p(x)$
  $y_p(x) = -\frac{2\sqrt{3}\,e^{-\frac{x}{2}}\left(\cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int e^{\frac{x}{2}}\sin(x)\sin\left(\frac{\sqrt{3}\,x}{2}\right)dx\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int e^{\frac{x}{2}}\sin(x)\cos\left(\frac{\sqrt{3}\,x}{2}\right)dx\right)\right)}{3}$
- ○ Compute integrals
  $y_p(x) = -\cos(x)$

- Substitute particular solution into general solution to ODE
  $y(x) = C1\,e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) + C2\,e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right) - \cos(x)$

□ Check validity of solution $y(x) = \_C1e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) + \_C2e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right) - \cos(x)$

- ○ Use initial condition $y(2) = 0$
  $0 = \_C1\,e^{-1}\cos(\sqrt{3}) + \_C2\,e^{-1}\sin(\sqrt{3}) - \cos(2)$
- ○ Compute derivative of the solution
  $\frac{d}{dx}y(x) = -\frac{\_C1\,e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\_C1\,e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} - \frac{\_C2e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\_C2e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} + \mathrm{si}$
- ○ Use the initial condition $\left(\frac{d}{dx}y(x)\right)\Big|_{\{x=1\}} = 0$
  $0 = -\frac{\_C1\,e^{-\frac{1}{2}}\cos\left(\frac{\sqrt{3}}{2}\right)}{2} - \frac{\_C1\,e^{-\frac{1}{2}}\sin\left(\frac{\sqrt{3}}{2}\right)\sqrt{3}}{2} - \frac{\_C2e^{-\frac{1}{2}}\sin\left(\frac{\sqrt{3}}{2}\right)}{2} + \frac{\_C2e^{-\frac{1}{2}}\cos\left(\frac{\sqrt{3}}{2}\right)\sqrt{3}}{2} + \sin(1)$
- ○ Solve for $\_C1$ and $\_C2$
  $\left\{\_C1 = \frac{\sqrt{3}\,e^{-\frac{1}{2}}\cos\left(\frac{\sqrt{3}}{2}\right)\cos(2) + 2\sin(1)e^{-1}\sin\left(\sqrt{3}\right) - e^{-\frac{1}{2}}\sin\left(\frac{\sqrt{3}}{2}\right)\cos(2)}{e^{-\frac{1}{2}}\left(\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}\right)\cos\left(\sqrt{3}\right) + \sqrt{3}\sin\left(\sqrt{3}\right)\sin\left(\frac{\sqrt{3}}{2}\right) - \sin\left(\frac{\sqrt{3}}{2}\right)\cos\left(\sqrt{3}\right) + \sin\left(\sqrt{3}\right)\cos\left(\frac{\sqrt{3}}{2}\right)\right)e^{-1}}, \_C2 = \frac{}{e^{-\frac{1}{2}}\left(\sqrt{3}\right.}\right.$

- Substitute constant values into general solution and simplify

$$y(x) = \frac{2\sin(1)\left(\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sin\left(\sqrt{3}\right)-\sin\left(\frac{\sqrt{3}\,x}{2}\right)\cos\left(\sqrt{3}\right)\right)\mathrm{e}^{-\frac{x}{2}+\frac{1}{2}}-\left(\left(-\sqrt{3}\,\cos\left(\frac{\sqrt{3}}{2}\right)+\sin\left(\frac{\sqrt{3}}{2}\right)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(\sqrt{3}\,\mathrm{si}\right.\right.}{\sin\left(\frac{\sqrt{3}}{2}\right)+\sqrt{3}\,\cos\left(\frac{\sqrt{3}}{2}\right)}$$

- Solution to the IVP

$$y(x) = \frac{2\sin(1)\left(\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sin\left(\sqrt{3}\right)-\sin\left(\frac{\sqrt{3}\,x}{2}\right)\cos\left(\sqrt{3}\right)\right)\mathrm{e}^{-\frac{x}{2}+\frac{1}{2}}-\left(\left(-\sqrt{3}\,\cos\left(\frac{\sqrt{3}}{2}\right)+\sin\left(\frac{\sqrt{3}}{2}\right)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(\sqrt{3}\,\mathrm{si}\right.\right.}{\sin\left(\frac{\sqrt{3}}{2}\right)+\sqrt{3}\,\cos\left(\frac{\sqrt{3}}{2}\right)}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : `1.039` (sec)
Leaf size : 145

```
dsolve([diff(diff(y(x),x),x)+diff(y(x),x)+y(x) = sin(x),
       op([D(y)(1) = 0, y(2) = 0])],y(x),singsol=all)
```

$$y = \frac{2\sin(1)\left(\sin\left(\sqrt{3}\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\sin\left(\frac{\sqrt{3}\,x}{2}\right)\cos\left(\sqrt{3}\right)\right)\mathrm{e}^{-\frac{x}{2}+\frac{1}{2}}-\cos(2)\left(\left(-\sqrt{3}\,\cos\left(\frac{\sqrt{3}}{2}\right)+\sin\left(\frac{\sqrt{3}}{2}\right)\right)\mathrm{co}\right.\right.}{\sin\left(\frac{\sqrt{3}}{2}\right)+\sqrt{3}\,\mathrm{c}}$$

Mathematica DSolve solution

Solving time : 0.892 (sec)
Leaf size : 12765

```
DSolve[{D[y[x],{x,3}]+D[y[x],x]+y[x]==Sin[x],{Derivative[1][y][1] == 0,y[2]==0}},
       y[x],x,IncludeSingularSolutions->True]
```

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2.3.12 problem 12

Internal problem ID [8546]
Book : Own collection of miscellaneous problems
Section : section 3.0
Problem number : 12
Date solved : `Thursday, December 12, 2024 at 09:28:23 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' + y' + y = \sin(x)$$

With initial conditions

$$y'(1) = 0$$

Solved as second order linear constant coeff ode

Time used: 0.300 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 1, f(x) = \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + \lambda\, e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(1)}\\ &= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2}\\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2}\\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right), e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \sin(x) + A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\cos(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right)\right) + (-\cos(x)) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\cos(x) + e^{-\frac{x}{2}} \left(\frac{\left(c_2\sqrt{3}\,\cos\left(\frac{\sqrt{3}}{2}\right) + 2\sin(1)\,e^{\frac{1}{2}} - c_2\sin\left(\frac{\sqrt{3}}{2}\right)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{\sqrt{3}\,\sin\left(\frac{\sqrt{3}}{2}\right) + \cos\left(\frac{\sqrt{3}}{2}\right)} + c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

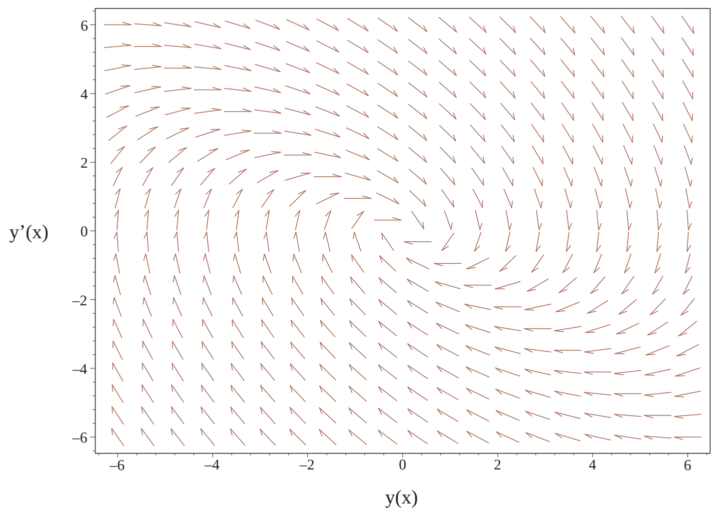

Figure 2.230: Slope field plot
$y'' + y' + y = \sin(x)$

Solved as second order ode using Kovacic algorithm

Time used: 0.344 (sec)

Writing the ode as

$$y'' + y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.138: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{2\sqrt{3}\, \tan\left(\frac{\sqrt{3}\, x}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) \left(\frac{2\sqrt{3}\, \tan\left(\frac{\sqrt{3}\, x}{2}\right)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1\, \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2c_2\, \mathrm{e}^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\sin(x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(x), \sin(x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right), \frac{2\,e^{-\frac{x}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \sin(x) + A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\cos(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\,e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + \frac{2c_2\,e^{-\frac{x}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right) + (-\cos(x)) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$\begin{aligned} y = {} & \frac{\left(-2c_2\sqrt{3}\,\sin\left(\frac{\sqrt{3}}{2}\right) + 6c_2 \cos\left(\frac{\sqrt{3}}{2}\right) + 6\sin(1)\,e^{\frac{1}{2}}\right) e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{3\sqrt{3}\,\sin\left(\frac{\sqrt{3}}{2}\right) + 3\cos\left(\frac{\sqrt{3}}{2}\right)} \\ & + \frac{2c_2\,e^{-\frac{x}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{3} - \cos(x) \end{aligned}$$

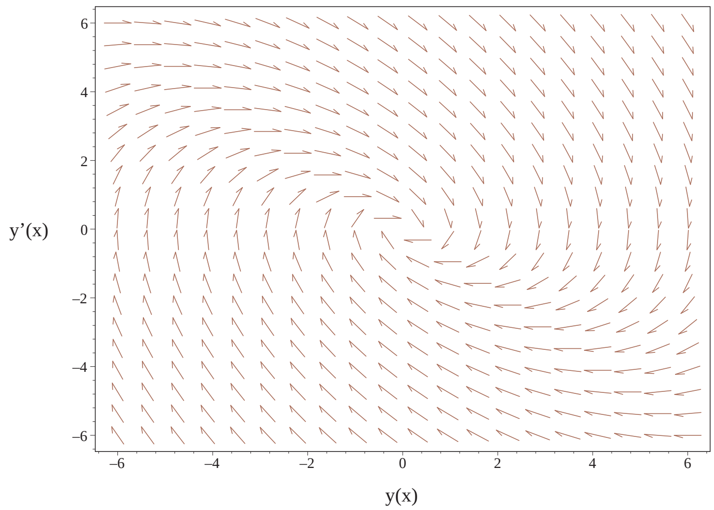

Figure 2.231: Slope field plot
$y'' + y' + y = \sin(x)$

Maple step by step solution

Let's solve

$\left[\frac{d^2}{dx^2}y(x) + \frac{d}{dx}y(x) + y(x) = \sin(x), \left(\frac{d}{dx}y(x)\right)\Big|_{\{x=1\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, -\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)$

- General solution of the ODE

  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

$$y(x) = C1\,e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + C2\,e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) + y_p(x)$$

□ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $$\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(x)\right]$$

- Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) & e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) \\ -\frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} & -\frac{e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \end{bmatrix}$$

- Compute Wronskian

  $$W(y_1(x), y_2(x)) = \frac{\sqrt{3}\,e^{-x}}{2}$$

- Substitute functions into equation for $y_p(x)$

  $$y_p(x) = -\frac{2\sqrt{3}\,e^{-\frac{x}{2}}\left(\cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int e^{\frac{x}{2}} \sin(x) \sin\left(\frac{\sqrt{3}\,x}{2}\right) dx\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int e^{\frac{x}{2}} \sin(x) \cos\left(\frac{\sqrt{3}\,x}{2}\right) dx\right)\right)}{3}$$

- Compute integrals

  $$y_p(x) = -\cos(x)$$

• Substitute particular solution into general solution to ODE

$$y(x) = C1\,e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + C2\,e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) - \cos(x)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : `0.343` (sec)
Leaf size : 110

```
dsolve([diff(diff(y(x),x),x)+diff(y(x),x)+y(x) = sin(x),
       op([D(y)(1) = 0])],y(x),singsol=all)
```

$$y = \frac{2\cos\left(\frac{\sqrt{3}\,x}{2}\right) \mathrm{e}^{-\frac{x}{2}+\frac{1}{2}} \sin\left(1\right) + \left(\sqrt{3}\,\cos\left(\frac{\sqrt{3}}{2}\right) - \sin\left(\frac{\sqrt{3}}{2}\right)\right) \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) c_2 + \left(\sqrt{3}\,\sin\left(\frac{\sqrt{3}}{2}\right) + \cos\left(\frac{\sqrt{3}}{2}\right)\right.}{\sqrt{3}\,\sin\left(\frac{\sqrt{3}}{2}\right) + \cos\left(\frac{\sqrt{3}}{2}\right)}$$

Mathematica DSolve solution

Solving time : `0.425` (sec)
Leaf size : 4176

```
DSolve[{D[y[x],{x,3}]+D[y[x],x]+y[x]==Sin[x],{Derivative[1][y][1] == 0}},
       y[x],x,IncludeSingularSolutions->True]
```

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2.3.13 problem 13

Internal problem ID [8547]
Book : Own collection of miscellaneous problems
Section : section 3.0
Problem number : 13
Date solved : `Thursday, December 12, 2024 at 09:29:05 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' + y' + y = \sin(x)$$

With initial conditions

$$y'(1) = 0$$
$$y(2) = 0$$

Solved as second order linear constant coeff ode

Time used: 0.782 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 1, f(x) = \sin(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + \lambda e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(1)}\\ &= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right), e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \sin(x) + A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\cos(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right)\right) + (-\cos(x)) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
\begin{aligned}
y = {} & -\cos\left(x\right) \\
& + e^{-\frac{x}{2}} \left(\frac{e^{\frac{1}{2}} \left(\cos\left(\frac{\sqrt{3}}{2}\right) e^{\frac{1}{2}} \cos\left(2\right) \sqrt{3} - \sin\left(\frac{\sqrt{3}}{2}\right) e^{\frac{1}{2}} \cos\left(2\right) + 2\sin\left(1\right) \sin\left(\sqrt{3}\right)\right) \cos\left(\frac{\sqrt{3}\, x}{2}\right)}{\sqrt{3}\, \cos\left(\frac{\sqrt{3}}{2}\right) + \sin\left(\frac{\sqrt{3}}{2}\right)} \right. \\
& \left. + \frac{e^{\frac{1}{2}} \left(\sin\left(\frac{\sqrt{3}}{2}\right) \sqrt{3}\, \cos\left(2\right) e^{\frac{1}{2}} + \cos\left(\frac{\sqrt{3}}{2}\right) \cos\left(2\right) e^{\frac{1}{2}} - 2\cos\left(\sqrt{3}\right) \sin\left(1\right)\right) \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{\sqrt{3}\, \cos\left(\frac{\sqrt{3}}{2}\right) + \sin\left(\frac{\sqrt{3}}{2}\right)} \right)
\end{aligned}
$$

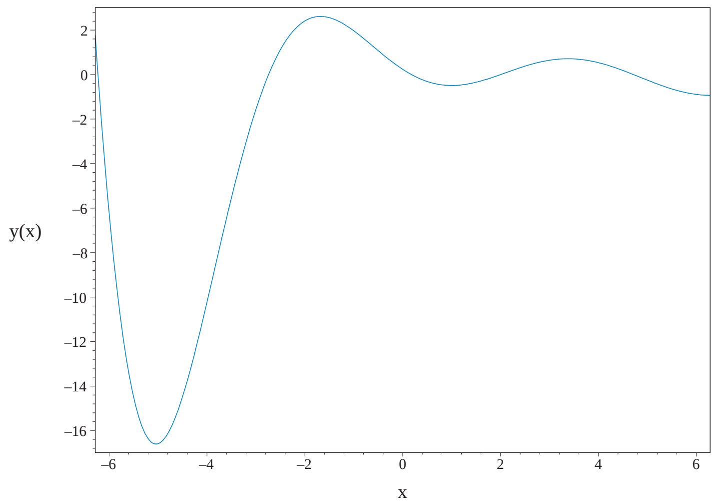

Figure 2.232: Solution plot

$$
y = -\cos\left(x\right) + e^{-\frac{x}{2}} \left(\frac{e^{\frac{1}{2}} \left(\cos\left(\frac{\sqrt{3}}{2}\right) e^{\frac{1}{2}} \cos(2)\sqrt{3} - \sin\left(\frac{\sqrt{3}}{2}\right) e^{\frac{1}{2}} \cos(2) + 2\sin(1)\sin\left(\sqrt{3}\right)\right) \cos\left(\frac{\sqrt{3}\, x}{2}\right)}{\sqrt{3}\, \cos\left(\frac{\sqrt{3}}{2}\right) + \sin\left(\frac{\sqrt{3}}{2}\right)} + \frac{e^{\frac{1}{2}} \left(\sin\left(\frac{\sqrt{3}}{2}\right) \sqrt{3}\, \cos(2) e^{\frac{1}{2}} + \cos\left(\frac{\sqrt{3}}{2}\right. \cdots}{\sqrt{3}\, \cos\left(\cdots \right.} \right.
$$

Solved as second order ode using Kovacic algorithm

Time used: 0.733 (sec)

Writing the ode as

$$
y'' + y' + y = 0 \tag{1}
$$

$$
Ay'' + By' + Cy = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 1 \qquad (3) \\
C &= 1
\end{aligned}
$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.140: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + \frac{2c_2\,\mathrm{e}^{-\frac{x}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right), \frac{2\, e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(x) + A_2 \sin(x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-A_1 \sin(x) + A_2 \cos(x) = \sin(x)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 0]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\cos(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\, e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) + \frac{2 c_2\, e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{3} \right) + (-\cos(x)) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$\begin{aligned} y &= \frac{e^{\frac{1}{2}} \left(-\sin\left(\frac{\sqrt{3}}{2}\right) \sqrt{3}\, \cos(2)\, e^{\frac{1}{2}} + 2 \sin\left(\sqrt{3}\right) \sin(1) \sqrt{3} + 3 \cos\left(\frac{\sqrt{3}}{2}\right) \cos(2)\, e^{\frac{1}{2}}\right) e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)}{\sqrt{3}\, \sin\left(\frac{\sqrt{3}}{2}\right) + 3 \cos\left(\frac{\sqrt{3}}{2}\right)} \\ &\quad + \frac{2\, e^{\frac{1}{2}} \left(\sin\left(\frac{\sqrt{3}}{2}\right) \sqrt{3}\, \cos(2)\, e^{\frac{1}{2}} + \cos\left(\frac{\sqrt{3}}{2}\right) \cos(2)\, e^{\frac{1}{2}} - 2 \cos\left(\sqrt{3}\right) \sin(1)\right) e^{-\frac{x}{2}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{2\sqrt{3}\, \sin\left(\frac{\sqrt{3}}{2}\right) + 6 \cos\left(\frac{\sqrt{3}}{2}\right)} \\ &\quad - \cos(x) \end{aligned}$$

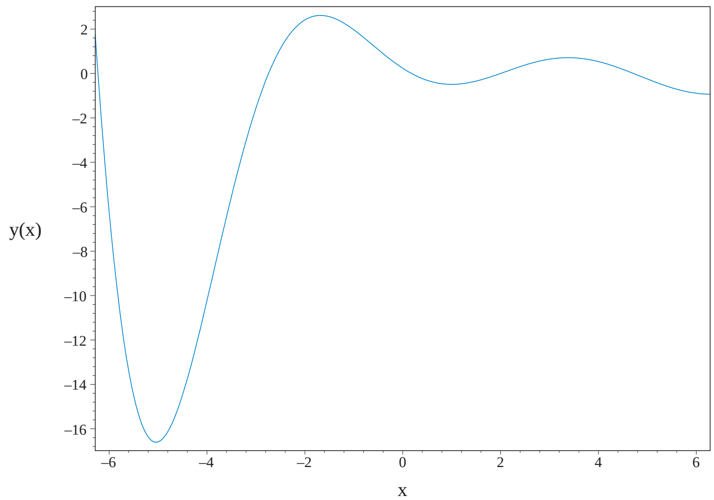

Figure 2.233: Solution plot

$$y = \frac{e^{\frac{1}{2}}\left(-\sin\left(\frac{\sqrt{3}}{2}\right)\sqrt{3}\,\cos(2)e^{\frac{1}{2}}+2\sin\left(\sqrt{3}\right)\sin(1)\sqrt{3}+3\cos\left(\frac{\sqrt{3}}{2}\right)\cos(2)e^{\frac{1}{2}}\right)e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{\sqrt{3}\,\sin\left(\frac{\sqrt{3}}{2}\right)+3\cos\left(\frac{\sqrt{3}}{2}\right)} + \frac{2\,e^{\frac{1}{2}}\left(\sin\left(\frac{\sqrt{3}}{2}\right)\sqrt{3}\,\cos(2)e^{\frac{1}{2}}+\cos\left(\frac{\sqrt{3}}{2}\right)\cos(2)e^{\frac{1}{2}}-2\cos\left(\sqrt{3}\right)\sin(1)\right)e^{-\frac{x}{2}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2\sqrt{3}\,\sin\left(\frac{\sqrt{3}}{2}\right)+6\cos\left(\frac{\sqrt{3}}{2}\right)} - \cos(x)$$

Maple step by step solution

Let's solve

$$\left[\frac{d^2}{dx^2}y(x)+\frac{d}{dx}y(x)+y(x)=\sin(x), \left(\frac{d}{dx}y(x)\right)\bigg|_{\{x=1\}}=0, y(2)=0\right]$$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Characteristic polynomial of homogeneous ODE

  $r^2+r+1=0$

- Use quadratic formula to solve for r

  $r=\frac{(-1)\pm\left(\sqrt{-3}\right)}{2}$

- Roots of the characteristic polynomial

  $r=\left(-\frac{1}{2}-\frac{\mathrm{I}\sqrt{3}}{2}, -\frac{1}{2}+\frac{\mathrm{I}\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x)=e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x)=e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)$

- General solution of the ODE
  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE
  $y(x) = C1\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + C2\,\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) + y_p(x)$
- □ Find a particular solution $y_p(x)$ of the ODE
  - Use variation of parameters to find y_p here $f(x)$ is the forcing function
    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \sin(x)\right]$
  - Wronskian of solutions of the homogeneous equation
    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) & \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) \\ -\frac{\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} & -\frac{\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \end{bmatrix}$
  - Compute Wronskian
    $W(y_1(x), y_2(x)) = \frac{\sqrt{3}\,\mathrm{e}^{-x}}{2}$
  - Substitute functions into equation for $y_p(x)$
    $y_p(x) = -\frac{2\sqrt{3}\,\mathrm{e}^{-\frac{x}{2}}\left(\cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int \mathrm{e}^{\frac{x}{2}} \sin(x) \sin\left(\frac{\sqrt{3}\,x}{2}\right)dx\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int \mathrm{e}^{\frac{x}{2}} \sin(x) \cos\left(\frac{\sqrt{3}\,x}{2}\right)dx\right)\right)}{3}$
  - Compute integrals
    $y_p(x) = -\cos(x)$
- Substitute particular solution into general solution to ODE
  $y(x) = C1\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + C2\,\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) - \cos(x)$
- □ Check validity of solution $y(x) = \_C1 \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + \_C2 \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) - \cos(x)$
  - Use initial condition $y(2) = 0$
    $0 = \_C1\,\mathrm{e}^{-1} \cos(\sqrt{3}) + \_C2\,\mathrm{e}^{-1} \sin(\sqrt{3}) - \cos(2)$
  - Compute derivative of the solution
    $\frac{d}{dx}y(x) = -\frac{\_C1\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\_C1\,\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} - \frac{\_C2\,\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\_C2\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} + \mathrm{si}$
  - Use the initial condition $\left(\frac{d}{dx}y(x)\right)\Big|_{\{x=1\}} = 0$
    $0 = -\frac{\_C1\,\mathrm{e}^{-\frac{1}{2}} \cos\left(\frac{\sqrt{3}}{2}\right)}{2} - \frac{\_C1\,\mathrm{e}^{-\frac{1}{2}} \sin\left(\frac{\sqrt{3}}{2}\right)\sqrt{3}}{2} - \frac{\_C2\,\mathrm{e}^{-\frac{1}{2}} \sin\left(\frac{\sqrt{3}}{2}\right)}{2} + \frac{\_C2\,\mathrm{e}^{-\frac{1}{2}} \cos\left(\frac{\sqrt{3}}{2}\right)\sqrt{3}}{2} + \sin(1)$
  - Solve for $\_C1$ and $\_C2$
    $\left\{\_C1 = \frac{\sqrt{3}\,\mathrm{e}^{-\frac{1}{2}} \cos\left(\frac{\sqrt{3}}{2}\right) \cos(2) + 2\sin(1)\mathrm{e}^{-1} \sin\left(\sqrt{3}\right) - \mathrm{e}^{-\frac{1}{2}} \sin\left(\frac{\sqrt{3}}{2}\right) \cos(2)}{\mathrm{e}^{-\frac{1}{2}}\left(\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}\right)\cos\left(\sqrt{3}\right) + \sqrt{3}\sin\left(\sqrt{3}\right)\sin\left(\frac{\sqrt{3}}{2}\right) - \sin\left(\frac{\sqrt{3}}{2}\right)\cos\left(\sqrt{3}\right) + \sin\left(\sqrt{3}\right)\cos\left(\frac{\sqrt{3}}{2}\right)\right)\mathrm{e}^{-1}}, \_C2 = \frac{}{\mathrm{e}^{-\frac{1}{2}}\left(\sqrt{3}\right.}\right.$

- Substitute constant values into general solution and simplify

$$y(x) = \frac{2\sin(1)\left(\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sin\left(\sqrt{3}\right)-\sin\left(\frac{\sqrt{3}\,x}{2}\right)\cos\left(\sqrt{3}\right)\right)\mathrm{e}^{-\frac{x}{2}+\frac{1}{2}}-\left(\left(-\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}\right)+\sin\left(\frac{\sqrt{3}}{2}\right)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(\sqrt{3}\,\mathrm{si}\right.\right.}{\sin\left(\frac{\sqrt{3}}{2}\right)+\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}\right)}$$

- Solution to the IVP

$$y(x) = \frac{2\sin(1)\left(\cos\left(\frac{\sqrt{3}\,x}{2}\right)\sin\left(\sqrt{3}\right)-\sin\left(\frac{\sqrt{3}\,x}{2}\right)\cos\left(\sqrt{3}\right)\right)\mathrm{e}^{-\frac{x}{2}+\frac{1}{2}}-\left(\left(-\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}\right)+\sin\left(\frac{\sqrt{3}}{2}\right)\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(\sqrt{3}\,\mathrm{si}\right.\right.}{\sin\left(\frac{\sqrt{3}}{2}\right)+\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}\right)}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.014 (sec)
Leaf size : 145

```
dsolve([diff(diff(y(x),x),x)+diff(y(x),x)+y(x) = sin(x),
       op([D(y)(1) = 0, y(2) = 0])],y(x),singsol=all)
```

$$y = \frac{2\sin(1)\left(\sin\left(\sqrt{3}\right)\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\sin\left(\frac{\sqrt{3}\,x}{2}\right)\cos\left(\sqrt{3}\right)\right)\mathrm{e}^{-\frac{x}{2}+\frac{1}{2}}-\cos(2)\left(\left(-\sqrt{3}\cos\left(\frac{\sqrt{3}}{2}\right)+\sin\left(\frac{\sqrt{3}}{2}\right)\right)\mathrm{co}\right.}{\sin\left(\frac{\sqrt{3}}{2}\right)+\sqrt{3}\,\mathrm{c}}$$

Mathematica DSolve solution

Solving time : 0.685 (sec)
Leaf size : 12765

```
DSolve[{D[y[x],{x,3}]+D[y[x],x]+y[x]==Sin[x],{Derivative[1][y][1] == 0,y[2]==0}},
       y[x],x,IncludeSingularSolutions->True]
```

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2.3.14 problem 14

Internal problem ID [8548]
Book : Own collection of miscellaneous problems
Section : section 3.0
Problem number : 14
Date solved : `Thursday, December 12, 2024 at 09:29:55 AM`
CAS classification : `[[_3rd_order, _with_linear_symmetries]]`

Solve

$$y''' + y' + y = x$$

With initial conditions

$$\begin{aligned} y'(0) &= 0 \\ y(0) &= 0 \\ y''(0) &= 1 \end{aligned}$$

Solved as higher order constant coeff ode

Time used: 0.150 (sec)

The characteristic equation is

$$\lambda^3 + \lambda + 1 = 0$$

The roots of the above equation are

$$\lambda_1 = -\frac{\left(108+12\sqrt{93}\right)^{1/3}}{6} + \frac{2}{\left(108+12\sqrt{93}\right)^{1/3}}$$

$$\lambda_2 = \frac{\left(108+12\sqrt{93}\right)^{1/3}}{12} - \frac{1}{\left(108+12\sqrt{93}\right)^{1/3}} + \frac{i\sqrt{3}\left(-\frac{\left(108+12\sqrt{93}\right)^{1/3}}{6} - \frac{2}{\left(108+12\sqrt{93}\right)^{1/3}}\right)}{2}$$

$$\lambda_3 = \frac{\left(108+12\sqrt{93}\right)^{1/3}}{12} - \frac{1}{\left(108+12\sqrt{93}\right)^{1/3}} - \frac{i\sqrt{3}\left(-\frac{\left(108+12\sqrt{93}\right)^{1/3}}{6} - \frac{2}{\left(108+12\sqrt{93}\right)^{1/3}}\right)}{2}$$

Therefore the homogeneous solution is

$$y_h(x) = \mathrm{e}^{\left(\frac{\left(108+12\sqrt{93}\right)^{1/3}}{12}-\frac{1}{\left(108+12\sqrt{93}\right)^{1/3}}+\frac{i\sqrt{3}\left(-\frac{\left(108+12\sqrt{93}\right)^{1/3}}{6}-\frac{2}{\left(108+12\sqrt{93}\right)^{1/3}}\right)}{2}\right)x} c_1+\mathrm{e}^{\left(\frac{\left(108+12\sqrt{93}\right)^{1/3}}{12}-\frac{1}{\left(108+12\sqrt{93}\right)^{1/3}}-\right.}$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = \mathrm{e}^{\left(\frac{\left(108+12\sqrt{93}\right)^{1/3}}{12}-\frac{1}{\left(108+12\sqrt{93}\right)^{1/3}}+\frac{i\sqrt{3}\left(-\frac{\left(108+12\sqrt{93}\right)^{1/3}}{6}-\frac{2}{\left(108+12\sqrt{93}\right)^{1/3}}\right)}{2}\right)x}$$

$$y_2 = \mathrm{e}^{\left(\frac{\left(108+12\sqrt{93}\right)^{1/3}}{12}-\frac{1}{\left(108+12\sqrt{93}\right)^{1/3}}-\frac{i\sqrt{3}\left(-\frac{\left(108+12\sqrt{93}\right)^{1/3}}{6}-\frac{2}{\left(108+12\sqrt{93}\right)^{1/3}}\right)}{2}\right)x}$$

$$y_3 = \mathrm{e}^{\left(-\frac{\left(108+12\sqrt{93}\right)^{1/3}}{6}+\frac{2}{\left(108+12\sqrt{93}\right)^{1/3}}\right)x}$$

This is higher order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE And y_p is a particular solution to the nonhomogeneous ODE. y_h is the solution to

$$y''' + y' + y = 0$$

Now the particular solution to the given ODE is found

$$y''' + y' + y = x$$

The particular solution is now found using the method of undetermined coefficients.

Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{\left(-\frac{\left(108+12\sqrt{93}\right)^{1/3}}{6}+\frac{2}{\left(108+12\sqrt{93}\right)^{1/3}}\right)x}, e^{\left(\frac{\left(108+12\sqrt{93}\right)^{1/3}}{12}-\frac{1}{\left(108+12\sqrt{93}\right)^{1/3}}-\frac{i\sqrt{3}\left(-\frac{\left(108+12\sqrt{93}\right)^{1/3}}{6}-\frac{2}{\left(108+12\sqrt{93}\right)^{1/3}}\right)}{2}\right)x}, e^{\left(\frac{(108+\right.}{}}\right.$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_2x + A_1 + A_2 = x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -1, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x - 1$$

Therefore the general solution is

$$y = y_h + y_p$$

$$= \left(e^{\left(\frac{(108+12\sqrt{93})^{1/3}}{12} - \frac{1}{(108+12\sqrt{93})^{1/3}} + \frac{i\sqrt{3}\left(-\frac{(108+12\sqrt{93})^{1/3}}{6} - \frac{2}{(108+12\sqrt{93})^{1/3}} \right)}{2} \right) x} c_1 + e^{\left(\frac{(108+12\sqrt{93})^{1/3}}{12} - \frac{1}{(108+12\sqrt{93})^{1/3}} - \frac{i\sqrt{3}\left(-\frac{(108+12\sqrt{93})^{1/3}}{6} - \frac{2}{(108+12\sqrt{93})^{1/3}} \right)}{2} \right) x} c_2 + e^{\left(-\frac{(108+12\sqrt{93})^{1/3}}{6} + \frac{2}{(108+12\sqrt{93})^{1/3}} \right) x} c_3 \right) + (x-1)$$

Solving for constants of integration using given initial conditions, the solution becomes

$$y = e^{\left(\frac{(108+12\sqrt{93})^{1/3}}{12} - \frac{1}{(108+12\sqrt{93})^{1/3}} + \frac{i\sqrt{3}\left(-\frac{(108+12\sqrt{93})^{1/3}}{6} - \frac{2}{(108+12\sqrt{93})^{1/3}} \right)}{2} \right) x} \left(\frac{19\left(\sqrt{93} - \frac{93}{19}\right)\left(i\sqrt{3}-1\right)\left(108+12\sqrt{9\right.}}{2232} \right.$$

Maple step by step solution

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

Maple dsolve solution

Solving time : `0.892` (sec)
Leaf size : 447

```
dsolve([diff(diff(diff(y(x),x),x),x)+diff(y(x),x)+y(x) = x,
       op([D(y)(0) = 0, y(0) = 0, (D@@2)(y)(0) = 1])],y(x),singsol=all)
```

$$y = \frac{\frac{10\,\mathrm{e}^{-\frac{\left(108+12\sqrt{93}\right)^{1/3} x \left(-12+\left(-9+\sqrt{93}\right)\left(108+12\sqrt{93}\right)^{1/3}\right)}{144}} \left(\left(\left(108+12\sqrt{3}\,\sqrt{31}\right)^{1/3}\sqrt{3}\,\sqrt{31}+\frac{3\sqrt{3}\,\sqrt{31}\left(108+12\sqrt{3}\,\sqrt{31}\right)^{2/3}}{5}-\frac{6\sqrt{3}\,\sqrt{31}}{5}-\frac{39\left(108+12\sqrt{3}\cdots\right.}{5}\cdots\right.\right.}{3}}{\cdots}$$

Mathematica DSolve solution

Solving time : `0.021` (sec)
Leaf size : 1546

```
DSolve[{D[y[x],{x,3}]+D[y[x],x]+y[x]==x,{Derivative[1][y][1] == 0,y[0]==0,Derivative[2][y][0
       y[x],x,IncludeSingularSolutions->True]
```

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2.3.15 problem 15

Internal problem ID [8549]
Book : Own collection of miscellaneous problems
Section : section 3.0
Problem number : 15
Date solved : `Thursday, December 12, 2024 at 09:29:56 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^4y'' + x^3y' - 4x^2y = 1$$

Solved as second order ode using change of variable on x method 2

Time used: 0.398 (sec)

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^4y'' + x^3y' - 4x^2y = 0$$

In normal form the ode

$$x^4y'' + x^3y' - 4x^2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{4}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\, \tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int e^{-\int p(x) dx} dx \\ &= \int e^{-\int \frac{1}{x} dx} dx \\ &= \int e^{-\ln(x)}\, dx \\ &= \int \frac{1}{x} dx \\ &= \ln(x) \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{-\frac{4}{x^2}}{\frac{1}{x^2}} \\ &= -4 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - 4y(\tau) &= 0
\end{aligned}
$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Ay''(\tau) + By'(\tau) + Cy(\tau) = 0
$$

Where in the above $A = 1, B = 0, C = -4$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{\tau\lambda} - 4\,\mathrm{e}^{\tau\lambda} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$
\lambda^2 - 4 = 0 \tag{2}
$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$
\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}
$$

Substituting $A = 1, B = 0, C = -4$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-4)} \\
&= \pm 2
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= +2 \\
\lambda_2 &= -2
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= 2 \\
\lambda_2 &= -2
\end{aligned}
$$

Since roots are real and distinct, then the solution is

$$
\begin{aligned}
y(\tau) &= c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau} \\
y(\tau) &= c_1 e^{(2)\tau} + c_2 e^{(-2)\tau}
\end{aligned}
$$

Or

$$y(\tau) = c_1 \,\mathrm{e}^{2\tau} + c_2 \,\mathrm{e}^{-2\tau}$$

Will add steps showing solving for IC soon.

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \, x^2 + \frac{c_2}{x^2}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \, x^2 + \frac{c_2}{x^2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x^2}$$
$$y_2 = x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & x^2 \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & x^2 \\ -\frac{2}{x^3} & 2x \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)(2x) - (x^2)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{4}{x}$$

Which simplifies to

$$W = \frac{4}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^2}{4x^3}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{1}{4x}dx$$

Hence

$$u_1 = -\frac{\ln(x)}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1}{x^2}}{4x^3}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{4x^5}dx$$

Hence

$$u_2 = -\frac{1}{16x^4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln(x)}{4x^2} - \frac{1}{16x^2}$$

Which simplifies to

$$y_p(x) = \frac{-1 - 4\ln(x)}{16x^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 x^2 + \frac{c_2}{x^2}\right) + \left(\frac{-1 - 4\ln(x)}{16x^2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{-1 - 4\ln(x)}{16x^2} + c_1 x^2 + \frac{c_2}{x^2}$$

Solved as second order ode using change of variable on x method 1

Time used: 0.606 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^4, B = x^3, C = -4x^2, f(x) = 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^4y'' + x^3y' - 4x^2y = 0$$

In normal form the ode

$$x^4y'' + x^3y' - 4x^2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{4}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1\left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{-\frac{1}{x^2}}}{c} \\ \tau'' &= \frac{2}{c\sqrt{-\frac{1}{x^2}}\, x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \\ &= \frac{\frac{2}{c\sqrt{-\frac{1}{x^2}}\, x^3} + \frac{1}{x} \frac{2\sqrt{-\frac{1}{x^2}}}{c}}{\left(\frac{2\sqrt{-\frac{1}{x^2}}}{c}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c} \sqrt{q} \, dx \\
&= \frac{\int 2\sqrt{-\frac{1}{x^2}} dx}{c} \\
&= \frac{2\sqrt{-\frac{1}{x^2}}\, x \ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 \cos\left(2\sqrt{-\frac{1}{x^2}}\, x \ln(x)\right) + c_2 \sin\left(2\sqrt{-\frac{1}{x^2}}\, x \ln(x)\right)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \cos\left(2\sqrt{-\frac{1}{x^2}}\, x \ln(x)\right) \\
y_2 &= \sin\left(2\sqrt{-\frac{1}{x^2}}\, x \ln(x)\right)
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos\left(2\sqrt{-\frac{1}{x^2}}\, x\ln(x)\right) & \sin\left(2\sqrt{-\frac{1}{x^2}}\, x\ln(x)\right) \\ \frac{d}{dx}\left(\cos\left(2\sqrt{-\frac{1}{x^2}}\, x\ln(x)\right)\right) & \frac{d}{dx}\left(\sin\left(2\sqrt{-\frac{1}{x^2}}\, x\ln(x)\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos\left(2\sqrt{-\frac{1}{x^2}}\, x\ln(x)\right) & \sin\left(2\sqrt{-\frac{1}{x^2}}\, x \right. \\ -\left(\frac{2\ln(x)}{\sqrt{-\frac{1}{x^2}}\, x^2} + 2\sqrt{-\frac{1}{x^2}}\, \ln(x) + 2\sqrt{-\frac{1}{x^2}}\right)\sin\left(2\sqrt{-\frac{1}{x^2}}\, x\ln(x)\right) & \left(\frac{2\ln(x)}{\sqrt{-\frac{1}{x^2}}\, x^2} + 2\sqrt{-\frac{1}{x^2}}\, \ln(x) + 2\sqrt{-\right. \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & \left(\cos\left(2\sqrt{-\frac{1}{x^2}}\, x\ln(x)\right)\right)\left(\left(\frac{2\ln(x)}{\sqrt{-\frac{1}{x^2}}\, x^2} + 2\sqrt{-\frac{1}{x^2}}\, \ln(x)\right.\right. \\ & \left.\left. + 2\sqrt{-\frac{1}{x^2}}\right)\cos\left(2\sqrt{-\frac{1}{x^2}}\, x\ln(x)\right)\right) \\ & - \left(\sin\left(2\sqrt{-\frac{1}{x^2}}\, x\ln(x)\right)\right)\left(-\left(\frac{2\ln(x)}{\sqrt{-\frac{1}{x^2}}\, x^2} + 2\sqrt{-\frac{1}{x^2}}\, \ln(x)\right.\right. \\ & \left.\left. + 2\sqrt{-\frac{1}{x^2}}\right)\sin\left(2\sqrt{-\frac{1}{x^2}}\, x\ln(x)\right)\right) \end{aligned}$$

Which simplifies to

$$W = -\frac{2\left(\cos\left(2\sqrt{-\frac{1}{x^2}}\, x\ln(x)\right)^2 + \sin\left(2\sqrt{-\frac{1}{x^2}}\, x\ln(x)\right)^2\right)}{\sqrt{-\frac{1}{x^2}}\, x^2}$$

Which simplifies to

$$W=-\frac{2}{\sqrt{-\frac{1}{x^2}}\,x^2}$$

Therefore Eq. (2) becomes

$$u_1=-\int\frac{\sin\left(2\sqrt{-\frac{1}{x^2}}\,x\ln\left(x\right)\right)}{-\frac{2x^2}{\sqrt{-\frac{1}{x^2}}}}\,dx$$

Which simplifies to

$$u_1=-\int-\frac{\sin\left(2\sqrt{-\frac{1}{x^2}}\,x\ln\left(x\right)\right)\sqrt{-\frac{1}{x^2}}}{2x^2}dx$$

Hence

$$u_1=-\int_0^x-\frac{\sin\left(2\sqrt{-\frac{1}{\alpha^2}}\,\alpha\ln\left(\alpha\right)\right)\sqrt{-\frac{1}{\alpha^2}}}{2\alpha^2}d\alpha$$

And Eq. (3) becomes

$$u_2=\int\frac{\cos\left(2\sqrt{-\frac{1}{x^2}}\,x\ln\left(x\right)\right)}{-\frac{2x^2}{\sqrt{-\frac{1}{x^2}}}}\,dx$$

Which simplifies to

$$u_2=\int-\frac{\cos\left(2\sqrt{-\frac{1}{x^2}}\,x\ln\left(x\right)\right)\sqrt{-\frac{1}{x^2}}}{2x^2}dx$$

Hence

$$u_2=\int_0^x-\frac{\cos\left(2\sqrt{-\frac{1}{\alpha^2}}\,\alpha\ln\left(\alpha\right)\right)\sqrt{-\frac{1}{\alpha^2}}}{2\alpha^2}d\alpha$$

Therefore the particular solution, from equation (1) is

$$
\begin{aligned}
y_p(x) = &-\int_0^x -\frac{\sin\left(2\sqrt{-\frac{1}{\alpha^2}}\,\alpha\ln(\alpha)\right)\sqrt{-\frac{1}{\alpha^2}}}{2\alpha^2}d\alpha\cos\left(2\sqrt{-\frac{1}{x^2}}\,x\ln(x)\right) \\
&+\sin\left(2\sqrt{-\frac{1}{x^2}}\,x\ln(x)\right)\int_0^x -\frac{\cos\left(2\sqrt{-\frac{1}{\alpha^2}}\,\alpha\ln(\alpha)\right)\sqrt{-\frac{1}{\alpha^2}}}{2\alpha^2}d\alpha
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
y_p(x) = &\frac{\int_0^x \frac{\sin\left(2\sqrt{-\frac{1}{\alpha^2}}\,\alpha\ln(\alpha)\right)\sqrt{-\frac{1}{\alpha^2}}}{\alpha^2}d\alpha\cos\left(2\sqrt{-\frac{1}{x^2}}\,x\ln(x)\right)}{2} \\
&-\frac{\sin\left(2\sqrt{-\frac{1}{x^2}}\,x\ln(x)\right)\int_0^x \frac{\cos\left(2\sqrt{-\frac{1}{\alpha^2}}\,\alpha\ln(\alpha)\right)\sqrt{-\frac{1}{\alpha^2}}}{\alpha^2}d\alpha}{2}
\end{aligned}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1\cos\left(2\sqrt{-\frac{1}{x^2}}\,x\ln(x)\right) + c_2\sin\left(2\sqrt{-\frac{1}{x^2}}\,x\ln(x)\right)\right) \\
&\quad+\left(\frac{\int_0^x \frac{\sin\left(2\sqrt{-\frac{1}{\alpha^2}}\,\alpha\ln(\alpha)\right)\sqrt{-\frac{1}{\alpha^2}}}{\alpha^2}d\alpha\cos\left(2\sqrt{-\frac{1}{x^2}}\,x\ln(x)\right)}{2}\right. \\
&\qquad\left.-\frac{\sin\left(2\sqrt{-\frac{1}{x^2}}\,x\ln(x)\right)\int_0^x \frac{\cos\left(2\sqrt{-\frac{1}{\alpha^2}}\,\alpha\ln(\alpha)\right)\sqrt{-\frac{1}{\alpha^2}}}{\alpha^2}d\alpha}{2}\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \cos\left(2\sqrt{-\frac{1}{x^2}}\, x \ln(x)\right) + c_2 \sin\left(2\sqrt{-\frac{1}{x^2}}\, x \ln(x)\right) + \frac{\int_0^x \frac{\sin\left(2\sqrt{-\frac{1}{\alpha^2}}\, \alpha \ln(\alpha)\right)\sqrt{-\frac{1}{\alpha^2}}}{\alpha^2} d\alpha \cos\left(2\sqrt{-\frac{1}{x^2}}\, x \ln(x)\right)}{2} - \frac{\sin\left(2\sqrt{-\frac{1}{x^2}}\, x \ln(x)\right) \int_0^x \frac{\cos\left(2\sqrt{-\frac{1}{\alpha^2}}\, \alpha \ln(\alpha)\right)\sqrt{-\frac{1}{\alpha^2}}}{\alpha^2} d\alpha}{2}$$

Solved as second order ode using change of variable on y method 2

Time used: 0.170 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^4, B = x^3, C = -4x^2, f(x) = 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^4y'' + x^3y' - 4x^2y = 0$$

In normal form the ode

$$x^4y'' + x^3y' - 4x^2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{4}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^2} - \frac{4}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{5v'(x)}{x} &= 0 \\ v''(x) + \frac{5v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{5u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form a linear first order is

$$u'(x) + q(x)u(x) = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= \frac{5}{x} \\ p(x) &= 0 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dx} \\ &= \mathrm{e}^{\int \frac{5}{x}dx} \\ &= x^5\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mu u &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(u\,x^5\right) &= 0\end{aligned}$$

Integrating gives

$$\begin{aligned}u\,x^5 &= \int 0\,dx + c_1 \\ &= c_1\end{aligned}$$

Dividing throughout by the integrating factor x^5 gives the final solution

$$u(x) = \frac{c_1}{x^5}$$

Now that $u(x)$ is known, then

$$\begin{aligned}v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= -\frac{c_1}{4x^4} + c_2\end{aligned}$$

Hence

$$\begin{aligned}y &= v(x)\,x^n \\ &= \left(-\frac{c_1}{4x^4} + c_2\right)x^2 \\ &= \frac{4c_2\,x^4 - c_1}{4x^2}\end{aligned}$$

Now the particular solution to this ODE is found

$$x^4y'' + x^3y' - 4x^2y = 1$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x^2}$$
$$y_2 = x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & x^2 \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & x^2 \\ -\frac{2}{x^3} & 2x \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)(2x) - \left(x^2\right)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{4}{x}$$

Which simplifies to

$$W = \frac{4}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^2}{4x^3}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{1}{4x}dx$$

Hence

$$u_1 = -\frac{\ln(x)}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1}{x^2}}{4x^3}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{4x^5}dx$$

Hence

$$u_2 = -\frac{1}{16x^4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln(x)}{4x^2} - \frac{1}{16x^2}$$

Which simplifies to

$$y_p(x) = \frac{-1 - 4\ln(x)}{16x^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{4x^4} + c_2\right)x^2\right) + \left(\frac{-1 - 4\ln(x)}{16x^2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \left(-\frac{c_1}{4x^4} + c_2\right)x^2 + \frac{-1 - 4\ln(x)}{16x^2}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.240 (sec)

Writing the ode as

$$x^4y'' + x^3y' - 4x^2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4 \\ B &= x^3 \\ C &= -4x^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.142: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{3}{2} - \left(-\frac{3}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{3}{2x} + (-)\,(0) \\ &= -\frac{3}{2x} \\ &= -\frac{3}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{3}{2x}\right)(0) + \left(\left(\frac{3}{2x^2}\right) + \left(-\frac{3}{2x}\right)^2 - \left(\frac{15}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= pe^{\int \omega\, dx} \\ &= \mathrm{e}^{\int -\frac{3}{2x}dx} \\ &= \frac{1}{x^{3/2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^3}{x^4}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{1}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^3}{x^4}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{x^4}{4}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x^2}\right) + c_2\left(\frac{1}{x^2}\left(\frac{x^4}{4}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$x^4y''+x^3y'-4x^2y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1}{x^2}+\frac{c_2\,x^2}{4}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1+u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x^2}$$
$$y_2 = \frac{x^2}{4}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{x^2}{4} \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}\left(\frac{x^2}{4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{x^2}{4} \\ -\frac{2}{x^3} & \frac{x}{2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)\left(\frac{x}{2}\right) - \left(\frac{x^2}{4}\right)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{1}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^2}{4}}{x^3}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{1}{4x}dx$$

Hence

$$u_1 = -\frac{\ln(x)}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1}{x^2}}{x^3}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x^5}dx$$

Hence

$$u_2 = -\frac{1}{4x^4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln(x)}{4x^2} - \frac{1}{16x^2}$$

Which simplifies to

$$y_p(x) = \frac{-1 - 4\ln(x)}{16x^2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x^2} + \frac{c_2\, x^2}{4}\right) + \left(\frac{-1 - 4\ln(x)}{16x^2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{c_1}{x^2} + \frac{c_2\, x^2}{4} + \frac{-1 - 4\ln(x)}{16x^2}$$

Solved as second order ode adjoint method

Time used: 0.194 (sec)

In normal form the ode

$$x^4 y'' + x^3 y' - 4x^2 y = 1 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{4}{x^2} \\ r(x) &= \frac{1}{x^4} \end{aligned}$$

The Lagrange adjoint ode is given by

$$
\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - \left(\frac{\xi(x)}{x}\right)' + \left(-\frac{4\xi(x)}{x^2}\right) &= 0 \\
\xi''(x) - \frac{3\xi(x)}{x^2} - \frac{\xi'(x)}{x} &= 0
\end{aligned}
$$

Which is solved for $\xi(x)$. This is Euler second order ODE. Let the solution be $\xi = x^r$, then $\xi' = rx^{r-1}$ and $\xi'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$
x^2(r(r-1))x^{r-2} - xrx^{r-1} - 3x^r = 0
$$

Simplifying gives

$$
r(r-1)\,x^r - r\,x^r - 3x^r = 0
$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$
r(r-1) - r - 3 = 0
$$

Or

$$
r^2 - 2r - 3 = 0 \tag{1}
$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= -1 \\
r_2 &= 3
\end{aligned}
$$

Since the roots are real and distinct, then the general solution is

$$
\xi = c_1\xi_1 + c_2\xi_2
$$

Where $\xi_1 = x^{r_1}$ and $\xi_2 = x^{r_2}$. Hence

$$
\xi = \frac{c_1}{x} + c_2\, x^3
$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$
\begin{aligned}
\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi\,(x)}
\end{aligned}
$$

Or

$$y'+y\left(\frac{1}{x}-\frac{-\frac{c_1}{x^2}+3c_2\,x^2}{\frac{c_1}{x}+c_2\,x^3}\right)=\frac{-\frac{c_1}{4x^4}+c_2\ln\left(x\right)}{\frac{c_1}{x}+c_2\,x^3}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y'+q(x)y=p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned}q(x)&=-\frac{2c_2\,x^4-2c_1}{x\left(c_2\,x^4+c_1\right)}\\p(x)&=\frac{4c_2\ln\left(x\right)x^4-c_1}{4x^3\left(c_2\,x^4+c_1\right)}\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}\mu&=e^{\int q\,dx}\\&=\mathrm{e}^{\int-\frac{2c_2\,x^4-2c_1}{x\left(c_2\,x^4+c_1\right)}dx}\\&=\frac{x^2}{c_2\,x^4+c_1}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y)&=\mu p\\\frac{\mathrm{d}}{\mathrm{d}x}(\mu y)&=(\mu)\left(\frac{4c_2\ln\left(x\right)x^4-c_1}{4x^3\left(c_2\,x^4+c_1\right)}\right)\\\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y\,x^2}{c_2\,x^4+c_1}\right)&=\left(\frac{x^2}{c_2\,x^4+c_1}\right)\left(\frac{4c_2\ln\left(x\right)x^4-c_1}{4x^3\left(c_2\,x^4+c_1\right)}\right)\\\mathrm{d}\left(\frac{y\,x^2}{c_2\,x^4+c_1}\right)&=\left(\frac{4c_2\ln\left(x\right)x^4-c_1}{4x\left(c_2\,x^4+c_1\right)^2}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y\,x^2}{c_2\,x^4+c_1}&=\int\frac{4c_2\ln\left(x\right)x^4-c_1}{4x\left(c_2\,x^4+c_1\right)^2}\,dx\\&=-\frac{1}{16\left(c_2\,x^4+c_1\right)}-\frac{\ln\left(x\right)}{4c_1}+\frac{c_2\ln\left(x\right)x^4}{4c_1\left(c_2\,x^4+c_1\right)}+c_3\end{aligned}$$

Dividing throughout by the integrating factor $\frac{x^2}{c_2 x^4+c_1}$ gives the final solution

$$y = \frac{-1-4\ln(x)+16(c_2 x^4+c_1) c_3}{16x^2}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{-1-4\ln(x)+16(c_2 x^4+c_1) c_3}{16x^2}$$

The constants can be merged to give

$$y = \frac{-1-4\ln(x)+16c_2 x^4+16c_1}{16x^2}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{-1-4\ln(x)+16c_2 x^4+16c_1}{16x^2}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : `0.010` (sec)
Leaf size : 24

```
dsolve(x^4*diff(diff(y(x),x),x)+diff(y(x),x)*x^3-4*x^2*y(x) = 1,
       y(x),singsol=all)
```

$$y = \frac{16c_1x^4 - 4\ln(x) + 16c_2 - 1}{16x^2}$$

Mathematica DSolve solution

Solving time : `0.017` (sec)
Leaf size : 29

```
DSolve[{x^4*D[y[x],{x,2}]+x^3*D[y[x],x]-4*x^2*y[x]==1,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{16c_2x^4 - 4\log(x) - 1 + 16c_1}{16x^2}$$

2.3.16 problem 16

Internal problem ID [8550]
Book : Own collection of miscellaneous problems
Section : section 3.0
Problem number : 16
Date solved : `Thursday, December 12, 2024 at 09:29:58 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^4y'' + x^3y' - 4x^2y = x$$

Solved as second order ode using change of variable on x method 2

Time used: 0.383 (sec)

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^4y'' + x^3y' - 4x^2y = 0$$

In normal form the ode

$$x^4y'' + x^3y' - 4x^2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned}p(x) &= \frac{1}{x}\\ q(x) &= -\frac{4}{x^2}\end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}\tau &= \int e^{-\int p(x)dx}dx\\ &= \int e^{-\int \frac{1}{x}dx}dx\\ &= \int e^{-\ln(x)}\,dx\\ &= \int \frac{1}{x}dx\\ &= \ln(x)\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}q_1(\tau) &= \frac{q(x)}{\tau'(x)^2}\\ &= \frac{-\frac{4}{x^2}}{\frac{1}{x^2}}\\ &= -4\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0\\ \frac{d^2}{d\tau^2}y(\tau) - 4y(\tau) &= 0\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -4$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\tau\lambda} - 4\,\mathrm{e}^{\tau\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-4)}\\ &= \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2\\ \lambda_2 &= -2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y(\tau) &= c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau}\\ y(\tau) &= c_1 e^{(2)\tau} + c_2 e^{(-2)\tau}\end{aligned}$$

Or

$$y(\tau) = c_1 \,\mathrm{e}^{2\tau} + c_2 \,\mathrm{e}^{-2\tau}$$

Will add steps showing solving for IC soon.

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \, x^2 + \frac{c_2}{x^2}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \, x^2 + \frac{c_2}{x^2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x^2}$$
$$y_2 = x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & x^2 \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}(x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & x^2 \\ -\frac{2}{x^3} & 2x \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)(2x) - (x^2)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{4}{x}$$

Which simplifies to

$$W = \frac{4}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^3}{4x^3}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{1}{4}dx$$

Hence

$$u_1 = -\frac{x}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1}{x}}{4x^3}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{4x^4}dx$$

Hence

$$u_2 = -\frac{1}{12x^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{1}{3x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\,x^2 + \frac{c_2}{x^2}\right) + \left(-\frac{1}{3x}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{1}{3x} + c_1\,x^2 + \frac{c_2}{x^2}$$

Solved as second order ode using change of variable on x method 1

Time used: 0.596 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^4, B = x^3, C = -4x^2, f(x) = x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^4y'' + x^3y' - 4x^2y = 0$$

In normal form the ode

$$x^4y'' + x^3y' - 4x^2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= -\frac{4}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) results

$$
\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}
$$

Where τ is the new independent variable, and

$$
p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}
$$

$$
q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}
$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$
\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{2\sqrt{-\frac{1}{x^2}}}{c} \\
\tau'' &= \frac{2}{c\sqrt{-\frac{1}{x^2}}\,x^3}
\end{aligned} \tag{6}
$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{\frac{2}{c\sqrt{-\frac{1}{x^2}}\,x^3} + \frac{1}{x}\frac{2\sqrt{-\frac{1}{x^2}}}{c}}{\left(\frac{2\sqrt{-\frac{1}{x^2}}}{c}\right)^2} \\
&= 0
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\ &= \frac{\int 2\sqrt{-\frac{1}{x^2}}dx}{c} \\ &= \frac{2\sqrt{-\frac{1}{x^2}}\,x\ln(x)}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos\left(2\sqrt{-\frac{1}{x^2}}\,x\ln(x)\right) + c_2 \sin\left(2\sqrt{-\frac{1}{x^2}}\,x\ln(x)\right)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1 &= \cos\left(2\sqrt{-\frac{1}{x^2}}\,x\ln(x)\right) \\ y_2 &= \sin\left(2\sqrt{-\frac{1}{x^2}}\,x\ln(x)\right)\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos\left(2\sqrt{-\frac{1}{x^2}}\, x\ln(x)\right) & \sin\left(2\sqrt{-\frac{1}{x^2}}\, x\ln(x)\right) \\ \frac{d}{dx}\left(\cos\left(2\sqrt{-\frac{1}{x^2}}\, x\ln(x)\right)\right) & \frac{d}{dx}\left(\sin\left(2\sqrt{-\frac{1}{x^2}}\, x\ln(x)\right)\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos\left(2\sqrt{-\frac{1}{x^2}}\, x\ln(x)\right) & \sin\left(2\sqrt{-\frac{1}{x^2}}\, x \right. \\ -\left(\frac{2\ln(x)}{\sqrt{-\frac{1}{x^2}}\, x^2} + 2\sqrt{-\frac{1}{x^2}}\, \ln(x) + 2\sqrt{-\frac{1}{x^2}}\right)\sin\left(2\sqrt{-\frac{1}{x^2}}\, x\ln(x)\right) & \left(\frac{2\ln(x)}{\sqrt{-\frac{1}{x^2}}\, x^2} + 2\sqrt{-\frac{1}{x^2}}\, \ln(x) + 2\sqrt{-\right. \end{vmatrix}$$

Therefore

$$\begin{aligned} W = {} & \left(\cos\left(2\sqrt{-\frac{1}{x^2}}\, x\ln(x)\right)\right)\left(\left(\frac{2\ln(x)}{\sqrt{-\frac{1}{x^2}}\, x^2} + 2\sqrt{-\frac{1}{x^2}}\, \ln(x)\right.\right. \\ & \left.\left. + 2\sqrt{-\frac{1}{x^2}}\right)\cos\left(2\sqrt{-\frac{1}{x^2}}\, x\ln(x)\right)\right) \\ & - \left(\sin\left(2\sqrt{-\frac{1}{x^2}}\, x\ln(x)\right)\right)\left(-\left(\frac{2\ln(x)}{\sqrt{-\frac{1}{x^2}}\, x^2} + 2\sqrt{-\frac{1}{x^2}}\, \ln(x)\right.\right. \\ & \left.\left. + 2\sqrt{-\frac{1}{x^2}}\right)\sin\left(2\sqrt{-\frac{1}{x^2}}\, x\ln(x)\right)\right) \end{aligned}$$

Which simplifies to

$$W = -\frac{2\left(\cos\left(2\sqrt{-\frac{1}{x^2}}\, x\ln(x)\right)^2 + \sin\left(2\sqrt{-\frac{1}{x^2}}\, x\ln(x)\right)^2\right)}{\sqrt{-\frac{1}{x^2}}\, x^2}$$

Which simplifies to

$$W = -\frac{2}{\sqrt{-\frac{1}{x^2}}\, x^2}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin\left(2\sqrt{-\frac{1}{x^2}}\, x\ln(x)\right) x}{-\frac{2x^2}{\sqrt{-\frac{1}{x^2}}}} dx$$

Which simplifies to

$$u_1 = -\int -\frac{\sin\left(2\sqrt{-\frac{1}{x^2}}\, x\ln(x)\right)\sqrt{-\frac{1}{x^2}}}{2x} dx$$

Hence

$$u_1 = -\int_0^x -\frac{\sin\left(2\sqrt{-\frac{1}{\alpha^2}}\, \alpha\ln(\alpha)\right)\sqrt{-\frac{1}{\alpha^2}}}{2\alpha} d\alpha$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos\left(2\sqrt{-\frac{1}{x^2}}\, x\ln(x)\right) x}{-\frac{2x^2}{\sqrt{-\frac{1}{x^2}}}} dx$$

Which simplifies to

$$u_2 = \int -\frac{\cos\left(2\sqrt{-\frac{1}{x^2}}\, x\ln(x)\right)\sqrt{-\frac{1}{x^2}}}{2x} dx$$

Hence

$$u_2 = \int_0^x -\frac{\cos\left(2\sqrt{-\frac{1}{\alpha^2}}\, \alpha\ln(\alpha)\right)\sqrt{-\frac{1}{\alpha^2}}}{2\alpha} d\alpha$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = &-\int_0^x -\frac{\sin\left(2\sqrt{-\frac{1}{\alpha^2}}\, \alpha\ln(\alpha)\right)\sqrt{-\frac{1}{\alpha^2}}}{2\alpha} d\alpha \cos\left(2\sqrt{-\frac{1}{x^2}}\, x\ln(x)\right) \\ &+ \sin\left(2\sqrt{-\frac{1}{x^2}}\, x\ln(x)\right)\int_0^x -\frac{\cos\left(2\sqrt{-\frac{1}{\alpha^2}}\, \alpha\ln(\alpha)\right)\sqrt{-\frac{1}{\alpha^2}}}{2\alpha} d\alpha \end{aligned}$$

Which simplifies to

$$
y_p(x) = \frac{\int_0^x \frac{\sin\left(2\sqrt{-\frac{1}{\alpha^2}}\,\alpha\ln(\alpha)\right)\sqrt{-\frac{1}{\alpha^2}}}{\alpha}d\alpha\cos\left(2\sqrt{-\frac{1}{x^2}}\,x\ln(x)\right)}{2} - \frac{\sin\left(2\sqrt{-\frac{1}{x^2}}\,x\ln(x)\right)\int_0^x \frac{\cos\left(2\sqrt{-\frac{1}{\alpha^2}}\,\alpha\ln(\alpha)\right)\sqrt{-\frac{1}{\alpha^2}}}{\alpha}d\alpha}{2}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1\cos\left(2\sqrt{-\frac{1}{x^2}}\,x\ln(x)\right) + c_2\sin\left(2\sqrt{-\frac{1}{x^2}}\,x\ln(x)\right)\right) \\
&\quad + \left(\frac{\int_0^x \frac{\sin\left(2\sqrt{-\frac{1}{\alpha^2}}\,\alpha\ln(\alpha)\right)\sqrt{-\frac{1}{\alpha^2}}}{\alpha}d\alpha\cos\left(2\sqrt{-\frac{1}{x^2}}\,x\ln(x)\right)}{2} - \frac{\sin\left(2\sqrt{-\frac{1}{x^2}}\,x\ln(x)\right)\int_0^x \frac{\cos\left(2\sqrt{-\frac{1}{\alpha^2}}\,\alpha\ln(\alpha)\right)\sqrt{-\frac{1}{\alpha^2}}}{\alpha}d\alpha}{2}\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
\begin{aligned}
y &= c_1\cos\left(2\sqrt{-\frac{1}{x^2}}\,x\ln(x)\right) + c_2\sin\left(2\sqrt{-\frac{1}{x^2}}\,x\ln(x)\right) \\
&\quad + \frac{\int_0^x \frac{\sin\left(2\sqrt{-\frac{1}{\alpha^2}}\,\alpha\ln(\alpha)\right)\sqrt{-\frac{1}{\alpha^2}}}{\alpha}d\alpha\cos\left(2\sqrt{-\frac{1}{x^2}}\,x\ln(x)\right)}{2} \\
&\quad - \frac{\sin\left(2\sqrt{-\frac{1}{x^2}}\,x\ln(x)\right)\int_0^x \frac{\cos\left(2\sqrt{-\frac{1}{\alpha^2}}\,\alpha\ln(\alpha)\right)\sqrt{-\frac{1}{\alpha^2}}}{\alpha}d\alpha}{2}
\end{aligned}
$$

Solved as second order ode using change of variable on y method 2

Time used: 0.168 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^4, B = x^3, C = -4x^2, f(x) = x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^4y'' + x^3y' - 4x^2y = 0$$

In normal form the ode

$$x^4y'' + x^3y' - 4x^2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{4}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^2} - \frac{4}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{5v'(x)}{x} &= 0 \\ v''(x) + \frac{5v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{5u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form a linear first order is

$$u'(x) + q(x)u(x) = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= \frac{5}{x} \\ p(x) &= 0 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q \, dx} \\ &= e^{\int \frac{5}{x} dx} \\ &= x^5 \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mu u &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(u\, x^5\right) &= 0 \end{aligned}$$

Integrating gives

$$
\begin{aligned}
u\,x^5 &= \int 0\,dx + c_1 \\
&= c_1
\end{aligned}
$$

Dividing throughout by the integrating factor x^5 gives the final solution

$$
u(x) = \frac{c_1}{x^5}
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= -\frac{c_1}{4x^4} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\,x^n \\
&= \left(-\frac{c_1}{4x^4} + c_2\right) x^2 \\
&= \frac{4c_2\,x^4 - c_1}{4x^2}
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
x^4y'' + x^3y' - 4x^2y = x
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \frac{1}{x^2} \\
y_2 &= x^2
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & x^2 \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}\left(x^2\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & x^2 \\ -\frac{2}{x^3} & 2x \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)(2x) - \left(x^2\right)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{4}{x}$$

Which simplifies to

$$W = \frac{4}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^3}{4x^3}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{1}{4} dx$$

Hence

$$u_1 = -\frac{x}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1}{x}}{4x^3}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{4x^4} dx$$

Hence

$$u_2 = -\frac{1}{12x^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{1}{3x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{4x^4} + c_2\right) x^2\right) + \left(-\frac{1}{3x}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \left(-\frac{c_1}{4x^4} + c_2\right) x^2 - \frac{1}{3x}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.238 (sec)

Writing the ode as

$$x^4y'' + x^3y' - 4x^2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^4 \\ B &= x^3 \\ C &= -4x^2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.143: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = \frac{15}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = \frac{15}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = -\frac{3}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= -\frac{3}{2} - \left(-\frac{3}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{3}{2x} + (-)(0) \\ &= -\frac{3}{2x} \\ &= -\frac{3}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{3}{2x}\right)(0) + \left(\left(\frac{3}{2x^2}\right) + \left(-\frac{3}{2x}\right)^2 - \left(\frac{15}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{3}{2x} dx} \\ &= \frac{1}{x^{3/2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x^3}{x^4}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1\left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x^2}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x^3}{x^4}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{x^4}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x^2}\right) + c_2\left(\frac{1}{x^2}\left(\frac{x^4}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$x^4y''+x^3y'-4x^2y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h=\frac{c_1}{x^2}+\frac{c_2\,x^2}{4}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1,u_2 to be determined, and y_1,y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}y_1&=\frac{1}{x^2}\\ y_2&=\frac{x^2}{4}\end{aligned}$$

In the Variation of parameters u_1,u_2 are found using

$$u_1=-\int\frac{y_2f(x)}{aW(x)} \tag{2}$$

$$u_2=\int\frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W=\begin{vmatrix}y_1&y_2\\y_1'&y_2'\end{vmatrix}$. Hence

$$W=\begin{vmatrix}\frac{1}{x^2}&\frac{x^2}{4}\\ \frac{d}{dx}\left(\frac{1}{x^2}\right)&\frac{d}{dx}\left(\frac{x^2}{4}\right)\end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{x^2}{4} \\ -\frac{2}{x^3} & \frac{x}{2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)\left(\frac{x}{2}\right) - \left(\frac{x^2}{4}\right)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{1}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^3}{4}}{x^3}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{1}{4} dx$$

Hence

$$u_1 = -\frac{x}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1}{x}}{x^3}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x^4} dx$$

Hence

$$u_2 = -\frac{1}{3x^3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{1}{3x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x^2} + \frac{c_2\, x^2}{4}\right) + \left(-\frac{1}{3x}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{c_1}{x^2} + \frac{c_2\, x^2}{4} - \frac{1}{3x}$$

Solved as second order ode adjoint method

Time used: 0.174 (sec)

In normal form the ode

$$x^4 y'' + x^3 y' - 4x^2 y = x \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{4}{x^2} \\ r(x) &= \frac{1}{x^3} \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - \left(\frac{\xi(x)}{x}\right)' + \left(-\frac{4\xi(x)}{x^2}\right) &= 0 \\ \xi''(x) - \frac{3\xi(x)}{x^2} - \frac{\xi'(x)}{x} &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is Euler second order ODE. Let the solution be $\xi = x^r$, then $\xi' = rx^{r-1}$ and $\xi'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - xrx^{r-1} - 3x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - r\,x^r - 3x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - r - 3 = 0$$

Or

$$r^2 - 2r - 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -1 \\ r_2 &= 3 \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$\xi = c_1\xi_1 + c_2\xi_2$$

Where $\xi_1 = x^{r_1}$ and $\xi_2 = x^{r_2}$. Hence

$$\xi = \frac{c_1}{x} + c_2\,x^3$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned} \xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y &= \int \xi(x)\,r(x)\,dx \\ y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi\,(x)} \end{aligned}$$

Or

$$y' + y\left(\frac{1}{x} - \frac{-\frac{c_1}{x^2} + 3c_2\,x^2}{\frac{c_1}{x} + c_2\,x^3}\right) = \frac{c_2x - \frac{c_1}{3x^3}}{\frac{c_1}{x} + c_2\,x^3}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= -\frac{2c_2\,x^4 - 2c_1}{x\,(c_2\,x^4 + c_1)} \\
p(x) &= \frac{3c_2\,x^4 - c_1}{3x^2\,(c_2\,x^4 + c_1)}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= \mathrm{e}^{\int -\frac{2c_2\,x^4 - 2c_1}{x\left(c_2\,x^4 + c_1\right)}dx} \\
&= \frac{x^2}{c_2\,x^4 + c_1}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{3c_2\,x^4 - c_1}{3x^2\,(c_2\,x^4 + c_1)}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y\,x^2}{c_2\,x^4 + c_1}\right) &= \left(\frac{x^2}{c_2\,x^4 + c_1}\right)\left(\frac{3c_2\,x^4 - c_1}{3x^2\,(c_2\,x^4 + c_1)}\right) \\
\mathrm{d}\left(\frac{y\,x^2}{c_2\,x^4 + c_1}\right) &= \left(\frac{3c_2\,x^4 - c_1}{3\,(c_2\,x^4 + c_1)^2}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y\,x^2}{c_2\,x^4 + c_1} &= \int \frac{3c_2\,x^4 - c_1}{3\,(c_2\,x^4 + c_1)^2}\,dx \\
&= -\frac{x}{3\,(c_2\,x^4 + c_1)} + c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{x^2}{c_2\,x^4 + c_1}$ gives the final solution

$$
y = \frac{3c_2c_3\,x^4 + 3c_1c_3 - x}{3x^2}
$$

Hence, the solution found using Lagrange adjoint equation method is

$$
y = \frac{3c_2c_3\,x^4 + 3c_1c_3 - x}{3x^2}
$$

The constants can be merged to give

$$y = \frac{3c_2 x^4 + 3c_1 - x}{3x^2}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{3c_2 x^4 + 3c_1 - x}{3x^2}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : `0.012` (sec)
Leaf size : 22

```
dsolve(x^4*diff(diff(y(x),x),x)+diff(y(x),x)*x^3-4*x^2*y(x) = x,
       y(x),singsol=all)
```

$$y = \frac{3c_1 x^4 + 3c_2 - x}{3x^2}$$

Mathematica DSolve solution

Solving time : 0.016 (sec)
Leaf size : 25

```
DSolve[{x^4*D[y[x],{x,2}]+x^3*D[y[x],x]-4*x^2*y[x]==x,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2 x^2 + \frac{c_1}{x^2} - \frac{1}{3x}$$

2.3.17 problem 17

Internal problem ID [8551]
Book : Own collection of miscellaneous problems
Section : section 3.0
Problem number : 17
Date solved : `Thursday, December 12, 2024 at 09:30:00 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' - 4y = x$$

Solved as second order Euler type ode

Time used: 0.156 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = -4, f(x) = x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + xy' - 4y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} - 4x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r - 4x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + r - 4 = 0$$

Or

$$r^2 - 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -2 \\ r_2 &= 2 \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x^2} + c_2\,x^2$$

Next, we find the particular solution to the ODE

$$x^2 y'' + xy' - 4y = x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \frac{1}{x^2} \\ y_2 &= x^2 \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & x^2 \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}\left(x^2\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & x^2 \\ -\frac{2}{x^3} & 2x \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)(2x) - \left(x^2\right)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{4}{x}$$

Which simplifies to

$$W = \frac{4}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^3}{4x}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{x^2}{4}dx$$

Hence

$$u_1 = -\frac{x^3}{12}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1}{x}}{4x} \, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{4x^2} dx$$

Hence

$$u_2 = -\frac{1}{4x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{c_1}{x^2} + c_2 \, x^2 - \frac{x}{3} \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{c_1}{x^2} + c_2 \, x^2 - \frac{x}{3}$$

Solved as second order ode using change of variable on x method 2

Time used: 0.412 (sec)

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + xy' - 4y = 0$$

In normal form the ode

$$x^2y'' + xy' - 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{4}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\int p(x)dx} dx \\ &= \int \mathrm{e}^{-\int \frac{1}{x}dx} dx \\ &= \int e^{-\ln(x)}\, dx \\ &= \int \frac{1}{x} dx \\ &= \ln(x) \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{-\frac{4}{x^2}}{\frac{1}{x^2}} \\ &= -4 \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) - 4y(\tau) = 0$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -4$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 e^{\tau\lambda} - 4\, e^{\tau\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -4$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-4)} \\ &= \pm 2 \end{aligned}$$

Hence

$$\lambda_1 = +2$$
$$\lambda_2 = -2$$

Which simplifies to

$$\lambda_1 = 2$$
$$\lambda_2 = -2$$

Since roots are real and distinct, then the solution is

$$y(\tau) = c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau}$$
$$y(\tau) = c_1 e^{(2)\tau} + c_2 e^{(-2)\tau}$$

Or

$$y(\tau) = c_1 \,\mathrm{e}^{2\tau} + c_2 \,\mathrm{e}^{-2\tau}$$

Will add steps showing solving for IC soon.

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \,x^2 + \frac{c_2}{x^2}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \,x^2 + \frac{c_2}{x^2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{1}{x^2}, x^2\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_2x - 4A_1 = x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = -\frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\,x^2 + \frac{c_2}{x^2}\right) + \left(-\frac{x}{3}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{x}{3} + c_1\,x^2 + \frac{c_2}{x^2}$$

Solved as second order ode using change of variable on x method 1

Time used: 0.140 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = -4, f(x) = x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + xy' - 4y = 0$$

In normal form the ode

$$x^2y'' + xy' - 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned}p(x) &= \frac{1}{x}\\ q(x) &= -\frac{4}{x^2}\end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q}\\ &= \frac{2\sqrt{-\frac{1}{x^2}}}{c}\\ \tau'' &= \frac{2}{c\sqrt{-\frac{1}{x^2}}\,x^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2}\\ &= \frac{\frac{2}{c\sqrt{-\frac{1}{x^2}}\,x^3} + \frac{1}{x}\frac{2\sqrt{-\frac{1}{x^2}}}{c}}{\left(\frac{2\sqrt{-\frac{1}{x^2}}}{c}\right)^2}\\ &= 0\end{aligned}$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c} \sqrt{q}\, dx \\
&= \frac{\int 2\sqrt{-\frac{1}{x^2}}dx}{c} \\
&= \frac{2\sqrt{-\frac{1}{x^2}}\, x \ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 \cos\left(2\sqrt{-\frac{1}{x^2}}\, x \ln(x)\right) + c_2 \sin\left(2\sqrt{-\frac{1}{x^2}}\, x \ln(x)\right)
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1, x\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\cos\left(2\sqrt{-\frac{1}{x^2}}\, x \ln(x)\right), \sin\left(2\sqrt{-\frac{1}{x^2}}\, x \ln(x)\right)\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_2 x + A_1
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_2x - 4A_1 = x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = -\frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos\left(2\sqrt{-\frac{1}{x^2}}\, x \ln(x)\right) + c_2 \sin\left(2\sqrt{-\frac{1}{x^2}}\, x \ln(x)\right)\right) + \left(-\frac{x}{3}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \cos\left(2\sqrt{-\frac{1}{x^2}}\, x \ln(x)\right) + c_2 \sin\left(2\sqrt{-\frac{1}{x^2}}\, x \ln(x)\right) - \frac{x}{3}$$

Solved as second order ode using change of variable on y method 2

Time used: 0.220 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = -4, f(x) = x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + xy' - 4y = 0$$

In normal form the ode

$$x^2y'' + xy' - 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{4}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^2} - \frac{4}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{5v'(x)}{x} &= 0 \\ v''(x) + \frac{5v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{5u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form a linear first order is

$$u'(x) + q(x)u(x) = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= \frac{5}{x} \\ p(x) &= 0 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= \mathrm{e}^{\int \frac{5}{x}dx} \\ &= x^5 \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mu u &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(u\,x^5\right) &= 0 \end{aligned}$$

Integrating gives

$$\begin{aligned} u\,x^5 &= \int 0\,dx + c_1 \\ &= c_1 \end{aligned}$$

Dividing throughout by the integrating factor x^5 gives the final solution

$$u(x) = \frac{c_1}{x^5}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= -\frac{c_1}{4x^4} + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{c_1}{4x^4} + c_2\right) x^2 \\
&= \frac{4c_2\, x^4 - c_1}{4x^2}
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
x^2y'' + xy' - 4y = x
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \frac{1}{x^2} \\
y_2 &= x^2
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} \frac{1}{x^2} & x^2 \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}\left(x^2\right) \end{vmatrix}
$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & x^2 \\ -\frac{2}{x^3} & 2x \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)(2x) - \left(x^2\right)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{4}{x}$$

Which simplifies to

$$W = \frac{4}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^3}{4x}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{x^2}{4} dx$$

Hence

$$u_1 = -\frac{x^3}{12}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1}{x}}{4x}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{4x^2} dx$$

Hence

$$u_2 = -\frac{1}{4x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{4x^4} + c_2\right) x^2\right) + \left(-\frac{x}{3}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \left(-\frac{c_1}{4x^4} + c_2\right) x^2 - \frac{x}{3}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.231 (sec)

Writing the ode as

$$x^2 y'' + x y' - 4y = 0 \tag{1}$$
$$A y'' + B y' + C y = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{15}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 15 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{15}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.144: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{15}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{15}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{15}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{5}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{3}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{15}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{5}{2}$	$-\frac{3}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{3}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{3}{2} - \left(-\frac{3}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{3}{2x} + (-)\,(0) \\
&= -\frac{3}{2x} \\
&= -\frac{3}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{3}{2x}\right)(0) + \left(\left(\frac{3}{2x^2}\right) + \left(-\frac{3}{2x}\right)^2 - \left(\frac{15}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{3}{2x} dx} \\ &= \frac{1}{x^{3/2}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x}{x^2} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^2}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{x^4}{4}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x^2}\right) + c_2\left(\frac{1}{x^2}\left(\frac{x^4}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2y'' + xy' - 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \frac{c_1}{x^2} + \frac{c_2\, x^2}{4}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \frac{1}{x^2} \\
y_2 &= \frac{x^2}{4}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{x^2}{4} \\ \frac{d}{dx}\left(\frac{1}{x^2}\right) & \frac{d}{dx}\left(\frac{x^2}{4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x^2} & \frac{x^2}{4} \\ -\frac{2}{x^3} & \frac{x}{2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x^2}\right)\left(\frac{x}{2}\right) - \left(\frac{x^2}{4}\right)\left(-\frac{2}{x^3}\right)$$

Which simplifies to

$$W = \frac{1}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{x^3}{4}}{x}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{x^2}{4} dx$$

Hence

$$u_1 = -\frac{x^3}{12}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\frac{1}{x}}{x}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x^2} dx$$

Hence

$$u_2 = -\frac{1}{x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x^2} + \frac{c_2\, x^2}{4}\right) + \left(-\frac{x}{3}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{c_1}{x^2} + \frac{c_2\, x^2}{4} - \frac{x}{3}$$

Solved as second order ode adjoint method

Time used: 0.171 (sec)

In normal form the ode

$$x^2 y'' + x y' - 4y = x \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{4}{x^2} \\ r(x) &= \frac{1}{x} \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - \left(\frac{\xi(x)}{x}\right)' + \left(-\frac{4\xi(x)}{x^2}\right) &= 0 \\
\xi''(x) - \frac{3\xi(x)}{x^2} - \frac{\xi'(x)}{x} &= 0
\end{aligned}$$

Which is solved for $\xi(x)$. This is Euler second order ODE. Let the solution be $\xi = x^r$, then $\xi' = r x^{r-1}$ and $\xi'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - x r x^{r-1} - 3x^r = 0$$

Simplifying gives

$$r(r-1)\, x^r - r\, x^r - 3x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - r - 3 = 0$$

Or

$$r^2 - 2r - 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned}
r_1 &= -1 \\
r_2 &= 3
\end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$\xi = c_1 \xi_1 + c_2 \xi_2$$

Where $\xi_1 = x^{r_1}$ and $\xi_2 = x^{r_2}$. Hence

$$\xi = \frac{c_1}{x} + c_2\, x^3$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}
\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)}
\end{aligned}$$

Or

$$y' + y\left(\frac{1}{x} - \frac{-\frac{c_1}{x^2} + 3c_2\,x^2}{\frac{c_1}{x} + c_2\,x^3}\right) = \frac{\frac{c_2\,x^3}{3} - \frac{c_1}{x}}{\frac{c_1}{x} + c_2\,x^3}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = -\frac{2c_2\,x^4 - 2c_1}{x\,(c_2\,x^4 + c_1)}$$
$$p(x) = \frac{c_2\,x^4 - 3c_1}{3c_2\,x^4 + 3c_1}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dx}\\ &= \mathrm{e}^{\int -\frac{2c_2\,x^4 - 2c_1}{x\left(c_2\,x^4 + c_1\right)}dx}\\ &= \frac{x^2}{c_2\,x^4 + c_1}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p\\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_2\,x^4 - 3c_1}{3c_2\,x^4 + 3c_1}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y\,x^2}{c_2\,x^4 + c_1}\right) &= \left(\frac{x^2}{c_2\,x^4 + c_1}\right)\left(\frac{c_2\,x^4 - 3c_1}{3c_2\,x^4 + 3c_1}\right)\\ \mathrm{d}\left(\frac{y\,x^2}{c_2\,x^4 + c_1}\right) &= \left(\frac{(c_2\,x^4 - 3c_1)\,x^2}{(3c_2\,x^4 + 3c_1)\,(c_2\,x^4 + c_1)}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y\,x^2}{c_2\,x^4 + c_1} &= \int \frac{(c_2\,x^4 - 3c_1)\,x^2}{(3c_2\,x^4 + 3c_1)\,(c_2\,x^4 + c_1)}\,dx\\ &= -\frac{x^3}{3\,(c_2\,x^4 + c_1)} + c_3\end{aligned}$$

Dividing throughout by the integrating factor $\frac{x^2}{c_2 x^4+c_1}$ gives the final solution

$$y = \frac{3c_2c_3 x^4 - x^3 + 3c_1c_3}{3x^2}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{3c_2c_3 x^4 - x^3 + 3c_1c_3}{3x^2}$$

The constants can be merged to give

$$y = \frac{3c_2 x^4 - x^3 + 3c_1}{3x^2}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{3c_2 x^4 - x^3 + 3c_1}{3x^2}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 18

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x-4*y(x) = x,
       y(x),singsol=all)
```

$$y = \frac{c_2}{x^2} + c_1 x^2 - \frac{x}{3}$$

Mathematica DSolve solution

Solving time : 0.016 (sec)
Leaf size : 23

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]-4*y[x] == x,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_2 x^2 + \frac{c_1}{x^2} - \frac{x}{3}$$

2.3.18 problem 18

Internal problem ID [8552]
Book : Own collection of miscellaneous problems
Section : section 3.0
Problem number : 18
Date solved : `Thursday, December 12, 2024 at 09:30:02 AM`
CAS classification : `[[_3rd_order, _with_linear_symmetries]]`

Solve

$$x^4y''' + x^3y'' + x^2y' + xy = 0$$

Solved as higher order Euler type ode

Time used: 0.169 (sec)

The ode can be normalized and rewritten as Euler ode.

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned} y' &= \lambda\, x^{\lambda-1} \\ y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\ y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \end{aligned}$$

Substituting these back into

$$x^4y''' + x^3y'' + x^2y' + xy = 0$$

gives

$$x\lambda\, x^{\lambda-1} + x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} + x^\lambda = 0$$

Which simplifies to

$$\lambda\, x^\lambda + \lambda(\lambda-1)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\, x^\lambda + x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$\lambda + \lambda(\lambda-1) + \lambda(\lambda-1)(\lambda-2) + 1 = 0$$

Simplifying gives the characteristic equation as

$$\lambda^3 - 2\lambda^2 + 2\lambda + 1 = 0$$

Solving the above gives the following roots

$$\lambda_1 = -\frac{\left(188+12\sqrt{249}\right)^{1/3}}{6} + \frac{4}{3\left(188+12\sqrt{249}\right)^{1/3}} + \frac{2}{3}$$

$$\lambda_2 = \frac{\left(188+12\sqrt{249}\right)^{1/3}}{12} - \frac{2}{3\left(188+12\sqrt{249}\right)^{1/3}} + \frac{2}{3} + \frac{i\sqrt{3}\left(-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{6} - \frac{4}{3\left(188+12\sqrt{249}\right)^{1/3}}\right)}{2}$$

$$\lambda_3 = \frac{\left(188+12\sqrt{249}\right)^{1/3}}{12} - \frac{2}{3\left(188+12\sqrt{249}\right)^{1/3}} + \frac{2}{3} - \frac{i\sqrt{3}\left(-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{6} - \frac{4}{3\left(188+12\sqrt{249}\right)^{1/3}}\right)}{2}$$

This table summarises the result

root	multiplicity	type of ro
$\frac{\left(188+12\sqrt{249}\right)^{1/3}}{12} - \frac{2}{3\left(188+12\sqrt{249}\right)^{1/3}} + \frac{2}{3} \pm -\frac{\sqrt{3}\left(-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{6} - \frac{4}{3\left(188+12\sqrt{249}\right)^{1/3}}\right)}{2} i$	1	complex c
$-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{6} + \frac{4}{3\left(188+12\sqrt{249}\right)^{1/3}} + \frac{2}{3}$	1	real root

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a $c_1 x^\lambda$ basis solution. Each real root of multiplicty two, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ and $c_3 x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2 x^\alpha(c_1 \cos(\beta \ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = x^{\frac{(188+12\sqrt{249})^{1/3}}{12}-\frac{2}{3(188+12\sqrt{249})^{1/3}}+\frac{2}{3}}\left(c_1\cos\left(\frac{\sqrt{3}\left(-\frac{(188+12\sqrt{249})^{1/3}}{6}-\frac{4}{3(188+12\sqrt{249})^{1/3}}\right)\ln(x)}{2}\right)-c_2\sin\right.$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = x^{\frac{(188+12\sqrt{249})^{1/3}}{12}-\frac{2}{3(188+12\sqrt{249})^{1/3}}+\frac{2}{3}}\cos\left(\frac{\sqrt{3}\left(-\frac{(188+12\sqrt{249})^{1/3}}{6}-\frac{4}{3(188+12\sqrt{249})^{1/3}}\right)\ln(x)}{2}\right)$$

$$y_2 = -x^{\frac{(188+12\sqrt{249})^{1/3}}{12}-\frac{2}{3(188+12\sqrt{249})^{1/3}}+\frac{2}{3}}\sin\left(\frac{\sqrt{3}\left(-\frac{(188+12\sqrt{249})^{1/3}}{6}-\frac{4}{3(188+12\sqrt{249})^{1/3}}\right)\ln(x)}{2}\right)$$

$$y_3 = x^{-\frac{(188+12\sqrt{249})^{1/3}}{6}+\frac{4}{3(188+12\sqrt{249})^{1/3}}+\frac{2}{3}}$$

Maple step by step solution

Let's solve

$x^4\left(\frac{d^3}{dx^3}y(x)\right)+\left(\frac{d^2}{dx^2}y(x)\right)x^3+x^2\left(\frac{d}{dx}y(x)\right)+xy(x)=0$

- Highest derivative means the order of the ODE is 3

  $\frac{d^3}{dx^3}y(x)$

- Isolate 3rd derivative

  $\frac{d^3}{dx^3}y(x)=-\frac{y(x)}{x^3}-\frac{\left(\frac{d^2}{dx^2}y(x)\right)x+\frac{d}{dx}y(x)}{x^2}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^3}{dx^3}y(x)+\frac{\frac{d^2}{dx^2}y(x)}{x}+\frac{\frac{d}{dx}y(x)}{x^2}+\frac{y(x)}{x^3}=0$

- Multiply by denominators of the ODE

  $x^3\left(\frac{d^3}{dx^3}y(x)\right)+x^2\left(\frac{d^2}{dx^2}y(x)\right)+x\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Make a change of variables
  $t = \ln(x)$
- □ Substitute the change of variables back into the ODE
  - ○ Calculate the 1st derivative of y with respect to x , using the chain rule
    $\frac{d}{dx}y(x) = \left(\frac{d}{dt}y(t)\right)\left(\frac{d}{dx}t(x)\right)$
  - ○ Compute derivative
    $\frac{d}{dx}y(x) = \frac{\frac{d}{dt}y(t)}{x}$
  - ○ Calculate the 2nd derivative of y with respect to x , using the chain rule
    $\frac{d^2}{dx^2}y(x) = \left(\frac{d^2}{dt^2}y(t)\right)\left(\frac{d}{dx}t(x)\right)^2 + \left(\frac{d^2}{dx^2}t(x)\right)\left(\frac{d}{dt}y(t)\right)$
  - ○ Compute derivative
    $\frac{d^2}{dx^2}y(x) = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$
  - ○ Calculate the 3rd derivative of y with respect to x , using the chain rule
    $\frac{d^3}{dx^3}y(x) = \left(\frac{d^3}{dt^3}y(t)\right)\left(\frac{d}{dx}t(x)\right)^3 + 3\left(\frac{d}{dx}t(x)\right)\left(\frac{d^2}{dx^2}t(x)\right)\left(\frac{d^2}{dt^2}y(t)\right) + \left(\frac{d^3}{dx^3}t(x)\right)\left(\frac{d}{dt}y(t)\right)$
  - ○ Compute derivative
    $\frac{d^3}{dx^3}y(x) = \frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}$

  Substitute the change of variables back into the ODE
  $x^3\left(\frac{\frac{d^3}{dt^3}y(t)}{x^3} - \frac{3\left(\frac{d^2}{dt^2}y(t)\right)}{x^3} + \frac{2\left(\frac{d}{dt}y(t)\right)}{x^3}\right) + x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + \frac{d}{dt}y(t) + y(t) = 0$
- Simplify
  $\frac{d^3}{dt^3}y(t) - 2\frac{d^2}{dt^2}y(t) + 2\frac{d}{dt}y(t) + y(t) = 0$
- □ Convert linear ODE into a system of first order ODEs
  - ○ Define new variable $y_1(t)$
    $y_1(t) = y(t)$
  - ○ Define new variable $y_2(t)$
    $y_2(t) = \frac{d}{dt}y(t)$
  - ○ Define new variable $y_3(t)$
    $y_3(t) = \frac{d^2}{dt^2}y(t)$
  - ○ Isolate for $\frac{d}{dt}y_3(t)$ using original ODE
    $\frac{d}{dt}y_3(t) = 2y_3(t) - 2y_2(t) - y_1(t)$

  Convert linear ODE into a system of first order ODEs
  $\left[y_2(t) = \frac{d}{dt}y_1(t), y_3(t) = \frac{d}{dt}y_2(t), \frac{d}{dt}y_3(t) = 2y_3(t) - 2y_2(t) - y_1(t)\right]$
- Define vector

$$\vec{y}(t) = \left[\begin{array}{c} y_1(t) \\ y_2(t) \\ y_3(t) \end{array}\right]$$

- System to solve

$$\frac{d}{dt}\vec{y}(t) = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & 2 \end{array}\right] \cdot \vec{y}(t)$$

- Define the coefficient matrix

$$A = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & 2 \end{array}\right]$$

- Rewrite the system as

$$\frac{d}{dt}\vec{y}(t) = A \cdot \vec{y}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{6} + \frac{4}{3\left(188+12\sqrt{249}\right)^{1/3}} + \frac{2}{3}, \left[\begin{array}{c} \frac{1}{\left(-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{6}+\frac{4}{3\left(188+12\sqrt{249}\right)^{1/3}}+\frac{2}{3}\right)^2} \\ \frac{1}{-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{6}+\frac{4}{3\left(188+12\sqrt{249}\right)^{1/3}}+\frac{2}{3}} \\ 1 \end{array}\right]\right], \left[\frac{\left(188+12\sqrt{}\right.}{12}\right.\right.$$

- Consider eigenpair

$$\left[-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{6} + \frac{4}{3\left(188+12\sqrt{249}\right)^{1/3}} + \frac{2}{3}, \left[\begin{array}{c} \frac{1}{\left(-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{6}+\frac{4}{3\left(188+12\sqrt{249}\right)^{1/3}}+\frac{2}{3}\right)^2} \\ \frac{1}{-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{6}+\frac{4}{3\left(188+12\sqrt{249}\right)^{1/3}}+\frac{2}{3}} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{y}_1 = \mathrm{e}^{\left(-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{6}+\frac{4}{3\left(188+12\sqrt{249}\right)^{1/3}}+\frac{2}{3}\right)t} \cdot \left[\begin{array}{c} \frac{1}{\left(-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{6}+\frac{4}{3\left(188+12\sqrt{249}\right)^{1/3}}+\frac{2}{3}\right)^{2}} \\ \frac{1}{-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{6}+\frac{4}{3\left(188+12\sqrt{249}\right)^{1/3}}+\frac{2}{3}} \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[\frac{\left(188+12\sqrt{249}\right)^{1/3}}{12}-\frac{2}{3\left(188+12\sqrt{249}\right)^{1/3}}+\frac{2}{3}-\frac{\mathrm{I}\sqrt{3}\left(-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{6}-\frac{4}{3\left(188+12\sqrt{249}\right)^{1/3}}\right)}{2}, \left[\begin{array}{c} \frac{\quad}{\left(\frac{\left(188+12\sqrt{249}\right)^{1/3}}{12}- \cdots\right.} \\ \frac{\quad}{\frac{\left(188+12\sqrt{249}\right)^{1/3}}{12}- \cdots} \end{array}\right.\right.$$

- Solution from eigenpair

$$\mathrm{e}^{\left(\frac{\left(188+12\sqrt{249}\right)^{1/3}}{12}-\frac{2}{3\left(188+12\sqrt{249}\right)^{1/3}}+\frac{2}{3}-\frac{\mathrm{I}\sqrt{3}\left(-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{6}-\frac{4}{3\left(188+12\sqrt{249}\right)^{1/3}}\right)}{2}\right)t} \cdot \left[\begin{array}{c} \frac{\quad}{\left(\frac{\left(188+12\sqrt{249}\right)^{1/3}}{12}-\frac{\cdots}{3\left(188+12 \cdots\right.}\right.} \\ \frac{\quad}{\frac{\left(188+12\sqrt{249}\right)^{1/3}}{12}-\frac{\cdots}{3\left(188+1 \cdots\right.}} \end{array}\right.$$

- Use Euler identity to write solution in terms of sin and cos

$$\mathrm{e}^{\left(\frac{\left(188+12\sqrt{249}\right)^{1/3}}{12}-\frac{2}{3\left(188+12\sqrt{249}\right)^{1/3}}+\frac{2}{3}\right)t} \cdot \left(\cos\left(\frac{\sqrt{3}\left(-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{6}-\frac{4}{3\left(188+12\sqrt{249}\right)^{1/3}}\right)t}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\left(- \cdots\right.}{\quad}\right.\right.$$

- Simplify expression

$$
e^{\left(\frac{\left(188+12\sqrt{249}\right)^{1/3}}{12}-\frac{2}{3\left(188+12\sqrt{249}\right)^{1/3}}+\frac{2}{3}\right)t}\cdot\left[\begin{array}{c}
\frac{\cos\left(\frac{\sqrt{3}\left(-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{6}-\frac{4}{3\left(188+12\sqrt{249}\right)^{1/3}}\right)t}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\left(-\frac{(188}{}\right.}{}\right.}{\left(\frac{\left(188+12\sqrt{249}\right)^{1/3}}{12}-\frac{2}{3\left(188+12\sqrt{249}\right)^{1/3}}+\frac{2}{3}-\frac{\mathrm{I}\sqrt{3}\left(-\frac{\left(188+12\sqrt{249}\right.}{6}\right.}{}\right.} \\
\frac{\cos\left(\frac{\sqrt{3}\left(-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{6}-\frac{4}{3\left(188+12\sqrt{249}\right)^{1/3}}\right)t}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\left(-\frac{(188}{}\right.}{}\right.}{\frac{\left(188+12\sqrt{249}\right)^{1/3}}{12}-\frac{2}{3\left(188+12\sqrt{249}\right)^{1/3}}+\frac{2}{3}-\frac{\mathrm{I}\sqrt{3}\left(-\frac{\left(188+12\sqrt{24\right.}{6}\right.}{}} \\
\cos\left(\frac{\sqrt{3}\left(-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{6}-\frac{4}{3\left(188+12\sqrt{249}\right)^{1/3}}\right)t}{2}\right)-\mathrm{I}\sin\left(\frac{\sqrt{3}\left(-\right.}{}\right.
\end{array}\right.
$$

- Both real and imaginary parts are solutions to the homogeneous system

$$
\left[\vec{y}_2(t)=e^{\left(\frac{\left(188+12\sqrt{249}\right)^{1/3}}{12}-\frac{2}{3\left(188+12\sqrt{249}\right)^{1/3}}+\frac{2}{3}\right)t}\cdot\left[\frac{18\left(188+12\sqrt{249}\right)^{2/3}\left(\sqrt{3}\left(188+12\sqrt{249}\right)^{4/3}\sin\left(\frac{\sqrt{3}\left(\left(188+12\sqrt{\right.\right.}{12\left(188+12\right.}\right.\right.}{}\right.\right.
$$

- General solution to the system of ODEs

$\vec{y}=C1\vec{y}_1+C2\vec{y}_2(t)+C3\vec{y}_3(t)$

- Substitute solutions into the general solution

$$\vec{y} = C1\, \mathrm{e}^{\left(-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{6}+\frac{4}{3\left(188+12\sqrt{249}\right)^{1/3}}+\frac{2}{3}\right)t} \cdot \left[\begin{array}{c} \frac{1}{\left(-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{6}+\frac{4}{3\left(188+12\sqrt{249}\right)^{1/3}}+\frac{2}{3}\right)^{2}} \\ \frac{1}{-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{6}+\frac{4}{3\left(188+12\sqrt{249}\right)^{1/3}}+\frac{2}{3}} \\ 1 \end{array}\right] + C2\, \mathrm{e}^{\left(\right.}$$

- First component of the vector is the solution to the ODE

$$y(t) = \frac{96\left(188+12\sqrt{83}\,\sqrt{3}\right)^{2/3}\left(\left(\left(\left(\left(-\frac{7\sqrt{3}\,C2}{96}-\frac{7C3}{32}\right)\sqrt{83}-\frac{115C2}{96}-\frac{115C3\sqrt{3}}{96}\right)\left(188+12\sqrt{83}\,\sqrt{3}\right)^{2/3}+C2\left(\sqrt{83}\,\sqrt{3}+\frac{47}{3}\right)\left(188+1\right.\right.\right.\right.\right.}{}$$

- Change variables back using $t = \ln(x)$

$$y(x) = \frac{96\left(188+12\sqrt{83}\,\sqrt{3}\right)^{2/3}\left(\left(\left(\left(\left(-\frac{7\sqrt{3}\,C2}{96}-\frac{7C3}{32}\right)\sqrt{83}-\frac{115C2}{96}-\frac{115C3\sqrt{3}}{96}\right)\left(188+12\sqrt{83}\,\sqrt{3}\right)^{2/3}+C2\left(\sqrt{83}\,\sqrt{3}+\frac{47}{3}\right)\left(188+\right.\right.\right.\right.\right.}{}$$

- Simplify

$$y(x) = -\frac{7\left(\left(\left(\left(\sqrt{3}\,C2+3C3\right)\sqrt{83}+\frac{115C2}{7}+\frac{115C3\sqrt{3}}{7}\right)\left(188+12\sqrt{83}\,\sqrt{3}\right)^{2/3}-\frac{96C2\left(\sqrt{83}\,\sqrt{3}+\frac{47}{3}\right)\left(188+12\sqrt{83}\,\sqrt{3}\right)^{1/3}}{7}+\frac{320\left(\sqrt{3}\,C2\right.}{7}\right.\right.}{}$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 184

```
dsolve(x^4*diff(diff(diff(y(x),x),x),x)+x^3*diff(diff(y(x),x),x)+diff(y(x),x)*x^2+x*y(
       y(x),singsol=all)
```

$$y = c_1 x^{-\frac{\left(188+12\sqrt{249}\right)^{2/3}-4\left(188+12\sqrt{249}\right)^{1/3}-8}{6\left(188+12\sqrt{249}\right)^{1/3}}} + c_2 x^{\frac{-8+\left(188+12\sqrt{249}\right)^{2/3}+8\left(188+12\sqrt{249}\right)^{1/3}}{12\left(188+12\sqrt{249}\right)^{1/3}}} \sin\left(\frac{\sqrt{3}\left(\left(188+12\sqrt{3}\,\sqrt{83}\right)^{2/3}+8\right)\ln\left(x\right)}{12\left(188+12\sqrt{3}\,\sqrt{83}\right)^{1/3}}\right) + c_3\, x^{\frac{-8+\left(188+12\sqrt{249}\right)^{2/3}}{12\left(188+12\sqrt{\right.}}}$$

Mathematica DSolve solution

Solving time : 0.005 (sec)
Leaf size : 81

```
DSolve[{x^4*D[y[x],{x,3}]+x^3*D[y[x],{x,2}]+x^2*D[y[x],x]+x*y[x]== 0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 x^{\text{Root}\left[\#1^3-2\#1^2+2\#1+1\&,1\right]} + c_3 x^{\text{Root}\left[\#1^3-2\#1^2+2\#1+1\&,3\right]} + c_2 x^{\text{Root}\left[\#1^3-2\#1^2+2\#1+1\&,2\right]}$$

2.3.19 problem 19

Internal problem ID [8553]
Book : Own collection of miscellaneous problems
Section : section 3.0
Problem number : 19
Date solved : `Thursday, December 12, 2024 at 09:30:02 AM`
CAS classification : `[[_3rd_order, _with_linear_symmetries]]`

Solve

$$x^4y''' + x^3y'' + x^2y' + xy = x$$

Solved as higher order Euler type ode

Time used: 3.369 (sec)

The ode can be normalized and rewritten as Euler ode.

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned}
y' &= \lambda\, x^{\lambda-1} \\
y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\
y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3}
\end{aligned}$$

Substituting these back into

$$x^4y''' + x^3y'' + x^2y' + xy = x$$

gives

$$x\lambda\, x^{\lambda-1} + x^2\lambda(\lambda-1)\, x^{\lambda-2} + x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} + x^\lambda = 0$$

Which simplifies to

$$\lambda\, x^\lambda + \lambda(\lambda-1)\, x^\lambda + \lambda(\lambda-1)\,(\lambda-2)\, x^\lambda + x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$\lambda + \lambda(\lambda - 1) + \lambda(\lambda - 1)(\lambda - 2) + 1 = 0$$

Simplifying gives the characteristic equation as

$$\lambda^3 - 2\lambda^2 + 2\lambda + 1 = 0$$

Solving the above gives the following roots

$$\lambda_1 = -\frac{(188 + 12\sqrt{249})^{1/3}}{6} + \frac{4}{3(188 + 12\sqrt{249})^{1/3}} + \frac{2}{3}$$

$$\lambda_2 = \frac{(188 + 12\sqrt{249})^{1/3}}{12} - \frac{2}{3(188 + 12\sqrt{249})^{1/3}} + \frac{2}{3} + \frac{i\sqrt{3}\left(-\frac{(188+12\sqrt{249})^{1/3}}{6} - \frac{4}{3(188+12\sqrt{249})^{1/3}}\right)}{2}$$

$$\lambda_3 = \frac{(188 + 12\sqrt{249})^{1/3}}{12} - \frac{2}{3(188 + 12\sqrt{249})^{1/3}} + \frac{2}{3} - \frac{i\sqrt{3}\left(-\frac{(188+12\sqrt{249})^{1/3}}{6} - \frac{4}{3(188+12\sqrt{249})^{1/3}}\right)}{2}$$

This table summarises the result

root	multiplicity	type of ro
$-\frac{(188+12\sqrt{249})^{1/3}}{6} + \frac{4}{3(188+12\sqrt{249})^{1/3}} + \frac{2}{3}$	1	real root
$\frac{(188+12\sqrt{249})^{1/3}}{12} - \frac{2}{3(188+12\sqrt{249})^{1/3}} + \frac{2}{3} \pm -\frac{\sqrt{3}\left(-\frac{(188+12\sqrt{249})^{1/3}}{6} - \frac{4}{3(188+12\sqrt{249})^{1/3}}\right)}{2}i$	1	complex c

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a $c_1 x^\lambda$ basis solution. Each real root of multiplicty two, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates $c_1 x^\lambda$ and $c_2 x^\lambda \ln(x)$ and $c_3 x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2 \sin(\beta \ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2\, x^\alpha(c_1 \cos(\beta \ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

$$y = c_1 x^{-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{6}+\frac{4}{3\left(188+12\sqrt{249}\right)^{1/3}}+\frac{2}{3}} + x^{\frac{\left(188+12\sqrt{249}\right)^{1/3}}{12}-\frac{2}{3\left(188+12\sqrt{249}\right)^{1/3}}+\frac{2}{3}} \left(c_2 \cos\left(\frac{\sqrt{3}\left(-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{6} \right.}{}\right.\right.$$

The fundamental set of solutions for the homogeneous solution are the following

$$y_1 = x^{-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{6}+\frac{4}{3\left(188+12\sqrt{249}\right)^{1/3}}+\frac{2}{3}}$$

$$y_2 = x^{\frac{\left(188+12\sqrt{249}\right)^{1/3}}{12}-\frac{2}{3\left(188+12\sqrt{249}\right)^{1/3}}+\frac{2}{3}} \cos\left(\frac{\sqrt{3}\left(-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{6}-\frac{4}{3\left(188+12\sqrt{249}\right)^{1/3}}\right)\ln(x)}{2}\right)$$

$$y_3 = -x^{\frac{\left(188+12\sqrt{249}\right)^{1/3}}{12}-\frac{2}{3\left(188+12\sqrt{249}\right)^{1/3}}+\frac{2}{3}} \sin\left(\frac{\sqrt{3}\left(-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{6}-\frac{4}{3\left(188+12\sqrt{249}\right)^{1/3}}\right)\ln(x)}{2}\right)$$

This is higher order nonhomogeneous Euler type ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous Euler ODE And y_p is a particular solution to the nonhomogeneous Euler ODE. y_h is the solution to

$$y'x + y''x^2 + y'''x^3 + y = 0$$

Now the particular solution to the given ODE is found

$$y'x + y''x^2 + y'''x^3 + y = x$$

Let the particular solution be

$$y_p = U_1 y_1 + U_2 y_2 + U_3 y_3$$

Where y_i are the basis solutions found above for the homogeneous solution y_h and $U_i(x)$ are functions to be determined as follows

$$U_i = (-1)^{n-i} \int \frac{F(x)W_i(x)}{aW(x)}\, dx$$

Where $W(x)$ is the Wronskian and $W_i(x)$ is the Wronskian that results after deleting the last row and the i-th column of the determinant and n is the order of the ODE or equivalently, the number of basis solutions, and a is the coefficient of the leading derivative in the ODE, and $F(x)$ is the RHS of the ODE. Therefore, the first step is to find the Wronskian $W(x)$. This is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Substituting the fundamental set of solutions y_i found above in the Wronskian gives

$$W = \text{Expression too large to display}$$
$$|W| = \text{Expression too large to display}$$

The determinant simplifies to

$$|W| = \frac{\sqrt{83}}{2x}$$

Now we determine W_i for each U_i.

$$W_1(x) = \det \begin{bmatrix} \cos\left(\frac{\sqrt{3}\left(\left(188+12\sqrt{3}\sqrt{83}\right)^{2/3}+8\right)\ln(x)}{12\left(188+12\sqrt{3}\sqrt{83}\right)^{1/3}}\right) x^{\frac{\left(188+12\ldots\right.}{}} \\ -\frac{x^{\frac{\left(188+12\sqrt{3}\sqrt{83}\right)^{2/3}-4\left(188+12\sqrt{3}\sqrt{83}\right)^{1/3}-8}{12\left(188+12\sqrt{3}\sqrt{83}\right)^{1/3}}}\left(\left(-\left(188+12\sqrt{3}\sqrt{83}\right)^{2/3}-8\left(188+12\sqrt{3}\sqrt{83}\right)^{1/3}+8\right)\cos\left(\frac{\sqrt{3}\left(\left(188+12\ldots\right.\right.}{12\left(188+\ldots\right.}\right.\right.}{12\left(188+12\sqrt{3}\sqrt{83}\right)^{1/3}} \end{bmatrix}$$

$$= \frac{\sqrt{3}\left(\left(188+12\sqrt{3}\sqrt{83}\right)^{2/3}+8\right) x^{\frac{\left(188+12\sqrt{3}\sqrt{83}\right)^{1/3}}{6}+\frac{1}{3}-\frac{4}{3\left(188+12\sqrt{3}\sqrt{83}\right)^{1/3}}}}{12\left(188+12\sqrt{3}\sqrt{83}\right)^{1/3}}$$

$$W_2(x) = \det \left[\begin{array}{ll} x^{-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{6}+\frac{4}{3\left(188+12\sqrt{249}\right)^{1/3}}+\frac{2}{3}} & \\ \frac{x^{-\frac{\left(\left(188+12\sqrt{249}\right)^{1/3}+4\right)\left(\left(188+12\sqrt{249}\right)^{1/3}-2\right)}{6\left(188+12\sqrt{249}\right)^{1/3}}}\left(2-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{2}+\frac{4}{\left(188+12\sqrt{249}\right)^{1/3}}\right)}{3} & \frac{x^{\frac{\left(188+12\sqrt{3}\,\sqrt{83}\right)^{2/3}-4\left(188\cdots\right.}{12\left(188+12\sqrt{3}\,\sqrt{8\cdots}\right.}}}{\cdots} \end{array} \right.$$

$$= -\frac{3\left(\left(\left(\frac{2i}{3}-\frac{2\sqrt{3}}{9}\right)\left(188+12\sqrt{3}\,\sqrt{83}\right)^{1/3}-i\sqrt{3}\,\sqrt{83}-\frac{47i}{3}-\frac{47\sqrt{3}}{9}-\sqrt{83}\right)x^{\frac{-2i\sqrt{3}\,\left(188+12\sqrt{3}\,\sqrt{83}\right)^{1/3}+\left(\cdots\right.}{\cdots}}\right.}{\cdots}$$

$$W_3(x) = \det \left[\begin{array}{ll} x^{-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{6}+\frac{4}{3\left(188+12\sqrt{249}\right)^{1/3}}+\frac{2}{3}} & \\ \frac{x^{-\frac{\left(\left(188+12\sqrt{249}\right)^{1/3}+4\right)\left(\left(188+12\sqrt{249}\right)^{1/3}-2\right)}{6\left(188+12\sqrt{249}\right)^{1/3}}}\left(2-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{2}+\frac{4}{\left(188+12\sqrt{249}\right)^{1/3}}\right)}{3} & -\frac{x^{\frac{\left(188+12\sqrt{3}\,\sqrt{83}\right)^{2/3}-4\left(1\cdots\right.}{12\left(188+12\sqrt{3}\cdots\right.}}}{\cdots} \end{array} \right.$$

$$= \frac{\left(\frac{141}{2}+\left(-i\sqrt{3}-3\right)\left(188+12\sqrt{3}\,\sqrt{83}\right)^{1/3}+\frac{\left(-47i+9\sqrt{83}\right)\sqrt{3}}{2}-\frac{9i\sqrt{83}}{2}\right)x^{\frac{-2i\sqrt{3}\,\left(188+12\sqrt{3}\,\sqrt{83}\right)^{1/3}+\left(188+12\sqrt{\cdots}\right.}{\cdots}}}{\cdots}$$

Now we are ready to evaluate each $U_i(x)$.

$$
\begin{aligned}
U_1 &= (-1)^{3-1} \int \frac{F(x)W_1(x)}{aW(x)}\, dx \\
&= (-1)^2 \int \frac{(x)\left(\frac{\sqrt{3}\left(\left(188+12\sqrt{3}\,\sqrt{83}\right)^{2/3}+8\right)x^{\frac{\left(188+12\sqrt{3}\,\sqrt{83}\right)^{1/3}}{6}+\frac{1}{3}-\frac{4}{3\left(188+12\sqrt{3}\,\sqrt{83}\right)^{1/3}}}}{12\left(188+12\sqrt{3}\,\sqrt{83}\right)^{1/3}}\right)}{(x^3)\left(\frac{\sqrt{83}}{2x}\right)}\, dx \\
&= \int \frac{\frac{x\sqrt{3}\left(\left(188+12\sqrt{3}\,\sqrt{83}\right)^{2/3}+8\right)x^{\frac{\left(188+12\sqrt{3}\,\sqrt{83}\right)^{1/3}}{6}+\frac{1}{3}-\frac{4}{3\left(188+12\sqrt{3}\,\sqrt{83}\right)^{1/3}}}}{12\left(188+12\sqrt{3}\,\sqrt{83}\right)^{1/3}}}{\frac{x^2\sqrt{83}}{2}}\, dx \\
&= \int \left(\frac{x^{\frac{\left(188+12\sqrt{249}\right)^{2/3}-4\left(188+12\sqrt{249}\right)^{1/3}-8}{6\left(188+12\sqrt{249}\right)^{1/3}}}\left(\left(188+12\sqrt{249}\right)^{2/3}+8\right)\sqrt{249}}{498\left(188+12\sqrt{249}\right)^{1/3}}\right) dx \\
&= \frac{x^{1+\frac{\left(188+12\sqrt{249}\right)^{2/3}-4\left(188+12\sqrt{249}\right)^{1/3}-8}{6\left(188+12\sqrt{249}\right)^{1/3}}}\sqrt{249}\left(\left(188+12\sqrt{249}\right)^{2/3}+8\right)\left(\sqrt{249}\left(188+12\sqrt{249}\right)^{2/3}-13(18\right.}{95616\left(188+12\sqrt{249}\right)^{1/}} \\
&= \frac{x^{1+\frac{\left(188+12\sqrt{249}\right)^{2/3}-4\left(188+12\sqrt{249}\right)^{1/3}-8}{6\left(188+12\sqrt{249}\right)^{1/3}}}\sqrt{249}\left(\left(188+12\sqrt{249}\right)^{2/3}+8\right)\left(\sqrt{249}\left(188+12\sqrt{249}\right)^{2/3}-13(18\right.}{95616\left(188+12\sqrt{249}\right)^{1/}}
\end{aligned}
$$

$$U_2 = (-1)^{3-2} \int \frac{F(x)W_2(x)}{aW(x)}\,dx$$

$$= (-1)^1 \int \frac{(x)\left(-\frac{3\left(\left(\left(\frac{2i}{3}-\frac{2\sqrt{3}}{9}\right)\left(188+12\sqrt{3}\sqrt{83}\right)^{1/3}-i\sqrt{3}\sqrt{83}-\frac{47i}{3}-\frac{47\sqrt{3}}{9}-\sqrt{83}\right)x^{\frac{-2i\sqrt{3}\left(188+12\sqrt{3}\sqrt{83}\right)^{1/3}+\left(188+12\sqrt{3}\sqrt{83}\right)^{2/}}{3\left(188+\right.}}}{}\right.}{}$$

$$= -\int \frac{-\frac{3x\left(\left(\left(\frac{2i}{3}-\frac{2\sqrt{3}}{9}\right)\left(188+12\sqrt{3}\sqrt{83}\right)^{1/3}-i\sqrt{3}\sqrt{83}-\frac{47i}{3}-\frac{47\sqrt{3}}{9}-\sqrt{83}\right)x^{\frac{-2i\sqrt{3}\left(188+12\sqrt{3}\sqrt{83}\right)^{1/3}+\left(188+12\sqrt{3}\sqrt{83}\right)^{2/3}-3\sqrt{3}\sqrt{83}-}{3\left(188+12\sqrt{3}\sqrt{83}\right)^{2/3}}}}{}}{}$$

$$= -\int \left(-\frac{3\left(\left(\left(\frac{2i}{3}-\frac{2\sqrt{3}}{9}\right)\left(188+12\sqrt{3}\sqrt{83}\right)^{1/3}-i\sqrt{3}\sqrt{83}-\frac{47i}{3}-\frac{47\sqrt{3}}{9}-\sqrt{83}\right)x^{\frac{-2i\sqrt{3}\left(188+12\sqrt{3}\sqrt{83}\right)^{1/}}{}}}{}\right.$$

$$= -\frac{\left(-\frac{3i\sqrt{83}}{332}-\frac{\sqrt{3}\sqrt{83}}{332}-\frac{i\sqrt{3}}{12}-\frac{1}{12}-\frac{\left(188+12\sqrt{3}\sqrt{83}\right)^{1/3}}{6}-\frac{i\sqrt{3}\left(188+12\sqrt{3}\sqrt{83}\right)^{2/3}}{96}+\frac{3i\left(188+12\sqrt{3}\sqrt{83}\right)^{2/3}\sqrt{83}}{2656}+\right.}{}$$

$$
U_3 = (-1)^{3-3} \int \frac{F(x)W_3(x)}{aW(x)}\,dx
$$

$$
= (-1)^0 \int \frac{(x)\left(\left(\frac{141}{2}+\left(-i\sqrt{3}-3\right)\left(188+12\sqrt{3}\sqrt{83}\right)^{1/3}+\frac{\left(-47i+9\sqrt{83}\right)\sqrt{3}}{2}-\frac{9i\sqrt{83}}{2}\right)x^{\frac{-2i\sqrt{3}\left(188+12\sqrt{3}\sqrt{83}\right)^{1/3}+\left(188+12\sqrt{3}\sqrt{83}\right)^{2/3}-}{3\left(188+12\sqrt{\ }\right.}}}{\ }
$$

$$
= \int \frac{x\left(\left(\frac{141}{2}+\left(-i\sqrt{3}-3\right)\left(188+12\sqrt{3}\sqrt{83}\right)^{1/3}+\frac{\left(-47i+9\sqrt{83}\right)\sqrt{3}}{2}-\frac{9i\sqrt{83}}{2}\right)x^{\frac{-2i\sqrt{3}\left(188+12\sqrt{3}\sqrt{83}\right)^{1/3}+\left(188+12\sqrt{3}\sqrt{83}\right)^{2/3}-3\sqrt{3}\sqrt{83}-47i}{3\left(188+12\sqrt{3}\sqrt{83}\right)^{2/3}}}}{\ }
$$

$$
= \int \left(-\frac{2\sqrt{83}\left(\left(\left(3+i\sqrt{3}\right)\left(188+12\sqrt{249}\right)^{1/3}+\frac{47i\sqrt{3}}{2}+\frac{9i\sqrt{83}}{2}-\frac{9\sqrt{249}}{2}-\frac{141}{2}\right)x^{\frac{-2i\sqrt{3}\left(188+12\sqrt{3}\sqrt{83}\right)^{1/3}+(}{\ }}}{\ }\right.
$$

$$
= \frac{\left(\frac{3\sqrt{83}}{332}-\frac{i\sqrt{3}\sqrt{83}}{332}+\frac{\sqrt{3}}{12}-\frac{i}{12}-\frac{i\left(188+12\sqrt{3}\sqrt{83}\right)^{1/3}}{6}+\frac{i\left(188+12\sqrt{3}\sqrt{83}\right)^{2/3}}{96}+\frac{\sqrt{3}\left(188+12\sqrt{3}\sqrt{83}\right)^{2/3}}{96}-\frac{3\left(188+12\sqrt{\ }\right.}{2}\right.}{\ }
$$

$$
= \frac{\left(\frac{3\sqrt{83}}{332}-\frac{i\sqrt{3}\sqrt{83}}{332}+\frac{\sqrt{3}}{12}-\frac{i}{12}-\frac{i\left(188+12\sqrt{3}\sqrt{83}\right)^{1/3}}{6}+\frac{i\left(188+12\sqrt{3}\sqrt{83}\right)^{2/3}}{96}+\frac{\sqrt{3}\left(188+12\sqrt{3}\sqrt{83}\right)^{2/3}}{96}-\frac{3\left(188+12\sqrt{\ }\right.}{2}\right.}{\ }
$$

Now that all the U_i functions have been determined, the particular solution is found from

$$
y_p = U_1y_1 + U_2y_2 + U_3y_3
$$

Hence

$$y_p = \left(\frac{x^{1+\frac{\left(188+12\sqrt{249}\right)^{2/3}-4\left(188+12\sqrt{249}\right)^{1/3}-8}{6\left(188+12\sqrt{249}\right)^{1/3}}} \sqrt{249}\left(\left(188+12\sqrt{249}\right)^{2/3}+8\right)\left(\sqrt{249}\left(188+12\sqrt{249}\right)^{2/3}-13(\cdots\right.}{95616\left(188+12\sqrt{249}\right)\cdots} \right.$$

$$+ \left(-\frac{\left(-\frac{3i\sqrt{83}}{332}-\frac{\sqrt{3}\,\sqrt{83}}{332}-\frac{i\sqrt{3}}{12}-\frac{1}{12}-\frac{\left(188+12\sqrt{3}\,\sqrt{83}\right)^{1/3}}{6}-\frac{i\sqrt{3}\left(188+12\sqrt{3}\,\sqrt{83}\right)^{2/3}}{96}+\frac{3i\left(188+12\sqrt{3}\,\sqrt{83}\right)^{2/3}\sqrt{83}}{2656}+\cdots\right.}{\cdots} \right.$$

$$+ \left(\frac{\left(\frac{3\sqrt{83}}{332}-\frac{i\sqrt{3}\,\sqrt{83}}{332}+\frac{\sqrt{3}}{12}-\frac{i}{12}-\frac{i\left(188+12\sqrt{3}\,\sqrt{83}\right)^{1/3}}{6}+\frac{i\left(188+12\sqrt{3}\,\sqrt{83}\right)^{2/3}}{96}+\frac{\sqrt{3}\left(188+12\sqrt{3}\,\sqrt{83}\right)^{2/3}}{96}-\frac{3\left(188+12\cdots\right.}{\cdots}\right.}{\cdots} \right.$$

Therefore the particular solution is

$$y_p = \frac{x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\, x^{-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{6}+\frac{4}{3\left(188+12\sqrt{249}\right)^{1/3}}+\frac{2}{3}} \right. \\ &\quad + x^{\frac{\left(188+12\sqrt{249}\right)^{1/3}}{12}-\frac{2}{3\left(188+12\sqrt{249}\right)^{1/3}}+\frac{2}{3}} \left(c_2 \cos\left(\frac{\sqrt{3}\left(-\frac{\left(188+12\sqrt{249}\right)^{1/3}}{6}-\frac{4}{3\left(188+12\sqrt{249}\right)^{1/3}}\right)\ln(x)}{2} \right) - c_3 \sin \cdots \right. \end{aligned}$$

Maple step by step solution

Let's solve

$$x^4\left(\frac{d^3}{dx^3}y(x)\right)+\left(\frac{d^2}{dx^2}y(x)\right)x^3+x^2\left(\frac{d}{dx}y(x)\right)+xy(x)=x$$

- Highest derivative means the order of the ODE is 3

$$\frac{d^3}{dx^3}y(x)$$

Maple trace

```
`Methods for third order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 3; linear nonhomogeneous with symmetry [0,1]
trying high order linear exact nonhomogeneous
trying differential order: 3; missing the dependent variable
checking if the LODE is of Euler type
<- LODE of Euler type successful
Euler equation successful`
```

Maple dsolve solution

Solving time : 0.013 (sec)
Leaf size : 223

```
dsolve(x^4*diff(diff(diff(y(x),x),x),x)+x^3*diff(diff(y(x),x),x)+diff(y(x),x)*x^2+x*y(
       y(x),singsol=all)
```

$$y = c_2 x^{\frac{(47-3\sqrt{249})(188+12\sqrt{249})^{2/3}}{192}+\frac{(188+12\sqrt{249})^{1/3}}{12}+\frac{2}{3}} \cos\left(\frac{(188+12\sqrt{3}\sqrt{83})^{1/3}\sqrt{3}\left(3(188+12\sqrt{3}\sqrt{83})^{1/3}\sqrt{3}\sqrt{8}\right.}{192}\right.$$

Mathematica DSolve solution

Solving time : 0.004 (sec)
Leaf size : 82

```
DSolve[{x^4*D[y[x],{x,3}]+x^3*D[y[x],{x,2}]+x^2*D[y[x],x]+x*y[x]== x,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
y(x) \to\ & c_1 x^{\text{Root}\left[\#1^3-2\#1^2+2\#1+1\&,1\right]}+c_3 x^{\text{Root}\left[\#1^3-2\#1^2+2\#1+1\&,3\right]} \\
& + c_2 x^{\text{Root}\left[\#1^3-2\#1^2+2\#1+1\&,2\right]}+1
\end{aligned}
$$

2.3.20 problem 20

Internal problem ID [8554]
Book : Own collection of miscellaneous problems
Section : section 3.0
Problem number : 20
Date solved : `Thursday, December 12, 2024 at 09:30:06 AM`
CAS classification : `[[_high_order, _with_linear_symmetries]]`

Solve

$$5x^5y'''' + 4x^4y''' + x^2y' + xy = 0$$

Solved as higher order Euler type ode

Time used: 0.271 (sec)

The ode can be normalized and rewritten as Euler ode.

This is Euler ODE of higher order. Let $y = x^\lambda$. Hence

$$\begin{aligned}
y' &= \lambda\, x^{\lambda-1} \\
y'' &= \lambda(\lambda-1)\, x^{\lambda-2} \\
y''' &= \lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} \\
y'''' &= \lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^{\lambda-4}
\end{aligned}$$

Substituting these back into

$$5x^5y'''' + 4x^4y''' + x^2y' + xy = 0$$

gives

$$x\lambda\, x^{\lambda-1} + 4x^3\lambda(\lambda-1)\,(\lambda-2)\, x^{\lambda-3} + 5x^4\lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^{\lambda-4} + x^\lambda = 0$$

Which simplifies to

$$\lambda\, x^\lambda + 4\lambda(\lambda-1)\,(\lambda-2)\, x^\lambda + 5\lambda(\lambda-1)\,(\lambda-2)\,(\lambda-3)\, x^\lambda + x^\lambda = 0$$

And since $x^\lambda \neq 0$ then dividing through by x^λ, the above becomes

$$\lambda + 4\lambda(\lambda-1)(\lambda-2) + 5\lambda(\lambda-1)(\lambda-2)(\lambda-3) + 1 = 0$$

Simplifying gives the characteristic equation as

$$5\lambda^4 - 26\lambda^3 + 43\lambda^2 - 21\lambda + 1 = 0$$

Solving the above gives the following roots

$$\lambda_1 = \frac{13}{10} + \frac{\sqrt{6}\sqrt{\frac{5\left(40076+12\sqrt{2307813}\right)^{2/3}+154\left(40076+12\sqrt{2307813}\right)^{1/3}+5420}{\left(40076+12\sqrt{2307813}\right)^{1/3}}}}{60} + \frac{i\sqrt{6}\sqrt{5\sqrt{\frac{5\left(40076+12\sqrt{2307813}\right)^{2/3}+154\left(40076+1\cdots\right.}{\left(40076+12\sqrt{2307813}\right)\cdots}}\cdots}}{\cdots}$$

$$\lambda_2 = \frac{13}{10} + \frac{\sqrt{6}\sqrt{\frac{5\left(40076+12\sqrt{2307813}\right)^{2/3}+154\left(40076+12\sqrt{2307813}\right)^{1/3}+5420}{\left(40076+12\sqrt{2307813}\right)^{1/3}}}}{60} - \frac{i\sqrt{6}\sqrt{5\sqrt{\frac{5\left(40076+12\sqrt{2307813}\right)^{2/3}+154\left(40076+1\cdots\right.}{\left(40076+12\sqrt{2307813}\right)\cdots}}\cdots}}{\cdots}$$

$$\lambda_3 = \frac{13}{10} - \frac{\sqrt{6}\sqrt{\frac{5\left(40076+12\sqrt{2307813}\right)^{2/3}+154\left(40076+12\sqrt{2307813}\right)^{1/3}+5420}{\left(40076+12\sqrt{2307813}\right)^{1/3}}}}{60} + \frac{\sqrt{6}\sqrt{-5\sqrt{\frac{5\left(40076+12\sqrt{2307813}\right)^{2/3}+154\left(40076+\cdots\right.}{\left(40076+12\sqrt{2307813}\right)\cdots}}\cdots}}{\cdots}$$

$$\lambda_4 = \frac{13}{10} - \frac{\sqrt{6}\sqrt{\frac{5\left(40076+12\sqrt{2307813}\right)^{2/3}+154\left(40076+12\sqrt{2307813}\right)^{1/3}+5420}{\left(40076+12\sqrt{2307813}\right)^{1/3}}}}{60} - \frac{\sqrt{6}\sqrt{-5\sqrt{\frac{5\left(40076+12\sqrt{2307813}\right)^{2/3}+154\left(40076+\cdots\right.}{\left(40076+12\sqrt{2307813}\right)\cdots}}\cdots}}{\cdots}$$

This table summarises the result

root
$\frac{13}{10}-\frac{\sqrt{6}\sqrt{\frac{5\left(40076+12\sqrt{2307813}\right)^{2/3}+154\left(40076+12\sqrt{2307813}\right)^{1/3}+5420}{\left(40076+12\sqrt{2307813}\right)^{1/3}}}}{60}+\frac{\sqrt{6}\sqrt{-5\sqrt{\frac{5\left(40076+12\sqrt{2307813}\right)^{2/3}+154\left(40076+12\sqrt{2307813}\right)^{1/3}+5420}{\left(40076+12\sqrt{2307813}\right)^{1/3}}}}}{}$
$\frac{13}{10}+\frac{\sqrt{6}\sqrt{\frac{5\left(40076+12\sqrt{2307813}\right)^{2/3}+154\left(40076+12\sqrt{2307813}\right)^{1/3}+5420}{\left(40076+12\sqrt{2307813}\right)^{1/3}}}}{60}\pm\frac{\sqrt{6}\sqrt{5\sqrt{\frac{5\left(40076+12\sqrt{2307813}\right)^{2/3}+154\left(40076+12\sqrt{2307813}\right)^{1/3}+5420}{\left(40076+12\sqrt{2307813}\right)^{1/3}}}\left(\right.}}{}$
$\frac{13}{10}-\frac{\sqrt{6}\sqrt{\frac{5\left(40076+12\sqrt{2307813}\right)^{2/3}+154\left(40076+12\sqrt{2307813}\right)^{1/3}+5420}{\left(40076+12\sqrt{2307813}\right)^{1/3}}}}{60}-\frac{\sqrt{6}\sqrt{-5\sqrt{\frac{5\left(40076+12\sqrt{2307813}\right)^{2/3}+154\left(40076+12\sqrt{2307813}\right)^{1/3}+5420}{\left(40076+12\sqrt{2307813}\right)^{1/3}}}}}{}$

The solution is generated by going over the above table. For each real root λ of multiplicity one generates a c_1x^λ basis solution. Each real root of multiplicty two, generates c_1x^λ and $c_2x^\lambda \ln(x)$ basis solutions. Each real root of multiplicty three, generates c_1x^λ and $c_2x^\lambda \ln(x)$ and $c_3x^\lambda \ln(x)^2$ basis solutions, and so on. Each complex root $\alpha \pm i\beta$ of multiplicity one generates $x^\alpha(c_1\cos(\beta\ln(x)) + c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity two generates $\ln(x)\, x^\alpha(c_1\cos(\beta\ln(x)) + c_2\sin(\beta\ln(x)))$ basis solutions. And each complex root $\alpha \pm i\beta$ of multiplicity three generates $\ln(x)^2\, x^\alpha(c_1\cos(\beta\ln(x)) + c_2$ basis solutions. And so on. Using the above show that the solution is

Expression too large to display

The fundamental set of solutions for the homogeneous solution are the following

$$
y_1 = x^{\frac{13}{10}-\frac{\sqrt{6}\sqrt{\frac{5\left(40076+12\sqrt{2307813}\right)^{2/3}+154\left(40076+12\sqrt{2307813}\right)^{1/3}+5420}{\left(40076+12\sqrt{2307813}\right)^{1/3}}}}{60}+\frac{\sqrt{6}\sqrt{-5\sqrt{\frac{5\left(40076+12\sqrt{2307813}\right)^{2/3}+154\left(40076+12\sqrt{2307813}\right)^{1/3}+5420}{\left(40076+12\sqrt{2307813}\right)^{1/3}}}\left(40\right.}}{}}
$$

y_2

$$= x^{\frac{13}{10}+\frac{\sqrt{6}\,\sqrt{\frac{5\left(40076+12\sqrt{2307813}\right)^{2/3}+154\left(40076+12\sqrt{2307813}\right)^{1/3}+5420}{\left(40076+12\sqrt{2307813}\right)^{1/3}}}}{60}} \cos\left(\frac{\sqrt{6}\,\sqrt{5\sqrt{\frac{5\left(40076+12\sqrt{2307813}\right)^{2/3}+154\left(40076+12\sqrt{2307813}\right)^{1/3}\cdots}{\left(40076+12\sqrt{2307813}\right)^{1/3}}}\cdots}}{\cdots}\cdots\right)$$

y_3

$$= x^{\frac{13}{10}+\frac{\sqrt{6}\,\sqrt{\frac{5\left(40076+12\sqrt{2307813}\right)^{2/3}+154\left(40076+12\sqrt{2307813}\right)^{1/3}+5420}{\left(40076+12\sqrt{2307813}\right)^{1/3}}}}{60}} \sin\left(\frac{\sqrt{6}\,\sqrt{5\sqrt{\frac{5\left(40076+12\sqrt{2307813}\right)^{2/3}+154\left(40076+12\sqrt{2307813}\right)^{1/3}\cdots}{\left(40076+12\sqrt{2307813}\right)^{1/3}}}\cdots}}{\cdots}\cdots\right)$$

y_4

$$= x^{\frac{13}{10}-\frac{\sqrt{6}\,\sqrt{\frac{5\left(40076+12\sqrt{2307813}\right)^{2/3}+154\left(40076+12\sqrt{2307813}\right)^{1/3}+5420}{\left(40076+12\sqrt{2307813}\right)^{1/3}}}}{60}-\frac{\sqrt{6}\,\sqrt{-5\sqrt{\frac{5\left(40076+12\sqrt{2307813}\right)^{2/3}+154\left(40076+12\sqrt{2307813}\right)^{1/3}+5420}{\left(40076+12\sqrt{2307813}\right)^{1/3}}}\left(40\cdots\right.}}{\cdots}}$$

Maple step by step solution

Maple trace

```
`Methods for high order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 38

```
dsolve(5*x^5*diff(diff(diff(diff(y(x),x),x),x),x)+4*x^4*diff(diff(diff(y(x),x),x),x)+d
       y(x),singsol=all)
```

$$y = \sum_{\_a=1}^{4} x^{\mathrm{RootOf}(5\_Z^4-26\_Z^3+43\_Z^2-21\_Z+1,\mathrm{index}=\_a)} \_C\_a$$

Mathematica DSolve solution

Solving time : 1.419 (sec)
Leaf size : 1931

```
DSolve[{5*x^5*D[y[x],{x,4}]+4*x^4*D[y[x],{x,3}]+x^2*D[y[x],x]+x*y[x]== Sin[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

Too large to display

2.3.21 problem 21

Internal problem ID [8555]
Book : Own collection of miscellaneous problems
Section : section 3.0
Problem number : 21
Date solved : `Thursday, December 12, 2024 at 09:30:07 AM`
CAS classification :
`[[_2nd_order, _missing_y], [_2nd_order, _reducible, _mu_y_y1]]`

Solve

$$\left(x^2+1\right) y''+1+y'^2=0$$

Solved as second order missing y ode

Time used: 0.679 (sec)

This is second order ode with missing dependent variable y. Let

$$p(x)=y'$$

Then

$$p'(x)=y''$$

Hence the ode becomes

$$\left(x^2+1\right) p'(x)+1+p(x)^2=0$$

Which is now solve for $p(x)$ as first order ode. The ode $p'(x)=-\frac{p(x)^2+1}{x^2+1}$ is separable as it can be written as

$$\begin{aligned} p'(x) &= -\frac{p(x)^2+1}{x^2+1} \\ &= f(x)g(p) \end{aligned}$$

Where

$$
\begin{aligned}
f(x) &= \frac{1}{x^2+1} \\
g(p) &= -p^2-1
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(p)}\,dp &= \int f(x)\,dx \\
\int \frac{1}{-p^2-1}\,dp &= \int \frac{1}{x^2+1}\,dx \\
-\arctan(p(x)) &= \arctan(x)+c_1
\end{aligned}
$$

We now need to find the singular solutions, these are found by finding for what values $g(p)$ is zero, since we had to divide by this above. Solving $g(p)=0$ or $-p^2-1=0$ for $p(x)$ gives

$$
\begin{aligned}
p(x) &= -i \\
p(x) &= i
\end{aligned}
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
-\arctan(p(x)) &= \arctan(x)+c_1 \\
p(x) &= -i \\
p(x) &= i
\end{aligned}
$$

Solving for $p(x)$ gives

$$
p(x) = -\tan(\arctan(x)+c_1)
$$

For solution (1) found earlier, since $p=y'$ then we now have a new first order ode to solve which is

$$
y' = -i
$$

Since the ode has the form $y'=f(x)$, then we only need to integrate $f(x)$.

$$
\begin{aligned}
\int dy &= \int -i\,dx \\
y &= -ix+c_2
\end{aligned}
$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = i$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int i\,dx$$
$$y = ix + c_3$$

For solution (3) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\tan\left(\arctan\left(x\right) + c_1\right)$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int -\tan\left(\arctan\left(x\right) + c_1\right)\,dx$$
$$y = \frac{i\mathrm{e}^{4ic_1}x}{\left(\mathrm{e}^{2ic_1} - 1\right)^2} - \frac{4\,\mathrm{e}^{2ic_1}\ln\left(\left(-\mathrm{e}^{2ic_1} + 1\right)x + i\mathrm{e}^{2ic_1} + i\right)}{\left(\mathrm{e}^{2ic_1} - 1\right)^2} - \frac{ix}{\left(\mathrm{e}^{2ic_1} - 1\right)^2} + c_4$$

$$y = \frac{-4\,\mathrm{e}^{2ic_1}\ln\left(\left(-x + i\right)\mathrm{e}^{2ic_1} + i + x\right) - 2\,\mathrm{e}^{2ic_1}c_4 + \left(ix + c_4\right)\mathrm{e}^{4ic_1} - ix + c_4}{\left(\mathrm{e}^{2ic_1} - 1\right)^2}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{-4\,\mathrm{e}^{2ic_1}\ln\left(\left(-x + i\right)\mathrm{e}^{2ic_1} + i + x\right) - 2\,\mathrm{e}^{2ic_1}c_4 + \left(ix + c_4\right)\mathrm{e}^{4ic_1} - ix + c_4}{\left(\mathrm{e}^{2ic_1} - 1\right)^2}$$
$$y = -ix + c_2$$
$$y = ix + c_3$$

Maple step by step solution

Let's solve

$\left(x^2 + 1\right)\left(\frac{d^2}{dx^2}y(x)\right) + 1 + \left(\frac{d}{dx}y(x)\right)^2 = 0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Make substitution $u = \frac{d}{dx}y(x)$ to reduce order of ODE

  $(x^2+1)\left(\frac{d}{dx}u(x)\right) + 1 + u(x)^2 = 0$
- Solve for the highest derivative

  $\frac{d}{dx}u(x) = \frac{-1-u(x)^2}{x^2+1}$
- Separate variables

  $\frac{\frac{d}{dx}u(x)}{-1-u(x)^2} = \frac{1}{x^2+1}$
- Integrate both sides with respect to x

  $\int \frac{\frac{d}{dx}u(x)}{-1-u(x)^2}dx = \int \frac{1}{x^2+1}dx + C1$
- Evaluate integral

  $-\arctan(u(x)) = \arctan(x) + C1$
- Solve for $u(x)$

  $u(x) = -\tan(\arctan(x) + C1)$
- Solve 1st ODE for $u(x)$

  $u(x) = -\tan(\arctan(x) + C1)$
- Make substitution $u = \frac{d}{dx}y(x)$

  $\frac{d}{dx}y(x) = -\tan(\arctan(x) + C1)$
- Integrate both sides to solve for $y(x)$

  $\int \left(\frac{d}{dx}y(x)\right)dx = \int -\tan(\arctan(x) + C1)\,dx + C2$
- Compute integrals

  $y(x) = \frac{\mathrm{I}\,e^{4\mathrm{I}C1}x}{(e^{2\mathrm{I}C1}-1)^2} - \frac{\mathrm{I}x}{(e^{2\mathrm{I}C1}-1)^2} - \frac{4\,e^{2\mathrm{I}C1}\ln\left((-e^{2\mathrm{I}C1}+1)x+\mathrm{I}\,e^{2\mathrm{I}C1}+\mathrm{I}\right)}{(e^{2\mathrm{I}C1}-1)^2} + C2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
```

```
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(1+_b(_a)^2)/(_a^2+1), _b(_a)`
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   trying separable
   <- separable successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

Maple dsolve solution

Solving time : 0.020 (sec)
Leaf size : 33

```
dsolve((x^2+1)*diff(diff(y(x),x),x)+1+diff(y(x),x)^2 = 0,
       y(x),singsol=all)
```

$$y = \frac{\ln\left(c_1 x - 1\right) c_1^2 + c_2 c_1^2 + c_1 x + \ln\left(c_1 x - 1\right)}{c_1^2}$$

Mathematica DSolve solution

Solving time : 6.907 (sec)
Leaf size : 33

```
DSolve[{(1+x^2)*D[y[x],{x,2}]+1+(D[y[x],x])^2==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -x\cot(c_1) + \csc^2(c_1)\log(-x\sin(c_1) - \cos(c_1)) + c_2$$

2.3.22 problem 22

Internal problem ID [8556]
Book : Own collection of miscellaneous problems
Section : section 3.0
Problem number : 22
Date solved : `Wednesday, December 18, 2024 at 01:53:23 AM`
CAS classification : `[[_2nd_order, _missing_y]]`

Solve

$$\left(x^2+1\right) y''+1+y'^2=x$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(_b(_a)^2-_a+1)/(_a^2+1), _b(_a)
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   trying separable
   trying inverse linear
```

```
   trying homogeneous types:
   trying Chini
   differential order: 1; looking for linear symmetries
   trying exact
   Looking for potential symmetries
   trying Riccati
   trying Riccati sub-methods:
      <- Abel AIR successful: ODE belongs to the 2F1 2-parameter class
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

Maple dsolve solution

Solving time : 0.021 (sec)
Leaf size : 377

```
dsolve((x^2+1)*diff(diff(y(x),x),x)+1+diff(y(x),x)^2 = x,
       y(x),singsol=all)
```

$$y = \int \frac{\sqrt{-1+i}\left(\frac{1}{2}-\frac{ix}{2}\right)^{\frac{i\sqrt{-2+2\sqrt{2}}}{2}}(x+i)\left(\frac{ix}{2}+\frac{1}{2}\right)^{i\sqrt{-1+i}}\operatorname{hypergeom}\left(\left[\frac{i\sqrt{-2+2\sqrt{2}}}{2}, \frac{i\sqrt{-1+i}}{2}+\frac{\sqrt{1+i}}{2}+1\right], \left[i\sqrt{-1\right.\right.}{\left(4\left(-\frac{1}{2}+\frac{ix}{2}\right)^{i\sqrt{-1+i}}\left(\frac{1}{2}-\right.\right.} + c_2$$

Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{(1+x^2)*D[y[x],{x,2}]+1+(D[y[x],x])^2==x,{}},
       y[x],x,IncludeSingularSolutions->True]
```

Not solved

2.3.23 problem 23

Internal problem ID [8557]
Book : Own collection of miscellaneous problems
Section : section 3.0
Problem number : 23
Date solved : `Thursday, December 12, 2024 at 09:30:53 AM`
CAS classification :
`[[_2nd_order, _missing_y], [_2nd_order, _reducible, _mu_y_y1]]`

Solve

$$\left(x^2+1\right) y''+1+x{y'}^2=1$$

Solved as second order missing y ode

Time used: 0.710 (sec)

This is second order ode with missing dependent variable y. Let

$$p(x)=y'$$

Then

$$p'(x)=y''$$

Hence the ode becomes

$$\left(x^2+1\right) p'(x)+x p(x)^2=0$$

Which is now solve for $p(x)$ as first order ode. The ode $p'(x)=-\frac{x p(x)^2}{x^2+1}$ is separable as it can be written as

$$\begin{aligned} p'(x) &= -\frac{x p(x)^2}{x^2+1} \\ &= f(x) g(p) \end{aligned}$$

Where

$$
\begin{aligned}
f(x) &= -\frac{x}{x^2+1} \\
g(p) &= p^2
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(p)}\, dp &= \int f(x)\, dx \\
\int \frac{1}{p^2}\, dp &= \int -\frac{x}{x^2+1}\, dx \\
-\frac{1}{p(x)} &= \ln\left(\frac{1}{\sqrt{x^2+1}}\right) + c_1
\end{aligned}
$$

We now need to find the singular solutions, these are found by finding for what values $g(p)$ is zero, since we had to divide by this above. Solving $g(p) = 0$ or $p^2 = 0$ for $p(x)$ gives

$$
p(x) = 0
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
-\frac{1}{p(x)} &= \ln\left(\frac{1}{\sqrt{x^2+1}}\right) + c_1 \\
p(x) &= 0
\end{aligned}
$$

Solving for $p(x)$ gives

$$
p(x) = -\frac{1}{\ln\left(\frac{1}{\sqrt{x^2+1}}\right) + c_1}
$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
y' = 0
$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$
\begin{aligned}
\int dy &= \int 0\, dx + c_2 \\
y &= c_2
\end{aligned}
$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\frac{1}{-\frac{\ln(x^2+1)}{2} + c_1}$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\begin{aligned}\int dy &= \int \frac{2}{\ln(x^2+1) - 2c_1}\,dx\\ y &= \int \frac{2}{\ln(x^2+1) - 2c_1}dx + c_3\end{aligned}$$

$$y = \int \frac{2}{\ln(x^2+1) - 2c_1}dx + c_3$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$\begin{aligned}y &= c_2\\ y &= 2\int \frac{1}{\ln(x^2+1) - 2c_1}dx + c_3\end{aligned}$$

Maple step by step solution

Let's solve

$(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + 1 + x\left(\frac{d}{dx}y(x)\right)^2 = 1$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Make substitution $u = \frac{d}{dx}y(x)$ to reduce order of ODE

  $(x^2+1)\left(\frac{d}{dx}u(x)\right) + 1 + xu(x)^2 = 1$

- Solve for the highest derivative

  $\frac{d}{dx}u(x) = -\frac{xu(x)^2}{x^2+1}$

- Separate variables

  $\frac{\frac{d}{dx}u(x)}{u(x)^2} = -\frac{x}{x^2+1}$

- Integrate both sides with respect to x

$$\int \frac{\frac{d}{dx}u(x)}{u(x)^2}dx = \int -\frac{x}{x^2+1}dx + C1$$

- Evaluate integral

$$-\frac{1}{u(x)} = -\frac{\ln(x^2+1)}{2} + C1$$

- Solve for $u(x)$

$$u(x) = \frac{2}{\ln(x^2+1)-2C1}$$

- Solve 1st ODE for $u(x)$

$$u(x) = \frac{2}{\ln(x^2+1)-2C1}$$

- Make substitution $u = \frac{d}{dx}y(x)$

$$\frac{d}{dx}y(x) = \frac{2}{\ln(x^2+1)-2C1}$$

- Integrate both sides to solve for $y(x)$

$$\int \left(\frac{d}{dx}y(x)\right)dx = \int \frac{2}{\ln(x^2+1)-2C1}dx + C2$$

- Compute integrals

$$y(x) = \int \frac{2}{\ln(x^2+1)-2C1}dx + C2$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -_b(_a)^2*_a/(_a^2+1), _b(_a)`
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   <- Bernoulli successful
<- differential order: 2; canonical coordinates successful
```

```
<- differential order 2; missing variables successful`
```

Maple dsolve solution

Solving time : `0.018` (sec)
Leaf size : 22

```
dsolve((x^2+1)*diff(diff(y(x),x),x)+1+x*diff(y(x),x)^2 = 1,
       y(x),singsol=all)
```

$$y = 2\left(\int \frac{1}{\ln\left(x^{2}+1\right)+2c_{1}}dx\right)+c_{2}$$

Mathematica DSolve solution

Solving time : `60.266` (sec)
Leaf size : 33

```
DSolve[{(1+x^2)*D[y[x],{x,2}]+1+x*(D[y[x],x])^2==1,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \int_{1}^{x} -\frac{2}{2c_{1}-\log\left(K[1]^{2}+1\right)}dK[1]+c_{2}$$

2.3.24 problem 24

Internal problem ID [8558]
Book : Own collection of miscellaneous problems
Section : section 3.0
Problem number : 24
Date solved : Thursday, December 12, 2024 at 09:30:54 AM
CAS classification : [NONE]

Solve

$$\left(x^2+1\right) y''+y{y'}^2=0$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
-> trying 2nd order, dynamical_symmetries, fully reducible to Abel through one integra
   --- trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods for
   -> trying 2nd order, dynamical_symmetries, fully reducible to Abel through one inte
trying 2nd order, integrating factors of the form mu(x,y)/(y)^n, only the singular cas
trying symmetries linear in x and y(x)
trying differential order: 2; exact nonlinear
trying 2nd order, integrating factor of the form mu(y)
trying 2nd order, integrating factor of the form mu(x,y)
-> Calling odsolve with the ODE`, -(_y1^2*x^2+_y1^2-4*x^2)*y(x)/((x^2+1)*_y1^2)+(2*x^3
   Methods for first order ODEs:
```

```
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
trying 2nd order, integrating factor of the form mu(x,y)/(y)^n, only the general case
trying 2nd order, integrating factor of the form mu(y,y)
trying differential order: 2; mu polynomial in y
trying 2nd order, integrating factor of the form mu(x,y)
differential order: 2; looking for linear symmetries
-> trying 2nd order, the S-function method
   -> trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods for
   -> trying 2nd order, the S-function method
   -> trying 2nd order, No Point Symmetries Class V
      --- trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods
      -> trying 2nd order, No Point Symmetries Class V
   -> trying 2nd order, No Point Symmetries Class V
      --- trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods
      -> trying 2nd order, No Point Symmetries Class V
   -> trying 2nd order, No Point Symmetries Class V
      --- trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods
      -> trying 2nd order, No Point Symmetries Class V
trying 2nd order, integrating factor of the form mu(x,y)/(y)^n, only the general case
-> trying 2nd order, dynamical_symmetries, only a reduction of order through one integ
   --- trying a change of variables {x -> y(x), y(x) -> x} and re-entering methods for
   -> trying 2nd order, dynamical_symmetries, only a reduction of order through one in
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 5
`, `-> Computing symmetries using: way = formal`
```

Maple dsolve solution

Solving time : 0.172 (sec)
Leaf size : maple_leaf_size

```
dsolve((x^2+1)*diff(diff(y(x),x),x)+y(x)*diff(y(x),x)^2 = 0,
       y(x),singsol=all)
```

No solution found

Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{(1+x^2)*D[y[x],{x,2}]+y[x]*(D[y[x],x])^2==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

Not solved

2.3.25 problem 25

Internal problem ID [8559]
Book : Own collection of miscellaneous problems
Section : section 3.0
Problem number : 25
Date solved : `Thursday, December 12, 2024 at 09:30:55 AM`
CAS classification :
`[[_2nd_order, _missing_y], [_2nd_order, _reducible, _mu_y_y1]]`

Solve

$$\left(x^2+1\right) y''+y'^2=0$$

Solved as second order missing y ode

Time used: 0.503 (sec)

This is second order ode with missing dependent variable y. Let

$$p(x)=y'$$

Then

$$p'(x)=y''$$

Hence the ode becomes

$$\left(x^2+1\right) p'(x)+p(x)^2=0$$

Which is now solve for $p(x)$ as first order ode. The ode $p'(x)=-\frac{p(x)^2}{x^2+1}$ is separable as it can be written as

$$\begin{aligned} p'(x) &= -\frac{p(x)^2}{x^2+1} \\ &= f(x)g(p) \end{aligned}$$

Where

$$
\begin{aligned}
f(x) &= -\frac{1}{x^2+1} \\
g(p) &= p^2
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(p)}\,dp &= \int f(x)\,dx \\
\int \frac{1}{p^2}\,dp &= \int -\frac{1}{x^2+1}\,dx \\
\frac{1}{p(x)} &= \arctan(x) - c_1
\end{aligned}
$$

We now need to find the singular solutions, these are found by finding for what values $g(p)$ is zero, since we had to divide by this above. Solving $g(p) = 0$ or $p^2 = 0$ for $p(x)$ gives

$$
p(x) = 0
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
\frac{1}{p(x)} &= \arctan(x) - c_1 \\
p(x) &= 0
\end{aligned}
$$

Solving for $p(x)$ gives

$$
p(x) = \frac{1}{\arctan(x) - c_1}
$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$
y' = 0
$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$
\begin{aligned}
\int dy &= \int 0\,dx + c_2 \\
y &= c_2
\end{aligned}
$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{1}{\arctan(x) - c_1}$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\begin{aligned} \int dy &= \int \frac{1}{\arctan(x) - c_1}\, dx \\ y &= \int \frac{1}{\arctan(x) - c_1} dx + c_3 \end{aligned}$$

$$y = \int \frac{1}{\arctan(x) - c_1} dx + c_3$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$\begin{aligned} y &= c_2 \\ y &= \int \frac{1}{\arctan(x) - c_1} dx + c_3 \end{aligned}$$

Maple step by step solution

Let's solve
$(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right) + \left(\frac{d}{dx}y(x)\right)^2 = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Make substitution $u = \frac{d}{dx}y(x)$ to reduce order of ODE
  $(x^2+1)\left(\frac{d}{dx}u(x)\right) + u(x)^2 = 0$
- Solve for the highest derivative
  $\frac{d}{dx}u(x) = -\frac{u(x)^2}{x^2+1}$
- Separate variables
  $\frac{\frac{d}{dx}u(x)}{u(x)^2} = -\frac{1}{x^2+1}$
- Integrate both sides with respect to x

$\int \frac{\frac{d}{dx}u(x)}{u(x)^2}dx = \int -\frac{1}{x^2+1}dx + C1$

- Evaluate integral

  $-\frac{1}{u(x)} = -\arctan(x) + C1$

- Solve for $u(x)$

  $u(x) = \frac{1}{\arctan(x) - C1}$

- Solve 1st ODE for $u(x)$

  $u(x) = \frac{1}{\arctan(x) - C1}$

- Make substitution $u = \frac{d}{dx}y(x)$

  $\frac{d}{dx}y(x) = \frac{1}{\arctan(x) - C1}$

- Integrate both sides to solve for $y(x)$

  $\int \left(\frac{d}{dx}y(x)\right) dx = \int \frac{1}{\arctan(x) - C1}dx + C2$

- Compute integrals

  $y(x) = \int \frac{1}{\arctan(x) - C1}dx + C2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -_b(_a)^2/(_a^2+1), _b(_a)`   ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   <- Bernoulli successful
<- differential order: 2; canonical coordinates successful
```

```
<- differential order 2; missing variables successful`
```

Maple dsolve solution

Solving time : 0.020 (sec)
Leaf size : 14

```
dsolve((x^2+1)*diff(diff(y(x),x),x)+diff(y(x),x)^2 = 0,
       y(x),singsol=all)
```

$$y = \int \frac{1}{\arctan(x) + c_1} dx + c_2$$

Mathematica DSolve solution

Solving time : 60.284 (sec)
Leaf size : 25

```
DSolve[{(1+x^2)*D[y[x],{x,2}]+(D[y[x],x])^2==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \int_1^x \frac{1}{\arctan(K[1]) - c_1} dK[1] + c_2$$

2.3.26 problem 26

Internal problem ID [8560]
Book : Own collection of miscellaneous problems
Section : section 3.0
Problem number : 26
Date solved : `Thursday, December 12, 2024 at 09:30:56 AM`
CAS classification :
`[[_2nd_order, _missing_x], _Liouville, [_2nd_order, _reducible, _mu_x_y1], [_2nd_order`

Solve

$$y'' + \sin(y)\,y'^2 = 0$$

Solved as second order missing x ode

Time used: 0.282 (sec)

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dp}{dy}\frac{dy}{dx} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) + \sin(y)\,p(y)^2 = 0$$

Which is now solved as first order ode for $p(y)$.

In canonical form a linear first order is

$$p' + q(y)p = p(y)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(y) &= \sin(y) \\ p(y) &= 0 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\, dy} \\ &= \mathrm{e}^{\int \sin(y) dy} \\ &= \mathrm{e}^{-\cos(y)} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}y}\mu p &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}y}\left(p\,\mathrm{e}^{-\cos(y)}\right) &= 0 \end{aligned}$$

Integrating gives

$$\begin{aligned} p\,\mathrm{e}^{-\cos(y)} &= \int 0\, dy + c_1 \\ &= c_1 \end{aligned}$$

Dividing throughout by the integrating factor $\mathrm{e}^{-\cos(y)}$ gives the final solution

$$p = \mathrm{e}^{\cos(y)} c_1$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \mathrm{e}^{\cos(y)} c_1$$

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} \frac{\mathrm{e}^{-\cos(\tau)}}{c_1} d\tau = x + c_2$$

Will add steps showing solving for IC soon.

<u>Summary of solutions found</u>

$$\int^{y} \frac{\mathrm{e}^{-\cos(\tau)}}{c_1} d\tau = x + c_2$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
<- 2nd_order Liouville successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 21

```
dsolve(diff(diff(y(x),x),x)+sin(y(x))*diff(y(x),x)^2 = 0,
       y(x),singsol=all)
```

$$\int^{y} \mathrm{e}^{-\cos(\_a)} d\_a - c_1 x - c_2 = 0$$

Mathematica DSolve solution

Solving time : 1.302 (sec)
Leaf size : 111

```
DSolve[{D[y[x],{x,2}]+y[x]*Sin[y[x]](D[y[x],x])^2==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \rightarrow \text{InverseFunction}\left[\int_1^{\#1} \frac{e^{\sin(K[1]) - \cos(K[1])K[1]}}{c_1} dK[1]\&\right][x + c_2]$$
$$y(x) \rightarrow \text{InverseFunction}\left[\int_1^{\#1} -\frac{e^{\sin(K[1]) - \cos(K[1])K[1]}}{c_1} dK[1]\&\right][x + c_2]$$
$$y(x) \rightarrow \text{InverseFunction}\left[\int_1^{\#1} \frac{e^{\sin(K[1]) - \cos(K[1])K[1]}}{c_1} dK[1]\&\right][x + c_2]$$

2.3.27 problem 27

Internal problem ID [8561]
Book : Own collection of miscellaneous problems
Section : section 3.0
Problem number : 27
Date solved : Thursday, December 12, 2024 at 09:30:57 AM
CAS classification :
[[_2nd_order, _missing_y], [_2nd_order, _reducible, _mu_y_y1]]

Solve

$$\left(x^2+1\right) y''+y'^3=0$$

Solved as second order missing y ode

Time used: 0.994 (sec)

This is second order ode with missing dependent variable y. Let

$$p(x)=y'$$

Then

$$p'(x)=y''$$

Hence the ode becomes

$$\left(x^2+1\right) p'(x)+p(x)^3=0$$

Which is now solve for $p(x)$ as first order ode. The ode $p'(x)=-\frac{p(x)^3}{x^2+1}$ is separable as it can be written as

$$\begin{aligned}p'(x)&=-\frac{p(x)^3}{x^2+1}\\&=f(x)g(p)\end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x^2+1} \\ g(p) &= p^3 \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(p)}\, dp &= \int f(x)\, dx \\ \int \frac{1}{p^3}\, dp &= \int -\frac{1}{x^2+1}\, dx \\ \frac{1}{2p(x)^2} &= \arctan(x) - c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(p)$ is zero, since we had to divide by this above. Solving $g(p) = 0$ or $p^3 = 0$ for $p(x)$ gives

$$p(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} \frac{1}{2p(x)^2} &= \arctan(x) - c_1 \\ p(x) &= 0 \end{aligned}$$

Solving for $p(x)$ gives

$$\begin{aligned} p(x) &= \frac{1}{\sqrt{2\arctan(x) - 2c_1}} \\ p(x) &= -\frac{1}{\sqrt{2\arctan(x) - 2c_1}} \end{aligned}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = 0$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\begin{aligned} \int dy &= \int 0\, dx + c_2 \\ y &= c_2 \end{aligned}$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{1}{\sqrt{2\arctan(x) - 2c_1}}$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\begin{aligned}
\int dy &= \int \frac{1}{\sqrt{2\arctan(x) - 2c_1}}\, dx \\
y &= \int \frac{1}{\sqrt{2\arctan(x) - 2c_1}} dx + c_3
\end{aligned}$$

$$y = \int \frac{1}{\sqrt{2\arctan(x) - 2c_1}} dx + c_3$$

For solution (3) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\frac{1}{\sqrt{2\arctan(x) - 2c_1}}$$

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\begin{aligned}
\int dy &= \int -\frac{1}{\sqrt{2\arctan(x) - 2c_1}}\, dx \\
y &= \int -\frac{1}{\sqrt{2\arctan(x) - 2c_1}} dx + c_4
\end{aligned}$$

$$y = \int -\frac{1}{\sqrt{2\arctan(x) - 2c_1}} dx + c_4$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$\begin{aligned}
y &= c_2 \\
y &= -\int \frac{1}{\sqrt{2\arctan(x) - 2c_1}} dx + c_4 \\
y &= \int \frac{1}{\sqrt{2\arctan(x) - 2c_1}} dx + c_3
\end{aligned}$$

Maple step by step solution

Let's solve

$(x^2+1)\left(\frac{d^2}{dx^2}y(x)\right)+\left(\frac{d}{dx}y(x)\right)^3=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Make substitution $u=\frac{d}{dx}y(x)$ to reduce order of ODE

  $(x^2+1)\left(\frac{d}{dx}u(x)\right)+u(x)^3=0$
- Solve for the highest derivative

  $\frac{d}{dx}u(x)=-\frac{u(x)^3}{x^2+1}$
- Separate variables

  $\frac{\frac{d}{dx}u(x)}{u(x)^3}=-\frac{1}{x^2+1}$
- Integrate both sides with respect to x

  $\int \frac{\frac{d}{dx}u(x)}{u(x)^3}dx=\int -\frac{1}{x^2+1}dx+C1$
- Evaluate integral

  $-\frac{1}{2u(x)^2}=-\arctan(x)+C1$
- Solve for $u(x)$

  $\left\{u(x)=\frac{1}{\sqrt{-2C1+2\arctan(x)}}, u(x)=-\frac{1}{\sqrt{-2C1+2\arctan(x)}}\right\}$
- Solve 1st ODE for $u(x)$

  $u(x)=\frac{1}{\sqrt{-2C1+2\arctan(x)}}$
- Make substitution $u=\frac{d}{dx}y(x)$

  $\frac{d}{dx}y(x)=\frac{1}{\sqrt{-2C1+2\arctan(x)}}$
- Integrate both sides to solve for $y(x)$

  $\int \left(\frac{d}{dx}y(x)\right)dx=\int \frac{1}{\sqrt{-2C1+2\arctan(x)}}dx+C2$
- Compute integrals

  $y(x)=\int \frac{1}{\sqrt{-2C1+2\arctan(x)}}dx+C2$
- Solve 2nd ODE for $u(x)$

  $u(x)=-\frac{1}{\sqrt{-2C1+2\arctan(x)}}$
- Make substitution $u=\frac{d}{dx}y(x)$

  $\frac{d}{dx}y(x)=-\frac{1}{\sqrt{-2C1+2\arctan(x)}}$

- Integrate both sides to solve for $y(x)$
  $\int \left(\frac{d}{dx}y(x)\right) dx = \int -\frac{1}{\sqrt{-2C1+2\arctan(x)}}dx + C2$
- Compute integrals
  $y(x) = \int -\frac{1}{\sqrt{-2C1+2\arctan(x)}}dx + C2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = exp_sym
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -_b(_a)^3/(_a^2+1), _b(_a)`   ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   <- Bernoulli successful
<- differential order: 2; canonical coordinates successful
<- differential order 2; missing variables successful`
```

Maple dsolve solution

Solving time : `0.021` (sec)
Leaf size : 33

```
dsolve((x^2+1)*diff(diff(y(x),x),x)+diff(y(x),x)^3 = 0,
       y(x),singsol=all)
```

$$y = \int \frac{1}{\sqrt{c_1 + 2\arctan(x)}}dx + c_2$$
$$y = -\left(\int \frac{1}{\sqrt{c_1 + 2\arctan(x)}}dx\right) + c_2$$

Mathematica DSolve solution

Solving time : `62.184` (sec)
Leaf size : 59

```
DSolve[{(1+x^2)*D[y[x],{x,2}]+D[y[x],x]^3==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \int_1^x -\frac{1}{\sqrt{2\arctan(K[1]) - 2c_1}}dK[1] + c_2$$
$$y(x) \to \int_1^x \frac{1}{\sqrt{2\arctan(K[2]) - 2c_1}}dK[2] + c_2$$

2.3.28 problem 28

Internal problem ID [8562]
Book : Own collection of miscellaneous problems
Section : section 3.0
Problem number : 28
Date solved : `Tuesday, December 17, 2024 at 12:56:01 PM`
CAS classification : `[[_homogeneous, ‘class A‘], _dAlembert]`

Solve

$$y' = \mathrm{e}^{-\frac{y}{x}}$$

Solved as first order homogeneous class A ode

Time used: 0.299 (sec)

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \mathrm{e}^{-\frac{y}{x}} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = \mathrm{e}^{-\frac{y}{x}}$ and $N = 1$ are both homogeneous and of the same order $n = 0$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\frac{du}{dx}x + u = e^{-u}$$
$$\frac{du}{dx} = \frac{e^{-u(x)} - u(x)}{x}$$

Or

$$u'(x) - \frac{e^{-u(x)} - u(x)}{x} = 0$$

Or

$$u'(x)\,e^{u(x)}x + u(x)\,e^{u(x)} - 1 = 0$$

Which is now solved as separable in $u(x)$.

The ode $u'(x) = -\frac{(u(x)e^{u(x)}-1)e^{-u(x)}}{x}$ is separable as it can be written as

$$\begin{aligned}
u'(x) &= -\frac{\left(u(x)\,e^{u(x)} - 1\right)e^{-u(x)}}{x} \\
&= f(x)g(u)
\end{aligned}$$

Where

$$\begin{aligned}
f(x) &= -\frac{1}{x} \\
g(u) &= (u\,e^{u} - 1)\,e^{-u}
\end{aligned}$$

Integrating gives

$$\int \frac{1}{g(u)}\,du = \int f(x)\,dx$$
$$\int \frac{e^{u}}{u\,e^{u} - 1}\,du = \int -\frac{1}{x}\,dx$$
$$\int^{u(x)} \frac{e^{\tau}}{\tau\,e^{\tau} - 1}d\tau = \ln\left(\frac{1}{x}\right) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $(u\,e^{u} - 1)\,e^{-u} = 0$ for $u(x)$ gives

$$u(x) = \operatorname{LambertW}(1)$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\int^{u(x)} \frac{e^{\tau}}{\tau e^{\tau}-1}d\tau = \ln\left(\frac{1}{x}\right)+c_1$$
$$u(x) = \mathrm{LambertW}(1)$$

Converting $\int^{u(x)} \frac{e^{\tau}}{\tau e^{\tau}-1}d\tau = \ln\left(\frac{1}{x}\right)+c_1$ back to y gives

$$\int^{\frac{y}{x}} \frac{e^{\tau}}{\tau e^{\tau}-1}d\tau = \ln\left(\frac{1}{x}\right)+c_1$$

Converting $u(x) = \mathrm{LambertW}(1)$ back to y gives

$$y = x\,\mathrm{LambertW}(1)$$

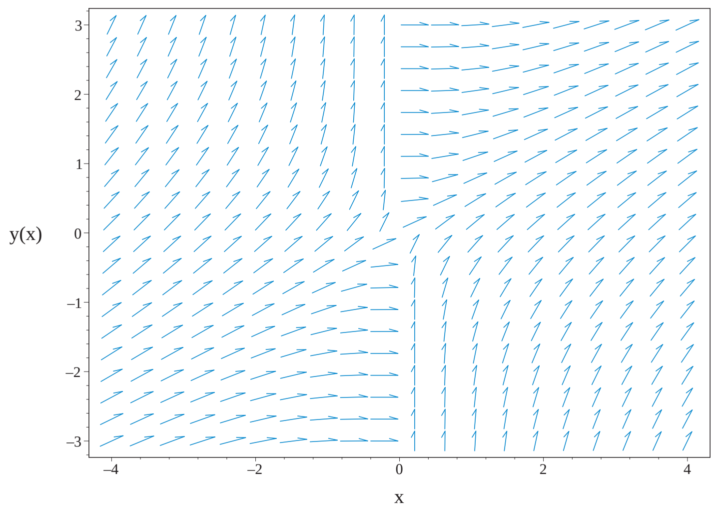


Figure 2.234: Slope field plot
$y' = e^{-\frac{y}{x}}$

Summary of solutions found

$$\int^{\frac{y}{x}} \frac{e^{\tau}}{\tau e^{\tau}-1}d\tau = \ln\left(\frac{1}{x}\right)+c_1$$
$$y = x\,\mathrm{LambertW}(1)$$

Solved as first order homogeneous class D2 ode

Time used: 0.139 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$u'(x)\,x + u(x) = \mathrm{e}^{-u(x)}$$

Which is now solved The ode $u'(x) = -\frac{u(x)-\mathrm{e}^{-u(x)}}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x) - \mathrm{e}^{-u(x)}}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{1}{x} \\ g(u) &= -u + \mathrm{e}^{-u} \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\ \int \frac{1}{-u + \mathrm{e}^{-u}}\,du &= \int \frac{1}{x}\,dx \\ \int^{u(x)} \frac{1}{-\tau + \mathrm{e}^{-\tau}} d\tau &= \ln(x) + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $-u + \mathrm{e}^{-u} = 0$ for $u(x)$ gives

$$u(x) = \mathrm{LambertW}(1)$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} \int^{u(x)} \frac{1}{-\tau + \mathrm{e}^{-\tau}} d\tau &= \ln(x) + c_1 \\ u(x) &= \mathrm{LambertW}(1) \end{aligned}$$

Converting $\int^{u(x)} \frac{1}{-\tau+\mathrm{e}^{-\tau}}d\tau = \ln(x) + c_1$ back to y gives

$$\int^{\frac{y}{x}} \frac{1}{-\tau+\mathrm{e}^{-\tau}}d\tau = \ln(x) + c_1$$

Converting $u(x) = \mathrm{LambertW}(1)$ back to y gives

$$y = x\,\mathrm{LambertW}(1)$$

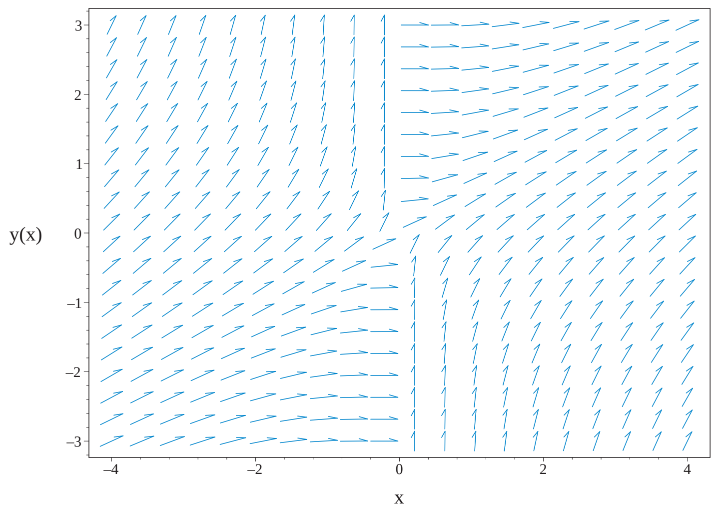


Figure 2.235: Slope field plot
$y' = \mathrm{e}^{-\frac{y}{x}}$

Summary of solutions found

$$\int^{\frac{y}{x}} \frac{1}{-\tau+\mathrm{e}^{-\tau}}d\tau = \ln(x) + c_1$$

$$y = x\,\mathrm{LambertW}(1)$$

Solved as first order isobaric ode

Time used: 0.102 (sec)

Solving for y' gives

$$y' = \mathrm{e}^{-\frac{y}{x}} \tag{1}$$

Each of the above ode's is now solved An ode $y' = f(x, y)$ is isobaric if

$$f(tx, t^m y) = t^{m-1} f(x, y) \tag{1}$$

Where here

$$f(x,y) = e^{-\frac{y}{x}} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m = 1$$

Since the ode is isobaric of order $m = 1$, then the substitution

$$\begin{aligned} y &= u x^m \\ &= ux \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$u(x) + x u'(x) = e^{-u(x)}$$

The ode $u'(x) = -\frac{u(x)-e^{-u(x)}}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x) - e^{-u(x)}}{x} \\ &= f(x) g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{1}{x} \\ g(u) &= -u + e^{-u} \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\, du &= \int f(x)\, dx \\ \int \frac{1}{-u + e^{-u}}\, du &= \int \frac{1}{x}\, dx \\ \int^{u(x)} \frac{1}{-\tau + e^{-\tau}} d\tau &= \ln(x) + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $-u + e^{-u} = 0$ for $u(x)$ gives

$$u(x) = \operatorname{LambertW}(1)$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\int^{u(x)} \frac{1}{-\tau + \mathrm{e}^{-\tau}} d\tau = \ln(x) + c_1$$
$$u(x) = \mathrm{LambertW}(1)$$

Converting $\int^{u(x)} \frac{1}{-\tau+\mathrm{e}^{-\tau}} d\tau = \ln(x) + c_1$ back to y gives

$$\int^{\frac{y}{x}} \frac{1}{-\tau + \mathrm{e}^{-\tau}} d\tau = \ln(x) + c_1$$

Converting $u(x) = \mathrm{LambertW}(1)$ back to y gives

$$\frac{y}{x} = \mathrm{LambertW}(1)$$

Solving for y gives

$$\int^{\frac{y}{x}} \frac{1}{-\tau + \mathrm{e}^{-\tau}} d\tau = \ln(x) + c_1$$
$$y = x \,\mathrm{LambertW}(1)$$

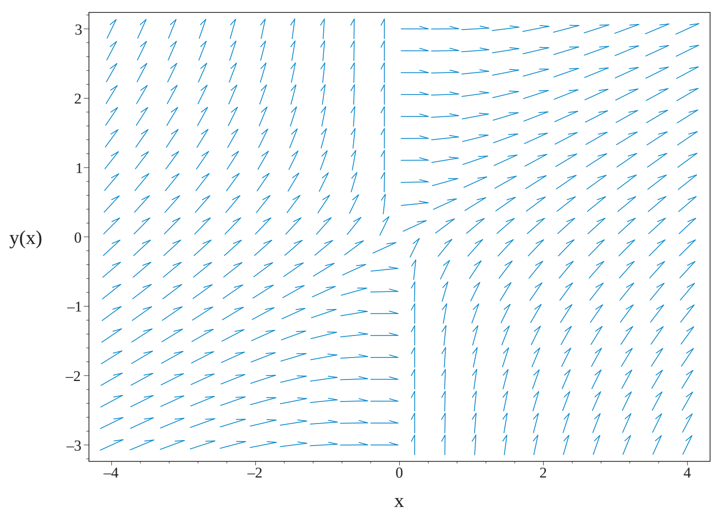


Figure 2.236: Slope field plot
$y' = \mathrm{e}^{-\frac{y}{x}}$

Summary of solutions found

$$\int^{\frac{y}{x}} \frac{1}{-\tau + \mathrm{e}^{-\tau}} d\tau = \ln(x) + c_1$$
$$y = x \,\mathrm{LambertW}(1)$$

Solved using Lie symmetry for first order ode

Time used: 0.615 (sec)

Writing the ode as

$$y' = \mathrm{e}^{-\frac{y}{x}}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \mathrm{e}^{-\frac{y}{x}}(b_3 - a_2) - \mathrm{e}^{-\frac{2y}{x}} a_3 - \frac{y\,\mathrm{e}^{-\frac{y}{x}}(x a_2 + y a_3 + a_1)}{x^2} + \frac{\mathrm{e}^{-\frac{y}{x}}(x b_2 + y b_3 + b_1)}{x} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{\mathrm{e}^{-\frac{2y}{x}} a_3 x^2 + \mathrm{e}^{-\frac{y}{x}} x^2 a_2 - \mathrm{e}^{-\frac{y}{x}} x^2 b_2 - \mathrm{e}^{-\frac{y}{x}} x^2 b_3 + \mathrm{e}^{-\frac{y}{x}} x y a_2 - \mathrm{e}^{-\frac{y}{x}} x y b_3 + \mathrm{e}^{-\frac{y}{x}} y^2 a_3 - \mathrm{e}^{-\frac{y}{x}} x b_1 + \mathrm{e}^{-\frac{y}{x}} y a_1 - b_2 x^2}{x^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-\mathrm{e}^{-\frac{2y}{x}} a_3 x^2 - \mathrm{e}^{-\frac{y}{x}} x^2 a_2 + \mathrm{e}^{-\frac{y}{x}} x^2 b_2 + \mathrm{e}^{-\frac{y}{x}} x^2 b_3 - \mathrm{e}^{-\frac{y}{x}} x y a_2 \\ &+ \mathrm{e}^{-\frac{y}{x}} x y b_3 - \mathrm{e}^{-\frac{y}{x}} y^2 a_3 + \mathrm{e}^{-\frac{y}{x}} x b_1 - \mathrm{e}^{-\frac{y}{x}} y a_1 + b_2 x^2 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$
\begin{aligned}
&-\mathrm{e}^{-\frac{2y}{x}} a_3 x^2 - \mathrm{e}^{-\frac{y}{x}} x^2 a_2 + \mathrm{e}^{-\frac{y}{x}} x^2 b_2 + \mathrm{e}^{-\frac{y}{x}} x^2 b_3 - \mathrm{e}^{-\frac{y}{x}} x y a_2 \\
&+ \mathrm{e}^{-\frac{y}{x}} x y b_3 - \mathrm{e}^{-\frac{y}{x}} y^2 a_3 + \mathrm{e}^{-\frac{y}{x}} x b_1 - \mathrm{e}^{-\frac{y}{x}} y a_1 + b_2 x^2 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \mathrm{e}^{-\frac{2y}{x}}, \mathrm{e}^{-\frac{y}{x}}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x = v_1, y = v_2, \mathrm{e}^{-\frac{2y}{x}} = v_3, \mathrm{e}^{-\frac{y}{x}} = v_4\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-v_4 v_1^2 a_2 - v_4 v_1 v_2 a_2 - v_3 a_3 v_1^2 - v_4 v_2^2 a_3 + v_4 v_1^2 b_2 \\
&+ v_4 v_1^2 b_3 + v_4 v_1 v_2 b_3 - v_4 v_2 a_1 + v_4 v_1 b_1 + b_2 v_1^2 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3, v_4\}
$$

Equation (7E) now becomes

$$
-v_3 a_3 v_1^2 + (-a_2 + b_2 + b_3) v_1^2 v_4 + b_2 v_1^2 + (b_3 - a_2) v_1 v_2 v_4 + v_4 v_1 b_1 - v_4 v_2^2 a_3 - v_4 v_2 a_1 = 0 \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_1 &= 0 \\
b_2 &= 0 \\
-a_1 &= 0 \\
-a_3 &= 0 \\
b_3 - a_2 &= 0 \\
-a_2 + b_2 + b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{y}{x} \\
&= \frac{y}{x}
\end{aligned}
$$

This is easily solved to give

$$
y = c_1 x
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = \frac{y}{x}
$$

And S is found from

$$\begin{aligned} dS &= \frac{dx}{\xi} \\ &= \frac{dx}{x} \end{aligned}$$

Integrating gives

$$\begin{aligned} S &= \int \frac{dx}{T} \\ &= \ln(x) \end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \mathrm{e}^{-\frac{y}{x}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= -\frac{y}{x^2} \\ R_y &= \frac{1}{x} \\ S_x &= \frac{1}{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{x}{\mathrm{e}^{-\frac{y}{x}}x - y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{\mathrm{e}^{-R} - R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

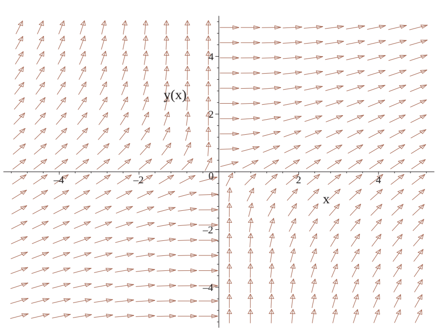
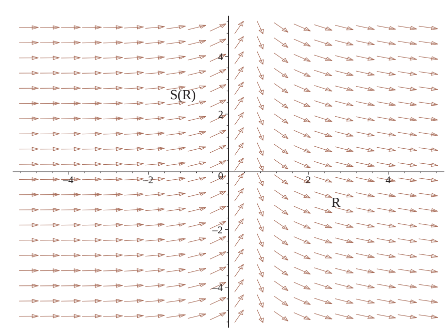
$$
\begin{aligned}
\int dS &= \int \frac{1}{\mathrm{e}^{-R} - R}\, dR \\
S(R) &= \int \frac{1}{\mathrm{e}^{-R} - R} dR + c_2
\end{aligned}
$$

$$
S(R) = \int \frac{1}{\mathrm{e}^{-R} - R} dR + c_2
$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$
\ln(x) = \int^{\frac{y}{x}} \frac{1}{\mathrm{e}^{-\_a} - \_a} d\_a + c_2
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \mathrm{e}^{-\frac{y}{x}}$ 	$R = \frac{y}{x}$ $S = \ln(x)$	$\frac{dS}{dR} = \frac{1}{\mathrm{e}^{-R} - R}$ 

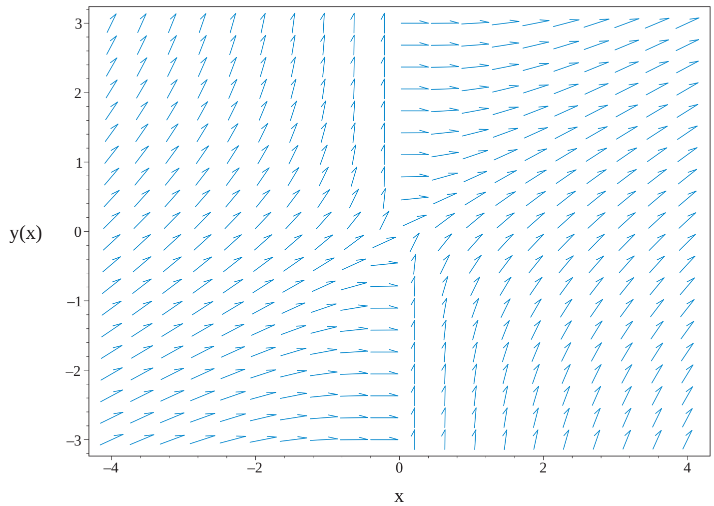


Figure 2.237: Slope field plot
$y' = \mathrm{e}^{-\frac{y}{x}}$

Summary of solutions found

$$\ln(x) = \int^{\frac{y}{x}} \frac{1}{\mathrm{e}^{-\_a} - \_a} d\_a + c_2$$

Solved as first order ode of type dAlembert

Time used: 0.210 (sec)

Let $p = y'$ the ode becomes

$$p = \mathrm{e}^{-\frac{y}{x}}$$

Solving for y from the above results in

$$y = -\ln(p)\, x \tag{1}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = -\ln (p)$$
$$g = 0$$

Hence (2) becomes

$$p + \ln (p) = -\frac{xp'(x)}{p} \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p + \ln (p) = 0$$

Solving the above for p results in

$$p_1 = \text{LambertW}(1)$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$y = x \,\text{LambertW}(1)$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = -\frac{(p(x) + \ln (p(x)))\, p(x)}{x} \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed. The ode $p'(x) = -\frac{(p(x)+\ln(p(x)))p(x)}{x}$ is separable as it can be written as

$$\begin{aligned} p'(x) &= -\frac{(p(x) + \ln (p(x)))\, p(x)}{x} \\ &= f(x)g(p) \end{aligned}$$

Where

$$f(x) = -\frac{1}{x}$$
$$g(p) = (p + \ln (p))\, p$$

Integrating gives

$$\int \frac{1}{g(p)}\, dp = \int f(x)\, dx$$
$$\int \frac{1}{(p + \ln (p))\, p}\, dp = \int -\frac{1}{x}\, dx$$
$$\int^{p(x)} \frac{1}{(\tau + \ln (\tau))\, \tau} d\tau = \ln \left(\frac{1}{x}\right) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(p)$ is zero, since we had to divide by this above. Solving $g(p) = 0$ or $(p + \ln(p))\, p = 0$ for $p(x)$ gives

$$p(x) = \mathrm{LambertW}(1)$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\int^{p(x)} \frac{1}{(\tau + \ln(\tau))\,\tau} d\tau = \ln\left(\frac{1}{x}\right) + c_1$$
$$p(x) = \mathrm{LambertW}(1)$$

Substituing the above solution for p in (2A) gives

$$y = -\ln\left(\mathrm{RootOf}\left(-\int^{\_Z} \frac{1}{(\tau + \ln(\tau))\,\tau} d\tau + \ln\left(\frac{1}{x}\right) + c_1\right)\right) x$$
$$y = -\ln(\mathrm{LambertW}(1))\, x$$

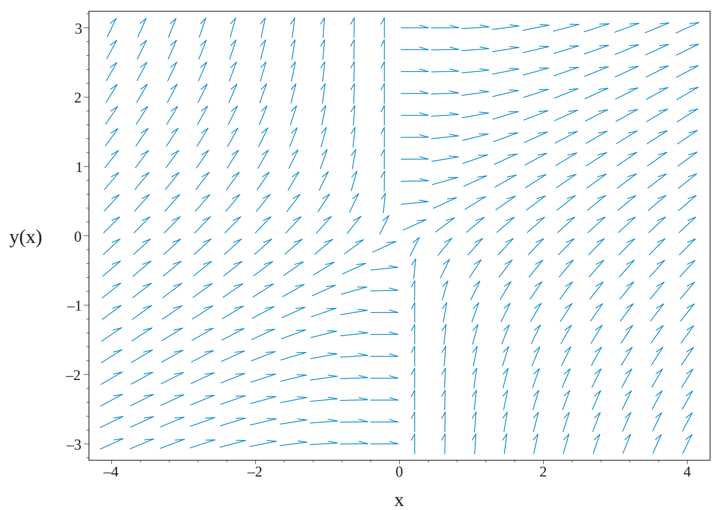


Figure 2.238: Slope field plot
$y' = \mathrm{e}^{-\frac{y}{x}}$

Summary of solutions found

$$y = x\,\mathrm{LambertW}(1)$$

$$y = -\ln\left(\mathrm{RootOf}\left(-\int^{\_Z}\frac{1}{(\tau+\ln(\tau))\,\tau}d\tau+\ln\left(\frac{1}{x}\right)+c_1\right)\right)x$$
$$y = -\ln(\mathrm{LambertW}(1))\,x$$

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) = \mathrm{e}^{-\frac{y(x)}{x}}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = \mathrm{e}^{-\frac{y(x)}{x}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

Maple dsolve solution

Solving time : `0.008` (sec)
Leaf size : 29

```
dsolve(diff(y(x),x) = exp(-y(x)/x),
       y(x),singsol=all)
```

$$y = \mathrm{RootOf}\left(-\left(\int^{\_Z}-\frac{1}{-\mathrm{e}^{-\_a}+\_a}d\_a\right)+\ln(x)+c_1\right)x$$

Mathematica DSolve solution

Solving time : `0.44` (sec)
Leaf size : 39

```
DSolve[{D[y[x],x]==Exp[-y[x]/x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$\text{Solve}\left[\int_1^{\frac{y(x)}{x}} \frac{e^{K[1]}}{e^{K[1]}K[1]-1} dK[1] = -\log(x) + c_1, y(x)\right]$$

2.3.29 problem 29

Internal problem ID [8563]
Book : Own collection of miscellaneous problems
Section : section 3.0
Problem number : 29
Date solved : `Tuesday, December 17, 2024 at 12:56:03 PM`
CAS classification : `[[_homogeneous, ‘class D‘]]`

Solve

$$y' = 2x^2 \sin\left(\frac{y}{x}\right)^2 + \frac{y}{x}$$

Solved as first order homogeneous class D ode

Time used: 0.233 (sec)

Writing the ode as

$$y' = 2x^2 \sin\left(\frac{y}{x}\right)^2 + \frac{y}{x} \tag{A}$$

The given ode has the form

$$y' = \frac{y}{x} + g(x)\, f\left(b\frac{y}{x}\right)^{\frac{n}{m}} \tag{1}$$

Where b is scalar and $g(x)$ is function of x and n, m are integers. The solution is given in Kamke page 20. Using the substitution $y(x) = u(x)\, x$ then

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Hence the given ode becomes

$$\begin{aligned} \frac{du}{dx}x + u &= u + g(x)\, f(bu)^{\frac{n}{m}} \\ u' &= \frac{1}{x} g(x)\, f(bu)^{\frac{n}{m}} \end{aligned} \tag{2}$$

The above ode is always separable. This is easily solved for u assuming the integration can be resolved, and then the solution to the original ode becomes $y = ux$. Comapring the given ode (A) with the form (1) shows that

$$
\begin{aligned}
g(x) &= 2x^2 \\
b &= 1 \\
f\left(\frac{bx}{y}\right) &= \sin\left(\frac{y}{x}\right)
\end{aligned}
$$

Substituting the above in (2) results in the $u(x)$ ode as

$$
u'(x) = 2x \sin(u(x))^2
$$

Which is now solved as separable The ode $u'(x) = 2x \sin(u(x))^2$ is separable as it can be written as

$$
\begin{aligned}
u'(x) &= 2x \sin(u(x))^2 \\
&= f(x)g(u)
\end{aligned}
$$

Where

$$
\begin{aligned}
f(x) &= 2x \\
g(u) &= \sin(u)^2
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(u)}\, du &= \int f(x)\, dx \\
\int \frac{1}{\sin(u)^2}\, du &= \int 2x\, dx \\
-\cot(u(x)) &= x^2 + c_1
\end{aligned}
$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $\sin(u)^2 = 0$ for $u(x)$ gives

$$
u(x) = 0
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
-\cot(u(x)) &= x^2 + c_1 \\
u(x) &= 0
\end{aligned}
$$

Solving for $u(x)$ gives

$$u(x) = 0$$
$$u(x) = \pi - \operatorname{arccot}\left(x^2 + c_1\right)$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = \pi - \operatorname{arccot}\left(x^2 + c_1\right)$ back to y gives

$$y = x\left(\pi - \operatorname{arccot}\left(x^2 + c_1\right)\right)$$

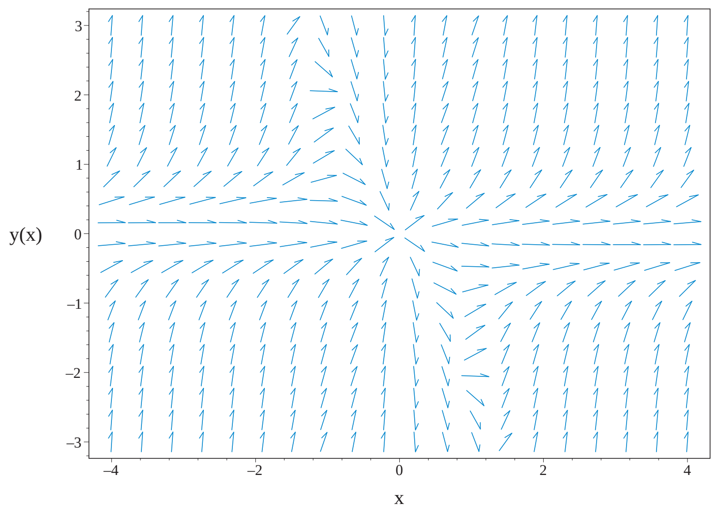


Figure 2.239: Slope field plot
$y' = 2x^2 \sin\left(\frac{y}{x}\right)^2 + \frac{y}{x}$

Summary of solutions found

$$y = 0$$
$$y = x\left(\pi - \operatorname{arccot}\left(x^2 + c_1\right)\right)$$

Solved as first order homogeneous class D2 ode

Time used: 0.077 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$u'(x)\,x + u(x) = 2x^2 \sin\left(u(x)\right)^2 + u(x)$$

Which is now solved The ode $u'(x) = 2x \sin\left(u(x)\right)^2$ is separable as it can be written as

$$\begin{aligned} u'(x) &= 2x \sin\left(u(x)\right)^2 \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= 2x \\ g(u) &= \sin\left(u\right)^2 \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\ \int \frac{1}{\sin\left(u\right)^2}\,du &= \int 2x\,dx \\ -\cot\left(u(x)\right) &= x^2 + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $\sin\left(u\right)^2 = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} -\cot\left(u(x)\right) &= x^2 + c_1 \\ u(x) &= 0 \end{aligned}$$

Solving for $u(x)$ gives

$$\begin{aligned} u(x) &= 0 \\ u(x) &= \pi - \operatorname{arccot}\left(x^2 + c_1\right) \end{aligned}$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = \pi - \operatorname{arccot}(x^2 + c_1)$ back to y gives

$$y = x\left(\pi - \operatorname{arccot}\left(x^2 + c_1\right)\right)$$

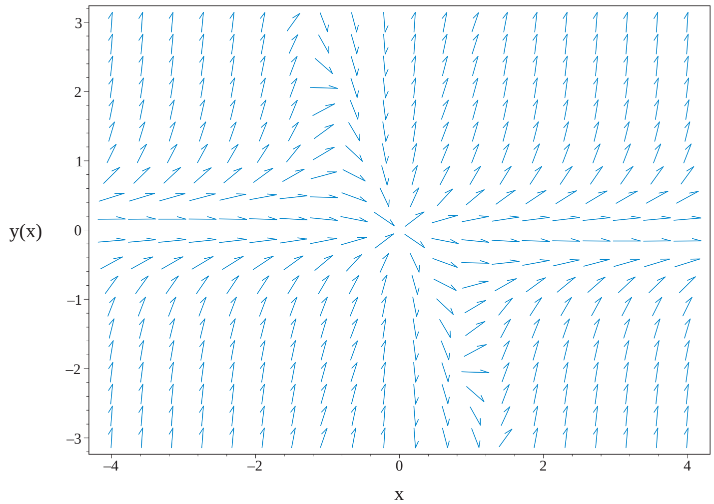


Figure 2.240: Slope field plot
$y' = 2x^2 \sin\left(\frac{y}{x}\right)^2 + \frac{y}{x}$

Summary of solutions found

$$y = 0$$
$$y = x\left(\pi - \operatorname{arccot}\left(x^2 + c_1\right)\right)$$

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) = 2x^2 \sin\left(\frac{y(x)}{x}\right)^2 + \frac{y(x)}{x}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = 2x^2 \sin\left(\frac{y(x)}{x}\right)^2 + \frac{y(x)}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 18

```
dsolve(diff(y(x),x) = 2*x^2*sin(y(x)/x)^2+y(x)/x,
       y(x),singsol=all)
```

$$y = \left(\frac{\pi}{2} + \arctan\left(x^2 + 2c_1\right)\right) x$$

Mathematica DSolve solution

Solving time : 0.307 (sec)
Leaf size : 22

```
DSolve[{D[y[x],x]== 2*x^2 * Sin[y[x]/x]^2 + y[x]/x,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -x \cot^{-1}\left(x^2 - 2c_1\right)$$
$$y(x) \to 0$$

2.3.30 problem 30

Internal problem ID [8564]
Book : Own collection of miscellaneous problems
Section : section 3.0
Problem number : 30
Date solved : `Thursday, December 12, 2024 at 09:31:04 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$4x^2y'' + y = 8\sqrt{x}\,(1 + \ln(x))$$

Solved as second order ode using Kovacic algorithm

Time used: 0.188 (sec)

Writing the ode as

$$4x^2y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned}
\tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-1}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -1 \\
t &= 4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.153: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 2 - 0\\ &= 2\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0\\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_\infty &= 0\\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2}\\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\,(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \sqrt{x} \end{aligned}$$

Which simplifies to

$$y_1 = \sqrt{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\
&= \sqrt{x} \int \frac{1}{x} \, dx \\
&= \sqrt{x}(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\right) + c_2\left(\sqrt{x}(\ln(x))\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
4x^2y'' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1\sqrt{x} + c_2\sqrt{x}\, \ln(x)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \sqrt{x} \\
y_2 &= \sqrt{x}\, \ln(x)
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x} & \sqrt{x}\,\ln(x) \\ \frac{d}{dx}(\sqrt{x}) & \frac{d}{dx}(\sqrt{x}\,\ln(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x} & \sqrt{x}\,\ln(x) \\ \frac{1}{2\sqrt{x}} & \frac{\ln(x)}{2\sqrt{x}} + \frac{1}{\sqrt{x}} \end{vmatrix}$$

Therefore

$$W = (\sqrt{x})\left(\frac{\ln(x)}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\right) - (\sqrt{x}\,\ln(x))\left(\frac{1}{2\sqrt{x}}\right)$$

Which simplifies to

$$W = 1$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{8x\ln(x)(1+\ln(x))}{4x^2}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{2\ln(x)(1+\ln(x))}{x}\,dx$$

Hence

$$u_1 = -\frac{2\ln(x)^3}{3} - \ln(x)^2$$

And Eq. (3) becomes

$$u_2 = \int \frac{8x(1 + \ln(x))}{4x^2}\, dx$$

Which simplifies to

$$u_2 = \int \frac{2\ln(x) + 2}{x} dx$$

Hence

$$u_2 = 2\ln(x) + \ln(x)^2$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{2\ln(x)^3}{3} - \ln(x)^2\right)\sqrt{x} + \sqrt{x}\,\ln(x)\left(2\ln(x) + \ln(x)^2\right)$$

Which simplifies to

$$y_p(x) = \frac{\ln(x)^2\,(\ln(x) + 3)\sqrt{x}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\sqrt{x} + c_2\sqrt{x}\,\ln(x)\right) + \left(\frac{\ln(x)^2\,(\ln(x) + 3)\sqrt{x}}{3}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1\sqrt{x} + c_2\sqrt{x}\,\ln(x) + \frac{\ln(x)^2\,(\ln(x) + 3)\sqrt{x}}{3}$$

Solved as second order ode adjoint method

Time used: 0.597 (sec)

In normal form the ode

$$4x^2y'' + y = 8\sqrt{x}\,(1 + \ln(x)) \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}$$

Where

$$\begin{aligned}
p(x) &= 0 \\
q(x) &= \frac{1}{4x^2} \\
r(x) &= \frac{2\ln(x) + 2}{x^{3/2}}
\end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - (0)' + \left(\frac{\xi(x)}{4x^2}\right) &= 0 \\
\xi''(x) + \frac{\xi(x)}{4x^2} &= 0
\end{aligned}$$

Which is solved for $\xi(x)$. Writing the ode as

$$\xi'' + \frac{\xi}{4x^2} = 0 \tag{1}$$

$$A\xi'' + B\xi' + C\xi = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= \frac{1}{4x^2}
\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = \xi e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then ξ is found using the inverse transformation

$$\xi = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.154: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) &= 0\\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx}\\ &= \mathrm{e}^{\int \frac{1}{2x} dx}\\ &= \sqrt{x}\end{aligned}$$

The first solution to the original ode in ξ is found from

$$\xi_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}\xi_1 &= z_1\\ &= \sqrt{x}\end{aligned}$$

Which simplifies to

$$\xi_1 = \sqrt{x}$$

The second solution ξ_2 to the original ode is found using reduction of order

$$\xi_2 = \xi_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{\xi_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
\xi_2 &= \xi_1 \int \frac{1}{\xi_1^2}\,dx \\
&= \sqrt{x} \int \frac{1}{x}\,dx \\
&= \sqrt{x}(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
\xi &= c_1\xi_1 + c_2\xi_2 \\
&= c_1\left(\sqrt{x}\right) + c_2\left(\sqrt{x}(\ln(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$
\begin{aligned}
\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y &= \int \xi(x)\,r(x)\,dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi(x)}
\end{aligned}
$$

Or

$$
y' - \frac{y\left(\frac{c_1}{2\sqrt{x}} + \frac{c_2\ln(x)}{2\sqrt{x}} + \frac{c_2}{\sqrt{x}}\right)}{c_1\sqrt{x} + c_2\sqrt{x}\,\ln(x)} = \frac{\frac{2c_2\ln(x)^3}{3} + (c_1 + c_2)\ln(x)^2 + 2\ln(x)\,c_1}{c_1\sqrt{x} + c_2\sqrt{x}\,\ln(x)}
$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$
y' + q(x)y = p(x)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= -\frac{c_2\ln(x) + c_1 + 2c_2}{2x\,(c_2\ln(x) + c_1)} \\
p(x) &= \frac{\ln(x)\left(\frac{2\ln(x)^2c_2}{3} + (c_1 + c_2)\ln(x) + 2c_1\right)}{\sqrt{x}\,(c_2\ln(x) + c_1)}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= \mathrm{e}^{\int -\frac{c_2 \ln(x)+c_1+2c_2}{2x(c_2\ln(x)+c_1)}dx} \\
&= \frac{1}{\sqrt{x}\,\left(c_2 \ln\left(x\right)+c_1\right)}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{\ln\left(x\right)\left(\frac{2\ln(x)^2 c_2}{3}+\left(c_1+c_2\right)\ln\left(x\right)+2c_1\right)}{\sqrt{x}\,\left(c_2\ln\left(x\right)+c_1\right)}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{\sqrt{x}\,\left(c_2\ln\left(x\right)+c_1\right)}\right) \\
&= \left(\frac{1}{\sqrt{x}\,\left(c_2\ln\left(x\right)+c_1\right)}\right)\left(\frac{\ln\left(x\right)\left(\frac{2\ln(x)^2 c_2}{3}+\left(c_1+c_2\right)\ln\left(x\right)+2c_1\right)}{\sqrt{x}\,\left(c_2\ln\left(x\right)+c_1\right)}\right) \\
\mathrm{d}\left(\frac{y}{\sqrt{x}\,\left(c_2\ln\left(x\right)+c_1\right)}\right) &= \left(\frac{\ln\left(x\right)\left(\frac{2\ln(x)^2 c_2}{3}+\left(c_1+c_2\right)\ln\left(x\right)+2c_1\right)}{x\left(c_2\ln\left(x\right)+c_1\right)^2}\right)\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{\sqrt{x}\,\left(c_2\ln\left(x\right)+c_1\right)} &= \int \frac{\ln\left(x\right)\left(\frac{2\ln(x)^2 c_2}{3}+\left(c_1+c_2\right)\ln\left(x\right)+2c_1\right)}{x\left(c_2\ln\left(x\right)+c_1\right)^2}\,dx \\
&= \frac{\ln\left(x\right)^2 c_2-\ln\left(x\right)c_1+3c_2\ln\left(x\right)}{3c_2^2}-\frac{c_1^2(c_1-3c_2)}{3c_2^3\left(c_2\ln\left(x\right)+c_1\right)}+c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{1}{\sqrt{x}\,(c_2\ln(x)+c_1)}$ gives the final solution

$$
y = -\frac{\sqrt{x}\left(-\ln\left(x\right)^3 c_2^3-3\ln\left(x\right)^2 c_2^3+c_2(-3c_3\,c_2^3+c_1^2-3c_1c_2)\ln\left(x\right)+c_1(-3c_3\,c_2^3+c_1^2-3c_1c_2)\right)}{3c_2^3}
$$

Hence, the solution found using Lagrange adjoint equation method is

$$
y = -\frac{\sqrt{x}\left(-\ln\left(x\right)^3 c_2^3-3\ln\left(x\right)^2 c_2^3+c_2(-3c_3\,c_2^3+c_1^2-3c_1c_2)\ln\left(x\right)+c_1(-3c_3\,c_2^3+c_1^2-3c_1c_2)\right)}{3c_2^3}
$$

The constants can be merged to give

$$y = -\frac{\sqrt{x}\,\left(-\ln\left(x\right)^{3} c_2^{3}-3\ln\left(x\right)^{2} c_2^{3}+c_2(-3c_2^{3}+c_1^{2}-3c_1c_2)\ln\left(x\right)+c_1(-3c_2^{3}+c_1^{2}-3c_1c_2)\right)}{3c_2^{3}}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{\sqrt{x}\,\left(-\ln\left(x\right)^{3} c_2^{3}-3\ln\left(x\right)^{2} c_2^{3}+c_2(-3c_2^{3}+c_1^{2}-3c_1c_2)\ln\left(x\right)+c_1(-3c_2^{3}+c_1^{2}-3c_1c_2)\right)}{3c_2^{3}}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : 24

```
dsolve(4*x^2*diff(diff(y(x),x),x)+y(x) = 8*x^(1/2)*(1+ln(x)),
       y(x),singsol=all)
```

$$y = \left(c_2 + c_1\ln\left(x\right) + \frac{\ln\left(x\right)^{3}}{3} + \ln\left(x\right)^{2}\right)\sqrt{x}$$

Mathematica DSolve solution

Solving time : 0.037 (sec)
Leaf size : 37

```
DSolve[{4*x^2*D[y[x],{x,2}]+y[x] == 8*Sqrt[x]*(1+Log[x]),{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{6}\sqrt{x}\left(2\log^3(x) + 6\log^2(x) + 3c_2\log(x) + 6c_1\right)$$

2.3.31 problem 31

Internal problem ID [8565]
Book : Own collection of miscellaneous problems
Section : section 3.0
Problem number : 31
Date solved : `Tuesday, December 17, 2024 at 12:56:06 PM`
CAS classification : `[_rational, _Bernoulli]`

Solve

$$vv' = \frac{2v^2}{r^3} + \frac{\lambda r}{3}$$

Solved as first order Bernoulli ode

Time used: 0.369 (sec)

In canonical form, the ODE is

$$\begin{aligned} v' &= F(r, v) \\ &= \frac{\lambda\, r^4 + 6v^2}{3v\, r^3} \end{aligned}$$

This is a Bernoulli ODE.

$$v' = \left(\frac{2}{r^3}\right) v + \left(\frac{\lambda r}{3}\right) \frac{1}{v} \tag{1}$$

The standard Bernoulli ODE has the form

$$v' = f_0(r)v + f_1(r)v^n \tag{2}$$

Comparing this to (1) shows that

$$\begin{aligned} f_0 &= \frac{2}{r^3} \\ f_1 &= \frac{\lambda r}{3} \end{aligned}$$

The first step is to divide the above equation by v^n which gives

$$\frac{v'}{v^n} = f_0(r)v^{1-n} + f_1(r) \tag{3}$$

The next step is use the substitution $v = v^{1-n}$ in equation (3) which generates a new ODE in $v(r)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $v(r)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(r) &= \frac{2}{r^3} \\ f_1(r) &= \frac{\lambda r}{3} \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $v^n = \frac{1}{v}$ gives

$$v'v = \frac{2v^2}{r^3} + \frac{\lambda r}{3} \tag{4}$$

Let

$$\begin{aligned} v &= v^{1-n} \\ &= v^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t r gives

$$v' = 2vv' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{v'(r)}{2} &= \frac{2v(r)}{r^3} + \frac{\lambda r}{3} \\ v' &= \frac{4v}{r^3} + \frac{2\lambda r}{3} \end{aligned} \tag{7}$$

The above now is a linear ODE in $v(r)$ which is now solved.

In canonical form a linear first order is

$$v'(r) + q(r)v(r) = p(r)$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(r) &= -\frac{4}{r^3} \\
p(r) &= \frac{2\lambda r}{3}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q \, dr} \\
&= \mathrm{e}^{\int -\frac{4}{r^3} dr} \\
&= \mathrm{e}^{\frac{2}{r^2}}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r}(\mu v) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}r}(\mu v) &= (\mu)\left(\frac{2\lambda r}{3}\right) \\
\frac{\mathrm{d}}{\mathrm{d}r}\left(v\,\mathrm{e}^{\frac{2}{r^2}}\right) &= \left(\mathrm{e}^{\frac{2}{r^2}}\right)\left(\frac{2\lambda r}{3}\right) \\
\mathrm{d}\left(v\,\mathrm{e}^{\frac{2}{r^2}}\right) &= \left(\frac{2\lambda r\,\mathrm{e}^{\frac{2}{r^2}}}{3}\right) \mathrm{d}r
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
v\,\mathrm{e}^{\frac{2}{r^2}} &= \int \frac{2\lambda r\,\mathrm{e}^{\frac{2}{r^2}}}{3}\, dr \\
&= \frac{2\lambda\left(\frac{\mathrm{e}^{\frac{2}{r^2}} r^2}{2} + \mathrm{Ei}_1\left(-\frac{2}{r^2}\right)\right)}{3} + c_1
\end{aligned}
$$

Dividing throughout by the integrating factor $\mathrm{e}^{\frac{2}{r^2}}$ gives the final solution

$$
v(r) = \frac{\lambda r^2}{3} + \frac{2\lambda\,\mathrm{Ei}_1\left(-\frac{2}{r^2}\right)\mathrm{e}^{-\frac{2}{r^2}}}{3} + c_1\,\mathrm{e}^{-\frac{2}{r^2}}
$$

The substitution $v = v^{1-n}$ is now used to convert the above solution back to v which results in

$$
v^2 = \frac{\lambda r^2}{3} + \frac{2\lambda\,\mathrm{Ei}_1\left(-\frac{2}{r^2}\right)\mathrm{e}^{-\frac{2}{r^2}}}{3} + c_1\,\mathrm{e}^{-\frac{2}{r^2}}
$$

Solving for v gives

$$v = -\frac{\sqrt{6\lambda\,\mathrm{Ei}_1\left(-\frac{2}{r^2}\right)\mathrm{e}^{-\frac{2}{r^2}} + 3\lambda\,r^2 + 9c_1\,\mathrm{e}^{-\frac{2}{r^2}}}}{3}$$

$$v = \frac{\sqrt{6\lambda\,\mathrm{Ei}_1\left(-\frac{2}{r^2}\right)\mathrm{e}^{-\frac{2}{r^2}} + 3\lambda\,r^2 + 9c_1\,\mathrm{e}^{-\frac{2}{r^2}}}}{3}$$

Summary of solutions found

$$v = -\frac{\sqrt{6\lambda\,\mathrm{Ei}_1\left(-\frac{2}{r^2}\right)\mathrm{e}^{-\frac{2}{r^2}} + 3\lambda\,r^2 + 9c_1\,\mathrm{e}^{-\frac{2}{r^2}}}}{3}$$

$$v = \frac{\sqrt{6\lambda\,\mathrm{Ei}_1\left(-\frac{2}{r^2}\right)\mathrm{e}^{-\frac{2}{r^2}} + 3\lambda\,r^2 + 9c_1\,\mathrm{e}^{-\frac{2}{r^2}}}}{3}$$

Solved as first order Exact ode

Time used: 0.262 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(r,v)\,\mathrm{d}r + N(r,v)\,\mathrm{d}v = 0 \tag{1A}$$

Therefore

$$(v)\,\mathrm{d}v = \left(\frac{2v^2}{r^3} + \frac{\lambda r}{3}\right)\mathrm{d}r$$

$$\left(-\frac{2v^2}{r^3} - \frac{\lambda r}{3}\right)\mathrm{d}r + (v)\,\mathrm{d}v = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(r,v) &= -\frac{2v^2}{r^3} - \frac{\lambda r}{3} \\ N(r,v) &= v \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial v} = \frac{\partial N}{\partial r}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial v} &= \frac{\partial}{\partial v}\left(-\frac{2v^2}{r^3} - \frac{\lambda r}{3}\right) \\ &= -\frac{4v}{r^3} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial r} &= \frac{\partial}{\partial r}(v) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial v} \neq \frac{\partial N}{\partial r}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial v} - \frac{\partial N}{\partial r}\right) \\ &= \frac{1}{v}\left(\left(-\frac{4v}{r^3}\right) - (0)\right) \\ &= -\frac{4}{r^3} \end{aligned}$$

Since A does not depend on v, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}r}\\ &= e^{\int -\frac{4}{r^3}\,\mathrm{d}r}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\frac{2}{r^2}}\\ &= \mathrm{e}^{\frac{2}{r^2}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{\frac{2}{r^2}}\left(-\frac{2v^2}{r^3}-\frac{\lambda r}{3}\right)\\ &= -\frac{(\lambda\, r^4+6v^2)\,\mathrm{e}^{\frac{2}{r^2}}}{3r^3}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{\frac{2}{r^2}}(v)\\ &= v\,\mathrm{e}^{\frac{2}{r^2}}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M}+\overline{N}\frac{\mathrm{d}v}{\mathrm{d}r} &= 0\\ \left(-\frac{(\lambda\, r^4+6v^2)\,\mathrm{e}^{\frac{2}{r^2}}}{3r^3}\right)+\left(v\,\mathrm{e}^{\frac{2}{r^2}}\right)\frac{\mathrm{d}v}{\mathrm{d}r} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(r,v)$

$$\frac{\partial\phi}{\partial r} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial v} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. r gives

$$\int \frac{\partial \phi}{\partial r}\, \mathrm{d}r = \int \overline{M}\, \mathrm{d}r$$

$$\int \frac{\partial \phi}{\partial r}\, \mathrm{d}r = \int -\frac{(\lambda\, r^4 + 6v^2)\, \mathrm{e}^{\frac{2}{r^2}}}{3r^3}\, \mathrm{d}r$$

$$\phi = -\frac{\lambda\, \mathrm{Ei}_1\left(-\frac{2}{r^2}\right)}{3} - \frac{\mathrm{e}^{\frac{2}{r^2}}(\lambda\, r^2 - 3v^2)}{6} + f(v) \tag{3}$$

Where $f(v)$ is used for the constant of integration since ϕ is a function of both r and v. Taking derivative of equation (3) w.r.t v gives

$$\frac{\partial \phi}{\partial v} = v\, \mathrm{e}^{\frac{2}{r^2}} + f'(v) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial v} = v\, \mathrm{e}^{\frac{2}{r^2}}$. Therefore equation (4) becomes

$$v\, \mathrm{e}^{\frac{2}{r^2}} = v\, \mathrm{e}^{\frac{2}{r^2}} + f'(v) \tag{5}$$

Solving equation (5) for $f'(v)$ gives

$$f'(v) = 0$$

Therefore

$$f(v) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(v)$ into equation (3) gives ϕ

$$\phi = -\frac{\lambda\, \mathrm{Ei}_1\left(-\frac{2}{r^2}\right)}{3} - \frac{\mathrm{e}^{\frac{2}{r^2}}(\lambda\, r^2 - 3v^2)}{6} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -\frac{\lambda\, \mathrm{Ei}_1\left(-\frac{2}{r^2}\right)}{3} - \frac{\mathrm{e}^{\frac{2}{r^2}}(\lambda\, r^2 - 3v^2)}{6}$$

Solving for v gives

$$v = -\frac{e^{-\frac{2}{r^2}}\sqrt{3}\,\sqrt{e^{\frac{2}{r^2}}\left(\lambda\,e^{\frac{2}{r^2}}r^2 + 2\lambda\,\mathrm{Ei}_1\left(-\frac{2}{r^2}\right) + 6c_1\right)}}{3}$$

$$v = \frac{e^{-\frac{2}{r^2}}\sqrt{3}\,\sqrt{e^{\frac{2}{r^2}}\left(\lambda\,e^{\frac{2}{r^2}}r^2 + 2\lambda\,\mathrm{Ei}_1\left(-\frac{2}{r^2}\right) + 6c_1\right)}}{3}$$

Summary of solutions found

$$v = -\frac{e^{-\frac{2}{r^2}}\sqrt{3}\,\sqrt{e^{\frac{2}{r^2}}\left(\lambda\,e^{\frac{2}{r^2}}r^2 + 2\lambda\,\mathrm{Ei}_1\left(-\frac{2}{r^2}\right) + 6c_1\right)}}{3}$$

$$v = \frac{e^{-\frac{2}{r^2}}\sqrt{3}\,\sqrt{e^{\frac{2}{r^2}}\left(\lambda\,e^{\frac{2}{r^2}}r^2 + 2\lambda\,\mathrm{Ei}_1\left(-\frac{2}{r^2}\right) + 6c_1\right)}}{3}$$

Maple step by step solution

Let's solve

$v(r)\left(\frac{d}{dr}v(r)\right) = \frac{2v(r)^2}{r^3} + \frac{\lambda r}{3}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dr}v(r)$

- Solve for the highest derivative

  $\frac{d}{dr}v(r) = \frac{\frac{2v(r)^2}{r^3} + \frac{\lambda r}{3}}{v(r)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

Maple dsolve solution

Solving time : 0.035 (sec)
Leaf size : 97

```
dsolve(v(r)*diff(v(r),r) = 2*v(r)^2/r^3+1/3*lambda*r,
       v(r),singsol=all)
```

$$v(r) = -\frac{\sqrt{3}\,\sqrt{\mathrm{e}^{\frac{2}{r^2}}\left(\lambda\,\mathrm{e}^{\frac{2}{r^2}} r^2+2\,\mathrm{Ei}_1\left(-\frac{2}{r^2}\right)\lambda+3c_1\right)}\,\mathrm{e}^{-\frac{2}{r^2}}}{3}$$
$$v(r) = \frac{\sqrt{3}\,\sqrt{\mathrm{e}^{\frac{2}{r^2}}\left(\lambda\,\mathrm{e}^{\frac{2}{r^2}} r^2+2\,\mathrm{Ei}_1\left(-\frac{2}{r^2}\right)\lambda+3c_1\right)}\,\mathrm{e}^{-\frac{2}{r^2}}}{3}$$

Mathematica DSolve solution

Solving time : 8.578 (sec)
Leaf size : 98

```
DSolve[{v[r]*D[v[r],r]==2*v[r]^2/r^3+1/3*\[Lambda]*r,{}},
       v[r],r,IncludeSingularSolutions->True]
```

$$v(r) \to -\frac{\sqrt{e^{-\frac{2}{r^2}}\left(-2\lambda\,\text{ExpIntegralEi}\left(\frac{2}{r^2}\right)+\lambda e^{\frac{2}{r^2}} r^2+3c_1\right)}}{\sqrt{3}}$$
$$v(r) \to \frac{\sqrt{e^{-\frac{2}{r^2}}\left(-2\lambda\,\text{ExpIntegralEi}\left(\frac{2}{r^2}\right)+\lambda e^{\frac{2}{r^2}} r^2+3c_1\right)}}{\sqrt{3}}$$

2.4 section 4.0

2.4.1 problem 1

Internal problem ID [8566]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 1
Date solved : `Thursday, December 12, 2024 at 09:31:07 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2y'' - xy' + \left(-x^2 + 1\right) y = 0$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' - xy' + \left(-x^2 + 1\right) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{2x}$$
$$q(x) = -\frac{x^2 - 1}{2x^2}$$

Table 2.156: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{x^2-1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' - xy' + (-x^2 + 1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &2x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\ &- x\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (-x^2+1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\begin{aligned} &\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)\,(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) \\ &+ \sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) = \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$2x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 3r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = \frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as $\left[1, \frac{1}{2}\right]$.

Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) - a_n(n+r) - a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-2}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \ldots\right) \\ &= x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - b_n(n+r) - b_{n-2} + b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = \frac{b_{n-2}}{n(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = \frac{1}{168}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \ldots\right) \\ &= \sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) + c_2\sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) + c_2\sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right) \end{aligned}$$

Maple step by step solution

Let's solve

$2x^2\left(\frac{d^2}{dx^2}y(x)\right) - x\left(\frac{d}{dx}y(x)\right) + (-x^2+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{(x^2-1)y(x)}{2x^2} + \frac{\frac{d}{dx}y(x)}{2x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{\frac{d}{dx}y(x)}{2x} - \frac{(x^2-1)y(x)}{2x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = -\frac{1}{2x}, P_3(x) = -\frac{x^2-1}{2x^2}\right]$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = -\frac{1}{2}$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = \frac{1}{2}$

- ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $2x^2\left(\frac{d^2}{dx^2}y(x)\right) - x\left(\frac{d}{dx}y(x)\right) + (-x^2+1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

- ◦ Shift index using $k- >k-m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$
- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(-1+2r)\,(-1+r)\, x^r+a_1(1+2r)\, r\, x^{1+r}+\left(\sum_{k=2}^{\infty} \left(a_k(2k+2r-1)\,(k+r-1)-a_{k-2}\right) x^{k+r}\right) =$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(-1+2r)\,(-1+r)=0$
- Values of r that satisfy the indicial equation

  $r \in \left\{1, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(1+2r)\, r=0$
- Solve for the dependent coefficient(s)

  $a_1=0$
- Each term in the series must be 0, giving the recursion relation

  $2\left(k+r-\frac{1}{2}\right)(k+r-1)\, a_k-a_{k-2}=0$
- Shift index using $k->k+2$

  $2\left(k+\frac{3}{2}+r\right)(k+1+r)\, a_{k+2}-a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+2}=\frac{a_k}{(2k+3+2r)(k+1+r)}$
- Recursion relation for $r=1$

  $a_{k+2}=\frac{a_k}{(2k+5)(k+2)}$
- Solution for $r=1$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+2}=\frac{a_k}{(2k+5)(k+2)}, a_1=0\right]$
- Recursion relation for $r=\frac{1}{2}$

  $a_{k+2}=\frac{a_k}{(2k+4)\left(k+\frac{3}{2}\right)}$
- Solution for $r=\frac{1}{2}$

  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2}=\frac{a_k}{(2k+4)\left(k+\frac{3}{2}\right)}, a_1=0\right]$
- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = \frac{a_k}{(2k+5)(k+2)}, a_1 = 0, b_{k+2} = \frac{b_k}{(2k+4)\left(k+\frac{3}{2}\right)}, b_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

Maple dsolve solution

Solving time : 0.019 (sec)
Leaf size : 33

```
dsolve(2*x^2*diff(diff(y(x),x),x)-diff(y(x),x)*x+(-x^2+1)*y(x) = 0,y(x),
       series,x=0)
```

$$y = c_1\sqrt{x}\left(1+\frac{1}{6}x^2+\frac{1}{168}x^4+\mathrm{O}\left(x^6\right)\right)+c_2x\left(1+\frac{1}{10}x^2+\frac{1}{360}x^4+\mathrm{O}\left(x^6\right)\right)$$

Mathematica DSolve solution

Solving time : 0.008 (sec)
Leaf size : 48

```
AsymptoticDSolveValue[{2*x^2*D[y[x],{x,2}] - x*D[y[x],x] + (1-x^2 )*y[x] ==0,{}},
       y[x],{x,0,5}]
```

$$y(x) \to c_1 x \left(\frac{x^4}{360} + \frac{x^2}{10} + 1\right) + c_2 \sqrt{x} \left(\frac{x^4}{168} + \frac{x^2}{6} + 1\right)$$

2.4.2 problem 2

Internal problem ID [8567]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 2
Date solved : `Thursday, December 12, 2024 at 09:31:08 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$2x^2y'' - xy' + \left(-x^2+1\right) y = 1$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' - xy' + \left(-x^2+1\right) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{2x}$$
$$q(x) = -\frac{x^2-1}{2x^2}$$

Table 2.158: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{x^2-1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' - xy' + \left(-x^2 + 1\right) y = 1$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $2x^2y'' - xy' + (-x^2 + 1) y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &2x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) \\ &- x \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + \left(-x^2 + 1\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n(n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) = \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n(n+r)\right) + \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n(n+r)(n+r-1) - x^{n+r} a_n(n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 3r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = \frac{1}{2}$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$(2x^m m(-1+m) - x^m m + x^m)\, c_0 = 1$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as $\left[1, \frac{1}{2}\right]$.

Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) - a_n(n+r) - a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-2}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - b_n(n+r) - b_{n-2} + b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = \frac{b_{n-2}}{n(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = \frac{1}{168}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) + c_2\sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right) \end{aligned}$$

The particular solution is found by solving for c, m the balance equation

$$\left(2x^m m(-1+m) - x^m m + x^m\right) c_0 = F$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$\begin{aligned} a_1 &= 0 \\ a_2 &= \frac{a_0}{2r^2+5r+3} \\ a_3 &= 0 \\ a_4 &= \frac{a_0}{(2r^2+5r+3)(2r^2+13r+21)} \\ a_5 &= 0 \end{aligned}$$

Now we determine the particular solution y_p associated with $F = 1$ by solving the balance equation

$$\left(2x^m m(-1+m) - x^m m + x^m\right) c_0 = 1$$

For c_0 and x. This results in

$$\begin{aligned} c_0 &= 1 \\ m &= 0 \end{aligned}$$

The particular solution is therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= \sum_{n=0}^{\infty} c_n x^{n+0} \end{aligned}$$

Where in the above $c_0 = 1$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 0$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = 1$ and $r = m$ or $r = 0$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = 1$
$c_1 = 0$
$c_2 = \frac{1}{3}$
$c_3 = 0$
$c_4 = \frac{1}{63}$
$c_5 = 0$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
&= 1 \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= 1\left(1 + \frac{1}{3}x^2 + \frac{1}{63}x^4\right) \\
&= 1 + \frac{1}{3}x^2 + \frac{1}{63}x^4
\end{aligned}
$$

Adding all the above particular solution(s) gives

$$
y_p = 1 + \frac{x^2}{3} + \frac{x^4}{63} + O(x^6)
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= 1+\frac{x^2}{3}+\frac{x^4}{63}+O(x^6)+c_1 x\left(1+\frac{x^2}{10}+\frac{x^4}{360}+O(x^6)\right)+c_2\sqrt{x}\left(1+\frac{x^2}{6}+\frac{x^4}{168}+O(x^6)\right)
\end{aligned}
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.018 (sec)
Leaf size : 43

```
dsolve(2*x^2*diff(diff(y(x),x),x)-diff(y(x),x)*x+(-x^2+1)*y(x) = 1,y(x),
       series,x=0)
```

$$
\begin{aligned}
y = {} & c_1\sqrt{x}\left(1+\frac{1}{6}x^2+\frac{1}{168}x^4+\mathrm{O}\left(x^6\right)\right) \\
& + c_2 x\left(1+\frac{1}{10}x^2+\frac{1}{360}x^4+\mathrm{O}\left(x^6\right)\right)+\left(1+\frac{1}{3}x^2+\frac{1}{63}x^4+\mathrm{O}\left(x^6\right)\right)
\end{aligned}
$$

Mathematica DSolve solution

Solving time : 0.028 (sec)
Leaf size : 176

```
AsymptoticDSolveValue[{2*x^2*D[y[x],{x,2}] - x*D[y[x],x] + (1-x^2 )*y[x] ==1,{}},
       y[x],{x,0,5}]
```

$$y(x) \to c_2 x\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right) + c_1\sqrt{x}\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right) + \sqrt{x}\left(-\frac{x^{11/2}}{154440}-\frac{x^{7/2}}{1260}-\frac{x^{3/2}}{15} + \frac{2}{\sqrt{x}}\right)\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)+x\left(\frac{x^5}{55440}+\frac{x^3}{504}+\frac{x}{6}-\frac{1}{x}\right)\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right)$$

2.4.3 problem 3

Internal problem ID [8568]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 3
Date solved : `Thursday, December 12, 2024 at 09:31:09 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$2x^2y'' - xy' + \left(-x^2+1\right)y = 1+x$$

Using series expansion around $x=0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' - xy' + \left(-x^2+1\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{2x}$$
$$q(x) = -\frac{x^2-1}{2x^2}$$

Table 2.159: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{2x}$	
singularity	type
$x=0$	"regular"

$q(x) = -\frac{x^2-1}{2x^2}$	
singularity	type
$x=0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' - xy' + (-x^2 + 1)\, y = 1 + x$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $2x^2y'' - xy' + (-x^2 + 1)\, y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &2x^2\left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) \\ &- x\left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (-x^2 + 1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n(n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) = \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n(n+r)\right) + \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n(n+r)(n+r-1) - x^{n+r} a_n(n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 3r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 1 \\ r_2 &= \frac{1}{2} \end{aligned}$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$(2x^m m(-1+m) - x^m m + x^m)\, c_0 = 1 + x$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as $\left[1, \frac{1}{2}\right]$.

Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+1} \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}} \end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) - a_n(n+r) - a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-2}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - b_n(n+r) - b_{n-2} + b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = \frac{b_{n-2}}{n(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = \frac{1}{168}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) + c_2\sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)
\end{aligned}$$

The particular solution is found by solving for c, m the balance equation

$$(2x^m m(-1+m) - x^m m + x^m)\, c_0 = F$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$\boxed{\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{a_0}{2r^2+5r+3} \\
a_3 &= 0 \\
a_4 &= \frac{a_0}{(2r^2+5r+3)(2r^2+13r+21)} \\
a_5 &= 0
\end{aligned}}$$

Since the $F = 1 + x$ has more than one term then we find a particular solution for each term and add the result to find the particular solution to the ode.

Now we determine the particular solution y_p associated with $F = 1$ by solving the balance equation

$$(2x^m m(-1+m) - x^m m + x^m)\, c_0 = 1$$

For c_0 and x. This results in

$$\begin{aligned}
c_0 &= 1 \\
m &= 0
\end{aligned}$$

The particular solution is therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= \sum_{n=0}^{\infty} c_n x^{n+0} \end{aligned}$$

Where in the above $c_0 = 1$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 0$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = 1$ and $r = m$ or $r = 0$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = 1$
$c_1 = 0$
$c_2 = \frac{1}{3}$
$c_3 = 0$
$c_4 = \frac{1}{63}$
$c_5 = 0$

The particular solution is now found using

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= 1 \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$\begin{aligned} y_p &= 1\left(1 + \frac{1}{3}x^2 + \frac{1}{63}x^4\right) \\ &= 1 + \frac{1}{3}x^2 + \frac{1}{63}x^4 \end{aligned}$$

Unable to solve the balance equation $(2x^m m(-1+m) - x^m m + x^m)\, c_0$ for c_0 and x. No particular solution exists.

Failed to convert RHS $1 + x$ to series in order to find particular solution. Unable to solve. Terminating Unable to find the particular solution or no solution exists.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.040 (sec)
Leaf size : maple_leaf_size

```
dsolve(2*x^2*diff(diff(y(x),x),x)-diff(y(x),x)*x+(-x^2+1)*y(x) = x+1,y(x),
       series,x=0)
```

No solution found

Mathematica DSolve solution

Solving time : `0.042` (sec)
Leaf size : 224

```
AsymptoticDSolveValue[{2*x^2*D[y[x],{x,2}] - x*D[y[x],x] + (1-x^2 )*y[x] ==1+x,{}},
       y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to{} & c_1\sqrt{x}\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)+c_2x\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right) \\
& +\sqrt{x}\left(-\frac{x^{11/2}}{154440}-\frac{x^{9/2}}{1620}-\frac{x^{7/2}}{1260}-\frac{x^{5/2}}{25}-\frac{x^{3/2}}{15}-2\sqrt{x}\right. \\
& \left.+\frac{2}{\sqrt{x}}\right)\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)+x\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right)\left(\frac{x^6}{66528}+\frac{x^5}{55440}+\frac{x^4}{672}+\frac{x^3}{504}+\frac{x^2}{12}+\frac{x}{6}-\frac{1}{x}+\right.
\end{aligned}
$$

2.4.4 problem 4

Internal problem ID [8569]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 4
Date solved : `Thursday, December 12, 2024 at 09:31:10 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$2x^2y'' - xy' + \left(-x^2 + 1\right) y = x$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' - xy' + \left(-x^2 + 1\right) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{2x}$$
$$q(x) = -\frac{x^2 - 1}{2x^2}$$

Table 2.160: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{x^2-1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' - xy' + \left(-x^2+1\right)y = x$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $2x^2y'' - xy' + (-x^2+1)\,y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &2x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) \\ &- x\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + \left(-x^2+1\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r}a_n(n+r)\,(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-x^{n+r}a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-x^{n+r+2}a_n\right)+\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-x^{n+r+2}a_n\right)=\sum_{n=2}^{\infty}\left(-a_{n-2}x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r}a_n(n+r)\,(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-x^{n+r}a_n(n+r)\right)+\sum_{n=2}^{\infty}\left(-a_{n-2}x^{n+r}\right)+\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r}a_n(n+r)\,(n+r-1)-x^{n+r}a_n(n+r)+a_nx^{n+r}=0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r)-x^r a_0 r+a_0 x^r=0$$

Or

$$\left(2x^r r(-1+r)-x^r r+x^r\right)a_0=0$$

Since $a_0\neq 0$ then the above simplifies to

$$\left(2r^2-3r+1\right)x^r=0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2-3r+1=0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = \frac{1}{2}$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$(2x^m m(-1+m) - x^m m + x^m)\, c_0 = x$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$(2r^2 - 3r + 1)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as $\left[1, \frac{1}{2}\right]$.

Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) - a_n(n+r) - a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-2}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - b_n(n+r) - b_{n-2} + b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = \frac{b_{n-2}}{n(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = \frac{1}{168}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) + c_2 \sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)
\end{aligned}$$

The particular solution is found by solving for c, m the balance equation

$$(2x^m m(-1+m) - x^m m + x^m)\, c_0 = F$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$\boxed{\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{a_0}{2r^2+5r+3} \\
a_3 &= 0 \\
a_4 &= \frac{a_0}{(2r^2+5r+3)(2r^2+13r+21)} \\
a_5 &= 0
\end{aligned}}$$

Unable to solve the balance equation $(2x^m m(-1+m) - x^m m + x^m)\, c_0$ for c_0 and x. No particular solution exists.

Failed to convert RHS x to series in order to find particular solution. Unable to solve. Terminating Unable to find the particular solution or no solution exists.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.039 (sec)
Leaf size : maple_leaf_size

```
dsolve(2*x^2*diff(diff(y(x),x),x)-diff(y(x),x)*x+(-x^2+1)*y(x) = x,y(x),
       series,x=0)
```

No solution found

Mathematica DSolve solution

Solving time : 0.057 (sec)
Leaf size : 166

```
AsymptoticDSolveValue[{2*x^2*D[y[x],{x,2}] - x*D[y[x],x] + (1-x^2 )*y[x] ==x,{}},
       y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to\ & x\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right)\left(\frac{x^6}{66528}+\frac{x^4}{672}+\frac{x^2}{12}+\log(x)\right) \\
& + c_1\sqrt{x}\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)+c_2x\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right) \\
& + \sqrt{x}\left(-\frac{x^{9/2}}{1620}-\frac{x^{5/2}}{25}-2\sqrt{x}\right)\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)
\end{aligned}
$$

2.4.5 problem 5

Internal problem ID [8570]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 5
Date solved : `Thursday, December 12, 2024 at 09:31:12 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$2x^2y'' - xy' + \left(-x^2+1\right)y = x^2+x+1$$

Using series expansion around $x=0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' - xy' + \left(-x^2+1\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{2x} \\ q(x) &= -\frac{x^2-1}{2x^2} \end{aligned}$$

Table 2.161: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{2x}$	
singularity	type
$x=0$	"regular"

$q(x) = -\frac{x^2-1}{2x^2}$	
singularity	type
$x=0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' - xy' + \left(-x^2 + 1\right)y = x^2 + x + 1$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $2x^2y'' - xy' + (-x^2 + 1)\,y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned}
&2x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\
&- x\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(-x^2+1\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0
\end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r}a_n(n+r)(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-x^{n+r}a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-x^{n+r+2}a_n\right)+\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-x^{n+r+2}a_n\right)=\sum_{n=2}^{\infty}\left(-a_{n-2}x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r}a_n(n+r)(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-x^{n+r}a_n(n+r)\right)+\sum_{n=2}^{\infty}\left(-a_{n-2}x^{n+r}\right)+\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r}a_n(n+r)(n+r-1)-x^{n+r}a_n(n+r)+a_n x^{n+r}=0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r)-x^r a_0 r+a_0 x^r=0$$

Or

$$\left(2x^r r(-1+r)-x^r r+x^r\right)a_0=0$$

Since $a_0\neq 0$ then the above simplifies to

$$\left(2r^2-3r+1\right)x^r=0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2-3r+1=0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = \frac{1}{2}$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$\left(2x^m m(-1+m) - x^m m + x^m\right) c_0 = x^2 + x + 1$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as $\left[1, \frac{1}{2}\right]$.

Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) - a_n(n+r) - a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-2}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - b_n(n+r) - b_{n-2} + b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = \frac{b_{n-2}}{n(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = \frac{1}{168}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) + c_2\sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)
\end{aligned}
$$

The particular solution is found by solving for c, m the balance equation

$$
\left(2x^m m(-1+m) - x^m m + x^m\right) c_0 = F
$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$
\boxed{
\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{a_0}{2r^2+5r+3} \\
a_3 &= 0 \\
a_4 &= \frac{a_0}{(2r^2+5r+3)(2r^2+13r+21)} \\
a_5 &= 0
\end{aligned}
}
$$

Since the $F = x^2 + x + 1$ has more than one term then we find a particular solution for each term and add the result to find the particular solution to the ode.

Now we determine the particular solution y_p associated with $F = x^2$ by solving the balance equation

$$
\left(2x^m m(-1+m) - x^m m + x^m\right) c_0 = x^2
$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= \frac{1}{3} \\
m &= 2
\end{aligned}
$$

The particular solution is therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= \sum_{n=0}^{\infty} c_n x^{n+2} \end{aligned}$$

Where in the above $c_0 = \frac{1}{3}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 2$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{3}$ and $r = m$ or $r = 2$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = \frac{1}{3}$
$c_1 = 0$
$c_2 = \frac{1}{63}$
$c_3 = 0$
$c_4 = \frac{1}{3465}$
$c_5 = 0$

The particular solution is now found using

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= x^2 \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$\begin{aligned} y_p &= x^2 \left(\frac{1}{3} + \frac{1}{63} x^2 + \frac{1}{3465} x^4 \right) \\ &= \frac{1}{3} x^2 + \frac{1}{63} x^4 + \frac{1}{3465} x^6 \end{aligned}$$

Unable to solve the balance equation $(2x^m m(-1+m) - x^m m + x^m)\, c_0$ for c_0 and x. No particular solution exists.

Failed to convert RHS $x^2 + x + 1$ to series in order to find particular solution. Unable to solve. Terminating Unable to find the particular solution or no solution exists.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.038 (sec)
Leaf size : maple_leaf_size

```
dsolve(2*x^2*diff(diff(y(x),x),x)-diff(y(x),x)*x+(-x^2+1)*y(x) = x^2+x+1,y(x),
       series,x=0)
```

No solution found

Mathematica DSolve solution

Solving time : 0.045 (sec)
Leaf size : 224

```
AsymptoticDSolveValue[{2*x^2*D[y[x],{x,2}] - x*D[y[x],x] + (1-x^2 )*y[x] ==1+x+x^2,{}},
       y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt{x}\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)+c_2x\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right)$$
$$+\sqrt{x}\left(-\frac{79x^{11/2}}{154440}-\frac{x^{9/2}}{1620}-\frac{37x^{7/2}}{1260}-\frac{x^{5/2}}{25}-\frac{11x^{3/2}}{15}-2\sqrt{x}\right.$$
$$\left.+\frac{2}{\sqrt{x}}\right)\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)+x\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right)\left(\frac{x^6}{66528}+\frac{67x^5}{55440}+\frac{x^4}{672}+\frac{29x^3}{504}+\frac{x^2}{12}+\frac{7x}{6}-\ldots\right.$$

2.4.6 problem 6

Internal problem ID [8571]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 6
Date solved : `Thursday, December 12, 2024 at 09:31:13 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$2x^2y'' - xy' + \left(-x^2+1\right)y = x^2$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' - xy' + \left(-x^2+1\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{2x}$$
$$q(x) = -\frac{x^2-1}{2x^2}$$

Table 2.162: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{x^2-1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' - xy' + \left(-x^2+1\right)y = x^2$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $2x^2y'' - xy' + (-x^2+1)\,y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} 2x^2 &\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\ &- x\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(-x^2+1\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) \\
&+ \sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) = \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right)
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) \\
&+ \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
2x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0
$$

When $n=0$ the above becomes

$$
2x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0
$$

Or

$$
\left(2x^r r(-1+r) - x^r r + x^r\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
\left(2r^2 - 3r + 1\right) x^r = 0
$$

Since the above is true for all x then the indicial equation becomes

$$
2r^2 - 3r + 1 = 0
$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = \frac{1}{2}$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$(2x^m m(-1+m) - x^m m + x^m)\, c_0 = x^2$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$(2r^2 - 3r + 1)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as $\left[1, \frac{1}{2}\right]$.

Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) - a_n(n+r) - a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-2}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - b_n(n+r) - b_{n-2} + b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = \frac{b_{n-2}}{n(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = \frac{1}{168}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) + c_2\sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)
\end{aligned}
$$

The particular solution is found by solving for c, m the balance equation

$$
\left(2x^m m(-1+m) - x^m m + x^m\right) c_0 = F
$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$
\boxed{
\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{a_0}{2r^2+5r+3} \\
a_3 &= 0 \\
a_4 &= \frac{a_0}{(2r^2+5r+3)(2r^2+13r+21)} \\
a_5 &= 0
\end{aligned}
}
$$

Now we determine the particular solution y_p associated with $F = x^2$ by solving the balance equation

$$
\left(2x^m m(-1+m) - x^m m + x^m\right) c_0 = x^2
$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= \frac{1}{3} \\
m &= 2
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= \sum_{n=0}^{\infty} c_n x^{n+2}
\end{aligned}
$$

Where in the above $c_0 = \frac{1}{3}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 2$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{3}$ and $r = m$ or $r = 2$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = \frac{1}{3}$
$c_1 = 0$
$c_2 = \frac{1}{63}$
$c_3 = 0$
$c_4 = \frac{1}{3465}$
$c_5 = 0$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
&= x^2 \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x^2 \left(\frac{1}{3} + \frac{1}{63} x^2 + \frac{1}{3465} x^4\right) \\
&= \frac{1}{3} x^2 + \frac{1}{63} x^4 + \frac{1}{3465} x^6
\end{aligned}
$$

Adding all the above particular solution(s) gives

$$
y_p = \frac{x^2}{3} + \frac{x^4}{63} + \frac{x^6}{3465} + O(x^6)
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \frac{x^2}{3}+\frac{x^4}{63}+\frac{x^6}{3465}+O(x^6)+c_1x\left(1+\frac{x^2}{10}+\frac{x^4}{360}+O(x^6)\right)+c_2\sqrt{x}\left(1+\frac{x^2}{6}+\frac{x^4}{168}+O(x^6)\right)
\end{aligned}
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.017 (sec)
Leaf size : 45

```
dsolve(2*x^2*diff(diff(y(x),x),x)-diff(y(x),x)*x+(-x^2+1)*y(x) = x^2,y(x),
       series,x=0)
```

$$y = c_1\sqrt{x}\left(1+\frac{1}{6}x^2+\frac{1}{168}x^4+\mathrm{O}\left(x^6\right)\right) + c_2x\left(1+\frac{1}{10}x^2+\frac{1}{360}x^4+\mathrm{O}\left(x^6\right)\right)+x^2\left(\frac{1}{3}+\frac{1}{63}x^2+\mathrm{O}\left(x^4\right)\right)$$

Mathematica DSolve solution

Solving time : 0.029 (sec)
Leaf size : 160

```
AsymptoticDSolveValue[{2*x^2*D[y[x],{x,2}] - x*D[y[x],x] + (1-x^2 )*y[x] ==x^2,{}},
      y[x],{x,0,5}]
```

$$y(x)\to c_2x\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right)+c_1\sqrt{x}\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)+\sqrt{x}\left(-\frac{x^{11/2}}{1980}-\frac{x^{7/2}}{35}-\frac{2x^{3/2}}{3}\right)\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)+x\left(\frac{x^5}{840}+\frac{x^3}{18}+x\right)\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right)$$

2.4.7 problem 7

Internal problem ID [8572]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 7
Date solved : `Thursday, December 12, 2024 at 09:31:14 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$2x^2y'' - xy' + \left(-x^2+1\right)y = x^2+1$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' - xy' + \left(-x^2+1\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{2x}$$
$$q(x) = -\frac{x^2-1}{2x^2}$$

Table 2.163: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{x^2-1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' - xy' + \left(-x^2 + 1\right) y = x^2 + 1$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $2x^2y'' - xy' + (-x^2 + 1) y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) - x \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + \left(-x^2 + 1\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) \\
&+ \sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) = \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right)
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) \\
&+ \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
2x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0
$$

When $n=0$ the above becomes

$$
2x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0
$$

Or

$$
\left(2x^r r(-1+r) - x^r r + x^r\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
\left(2r^2 - 3r + 1\right) x^r = 0
$$

Since the above is true for all x then the indicial equation becomes

$$
2r^2 - 3r + 1 = 0
$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = \frac{1}{2}$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$(2x^m m(-1+m) - x^m m + x^m)\, c_0 = x^2 + 1$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$(2r^2 - 3r + 1)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as $\left[1, \frac{1}{2}\right]$.

Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) - a_n(n+r) - a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-2}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - b_n(n+r) - b_{n-2} + b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = \frac{b_{n-2}}{n(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = \frac{1}{168}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O\left(x^6\right)\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O\left(x^6\right)\right) + c_2\sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O\left(x^6\right)\right)
\end{aligned}$$

The particular solution is found by solving for c, m the balance equation

$$\left(2x^m m(-1+m) - x^m m + x^m\right) c_0 = F$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$\boxed{\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{a_0}{2r^2+5r+3} \\
a_3 &= 0 \\
a_4 &= \frac{a_0}{(2r^2+5r+3)(2r^2+13r+21)} \\
a_5 &= 0
\end{aligned}}$$

Since the $F = x^2 + 1$ has more than one term then we find a particular solution for each term and add the result to find the particular solution to the ode.

Now we determine the particular solution y_p associated with $F = x^2$ by solving the balance equation

$$\left(2x^m m(-1+m) - x^m m + x^m\right) c_0 = x^2$$

For c_0 and x. This results in

$$\begin{aligned}
c_0 &= \frac{1}{3} \\
m &= 2
\end{aligned}$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= \sum_{n=0}^{\infty} c_n x^{n+2}
\end{aligned}
$$

Where in the above $c_0 = \frac{1}{3}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 2$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{3}$ and $r = m$ or $r = 2$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = \frac{1}{3}$
$c_1 = 0$
$c_2 = \frac{1}{63}$
$c_3 = 0$
$c_4 = \frac{1}{3465}$
$c_5 = 0$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
&= x^2 \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x^2 \left(\frac{1}{3} + \frac{1}{63} x^2 + \frac{1}{3465} x^4\right) \\
&= \frac{1}{3} x^2 + \frac{1}{63} x^4 + \frac{1}{3465} x^6
\end{aligned}
$$

Now we determine the particular solution y_p associated with $F = 1$ by solving the balance equation

$$
\left(2x^m m(-1+m) - x^m m + x^m\right) c_0 = 1
$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= 1 \\
m &= 0
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= \sum_{n=0}^{\infty} c_n x^{n+0}
\end{aligned}
$$

Where in the above $c_0 = 1$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 0$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = 1$ and $r = m$ or $r = 0$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = 1$
$c_1 = 0$
$c_2 = \frac{1}{3}$
$c_3 = 0$
$c_4 = \frac{1}{63}$
$c_5 = 0$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
&= 1 \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= 1\left(1 + \frac{1}{3}x^2 + \frac{1}{63}x^4\right) \\
&= 1 + \frac{1}{3}x^2 + \frac{1}{63}x^4
\end{aligned}
$$

Adding all the above particular solution(s) gives

$$y_p = 1 + \frac{2x^2}{3} + \frac{2x^4}{63} + \frac{x^6}{3465} + O(x^6)$$

Hence the final solution is

$$\begin{aligned} y &= y_h + y_p \\ &= 1 + \frac{2x^2}{3} + \frac{2x^4}{63} + \frac{x^6}{3465} + O(x^6) + c_1 x \left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) \\ &\quad + c_2 \sqrt{x} \left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right) \end{aligned}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.018 (sec)
Leaf size : 43

```
dsolve(2*x^2*diff(diff(y(x),x),x)-diff(y(x),x)*x+(-x^2+1)*y(x) = x^2+1,y(x),
       series,x=0)
```

$$y = c_1\sqrt{x}\left(1+\frac{1}{6}x^2+\frac{1}{168}x^4+\mathrm{O}\left(x^6\right)\right) + c_2x\left(1+\frac{1}{10}x^2+\frac{1}{360}x^4+\mathrm{O}\left(x^6\right)\right)+\left(1+\frac{2}{3}x^2+\frac{2}{63}x^4+\mathrm{O}\left(x^6\right)\right)$$

Mathematica DSolve solution

Solving time : 0.04 (sec)
Leaf size : 176

```
AsymptoticDSolveValue[{2*x^2*D[y[x],{x,2}] - x*D[y[x],x] + (1-x^2 )*y[x] ==1+x^2,{}},
      y[x],{x,0,5}]
```

$$y(x) \to c_2x\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right) + c_1\sqrt{x}\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)+\sqrt{x}\left(-\frac{79x^{11/2}}{154440}-\frac{37x^{7/2}}{1260}-\frac{11x^{3/2}}{15} + \frac{2}{\sqrt{x}}\right)\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)+x\left(\frac{67x^5}{55440}+\frac{29x^3}{504}+\frac{7x}{6}-\frac{1}{x}\right)\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right)$$

2.4.8 problem 8

Internal problem ID [8573]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 8
Date solved : `Thursday, December 12, 2024 at 09:31:16 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$2x^2y'' - xy' + \left(-x^2+1\right)y = x^4$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' - xy' + \left(-x^2+1\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{2x}$$
$$q(x) = -\frac{x^2-1}{2x^2}$$

Table 2.164: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{x^2-1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' - xy' + \left(-x^2 + 1\right) y = x^4$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $2x^2y'' - xy' + (-x^2 + 1)\, y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &2x^2\left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) \\ &- x\left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + \left(-x^2 + 1\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) \\
&\quad + \sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) = \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right)
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) \\
&\quad + \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
2x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0
$$

When $n=0$ the above becomes

$$
2x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0
$$

Or

$$
\left(2x^r r(-1+r) - x^r r + x^r\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
\left(2r^2 - 3r + 1\right) x^r = 0
$$

Since the above is true for all x then the indicial equation becomes

$$
2r^2 - 3r + 1 = 0
$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = \frac{1}{2}$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$(2x^m m(-1+m) - x^m m + x^m)\, c_0 = x^4$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as $\left[1, \frac{1}{2}\right]$.

Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) - a_n(n+r) - a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-2}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - b_n(n+r) - b_{n-2} + b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = \frac{b_{n-2}}{n(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = \frac{1}{168}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x \left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) + c_2 \sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)
\end{aligned}$$

The particular solution is found by solving for c, m the balance equation

$$(2x^m m(-1+m) - x^m m + x^m)\, c_0 = F$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$\boxed{\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{a_0}{2r^2+5r+3} \\
a_3 &= 0 \\
a_4 &= \frac{a_0}{(2r^2+5r+3)(2r^2+13r+21)} \\
a_5 &= 0
\end{aligned}}$$

Now we determine the particular solution y_p associated with $F = x^4$ by solving the balance equation

$$(2x^m m(-1+m) - x^m m + x^m)\, c_0 = x^4$$

For c_0 and x. This results in

$$\begin{aligned}
c_0 &= \frac{1}{21} \\
m &= 4
\end{aligned}$$

The particular solution is therefore

$$\begin{aligned}y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= \sum_{n=0}^{\infty} c_n x^{n+4}\end{aligned}$$

Where in the above $c_0 = \frac{1}{21}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 4$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{21}$ and $r = m$ or $r = 4$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = \frac{1}{21}$
$c_1 = 0$
$c_2 = \frac{1}{1155}$
$c_3 = 0$
$c_4 = \frac{1}{121275}$
$c_5 = 0$

The particular solution is now found using

$$\begin{aligned}y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= x^4 \sum_{n=0}^{\infty} c_n x^n\end{aligned}$$

Using the values found above for c_n into the above sum gives

$$\begin{aligned}y_p &= x^4 \left(\frac{1}{21} + \frac{1}{1155} x^2 + \frac{1}{121275} x^4\right) \\ &= \frac{1}{21} x^4 + \frac{1}{1155} x^6 + \frac{1}{121275} x^8\end{aligned}$$

Adding all the above particular solution(s) gives

$$y_p = \frac{x^4}{21} + \frac{x^6}{1155} + \frac{x^8}{121275} + O(x^6)$$

Truncating the particular solution to the order of series requested gives

$$y_p = \frac{x^4}{21} + O(x^6)$$

Hence the final solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{x^4}{21} + O(x^6) + c_1 x \left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) + c_2 \sqrt{x} \left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right) \end{aligned}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.018 (sec)
Leaf size : 43

```
dsolve(2*x^2*diff(diff(y(x),x),x)-diff(y(x),x)*x+(-x^2+1)*y(x) = x^4,y(x),
       series,x=0)
```

$$
\begin{aligned}
y = c_1\sqrt{x}\left(1+\frac{1}{6}x^2+\frac{1}{168}x^4+\mathrm{O}\left(x^6\right)\right) \\
+ c_2x\left(1+\frac{1}{10}x^2+\frac{1}{360}x^4+\mathrm{O}\left(x^6\right)\right)+x^4\left(\frac{1}{21}+\mathrm{O}\left(x^2\right)\right)
\end{aligned}
$$

Mathematica DSolve solution

Solving time : 0.032 (sec)
Leaf size : 150

```
AsymptoticDSolveValue[{2*x^2*D[y[x],{x,2}] - x*D[y[x],x] + (1-x^2 )*y[x] ==x^4,{}},
      y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to c_2x\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right)+c_1\sqrt{x}\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)+\sqrt{x}\left(-\frac{x^{11/2}}{55}\right. \\
\left.-\frac{2x^{7/2}}{7}\right)\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)+x\left(\frac{x^5}{30}+\frac{x^3}{3}\right)\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right)
\end{aligned}
$$

2.4.9 problem 9

Internal problem ID [8574]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 9
Date solved : `Thursday, December 12, 2024 at 09:31:17 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$2x^2y'' - xy' + \left(-x^2+1\right) y = \sin\left(x\right)$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' - xy' + \left(-x^2+1\right) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{2x}$$
$$q(x) = -\frac{x^2-1}{2x^2}$$

Table 2.165: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{x^2-1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' - xy' + \left(-x^2 + 1\right) y = \sin\left(x\right)$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $2x^2y'' - xy' + \left(-x^2 + 1\right) y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} \left(n + r\right) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} \left(n + r\right) \left(n + r - 1\right) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &2x^2 \left(\sum_{n=0}^{\infty} \left(n + r\right) \left(n + r - 1\right) a_n x^{n+r-2}\right) \\ &- x \left(\sum_{n=0}^{\infty} \left(n + r\right) a_n x^{n+r-1}\right) + \left(-x^2 + 1\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n(n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) = \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n(n+r)\right) + \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$2x^{n+r} a_n(n+r)(n+r-1) - x^{n+r} a_n(n+r) + a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$2x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 3r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = \frac{1}{2}$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$(2x^m m(-1+m) - x^m m + x^m)\, c_0 = \sin(x)$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as $\left[1, \frac{1}{2}\right]$.

Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) - a_n(n+r) - a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-2}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O\left(x^6\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - b_n(n+r) - b_{n-2} + b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = \frac{b_{n-2}}{n(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = \frac{1}{168}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) + c_2\sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right) \end{aligned}$$

The particular solution is found by solving for c, m the balance equation

$$\left(2x^m m(-1+m) - x^m m + x^m\right) c_0 = F$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$\begin{aligned} a_1 &= 0 \\ a_2 &= \frac{a_0}{2r^2+5r+3} \\ a_3 &= 0 \\ a_4 &= \frac{a_0}{(2r^2+5r+3)(2r^2+13r+21)} \\ a_5 &= 0 \end{aligned}$$

Expanding the rhs of the ode $\sin(x)$ in series gives

$$\sin(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

Since the $F = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$ has more than one term then we find a particular solution for each term and add the result to find the particular solution to the ode.

Unable to solve the balance equation $\left(2x^m m(-1+m) - x^m m + x^m\right) c_0$ for c_0 and x. No particular solution exists.

Failed to convert RHS $\sin(x)$ to series in order to find particular solution. Unable to solve. Terminating Unable to find the particular solution or no solution exists.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.019 (sec)
Leaf size : maple_leaf_size

```
dsolve(2*x^2*diff(diff(y(x),x),x)-diff(y(x),x)*x+(-x^2+1)*y(x) = sin(x),y(x),
       series,x=0)
```

No solution found

Mathematica DSolve solution

Solving time : 0.048 (sec)
Leaf size : 159

```
AsymptoticDSolveValue[{2*x^2*D[y[x],{x,2}] - x*D[y[x],x] + (1-x^2 )*y[x] ==Sin[x],{}},
       y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to\ & x\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right)\left(\frac{x^6}{20790}-\frac{17x^4}{5040}+\log(x)\right) \\
& + c_1\sqrt{x}\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)+c_2x\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right) \\
& + \sqrt{x}\left(\frac{x^{9/2}}{810}+\frac{2x^{5/2}}{75}-2\sqrt{x}\right)\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)
\end{aligned}
$$

2.4.10 problem 10

Internal problem ID [8575]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 10
Date solved : `Thursday, December 12, 2024 at 09:31:18 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$2x^2y'' - xy' + \left(-x^2+1\right)y = 1+\sin\left(x\right)$$

Using series expansion around $x=0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' - xy' + \left(-x^2+1\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{2x}$$
$$q(x) = -\frac{x^2-1}{2x^2}$$

Table 2.166: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{2x}$	
singularity	type
$x=0$	"regular"

$q(x) = -\frac{x^2-1}{2x^2}$	
singularity	type
$x=0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x=0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' - xy' + \left(-x^2+1\right)y = 1+\sin\left(x\right)$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $2x^2y'' - xy' + \left(-x^2+1\right)y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} \left(n+r\right) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} \left(n+r\right)\left(n+r-1\right) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &2x^2\left(\sum_{n=0}^{\infty} \left(n+r\right)\left(n+r-1\right) a_n x^{n+r-2}\right) \\ &- x\left(\sum_{n=0}^{\infty} \left(n+r\right) a_n x^{n+r-1}\right) + \left(-x^2+1\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) = \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 3r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= 1 \\
r_2 &= \frac{1}{2}
\end{aligned}
$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$
(2x^m m(-1+m) - x^m m + x^m)\, c_0 = 1 + \sin(x)
$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$
(2r^2 - 3r + 1)\, x^r = 0
$$

Solving for r gives the roots of the indicial equation as $\left[1, \frac{1}{2}\right]$.

Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+1} \\
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}
\end{aligned}
$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
2a_n(n+r)(n+r-1) - a_n(n+r) - a_{n-2} + a_n = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-2}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - b_n(n+r) - b_{n-2} + b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = \frac{b_{n-2}}{n(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = \frac{1}{168}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x\left(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots\right) \\ &= \sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) + c_2\sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right) \end{aligned}$$

The particular solution is found by solving for c, m the balance equation

$$(2x^m m(-1+m) - x^m m + x^m)\, c_0 = F$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$\boxed{\begin{aligned} a_1 &= 0 \\ a_2 &= \frac{a_0}{2r^2+5r+3} \\ a_3 &= 0 \\ a_4 &= \frac{a_0}{(2r^2+5r+3)(2r^2+13r+21)} \\ a_5 &= 0 \end{aligned}}$$

Expanding the rhs of the ode $1 + \sin(x)$ in series gives

$$1 + \sin(x) = 1 + x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

Since the $F = 1 + x - \frac{1}{6}x^3 + \frac{1}{120}x^5$ has more than one term then we find a particular solution for each term and add the result to find the particular solution to the ode.

Now we determine the particular solution y_p associated with $F = 1$ by solving the balance equation

$$(2x^m m(-1+m) - x^m m + x^m)\, c_0 = 1$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= 1 \\
m &= 0
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= \sum_{n=0}^{\infty} c_n x^{n+0}
\end{aligned}
$$

Where in the above $c_0 = 1$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 0$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = 1$ and $r = m$ or $r = 0$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = 1$
$c_1 = 0$
$c_2 = \frac{1}{3}$
$c_3 = 0$
$c_4 = \frac{1}{63}$
$c_5 = 0$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
&= 1 \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= 1\left(1 + \frac{1}{3}x^2 + \frac{1}{63}x^4\right) \\
&= 1 + \frac{1}{3}x^2 + \frac{1}{63}x^4
\end{aligned}
$$

Unable to solve the balance equation $(2x^m m(-1+m) - x^m m + x^m)\, c_0$ for c_0 and x. No particular solution exists.

Failed to convert RHS $1 + \sin(x)$ to series in order to find particular solution. Unable to solve. Terminating Unable to find the particular solution or no solution exists.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.018 (sec)
Leaf size : maple_leaf_size

```
dsolve(2*x^2*diff(diff(y(x),x),x)-diff(y(x),x)*x+(-x^2+1)*y(x) = sin(x)+1,y(x),
       series,x=0)
```

No solution found

Mathematica DSolve solution

Solving time : 0.098 (sec)
Leaf size : 217

```
AsymptoticDSolveValue[{2*x^2*D[y[x],{x,2}] - x*D[y[x],x] + (1-x^2 )*y[x] ==1+Sin[x],{}},
       y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt{x}\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)+c_2x\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right)$$
$$+\sqrt{x}\left(-\frac{x^{11/2}}{154440}+\frac{x^{9/2}}{810}-\frac{x^{7/2}}{1260}+\frac{2x^{5/2}}{75}-\frac{x^{3/2}}{15}-2\sqrt{x}\right.$$
$$\left.+\frac{2}{\sqrt{x}}\right)\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)+x\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right)\left(\frac{x^6}{20790}+\frac{x^5}{55440}-\frac{17x^4}{5040}+\frac{x^3}{504}+\frac{x}{6}-\frac{1}{x}+\log\right.$$

2.4.11 problem 11

Internal problem ID [8576]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 11
Date solved : `Thursday, December 12, 2024 at 09:31:19 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$2x^2y'' - xy' + \left(-x^2+1\right)y = x\sin\left(x\right)$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' - xy' + \left(-x^2+1\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{2x}$$
$$q(x) = -\frac{x^2-1}{2x^2}$$

Table 2.167: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{x^2-1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' - xy' + \left(-x^2 + 1\right) y = x \sin \left(x\right)$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $2x^2y'' - xy' + \left(-x^2 + 1\right) y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} \left(n+r\right) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} \left(n+r\right) \left(n+r-1\right) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &2x^2 \left(\sum_{n=0}^{\infty} \left(n+r\right) \left(n+r-1\right) a_n x^{n+r-2}\right) \\ &- x \left(\sum_{n=0}^{\infty} \left(n+r\right) a_n x^{n+r-1}\right) + \left(-x^2 + 1\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) = \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 3r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = \frac{1}{2}$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$(2x^m m(-1+m) - x^m m + x^m)\, c_0 = x \sin(x)$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as $\left[1, \frac{1}{2}\right]$.

Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) - a_n(n+r) - a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-2}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - b_n(n+r) - b_{n-2} + b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = \frac{b_{n-2}}{n(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = \frac{1}{168}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) + c_2\sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)\end{aligned}$$

The particular solution is found by solving for c, m the balance equation

$$\left(2x^m m(-1+m) - x^m m + x^m\right) c_0 = F$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$a_1 = 0$
$a_2 = \frac{a_0}{2r^2+5r+3}$
$a_3 = 0$
$a_4 = \frac{a_0}{(2r^2+5r+3)(2r^2+13r+21)}$
$a_5 = 0$

Expanding the rhs of the ode $x\sin(x)$ in series gives

$$x\sin(x) = x^2 - \frac{1}{6}x^4$$

Since the $F = x^2 - \frac{1}{6}x^4$ has more than one term then we find a particular solution for each term and add the result to find the particular solution to the ode.

Now we determine the particular solution y_p associated with $F = x^2$ by solving the balance equation

$$\left(2x^m m(-1+m) - x^m m + x^m\right) c_0 = x^2$$

For c_0 and x. This results in

$$c_0 = \frac{1}{3}$$
$$m = 2$$

The particular solution is therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= \sum_{n=0}^{\infty} c_n x^{n+2} \end{aligned}$$

Where in the above $c_0 = \frac{1}{3}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 2$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{3}$ and $r = m$ or $r = 2$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = \frac{1}{3}$
$c_1 = 0$
$c_2 = \frac{1}{63}$
$c_3 = 0$
$c_4 = \frac{1}{3465}$
$c_5 = 0$

The particular solution is now found using

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= x^2 \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$\begin{aligned} y_p &= x^2 \left(\frac{1}{3} + \frac{1}{63} x^2 + \frac{1}{3465} x^4 \right) \\ &= \frac{1}{3} x^2 + \frac{1}{63} x^4 + \frac{1}{3465} x^6 \end{aligned}$$

Now we determine the particular solution y_p associated with $F = -\frac{x^4}{6}$ by solving the balance equation

$$(2x^m m(-1+m) - x^m m + x^m)\, c_0 = -\frac{x^4}{6}$$

For c_0 and x. This results in

$$c_0 = -\frac{1}{126}$$
$$m = 4$$

The particular solution is therefore

$$y_p = \sum_{n=0}^{\infty} c_n x^{n+m} = \sum_{n=0}^{\infty} c_n x^{n+4}$$

Where in the above $c_0 = -\frac{1}{126}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 4$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = -\frac{1}{126}$ and $r = m$ or $r = 4$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = -\frac{1}{126}$
$c_1 = 0$
$c_2 = -\frac{1}{6930}$
$c_3 = 0$
$c_4 = -\frac{1}{727650}$
$c_5 = 0$

The particular solution is now found using

$$y_p = x^m \sum_{n=0}^{\infty} c_n x^n = x^4 \sum_{n=0}^{\infty} c_n x^n$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x^4\left(-\frac{1}{126}-\frac{1}{6930}x^2-\frac{1}{727650}x^4\right) \\
&= -\frac{1}{126}x^4-\frac{1}{6930}x^6-\frac{1}{727650}x^8
\end{aligned}
$$

Adding all the above particular solution(s) gives

$$
y_p = \frac{x^2}{3}+\frac{x^4}{126}+\frac{x^6}{6930}-\frac{x^8}{727650}+O\left(x^6\right)
$$

Truncating the particular solution to the order of series requested gives

$$
y_p = \frac{x^2}{3}+\frac{x^4}{126}+O\left(x^6\right)
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \frac{x^2}{3}+\frac{x^4}{126}+O\left(x^6\right)+c_1 x\left(1+\frac{x^2}{10}+\frac{x^4}{360}+O\left(x^6\right)\right)+c_2\sqrt{x}\left(1+\frac{x^2}{6}+\frac{x^4}{168}+O\left(x^6\right)\right)
\end{aligned}
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
```

```
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.020 (sec)
Leaf size : 45

```
dsolve(2*x^2*diff(diff(y(x),x),x)-diff(y(x),x)*x+(-x^2+1)*y(x) = x*sin(x),y(x),
       series,x=0)
```

$$
\begin{aligned}
y &= c_1\sqrt{x}\left(1+\frac{1}{6}x^2+\frac{1}{168}x^4+\mathrm{O}\left(x^6\right)\right) \\
&\quad + c_2x\left(1+\frac{1}{10}x^2+\frac{1}{360}x^4+\mathrm{O}\left(x^6\right)\right)+x^2\left(\frac{1}{3}+\frac{1}{126}x^2+\mathrm{O}\left(x^4\right)\right)
\end{aligned}
$$

Mathematica DSolve solution

Solving time : 0.026 (sec)
Leaf size : 167

```
AsymptoticDSolveValue[{2*x^2*D[y[x],{x,2}] - x*D[y[x],x] + (1-x^2 )*y[x] ==x*sin(x),{}},
       y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) &\to c_2x\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right)+c_1\sqrt{x}\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right) \\
&\quad +\sqrt{x}\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)\left(-\frac{x^{11/2}\sin}{1980}-\frac{1}{35}x^{7/2}\sin\right. \\
&\quad \left.-\frac{2}{3}x^{3/2}\sin\right)+x\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right)\left(\frac{x^5\sin}{840}+\frac{x^3\sin}{18}+x\sin\right)
\end{aligned}
$$

2.4.12 problem 12

Internal problem ID [8577]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 12
Date solved : `Thursday, December 12, 2024 at 09:31:21 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$2x^2y'' - xy' + \left(-x^2+1\right)y = \sin(x) + \cos(x)$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' - xy' + \left(-x^2+1\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{2x}$$
$$q(x) = -\frac{x^2-1}{2x^2}$$

Table 2.168: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{x^2-1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' - xy' + \left(-x^2 + 1\right) y = \sin\left(x\right) + \cos\left(x\right)$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $2x^2y'' - xy' + (-x^2 + 1)\, y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &2x^2\left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) \\ &- x\left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + \left(-x^2 + 1\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-x^{n+r} a_n(n+r)\right) \\ +\sum_{n=0}^{\infty}\left(-x^{n+r+2} a_n\right)+\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-x^{n+r+2} a_n\right)=\sum_{n=2}^{\infty}\left(-a_{n-2} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-x^{n+r} a_n(n+r)\right) \\ +\sum_{n=2}^{\infty}\left(-a_{n-2} x^{n+r}\right)+\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n(n+r)(n+r-1)-x^{n+r} a_n(n+r)+a_n x^{n+r}=0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r)-x^r a_0 r+a_0 x^r=0$$

Or

$$\left(2x^r r(-1+r)-x^r r+x^r\right) a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2-3r+1\right) x^r=0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2-3r+1=0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = \frac{1}{2}$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$(2x^m m(-1+m) - x^m m + x^m)\, c_0 = \sin(x) + \cos(x)$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as $\left[1, \frac{1}{2}\right]$.

Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) - a_n(n+r) - a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-2}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - b_n(n+r) - b_{n-2} + b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = \frac{b_{n-2}}{n(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = \frac{1}{168}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \ldots\right) \\
&= \sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) + c_2\sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)
\end{aligned}
$$

The particular solution is found by solving for c, m the balance equation

$$
\left(2x^m m(-1+m) - x^m m + x^m\right) c_0 = F
$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$a_1 = 0$
$a_2 = \frac{a_0}{2r^2+5r+3}$
$a_3 = 0$
$a_4 = \frac{a_0}{(2r^2+5r+3)(2r^2+13r+21)}$
$a_5 = 0$

Expanding the rhs of the ode $\sin(x) + \cos(x)$ in series gives

$$
\sin(x) + \cos(x) = 1 + x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5
$$

Since the $F = 1 + x - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5$ has more than one term then we find a particular solution for each term and add the result to find the particular solution to the ode.

Now we determine the particular solution y_p associated with $F = 1$ by solving the balance equation

$$
\left(2x^m m(-1+m) - x^m m + x^m\right) c_0 = 1
$$

For c_0 and x. This results in

$$\begin{aligned} c_0 &= 1 \\ m &= 0 \end{aligned}$$

The particular solution is therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= \sum_{n=0}^{\infty} c_n x^{n+0} \end{aligned}$$

Where in the above $c_0 = 1$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 0$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = 1$ and $r = m$ or $r = 0$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$$\boxed{\begin{aligned} &c_0 = 1 \\ &c_1 = 0 \\ &c_2 = \tfrac{1}{3} \\ &c_3 = 0 \\ &c_4 = \tfrac{1}{63} \\ &c_5 = 0 \end{aligned}}$$

The particular solution is now found using

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= 1 \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$\begin{aligned} y_p &= 1\left(1 + \frac{1}{3}x^2 + \frac{1}{63}x^4\right) \\ &= 1 + \frac{1}{3}x^2 + \frac{1}{63}x^4 \end{aligned}$$

Unable to solve the balance equation $(2x^m m(-1+m) - x^m m + x^m) c_0$ for c_0 and x. No particular solution exists.

Failed to convert RHS $\sin(x) + \cos(x)$ to series in order to find particular solution. Unable to solve. Terminating Unable to find the particular solution or no solution exists.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.019 (sec)
Leaf size : maple_leaf_size

```
dsolve(2*x^2*diff(diff(y(x),x),x)-diff(y(x),x)*x+(-x^2+1)*y(x) = cos(x)+sin(x),y(x),
       series,x=0)
```

No solution found

Mathematica DSolve solution

Solving time : 0.054 (sec)
Leaf size : 217

```
AsymptoticDSolveValue[{2*x^2*D[y[x],{x,2}] - x*D[y[x],x] + (1-x^2 )*y[x] ==Sin[x]+Cos[x],{}},
       y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt{x}\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)+c_2x\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right) + \sqrt{x}\left(-\frac{x^{11/2}}{3861}+\frac{x^{9/2}}{810}+\frac{x^{7/2}}{630}+\frac{2x^{5/2}}{75}+\frac{4x^{3/2}}{15}-2\sqrt{x} + \frac{2}{\sqrt{x}}\right)\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)+x\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right)\left(\frac{x^6}{20790}+\frac{37x^5}{69300}-\frac{17x^4}{5040}-\frac{x^3}{84}-\frac{x}{3}-\frac{1}{x}+\log(\right.$$

2.4.13 problem 13

Internal problem ID [8578]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 13
Date solved : `Thursday, December 12, 2024 at 09:31:22 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2 y'' + (\cos(x) - 1) y' + e^x y = 0$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2 y'' + (\cos(x) - 1) y' + e^x y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{\cos(x) - 1}{x^2}$$
$$q(x) = \frac{e^x}{x^2}$$

Table 2.169: Table $p(x), q(x)$ singularites.

$p(x) = \frac{\cos(x)-1}{x^2}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{e^x}{x^2}$	
singularity	type
$x = 0$	"regular"
$x = \infty$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + (\cos(x) - 1)\, y' + \mathrm{e}^x y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (\cos(x) - 1) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \mathrm{e}^x \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Expanding $\cos(x) - 1$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}\cos(x) - 1 &= -\frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \dots \\ &= -\frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6\end{aligned}$$

Expanding e^x as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned}\mathrm{e}^x &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \dots \\ &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6\end{aligned}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\,(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-\frac{x^{n+r+5} a_n (n+r)}{720}\right) \\
&+ \left(\sum_{n=0}^{\infty} \frac{x^{n+r+3} a_n (n+r)}{24}\right) + \sum_{n=0}^{\infty} \left(-\frac{x^{1+n+r} a_n (n+r)}{2}\right) \\
&+ \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) + \left(\sum_{n=0}^{\infty} \frac{x^{n+r+2} a_n}{2}\right) \\
&+ \left(\sum_{n=0}^{\infty} \frac{x^{n+r+3} a_n}{6}\right) + \left(\sum_{n=0}^{\infty} \frac{x^{n+r+4} a_n}{24}\right) \\
&+ \left(\sum_{n=0}^{\infty} \frac{x^{n+r+5} a_n}{120}\right) + \left(\sum_{n=0}^{\infty} \frac{x^{n+r+6} a_n}{720}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} \left(-\frac{x^{n+r+5} a_n (n+r)}{720}\right) &= \sum_{n=5}^{\infty} \left(-\frac{a_{n-5} (n-5+r)\, x^{n+r}}{720}\right) \\
\sum_{n=0}^{\infty} \frac{x^{n+r+3} a_n (n+r)}{24} &= \sum_{n=3}^{\infty} \frac{a_{n-3} (n-3+r)\, x^{n+r}}{24} \\
\sum_{n=0}^{\infty} \left(-\frac{x^{1+n+r} a_n (n+r)}{2}\right) &= \sum_{n=1}^{\infty} \left(-\frac{a_{n-1} (n+r-1)\, x^{n+r}}{2}\right) \\
\sum_{n=0}^{\infty} x^{1+n+r} a_n &= \sum_{n=1}^{\infty} a_{n-1} x^{n+r} \\
\sum_{n=0}^{\infty} \frac{x^{n+r+2} a_n}{2} &= \sum_{n=2}^{\infty} \frac{a_{n-2} x^{n+r}}{2} \\
\sum_{n=0}^{\infty} \frac{x^{n+r+3} a_n}{6} &= \sum_{n=3}^{\infty} \frac{a_{n-3} x^{n+r}}{6} \\
\sum_{n=0}^{\infty} \frac{x^{n+r+4} a_n}{24} &= \sum_{n=4}^{\infty} \frac{a_{n-4} x^{n+r}}{24}
\end{aligned}
$$

$$\sum_{n=0}^{\infty} \frac{x^{n+r+5} a_n}{120} = \sum_{n=5}^{\infty} \frac{a_{n-5} x^{n+r}}{120}$$
$$\sum_{n=0}^{\infty} \frac{x^{n+r+6} a_n}{720} = \sum_{n=6}^{\infty} \frac{a_{n-6} x^{n+r}}{720}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=5}^{\infty}\left(-\frac{a_{n-5}(n-5+r)\, x^{n+r}}{720}\right) \\
&+ \left(\sum_{n=3}^{\infty} \frac{a_{n-3}(n-3+r)\, x^{n+r}}{24}\right) + \sum_{n=1}^{\infty}\left(-\frac{a_{n-1}(n+r-1)\, x^{n+r}}{2}\right) \\
&+ \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right) + \left(\sum_{n=2}^{\infty} \frac{a_{n-2} x^{n+r}}{2}\right) \\
&+ \left(\sum_{n=3}^{\infty} \frac{a_{n-3} x^{n+r}}{6}\right) + \left(\sum_{n=4}^{\infty} \frac{a_{n-4} x^{n+r}}{24}\right) \\
&+ \left(\sum_{n=5}^{\infty} \frac{a_{n-5} x^{n+r}}{120}\right) + \left(\sum_{n=6}^{\infty} \frac{a_{n-6} x^{n+r}}{720}\right) = 0
\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$r_2 = \frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as $\left[\frac{1}{2} + \frac{i\sqrt{3}}{2}, \frac{1}{2} - \frac{i\sqrt{3}}{2}\right]$.

Since the roots are complex conjugates, then two linearly independent solutions can be constructed using

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}+\frac{i\sqrt{3}}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}-\frac{i\sqrt{3}}{2}}$$

$y_1(x)$ is found first. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{r-2}{2r^2 + 2r + 2}$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = -\frac{r(r+5)}{4\left(r^2 + r + 1\right)\left(r^2 + 3r + 3\right)}$$

Substituting $n = 3$ in Eq. (2B) gives

$$a_3 = \frac{-r^5 - 8r^4 - 32r^3 - 55r^2 - 9r + 24}{24\left(r^2 + r + 1\right)\left(r^2 + 3r + 3\right)\left(r^2 + 5r + 7\right)}$$

Substituting $n = 4$ in Eq. (2B) gives

$$a_4 = \frac{-4r^6 - 38r^5 - 141r^4 - 196r^3 + 85r^2 + 408r + 192}{48\left(r^2+r+1\right)\left(r^2+3r+3\right)\left(r^2+5r+7\right)\left(r^2+7r+13\right)}$$

Substituting $n = 5$ in Eq. (2B) gives

$$a_5 = \frac{2r^9 + 20r^8 - 17r^7 - 757r^6 - 1964r^5 + 7667r^4 + 50216r^3 + 98979r^2 + 73140r + 9504}{1440\left(r^2+r+1\right)\left(r^2+3r+3\right)\left(r^2+5r+7\right)\left(r^2+7r+13\right)\left(r^2+9r+21\right)}$$

For $6 \leq n$ the recursive equation is

$$\begin{aligned} &a_n(n+r)(n+r-1) - \frac{a_{n-5}(n-5+r)}{720} + \frac{a_{n-3}(n-3+r)}{24} \\ &- \frac{a_{n-1}(n+r-1)}{2} + a_n + a_{n-1} + \frac{a_{n-2}}{2} + \frac{a_{n-3}}{6} + \frac{a_{n-4}}{24} + \frac{a_{n-5}}{120} + \frac{a_{n-6}}{720} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{na_{n-5} - 30na_{n-3} + 360na_{n-1} + ra_{n-5} - 30ra_{n-3} + 360ra_{n-1} - a_{n-6} - 11a_{n-5} - 30a_{n-4} - 30a_{n-3} -}{720n^2 + 1440nr + 720r^2 - 720n - 720r + 720} \tag{4}$$

Which for the root $r = \frac{1}{2} + \frac{i\sqrt{3}}{2}$ becomes

$$a_n = \frac{i(a_{n-5} - 30a_{n-3} + 360a_{n-1})\sqrt{3} + 2(a_{n-5} - 30a_{n-3} + 360a_{n-1})\,n - 2a_{n-6} - 21a_{n-5} - 60a_{n-4} - 90a_{n}}{1440n\left(i\sqrt{3}+n\right)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2} + \frac{i\sqrt{3}}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r-2}{2r^2+2r+2}$	$\frac{i\sqrt{3}}{4}$
a_2	$-\frac{r(r+5)}{4(r^2+r+1)(r^2+3r+3)}$	$\frac{-i\sqrt{3}-11}{32i\sqrt{3}+64}$
a_3	$\frac{-r^5-8r^4-32r^3-55r^2-9r+24}{24(r^2+r+1)(r^2+3r+3)(r^2+5r+7)}$	$\frac{\frac{55\sqrt{3}}{288}+\frac{55i}{96}}{\left(i-\sqrt{3}\right)\left(i\sqrt{3}+2\right)\left(i\sqrt{3}+3\right)}$
a_4	$\frac{-4r^6-38r^5-141r^4-196r^3+85r^2+408r+192}{48(r^2+r+1)(r^2+3r+3)(r^2+5r+7)(r^2+7r+13)}$	$\frac{112i\sqrt{3}+199}{384\left(-\sqrt{3}+2i\right)\left(-i+\sqrt{3}\right)\left(i\sqrt{3}+3\right)\left(i\sqrt{3}+4\right)}$
a_5	$\frac{2r^9+20r^8-17r^7-757r^6-1964r^5+7667r^4+50216r^3+98979r^2+73140r+9504}{1440(r^2+r+1)(r^2+3r+3)(r^2+5r+7)(r^2+7r+13)(r^2+9r+21)}$	$\frac{\frac{18491\sqrt{3}}{38400}+\frac{4387i}{12800}}{\left(-i+\sqrt{3}\right)\left(i\sqrt{3}+2\right)\left(i\sqrt{3}+3\right)\left(i\sqrt{3}+4\right)\left(i\sqrt{3}+5\right)}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}\left(1 + \frac{i\sqrt{3}\,x}{4} + \frac{\left(-i\sqrt{3}-11\right)x^2}{32i\sqrt{3}+64} + \frac{55\left(\sqrt{3}+3i\right)x^3}{288\left(i-\sqrt{3}\right)\left(i\sqrt{3}+2\right)\left(i\sqrt{3}+3\right)}\right. \\
&\quad + \frac{\left(112i\sqrt{3}+199\right)x^4}{384\left(-\sqrt{3}+2i\right)\left(-i+\sqrt{3}\right)\left(i\sqrt{3}+3\right)\left(i\sqrt{3}+4\right)} \\
&\quad \left. + \frac{41\left(451\sqrt{3}+321i\right)x^5}{38400\left(-i+\sqrt{3}\right)\left(i\sqrt{3}+2\right)\left(i\sqrt{3}+3\right)\left(i\sqrt{3}+4\right)\left(i\sqrt{3}+5\right)} + O\left(x^6\right)\right)
\end{aligned}
$$

The second solution $y_2(x)$ is found by taking the complex conjugate of $y_1(x)$ which gives

$$
\begin{aligned}
y_2(x) &= x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\left(1 - \frac{i\sqrt{3}\,x}{4} + \frac{\left(i\sqrt{3}-11\right)x^2}{-32i\sqrt{3}+64} + \frac{55\left(\sqrt{3}-3i\right)x^3}{288\left(-i-\sqrt{3}\right)\left(-i\sqrt{3}+2\right)\left(-i\sqrt{3}+3\right)}\right. \\
&\quad + \frac{\left(-112i\sqrt{3}+199\right)x^4}{384\left(-\sqrt{3}-2i\right)\left(\sqrt{3}+i\right)\left(-i\sqrt{3}+3\right)\left(-i\sqrt{3}+4\right)} \\
&\quad \left. + \frac{41\left(451\sqrt{3}-321i\right)x^5}{38400\left(\sqrt{3}+i\right)\left(-i\sqrt{3}+2\right)\left(-i\sqrt{3}+3\right)\left(-i\sqrt{3}+4\right)\left(-i\sqrt{3}+5\right)} + O\left(x^6\right)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
y_h(x) = c_1 y_1(x) + c_2 y_2(x)
$$

$$
\begin{aligned}
&= c_1 x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}} \left(1+\frac{i\sqrt{3}\,x}{4}+\frac{\left(-i\sqrt{3}-11\right)x^2}{32i\sqrt{3}+64}+\frac{55\left(\sqrt{3}+3i\right)x^3}{288\left(i-\sqrt{3}\right)\left(i\sqrt{3}+2\right)\left(i\sqrt{3}+3\right)}\right. \\
&\quad+\frac{\left(112i\sqrt{3}+199\right)x^4}{384\left(-\sqrt{3}+2i\right)\left(-i+\sqrt{3}\right)\left(i\sqrt{3}+3\right)\left(i\sqrt{3}+4\right)} \\
&\quad\left.+\frac{41\left(451\sqrt{3}+321i\right)x^5}{38400\left(-i+\sqrt{3}\right)\left(i\sqrt{3}+2\right)\left(i\sqrt{3}+3\right)\left(i\sqrt{3}+4\right)\left(i\sqrt{3}+5\right)}+O\left(x^6\right)\right) \\
&\quad+c_2 x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \left(1-\frac{i\sqrt{3}\,x}{4}+\frac{\left(i\sqrt{3}-11\right)x^2}{-32i\sqrt{3}+64}\right. \\
&\quad+\frac{55\left(\sqrt{3}-3i\right)x^3}{288\left(-i-\sqrt{3}\right)\left(-i\sqrt{3}+2\right)\left(-i\sqrt{3}+3\right)} \\
&\quad+\frac{\left(-112i\sqrt{3}+199\right)x^4}{384\left(-\sqrt{3}-2i\right)\left(\sqrt{3}+i\right)\left(-i\sqrt{3}+3\right)\left(-i\sqrt{3}+4\right)} \\
&\quad\left.+\frac{41\left(451\sqrt{3}-321i\right)x^5}{38400\left(\sqrt{3}+i\right)\left(-i\sqrt{3}+2\right)\left(-i\sqrt{3}+3\right)\left(-i\sqrt{3}+4\right)\left(-i\sqrt{3}+5\right)}+O\left(x^6\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}} \left(1+\frac{i\sqrt{3}\,x}{4}+\frac{\left(-i\sqrt{3}-11\right)x^2}{32i\sqrt{3}+64}+\frac{55\left(\sqrt{3}+3i\right)x^3}{288\left(i-\sqrt{3}\right)\left(i\sqrt{3}+2\right)\left(i\sqrt{3}+3\right)}\right. \\
&\quad+\frac{\left(112i\sqrt{3}+199\right)x^4}{384\left(-\sqrt{3}+2i\right)\left(-i+\sqrt{3}\right)\left(i\sqrt{3}+3\right)\left(i\sqrt{3}+4\right)} \\
&\quad\left.+\frac{41\left(451\sqrt{3}+321i\right)x^5}{38400\left(-i+\sqrt{3}\right)\left(i\sqrt{3}+2\right)\left(i\sqrt{3}+3\right)\left(i\sqrt{3}+4\right)\left(i\sqrt{3}+5\right)}+O\left(x^6\right)\right) \\
&\quad+c_2 x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}} \left(1-\frac{i\sqrt{3}\,x}{4}+\frac{\left(i\sqrt{3}-11\right)x^2}{-32i\sqrt{3}+64}+\frac{55\left(\sqrt{3}-3i\right)x^3}{288\left(-i-\sqrt{3}\right)\left(-i\sqrt{3}+2\right)\left(-i\sqrt{3}+3\right)}\right. \\
&\quad+\frac{\left(-112i\sqrt{3}+199\right)x^4}{384\left(-\sqrt{3}-2i\right)\left(\sqrt{3}+i\right)\left(-i\sqrt{3}+3\right)\left(-i\sqrt{3}+4\right)} \\
&\quad\left.+\frac{41\left(451\sqrt{3}-321i\right)x^5}{38400\left(\sqrt{3}+i\right)\left(-i\sqrt{3}+2\right)\left(-i\sqrt{3}+3\right)\left(-i\sqrt{3}+4\right)\left(-i\sqrt{3}+5\right)}+O\left(x^6\right)\right)
\end{aligned}
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      -> trying with_periodic_functions in the coefficients
         --- Trying Lie symmetry methods, 2nd order ---
         `, `-> Computing symmetries using: way = 5
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      -> trying with_periodic_functions in the coefficients
         --- Trying Lie symmetry methods, 2nd order ---
         `, `-> Computing symmetries using: way = 5`[0, u]
```

Maple dsolve solution

Solving time : 0.257 (sec)
Leaf size : 300

```
dsolve(x^2*diff(diff(y(x),x),x)+(cos(x)-1)*diff(y(x),x)+exp(x)*y(x) = 0,y(x),
      series,x=0)
```

$$y = \sqrt{x}\left(c_2 x^{\frac{i\sqrt{3}}{2}}\left(1+\frac{1}{4}i\sqrt{3}x+\frac{-i\sqrt{3}-11}{32i\sqrt{3}+64}x^2+\frac{\frac{55\sqrt{3}}{288}+\frac{55i}{96}}{\left(i-\sqrt{3}\right)\left(i\sqrt{3}+3\right)\left(i\sqrt{3}+2\right)}x^3+\frac{1}{384}\frac{112i\sqrt{3}+199}{\left(-\sqrt{3}+2i\right)\left(i\sqrt{3}+4\right)\left(i\sqrt{3}+3\right)\left(-i+\sqrt{3}\right)}x^4+\frac{\frac{18491\sqrt{3}}{38400}+\frac{4387i}{12800}}{\left(i\sqrt{3}+5\right)\left(i\sqrt{3}+4\right)\left(i\sqrt{3}+3\right)\left(i\sqrt{3}+2\right)\left(-i+\sqrt{3}\right)}x^5+\mathrm{O}\left(x^6\right)\right)+c_1 x^{-\frac{i\sqrt{3}}{2}}\left(1-\frac{1}{4}i\sqrt{3}x+\frac{-\sqrt{3}-11i}{64i+32\sqrt{3}}x^2+\frac{55\sqrt{3}-165i}{3456i-2304\sqrt{3}}x^3+\frac{199i+112\sqrt{3}}{-27648i+7680\sqrt{3}}x^4+\frac{\frac{18491\sqrt{3}}{38400}-\frac{4387i}{12800}}{\left(2i+\sqrt{3}\right)\left(\sqrt{3}+5i\right)\left(\sqrt{3}+4i\right)\left(\sqrt{3}+3i\right)\left(\sqrt{3}+i\right)}x^5+\mathrm{O}\left(x^6\right)\right)\right)$$

Mathematica DSolve solution

Solving time : 0.01 (sec)
Leaf size : 2502

```
AsymptoticDSolveValue[{x^2*D[y[x],{x,2}] + (Cos[x]-1)*D[y[x],x] + Exp[x]*y[x] ==0,{}},
      y[x],{x,0,5}]
```

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2.4.14 problem 14

Internal problem ID [8579]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 14
Date solved : `Thursday, December 12, 2024 at 09:31:24 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x-2)\,y''+\frac{y'}{x}+(x+1)\,y=0$$

Using series expansion around $x=0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$(x-2)\,y''+\frac{y'}{x}+(x+1)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{1}{x\,(x-2)}$$
$$q(x)=\frac{x+1}{x-2}$$

Table 2.170: Table $p(x),q(x)$ singularites.

$p(x)=\frac{1}{x(x-2)}$	
singularity	type
$x=0$	"regular"
$x=2$	"regular"

$q(x)=\frac{x+1}{x-2}$	
singularity	type
$x=2$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, 2]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x(x-2)\,y'' + y' + x(x+1)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned}x(x-2)&\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right)\\&+\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + x(x+1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0\end{aligned} \tag{1}$$

Which simplifies to

$$\begin{aligned}&\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\,(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-2x^{n+r-1} a_n (n+r)\,(n+r-1)\right)\\&+\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) = 0\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=3}^{\infty} a_{n-3} x^{n+r-1}$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\begin{gathered}\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}\right)+\sum_{n=0}^{\infty}\left(-2x^{n+r-1} a_n (n+r)(n+r-1)\right) \\ +\left(\sum_{n=0}^{\infty}(n+r)\, a_n x^{n+r-1}\right)+\left(\sum_{n=3}^{\infty} a_{n-3} x^{n+r-1}\right)+\left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}\right)=0\end{gathered} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$-2x^{n+r-1} a_n (n+r)(n+r-1)+(n+r)\, a_n x^{n+r-1}=0$$

When $n=0$ the above becomes

$$-2x^{-1+r} a_0 r(-1+r)+r a_0 x^{-1+r}=0$$

Or

$$\left(-2x^{-1+r} r(-1+r)+r\, x^{-1+r}\right) a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(-2r^2+3r\right) x^{-1+r}=0$$

Since the above is true for all x then the indicial equation becomes

$$-2r^2+3r=0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{3}{2}$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(-2r^2 + 3r\right) x^{-1+r} = 0$$

Solving for r gives the roots of the indicial equation as $\left[\frac{3}{2}, 0\right]$.

Since $r_1 - r_2 = \frac{3}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{r(-1+r)}{2r^2 + r - 1}$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = \frac{r^3 - r^2 + 2r - 1}{4r^3 + 8r^2 - r - 2}$$

For $3 \leq n$ the recursive equation is

$$a_{n-1}(n+r-1)(n+r-2) - 2a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-3} + a_{n-2} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{n^2 a_{n-1} + 2nra_{n-1} + r^2 a_{n-1} - 3na_{n-1} - 3ra_{n-1} + a_{n-3} + a_{n-2} + 2a_{n-1}}{2n^2 + 4nr + 2r^2 - 3n - 3r} \quad (4)$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_n = \frac{4n^2 a_{n-1} + 4a_{n-3} + 4a_{n-2} - a_{n-1}}{8n^2 + 12n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r(-1+r)}{2r^2+r-1}$	$\frac{3}{20}$
a_2	$\frac{r^3-r^2+2r-1}{4r^3+8r^2-r-2}$	$\frac{25}{224}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{r^5 + r^4 + 7r^3 + 5r^2 - 2r - 2}{8r^5 + 44r^4 + 70r^3 + 25r^2 - 18r - 9}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_3 = \frac{1361}{17280}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r(-1+r)}{2r^2+r-1}$	$\frac{3}{20}$
a_2	$\frac{r^3-r^2+2r-1}{4r^3+8r^2-r-2}$	$\frac{25}{224}$
a_3	$\frac{r^5+r^4+7r^3+5r^2-2r-2}{8r^5+44r^4+70r^3+25r^2-18r-9}$	$\frac{1361}{17280}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^7 + 5r^6 + 21r^5 + 52r^4 + 51r^3 + 2r^2 - 21r - 11}{(4r^3 + 8r^2 - r - 2)(1 + r)(2r + 3)(2r^2 + 13r + 20)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_4 = \frac{80753}{2365440}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r(-1+r)}{2r^2+r-1}$	$\frac{3}{20}$
a_2	$\frac{r^3-r^2+2r-1}{4r^3+8r^2-r-2}$	$\frac{25}{224}$
a_3	$\frac{r^5+r^4+7r^3+5r^2-2r-2}{8r^5+44r^4+70r^3+25r^2-18r-9}$	$\frac{1361}{17280}$
a_4	$\frac{r^7+5r^6+21r^5+52r^4+51r^3+2r^2-21r-11}{(4r^3+8r^2-r-2)(1+r)(2r+3)(2r^2+13r+20)}$	$\frac{80753}{2365440}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{r^9 + 11r^8 + 66r^7 + 262r^6 + 652r^5 + 936r^4 + 648r^3 - 11r^2 - 311r - 164}{(8r^5 + 44r^4 + 70r^3 + 25r^2 - 18r - 9)(2 + r)(2r + 5)(2r^2 + 17r + 35)}$$

Which for the root $r = \frac{3}{2}$ becomes

$$a_5 = \frac{616517}{38707200}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{r(-1+r)}{2r^2+r-1}$	$\frac{3}{20}$
a_2	$\frac{r^3-r^2+2r-1}{4r^3+8r^2-r-2}$	$\frac{25}{224}$
a_3	$\frac{r^5+r^4+7r^3+5r^2-2r-2}{8r^5+44r^4+70r^3+25r^2-18r-9}$	$\frac{1361}{17280}$
a_4	$\frac{r^7+5r^6+21r^5+52r^4+51r^3+2r^2-21r-11}{(4r^3+8r^2-r-2)(1+r)(2r+3)(2r^2+13r+20)}$	$\frac{80753}{2365440}$
a_5	$\frac{r^9+11r^8+66r^7+262r^6+652r^5+936r^4+648r^3-11r^2-311r-164}{(8r^5+44r^4+70r^3+25r^2-18r-9)(2+r)(2r+5)(2r^2+17r+35)}$	$\frac{616517}{38707200}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{3/2}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{3/2}\left(1 + \frac{3x}{20} + \frac{25x^2}{224} + \frac{1361x^3}{17280} + \frac{80753x^4}{2365440} + \frac{616517x^5}{38707200} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = \frac{r(-1 + r)}{2r^2 + r - 1}$$

Substituting $n = 2$ in Eq. (2B) gives

$$b_2 = \frac{r^3 - r^2 + 2r - 1}{4r^3 + 8r^2 - r - 2}$$

For $3 \leq n$ the recursive equation is

$$b_{n-1}(n+r-1)(n+r-2) - 2b_n(n+r)(n+r-1) + (n+r)\,b_n + b_{n-3} + b_{n-2} = 0 \quad (3)$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{n^2 b_{n-1} + 2nrb_{n-1} + r^2 b_{n-1} - 3nb_{n-1} - 3rb_{n-1} + b_{n-3} + b_{n-2} + 2b_{n-1}}{2n^2 + 4nr + 2r^2 - 3n - 3r} \quad (4)$$

Which for the root $r = 0$ becomes

$$b_n = \frac{(n^2 - 3n + 2)\,b_{n-1} + b_{n-3} + b_{n-2}}{2n^2 - 3n} \quad (5)$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r(-1+r)}{2r^2+r-1}$	0
b_2	$\frac{r^3-r^2+2r-1}{4r^3+8r^2-r-2}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{r^5 + r^4 + 7r^3 + 5r^2 - 2r - 2}{8r^5 + 44r^4 + 70r^3 + 25r^2 - 18r - 9}$$

Which for the root $r = 0$ becomes

$$b_3 = \frac{2}{9}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r(-1+r)}{2r^2+r-1}$	0
b_2	$\frac{r^3-r^2+2r-1}{4r^3+8r^2-r-2}$	$\frac{1}{2}$
b_3	$\frac{r^5+r^4+7r^3+5r^2-2r-2}{8r^5+44r^4+70r^3+25r^2-18r-9}$	$\frac{2}{9}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r^7 + 5r^6 + 21r^5 + 52r^4 + 51r^3 + 2r^2 - 21r - 11}{(4r^3 + 8r^2 - r - 2)(1 + r)(2r + 3)(2r^2 + 13r + 20)}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{11}{120}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r(-1+r)}{2r^2+r-1}$	0
b_2	$\frac{r^3-r^2+2r-1}{4r^3+8r^2-r-2}$	$\frac{1}{2}$
b_3	$\frac{r^5+r^4+7r^3+5r^2-2r-2}{8r^5+44r^4+70r^3+25r^2-18r-9}$	$\frac{2}{9}$
b_4	$\frac{r^7+5r^6+21r^5+52r^4+51r^3+2r^2-21r-11}{(4r^3+8r^2-r-2)(1+r)(2r+3)(2r^2+13r+20)}$	$\frac{11}{120}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{r^9 + 11r^8 + 66r^7 + 262r^6 + 652r^5 + 936r^4 + 648r^3 - 11r^2 - 311r - 164}{(8r^5 + 44r^4 + 70r^3 + 25r^2 - 18r - 9)(2 + r)(2r + 5)(2r^2 + 17r + 35)}$$

Which for the root $r = 0$ becomes

$$b_5 = \frac{82}{1575}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{r(-1+r)}{2r^2+r-1}$	0
b_2	$\frac{r^3-r^2+2r-1}{4r^3+8r^2-r-2}$	$\frac{1}{2}$
b_3	$\frac{r^5+r^4+7r^3+5r^2-2r-2}{8r^5+44r^4+70r^3+25r^2-18r-9}$	$\frac{2}{9}$
b_4	$\frac{r^7+5r^6+21r^5+52r^4+51r^3+2r^2-21r-11}{(4r^3+8r^2-r-2)(1+r)(2r+3)(2r^2+13r+20)}$	$\frac{11}{120}$
b_5	$\frac{r^9+11r^8+66r^7+262r^6+652r^5+936r^4+648r^3-11r^2-311r-164}{(8r^5+44r^4+70r^3+25r^2-18r-9)(2+r)(2r+5)(2r^2+17r+35)}$	$\frac{82}{1575}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots \\ &= 1 + \frac{x^2}{2} + \frac{2x^3}{9} + \frac{11x^4}{120} + \frac{82x^5}{1575} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1\, x^{3/2}\left(1 + \frac{3x}{20} + \frac{25x^2}{224} + \frac{1361x^3}{17280} + \frac{80753x^4}{2365440} + \frac{616517x^5}{38707200} + O(x^6)\right) \\ &+ c_2\left(1 + \frac{x^2}{2} + \frac{2x^3}{9} + \frac{11x^4}{120} + \frac{82x^5}{1575} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1\, x^{3/2}\left(1 + \frac{3x}{20} + \frac{25x^2}{224} + \frac{1361x^3}{17280} + \frac{80753x^4}{2365440} + \frac{616517x^5}{38707200} + O(x^6)\right) \\ &+ c_2\left(1 + \frac{x^2}{2} + \frac{2x^3}{9} + \frac{11x^4}{120} + \frac{82x^5}{1575} + O(x^6)\right) \end{aligned}$$

Maple step by step solution

Let's solve

$(x-2)\left(\frac{d^2}{dx^2}y(x)\right) + \frac{\frac{d}{dx}y(x)}{x} + (x+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x+1)y(x)}{x-2} - \frac{\frac{d}{dx}y(x)}{(x-2)x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{\frac{d}{dx}y(x)}{(x-2)x} + \frac{(x+1)y(x)}{x-2} = 0$

☐ Check to see if x_0 is a regular singular point

- ◦ Define functions

  $\left[P_2(x) = \frac{1}{x(x-2)}, P_3(x) = \frac{x+1}{x-2}\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$\left(x \cdot P_2(x)\right)\bigg|_{x=0} = -\frac{1}{2}$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$\left(x^2 \cdot P_3(x)\right)\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$x(x-2)\left(\frac{d^2}{dx^2}y(x)\right) + \frac{d}{dx}y(x) + x(x+1)\,y(x) = 0$$

- Assume series solution for $y(x)$

  $$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x^m \cdot y(x)$ to series expansion for $m = 1..2$

  $$x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$$

- Shift index using $k- > k - m$

  $$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $\frac{d}{dx}y(x)$ to series expansion

  $$\frac{d}{dx}y(x) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1}$$

- Shift index using $k- > k + 1$

  $$\frac{d}{dx}y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 1..2$

  $$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$$

- Shift index using $k- > k + 2 - m$

  $$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$$

  Rewrite ODE with series expansions

$$-a_0 r(-3+2r)\, x^{-1+r} + (-a_1(1+r)\,(-1+2r) + a_0 r(-1+r))\, x^r + (-a_2(2+r)\,(1+2r) + a_1(1$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $$-r(-3+2r) = 0$$
- Values of r that satisfy the indicial equation
  $$r \in \left\{0, \frac{3}{2}\right\}$$
- The coefficients of each power of x must be 0
  $$[-a_1(1+r)\,(-1+2r) + a_0 r(-1+r) = 0, -a_2(2+r)\,(1+2r) + a_1(1+r)\,r + a_0 = 0]$$
- Solve for the dependent coefficient(s)
  $$\left\{a_1 = \frac{a_0 r(-1+r)}{2r^2+r-1}, a_2 = \frac{a_0\left(r^3-r^2+2r-1\right)}{4r^3+8r^2-r-2}\right\}$$
- Each term in the series must be 0, giving the recursion relation
  $$-2(k+1+r)\left(k+r-\frac{1}{2}\right) a_{k+1} + a_k(k+r)\,(k+r-1) + a_{k-1} + a_{k-2} = 0$$
- Shift index using $k- >k+2$
  $$-2(k+3+r)\left(k+\frac{3}{2}+r\right) a_{k+3} + a_{k+2}(k+2+r)\,(k+1+r) + a_{k+1} + a_k = 0$$
- Recursion relation that defines series solution to ODE
  $$a_{k+3} = \frac{k^2 a_{k+2} + 2kr a_{k+2} + r^2 a_{k+2} + 3k a_{k+2} + 3r a_{k+2} + a_k + a_{k+1} + 2a_{k+2}}{(k+3+r)(2k+3+2r)}$$
- Recursion relation for $r = 0$
  $$a_{k+3} = \frac{k^2 a_{k+2} + 3k a_{k+2} + a_k + a_{k+1} + 2a_{k+2}}{(k+3)(2k+3)}$$
- Solution for $r = 0$
  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = \frac{k^2 a_{k+2} + 3k a_{k+2} + a_k + a_{k+1} + 2a_{k+2}}{(k+3)(2k+3)}, a_1 = 0, a_2 = \frac{a_0}{2}\right]$$
- Recursion relation for $r = \frac{3}{2}$
  $$a_{k+3} = \frac{k^2 a_{k+2} + 6k a_{k+2} + a_k + a_{k+1} + \frac{35}{4} a_{k+2}}{\left(k+\frac{9}{2}\right)(2k+6)}$$
- Solution for $r = \frac{3}{2}$
  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{3}{2}}, a_{k+3} = \frac{k^2 a_{k+2} + 6k a_{k+2} + a_k + a_{k+1} + \frac{35}{4} a_{k+2}}{\left(k+\frac{9}{2}\right)(2k+6)}, a_1 = \frac{3a_0}{20}, a_2 = \frac{25a_0}{224}\right]$$
- Combine solutions and rename parameters
  $$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+3} = \frac{k^2 a_{k+2} + 3k a_{k+2} + a_k + a_{k+1} + 2a_{k+2}}{(k+3)(2k+3)}, a_1 = 0, a_2 = \frac{a_0}{2}, b_{k+3} =\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
<- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <>
```

Maple dsolve solution

Solving time : `0.020` (sec)
Leaf size : 42

```
dsolve((x-2)*diff(diff(y(x),x),x)+1/x*diff(y(x),x)+y(x)*(x+1) = 0,y(x),
       series,x=0)
```

$$
\begin{aligned}
y = {} & c_1 x^{3/2}\left(1+\frac{3}{20}x+\frac{25}{224}x^2+\frac{1361}{17280}x^3+\frac{80753}{2365440}x^4+\frac{616517}{38707200}x^5+\operatorname{O}\left(x^6\right)\right) \\
& + c_2\left(1+\frac{1}{2}x^2+\frac{2}{9}x^3+\frac{11}{120}x^4+\frac{82}{1575}x^5+\operatorname{O}\left(x^6\right)\right)
\end{aligned}
$$

Mathematica DSolve solution

Solving time : 0.01 (sec)
Leaf size : 80

```
AsymptoticDSolveValue[{(x-2)*D[y[x],{x,2}] + 1/x*D[y[x],x] + (x+1)*y[x] ==0,{}},
       y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{82x^5}{1575}+\frac{11x^4}{120}+\frac{2x^3}{9}+\frac{x^2}{2}+1\right)+c_1\left(\frac{616517x^5}{38707200}+\frac{80753x^4}{2365440}+\frac{1361x^3}{17280}+\frac{25x^2}{224}+\frac{3x}{20}+1\right)x^{3/2}$$

2.4.15 problem 15

Internal problem ID [8580]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 15
Date solved : `Thursday, December 12, 2024 at 09:31:25 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x-2)\,y''+\frac{y'}{x}+(x+1)\,y=0$$

Using series expansion around $x=2$

The ode does not have its expansion point at $x=0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$\tau=x-2$$

The ode is converted to be in terms of the new independent variable τ. This results in

$$\tau\left(\frac{d^2}{d\tau^2}y(\tau)\right)+\frac{\frac{d}{d\tau}y(\tau)}{\tau+2}+(\tau+3)\,y(\tau)=0$$

With its expansion point and initial conditions now at $\tau=0$. The transformed ODE is now solved. The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\tau\left(\frac{d^2}{d\tau^2}y(\tau)\right)+\frac{\frac{d}{d\tau}y(\tau)}{\tau+2}+(\tau+3)\,y(\tau)=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$\frac{d^2}{d\tau^2}y(\tau)+p(\tau)\frac{d}{d\tau}y(\tau)+q(\tau)y(\tau)=0$$

Where

$$
\begin{aligned}
p(\tau) &= \frac{1}{(\tau+2)\,\tau} \\
q(\tau) &= \frac{\tau+3}{\tau}
\end{aligned}
$$

Table 2.172: Table $p(\tau), q(\tau)$ singularites.

$p(\tau) = \frac{1}{(\tau+2)\tau}$	
singularity	type
$\tau = -2$	"regular"
$\tau = 0$	"regular"

$q(\tau) = \frac{\tau+3}{\tau}$	
singularity	type
$\tau = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-2, 0]$

Irregular singular points : $[\infty]$

Since $\tau = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$
(\tau+2)\,\tau\left(\frac{d^{2}}{d\tau^{2}}y(\tau)\right)+\frac{d}{d\tau}y(\tau)+(\tau+2)\,(\tau+3)\,y(\tau)=0
$$

Let the solution be represented as Frobenius power series of the form

$$
y(\tau)=\sum_{n=0}^{\infty}a_{n}\tau^{n+r}
$$

Then

$$
\begin{aligned}
\frac{d}{d\tau}y(\tau) &= \sum_{n=0}^{\infty}(n+r)\,a_{n}\tau^{n+r-1} \\
\frac{d^{2}}{d\tau^{2}}y(\tau) &= \sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_{n}\tau^{n+r-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\begin{aligned}
(\tau+2)\,\tau\left(\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n\tau^{n+r-2}\right)& \\
+\left(\sum_{n=0}^{\infty}(n+r)\,a_n\tau^{n+r-1}\right)+(\tau+2)\,(\tau+3)\left(\sum_{n=0}^{\infty}a_n\tau^{n+r}\right)&=0
\end{aligned}
\tag{1}
$$

Which simplifies to

$$
\begin{aligned}
\left(\sum_{n=0}^{\infty}a_n\tau^{n+r}(n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty}2a_n\tau^{n+r-1}(n+r)\,(n+r-1)\right)& \\
+\left(\sum_{n=0}^{\infty}(n+r)\,a_n\tau^{n+r-1}\right)+\left(\sum_{n=0}^{\infty}a_n\tau^{n+r+2}\right)& \\
+\left(\sum_{n=0}^{\infty}5\tau^{1+n+r}a_n\right)+\left(\sum_{n=0}^{\infty}6a_n\tau^{n+r}\right)&=0
\end{aligned}
\tag{2A}
$$

The next step is to make all powers of τ be $n+r-1$ in each summation term. Going over each summation term above with power of τ in it which is not already τ^{n+r-1} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty}a_n\tau^{n+r}(n+r)\,(n+r-1)&=\sum_{n=1}^{\infty}a_{n-1}(n+r-1)\,(n+r-2)\,\tau^{n+r-1} \\
\sum_{n=0}^{\infty}a_n\tau^{n+r+2}&=\sum_{n=3}^{\infty}a_{n-3}\tau^{n+r-1} \\
\sum_{n=0}^{\infty}5\tau^{1+n+r}a_n&=\sum_{n=2}^{\infty}5a_{n-2}\tau^{n+r-1} \\
\sum_{n=0}^{\infty}6a_n\tau^{n+r}&=\sum_{n=1}^{\infty}6a_{n-1}\tau^{n+r-1}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers

of τ are the same and equal to $n+r-1$.

$$\begin{aligned}&\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\,\tau^{n+r-1}\right)\\&+\left(\sum_{n=0}^{\infty} 2a_n\tau^{n+r-1}(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty}(n+r)\,a_n\tau^{n+r-1}\right)\\&+\left(\sum_{n=3}^{\infty} a_{n-3}\tau^{n+r-1}\right)+\left(\sum_{n=2}^{\infty} 5a_{n-2}\tau^{n+r-1}\right)+\left(\sum_{n=1}^{\infty} 6a_{n-1}\tau^{n+r-1}\right)=0\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2a_n\tau^{n+r-1}(n+r)(n+r-1)+(n+r)\,a_n\tau^{n+r-1}=0$$

When $n=0$ the above becomes

$$2a_0\tau^{-1+r}r(-1+r)+ra_0\tau^{-1+r}=0$$

Or

$$\left(2\tau^{-1+r}r(-1+r)+r\,\tau^{-1+r}\right)a_0=0$$

Since $a_0\neq 0$ then the above simplifies to

$$r\,\tau^{-1+r}(2r-1)=0$$

Since the above is true for all τ then the indicial equation becomes

$$2r^2-r=0$$

Solving for r gives the roots of the indicial equation as

$$r_1=\frac{1}{2}$$
$$r_2=0$$

Since $a_0\neq 0$ then the indicial equation becomes

$$r\,\tau^{-1+r}(2r-1)=0$$

Solving for r gives the roots of the indicial equation as $\left[\frac{1}{2},0\right]$.

Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(\tau) = \tau^{r_1}\left(\sum_{n=0}^{\infty} a_n \tau^n\right)$$
$$y_2(\tau) = \tau^{r_2}\left(\sum_{n=0}^{\infty} b_n \tau^n\right)$$

Or

$$y_1(\tau) = \sum_{n=0}^{\infty} a_n \tau^{n+\frac{1}{2}}$$
$$y_2(\tau) = \sum_{n=0}^{\infty} b_n \tau^n$$

We start by finding $y_1(\tau)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{-r^2 + r - 6}{2r^2 + 3r + 1}$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = \frac{r^4 + r^2 - 15r + 31}{4r^4 + 20r^3 + 35r^2 + 25r + 6}$$

For $3 \leq n$ the recursive equation is

$$\begin{aligned} &a_{n-1}(n+r-1)(n+r-2) + 2a_n(n+r)(n+r-1) \\ &+ a_n(n+r) + a_{n-3} + 5a_{n-2} + 6a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{n^2 a_{n-1} + 2nr a_{n-1} + r^2 a_{n-1} - 3n a_{n-1} - 3r a_{n-1} + a_{n-3} + 5a_{n-2} + 8a_{n-1}}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = \frac{-4n^2 a_{n-1} + 8n a_{n-1} - 4a_{n-3} - 20a_{n-2} - 27a_{n-1}}{8n^2 + 4n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+r-6}{2r^2+3r+1}$	$-\frac{23}{12}$
a_2	$\frac{r^4+r^2-15r+31}{4r^4+20r^3+35r^2+25r+6}$	$\frac{127}{160}$

For $n=3$, using the above recursive equation gives

$$a_3 = \frac{-r^6-3r^5-3r^4+17r^3+26r^2+182r-74}{(2r^2+11r+15)(2r^2+3r+1)(2r^2+7r+6)}$$

Which for the root $r=\frac{1}{2}$ becomes

$$a_3 = \frac{1621}{40320}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+r-6}{2r^2+3r+1}$	$-\frac{23}{12}$
a_2	$\frac{r^4+r^2-15r+31}{4r^4+20r^3+35r^2+25r+6}$	$\frac{127}{160}$
a_3	$\frac{-r^6-3r^5-3r^4+17r^3+26r^2+182r-74}{(2r^2+11r+15)(2r^2+3r+1)(2r^2+7r+6)}$	$\frac{1621}{40320}$

For $n=4$, using the above recursive equation gives

$$a_4 = \frac{r^8+8r^7+24r^6+11r^5-53r^4-153r^3-75r^2-1458r-897}{(2r^2+15r+28)(4r^4+20r^3+35r^2+25r+6)(2r^2+11r+15)}$$

Which for the root $r=\frac{1}{2}$ becomes

$$a_4 = -\frac{426599}{5806080}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+r-6}{2r^2+3r+1}$	$-\frac{23}{12}$
a_2	$\frac{r^4+r^2-15r+31}{4r^4+20r^3+35r^2+25r+6}$	$\frac{127}{160}$
a_3	$\frac{-r^6-3r^5-3r^4+17r^3+26r^2+182r-74}{(2r^2+11r+15)(2r^2+3r+1)(2r^2+7r+6)}$	$\frac{1621}{40320}$
a_4	$\frac{r^8+8r^7+24r^6+11r^5-53r^4-153r^3-75r^2-1458r-897}{(2r^2+15r+28)(4r^4+20r^3+35r^2+25r+6)(2r^2+11r+15)}$	$-\frac{426599}{5806080}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-r^{10} - 15r^9 - 92r^8 - 270r^7 - 316r^6 + 276r^5 + 970r^4 + 207r^3 - 4303r^2 + 2370r + 13486}{(2r^2 + 19r + 45)(2r^2 + 15r + 28)(4r^4 + 20r^3 + 35r^2 + 25r + 6)(2r^2 + 11r + 15)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = \frac{4670443}{425779200}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r^2+r-6}{2r^2+3r+1}$	$-\frac{23}{12}$
a_2	$\frac{r^4+r^2-15r+31}{4r^4+20r^3+35r^2+25r+6}$	$\frac{127}{160}$
a_3	$\frac{-r^6-3r^5-3r^4+17r^3+26r^2+182r-74}{(2r^2+11r+15)(2r^2+3r+1)(2r^2+7r+6)}$	$\frac{1621}{40320}$
a_4	$\frac{r^8+8r^7+24r^6+11r^5-53r^4-153r^3-75r^2-1458r-897}{(2r^2+15r+28)(4r^4+20r^3+35r^2+25r+6)(2r^2+11r+15)}$	$-\frac{426599}{5806080}$
a_5	$\frac{-r^{10}-15r^9-92r^8-270r^7-316r^6+276r^5+970r^4+207r^3-4303r^2+2370r+13486}{(2r^2+19r+45)(2r^2+15r+28)(4r^4+20r^3+35r^2+25r+6)(2r^2+11r+15)}$	$\frac{4670443}{425779200}$

Using the above table, then the solution $y_1(\tau)$ is

$$\begin{aligned}
y_1(\tau) &= \sqrt{\tau}\left(a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3 + a_4\tau^4 + a_5\tau^5 + a_6\tau^6 \dots\right) \\
&= \sqrt{\tau}\left(1 - \frac{23\tau}{12} + \frac{127\tau^2}{160} + \frac{1621\tau^3}{40320} - \frac{426599\tau^4}{5806080} + \frac{4670443\tau^5}{425779200} + O(\tau^6)\right)
\end{aligned}$$

Now the second solution $y_2(\tau)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = \frac{-r^2 + r - 6}{2r^2 + 3r + 1}$$

Substituting $n = 2$ in Eq. (2B) gives

$$b_2 = \frac{r^4 + r^2 - 15r + 31}{4r^4 + 20r^3 + 35r^2 + 25r + 6}$$

For $3 \leq n$ the recursive equation is

$$\begin{aligned}
&b_{n-1}(n + r - 1)(n + r - 2) + 2b_n(n + r)(n + r - 1) \\
&+ (n + r) b_n + b_{n-3} + 5b_{n-2} + 6b_{n-1} = 0
\end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{n^2 b_{n-1} + 2nrb_{n-1} + r^2 b_{n-1} - 3nb_{n-1} - 3rb_{n-1} + b_{n-3} + 5b_{n-2} + 8b_{n-1}}{2n^2 + 4nr + 2r^2 - n - r} \tag{4}$$

Which for the root $r = 0$ becomes

$$b_n = \frac{-n^2 b_{n-1} + 3nb_{n-1} - b_{n-3} - 5b_{n-2} - 8b_{n-1}}{n\,(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+r-6}{2r^2+3r+1}$	-6
b_2	$\frac{r^4+r^2-15r+31}{4r^4+20r^3+35r^2+25r+6}$	$\frac{31}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{-r^6 - 3r^5 - 3r^4 + 17r^3 + 26r^2 + 182r - 74}{(2r^2 + 11r + 15)(2r^2 + 3r + 1)(2r^2 + 7r + 6)}$$

Which for the root $r = 0$ becomes

$$b_3 = -\frac{37}{45}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+r-6}{2r^2+3r+1}$	-6
b_2	$\frac{r^4+r^2-15r+31}{4r^4+20r^3+35r^2+25r+6}$	$\frac{31}{6}$
b_3	$\frac{-r^6-3r^5-3r^4+17r^3+26r^2+182r-74}{(2r^2+11r+15)(2r^2+3r+1)(2r^2+7r+6)}$	$-\frac{37}{45}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r^8 + 8r^7 + 24r^6 + 11r^5 - 53r^4 - 153r^3 - 75r^2 - 1458r - 897}{(2r^2 + 15r + 28)(4r^4 + 20r^3 + 35r^2 + 25r + 6)(2r^2 + 11r + 15)}$$

Which for the root $r = 0$ becomes

$$b_4 = -\frac{299}{840}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+r-6}{2r^2+3r+1}$	-6
b_2	$\frac{r^4+r^2-15r+31}{4r^4+20r^3+35r^2+25r+6}$	$\frac{31}{6}$
b_3	$\frac{-r^6-3r^5-3r^4+17r^3+26r^2+182r-74}{(2r^2+11r+15)(2r^2+3r+1)(2r^2+7r+6)}$	$-\frac{37}{45}$
b_4	$\frac{r^8+8r^7+24r^6+11r^5-53r^4-153r^3-75r^2-1458r-897}{(2r^2+15r+28)(4r^4+20r^3+35r^2+25r+6)(2r^2+11r+15)}$	$-\frac{299}{840}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{-r^{10} - 15r^9 - 92r^8 - 270r^7 - 316r^6 + 276r^5 + 970r^4 + 207r^3 - 4303r^2 + 2370r + 13486}{(2r^2 + 7r + 6)(2r^2 + 3r + 1)(2r^2 + 15r + 28)(2r^2 + 11r + 15)(2r^2 + 19r + 45)}$$

Which for the root $r = 0$ becomes

$$b_5 = \frac{6743}{56700}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r^2+r-6}{2r^2+3r+1}$	-6
b_2	$\frac{r^4+r^2-15r+31}{4r^4+20r^3+35r^2+25r+6}$	$\frac{31}{6}$
b_3	$\frac{-r^6-3r^5-3r^4+17r^3+26r^2+182r-74}{(2r^2+11r+15)(2r^2+3r+1)(2r^2+7r+6)}$	$-\frac{37}{45}$
b_4	$\frac{r^8+8r^7+24r^6+11r^5-53r^4-153r^3-75r^2-1458r-897}{(2r^2+15r+28)(4r^4+20r^3+35r^2+25r+6)(2r^2+11r+15)}$	$-\frac{299}{840}$
b_5	$\frac{-r^{10}-15r^9-92r^8-270r^7-316r^6+276r^5+970r^4+207r^3-4303r^2+2370r+13486}{(2r^2+7r+6)(2r^2+3r+1)(2r^2+15r+28)(2r^2+11r+15)(2r^2+19r+45)}$	$\frac{6743}{56700}$

Using the above table, then the solution $y_2(\tau)$ is

$$\begin{aligned}
y_2(\tau) &= b_0 + b_1\tau + b_2\tau^2 + b_3\tau^3 + b_4\tau^4 + b_5\tau^5 + b_6\tau^6 \dots \\
&= 1 - 6\tau + \frac{31\tau^2}{6} - \frac{37\tau^3}{45} - \frac{299\tau^4}{840} + \frac{6743\tau^5}{56700} + O(\tau^6)
\end{aligned}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(\tau) &= c_1 y_1(\tau) + c_2 y_2(\tau) \\
&= c_1\sqrt{\tau}\left(1-\frac{23\tau}{12}+\frac{127\tau^2}{160}+\frac{1621\tau^3}{40320}-\frac{426599\tau^4}{5806080}+\frac{4670443\tau^5}{425779200}+O(\tau^6)\right) \\
&+ c_2\left(1-6\tau+\frac{31\tau^2}{6}-\frac{37\tau^3}{45}-\frac{299\tau^4}{840}+\frac{6743\tau^5}{56700}+O(\tau^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y(\tau) &= y_h \\
&= c_1\sqrt{\tau}\left(1-\frac{23\tau}{12}+\frac{127\tau^2}{160}+\frac{1621\tau^3}{40320}-\frac{426599\tau^4}{5806080}+\frac{4670443\tau^5}{425779200}+O(\tau^6)\right) \\
&+ c_2\left(1-6\tau+\frac{31\tau^2}{6}-\frac{37\tau^3}{45}-\frac{299\tau^4}{840}+\frac{6743\tau^5}{56700}+O(\tau^6)\right)
\end{aligned}
$$

Replacing τ in the above with the original independent variable $x$using $\tau = x-2$ results in

$$
\begin{aligned}
y = c_1\sqrt{x-2}\,&\left(\frac{29}{6}-\frac{23x}{12}+\frac{127(x-2)^2}{160}+\frac{1621(x-2)^3}{40320}-\frac{426599(x-2)^4}{5806080}\right. \\
&\left.+\frac{4670443(x-2)^5}{425779200}+O\left((x-2)^6\right)\right)+c_2\left(13-6x+\frac{31(x-2)^2}{6}-\frac{37(x-2)^3}{45}\right. \\
&\left.-\frac{299(x-2)^4}{840}+\frac{6743(x-2)^5}{56700}+O\left((x-2)^6\right)\right)
\end{aligned}
$$

Maple step by step solution

Let's solve

$(x-2)\left(\frac{d^2}{dx^2}y(x)\right)+\frac{\frac{d}{dx}y(x)}{x}+(x+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x+1)y(x)}{x-2}-\frac{\frac{d}{dx}y(x)}{(x-2)x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{\frac{d}{dx}y(x)}{(x-2)x} + \frac{(x+1)y(x)}{x-2} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{1}{x(x-2)}, P_3(x) = \frac{x+1}{x-2}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $(x \cdot P_2(x))\Big|_{x=0} = -\frac{1}{2}$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x(x-2)\left(\frac{d^2}{dx^2}y(x)\right) + \frac{d}{dx}y(x) + x(x+1)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 1..2$

  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  ◦ Shift index using $k- >k - m$

  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  ◦ Convert $\frac{d}{dx}y(x)$ to series expansion

  $\frac{d}{dx}y(x) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1}$

  ◦ Shift index using $k- >k + 1$

  $\frac{d}{dx}y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,x^{k+r}$

  ◦ Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 1..2$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\,x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\,x^{k+r}$$

Rewrite ODE with series expansions

$$-a_0r(-3+2r)\,x^{-1+r} + (-a_1(1+r)(-1+2r) + a_0r(-1+r))\,x^r + (-a_2(2+r)(1+2r) + a_1(1$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$-r(-3+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, \frac{3}{2}\right\}$$

- The coefficients of each power of x must be 0

$$[-a_1(1+r)(-1+2r) + a_0r(-1+r) = 0, -a_2(2+r)(1+2r) + a_1(1+r)\,r + a_0 = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_1 = \frac{a_0r(-1+r)}{2r^2+r-1}, a_2 = \frac{a_0(r^3-r^2+2r-1)}{4r^3+8r^2-r-2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$-2(k+1+r)\left(k+r-\frac{1}{2}\right)a_{k+1} + a_k(k+r)(k+r-1) + a_{k-1} + a_{k-2} = 0$$

- Shift index using $k- >k+2$

$$-2(k+3+r)\left(k+\frac{3}{2}+r\right)a_{k+3} + a_{k+2}(k+2+r)(k+1+r) + a_{k+1} + a_k = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+3} = \frac{k^2a_{k+2}+2kra_{k+2}+r^2a_{k+2}+3ka_{k+2}+3ra_{k+2}+a_k+a_{k+1}+2a_{k+2}}{(k+3+r)(2k+3+2r)}$$

- Recursion relation for $r=0$

$$a_{k+3} = \frac{k^2a_{k+2}+3ka_{k+2}+a_k+a_{k+1}+2a_{k+2}}{(k+3)(2k+3)}$$

- Solution for $r=0$

$$\left[y(x) = \sum_{k=0}^{\infty} a_kx^k, a_{k+3} = \frac{k^2a_{k+2}+3ka_{k+2}+a_k+a_{k+1}+2a_{k+2}}{(k+3)(2k+3)}, a_1 = 0, a_2 = \frac{a_0}{2}\right]$$

- Recursion relation for $r=\frac{3}{2}$

$$a_{k+3} = \frac{k^2a_{k+2}+6ka_{k+2}+a_k+a_{k+1}+\frac{35}{4}a_{k+2}}{\left(k+\frac{9}{2}\right)(2k+6)}$$

- Solution for $r=\frac{3}{2}$

$$\left[y(x) = \sum_{k=0}^{\infty} a_kx^{k+\frac{3}{2}}, a_{k+3} = \frac{k^2a_{k+2}+6ka_{k+2}+a_k+a_{k+1}+\frac{35}{4}a_{k+2}}{\left(k+\frac{9}{2}\right)(2k+6)}, a_1 = \frac{3a_0}{20}, a_2 = \frac{25a_0}{224}\right]$$

- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{3}{2}}\right), a_{k+3}=\frac{k^2 a_{k+2}+3k a_{k+2}+a_k+a_{k+1}+2a_{k+2}}{(k+3)(2k+3)}, a_1=0, a_2=\frac{a_0}{2}, b_{k+3}=\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Kummer
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
<- Heun successful: received ODE is equivalent to the  HeunC  ODE, case  a <> 0, e <>
```

Maple dsolve solution

Solving time : 0.030 (sec)
Leaf size : 46

```
dsolve((x-2)*diff(diff(y(x),x),x)+1/x*diff(y(x),x)+y(x)*(x+1) = 0,y(x),
       series,x=2)
```

$$y = c_1\sqrt{x-2}\left(1-\frac{23}{12}(x-2)+\frac{127}{160}(x-2)^2+\frac{1621}{40320}(x-2)^3-\frac{426599}{5806080}(x-2)^4+\frac{4670443}{425779200}(x-2)^5+\mathrm{O}\left((x-2)^6\right)\right)+c_2\left(1-6(x-2)+\frac{31}{6}(x-2)^2-\frac{37}{45}(x-2)^3-\frac{299}{840}(x-2)^4+\frac{6743}{56700}(x-2)^5+\mathrm{O}\left((x-2)^6\right)\right)$$

Mathematica DSolve solution

Solving time : 0.011 (sec)
Leaf size : 105

```
AsymptoticDSolveValue[{(x-2)*D[y[x],{x,2}] + 1/x*D[y[x],x] + (x+1)*y[x] ==0,{}},
       y[x],{x,2,5}]
```

$$y(x) \to c_1\left(\frac{4670443(x-2)^5}{425779200}-\frac{426599(x-2)^4}{5806080}+\frac{1621(x-2)^3}{40320}+\frac{127}{160}(x-2)^2-\frac{23(x-2)}{12}+1\right)\sqrt{x-2}+c_2\left(\frac{6743(x-2)^5}{56700}-\frac{299}{840}(x-2)^4-\frac{37}{45}(x-2)^3+\frac{31}{6}(x-2)^2-6(x-2)+1\right)$$

2.4.16 problem 16

Internal problem ID [8581]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 16
Date solved : `Thursday, December 12, 2024 at 09:31:26 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$(x+1)(3x-1)y''+\cos(x)y'-3xy=0$$

Using series expansion around $x=0$

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots\\
&= y_0+xy'_0+\frac{x^2}{2}f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}f'|_{x_0,y_0,y'_0}+\cdots\\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (2.1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (2.2)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{align*}
F_0 &= f(x,y,y') \tag{4}\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \tag{6}
\end{align*}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{\cos(x)\, y' - 3xy}{(x+1)(3x-1)} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{\left(\cos(x)^2 + (6x+2)\cos(x) + 9(x+1)\left(x-\frac{1}{3}\right)\left(x+\frac{\sin(x)}{3}\right)\right) y' - 9yx^2 - 3\cos(x)\, yx - 3y}{(x+1)^2 (3x-1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(-\cos(x)^3 + (-18x-6)\cos(x)^2 + ((-9x^2-6x+3)\sin(x) + 9x^4 - 6x^3 - 68x^2 - 34x - 13)\cos(x}{} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(\cos(x)^4 + (36x+12)\cos(x)^3 + ((18x^2+12x-6)\sin(x) - 63x^4 - 57x^3 + 356x^2 + 235x + 61)\cos}{} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{\left(-\cos(x)^5 + (-60x-20)\cos(x)^4 + ((-30x^2-20x+10)\sin(x) + 225x^4 + 264x^3 - 1154x^2 - 808\right.}{}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= y'(0) \\
F_1 &= 3y'(0) - 3y(0) \\
F_2 &= 14y'(0) - 15y(0) \\
F_3 &= -159y(0) + 140y'(0) \\
F_4 &= 1711y'(0) - 1917y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1 - \frac{1}{2}x^3 - \frac{5}{8}x^4 - \frac{53}{40}x^5\right) y(0) + \left(x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{7}{12}x^4 + \frac{7}{6}x^5\right) y'(0) + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary point, then this can also be solved using the standard power series method. The ode is normalized to be

$$(3x^2 + 2x - 1) y'' + \cos(x) y' - 3xy = 0$$

Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \end{aligned}$$

Substituting the above back into the ode gives

$$(3x^2 + 2x - 1) \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \cos(x) \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 3x \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Expanding $\cos(x)$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} \cos(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \dots \\ &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 \end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\begin{aligned} &(3x^2 + 2x - 1) \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) \\ &+ \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6\right) \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - 3x \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \end{aligned}$$

Expanding the second term in (1) gives

$$
\begin{aligned}
&(3x^2+2x-1)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+\left(1\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)\right)-\frac{x^2}{2}\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)\\
&+\frac{x^4}{24}\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)-\frac{x^6}{720}\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)-3x\left(\sum_{n=0}^{\infty} a_n x^n\right)=0
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} 3x^n a_n n(n-1)\right)+\left(\sum_{n=2}^{\infty} 2n\, x^{n-1} a_n(n-1)\right)\\
&+\sum_{n=2}^{\infty}\left(-n(n-1)\, a_n x^{n-2}\right)+\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)+\sum_{n=1}^{\infty}\left(-\frac{n\, x^{1+n} a_n}{2}\right)\\
&+\left(\sum_{n=1}^{\infty}\frac{n\, x^{n+3} a_n}{24}\right)+\sum_{n=1}^{\infty}\left(-\frac{n\, x^{n+5} a_n}{720}\right)+\sum_{n=0}^{\infty}\left(-3x^{1+n} a_n\right)=0
\end{aligned}
\tag{2}
$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=2}^{\infty} 2n\, x^{n-1} a_n(n-1) &= \sum_{n=1}^{\infty} 2(1+n)\, a_{1+n} n\, x^n\\
\sum_{n=2}^{\infty}\left(-n(n-1)\, a_n x^{n-2}\right) &= \sum_{n=0}^{\infty}\left(-(n+2)\, a_{n+2}(1+n)\, x^n\right)\\
\sum_{n=1}^{\infty} n a_n x^{n-1} &= \sum_{n=0}^{\infty} (1+n)\, a_{1+n} x^n\\
\sum_{n=1}^{\infty}\left(-\frac{n\, x^{1+n} a_n}{2}\right) &= \sum_{n=2}^{\infty}\left(-\frac{(n-1)\, a_{n-1} x^n}{2}\right)\\
\sum_{n=1}^{\infty}\frac{n\, x^{n+3} a_n}{24} &= \sum_{n=4}^{\infty}\frac{(n-3)\, a_{n-3} x^n}{24}\\
\sum_{n=1}^{\infty}\left(-\frac{n\, x^{n+5} a_n}{720}\right) &= \sum_{n=6}^{\infty}\left(-\frac{(n-5)\, a_{n-5} x^n}{720}\right)
\end{aligned}
$$

$$\sum_{n=0}^{\infty} \left(-3x^{1+n} a_n\right) = \sum_{n=1}^{\infty} \left(-3a_{n-1} x^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\begin{aligned}
&\left(\sum_{n=2}^{\infty} 3x^n a_n n(n-1)\right) + \left(\sum_{n=1}^{\infty} 2(1+n)\, a_{1+n} n\, x^n\right) \\
&+ \sum_{n=0}^{\infty} \left(-(n+2)\, a_{n+2}(1+n)\, x^n\right) + \left(\sum_{n=0}^{\infty} (1+n)\, a_{1+n} x^n\right) \\
&+ \sum_{n=2}^{\infty} \left(-\frac{(n-1)\, a_{n-1} x^n}{2}\right) + \left(\sum_{n=4}^{\infty} \frac{(n-3)\, a_{n-3} x^n}{24}\right) \\
&+ \sum_{n=6}^{\infty} \left(-\frac{(n-5)\, a_{n-5} x^n}{720}\right) + \sum_{n=1}^{\infty} \left(-3a_{n-1} x^n\right) = 0
\end{aligned} \tag{3}$$

$n = 0$ gives

$$-2a_2 + a_1 = 0$$

$$a_2 = \frac{a_1}{2}$$

$n = 1$ gives

$$6a_2 - 6a_3 - 3a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = -\frac{a_0}{2} + \frac{a_1}{2}$$

$n = 2$ gives

$$6a_2 + 15a_3 - 12a_4 - \frac{7a_1}{2} = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = -\frac{5a_0}{8} + \frac{7a_1}{12}$$

$n = 3$ gives

$$18a_3 + 28a_4 - 20a_5 - 4a_2 = 0$$

Which after substituting earlier equations, simplifies to

$$-\frac{53a_0}{2} + \frac{70a_1}{3} - 20a_5 = 0$$

Or

$$a_5 = -\frac{53a_0}{40} + \frac{7a_1}{6}$$

$n = 4$ gives

$$36a_4 + 45a_5 - 30a_6 - \frac{9a_3}{2} + \frac{a_1}{24} = 0$$

Which after substituting earlier equations, simplifies to

$$a_6 = -\frac{213a_0}{80} + \frac{1711a_1}{720}$$

$n = 5$ gives

$$60a_5 + 66a_6 - 42a_7 - 5a_4 + \frac{a_2}{12} = 0$$

Which after substituting earlier equations, simplifies to

$$-\frac{2521a_0}{10} + \frac{6719a_1}{30} - 42a_7 = 0$$

Or

$$a_7 = -\frac{2521a_0}{420} + \frac{6719a_1}{1260}$$

For $6 \leq n$, the recurrence equation is

$$\begin{aligned} &3na_n(n-1) + 2(1+n)\,a_{1+n}n - (n+2)\,a_{n+2}(1+n) + (1+n)\,a_{1+n} \\ &- \frac{(n-1)\,a_{n-1}}{2} + \frac{(n-3)\,a_{n-3}}{24} - \frac{(n-5)\,a_{n-5}}{720} - 3a_{n-1} = 0 \end{aligned} \tag{4}$$

Solving for a_{n+2}, gives

$$
\begin{aligned}
a_{n+2} &= \frac{2160n^2a_n + 1440n^2a_{1+n} - 2160na_n + 2160na_{1+n} - na_{n-5} + 30na_{n-3} - 360na_{n-1} + 720a_{1+n} + 5a_{n-5} -}{720\,(n+2)\,(1+n)} \qquad (5)\\
&= \frac{(2160n^2 - 2160n)\,a_n}{720\,(n+2)\,(1+n)} + \frac{(1440n^2 + 2160n + 720)\,a_{1+n}}{720\,(n+2)\,(1+n)}\\
&\quad + \frac{(-n+5)\,a_{n-5}}{720\,(n+2)\,(1+n)} + \frac{(30n-90)\,a_{n-3}}{720\,(n+2)\,(1+n)} + \frac{(-360n-1800)\,a_{n-1}}{720\,(n+2)\,(1+n)}
\end{aligned}
$$

And so on. Therefore the solution is

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n\\
&= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
y = a_0 + a_1x + \frac{a_1x^2}{2} + \left(-\frac{a_0}{2} + \frac{a_1}{2}\right)x^3 + \left(-\frac{5a_0}{8} + \frac{7a_1}{12}\right)x^4 + \left(-\frac{53a_0}{40} + \frac{7a_1}{6}\right)x^5 + \dots
$$

Collecting terms, the solution becomes

$$
y = \left(1 - \frac{1}{2}x^3 - \frac{5}{8}x^4 - \frac{53}{40}x^5\right)a_0 + \left(x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{7}{12}x^4 + \frac{7}{6}x^5\right)a_1 + O(x^6) \qquad (3)
$$

At $x = 0$ the solution above becomes

$$
y = \left(1 - \frac{1}{2}x^3 - \frac{5}{8}x^4 - \frac{53}{40}x^5\right)c_1 + \left(x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{7}{12}x^4 + \frac{7}{6}x^5\right)c_2 + O(x^6)
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
```

```
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      -> trying with_periodic_functions in the coefficients
         --- Trying Lie symmetry methods, 2nd order ---
         `, `-> Computing symmetries using: way = 5
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
<- unable to find a useful change of variables
   trying a symmetry of the form [xi=0, eta=F(x)]
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying 2nd order, integrating factor of the form mu(x,y)
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moe
   -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx))
   -> Trying changes of variables to rationalize or make the ODE simpler
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), c
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
```

```
      trying to convert to a linear ODE with constant coefficients
      -> trying with_periodic_functions in the coefficients
    trying a symmetry of the form [xi=0, eta=F(x)]
    checking if the LODE is missing y
    -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @
    -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), d
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
      -> trying with_periodic_functions in the coefficients
    trying a symmetry of the form [xi=0, eta=F(x)]
    checking if the LODE is missing y
    -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @
    -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), d
      trying a symmetry of the form [xi=0, eta=F(x)]
      trying 2nd order exact linear
      trying symmetries linear in x and y(x)
      trying to convert to a linear ODE with constant coefficients
  <- unable to find a useful change of variables
    trying a symmetry of the form [xi=0, eta=F(x)]
  trying to convert to an ODE of Bessel type
  -> trying reduction of order to Riccati
    trying Riccati sub-methods:
      trying Riccati_symmetries
      -> trying a symmetry pattern of the form [F(x)*G(y), 0]
      -> trying a symmetry pattern of the form [0, F(x)*G(y)]
      -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
  -> trying with_periodic_functions in the coefficients
    --- Trying Lie symmetry methods, 2nd order ---
    `, `-> Computing symmetries using: way = 5
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3`[0, y]
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 54

```
dsolve((x+1)*(3*x-1)*diff(diff(y(x),x),x)+diff(y(x),x)*cos(x)-3*x*y(x) = 0,y(x),
       series,x=0)
```

$$y = \left(1 - \frac{1}{2}x^3 - \frac{5}{8}x^4 - \frac{53}{40}x^5\right) y(0) + \left(x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{7}{12}x^4 + \frac{7}{6}x^5\right) y'(0) + O(x^6)$$

Mathematica DSolve solution

Solving time : 0.004 (sec)
Leaf size : 63

```
AsymptoticDSolveValue[{(x+1)*(3*x-1)*D[y[x],{x,2}]+Cos[x]*D[y[x],x]-3*x*y[x]==0,{}},
       y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(-\frac{53x^5}{40} - \frac{5x^4}{8} - \frac{x^3}{2} + 1\right) + c_2 \left(\frac{7x^5}{6} + \frac{7x^4}{12} + \frac{x^3}{2} + \frac{x^2}{2} + x\right)$$

2.4.17 problem 17

Internal problem ID [8582]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 17
Date solved : `Thursday, December 12, 2024 at 09:31:29 AM`
CAS classification : `[_Lienard]`

Solve

$$xy'' + 2y' + xy = 0$$

With initial conditions

$$y(0) = 1$$
$$y'(0) = 0$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + 2y' + xy = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{2}{x}$$
$$q(x) = 1$$

Table 2.174: Table $p(x), q(x)$ singularites.

$p(x) = \frac{2}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = 1$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + 2y' + xy = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + 2\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + x\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2(n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n\, =0}^{\infty} x^{1+n+r} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n(n+r)(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right)+\left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n(n+r)(n+r-1)+2(n+r) a_n x^{n+r-1}=0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r)+2 r a_0 x^{-1+r}=0$$

Or

$$\left(x^{-1+r} r(-1+r)+2 r\, x^{-1+r}\right) a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(1+r)=0$$

Since the above is true for all x then the indicial equation becomes

$$r(1+r)=0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 0 \\ r_2 &= -1 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r}(1+r)=0$$

Solving for r gives the roots of the indicial equation as $[0,-1]$.

Since $r_1-r_2=1$ is an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= C y_1(x) \ln(x)+x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\ y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x} \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\ y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right) \end{aligned}$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 2a_n(n+r) + a_{n-2} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2 + n + r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{a_{n-2}}{n(1+n)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{r^2 + 5r + 6}$$

Which for the root $r = 0$ becomes

$$a_2 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{6}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{r^4 + 14r^3 + 71r^2 + 154r + 120}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{120}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{120}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \ldots \\ &= 1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= 0 \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} 0 &= \lim_{r \to -1} 0 \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-1} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + 2(n+r)b_n + b_{n-2} = 0 \tag{4}$$

Which for for the root $r = -1$ becomes

$$b_n(n-1)(n-2) + 2(n-1)b_n + b_{n-2} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{b_{n-2}}{n^2 + 2nr + r^2 + n + r} \tag{5}$$

Which for the root $r = -1$ becomes

$$b_n = -\frac{b_{n-2}}{n^2 - n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{1}{r^2 + 5r + 6}$$

Which for the root $r = -1$ becomes

$$b_2 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(r^2 + 5r + 6)(r^2 + 9r + 20)}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+5r+6}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{r^4+14r^3+71r^2+154r+120}$	$\frac{1}{24}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= 1\left(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots\right) \\ &= \frac{1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)}{x} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)}{x} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)}{x} \end{aligned}$$

$$y = c_1 - \frac{c_1\,x^2}{6} + \frac{c_1\,x^4}{120} + c_1O(x^6) + \frac{c_2}{x} - \frac{c_2x}{2} + \frac{c_2\,x^3}{24} + \frac{c_2O(x^6)}{x}$$

Maple step by step solution

Let's solve

$\left[\left(\frac{d^2}{dx^2}y(x)\right)x + 2\frac{d}{dx}y(x) + xy(x) = 0, y(0) = 1, \left(\frac{d}{dx}y(x)\right)\Big|_{\{x=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -y(x) - \frac{2\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{2\left(\frac{d}{dx}y(x)\right)}{x} + y(x) = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

$\left[P_2(x) = \frac{2}{x}, P_3(x) = 1\right]$

- $x \cdot P_2(x)$ is analytic at $x = 0$

  $$(x \cdot P_2(x))\bigg|_{x=0} = 2$$

- $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $$(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$$

- $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $$\left(\frac{d^2}{dx^2}y(x)\right)x + 2\frac{d}{dx}y(x) + xy(x) = 0$$

- Assume series solution for $y(x)$

  $$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$$

☐ Rewrite ODE with series expansions

- Convert $x \cdot y(x)$ to series expansion

  $$x \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+1}$$

- Shift index using $k- >k - 1$

  $$x \cdot y(x) = \sum_{k=1}^{\infty} a_{k-1} x^{k+r}$$

- Convert $\frac{d}{dx}y(x)$ to series expansion

  $$\frac{d}{dx}y(x) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$$

- Shift index using $k- >k + 1$

  $$\frac{d}{dx}y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\, x^{k+r}$$

- Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $$x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- >k + 1$

  $$x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+r+1)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

$$a_0 r(1+r)\, x^{-1+r} + a_1(1+r)\,(2+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(1+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-1, 0\}$
- Each term must be 0
  $a_1(1+r)\,(2+r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+r+1)\,(k+2+r) + a_{k-1} = 0$
- Shift index using $k- >k+1$
  $a_{k+2}(k+2+r)\,(k+3+r) + a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{a_k}{(k+2+r)(k+3+r)}$
- Recursion relation for $r = -1$
  $a_{k+2} = -\frac{a_k}{(k+1)(k+2)}$
- Solution for $r = -1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0\right]$
- Recursion relation for $r = 0$
  $a_{k+2} = -\frac{a_k}{(k+2)(k+3)}$
- Solution for $r = 0$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+2} = -\frac{a_k}{(k+2)(k+3)}, 2a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-1}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+2} = -\frac{a_k}{(k+1)(k+2)}, 0 = 0, b_{k+2} = -\frac{b_k}{(k+2)(k+3)}, 2b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
```

```
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.011` (sec)
Leaf size : 14

```
dsolve([x*diff(diff(y(x),x),x)+2*diff(y(x),x)+x*y(x) = 0,op([y(0) = 1, D(y)(0) = 0])],
       y(x),series,x=0)
```

$$y = 1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 + \mathrm{O}\left(x^6\right)$$

Mathematica DSolve solution

Solving time : `0.2` (sec)
Leaf size : 19

```
AsymptoticDSolveValue[{x*D[y[x],{x,2}]+2*D[y[x],x]+x*y[x]==0,{y[0]==1,Derivative[1][y][0] ==
       y[x],{x,0,5}]
```

$$y(x) \to \frac{x^4}{120} - \frac{x^2}{6} + 1$$

2.4.18 problem 18

Internal problem ID [8583]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 18
Date solved : `Thursday, December 12, 2024 at 09:31:30 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2y'' + 3xy' - xy = x^2 + 2x$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' + 3xy' - xy = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{3}{2x}$$
$$q(x) = -\frac{1}{2x}$$

Table 2.176: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' + 3xy' - xy = x^2 + 2x$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $2x^2y'' + 3xy' - xy = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} 2x^2 &\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\ &+ 3x\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - x\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n(n+r)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right) = 0 \quad \text{(2B)}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n(n+r)(n+r-1) + 3x^{n+r} a_n(n+r) = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) + 3x^r a_0 r = 0$$

Or

$$(2x^r r(-1+r) + 3x^r r) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 + r\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 + r = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -\frac{1}{2}$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$(2x^m m(-1+m) + 3x^m m) c_0 = x^2 + 2x$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 + r\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as $\left[0, -\frac{1}{2}\right]$.

Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n + r)(n + r - 1) + 3a_n(n + r) - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{2n^2 + 4nr + 2r^2 + n + r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-1}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = 0$ becomes

$$a_1 = \frac{1}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r^2+5r+3}$	$\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{4r^4 + 28r^3 + 71r^2 + 77r + 30}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{30}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r^2+5r+3}$	$\frac{1}{3}$
a_2	$\frac{1}{4r^4+28r^3+71r^2+77r+30}$	$\frac{1}{30}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{8r^6 + 108r^5 + 590r^4 + 1665r^3 + 2552r^2 + 2007r + 630}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{1}{630}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r^2+5r+3}$	$\frac{1}{3}$
a_2	$\frac{1}{4r^4+28r^3+71r^2+77r+30}$	$\frac{1}{30}$
a_3	$\frac{1}{8r^6+108r^5+590r^4+1665r^3+2552r^2+2007r+630}$	$\frac{1}{630}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16r^8 + 352r^7 + 3304r^6 + 17248r^5 + 54649r^4 + 107338r^3 + 127251r^2 + 82962r + 22680}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{22680}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r^2+5r+3}$	$\frac{1}{3}$
a_2	$\frac{1}{4r^4+28r^3+71r^2+77r+30}$	$\frac{1}{30}$
a_3	$\frac{1}{8r^6+108r^5+590r^4+1665r^3+2552r^2+2007r+630}$	$\frac{1}{630}$
a_4	$\frac{1}{16r^8+352r^7+3304r^6+17248r^5+54649r^4+107338r^3+127251r^2+82962r+22680}$	$\frac{1}{22680}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{32r^{10} + 1040r^9 + 14880r^8 + 123240r^7 + 653226r^6 + 2310945r^5 + 5514295r^4 + 8741785r^3 + 8786367}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{1}{1247400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2r^2+5r+3}$	$\frac{1}{3}$
a_2	$\frac{1}{4r^4+28r^3+71r^2+77r+30}$	$\frac{1}{30}$
a_3	$\frac{1}{8r^6+108r^5+590r^4+1665r^3+2552r^2+2007r+630}$	$\frac{1}{630}$
a_4	$\frac{1}{16r^8+352r^7+3304r^6+17248r^5+54649r^4+107338r^3+127251r^2+82962r+22680}$	$\frac{1}{22680}$
a_5	$\frac{1}{32r^{10}+1040r^9+14880r^8+123240r^7+653226r^6+2310945r^5+5514295r^4+8741785r^3+8786367r^2+5039190r+1247400}$	$\frac{1}{1247400}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\ &= 1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \frac{x^4}{22680} + \frac{x^5}{1247400} + O\left(x^6\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) + 3b_n(n+r) - b_{n-1} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-1}}{2n^2 + 4nr + 2r^2 + n + r} \tag{4}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_n = \frac{b_{n-1}}{n(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_1 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2r^2+5r+3}$	1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{4r^4 + 28r^3 + 71r^2 + 77r + 30}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2r^2+5r+3}$	1
b_2	$\frac{1}{4r^4+28r^3+71r^2+77r+30}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{1}{8r^6 + 108r^5 + 590r^4 + 1665r^3 + 2552r^2 + 2007r + 630}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_3 = \frac{1}{90}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2r^2+5r+3}$	1
b_2	$\frac{1}{4r^4+28r^3+71r^2+77r+30}$	$\frac{1}{6}$
b_3	$\frac{1}{8r^6+108r^5+590r^4+1665r^3+2552r^2+2007r+630}$	$\frac{1}{90}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{16r^8 + 352r^7 + 3304r^6 + 17248r^5 + 54649r^4 + 107338r^3 + 127251r^2 + 82962r + 22680}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = \frac{1}{2520}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2r^2+5r+3}$	1
b_2	$\frac{1}{4r^4+28r^3+71r^2+77r+30}$	$\frac{1}{6}$
b_3	$\frac{1}{8r^6+108r^5+590r^4+1665r^3+2552r^2+2007r+630}$	$\frac{1}{90}$
b_4	$\frac{1}{16r^8+352r^7+3304r^6+17248r^5+54649r^4+107338r^3+127251r^2+82962r+22680}$	$\frac{1}{2520}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{1}{32r^{10} + 1040r^9 + 14880r^8 + 123240r^7 + 653226r^6 + 2310945r^5 + 5514295r^4 + 8741785r^3 + 8786367}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_5 = \frac{1}{113400}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2r^2+5r+3}$	1
b_2	$\frac{1}{4r^4+28r^3+71r^2+77r+30}$	$\frac{1}{6}$
b_3	$\frac{1}{8r^6+108r^5+590r^4+1665r^3+2552r^2+2007r+630}$	$\frac{1}{90}$
b_4	$\frac{1}{16r^8+352r^7+3304r^6+17248r^5+54649r^4+107338r^3+127251r^2+82962r+22680}$	$\frac{1}{2520}$
b_5	$\frac{1}{32r^{10}+1040r^9+14880r^8+123240r^7+653226r^6+2310945r^5+5514295r^4+8741785r^3+8786367r^2+5039190r+1247400}$	$\frac{1}{113400}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 + x + \frac{x^2}{6} + \frac{x^3}{90} + \frac{x^4}{2520} + \frac{x^5}{113400} + O(x^6)}{\sqrt{x}} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1\left(1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \frac{x^4}{22680} + \frac{x^5}{1247400} + O(x^6)\right) \\ &\quad + \frac{c_2\left(1 + x + \frac{x^2}{6} + \frac{x^3}{90} + \frac{x^4}{2520} + \frac{x^5}{113400} + O(x^6)\right)}{\sqrt{x}} \end{aligned}$$

The particular solution is found by solving for c, m the balance equation

$$(2x^m m(-1+m) + 3x^m m)\, c_0 = F$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$
\begin{aligned}
a_1 &= \frac{a_0}{2r^2+5r+3} \\
a_2 &= \frac{a_0}{4r^4+28r^3+71r^2+77r+30} \\
a_3 &= \frac{a_0}{8r^6+108r^5+590r^4+1665r^3+2552r^2+2007r+630} \\
a_4 &= \frac{a_0}{16r^8+352r^7+3304r^6+17248r^5+54649r^4+107338r^3+127251r^2+82962r+22680} \\
a_5 &= \frac{a_0}{32r^{10}+1040r^9+14880r^8+123240r^7+653226r^6+2310945r^5+5514295r^4+8741785r^3+8786367r^2+5039190r+1247400}
\end{aligned}
$$

Since the $F = x^2 + 2x$ has more than one term then we find a particular solution for each term and add the result to find the particular solution to the ode.

Now we determine the particular solution y_p associated with $F = x^2$ by solving the balance equation

$$(2x^m m(-1+m) + 3x^m m)\, c_0 = x^2$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= \frac{1}{10} \\
m &= 2
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= \sum_{n=0}^{\infty} c_n x^{n+2}
\end{aligned}
$$

Where in the above $c_0 = \frac{1}{10}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 2$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{10}$ and $r = m$ or $r = 2$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = \frac{1}{10}$
$c_1 = \frac{1}{210}$
$c_2 = \frac{1}{7560}$
$c_3 = \frac{1}{415800}$
$c_4 = \frac{1}{32432400}$
$c_5 = \frac{1}{3405402000}$

The particular solution is now found using

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= x^2 \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$\begin{aligned} y_p &= x^2 \left(\frac{1}{10} + \frac{1}{210}x + \frac{1}{7560}x^2 + \frac{1}{415800}x^3 + \frac{1}{32432400}x^4 + \frac{1}{3405402000}x^5 \right) \\ &= \frac{1}{10}x^2 + \frac{1}{210}x^3 + \frac{1}{7560}x^4 + \frac{1}{415800}x^5 + \frac{1}{32432400}x^6 + \frac{1}{3405402000}x^7 \end{aligned}$$

Now we determine the particular solution y_p associated with $F = 2x$ by solving the balance equation

$$(2x^m m(-1+m) + 3x^m m)\, c_0 = 2x$$

For c_0 and x. This results in

$$\begin{aligned} c_0 &= \frac{2}{3} \\ m &= 1 \end{aligned}$$

The particular solution is therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= \sum_{n=0}^{\infty} c_n x^{n+1} \end{aligned}$$

Where in the above $c_0 = \frac{2}{3}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 1$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{2}{3}$ and $r = m$ or $r = 1$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = \frac{2}{3}$
$c_1 = \frac{1}{15}$
$c_2 = \frac{1}{315}$
$c_3 = \frac{1}{11340}$
$c_4 = \frac{1}{623700}$
$c_5 = \frac{1}{48648600}$

The particular solution is now found using

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= x \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$\begin{aligned} y_p &= x\left(\frac{2}{3} + \frac{1}{15}x + \frac{1}{315}x^2 + \frac{1}{11340}x^3 + \frac{1}{623700}x^4 + \frac{1}{48648600}x^5\right) \\ &= \frac{2}{3}x + \frac{1}{15}x^2 + \frac{1}{315}x^3 + \frac{1}{11340}x^4 + \frac{1}{623700}x^5 + \frac{1}{48648600}x^6 \end{aligned}$$

Adding all the above particular solution(s) gives

$$y_p = \frac{2x}{3} + \frac{x^2}{6} + \frac{x^3}{126} + \frac{x^4}{4536} + \frac{x^5}{249480} + \frac{x^6}{19459440} + \frac{x^7}{3405402000} + O(x^6)$$

Truncating the particular solution to the order of series requested gives

$$y_p = \frac{2x}{3} + \frac{x^2}{6} + \frac{x^3}{126} + \frac{x^4}{4536} + \frac{x^5}{249480} + O(x^6)$$

Hence the final solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{2x}{3} + \frac{x^2}{6} + \frac{x^3}{126} + \frac{x^4}{4536} + \frac{x^5}{249480} + O(x^6) \\ &\quad + c_1\left(1 + \frac{x}{3} + \frac{x^2}{30} + \frac{x^3}{630} + \frac{x^4}{22680} + \frac{x^5}{1247400} + O(x^6)\right) \\ &\quad + \frac{c_2\left(1 + x + \frac{x^2}{6} + \frac{x^3}{90} + \frac{x^4}{2520} + \frac{x^5}{113400} + O(x^6)\right)}{\sqrt{x}} \end{aligned}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Group is reducible or imprimitive
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.023 (sec)
Leaf size : 60

```
dsolve(2*x^2*diff(diff(y(x),x),x)+3*diff(y(x),x)*x-x*y(x) = x^2+2*x,y(x),
       series,x=0)
```

$$
\begin{aligned}
y = &\frac{c_1\left(1+x+\frac{1}{6}x^2+\frac{1}{90}x^3+\frac{1}{2520}x^4+\frac{1}{113400}x^5+\mathrm{O}\left(x^6\right)\right)}{\sqrt{x}} \\
&+ c_2\left(1+\frac{1}{3}x+\frac{1}{30}x^2+\frac{1}{630}x^3+\frac{1}{22680}x^4+\frac{1}{1247400}x^5+\mathrm{O}\left(x^6\right)\right) \\
&+ x\left(\frac{2}{3}+\frac{1}{6}x+\frac{1}{126}x^2+\frac{1}{4536}x^3+\frac{1}{249480}x^4+\mathrm{O}\left(x^5\right)\right)
\end{aligned}
$$

Mathematica DSolve solution

Solving time : 0.054 (sec)
Leaf size : 239

```
AsymptoticDSolveValue[{2*x^2*D[y[x],{x,2}]+3*x*D[y[x],x]-x*y[x]==x^2+2*x,{}},
       y[x],{x,0,5}]
```

$$
\begin{aligned}
&y(x) \\
&\to c_1\left(\frac{x^5}{1247400}+\frac{x^4}{22680}+\frac{x^3}{630}+\frac{x^2}{30}+\frac{x}{3}+1\right)+\frac{c_2\left(\frac{x^5}{113400}+\frac{x^4}{2520}+\frac{x^3}{90}+\frac{x^2}{6}+x+1\right)}{\sqrt{x}} \\
&+\frac{\left(\frac{x^5}{113400}+\frac{x^4}{2520}+\frac{x^3}{90}+\frac{x^2}{6}+x+1\right)\left(-\frac{19x^{11/2}}{62370}-\frac{23x^{9/2}}{2835}-\frac{4x^{7/2}}{35}-\frac{2x^{5/2}}{3}-\frac{4x^{3/2}}{3}\right)}{\sqrt{x}} \\
&+\left(\frac{x^5}{1247400}+\frac{x^4}{22680}+\frac{x^3}{630}+\frac{x^2}{30}+\frac{x}{3}+1\right)\left(\frac{47x^6}{680400}+\frac{x^5}{420}+\frac{17x^4}{360}+\frac{4x^3}{9}+\frac{3x^2}{2}+2x\right)
\end{aligned}
$$

2.4.19 problem 19

Internal problem ID [8584]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 19
Date solved : `Thursday, December 12, 2024 at 09:31:31 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$2x^2y'' - xy' + \left(-x^2+1\right)y = 1$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' - xy' + \left(-x^2+1\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{2x}$$
$$q(x) = -\frac{x^2-1}{2x^2}$$

Table 2.177: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{x^2-1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' - xy' + \left(-x^2 + 1\right) y = 1$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $2x^2y'' - xy' + \left(-x^2 + 1\right) y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} 2x^2 \left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2} \right) \\ - x \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1} \right) + \left(-x^2 + 1\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) = \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 3r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = \frac{1}{2}$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$(2x^m m(-1+m) - x^m m + x^m) c_0 = 1$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$(2r^2 - 3r + 1) x^r = 0$$

Solving for r gives the roots of the indicial equation as $\left[1, \frac{1}{2}\right]$.

Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) - a_n(n+r) - a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-2}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \ldots\right) \\ &= x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - b_n(n+r) - b_{n-2} + b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = \frac{b_{n-2}}{n(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = \frac{1}{168}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O\left(x^6\right)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O\left(x^6\right)\right) + c_2\sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O\left(x^6\right)\right) \end{aligned}$$

The particular solution is found by solving for c, m the balance equation

$$\left(2x^m m(-1+m) - x^m m + x^m\right) c_0 = F$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$\boxed{\begin{aligned} a_1 &= 0 \\ a_2 &= \frac{a_0}{2r^2+5r+3} \\ a_3 &= 0 \\ a_4 &= \frac{a_0}{(2r^2+5r+3)(2r^2+13r+21)} \\ a_5 &= 0 \end{aligned}}$$

Now we determine the particular solution y_p associated with $F = 1$ by solving the balance equation

$$\left(2x^m m(-1+m) - x^m m + x^m\right) c_0 = 1$$

For c_0 and x. This results in

$$\begin{aligned} c_0 &= 1 \\ m &= 0 \end{aligned}$$

The particular solution is therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= \sum_{n=0}^{\infty} c_n x^{n+0} \end{aligned}$$

Where in the above $c_0 = 1$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 0$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = 1$ and $r = m$ or $r = 0$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = 1$
$c_1 = 0$
$c_2 = \frac{1}{3}$
$c_3 = 0$
$c_4 = \frac{1}{63}$
$c_5 = 0$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
&= 1 \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= 1\left(1 + \frac{1}{3}x^2 + \frac{1}{63}x^4\right) \\
&= 1 + \frac{1}{3}x^2 + \frac{1}{63}x^4
\end{aligned}
$$

Adding all the above particular solution(s) gives

$$
y_p = 1 + \frac{x^2}{3} + \frac{x^4}{63} + O(x^6)
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= 1 + \frac{x^2}{3} + \frac{x^4}{63} + O(x^6) + c_1 x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) + c_2\sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)
\end{aligned}
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.019 (sec)
Leaf size : 43

```
dsolve(2*x^2*diff(diff(y(x),x),x)-diff(y(x),x)*x+(-x^2+1)*y(x) = 1,y(x),
       series,x=0)
```

$$
\begin{aligned}
y = c_1\sqrt{x}\left(1+\frac{1}{6}x^2+\frac{1}{168}x^4+\mathrm{O}\left(x^6\right)\right) \\
+ c_2 x\left(1+\frac{1}{10}x^2+\frac{1}{360}x^4+\mathrm{O}\left(x^6\right)\right)+\left(1+\frac{1}{3}x^2+\frac{1}{63}x^4+\mathrm{O}\left(x^6\right)\right)
\end{aligned}
$$

Mathematica DSolve solution

Solving time : `0.03` (sec)
Leaf size : 176

```
AsymptoticDSolveValue[{2*x^2*D[y[x],{x,2}]-x*D[y[x],x]+(1-x^2)*y[x]==1,{}},
       y[x],{x,0,5}]
```

$$y(x) \to c_2x\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right) + c_1\sqrt{x}\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right) + \sqrt{x}\left(-\frac{x^{11/2}}{154440}-\frac{x^{7/2}}{1260}-\frac{x^{3/2}}{15} + \frac{2}{\sqrt{x}}\right)\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)+x\left(\frac{x^5}{55440}+\frac{x^3}{504}+\frac{x}{6}-\frac{1}{x}\right)\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right)$$

2.4.20 problem 20

Internal problem ID [8585]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 20
Date solved : `Thursday, December 12, 2024 at 09:31:32 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$2x^2y'' + 2xy' - xy = 1$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' + 2xy' - xy = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{1}{2x}$$

Table 2.178: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' + 2xy' - xy = 1$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $2x^2y'' + 2xy' - xy = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &2x^2\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) \\ &+ 2x\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) - x\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r}a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r}a_n(n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r}a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right) = 0 \quad \text{(2B)}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n(n+r)(n+r-1) + 2x^{n+r} a_n(n+r) = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) + 2x^r a_0 r = 0$$

Or

$$(2x^r r(-1+r) + 2x^r r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$2x^r r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$(2x^m m(-1+m) + 2x^m m)\, c_0 = 1$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$2x^r r^2 = 0$$

Solving for r gives the roots of the indicial equation as $[0, 0]$.

Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + 2a_n(n+r) - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{2n^2 + 4nr + 2r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-1}}{2n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{2(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_1 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2(r+1)^2}$	$\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{4(r+1)^2(r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{16}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2(r+1)^2}$	$\frac{1}{2}$
a_2	$\frac{1}{4(r+1)^2(r+2)^2}$	$\frac{1}{16}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{8(r+1)^2(r+2)^2(r+3)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{1}{288}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2(r+1)^2}$	$\frac{1}{2}$
a_2	$\frac{1}{4(r+1)^2(r+2)^2}$	$\frac{1}{16}$
a_3	$\frac{1}{8(r+1)^2(r+2)^2(r+3)^2}$	$\frac{1}{288}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16\left(r+1\right)^2\left(r+2\right)^2\left(r+3\right)^2\left(r+4\right)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{9216}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2(r+1)^2}$	$\frac{1}{2}$
a_2	$\frac{1}{4(r+1)^2(r+2)^2}$	$\frac{1}{16}$
a_3	$\frac{1}{8(r+1)^2(r+2)^2(r+3)^2}$	$\frac{1}{288}$
a_4	$\frac{1}{16(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{1}{9216}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{32\left(r+1\right)^2\left(r+2\right)^2\left(r+3\right)^2\left(r+4\right)^2\left(r+5\right)^2}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{1}{460800}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2(r+1)^2}$	$\frac{1}{2}$
a_2	$\frac{1}{4(r+1)^2(r+2)^2}$	$\frac{1}{16}$
a_3	$\frac{1}{8(r+1)^2(r+2)^2(r+3)^2}$	$\frac{1}{288}$
a_4	$\frac{1}{16(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{1}{9216}$
a_5	$\frac{1}{32(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$\frac{1}{460800}$

Using the above table, then the first solution $y_1(x)$ becomes

$$
\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\
&= 1 + \frac{x}{2} + \frac{x^2}{16} + \frac{x^3}{288} + \frac{x^4}{9216} + \frac{x^5}{460800} + O(x^6)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r} \right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = 0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{1}{2(r+1)^2}$	$\frac{1}{2}$	$-\frac{1}{(r+1)^3}$	-1
b_2	$\frac{1}{4(r+1)^2(r+2)^2}$	$\frac{1}{16}$	$\frac{-2r-3}{2(r+1)^3(r+2)^3}$	$-\frac{3}{16}$
b_3	$\frac{1}{8(r+1)^2(r+2)^2(r+3)^2}$	$\frac{1}{288}$	$\frac{-3r^2-12r-11}{4(r+1)^3(r+2)^3(r+3)^3}$	$-\frac{11}{864}$
b_4	$\frac{1}{16(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{1}{9216}$	$\frac{-2r^3-15r^2-35r-25}{4(r+1)^3(r+2)^3(r+3)^3(r+4)^3}$	$-\frac{25}{55296}$
b_5	$\frac{1}{32(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$\frac{1}{460800}$	$\frac{-5r^4-60r^3-255r^2-450r-274}{16(r+1)^3(r+2)^3(r+3)^3(r+4)^3(r+5)^3}$	$-\frac{137}{13824000}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= \left(1 + \frac{x}{2} + \frac{x^2}{16} + \frac{x^3}{288} + \frac{x^4}{9216} + \frac{x^5}{460800} + O(x^6)\right)\ln(x) \\ &\quad - x - \frac{3x^2}{16} - \frac{11x^3}{864} - \frac{25x^4}{55296} - \frac{137x^5}{13824000} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1\left(1 + \frac{x}{2} + \frac{x^2}{16} + \frac{x^3}{288} + \frac{x^4}{9216} + \frac{x^5}{460800} + O(x^6)\right) \\ &\quad + c_2\left(\left(1 + \frac{x}{2} + \frac{x^2}{16} + \frac{x^3}{288} + \frac{x^4}{9216} + \frac{x^5}{460800} + O(x^6)\right)\ln(x) - x - \frac{3x^2}{16} - \frac{11x^3}{864}\right. \\ &\quad \left. - \frac{25x^4}{55296} - \frac{137x^5}{13824000} + O(x^6)\right) \end{aligned}$$

The particular solution is found by solving for c, m the balance equation

$$(2x^m m(-1+m) + 2x^m m)\,c_0 = F$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$a_1 = \frac{a_0}{2(r+1)^2}$
$a_2 = \frac{a_0}{4(r+1)^2(r+2)^2}$
$a_3 = \frac{a_0}{8(r+1)^2(r+2)^2(r+3)^2}$
$a_4 = \frac{a_0}{16(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$
$a_5 = \frac{a_0}{32(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$

Unable to solve the balance equation $(2x^m m(-1+m) + 2x^m m)\,c_0$ for c_0 and x. No particular solution exists.

Failed to convert RHS 1 to series in order to find particular solution. Unable to solve. Terminating Unable to find the particular solution or no solution exists.

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.040 (sec)
Leaf size : maple_leaf_size

```
dsolve(2*x^2*diff(diff(y(x),x),x)+2*diff(y(x),x)*x-x*y(x) = 1,y(x),
       series,x=0)
```

No solution found

Mathematica DSolve solution

Solving time : 0.154 (sec)
Leaf size : 360

```
AsymptoticDSolveValue[{2*x^2*D[y[x],{x,2}]+2*x*D[y[x],x]-x*y[x]==1,{}},
       y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to\ & c_2\left(\frac{x^5}{460800}+\frac{x^4}{9216}+\frac{x^3}{288}+\frac{x^2}{16}+\frac{x}{2}+1\right) \\
&+c_1\left(x^5\left(\frac{\log(x)}{460800}-\frac{107}{13824000}\right)+x^4\left(\frac{\log(x)}{9216}-\frac{19}{55296}\right)+x^3\left(\frac{\log(x)}{288}-\frac{1}{108}\right)\right. \\
&\left.+x^2\left(\frac{\log(x)}{16}-\frac{1}{8}\right)+x\left(\frac{\log(x)}{2}-\frac{1}{2}\right)+\log(x)+1\right) \\
&+\left(-\frac{137x^6}{1990656000}+\frac{x^5}{4608000}+\frac{x^4}{73728}+\frac{x^3}{1728}+\frac{x^2}{64}+\frac{x}{4}\right. \\
&\left.+\frac{\log(x)}{2}\right)\left(x^5\left(\frac{\log(x)}{460800}-\frac{107}{13824000}\right)+x^4\left(\frac{\log(x)}{9216}-\frac{19}{55296}\right)\right. \\
&\left.+x^3\left(\frac{\log(x)}{288}-\frac{1}{108}\right)+x^2\left(\frac{\log(x)}{16}-\frac{1}{8}\right)+x\left(\frac{\log(x)}{2}-\frac{1}{2}\right)+\log(x)+1\right) \\
&+\left(\frac{x^5}{460800}+\frac{x^4}{9216}+\frac{x^3}{288}+\frac{x^2}{16}+\frac{x}{2}+1\right)\left(\frac{137x^6(6\log(x)+5)}{11943936000}\right. \\
&+\frac{x^5(113-30\log(x))}{138240000}+\frac{x^4(41-12\log(x))}{884736}+\frac{x^3(3-\log(x))}{1728} \\
&\left.+\frac{1}{128}x^2(5-2\log(x))+\frac{1}{4}x(2-\log(x))-\frac{1}{4}\log(x)(\log(x)+2)\right)
\end{aligned}
$$

2.4.21 problem 21

Internal problem ID [8586]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 21
Date solved : `Thursday, December 12, 2024 at 09:31:34 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + (x - 6)\, y = 0$$

Using series expansion around $x = 0$

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0, y_0, y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (2.4)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (2.5)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} F_n|_{x_0,y_0,y_0'} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -(x-6)\,y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= (-x+6)\,y' - y \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -2y' + \left(x^2 - 12x + 36\right) y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= (x-6)\left((x-6)\,y' + 4y\right) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -(x-6)^3\,y + 6(x-6)\,y' + 4y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 6y(0) \\
F_1 &= 6y'(0) - y(0) \\
F_2 &= -2y'(0) + 36y(0) \\
F_3 &= 36y'(0) - 24y(0) \\
F_4 &= 220y(0) - 36y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + 3x^2 - \frac{1}{6}x^3 + \frac{3}{2}x^4 - \frac{1}{5}x^5\right) y(0) + \left(x + x^3 - \frac{1}{12}x^4 + \frac{3}{10}x^5\right) y'(0) + O\left(x^6\right)
$$

Since the expansion point $x = 0$ is an ordinary point, then this can also be solved using the standard power series method. Let the solution be represented as power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^n$$

Then

$$y'=\sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y''=\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+(x-6)\left(\sum_{n=0}^{\infty} a_n x^n\right)=0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+\left(\sum_{n=0}^{\infty} x^{1+n} a_n\right)+\sum_{n=0}^{\infty}\left(-6 a_n x^n\right)=0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)\, a_{n+2}(1+n)\, x^n$$
$$\sum_{n=0}^{\infty} x^{1+n} a_n=\sum_{n=1}^{\infty} a_{n-1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty}(n+2)\, a_{n+2}(1+n)\, x^n\right)+\left(\sum_{n=1}^{\infty} a_{n-1} x^n\right)+\sum_{n=0}^{\infty}\left(-6 a_n x^n\right)=0 \tag{3}$$

$n=0$ gives

$$2a_2-6a_0=0$$

$$a_2 = 3a_0$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\,a_{n+2}(1+n) + a_{n-1} - 6a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{-a_{n-1} + 6a_n}{(n+2)\,(1+n)} \\ &= \frac{6a_n}{(n+2)\,(1+n)} - \frac{a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + a_0 - 6a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_0}{6} + a_1$$

For $n = 2$ the recurrence equation gives

$$12a_4 + a_1 - 6a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_1}{12} + \frac{3a_0}{2}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + a_2 - 6a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_0}{5} + \frac{3a_1}{10}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + a_3 - 6a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{11a_0}{36} - \frac{a_1}{20}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + a_4 - 6a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{113a_1}{2520} - \frac{9a_0}{140}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + 3a_0 x^2 + \left(-\frac{a_0}{6} + a_1\right) x^3 + \left(-\frac{a_1}{12} + \frac{3a_0}{2}\right) x^4 + \left(-\frac{a_0}{5} + \frac{3a_1}{10}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + 3x^2 - \frac{1}{6}x^3 + \frac{3}{2}x^4 - \frac{1}{5}x^5\right) a_0 + \left(x + x^3 - \frac{1}{12}x^4 + \frac{3}{10}x^5\right) a_1 + O\left(x^6\right) \quad (3)$$

At $x = 0$ the solution above becomes

$$y = \left(1 + 3x^2 - \frac{1}{6}x^3 + \frac{3}{2}x^4 - \frac{1}{5}x^5\right) c_1 + \left(x + x^3 - \frac{1}{12}x^4 + \frac{3}{10}x^5\right) c_2 + O\left(x^6\right)$$

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + (-6+x)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

  $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

- ◦ Shift index using $k- >k-m$

  $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

- ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$

- ◦ Shift index using $k- >k+2$

  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$

Rewrite ODE with series expansions

$2a_2 - 6a_0 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - 6a_k + a_{k-1}\right) x^k\right) = 0$

- Each term must be 0

  $2a_2 - 6a_0 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\, a_{k+2} - 6a_k + a_{k-1} = 0$

- Shift index using $k- >k+1$

  $\left((k+1)^2 + 3k + 5\right) a_{k+3} - 6a_{k+1} + a_k = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = -\frac{-6a_{k+1}+a_k}{k^2+5k+6}, 2a_2 - 6a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 52

```
dsolve(diff(diff(y(x),x),x)+(x-6)*y(x) = 0,y(x),
       series,x=0)
```

$$y = \left(1+3x^2-\frac{1}{6}x^3+\frac{3}{2}x^4-\frac{1}{5}x^5\right)y(0)+\left(x+x^3-\frac{1}{12}x^4+\frac{3}{10}x^5\right)y'(0)+O\left(x^6\right)$$

Mathematica DSolve solution

Solving time : 0.001 (sec)
Leaf size : 57

```
AsymptoticDSolveValue[{D[y[x],{x,2}]+(x-6)*y[x]==0,{}},
       y[x],{x,0,5}]
```

$$y(x)\to c_2\left(\frac{3x^5}{10}-\frac{x^4}{12}+x^3+x\right)+c_1\left(-\frac{x^5}{5}+\frac{3x^4}{2}-\frac{x^3}{6}+3x^2+1\right)$$

2.4.22 problem 22

Internal problem ID [8587]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 22
Date solved : `Thursday, December 12, 2024 at 09:31:34 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + \left(3x^2 + 2x\right) y' - 2y = 0$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(3x^2 + 2x\right) y' - 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{3x + 2}{x}$$
$$q(x) = -\frac{2}{x^2}$$

Table 2.180: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3x+2}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{2}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + (3x^2 + 2x)\, y' - 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} & x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} \right) \\ & + (3x^2 + 2x) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \right) - 2 \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\begin{aligned} & \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\,(n+r-1) \right) + \left(\sum_{n=0}^{\infty} 3x^{1+n+r} a_n (n+r) \right) \\ & + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r) \right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r} \right) = 0 \end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 3x^{1+n+r} a_n(n+r) = \sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)\, x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\,(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)\, x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n(n+r)\,(n+r-1) + 2x^{n+r} a_n(n+r) - 2a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + 2x^r a_0 r - 2a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) + 2x^r r - 2x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 + r - 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 + r - 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = -2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 + r - 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as $[1, -2]$.

Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x)\ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{1+n}$$
$$y_2(x) = C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 3a_{n-1}(n+r-1) + 2a_n(n+r) - 2a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{3a_{n-1}}{n+r+2} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = -\frac{3a_{n-1}}{n+3} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{3}{3+r}$$

Which for the root $r = 1$ becomes

$$a_1 = -\frac{3}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{3+r}$	$-\frac{3}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{9}{(3+r)(4+r)}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{9}{20}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{3+r}$	$-\frac{3}{4}$
a_2	$\frac{9}{(3+r)(4+r)}$	$\frac{9}{20}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{27}{(3+r)(4+r)(5+r)}$$

Which for the root $r = 1$ becomes

$$a_3 = -\frac{9}{40}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{3+r}$	$-\frac{3}{4}$
a_2	$\frac{9}{(3+r)(4+r)}$	$\frac{9}{20}$
a_3	$-\frac{27}{(3+r)(4+r)(5+r)}$	$-\frac{9}{40}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{81}{(4+r)(5+r)(3+r)(6+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{27}{280}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{3+r}$	$-\frac{3}{4}$
a_2	$\frac{9}{(3+r)(4+r)}$	$\frac{9}{20}$
a_3	$-\frac{27}{(3+r)(4+r)(5+r)}$	$-\frac{9}{40}$
a_4	$\frac{81}{(4+r)(5+r)(3+r)(6+r)}$	$\frac{27}{280}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{243}{(5+r)(3+r)(6+r)(7+r)(4+r)}$$

Which for the root $r = 1$ becomes

$$a_5 = -\frac{81}{2240}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{3}{3+r}$	$-\frac{3}{4}$
a_2	$\frac{9}{(3+r)(4+r)}$	$\frac{9}{20}$
a_3	$-\frac{27}{(3+r)(4+r)(5+r)}$	$-\frac{9}{40}$
a_4	$\frac{81}{(4+r)(5+r)(3+r)(6+r)}$	$\frac{27}{280}$
a_5	$-\frac{243}{(5+r)(3+r)(6+r)(7+r)(4+r)}$	$-\frac{81}{2240}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x\left(1 - \frac{3x}{4} + \frac{9x^2}{20} - \frac{9x^3}{40} + \frac{27x^4}{280} - \frac{81x^5}{2240} + O(x^6)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_3 \\
&= -\frac{27}{(3+r)(4+r)(5+r)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r \to r_2} -\frac{27}{(3+r)(4+r)(5+r)} &= \lim_{r \to -2} -\frac{27}{(3+r)(4+r)(5+r)} \\
&= -\frac{9}{2}
\end{aligned}
$$

The limit is $-\frac{9}{2}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-2}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + 3b_{n-1}(n+r-1) + 2b_n(n+r) - 2b_n = 0 \tag{4}$$

Which for for the root $r = -2$ becomes

$$b_n(n-2)(n-3) + 3b_{n-1}(n-3) + 2b_n(n-2) - 2b_n = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{3b_{n-1}}{n+r+2} \tag{5}$$

Which for the root $r = -2$ becomes

$$b_n = -\frac{3b_{n-1}}{n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -2$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{3}{3+r}$$

Which for the root $r = -2$ becomes

$$b_1 = -3$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{3+r}$	-3

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{9}{(3+r)(4+r)}$$

Which for the root $r=-2$ becomes

$$b_2=\frac{9}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{3+r}$	-3
b_2	$\frac{9}{(3+r)(4+r)}$	$\frac{9}{2}$

For $n=3$, using the above recursive equation gives

$$b_3=-\frac{27}{(3+r)(4+r)(5+r)}$$

Which for the root $r=-2$ becomes

$$b_3=-\frac{9}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{3+r}$	-3
b_2	$\frac{9}{(3+r)(4+r)}$	$\frac{9}{2}$
b_3	$-\frac{27}{(3+r)(4+r)(5+r)}$	$-\frac{9}{2}$

For $n=4$, using the above recursive equation gives

$$b_4=\frac{81}{(4+r)(5+r)(3+r)(6+r)}$$

Which for the root $r=-2$ becomes

$$b_4=\frac{27}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{3+r}$	-3
b_2	$\frac{9}{(3+r)(4+r)}$	$\frac{9}{2}$
b_3	$-\frac{27}{(3+r)(4+r)(5+r)}$	$-\frac{9}{2}$
b_4	$\frac{81}{(4+r)(5+r)(3+r)(6+r)}$	$\frac{27}{8}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{243}{(5+r)(3+r)(6+r)(7+r)(4+r)}$$

Which for the root $r = -2$ becomes

$$b_5 = -\frac{81}{40}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{3}{3+r}$	-3
b_2	$\frac{9}{(3+r)(4+r)}$	$\frac{9}{2}$
b_3	$-\frac{27}{(3+r)(4+r)(5+r)}$	$-\frac{9}{2}$
b_4	$\frac{81}{(4+r)(5+r)(3+r)(6+r)}$	$\frac{27}{8}$
b_5	$-\frac{243}{(5+r)(3+r)(6+r)(7+r)(4+r)}$	$-\frac{81}{40}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x\left(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots\right) \\ &= \frac{1 - 3x + \frac{9x^2}{2} - \frac{9x^3}{2} + \frac{27x^4}{8} - \frac{81x^5}{40} + O(x^6)}{x^2} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1x\left(1 - \frac{3x}{4} + \frac{9x^2}{20} - \frac{9x^3}{40} + \frac{27x^4}{280} - \frac{81x^5}{2240} + O(x^6)\right) \\ &+ \frac{c_2\left(1 - 3x + \frac{9x^2}{2} - \frac{9x^3}{2} + \frac{27x^4}{8} - \frac{81x^5}{40} + O(x^6)\right)}{x^2} \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1 x\left(1-\frac{3x}{4}+\frac{9x^2}{20}-\frac{9x^3}{40}+\frac{27x^4}{280}-\frac{81x^5}{2240}+O(x^6)\right) \\
&\quad+\frac{c_2\left(1-3x+\frac{9x^2}{2}-\frac{9x^3}{2}+\frac{27x^4}{8}-\frac{81x^5}{40}+O(x^6)\right)}{x^2}
\end{aligned}
$$

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)+(3x^2+2x)\left(\frac{d}{dx}y(x)\right)-2y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{2y(x)}{x^2}-\frac{(3x+2)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{(3x+2)\left(\frac{d}{dx}y(x)\right)}{x}-\frac{2y(x)}{x^2}=0$

☐ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{3x+2}{x}, P_3(x)=-\frac{2}{x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=2$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=-2$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right)+x(3x+2)\left(\frac{d}{dx}y(x)\right)-2y(x)=0$

- Assume series solution for $y(x)$

$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- ○ Shift index using $k- >k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- ○ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)\,(-1+r)\, x^r + \left(\sum_{k=1}^{\infty} \left(a_k(k+r+2)\,(k+r-1) + 3a_{k-1}(k+r-1)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(-1+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 1\}$

- Each term in the series must be 0, giving the recursion relation

  $(k+r-1)\,(a_k(k+r+2) + 3a_{k-1}) = 0$

- Shift index using $k- >k+1$

  $(k+r)\,(a_{k+1}(k+3+r) + 3a_k) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{3a_k}{k+3+r}$

- Recursion relation for $r = -2$

  $a_{k+1} = -\frac{3a_k}{k+1}$

- Solution for $r = -2$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+1} = -\frac{3a_k}{k+1}\right]$

- Recursion relation for $r = 1$

  $a_{k+1} = -\frac{3a_k}{k+4}$

- Solution for $r = 1$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = -\frac{3a_k}{k+4}\right]$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+1} = -\frac{3a_k}{k+1}, b_{k+1} = -\frac{3b_k}{4+k}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.023 (sec)
Leaf size : 45

```
dsolve(x^2*diff(diff(y(x),x),x)+(3*x^2+2*x)*diff(y(x),x)-2*y(x) = 0,y(x),
       series,x=0)
```

$$y = c_1 x \left(1 - \frac{3}{4}x + \frac{9}{20}x^2 - \frac{9}{40}x^3 + \frac{27}{280}x^4 - \frac{81}{2240}x^5 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2 \left(12 - 36x + 54x^2 - 54x^3 + \frac{81}{2}x^4 - \frac{243}{10}x^5 + \mathrm{O}\left(x^6\right)\right)}{x^2}$$

Mathematica DSolve solution

Solving time : `0.028` (sec)
Leaf size : 64

```
AsymptoticDSolveValue[{x^2*D[y[x],{x,2}]+(2*x+3*x^2)*D[y[x],x]-2*y[x]==0,{}},
       y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{27x^2}{8}+\frac{1}{x^2}-\frac{9x}{2}-\frac{3}{x}+\frac{9}{2}\right)+c_2\left(\frac{27x^5}{280}-\frac{9x^4}{40}+\frac{9x^3}{20}-\frac{3x^2}{4}+x\right)$$

2.4.23 problem 23

Internal problem ID [8588]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 23
Date solved : `Thursday, December 12, 2024 at 09:31:36 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$2x^2y'' - xy' + \left(-x^2+1\right)y = x^2 + \cos\left(x\right)$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' - xy' + \left(-x^2+1\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{2x}$$
$$q(x) = -\frac{x^2-1}{2x^2}$$

Table 2.182: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{x^2-1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' - xy' + \left(-x^2 + 1\right) y = x^2 + \cos\left(x\right)$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $2x^2y'' - xy' + \left(-x^2 + 1\right) y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &2x^2\left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) \\ &- x\left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + \left(-x^2 + 1\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\begin{aligned}\left(\sum_{n=0}^{\infty} 2x^{n+r}a_n(n+r)(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-x^{n+r}a_n(n+r)\right)\\ +\sum_{n=0}^{\infty}\left(-x^{n+r+2}a_n\right)+\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)=0\end{aligned}\tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-x^{n+r+2}a_n\right)=\sum_{n=2}^{\infty}\left(-a_{n-2}x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\begin{aligned}\left(\sum_{n=0}^{\infty} 2x^{n+r}a_n(n+r)(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-x^{n+r}a_n(n+r)\right)\\ +\sum_{n=2}^{\infty}\left(-a_{n-2}x^{n+r}\right)+\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)=0\end{aligned}\tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r}a_n(n+r)(n+r-1)-x^{n+r}a_n(n+r)+a_nx^{n+r}=0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r)-x^r a_0 r+a_0x^r=0$$

Or

$$\left(2x^r r(-1+r)-x^r r+x^r\right)a_0=0$$

Since $a_0\neq 0$ then the above simplifies to

$$\left(2r^2-3r+1\right)x^r=0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2-3r+1=0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = \frac{1}{2}$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$(2x^m m(-1+m) - x^m m + x^m)\, c_0 = x^2 + \cos(x)$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$(2r^2 - 3r + 1)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as $\left[1, \frac{1}{2}\right]$.

Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) - a_n(n+r) - a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-2}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - b_n(n+r) - b_{n-2} + b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = \frac{b_{n-2}}{n(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = \frac{1}{168}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) + c_2\sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right) \end{aligned}$$

The particular solution is found by solving for c, m the balance equation

$$\left(2x^m m(-1+m) - x^m m + x^m\right) c_0 = F$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$\boxed{\begin{aligned} a_1 &= 0 \\ a_2 &= \frac{a_0}{2r^2+5r+3} \\ a_3 &= 0 \\ a_4 &= \frac{a_0}{(2r^2+5r+3)(2r^2+13r+21)} \\ a_5 &= 0 \end{aligned}}$$

Expanding the rhs of the ode $x^2 + \cos(x)$ in series gives

$$x^2 + \cos(x) = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4$$

Since the $F = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4$ has more than one term then we find a particular solution for each term and add the result to find the particular solution to the ode.

Now we determine the particular solution y_p associated with $F = 1$ by solving the balance equation

$$\left(2x^m m(-1+m) - x^m m + x^m\right) c_0 = 1$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= 1 \\
m &= 0
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= \sum_{n=0}^{\infty} c_n x^{n+0}
\end{aligned}
$$

Where in the above $c_0 = 1$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 0$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = 1$ and $r = m$ or $r = 0$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = 1$
$c_1 = 0$
$c_2 = \frac{1}{3}$
$c_3 = 0$
$c_4 = \frac{1}{63}$
$c_5 = 0$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
&= 1 \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= 1\left(1 + \frac{1}{3}x^2 + \frac{1}{63}x^4\right) \\
&= 1 + \frac{1}{3}x^2 + \frac{1}{63}x^4
\end{aligned}
$$

Now we determine the particular solution y_p associated with $F = \frac{x^2}{2}$ by solving the balance equation

$$(2x^m m(-1+m) - x^m m + x^m)\, c_0 = \frac{x^2}{2}$$

For c_0 and x. This results in

$$\begin{aligned} c_0 &= \frac{1}{6} \\ m &= 2 \end{aligned}$$

The particular solution is therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= \sum_{n=0}^{\infty} c_n x^{n+2} \end{aligned}$$

Where in the above $c_0 = \frac{1}{6}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 2$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{6}$ and $r = m$ or $r = 2$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = \frac{1}{6}$
$c_1 = 0$
$c_2 = \frac{1}{126}$
$c_3 = 0$
$c_4 = \frac{1}{6930}$
$c_5 = 0$

The particular solution is now found using

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= x^2 \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$\begin{aligned}y_p &= x^2\left(\frac{1}{6}+\frac{1}{126}x^2+\frac{1}{6930}x^4\right)\\ &= \frac{1}{6}x^2+\frac{1}{126}x^4+\frac{1}{6930}x^6\end{aligned}$$

Now we determine the particular solution y_p associated with $F = \frac{x^4}{24}$ by solving the balance equation

$$(2x^m m(-1+m) - x^m m + x^m)\, c_0 = \frac{x^4}{24}$$

For c_0 and x. This results in

$$\begin{aligned}c_0 &= \frac{1}{504}\\ m &= 4\end{aligned}$$

The particular solution is therefore

$$\begin{aligned}y_p &= \sum_{n=0}^{\infty} c_n x^{n+m}\\ &= \sum_{n=0}^{\infty} c_n x^{n+4}\end{aligned}$$

Where in the above $c_0 = \frac{1}{504}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 4$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{504}$ and $r = m$ or $r = 4$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = \frac{1}{504}$
$c_1 = 0$
$c_2 = \frac{1}{27720}$
$c_3 = 0$
$c_4 = \frac{1}{2910600}$
$c_5 = 0$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
&= x^4 \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x^4 \left(\frac{1}{504} + \frac{1}{27720} x^2 + \frac{1}{2910600} x^4\right) \\
&= \frac{1}{504} x^4 + \frac{1}{27720} x^6 + \frac{1}{2910600} x^8
\end{aligned}
$$

Adding all the above particular solution(s) gives

$$
y_p = 1 + \frac{x^2}{2} + \frac{13x^4}{504} + \frac{x^6}{5544} + \frac{x^8}{2910600} + O(x^6)
$$

Truncating the particular solution to the order of series requested gives

$$
y_p = 1 + \frac{x^2}{2} + \frac{13x^4}{504} + O(x^6)
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= 1 + \frac{x^2}{2} + \frac{13x^4}{504} + O(x^6) + c_1 x \left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) + c_2 \sqrt{x} \left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)
\end{aligned}
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
```

```
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : `0.018` (sec)
Leaf size : 43

```
dsolve(2*x^2*diff(diff(y(x),x),x)-diff(y(x),x)*x+(-x^2+1)*y(x) = x^2+cos(x),y(x),
       series,x=0)
```

$$
\begin{aligned}
y = {} & c_1\sqrt{x}\left(1+\frac{1}{6}x^2+\frac{1}{168}x^4+\mathrm{O}\left(x^6\right)\right) \\
& + c_2x\left(1+\frac{1}{10}x^2+\frac{1}{360}x^4+\mathrm{O}\left(x^6\right)\right)+\left(1+\frac{1}{2}x^2+\frac{13}{504}x^4+\mathrm{O}\left(x^6\right)\right)
\end{aligned}
$$

Mathematica DSolve solution

Solving time : `0.101` (sec)
Leaf size : 176

```
AsymptoticDSolveValue[{2*x^2*D[y[x],{x,2}]-x*D[y[x],x]+(1-x^2)*y[x]==x^2+Cos[x],{}},
       y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to\, & c_2 x\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right) \\
& + c_1\sqrt{x}\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)+\sqrt{x}\left(-\frac{59x^{11/2}}{77220}-\frac{17x^{7/2}}{630}-\frac{2x^{3/2}}{5}\right. \\
& \left.+\frac{2}{\sqrt{x}}\right)\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)+x\left(\frac{239x^5}{138600}+\frac{11x^3}{252}+\frac{2x}{3}-\frac{1}{x}\right)\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right)
\end{aligned}
$$

2.4.24 problem 24

Internal problem ID [8589]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 24
Date solved : `Thursday, December 12, 2024 at 09:31:37 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$2x^2y'' - xy' + \left(-x^2+1\right)y = \cos\left(x\right)$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' - xy' + \left(-x^2+1\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{2x}$$
$$q(x) = -\frac{x^2-1}{2x^2}$$

Table 2.183: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	“regular”

$q(x) = -\frac{x^2-1}{2x^2}$	
singularity	type
$x = 0$	“regular”

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' - xy' + \left(-x^2 + 1\right) y = \cos\left(x\right)$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $2x^2y'' - xy' + \left(-x^2 + 1\right) y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} \left(n + r\right) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} \left(n + r\right) \left(n + r - 1\right) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &2x^2\left(\sum_{n=0}^{\infty} \left(n + r\right) \left(n + r - 1\right) a_n x^{n+r-2}\right) \\ &- x\left(\sum_{n=0}^{\infty} \left(n + r\right) a_n x^{n+r-1}\right) + \left(-x^2 + 1\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) \\
&+ \sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$
\sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) = \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right)
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) \\
&+ \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0
\end{aligned} \tag{2B}
$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$
2x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0
$$

When $n=0$ the above becomes

$$
2x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0
$$

Or

$$
\left(2x^r r(-1+r) - x^r r + x^r\right) a_0 = 0
$$

Since $a_0 \neq 0$ then the above simplifies to

$$
\left(2r^2 - 3r + 1\right) x^r = 0
$$

Since the above is true for all x then the indicial equation becomes

$$
2r^2 - 3r + 1 = 0
$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = \frac{1}{2}$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$(2x^m m(-1+m) - x^m m + x^m)\, c_0 = \cos(x)$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as $\left[1, \frac{1}{2}\right]$.

Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) - a_n(n+r) - a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-2}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \ldots\right) \\
&= x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - b_n(n+r) - b_{n-2} + b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = \frac{b_{n-2}}{n(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = \frac{1}{168}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) + c_2\sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)
\end{aligned}
$$

The particular solution is found by solving for c, m the balance equation

$$
\left(2x^m m(-1+m) - x^m m + x^m\right) c_0 = F
$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$a_1 = 0$
$a_2 = \frac{a_0}{2r^2+5r+3}$
$a_3 = 0$
$a_4 = \frac{a_0}{(2r^2+5r+3)(2r^2+13r+21)}$
$a_5 = 0$

Expanding the rhs of the ode $\cos(x)$ in series gives

$$
\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4
$$

Since the $F = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$ has more than one term then we find a particular solution for each term and add the result to find the particular solution to the ode.

Now we determine the particular solution y_p associated with $F = 1$ by solving the balance equation

$$
\left(2x^m m(-1+m) - x^m m + x^m\right) c_0 = 1
$$

For c_0 and x. This results in

$$\begin{aligned} c_0 &= 1 \\ m &= 0 \end{aligned}$$

The particular solution is therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= \sum_{n=0}^{\infty} c_n x^{n+0} \end{aligned}$$

Where in the above $c_0 = 1$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 0$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = 1$ and $r = m$ or $r = 0$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = 1$
$c_1 = 0$
$c_2 = \frac{1}{3}$
$c_3 = 0$
$c_4 = \frac{1}{63}$
$c_5 = 0$

The particular solution is now found using

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= 1 \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$\begin{aligned} y_p &= 1\left(1 + \frac{1}{3}x^2 + \frac{1}{63}x^4\right) \\ &= 1 + \frac{1}{3}x^2 + \frac{1}{63}x^4 \end{aligned}$$

Now we determine the particular solution y_p associated with $F = -\frac{x^2}{2}$ by solving the balance equation

$$(2x^m m(-1+m) - x^m m + x^m) c_0 = -\frac{x^2}{2}$$

For c_0 and x. This results in

$$c_0 = -\frac{1}{6}$$
$$m = 2$$

The particular solution is therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= \sum_{n=0}^{\infty} c_n x^{n+2} \end{aligned}$$

Where in the above $c_0 = -\frac{1}{6}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 2$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = -\frac{1}{6}$ and $r = m$ or $r = 2$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = -\frac{1}{6}$
$c_1 = 0$
$c_2 = -\frac{1}{126}$
$c_3 = 0$
$c_4 = -\frac{1}{6930}$
$c_5 = 0$

The particular solution is now found using

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= x^2 \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x^2\left(-\frac{1}{6}-\frac{1}{126}x^2-\frac{1}{6930}x^4\right)\\
&= -\frac{1}{6}x^2-\frac{1}{126}x^4-\frac{1}{6930}x^6
\end{aligned}
$$

Now we determine the particular solution y_p associated with $F=\frac{x^4}{24}$ by solving the balance equation

$$
\left(2x^m m(-1+m)-x^m m+x^m\right)c_0=\frac{x^4}{24}
$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= \frac{1}{504}\\
m &= 4
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m}\\
&= \sum_{n=0}^{\infty} c_n x^{n+4}
\end{aligned}
$$

Where in the above $c_0=\frac{1}{504}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m=4$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0=c_0$ or $a_0=\frac{1}{504}$ and $r=m$ or $r=4$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0=\frac{1}{504}$
$c_1=0$
$c_2=\frac{1}{27720}$
$c_3=0$
$c_4=\frac{1}{2910600}$
$c_5=0$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
&= x^4 \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x^4 \left(\frac{1}{504} + \frac{1}{27720}x^2 + \frac{1}{2910600}x^4\right) \\
&= \frac{1}{504}x^4 + \frac{1}{27720}x^6 + \frac{1}{2910600}x^8
\end{aligned}
$$

Adding all the above particular solution(s) gives

$$
y_p = 1 + \frac{x^2}{6} + \frac{5x^4}{504} - \frac{x^6}{9240} + \frac{x^8}{2910600} + O(x^6)
$$

Truncating the particular solution to the order of series requested gives

$$
y_p = 1 + \frac{x^2}{6} + \frac{5x^4}{504} + O(x^6)
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= 1+\frac{x^2}{6}+\frac{5x^4}{504}+O(x^6)+c_1 x\left(1+\frac{x^2}{10}+\frac{x^4}{360}+O(x^6)\right)+c_2\sqrt{x}\left(1+\frac{x^2}{6}+\frac{x^4}{168}+O(x^6)\right)
\end{aligned}
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
```

```
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.018 (sec)
Leaf size : 43

```
dsolve(2*x^2*diff(diff(y(x),x),x)-diff(y(x),x)*x+(-x^2+1)*y(x) = cos(x),y(x),
       series,x=0)
```

$$
\begin{aligned}
y = {} & c_1\sqrt{x}\left(1+\frac{1}{6}x^2+\frac{1}{168}x^4+\mathrm{O}\left(x^6\right)\right) \\
& + c_2 x\left(1+\frac{1}{10}x^2+\frac{1}{360}x^4+\mathrm{O}\left(x^6\right)\right)+\left(1+\frac{1}{6}x^2+\frac{5}{504}x^4+\mathrm{O}\left(x^6\right)\right)
\end{aligned}
$$

Mathematica DSolve solution

Solving time : 0.049 (sec)
Leaf size : 176

```
AsymptoticDSolveValue[{2*x^2*D[y[x],{x,2}]-x*D[y[x],x]+(1-x^2)*y[x]==Cos[x],{}},
       y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) &\\
\rightarrow c_2 x\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right) &\\
+ c_1\sqrt{x}\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)+\sqrt{x}\left(-\frac{x^{11/2}}{3861}+\frac{x^{7/2}}{630}+\frac{4x^{3/2}}{15}\right. &\\
\left.+\frac{2}{\sqrt{x}}\right)\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)+x\left(\frac{37x^5}{69300}-\frac{x^3}{84}-\frac{x}{3}-\frac{1}{x}\right)\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right)
\end{aligned}
$$

2.4.25 problem 24

Internal problem ID [8590]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 24
Date solved : `Thursday, December 12, 2024 at 09:31:39 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$2x^2y'' - xy' + \left(-x^2+1\right)y = x^3 + \cos\left(x\right)$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' - xy' + \left(-x^2+1\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{2x}$$
$$q(x) = -\frac{x^2-1}{2x^2}$$

Table 2.184: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{x^2-1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' - xy' + \left(-x^2 + 1\right) y = x^3 + \cos\left(x\right)$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $2x^2y'' - xy' + (-x^2 + 1)\, y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2 \left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) - x \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + \left(-x^2 + 1\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) = \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 3r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = \frac{1}{2}$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$(2x^m m(-1+m) - x^m m + x^m)\, c_0 = x^3 + \cos(x)$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as $\left[1, \frac{1}{2}\right]$.

Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) - a_n(n+r) - a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-2}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - b_n(n+r) - b_{n-2} + b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = \frac{b_{n-2}}{n(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = \frac{1}{168}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x\left(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots\right) \\ &= \sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) + c_2\sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right) \end{aligned}$$

The particular solution is found by solving for c, m the balance equation

$$(2x^m m(-1+m) - x^m m + x^m)\, c_0 = F$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$a_1 = 0$
$a_2 = \frac{a_0}{2r^2+5r+3}$
$a_3 = 0$
$a_4 = \frac{a_0}{(2r^2+5r+3)(2r^2+13r+21)}$
$a_5 = 0$

Expanding the rhs of the ode $x^3 + \cos(x)$ in series gives

$$x^3 + \cos(x) = 1 - \frac{1}{2}x^2 + x^3 + \frac{1}{24}x^4$$

Since the $F = 1 - \frac{1}{2}x^2 + x^3 + \frac{1}{24}x^4$ has more than one term then we find a particular solution for each term and add the result to find the particular solution to the ode.

Now we determine the particular solution y_p associated with $F = 1$ by solving the balance equation

$$(2x^m m(-1+m) - x^m m + x^m)\, c_0 = 1$$

For c_0 and x. This results in

$$\begin{aligned} c_0 &= 1 \\ m &= 0 \end{aligned}$$

The particular solution is therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= \sum_{n=0}^{\infty} c_n x^{n+0} \end{aligned}$$

Where in the above $c_0 = 1$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 0$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = 1$ and $r = m$ or $r = 0$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = 1$
$c_1 = 0$
$c_2 = \frac{1}{3}$
$c_3 = 0$
$c_4 = \frac{1}{63}$
$c_5 = 0$

The particular solution is now found using

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= 1 \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$\begin{aligned} y_p &= 1\left(1 + \frac{1}{3}x^2 + \frac{1}{63}x^4\right) \\ &= 1 + \frac{1}{3}x^2 + \frac{1}{63}x^4 \end{aligned}$$

Now we determine the particular solution y_p associated with $F = -\frac{x^2}{2}$ by solving the balance equation

$$(2x^m m(-1+m) - x^m m + x^m)\, c_0 = -\frac{x^2}{2}$$

For c_0 and x. This results in

$$\begin{aligned} c_0 &= -\frac{1}{6} \\ m &= 2 \end{aligned}$$

The particular solution is therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= \sum_{n=0}^{\infty} c_n x^{n+2} \end{aligned}$$

Where in the above $c_0 = -\frac{1}{6}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 2$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = -\frac{1}{6}$ and $r = m$ or $r = 2$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = -\frac{1}{6}$
$c_1 = 0$
$c_2 = -\frac{1}{126}$
$c_3 = 0$
$c_4 = -\frac{1}{6930}$
$c_5 = 0$

The particular solution is now found using

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= x^2 \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x^2\left(-\frac{1}{6} - \frac{1}{126}x^2 - \frac{1}{6930}x^4\right) \\
&= -\frac{1}{6}x^2 - \frac{1}{126}x^4 - \frac{1}{6930}x^6
\end{aligned}
$$

Now we determine the particular solution y_p associated with $F = x^3$ by solving the balance equation

$$
(2x^m m(-1+m) - x^m m + x^m)\, c_0 = x^3
$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= \frac{1}{10} \\
m &= 3
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= \sum_{n=0}^{\infty} c_n x^{n+3}
\end{aligned}
$$

Where in the above $c_0 = \frac{1}{10}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 3$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{10}$ and $r = m$ or $r = 3$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = \frac{1}{10}$
$c_1 = 0$
$c_2 = \frac{1}{360}$
$c_3 = 0$
$c_4 = \frac{1}{28080}$
$c_5 = 0$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
&= x^3 \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x^3 \left(\frac{1}{10} + \frac{1}{360}x^2 + \frac{1}{28080}x^4\right) \\
&= \frac{1}{10}x^3 + \frac{1}{360}x^5 + \frac{1}{28080}x^7
\end{aligned}
$$

Now we determine the particular solution y_p associated with $F = \frac{x^4}{24}$ by solving the balance equation

$$
\left(2x^m m(-1+m) - x^m m + x^m\right) c_0 = \frac{x^4}{24}
$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= \frac{1}{504} \\
m &= 4
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= \sum_{n=0}^{\infty} c_n x^{n+4}
\end{aligned}
$$

Where in the above $c_0 = \frac{1}{504}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 4$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{504}$ and $r = m$ or $r = 4$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$$\boxed{\begin{aligned} c_0 &= \tfrac{1}{504} \\ c_1 &= 0 \\ c_2 &= \tfrac{1}{27720} \\ c_3 &= 0 \\ c_4 &= \tfrac{1}{2910600} \\ c_5 &= 0 \end{aligned}}$$

The particular solution is now found using

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= x^4 \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$\begin{aligned} y_p &= x^4 \left(\frac{1}{504} + \frac{1}{27720} x^2 + \frac{1}{2910600} x^4\right) \\ &= \frac{1}{504} x^4 + \frac{1}{27720} x^6 + \frac{1}{2910600} x^8 \end{aligned}$$

Adding all the above particular solution(s) gives

$$y_p = 1 + \frac{x^2}{6} + \frac{x^3}{10} + \frac{5x^4}{504} + \frac{x^5}{360} - \frac{x^6}{9240} + \frac{x^7}{28080} + \frac{x^8}{2910600} + O(x^6)$$

Truncating the particular solution to the order of series requested gives

$$y_p = 1 + \frac{x^2}{6} + \frac{x^3}{10} + \frac{5x^4}{504} + \frac{x^5}{360} + O(x^6)$$

Hence the final solution is

$$\begin{aligned} y &= y_h + y_p \\ &= 1 + \frac{x^2}{6} + \frac{x^3}{10} + \frac{5x^4}{504} + \frac{x^5}{360} + O(x^6) \\ &\quad + c_1 x \left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) + c_2 \sqrt{x} \left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right) \end{aligned}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.019 (sec)
Leaf size : 47

```
dsolve(2*x^2*diff(diff(y(x),x),x)-diff(y(x),x)*x+(-x^2+1)*y(x) = x^3+cos(x),y(x),
       series,x=0)
```

$$y = c_1\sqrt{x}\left(1+\frac{1}{6}x^2+\frac{1}{168}x^4+\mathrm{O}\left(x^6\right)\right)+c_2x\left(1+\frac{1}{10}x^2+\frac{1}{360}x^4+\mathrm{O}\left(x^6\right)\right) + \left(1+\frac{1}{6}x^2+\frac{1}{10}x^3+\frac{5}{504}x^4+\frac{1}{360}x^5+\mathrm{O}\left(x^6\right)\right)$$

Mathematica DSolve solution

Solving time : 0.046 (sec)
Leaf size : 176

```
AsymptoticDSolveValue[{2*x^2*D[y[x],{x,2}]-x*D[y[x],x]+(1-x^2)*y[x]==Cos[x],{}},
       y[x],{x,0,5}]
```

$$y(x) \rightarrow c_2 x\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right) + c_1\sqrt{x}\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)+\sqrt{x}\left(-\frac{x^{11/2}}{3861}+\frac{x^{7/2}}{630}+\frac{4x^{3/2}}{15} + \frac{2}{\sqrt{x}}\right)\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)+x\left(\frac{37x^5}{69300}-\frac{x^3}{84}-\frac{x}{3}-\frac{1}{x}\right)\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right)$$

2.4.26 problem 24

Internal problem ID [8591]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 24
Date solved : `Thursday, December 12, 2024 at 09:31:40 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$2x^2y'' - xy' + \left(-x^2+1\right) y = x^3 \cos\left(x\right)$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' - xy' + \left(-x^2+1\right) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned}p(x) &= -\frac{1}{2x}\\ q(x) &= -\frac{x^2-1}{2x^2}\end{aligned}$$

Table 2.185: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{x^2-1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' - xy' + (-x^2 + 1)\, y = x^3 \cos(x)$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $2x^2y'' - xy' + (-x^2 + 1)\, y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &2x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\ &- x\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (-x^2+1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) = \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 3r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = \frac{1}{2}$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$\left(2x^m m(-1+m) - x^m m + x^m\right) c_0 = x^3 \cos(x)$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as $\left[1, \frac{1}{2}\right]$.

Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) - a_n(n+r) - a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-2}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \ldots\right) \\
&= x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - b_n(n+r) - b_{n-2} + b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = \frac{b_{n-2}}{n(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = \frac{1}{168}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) + c_2\sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right)
\end{aligned}
$$

The particular solution is found by solving for c, m the balance equation

$$
(2x^m m(-1+m) - x^m m + x^m)\, c_0 = F
$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$
\boxed{\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{a_0}{2r^2+5r+3} \\
a_3 &= 0 \\
a_4 &= \frac{a_0}{(2r^2+5r+3)(2r^2+13r+21)} \\
a_5 &= 0
\end{aligned}}
$$

Expanding the rhs of the ode $x^3 \cos(x)$ in series gives

$$
x^3 \cos(x) = x^3 - \frac{1}{2}x^5
$$

Since the $F = x^3 - \frac{1}{2}x^5$ has more than one term then we find a particular solution for each term and add the result to find the particular solution to the ode.

Now we determine the particular solution y_p associated with $F = x^3$ by solving the balance equation

$$
(2x^m m(-1+m) - x^m m + x^m)\, c_0 = x^3
$$

For c_0 and x. This results in

$$c_0 = \frac{1}{10}$$
$$m = 3$$

The particular solution is therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= \sum_{n=0}^{\infty} c_n x^{n+3} \end{aligned}$$

Where in the above $c_0 = \frac{1}{10}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 3$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{10}$ and $r = m$ or $r = 3$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = \frac{1}{10}$
$c_1 = 0$
$c_2 = \frac{1}{360}$
$c_3 = 0$
$c_4 = \frac{1}{28080}$
$c_5 = 0$

The particular solution is now found using

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= x^3 \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$\begin{aligned} y_p &= x^3 \left(\frac{1}{10} + \frac{1}{360} x^2 + \frac{1}{28080} x^4 \right) \\ &= \frac{1}{10} x^3 + \frac{1}{360} x^5 + \frac{1}{28080} x^7 \end{aligned}$$

Now we determine the particular solution y_p associated with $F = -\frac{x^5}{2}$ by solving the balance equation

$$(2x^m m(-1+m) - x^m m + x^m)\, c_0 = -\frac{x^5}{2}$$

For c_0 and x. This results in

$$c_0 = -\frac{1}{72}$$
$$m = 5$$

The particular solution is therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= \sum_{n=0}^{\infty} c_n x^{n+5} \end{aligned}$$

Where in the above $c_0 = -\frac{1}{72}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 5$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = -\frac{1}{72}$ and $r = m$ or $r = 5$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = -\frac{1}{72}$
$c_1 = 0$
$c_2 = -\frac{1}{5616}$
$c_3 = 0$
$c_4 = -\frac{1}{763776}$
$c_5 = 0$

The particular solution is now found using

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= x^5 \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x^5\left(-\frac{1}{72}-\frac{1}{5616}x^2-\frac{1}{763776}x^4\right)\\
&= -\frac{1}{72}x^5-\frac{1}{5616}x^7-\frac{1}{763776}x^9
\end{aligned}
$$

Adding all the above particular solution(s) gives

$$
y_p = \frac{x^3}{10}-\frac{x^5}{90}-\frac{x^7}{7020}-\frac{x^9}{763776}+O(x^6)
$$

Truncating the particular solution to the order of series requested gives

$$
y_p = \frac{x^3}{10}-\frac{x^5}{90}+O(x^6)
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h + y_p\\
&= \frac{x^3}{10}-\frac{x^5}{90}+O(x^6)+c_1 x\left(1+\frac{x^2}{10}+\frac{x^4}{360}+O(x^6)\right)+c_2\sqrt{x}\left(1+\frac{x^2}{6}+\frac{x^4}{168}+O(x^6)\right)
\end{aligned}
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
```

```
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.018 (sec)
Leaf size : 45

```
dsolve(2*x^2*diff(diff(y(x),x),x)-diff(y(x),x)*x+(-x^2+1)*y(x) = cos(x)*x^3,y(x),
       series,x=0)
```

$$
y = c_1\sqrt{x}\left(1+\frac{1}{6}x^2+\frac{1}{168}x^4+\mathrm{O}\left(x^6\right)\right) + c_2x\left(1+\frac{1}{10}x^2+\frac{1}{360}x^4+\mathrm{O}\left(x^6\right)\right) + x^3\left(\frac{1}{10}-\frac{1}{90}x^2+\mathrm{O}\left(x^4\right)\right)
$$

Mathematica DSolve solution

Solving time : 0.141 (sec)
Leaf size : 215

```
AsymptoticDSolveValue[{2*x^2*D[y[x],{x,2}]-x*D[y[x],x]+(1-x^2)*y[x]==x^3+Cos[x],{}},
       y[x],{x,0,5}]
```

$$
y(x) \to c_1\sqrt{x}\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)+c_2x\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right) + \sqrt{x}\left(-\frac{x^{11/2}}{3861}-\frac{x^{9/2}}{45}+\frac{x^{7/2}}{630}-\frac{2x^{5/2}}{5}+\frac{4x^{3/2}}{15} + \frac{2}{\sqrt{x}}\right)\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)+x\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right)\left(\frac{x^6}{1008}+\frac{37x^5}{69300}+\frac{x^4}{24}-\frac{x^3}{84}+\frac{x^2}{2}-\frac{x}{3}-\frac{1}{x}\right)
$$

2.4.27 problem 24

Internal problem ID [8592]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 24
Date solved : `Thursday, December 12, 2024 at 09:31:42 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$2x^2y'' - xy' + \left(-x^2+1\right)y = x^3\cos\left(x\right) + \sin\left(x\right)^2$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' - xy' + \left(-x^2+1\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{2x} \\ q(x) &= -\frac{x^2-1}{2x^2} \end{aligned}$$

Table 2.186: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{x^2-1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' - xy' + (-x^2 + 1)\, y = x^3 \cos(x) + \sin(x)^2$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $2x^2y'' - xy' + (-x^2 + 1)\, y = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &2x^2 \left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) \\ &- x \left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (-x^2 + 1) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) = \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \sum_{n=2}^{\infty} \left(-a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n (n+r)(n+r-1) - x^{n+r} a_n (n+r) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) - x^r a_0 r + a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) - x^r r + x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - 3r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - 3r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 1 \\ r_2 &= \frac{1}{2} \end{aligned}$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$(2x^m m(-1+m) - x^m m + x^m)\, c_0 = x^3 \cos(x) + \sin(x)^2$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$(2r^2 - 3r + 1)\, x^r = 0$$

Solving for r gives the roots of the indicial equation as $\left[1, \frac{1}{2}\right]$.

Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+1} \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}} \end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) - a_n(n+r) - a_{n-2} + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-2}}{2n^2 + n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{360}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{360}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) - b_n(n+r) - b_{n-2} + b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-2}}{2n^2 + 4nr + 2r^2 - 3n - 3r + 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = \frac{b_{n-2}}{n(2n-1)} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{2r^2 + 5r + 3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{4r^4 + 36r^3 + 113r^2 + 144r + 63}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = \frac{1}{168}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{1}{2r^2+5r+3}$	$\frac{1}{6}$
b_3	0	0
b_4	$\frac{1}{4r^4+36r^3+113r^2+144r+63}$	$\frac{1}{168}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6)\right) + c_2\sqrt{x}\left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6)\right) \end{aligned}$$

The particular solution is found by solving for c, m the balance equation

$$(2x^m m(-1+m) - x^m m + x^m)\, c_0 = F$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$\boxed{\begin{aligned} a_1 &= 0 \\ a_2 &= \frac{a_0}{2r^2+5r+3} \\ a_3 &= 0 \\ a_4 &= \frac{a_0}{(2r^2+5r+3)(2r^2+13r+21)} \\ a_5 &= 0 \end{aligned}}$$

Expanding the rhs of the ode $x^3\cos(x) + \sin(x)^2$ in series gives

$$x^3\cos(x) + \sin(x)^2 = x^2 + x^3 - \frac{1}{3}x^4 - \frac{1}{2}x^5$$

Since the $F = x^2 + x^3 - \frac{1}{3}x^4 - \frac{1}{2}x^5$ has more than one term then we find a particular solution for each term and add the result to find the particular solution to the ode.

Now we determine the particular solution y_p associated with $F = x^2$ by solving the balance equation

$$(2x^m m(-1+m) - x^m m + x^m)\, c_0 = x^2$$

For c_0 and x. This results in

$$\begin{aligned} c_0 &= \frac{1}{3} \\ m &= 2 \end{aligned}$$

The particular solution is therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= \sum_{n=0}^{\infty} c_n x^{n+2} \end{aligned}$$

Where in the above $c_0 = \frac{1}{3}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 2$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{3}$ and $r = m$ or $r = 2$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$$\boxed{\begin{aligned} c_0 &= \frac{1}{3} \\ c_1 &= 0 \\ c_2 &= \frac{1}{63} \\ c_3 &= 0 \\ c_4 &= \frac{1}{3465} \\ c_5 &= 0 \end{aligned}}$$

The particular solution is now found using

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= x^2 \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$\begin{aligned} y_p &= x^2 \left(\frac{1}{3} + \frac{1}{63} x^2 + \frac{1}{3465} x^4\right) \\ &= \frac{1}{3} x^2 + \frac{1}{63} x^4 + \frac{1}{3465} x^6 \end{aligned}$$

Now we determine the particular solution y_p associated with $F = x^3$ by solving the balance equation

$$(2x^m m(-1+m) - x^m m + x^m)\, c_0 = x^3$$

For c_0 and x. This results in

$$c_0 = \frac{1}{10}$$
$$m = 3$$

The particular solution is therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= \sum_{n=0}^{\infty} c_n x^{n+3} \end{aligned}$$

Where in the above $c_0 = \frac{1}{10}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 3$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{10}$ and $r = m$ or $r = 3$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = \frac{1}{10}$
$c_1 = 0$
$c_2 = \frac{1}{360}$
$c_3 = 0$
$c_4 = \frac{1}{28080}$
$c_5 = 0$

The particular solution is now found using

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= x^3 \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x^3\left(\frac{1}{10} + \frac{1}{360}x^2 + \frac{1}{28080}x^4\right) \\
&= \frac{1}{10}x^3 + \frac{1}{360}x^5 + \frac{1}{28080}x^7
\end{aligned}
$$

Now we determine the particular solution y_p associated with $F = -\frac{x^4}{3}$ by solving the balance equation

$$
(2x^m m(-1+m) - x^m m + x^m)\, c_0 = -\frac{x^4}{3}
$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= -\frac{1}{63} \\
m &= 4
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= \sum_{n=0}^{\infty} c_n x^{n+4}
\end{aligned}
$$

Where in the above $c_0 = -\frac{1}{63}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 4$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = -\frac{1}{63}$ and $r = m$ or $r = 4$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = -\frac{1}{63}$
$c_1 = 0$
$c_2 = -\frac{1}{3465}$
$c_3 = 0$
$c_4 = -\frac{1}{363825}$
$c_5 = 0$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
&= x^4 \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x^4 \left(-\frac{1}{63} - \frac{1}{3465}x^2 - \frac{1}{363825}x^4\right) \\
&= -\frac{1}{63}x^4 - \frac{1}{3465}x^6 - \frac{1}{363825}x^8
\end{aligned}
$$

Now we determine the particular solution y_p associated with $F = -\frac{x^5}{2}$ by solving the balance equation

$$
\left(2x^m m(-1+m) - x^m m + x^m\right) c_0 = -\frac{x^5}{2}
$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= -\frac{1}{72} \\
m &= 5
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= \sum_{n=0}^{\infty} c_n x^{n+5}
\end{aligned}
$$

Where in the above $c_0 = -\frac{1}{72}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 5$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = -\frac{1}{72}$ and $r = m$ or $r = 5$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$$\boxed{\begin{aligned} c_0 &= -\tfrac{1}{72} \\ c_1 &= 0 \\ c_2 &= -\tfrac{1}{5616} \\ c_3 &= 0 \\ c_4 &= -\tfrac{1}{763776} \\ c_5 &= 0 \end{aligned}}$$

The particular solution is now found using

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= x^5 \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$\begin{aligned} y_p &= x^5 \left(-\frac{1}{72} - \frac{1}{5616} x^2 - \frac{1}{763776} x^4 \right) \\ &= -\frac{1}{72} x^5 - \frac{1}{5616} x^7 - \frac{1}{763776} x^9 \end{aligned}$$

Adding all the above particular solution(s) gives

$$y_p = \frac{x^2}{3} + \frac{x^3}{10} - \frac{x^5}{90} - \frac{x^7}{7020} - \frac{x^8}{363825} - \frac{x^9}{763776} + O(x^6)$$

Truncating the particular solution to the order of series requested gives

$$y_p = \frac{x^2}{3} + \frac{x^3}{10} - \frac{x^5}{90} + O(x^6)$$

Hence the final solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{x^2}{3} + \frac{x^3}{10} - \frac{x^5}{90} + O(x^6) + c_1 x \left(1 + \frac{x^2}{10} + \frac{x^4}{360} + O(x^6) \right) + c_2 \sqrt{x} \left(1 + \frac{x^2}{6} + \frac{x^4}{168} + O(x^6) \right) \end{aligned}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.020 (sec)
Leaf size : 47

```
dsolve(2*x^2*diff(diff(y(x),x),x)-diff(y(x),x)*x+(-x^2+1)*y(x) = cos(x)*x^3+sin(x)^2,y
       series,x=0)
```

$$
\begin{aligned}
y = {} & c_1\sqrt{x}\left(1+\frac{1}{6}x^2+\frac{1}{168}x^4+\mathrm{O}\left(x^6\right)\right) \\
& + c_2 x\left(1+\frac{1}{10}x^2+\frac{1}{360}x^4+\mathrm{O}\left(x^6\right)\right)+x^2\left(\frac{1}{3}+\frac{1}{10}x-\frac{1}{90}x^3+\mathrm{O}\left(x^4\right)\right)
\end{aligned}
$$

Mathematica DSolve solution

Solving time : 0.477 (sec)
Leaf size : 199

```
AsymptoticDSolveValue[{2*x^2*D[y[x],{x,2}]-x*D[y[x],x]+(1-x^2)*y[x]==x^3*Cos[x]+Sin[x]^2,{}},
       y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \rightarrow\ & c_2 x\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right) \\
& + c_1\sqrt{x}\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)+\sqrt{x}\left(-\frac{x^{11/2}}{396}+\frac{4x^{9/2}}{45}+\frac{x^{7/2}}{15}-\frac{2x^{5/2}}{5}\right. \\
& \left.-\frac{2x^{3/2}}{3}\right)\left(\frac{x^6}{11088}+\frac{x^4}{168}+\frac{x^2}{6}+1\right)+x\left(-\frac{x^6}{168}-\frac{13x^5}{12600}-\frac{x^4}{12}-\frac{x^3}{18}+\frac{x^2}{2}+x\right)\left(\frac{x^6}{28080}+\frac{x^4}{360}+\frac{x^2}{10}+1\right)
\end{aligned}
$$

2.4.28 problem 24

Internal problem ID [8593]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 24
Date solved : `Thursday, December 12, 2024 at 09:31:43 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$2x^2y'' - xy' + \left(-x^2+1\right)y = \ln\left(x\right)$$

Using series expansion around $x = 1$

The ode does not have its expansion point at $x = 0$, therefore to simplify the computation of power series expansion, change of variable is made on the independent variable to shift the initial conditions and the expasion point back to zero. The new ode is then solved more easily since the expansion point is now at zero. The solution converted back to the original independent variable. Let

$$\tau = x - 1$$

The ode is converted to be in terms of the new independent variable τ. This results in

$$2(\tau+1)^2\left(\frac{d^2}{d\tau^2}y(\tau)\right) - (\tau+1)\left(\frac{d}{d\tau}y(\tau)\right) + \left(-(\tau+1)^2+1\right)y(\tau) = \ln\left(\tau+1\right)$$

With its expansion point and initial conditions now at $\tau = 0$. The transformed ODE is now solved. Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$
\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y'_0 + \frac{x^2}{2} f|_{x_0, y_0, y'_0} + \frac{x^3}{3!} f'|_{x_0, y_0, y'_0} + \cdots \\
&= y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left. \frac{d^n f}{dx^n} \right|_{x_0, y_0, y'_0}
\end{aligned}
$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' && (2.7) \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (2.8) \\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right) y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right) f && (2) \\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right) y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right) f && (3) \\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$\begin{align*}
F_0 &= f(x,y,y') \tag{4}\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \tag{6}
\end{align*}$$

Therefore (6) can be used from now on along with

$$y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \frac{y(\tau)\,\tau^2 + \left(\frac{d}{d\tau}y(\tau)\right)\tau + 2y(\tau)\,\tau + \ln(\tau+1) + \frac{d}{d\tau}y(\tau)}{2(\tau+1)^2} \\
F_1 &= \frac{dF_0}{d\tau} \\
&= \frac{\partial F_0}{\partial \tau} + \frac{\partial F_0}{\partial y}\frac{d}{d\tau}y(\tau) + \frac{\partial F_0}{\partial \frac{d}{d\tau}y(\tau)}F_0 \\
&= \frac{-3\ln(\tau+1) + (2\tau^3+6\tau^2+3\tau-1)\left(\frac{d}{d\tau}y(\tau)\right) + 2 + (\tau^2+2\tau+4)\,y(\tau)}{4(\tau+1)^3} \\
F_2 &= \frac{dF_1}{d\tau} \\
&= \frac{\partial F_1}{\partial \tau} + \frac{\partial F_1}{\partial y}\frac{d}{d\tau}y(\tau) + \frac{\partial F_1}{\partial \frac{d}{d\tau}y(\tau)}F_1 \\
&= \frac{(2\tau^2+4\tau+17)\ln(\tau+1) + (4\tau^3+12\tau^2+27\tau+19)\left(\frac{d}{d\tau}y(\tau)\right) - 18 + (2\tau^4+8\tau^3+5\tau^2-6\tau-20)}{8(\tau+1)^4} \\
F_3 &= \frac{dF_2}{d\tau} \\
&= \frac{\partial F_2}{\partial \tau} + \frac{\partial F_2}{\partial y}\frac{d}{d\tau}y(\tau) + \frac{\partial F_2}{\partial \frac{d}{d\tau}y(\tau)}F_2 \\
&= \frac{(-4\tau^2-8\tau-109)\ln(\tau+1) + (4\tau^5+20\tau^4+22\tau^3-14\tau^2-139\tau-119)\left(\frac{d}{d\tau}y(\tau)\right) + (4\tau^4+16\tau^3+}{16(\tau+1)^5} \\
F_4 &= \frac{dF_3}{d\tau} \\
&= \frac{\partial F_3}{\partial \tau} + \frac{\partial F_3}{\partial y}\frac{d}{d\tau}y(\tau) + \frac{\partial F_3}{\partial \frac{d}{d\tau}y(\tau)}F_3 \\
&= \frac{(4\tau^4+16\tau^3+30\tau^2+28\tau+955)\ln(\tau+1) + (12\tau^5+60\tau^4+252\tau^3+516\tau^2+1401\tau+1089)\left(\frac{d}{d\tau}y(\right.}{32(\tau+1)^6}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $\tau = 0$ and $y(0) = y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= \frac{y'(0)}{2} \\
F_1 &= y(0) - \frac{y'(0)}{4} + \frac{1}{2} \\
F_2 &= -\frac{5y(0)}{2} + \frac{19y'(0)}{8} - \frac{9}{4} \\
F_3 &= \frac{37y(0)}{4} - \frac{119y'(0)}{16} + \frac{89}{8} \\
F_4 &= -\frac{323y(0)}{8} + \frac{1089y'(0)}{32} - \frac{991}{16}
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y(\tau) = &\left(1 + \frac{1}{6}\tau^3 - \frac{5}{48}\tau^4 + \frac{37}{480}\tau^5\right) y(0) \\
&+ \left(\tau + \frac{1}{4}\tau^2 - \frac{1}{24}\tau^3 + \frac{19}{192}\tau^4 - \frac{119}{1920}\tau^5\right) y'(0) + \frac{\tau^3}{12} - \frac{3\tau^4}{32} + \frac{89\tau^5}{960} + O(\tau^6)
\end{aligned}
$$

Since the expansion point $\tau = 0$ is an ordinary point, then this can also be solved using the standard power series method. The ode is normalized to be

$$
(2\tau^2 + 4\tau + 2)\left(\frac{d^2}{d\tau^2}y(\tau)\right) + (-\tau - 1)\left(\frac{d}{d\tau}y(\tau)\right) + (-\tau^2 - 2\tau)\, y(\tau) = \ln(\tau + 1)
$$

Let the solution be represented as power series of the form

$$
y(\tau) = \sum_{n=0}^{\infty} a_n \tau^n
$$

Then

$$
\begin{aligned}
\frac{d}{d\tau}y(\tau) &= \sum_{n=1}^{\infty} n a_n \tau^{n-1} \\
\frac{d^2}{d\tau^2}y(\tau) &= \sum_{n=2}^{\infty} n(n-1)\, a_n \tau^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
(2\tau^2 + 4\tau + 2)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n \tau^{n-2}\right) + (-\tau - 1)\left(\sum_{n=1}^{\infty} n a_n \tau^{n-1}\right) + (-\tau^2 - 2\tau)\left(\sum_{n=0}^{\infty} a_n \tau^n\right) = \ln(\tau + \tag{1}
$$

Expanding $\ln(\tau+1)$ as Taylor series around $\tau = 0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
\ln(\tau+1) &= \tau - \frac{1}{2}\tau^2 + \frac{1}{3}\tau^3 - \frac{1}{4}\tau^4 + \frac{1}{5}\tau^5 + \dots \\
&= \tau - \frac{1}{2}\tau^2 + \frac{1}{3}\tau^3 - \frac{1}{4}\tau^4 + \frac{1}{5}\tau^5
\end{aligned}
$$

Hence the ODE in Eq (1) becomes

$$
\begin{aligned}
&(2\tau^2 + 4\tau + 2)\left(\sum_{n=2}^{\infty} n(n-1)\, a_n \tau^{n-2}\right) + (-\tau - 1)\left(\sum_{n=1}^{\infty} n a_n \tau^{n-1}\right) \\
&+ (-\tau^2 - 2\tau)\left(\sum_{n=0}^{\infty} a_n \tau^n\right) = \tau - \frac{1}{2}\tau^2 + \frac{1}{3}\tau^3 - \frac{1}{4}\tau^4 + \frac{1}{5}\tau^5
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} 2\tau^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} 4n a_n \tau^{n-1}(n-1)\right) + \left(\sum_{n=2}^{\infty} 2n(n-1)\, a_n \tau^{n-2}\right) \\
&+ \sum_{n=1}^{\infty} (-n a_n \tau^n) + \sum_{n=1}^{\infty} (-n a_n \tau^{n-1}) + \sum_{n=0}^{\infty} (-a_n \tau^{n+2}) \\
&\quad + \sum_{n=0}^{\infty} (-2\tau^{1+n} a_n) = \tau - \frac{1}{2}\tau^2 + \frac{1}{3}\tau^3 - \frac{1}{4}\tau^4 + \frac{1}{5}\tau^5
\end{aligned}
\tag{2}
$$

The next step is to make all powers of τ be n in each summation term. Going over each summation term above with power of τ in it which is not already τ^n and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=2}^{\infty} 4n a_n \tau^{n-1}(n-1) &= \sum_{n=1}^{\infty} 4(1+n)\, a_{1+n} n\, \tau^n \\
\sum_{n=2}^{\infty} 2n(n-1)\, a_n \tau^{n-2} &= \sum_{n=0}^{\infty} 2(n+2)\, a_{n+2}(1+n)\, \tau^n \\
\sum_{n=1}^{\infty} (-n a_n \tau^{n-1}) &= \sum_{n=0}^{\infty} (-(1+n)\, a_{1+n} \tau^n) \\
\sum_{n=0}^{\infty} (-a_n \tau^{n+2}) &= \sum_{n=2}^{\infty} (-a_{n-2} \tau^n)
\end{aligned}
$$

$$\sum_{n=0}^{\infty}\left(-2\tau^{1+n} a_n\right)=\sum_{n=1}^{\infty}\left(-2a_{n-1}\tau^n\right)$$

Substituting all the above in Eq (2) gives the following equation where now all powers of τ are the same and equal to n.

$$\begin{aligned}
&\left(\sum_{n=2}^{\infty} 2\tau^n a_n n(n-1)\right)+\left(\sum_{n=1}^{\infty} 4(1+n)\, a_{1+n} n\, \tau^n\right) \\
&+\left(\sum_{n=0}^{\infty} 2(n+2)\, a_{n+2}(1+n)\, \tau^n\right)+\sum_{n=1}^{\infty}\left(-n a_n \tau^n\right)+\sum_{n=0}^{\infty}\left(-(1+n)\, a_{1+n}\tau^n\right) \\
&+\sum_{n=2}^{\infty}\left(-a_{n-2}\tau^n\right)+\sum_{n=1}^{\infty}\left(-2a_{n-1}\tau^n\right)=\tau-\frac{1}{2}\tau^2+\frac{1}{3}\tau^3-\frac{1}{4}\tau^4+\frac{1}{5}\tau^5
\end{aligned} \tag{3}$$

$n=0$ gives

$$4a_2-a_1=0$$

$$a_2=\frac{a_1}{4}$$

$n=1$ gives

$$\begin{aligned}
(6a_2+12a_3-a_1-2a_0)\,\tau&=\tau \\
6a_2+12a_3-a_1-2a_0&=1
\end{aligned}$$

Which after substituting earlier equations, simplifies to

$$a_3=\frac{a_0}{6}-\frac{a_1}{24}+\frac{1}{12}$$

For $2\leq n$, the recurrence equation is

$$\begin{aligned}
&\left(2n a_n(n-1)+4(1+n)\, a_{1+n} n+2(n+2)\, a_{n+2}(1+n)-n a_n\right. \\
&\left.-(1+n)\, a_{1+n}-a_{n-2}-2a_{n-1}\right)\tau^n=\tau-\frac{1}{2}\tau^2+\frac{1}{3}\tau^3-\frac{1}{4}\tau^4+\frac{1}{5}\tau^5
\end{aligned} \tag{4}$$

For $n=2$ the recurrence equation gives

$$\begin{aligned}
(2a_2+21a_3+24a_4-a_0-2a_1)\,\tau^2&=-\frac{\tau^2}{2} \\
2a_2+21a_3+24a_4-a_0-2a_1&=-\frac{1}{2}
\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{3}{32} + \frac{19a_1}{192} - \frac{5a_0}{48}$$

For $n = 3$ the recurrence equation gives

$$(9a_3 + 44a_4 + 40a_5 - a_1 - 2a_2)\,\tau^3 = \frac{\tau^3}{3}$$
$$9a_3 + 44a_4 + 40a_5 - a_1 - 2a_2 = \frac{1}{3}$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{89}{960} + \frac{37a_0}{480} - \frac{119a_1}{1920}$$

For $n = 4$ the recurrence equation gives

$$(20a_4 + 75a_5 + 60a_6 - a_2 - 2a_3)\,\tau^4 = -\frac{\tau^4}{4}$$
$$20a_4 + 75a_5 + 60a_6 - a_2 - 2a_3 = -\frac{1}{4}$$

Which after substituting the earlier terms found becomes

$$a_6 = -\frac{991}{11520} + \frac{121a_1}{2560} - \frac{323a_0}{5760}$$

For $n = 5$ the recurrence equation gives

$$(35a_5 + 114a_6 + 84a_7 - a_3 - 2a_4)\,\tau^5 = \frac{\tau^5}{5}$$
$$35a_5 + 114a_6 + 84a_7 - a_3 - 2a_4 = \frac{1}{5}$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{4261}{53760} + \frac{167a_0}{3840} - \frac{11761a_1}{322560}$$

And so on. Therefore the solution is

$$
\begin{aligned}
y(\tau) &= \sum_{n=0}^{\infty} a_n \tau^n \\
&= a_3\tau^3 + a_2\tau^2 + a_1\tau + a_0 + \dots
\end{aligned}
$$

Substituting the values for a_n found above, the solution becomes

$$
\begin{aligned}
y(\tau) = a_0 + a_1\tau + \frac{a_1\tau^2}{4} + \left(\frac{a_0}{6} - \frac{a_1}{24} + \frac{1}{12}\right)\tau^3 \\
+ \left(-\frac{3}{32} + \frac{19a_1}{192} - \frac{5a_0}{48}\right)\tau^4 + \left(\frac{89}{960} + \frac{37a_0}{480} - \frac{119a_1}{1920}\right)\tau^5 + \dots
\end{aligned}
$$

Collecting terms, the solution becomes

$$
\begin{aligned}
y(\tau) = \left(1 + \frac{1}{6}\tau^3 - \frac{5}{48}\tau^4 + \frac{37}{480}\tau^5\right) a_0 \\
+ \left(\tau + \frac{1}{4}\tau^2 - \frac{1}{24}\tau^3 + \frac{19}{192}\tau^4 - \frac{119}{1920}\tau^5\right) a_1 + \frac{\tau^3}{12} - \frac{3\tau^4}{32} + \frac{89\tau^5}{960} + O(\tau^6)
\end{aligned} \tag{3}
$$

At $\tau = 0$ the solution above becomes

$$
\begin{aligned}
y(\tau) = \left(1 + \frac{1}{6}\tau^3 - \frac{5}{48}\tau^4 + \frac{37}{480}\tau^5\right) c_1 + \left(\tau + \frac{1}{4}\tau^2 - \frac{1}{24}\tau^3 + \frac{19}{192}\tau^4 - \frac{119}{1920}\tau^5\right) c_2 \\
+ \frac{\tau^3}{12} - \frac{3\tau^4}{32} + \frac{89\tau^5}{960} + O(\tau^6)
\end{aligned}
$$

Replacing τ in the above with the original independent variable $x$using $\tau = x - 1$ results in

$$
\begin{aligned}
y = \left(1 + \frac{(x-1)^3}{6} - \frac{5(x-1)^4}{48} + \frac{37(x-1)^5}{480}\right) y(1) \\
+ \left(x - 1 + \frac{(x-1)^2}{4} - \frac{(x-1)^3}{24} + \frac{19(x-1)^4}{192} - \frac{119(x-1)^5}{1920}\right) y'(1) \\
+ \frac{(x-1)^3}{12} - \frac{3(x-1)^4}{32} + \frac{89(x-1)^5}{960} + O\left((x-1)^6\right)
\end{aligned}
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 90

```
dsolve(2*x^2*diff(diff(y(x),x),x)-diff(y(x),x)*x+(-x^2+1)*y(x) = ln(x),y(x),
       series,x=1)
```

$$
\begin{aligned}
y &= \left(1+\frac{(x-1)^{3}}{6}-\frac{5(x-1)^{4}}{48}+\frac{37(x-1)^{5}}{480}\right) y(1) \\
&\quad + \left(x-1+\frac{(x-1)^{2}}{4}-\frac{(x-1)^{3}}{24}+\frac{19(x-1)^{4}}{192}-\frac{119(x-1)^{5}}{1920}\right) y'(1) \\
&\quad + \frac{(x-1)^{3}}{12}-\frac{3(x-1)^{4}}{32}+\frac{89(x-1)^{5}}{960}+O\left(x^{6}\right)
\end{aligned}
$$

Mathematica DSolve solution

Solving time : 0.024 (sec)
Leaf size : 105

```
AsymptoticDSolveValue[{2*x^2*D[y[x],{x,2}]-x*D[y[x],x]+(1-x^2)*y[x]==Log[x],{}},
       y[x],{x,1,5}]
```

$$
\begin{aligned}
y(x) \rightarrow & \frac{89}{960}(x-1)^5-\frac{3}{32}(x-1)^4+\frac{1}{12}(x-1)^3 \\
& +c_1\left(\frac{37}{480}(x-1)^5-\frac{5}{48}(x-1)^4+\frac{1}{6}(x-1)^3+1\right) \\
& +c_2\left(-\frac{119(x-1)^5}{1920}+\frac{19}{192}(x-1)^4-\frac{1}{24}(x-1)^3+\frac{1}{4}(x-1)^2+x-1\right)
\end{aligned}
$$

2.4.29 problem 25

Internal problem ID [8594]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 25
Date solved : `Thursday, December 12, 2024 at 09:31:44 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2\left(x^2+x+1\right)y''+x\left(11x^2+11x+9\right)y'+\left(7x^2+10x+6\right)y=0$$

Using series expansion around $x=0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(2x^4+2x^3+2x^2\right)y''+\left(11x^3+11x^2+9x\right)y'+\left(7x^2+10x+6\right)y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{11x^2+11x+9}{2x\left(x^2+x+1\right)}$$
$$q(x)=\frac{7x^2+10x+6}{2x^2\left(x^2+x+1\right)}$$

Table 2.187: Table $p(x),q(x)$ singularites.

$p(x)=\frac{11x^2+11x+9}{2x(x^2+x+1)}$	
singularity	type
$x=0$	"regular"
$x=-\frac{1}{2}-\frac{i\sqrt{3}}{2}$	"regular"
$x=-\frac{1}{2}+\frac{i\sqrt{3}}{2}$	"regular"

$q(x)=\frac{7x^2+10x+6}{2x^2(x^2+x+1)}$	
singularity	type
$x=0$	"regular"
$x=-\frac{1}{2}-\frac{i\sqrt{3}}{2}$	"regular"
$x=-\frac{1}{2}+\frac{i\sqrt{3}}{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, -\frac{1}{2}-\frac{i\sqrt{3}}{2}, -\frac{1}{2}+\frac{i\sqrt{3}}{2}, \infty\right]$

Irregular singular points : $[]$

Since $x=0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2\left(x^2+x+1\right)y''+\left(11x^3+11x^2+9x\right)y'+\left(7x^2+10x+6\right)y=0$$

Let the solution be represented as Frobenius power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y'=\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}$$
$$y''=\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned}&2x^2\left(x^2+x+1\right)\left(\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r-2}\right)\\&+\left(11x^3+11x^2+9x\right)\left(\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\right)+\left(7x^2+10x+6\right)\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)=0\end{aligned}\tag{1}$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2x^{n+r+2}a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 2x^{1+n+r}a_n(n+r)\,(n+r-1)\right)\\
&+\left(\sum_{n=0}^{\infty} 2x^{n+r}a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 11x^{n+r+2}a_n(n+r)\right)\\
&+\left(\sum_{n=0}^{\infty} 11x^{1+n+r}a_n(n+r)\right)+\left(\sum_{n=0}^{\infty} 9x^{n+r}a_n(n+r)\right)\\
&+\left(\sum_{n=0}^{\infty} 7x^{n+r+2}a_n\right)+\left(\sum_{n=0}^{\infty} 10x^{1+n+r}a_n\right)+\left(\sum_{n=0}^{\infty} 6a_nx^{n+r}\right)=0
\end{aligned}\tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\begin{aligned}
\sum_{n=0}^{\infty} 2x^{n+r+2}a_n(n+r)\,(n+r-1) &= \sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)\,(n-3+r)\,x^{n+r}\\
\sum_{n=0}^{\infty} 2x^{1+n+r}a_n(n+r)\,(n+r-1) &= \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\,(n+r-2)\,x^{n+r}\\
\sum_{n=0}^{\infty} 11x^{n+r+2}a_n(n+r) &= \sum_{n=2}^{\infty} 11a_{n-2}(n+r-2)\,x^{n+r}\\
\sum_{n=0}^{\infty} 11x^{1+n+r}a_n(n+r) &= \sum_{n=1}^{\infty} 11a_{n-1}(n+r-1)\,x^{n+r}\\
\sum_{n=0}^{\infty} 7x^{n+r+2}a_n &= \sum_{n=2}^{\infty} 7a_{n-2}x^{n+r}\\
\sum_{n=0}^{\infty} 10x^{1+n+r}a_n &= \sum_{n=1}^{\infty} 10a_{n-1}x^{n+r}
\end{aligned}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers

of x are the same and equal to $n + r$.

$$
\begin{aligned}
&\left(\sum_{n=2}^{\infty} 2a_{n-2}(n+r-2)\,(n-3+r)\,x^{n+r}\right) \\
&+\left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)\,(n+r-2)\,x^{n+r}\right) \\
&+\left(\sum_{n=0}^{\infty} 2x^{n+r}a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=2}^{\infty} 11a_{n-2}(n+r-2)\,x^{n+r}\right) \\
&+\left(\sum_{n=1}^{\infty} 11a_{n-1}(n+r-1)\,x^{n+r}\right)+\left(\sum_{n=0}^{\infty} 9x^{n+r}a_n(n+r)\right) \\
&+\left(\sum_{n=2}^{\infty} 7a_{n-2}x^{n+r}\right)+\left(\sum_{n=1}^{\infty} 10a_{n-1}x^{n+r}\right)+\left(\sum_{n=0}^{\infty} 6a_nx^{n+r}\right)=0
\end{aligned}
\tag{2B}
$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$2x^{n+r}a_n(n+r)\,(n+r-1)+9x^{n+r}a_n(n+r)+6a_nx^{n+r}=0$$

When $n = 0$ the above becomes

$$2x^r a_0 r(-1+r)+9x^r a_0 r+6a_0x^r=0$$

Or

$$\left(2x^r r(-1+r)+9x^r r+6x^r\right)a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2+7r+6\right)x^r=0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2+7r+6=0$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= -\frac{3}{2} \\
r_2 &= -2
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2+7r+6\right)x^r=0$$

Solving for r gives the roots of the indicial equation as $\left[-\frac{3}{2}, -2\right]$.

Since $r_1 - r_2 = \frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-\frac{3}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-2}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{-r-2}{r+3}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &2a_{n-2}(n+r-2)(n-3+r)+2a_{n-1}(n+r-1)(n+r-2)+2a_n(n+r)(n+r-1) \\ &+11a_{n-2}(n+r-2)+11a_{n-1}(n+r-1)+9a_n(n+r)+7a_{n-2}+10a_{n-1}+6a_n=0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{na_{n-2} + na_{n-1} + ra_{n-2} + ra_{n-1} - a_{n-2} + a_{n-1}}{n+r+2} \tag{4}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$a_n = \frac{(-2a_{n-2} - 2a_{n-1})\, n + 5a_{n-2} + a_{n-1}}{2n+1} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -\frac{3}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r-2}{r+3}$	$-\frac{1}{3}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{4+r}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$a_2 = \frac{2}{5}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r-2}{r+3}$	$-\frac{1}{3}$
a_2	$\frac{1}{4+r}$	$\frac{2}{5}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{r^2+3r+1}{(r+3)(5+r)}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$a_3 = -\frac{5}{21}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r-2}{r+3}$	$-\frac{1}{3}$
a_2	$\frac{1}{4+r}$	$\frac{2}{5}$
a_3	$\frac{r^2+3r+1}{(r+3)(5+r)}$	$-\frac{5}{21}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{-r^3-8r^2-19r-13}{(6+r)(r+3)(4+r)}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$a_4 = \frac{7}{135}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r-2}{r+3}$	$-\frac{1}{3}$
a_2	$\frac{1}{4+r}$	$\frac{2}{5}$
a_3	$\frac{r^2+3r+1}{(r+3)(5+r)}$	$-\frac{5}{21}$
a_4	$\frac{-r^3-8r^2-19r-13}{(6+r)(r+3)(4+r)}$	$\frac{7}{135}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{2r^3 + 18r^2 + 52r + 49}{(r+3)(4+r)(5+r)(7+r)}$$

Which for the root $r = -\frac{3}{2}$ becomes

$$a_5 = \frac{76}{1155}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-r-2}{r+3}$	$-\frac{1}{3}$
a_2	$\frac{1}{4+r}$	$\frac{2}{5}$
a_3	$\frac{r^2+3r+1}{(r+3)(5+r)}$	$-\frac{5}{21}$
a_4	$\frac{-r^3-8r^2-19r-13}{(6+r)(r+3)(4+r)}$	$\frac{7}{135}$
a_5	$\frac{2r^3+18r^2+52r+49}{(r+3)(4+r)(5+r)(7+r)}$	$\frac{76}{1155}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \frac{1}{x^{3/2}}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= \frac{1 - \frac{x}{3} + \frac{2x^2}{5} - \frac{5x^3}{21} + \frac{7x^4}{135} + \frac{76x^5}{1155} + O(x^6)}{x^{3/2}} \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = \frac{-r-2}{r+3}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned}2b_{n-2}(n+r-2)(n-3+r)+2b_{n-1}(n+r-1)(n+r-2)+2b_n(n+r)(n+r-1) \\ +11b_{n-2}(n+r-2)+11b_{n-1}(n+r-1)+9b_n(n+r)+7b_{n-2}+10b_{n-1}+6b_n=0\end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{nb_{n-2}+nb_{n-1}+rb_{n-2}+rb_{n-1}-b_{n-2}+b_{n-1}}{n+r+2} \tag{4}$$

Which for the root $r = -2$ becomes

$$b_n = \frac{(-b_{n-2}-b_{n-1})\,n+3b_{n-2}+b_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -2$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r-2}{r+3}$	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{4+r}$$

Which for the root $r = -2$ becomes

$$b_2 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r-2}{r+3}$	0
b_2	$\frac{1}{4+r}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{r^2 + 3r + 1}{(r+3)(5+r)}$$

Which for the root $r = -2$ becomes

$$b_3 = -\frac{1}{3}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r-2}{r+3}$	0
b_2	$\frac{1}{4+r}$	$\frac{1}{2}$
b_3	$\frac{r^2+3r+1}{(r+3)(5+r)}$	$-\frac{1}{3}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{-r^3 - 8r^2 - 19r - 13}{(6+r)(r+3)(4+r)}$$

Which for the root $r = -2$ becomes

$$b_4 = \frac{1}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r-2}{r+3}$	0
b_2	$\frac{1}{4+r}$	$\frac{1}{2}$
b_3	$\frac{r^2+3r+1}{(r+3)(5+r)}$	$-\frac{1}{3}$
b_4	$\frac{-r^3-8r^2-19r-13}{(6+r)(r+3)(4+r)}$	$\frac{1}{8}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{2r^3 + 18r^2 + 52r + 49}{(r+3)(4+r)(5+r)(7+r)}$$

Which for the root $r = -2$ becomes

$$b_5 = \frac{1}{30}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-r-2}{r+3}$	0
b_2	$\frac{1}{4+r}$	$\frac{1}{2}$
b_3	$\frac{r^2+3r+1}{(r+3)(5+r)}$	$-\frac{1}{3}$
b_4	$\frac{-r^3-8r^2-19r-13}{(6+r)(r+3)(4+r)}$	$\frac{1}{8}$
b_5	$\frac{2r^3+18r^2+52r+49}{(r+3)(4+r)(5+r)(7+r)}$	$\frac{1}{30}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= \frac{1}{x^{3/2}}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{30} + O(x^6)}{x^2} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= \frac{c_1\left(1 - \frac{x}{3} + \frac{2x^2}{5} - \frac{5x^3}{21} + \frac{7x^4}{135} + \frac{76x^5}{1155} + O(x^6)\right)}{x^{3/2}} + \frac{c_2\left(1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{30} + O(x^6)\right)}{x^2} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= \frac{c_1\left(1 - \frac{x}{3} + \frac{2x^2}{5} - \frac{5x^3}{21} + \frac{7x^4}{135} + \frac{76x^5}{1155} + O(x^6)\right)}{x^{3/2}} + \frac{c_2\left(1 + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{30} + O(x^6)\right)}{x^2} \end{aligned}$$

Maple step by step solution

Let's solve

$2x^2(x^2+x+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(11x^2+11x+9)\left(\frac{d}{dx}y(x)\right)+(7x^2+10x+6)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(7x^2+10x+6)y(x)}{2x^2(x^2+x+1)}-\frac{(11x^2+11x+9)\left(\frac{d}{dx}y(x)\right)}{2x(x^2+x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{(11x^2+11x+9)\left(\frac{d}{dx}y(x)\right)}{2x(x^2+x+1)}+\frac{(7x^2+10x+6)y(x)}{2x^2(x^2+x+1)}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

  $\left[P_2(x)=\frac{11x^2+11x+9}{2x(x^2+x+1)},P_3(x)=\frac{7x^2+10x+6}{2x^2(x^2+x+1)}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=\frac{9}{2}$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=3$

- ○ $x=0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=0$

- Multiply by denominators

  $2x^2(x^2+x+1)\left(\frac{d^2}{dx^2}y(x)\right)+x(11x^2+11x+9)\left(\frac{d}{dx}y(x)\right)+(7x^2+10x+6)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ○ Shift index using $k->k-m$

$$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- > k+1-m$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- > k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(2+r)\,(3+2r)\, x^r + (a_1(3+r)\,(5+2r) + a_0(5+2r)\,(2+r))\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(k+r+2)\,(2k+$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
$$(2+r)\,(3+2r) = 0$$
- Values of r that satisfy the indicial equation
$$r \in \left\{-2, -\frac{3}{2}\right\}$$
- Each term must be 0
$$a_1(3+r)\,(5+2r) + a_0(5+2r)\,(2+r) = 0$$
- Solve for the dependent coefficient(s)
$$a_1 = -\frac{(2+r)a_0}{3+r}$$
- Each term in the series must be 0, giving the recursion relation
$$2\left(k+r+\frac{3}{2}\right)\left((a_k + a_{k-2} + a_{k-1})\, k + (a_k + a_{k-2} + a_{k-1})\, r + 2a_k - a_{k-2} + a_{k-1}\right) = 0$$
- Shift index using $k- > k+2$
$$2\left(k+\frac{7}{2}+r\right)\left((a_{k+2} + a_k + a_{k+1})\,(k+2) + (a_{k+2} + a_k + a_{k+1})\, r + 2a_{k+2} - a_k + a_{k+1}\right) = 0$$
- Recursion relation that defines series solution to ODE
$$a_{k+2} = -\frac{ka_k + ka_{k+1} + ra_k + ra_{k+1} + a_k + 3a_{k+1}}{k+4+r}$$
- Recursion relation for $r = -2$
$$a_{k+2} = -\frac{ka_k + ka_{k+1} - a_k + a_{k+1}}{k+2}$$
- Solution for $r = -2$

$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2}=-\frac{k a_k+k a_{k+1}-a_k+a_{k+1}}{k+2}, a_1=0\right]$$

- Recursion relation for $r=-\frac{3}{2}$

$$a_{k+2}=-\frac{k a_k+k a_{k+1}-\frac{1}{2}a_k+\frac{3}{2}a_{k+1}}{k+\frac{5}{2}}$$

- Solution for $r=-\frac{3}{2}$

$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k-\frac{3}{2}}, a_{k+2}=-\frac{k a_k+k a_{k+1}-\frac{1}{2}a_k+\frac{3}{2}a_{k+1}}{k+\frac{5}{2}}, a_1=-\frac{a_0}{3}\right]$$

- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k-\frac{3}{2}}\right), a_{k+2}=-\frac{k a_k+k a_{k+1}-a_k+a_{k+1}}{k+2}, a_1=0, b_{k+2}=-\frac{k b_k+k b_{k+1}-\frac{1}{2}}{k+\frac{5}{2}}\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
      -> Mathieu
         -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @
```

```
    <- Heun successful: received ODE is equivalent to the  HeunG  ODE, case  a <> 0,
  <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.038 (sec)
Leaf size : 45

```
dsolve(2*x^2*(x^2+x+1)*diff(diff(y(x),x),x)+x*(11*x^2+11*x+9)*diff(y(x),x)+(7*x^2+10*x
       series,x=0)
```

$$y = \frac{c_1\left(1+\frac{1}{2}x^2-\frac{1}{3}x^3+\frac{1}{8}x^4+\frac{1}{30}x^5+\mathrm{O}\left(x^6\right)\right)}{x^2} + \frac{c_2\left(1-\frac{1}{3}x+\frac{2}{5}x^2-\frac{5}{21}x^3+\frac{7}{135}x^4+\frac{76}{1155}x^5+\mathrm{O}\left(x^6\right)\right)}{x^{3/2}}$$

Mathematica DSolve solution

Solving time : 0.027 (sec)
Leaf size : 83

```
AsymptoticDSolveValue[{2*x^2*(1+x+x^2)*D[y[x],{x,2}] + x*(9+11*x+11*x^2)*D[y[x],x] + (6+10*x
       y[x],{x,0,5}]
```

$$y(x) \to \frac{c_2\left(\frac{x^5}{30}+\frac{x^4}{8}-\frac{x^3}{3}+\frac{x^2}{2}+1\right)}{x^2} + \frac{c_1\left(\frac{76x^5}{1155}+\frac{7x^4}{135}-\frac{5x^3}{21}+\frac{2x^2}{5}-\frac{x}{3}+1\right)}{x^{3/2}}$$

2.4.30 problem 26

Internal problem ID [8595]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 26
Date solved : `Thursday, December 12, 2024 at 09:31:46 AM`
CAS classification : `[[_2nd_order, _exact, _linear, _homogeneous]]`

Solve

$$x^2(3+x)\,y'' + 5x(1+x)\,y' - (1-4x)\,y = 0$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^3+3x^2\right)y'' + \left(5x^2+5x\right)y' + (4x-1)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{5+5x}{x\,(3+x)}$$
$$q(x) = \frac{4x-1}{x^2\,(3+x)}$$

Table 2.189: Table $p(x), q(x)$ singularites.

$p(x) = \frac{5+5x}{x(3+x)}$	
singularity	type
$x = -3$	"regular"
$x = 0$	"regular"

$q(x) = \frac{4x-1}{x^2(3+x)}$	
singularity	type
$x = -3$	"regular"
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-3, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(3+x)\, y'' + \left(5x^2+5x\right) y' + (4x-1)\, y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &x^2(3+x)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\ &+ \left(5x^2+5x\right)\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + (4x-1)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\begin{aligned} &\left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\,(n+r-1)\right) \\ &+ \left(\sum_{n=0}^{\infty} 5x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r} a_n (n+r)\right) \\ &+ \left(\sum_{n=0}^{\infty} 4x^{1+n+r} a_n\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n(n+r)(n+r-1) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 5x^{1+n+r} a_n(n+r) = \sum_{n=1}^{\infty} 5a_{n-1}(n+r-1)\, x^{n+r}$$

$$\sum_{n=0}^{\infty} 4x^{1+n+r} a_n = \sum_{n=1}^{\infty} 4a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\begin{aligned} &\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r}\right) \\ &+ \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 5a_{n-1}(n+r-1)\, x^{n+r}\right) \\ &+ \left(\sum_{n=0}^{\infty} 5x^{n+r} a_n(n+r)\right) + \left(\sum_{n=1}^{\infty} 4a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$3x^{n+r} a_n(n+r)(n+r-1) + 5x^{n+r} a_n(n+r) - a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$3x^r a_0 r(-1+r) + 5x^r a_0 r - a_0 x^r = 0$$

Or

$$(3x^r r(-1+r) + 5x^r r - x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(3r^2 + 2r - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$3r^2 + 2r - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{3}$$
$$r_2 = -1$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(3r^2 + 2r - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as $\left[\frac{1}{3}, -1\right]$.

Since $r_1 - r_2 = \frac{4}{3}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{3}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-1}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} a_{n-1}(n+r-1)(n+r-2) + 3a_n(n+r)(n+r-1) \\ + 5a_{n-1}(n+r-1) + 5a_n(n+r) + 4a_{n-1} - a_n = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(1+n+r)\, a_{n-1}}{3n+3r-1} \tag{4}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_n = -\frac{(4+3n)\, a_{n-1}}{9n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{3}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-2 - r}{2 + 3r}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_1 = -\frac{7}{9}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{2+3r}$	$-\frac{7}{9}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^2 + 5r + 6}{9r^2 + 21r + 10}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_2 = \frac{35}{81}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{2+3r}$	$-\frac{7}{9}$
a_2	$\frac{r^2+5r+6}{9r^2+21r+10}$	$\frac{35}{81}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-r^3 - 9r^2 - 26r - 24}{27r^3 + 135r^2 + 198r + 80}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_3 = -\frac{455}{2187}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{2+3r}$	$-\frac{7}{9}$
a_2	$\frac{r^2+5r+6}{9r^2+21r+10}$	$\frac{35}{81}$
a_3	$\frac{-r^3-9r^2-26r-24}{27r^3+135r^2+198r+80}$	$-\frac{455}{2187}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^4 + 14r^3 + 71r^2 + 154r + 120}{81r^4 + 702r^3 + 2079r^2 + 2418r + 880}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_4 = \frac{1820}{19683}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{2+3r}$	$-\frac{7}{9}$
a_2	$\frac{r^2+5r+6}{9r^2+21r+10}$	$\frac{35}{81}$
a_3	$\frac{-r^3-9r^2-26r-24}{27r^3+135r^2+198r+80}$	$-\frac{455}{2187}$
a_4	$\frac{r^4+14r^3+71r^2+154r+120}{81r^4+702r^3+2079r^2+2418r+880}$	$\frac{1820}{19683}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-r^5 - 20r^4 - 155r^3 - 580r^2 - 1044r - 720}{243r^5 + 3240r^4 + 16065r^3 + 36360r^2 + 36492r + 12320}$$

Which for the root $r = \frac{1}{3}$ becomes

$$a_5 = -\frac{6916}{177147}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{2+3r}$	$-\frac{7}{9}$
a_2	$\frac{r^2+5r+6}{9r^2+21r+10}$	$\frac{35}{81}$
a_3	$\frac{-r^3-9r^2-26r-24}{27r^3+135r^2+198r+80}$	$-\frac{455}{2187}$
a_4	$\frac{r^4+14r^3+71r^2+154r+120}{81r^4+702r^3+2079r^2+2418r+880}$	$\frac{1820}{19683}$
a_5	$\frac{-r^5-20r^4-155r^3-580r^2-1044r-720}{243r^5+3240r^4+16065r^3+36360r^2+36492r+12320}$	$-\frac{6916}{177147}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^{1/3}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^{1/3}\left(1 - \frac{7x}{9} + \frac{35x^2}{81} - \frac{455x^3}{2187} + \frac{1820x^4}{19683} - \frac{6916x^5}{177147} + O\left(x^6\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$\begin{aligned} b_{n-1}(n+r-1)(n+r-2) + 3b_n(n+r)(n+r-1) \\ + 5b_{n-1}(n+r-1) + 5b_n(n+r) + 4b_{n-1} - b_n = 0 \end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{(1+n+r)\, b_{n-1}}{3n+3r-1} \tag{4}$$

Which for the root $r = -1$ becomes

$$b_n = -\frac{n b_{n-1}}{3n-4} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{-2-r}{2+3r}$$

Which for the root $r = -1$ becomes

$$b_1 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2-r}{2+3r}$	1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{r^2+5r+6}{9r^2+21r+10}$$

Which for the root $r = -1$ becomes

$$b_2 = -1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2-r}{2+3r}$	1
b_2	$\frac{r^2+5r+6}{9r^2+21r+10}$	-1

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{-r^3-9r^2-26r-24}{27r^3+135r^2+198r+80}$$

Which for the root $r = -1$ becomes

$$b_3 = \frac{3}{5}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2-r}{2+3r}$	1
b_2	$\frac{r^2+5r+6}{9r^2+21r+10}$	-1
b_3	$\frac{-r^3-9r^2-26r-24}{27r^3+135r^2+198r+80}$	$\frac{3}{5}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r^4 + 14r^3 + 71r^2 + 154r + 120}{81r^4 + 702r^3 + 2079r^2 + 2418r + 880}$$

Which for the root $r = -1$ becomes

$$b_4 = -\frac{3}{10}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2-r}{2+3r}$	1
b_2	$\frac{r^2+5r+6}{9r^2+21r+10}$	-1
b_3	$\frac{-r^3-9r^2-26r-24}{27r^3+135r^2+198r+80}$	$\frac{3}{5}$
b_4	$\frac{r^4+14r^3+71r^2+154r+120}{81r^4+702r^3+2079r^2+2418r+880}$	$-\frac{3}{10}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{-r^5 - 20r^4 - 155r^3 - 580r^2 - 1044r - 720}{243r^5 + 3240r^4 + 16065r^3 + 36360r^2 + 36492r + 12320}$$

Which for the root $r = -1$ becomes

$$b_5 = \frac{3}{22}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{-2-r}{2+3r}$	1
b_2	$\frac{r^2+5r+6}{9r^2+21r+10}$	-1
b_3	$\frac{-r^3-9r^2-26r-24}{27r^3+135r^2+198r+80}$	$\frac{3}{5}$
b_4	$\frac{r^4+14r^3+71r^2+154r+120}{81r^4+702r^3+2079r^2+2418r+880}$	$-\frac{3}{10}$
b_5	$\frac{-r^5-20r^4-155r^3-580r^2-1044r-720}{243r^5+3240r^4+16065r^3+36360r^2+36492r+12320}$	$\frac{3}{22}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^{1/3}\left(b_0+b_1x+b_2x^2+b_3x^3+b_4x^4+b_5x^5+b_6x^6\dots\right) \\
&= \frac{1+x-x^2+\frac{3x^3}{5}-\frac{3x^4}{10}+\frac{3x^5}{22}+O(x^6)}{x}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x)+c_2y_2(x) \\
&= c_1\,x^{1/3}\left(1-\frac{7x}{9}+\frac{35x^2}{81}-\frac{455x^3}{2187}+\frac{1820x^4}{19683}-\frac{6916x^5}{177147}+O(x^6)\right) \\
&\quad+\frac{c_2\left(1+x-x^2+\frac{3x^3}{5}-\frac{3x^4}{10}+\frac{3x^5}{22}+O(x^6)\right)}{x}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\,x^{1/3}\left(1-\frac{7x}{9}+\frac{35x^2}{81}-\frac{455x^3}{2187}+\frac{1820x^4}{19683}-\frac{6916x^5}{177147}+O(x^6)\right) \\
&\quad+\frac{c_2\left(1+x-x^2+\frac{3x^3}{5}-\frac{3x^4}{10}+\frac{3x^5}{22}+O(x^6)\right)}{x}
\end{aligned}
$$

Maple step by step solution

Let's solve

$$x^2(x+3)\left(\frac{d^2}{dx^2}y(x)\right)+5x(x+1)\left(\frac{d}{dx}y(x)\right)-(1-4x)\,y(x)=0$$

- Highest derivative means the order of the ODE is 2

$$\frac{d^2}{dx^2}y(x)$$

- Isolate 2nd derivative

$$\frac{d^2}{dx^2}y(x)=-\frac{(4x-1)y(x)}{x^2(x+3)}-\frac{5(x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+3)}$$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$$\frac{d^2}{dx^2}y(x)+\frac{5(x+1)\left(\frac{d}{dx}y(x)\right)}{x(x+3)}+\frac{(4x-1)y(x)}{x^2(x+3)}=0$$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

$$\left[P_2(x)=\frac{5(x+1)}{x(x+3)},P_3(x)=\frac{4x-1}{x^2(x+3)}\right]$$

- ○ $(x+3)\cdot P_2(x)$ is analytic at $x=-3$

$$\left((x+3)\cdot P_2(x)\right)\bigg|_{x=-3}=\frac{10}{3}$$

- ○ $(x+3)^2\cdot P_3(x)$ is analytic at $x=-3$

$$\left((x+3)^2\cdot P_3(x)\right)\bigg|_{x=-3}=0$$

- ○ $x=-3$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=-3$

- Multiply by denominators

$$x^2(x+3)\left(\frac{d^2}{dx^2}y(x)\right)+5x(x+1)\left(\frac{d}{dx}y(x)\right)+(4x-1)\,y(x)=0$$

- Change variables using $x=u-3$ so that the regular singular point is at $u=0$

$$(u^3-6u^2+9u)\left(\frac{d^2}{du^2}y(u)\right)+(5u^2-25u+30)\left(\frac{d}{du}y(u)\right)+(4u-13)\,y(u)=0$$

- Assume series solution for $y(u)$

$$y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$$

□ Rewrite ODE with series expansions

- ○ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

$$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k- > k - m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du} y(u)\right)$ to series expansion for $m = 0..2$

$$u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, u^{k+r-1+m}$$

- Shift index using $k- > k + 1 - m$

$$u^m \cdot \left(\frac{d}{du} y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2} y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k- > k + 2 - m$

$$u^m \cdot \left(\frac{d^2}{du^2} y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$3a_0 r(7+3r)\, u^{-1+r} + (3a_1(1+r)\,(10+3r) - a_0(13+6r)\,(1+r))\, u^r + \left(\sum_{k=1}^{\infty} (3a_{k+1}(k+r+1)\,(3$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $3r(7+3r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \left\{0, -\frac{7}{3}\right\}$
- Each term must be 0
  $3a_1(1+r)\,(10+3r) - a_0(13+6r)\,(1+r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $-6\left(\left(a_k - \frac{a_{k-1}}{6} - \frac{3a_{k+1}}{2}\right) k + \left(a_k - \frac{a_{k-1}}{6} - \frac{3a_{k+1}}{2}\right) r + \frac{13a_k}{6} - \frac{a_{k-1}}{6} - 5a_{k+1}\right)(k+r+1) = 0$
- Shift index using $k- > k + 1$
  $-6\left(\left(a_{k+1} - \frac{a_k}{6} - \frac{3a_{k+2}}{2}\right)(k+1) + \left(a_{k+1} - \frac{a_k}{6} - \frac{3a_{k+2}}{2}\right) r + \frac{13a_{k+1}}{6} - \frac{a_k}{6} - 5a_{k+2}\right)(k+r+2) =$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{ka_k - 6ka_{k+1} + ra_k - 6ra_{k+1} + 2a_k - 19a_{k+1}}{3(3k+13+3r)}$
- Recursion relation for $r = 0$

$a_{k+2} = -\frac{k a_k - 6k a_{k+1} + 2a_k - 19a_{k+1}}{3(3k+13)}$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{k a_k - 6k a_{k+1} + 2a_k - 19a_{k+1}}{3(3k+13)}, 30a_1 - 13a_0 = 0\right]$$

- Revert the change of variables $u = x + 3$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k (x+3)^k, a_{k+2} = -\frac{k a_k - 6k a_{k+1} + 2a_k - 19a_{k+1}}{3(3k+13)}, 30a_1 - 13a_0 = 0\right]$$

- Recursion relation for $r = -\frac{7}{3}$

$a_{k+2} = -\frac{k a_k - 6k a_{k+1} - \frac{1}{3}a_k - 5a_{k+1}}{3(3k+6)}$

- Solution for $r = -\frac{7}{3}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{7}{3}}, a_{k+2} = -\frac{k a_k - 6k a_{k+1} - \frac{1}{3}a_k - 5a_{k+1}}{3(3k+6)}, -12a_1 - \frac{4a_0}{3} = 0\right]$$

- Revert the change of variables $u = x + 3$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k (x+3)^{k-\frac{7}{3}}, a_{k+2} = -\frac{k a_k - 6k a_{k+1} - \frac{1}{3}a_k - 5a_{k+1}}{3(3k+6)}, -12a_1 - \frac{4a_0}{3} = 0\right]$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k (x+3)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+3)^{k-\frac{7}{3}}\right), a_{k+2} = -\frac{k a_k - 6k a_{k+1} + 2a_k - 19a_{k+1}}{3(3k+13)}, 30a_1 - 13a_0 =$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
   One independent solution has integrals. Trying a hypergeometric solution free of in
   -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
   <- hyper3 successful: received ODE is equivalent to the 2F1 ODE
   -> Trying to convert hypergeometric functions to elementary form...
   <- elementary form for at least one hypergeometric solution is achieved - returning
<- linear_1 successful`
```

Maple dsolve solution

Solving time : 0.023 (sec)
Leaf size : 48

```
dsolve(x^2*(x+3)*diff(diff(y(x),x),x)+5*x*(x+1)*diff(y(x),x)-(1-4*x)*y(x) = 0,y(x),
       series,x=0)
```

$$y = \frac{c_2 x^{4/3}\left(1-\frac{7}{9}x+\frac{35}{81}x^2-\frac{455}{2187}x^3+\frac{1820}{19683}x^4-\frac{6916}{177147}x^5+\mathrm{O}\left(x^6\right)\right)+c_1\left(1+x-x^2+\frac{3}{5}x^3-\frac{3}{10}x^4+\frac{3}{22}x^5+\mathrm{O}\right.}{x}$$

Mathematica DSolve solution

Solving time : 0.018 (sec)
Leaf size : 82

```
AsymptoticDSolveValue[{x^2*(3+x)*D[y[x],{x,2}] + 5*x*(1+x)*D[y[x],x] - (1-4*x)*y[x] == 0,{}},
       y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt[3]{x}\left(-\frac{6916x^5}{177147}+\frac{1820x^4}{19683}-\frac{455x^3}{2187}+\frac{35x^2}{81}-\frac{7x}{9}+1\right) + \frac{c_2\left(\frac{3x^5}{22}-\frac{3x^4}{10}+\frac{3x^3}{5}-x^2+x+1\right)}{x}$$

2.4.31 problem 27

Internal problem ID [8596]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 27
Date solved : `Thursday, December 12, 2024 at 09:31:47 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(-x^2+2\right)y''-x\left(4x^2+3\right)y'+\left(-2x^2+2\right)y=0$$

Using series expansion around $x=0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(-x^4+2x^2\right)y''+\left(-4x^3-3x\right)y'+\left(-2x^2+2\right)y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{4x^2+3}{x\left(x^2-2\right)}$$
$$q(x)=\frac{2x^2-2}{x^2\left(x^2-2\right)}$$

Table 2.191: Table $p(x), q(x)$ singularites.

$p(x)=\frac{4x^2+3}{x(x^2-2)}$	
singularity	type
$x=0$	"regular"
$x=\sqrt{2}$	"regular"
$x=-\sqrt{2}$	"regular"

$q(x)=\frac{2x^2-2}{x^2(x^2-2)}$	
singularity	type
$x=0$	"regular"
$x=\sqrt{2}$	"regular"
$x=-\sqrt{2}$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $\left[0, \sqrt{2}, -\sqrt{2}, \infty\right]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$-x^2\left(x^2-2\right)y''+\left(-4x^3-3x\right)y'+\left(-2x^2+2\right)y=0$$

Let the solution be represented as Frobenius power series of the form

$$y=\sum_{n=0}^{\infty}a_n x^{n+r}$$

Then

$$y'=\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}$$
$$y''=\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$-x^2\left(x^2-2\right)\left(\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + \left(-4x^3-3x\right)\left(\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\right)+\left(-2x^2+2\right)\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)=0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty}\left(-x^{n+r+2}a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty}2x^{n+r}a_n(n+r)\,(n+r-1)\right) + \sum_{n=0}^{\infty}\left(-4x^{n+r+2}a_n(n+r)\right)+\sum_{n=0}^{\infty}\left(-3x^{n+r}a_n(n+r)\right) + \sum_{n=0}^{\infty}\left(-2x^{n+r+2}a_n\right)+\left(\sum_{n=0}^{\infty}2a_n x^{n+r}\right)=0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-x^{n+r+2} a_n(n+r)(n+r-1)\right)=\sum_{n=2}^{\infty}\left(-a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty}\left(-4x^{n+r+2} a_n(n+r)\right)=\sum_{n=2}^{\infty}\left(-4a_{n-2}(n+r-2)\, x^{n+r}\right)$$

$$\sum_{n=0}^{\infty}\left(-2x^{n+r+2} a_n\right)=\sum_{n=2}^{\infty}\left(-2a_{n-2} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\begin{gathered}\sum_{n=2}^{\infty}\left(-a_{n-2}(n+r-2)(n-3+r)\, x^{n+r}\right)+\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)(n+r-1)\right) \\ +\sum_{n=2}^{\infty}\left(-4a_{n-2}(n+r-2)\, x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-3x^{n+r} a_n(n+r)\right) \\ +\sum_{n=2}^{\infty}\left(-2a_{n-2} x^{n+r}\right)+\left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right)=0\end{gathered} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n(n+r)(n+r-1)-3x^{n+r} a_n(n+r)+2a_n x^{n+r}=0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r)-3x^r a_0 r+2a_0 x^r=0$$

Or

$$\left(2x^r r(-1+r)-3x^r r+2x^r\right) a_0=0$$

Since $a_0\neq 0$ then the above simplifies to

$$\left(2r^2-5r+2\right) x^r=0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2-5r+2=0$$

Solving for r gives the roots of the indicial equation as

$$
\begin{aligned}
r_1 &= 2 \\
r_2 &= \frac{1}{2}
\end{aligned}
$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$
(2r^2 - 5r + 2)\, x^r = 0
$$

Solving for r gives the roots of the indicial equation as $\left[2, \frac{1}{2}\right]$.

Since $r_1 - r_2 = \frac{3}{2}$ is not an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+2} \\
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}
\end{aligned}
$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
\begin{aligned}
-a_{n-2}(n + r - 2)\,(n - 3 + r) + 2a_n(n + r)\,(n + r - 1) \\
- 4a_{n-2}(n + r - 2) - 3a_n(n + r) - 2a_{n-2} + 2a_n = 0
\end{aligned} \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = \frac{a_{n-2}(n^2 + 2nr + r^2 - n - r)}{2n^2 + 4nr + 2r^2 - 5n - 5r + 2} \tag{4}
$$

Which for the root $r = 2$ becomes

$$a_n = \frac{a_{n-2}(n^2 + 3n + 2)}{2n^2 + 3n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^2 + 3r + 2}{2r^2 + 3r}$$

Which for the root $r = 2$ becomes

$$a_2 = \frac{6}{7}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{r^2+3r+2}{2r^2+3r}$	$\frac{6}{7}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{r^2+3r+2}{2r^2+3r}$	$\frac{6}{7}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^3 + 8r^2 + 19r + 12}{4r^3 + 20r^2 + 21r}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{45}{77}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{r^2+3r+2}{2r^2+3r}$	$\frac{6}{7}$
a_3	0	0
a_4	$\frac{r^3+8r^2+19r+12}{4r^3+20r^2+21r}$	$\frac{45}{77}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{r^2+3r+2}{2r^2+3r}$	$\frac{6}{7}$
a_3	0	0
a_4	$\frac{r^3+8r^2+19r+12}{4r^3+20r^2+21r}$	$\frac{45}{77}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^2\left(1 + \frac{6x^2}{7} + \frac{45x^4}{77} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned}-b_{n-2}(n+r-2)(n-3+r)+2b_n(n+r)(n+r-1)\\ -4b_{n-2}(n+r-2)-3b_n(n+r)-2b_{n-2}+2b_n=0\end{aligned} \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = \frac{b_{n-2}(n^2+2nr+r^2-n-r)}{2n^2+4nr+2r^2-5n-5r+2} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_n = \frac{4n^2 b_{n-2} - b_{n-2}}{8n^2 - 12n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{r^2+3r+2}{2r^2+3r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_2 = \frac{15}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{r^2+3r+2}{2r^2+3r}$	$\frac{15}{8}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{r^2+3r+2}{2r^2+3r}$	$\frac{15}{8}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{r^3 + 8r^2 + 19r + 12}{4r^3 + 20r^2 + 21r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$b_4 = \frac{189}{128}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{r^2+3r+2}{2r^2+3r}$	$\frac{15}{8}$
b_3	0	0
b_4	$\frac{r^3+8r^2+19r+12}{4r^3+20r^2+21r}$	$\frac{189}{128}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{r^2+3r+2}{2r^2+3r}$	$\frac{15}{8}$
b_3	0	0
b_4	$\frac{r^3+8r^2+19r+12}{4r^3+20r^2+21r}$	$\frac{189}{128}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}y_2(x) &= x^2\left(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots\right) \\ &= \sqrt{x}\left(1 + \frac{15x^2}{8} + \frac{189x^4}{128} + O(x^6)\right)\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1\, x^2\left(1 + \frac{6x^2}{7} + \frac{45x^4}{77} + O(x^6)\right) + c_2\sqrt{x}\left(1 + \frac{15x^2}{8} + \frac{189x^4}{128} + O(x^6)\right)\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= y_h \\ &= c_1\, x^2\left(1 + \frac{6x^2}{7} + \frac{45x^4}{77} + O(x^6)\right) + c_2\sqrt{x}\left(1 + \frac{15x^2}{8} + \frac{189x^4}{128} + O(x^6)\right)\end{aligned}$$

Maple step by step solution

Let's solve

$x^2(-x^2+2)\left(\frac{d^2}{dx^2}y(x)\right) - x(4x^2+3)\left(\frac{d}{dx}y(x)\right) + (-2x^2+2)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{2(x^2-1)y(x)}{x^2(x^2-2)} - \frac{(4x^2+3)\left(\frac{d}{dx}y(x)\right)}{x(x^2-2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{(4x^2+3)\left(\frac{d}{dx}y(x)\right)}{x(x^2-2)} + \frac{2(x^2-1)y(x)}{x^2(x^2-2)} = 0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = \frac{4x^2+3}{x(x^2-2)}, P_3(x) = \frac{2(x^2-1)}{x^2(x^2-2)}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = -\frac{3}{2}$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 1$

- ◦ $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x^2(x^2-2)\left(\frac{d^2}{dx^2}y(x)\right) + x(4x^2+3)\left(\frac{d}{dx}y(x)\right) + (2x^2-2)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions

  - ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$

  - ◦ Shift index using $k- >k - m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$

  - ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..3$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1+m}$

  - ◦ Shift index using $k- >k + 1 - m$

    $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\,x^{k+r}$

  - ◦ Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\,x^{k+r-2+m}$

  - ◦ Shift index using $k- >k + 2 - m$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\,x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0(-1+2r)\,(-2+r)\,x^r - a_1(1+2r)\,(-1+r)\,x^{1+r} + \left(\sum_{k=2}^{\infty}\left(-a_k(2k+2r-1)\,(k+r-2) + a_{k}\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $-(-1+2r)\,(-2+r) = 0$

- Values of r that satisfy the indicial equation
  $r \in \left\{2, \frac{1}{2}\right\}$
- Each term must be 0
  $-a_1(1+2r)(-1+r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $-2(k+r-2)\left(k+r-\frac{1}{2}\right)a_k + a_{k-2}(k+r)(k+r-1) = 0$
- Shift index using $k- >k+2$
  $-2(k+r)\left(k+\frac{3}{2}+r\right)a_{k+2} + a_k(k+r+2)(k+r+1) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = \frac{a_k(k+r+2)(k+r+1)}{(k+r)(2k+3+2r)}$
- Recursion relation for $r = 2$
  $a_{k+2} = \frac{a_k(k+4)(k+3)}{(k+2)(2k+7)}$
- Solution for $r = 2$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2} = \frac{a_k(k+4)(k+3)}{(k+2)(2k+7)}, a_1 = 0\right]$
- Recursion relation for $r = \frac{1}{2}$
  $a_{k+2} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k+\frac{3}{2}\right)}{\left(k+\frac{1}{2}\right)(2k+4)}$
- Solution for $r = \frac{1}{2}$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = \frac{a_k\left(k+\frac{5}{2}\right)\left(k+\frac{3}{2}\right)}{\left(k+\frac{1}{2}\right)(2k+4)}, a_1 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k+2}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = \frac{a_k(4+k)(k+3)}{(k+2)(2k+7)}, a_1 = 0, b_{k+2} = \frac{b_k\left(k+\frac{5}{2}\right)\left(k+\frac{3}{2}\right)}{\left(k+\frac{1}{2}\right)(2k+4)}, b_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
```

```
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
   Solution has integrals. Trying a special function solution free of integrals...
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      -> Kummer
         -> hyper3: Equivalence to 1F1 under a power @ Moebius
      -> hypergeometric
         -> heuristic approach
         <- heuristic approach successful
      <- hypergeometric successful
   <- special function solution successful
      -> Trying to convert hypergeometric functions to elementary form...
      <- elementary form for at least one hypergeometric solution is achieved - return
   <- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.018 (sec)
Leaf size : 35

```
dsolve(x^2*(-x^2+2)*diff(diff(y(x),x),x)-x*(4*x^2+3)*diff(y(x),x)+(-2*x^2+2)*y(x) = 0,
       series,x=0)
```

$$y = c_1\sqrt{x}\left(1+\frac{15}{8}x^2+\frac{189}{128}x^4+\mathrm{O}\left(x^6\right)\right)+c_2x^2\left(1+\frac{6}{7}x^2+\frac{45}{77}x^4+\mathrm{O}\left(x^6\right)\right)$$

Mathematica DSolve solution

Solving time : 0.024 (sec)
Leaf size : 50

```
AsymptoticDSolveValue[{x^2*(2-x^2)*D[y[x],{x,2}] - x*(3+4*x^2)*D[y[x],x] + (2-2*x^2)*y[x] ==
       y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{45x^4}{77}+\frac{6x^2}{7}+1\right)x^2+c_2\left(\frac{189x^4}{128}+\frac{15x^2}{8}+1\right)\sqrt{x}$$

2.4.32 problem 28

Internal problem ID [8597]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 28
Date solved : `Tuesday, December 17, 2024 at 12:56:08 PM`
CAS classification : `[‘y=_G(x,y’)‘]`

Solve

$$y'^2 + y^2 = \sec(x)^4$$

Solving for the derivative gives these ODE's to solve

$$y' = \frac{\sqrt{1 - y^2 \cos(x)^4}}{\cos(x)^2} \tag{1}$$

$$y' = -\frac{\sqrt{1 - y^2 \cos(x)^4}}{\cos(x)^2} \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Solving Eq. (2)

Maple step by step solution

Let's solve
$\left(\frac{d}{dx}y(x)\right)^2 + y(x)^2 = \sec(x)^4$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative

$$\left[\frac{d}{dx}y(x)=\frac{\sqrt{1-y(x)^2\cos(x)^4}}{\cos(x)^2}, \frac{d}{dx}y(x)=-\frac{\sqrt{1-y(x)^2\cos(x)^4}}{\cos(x)^2}\right]$$

- Solve the equation $\frac{d}{dx}y(x)=\frac{\sqrt{1-y(x)^2\cos(x)^4}}{\cos(x)^2}$
- Solve the equation $\frac{d}{dx}y(x)=-\frac{\sqrt{1-y(x)^2\cos(x)^4}}{\cos(x)^2}$
- Set of solutions
  $\{workingODE, workingODE\}$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying simple symmetries for implicit equations
Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying exact
   Looking for potential symmetries
   trying an equivalence to an Abel ODE
   trying 1st order ODE linearizable_by_differentiation
-> Solving 1st order ODE of high degree, Lie methods, 1st trial
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 2
-> Solving 1st order ODE of high degree, 2nd attempt. Trying parametric methods
trying dAlembert
-> Calling odsolve with the ODE`, diff(y(x), x) = -x^2/(-2*(x^2+y(x)^2)^(5/4)*(((x^2+y
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
```

```
   trying exact
   Looking for potential symmetries
   trying an equivalence to an Abel ODE
   trying 1st order ODE linearizable_by_differentiation
-> Calling odsolve with the ODE`, diff(y(x), x) = -x*(10+15*cos(2*y(x))+cos(6*y(x))+6*
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying exact
   Looking for potential symmetries
   trying an equivalence to an Abel ODE
   trying 1st order ODE linearizable_by_differentiation
-> Solving 1st order ODE of high degree, Lie methods, 2nd trial
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
`, `-> Computing symmetries using: way = 5`
```

Maple dsolve solution

Solving time : 0.139 (sec)
Leaf size : maple_leaf_size

```
dsolve(diff(y(x),x)^2+y(x)^2 = sec(x)^4,
       y(x),singsol=all)
```

No solution found

Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{D[y[x],x]^2+y[x]^2==Sec[x]^4,{}},
       y[x],x,IncludeSingularSolutions->True]
```

Not solved

2.4.33 problem 29

Internal problem ID [8598]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 29
Date solved : `Wednesday, December 18, 2024 at 02:12:06 AM`
CAS classification : `[[_1st_order, _with_linear_symmetries], _dAlembert]`

Solve

$$(y - 2xy')^2 = {y'}^3$$

Let $p = y'$ the ode becomes

$$(-2xp + y)^2 = p^3$$

Solving for y from the above results in

$$y = 2xp + p^{3/2} \tag{1}$$

$$y = 2xp - p^{3/2} \tag{2}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. Each of the above ode's is dAlembert ode which is now solved.

<u>Solving ode 1A</u>

Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$

$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = 2p$$
$$g = p^{3/2}$$

Hence (2) becomes

$$-p = \left(2x + \frac{3\sqrt{p}}{2}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$-p = 0$$

Solving the above for p results in

$$p_1 = 0$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$y = 0$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = -\frac{p(x)}{2x + \frac{3\sqrt{p(x)}}{2}} \tag{3}$$

Inverting the above ode gives

$$\frac{d}{dp}x(p) = -\frac{2x(p) + \frac{3\sqrt{p}}{2}}{p} \tag{4}$$

This ODE is now solved for $x(p)$. The integrating factor is

$$\mu = \mathrm{e}^{\int \frac{2}{p}dp}$$
$$\mu = p^2$$
$$\mu = p^2 \tag{5}$$

Integrating gives

$$\begin{aligned} x(p) &= \frac{1}{\mu}\left(\int \mu\left(-\frac{3}{2\sqrt{p}}\right) dp + c_1\right) \\ &= \frac{1}{\mu}\left(\frac{-3p^{5/2} + 5c_1}{5p^2} + c_1\right) \\ &= \frac{-3p^{5/2} + 5c_1}{5p^2} \end{aligned} \tag{5}$$

Now we need to eliminate p between the above solution and (1A). The first method is to solve for p from Eq. (1A) and substitute the result into Eq. (5). The Second method is to solve for p from Eq. (5) and substitute the result into (1A).

Eliminating p from the following two equations

$$
\begin{aligned}
x &= \frac{-3p^{5/2} + 5c_1}{5p^2} \\
y &= 2xp + p^{3/2}
\end{aligned}
$$

results in

$$
p = \operatorname{RootOf}\left(3\_Z^5 + 5x\_Z^4 - 5c_1\right)^2
$$

Substituting the above into Eq (1A) and simplifying gives

$$
y = 2x \operatorname{RootOf}\left(3\_Z^5 + 5x\_Z^4 - 5c_1\right)^2 + \left(\operatorname{RootOf}\left(3\_Z^5 + 5x\_Z^4 - 5c_1\right)^2\right)^{3/2}
$$

Solving ode 2A

Taking derivative of (*) w.r.t. x gives

$$
\begin{aligned}
p &= f + (xf' + g')\frac{dp}{dx} \\
p - f &= (xf' + g')\frac{dp}{dx} \qquad (2)
\end{aligned}
$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= 2p \\
g &= -p^{3/2}
\end{aligned}
$$

Hence (2) becomes

$$
-p = \left(2x - \frac{3\sqrt{p}}{2}\right) p'(x) \qquad (2A)
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
-p = 0
$$

Solving the above for p results in

$$
p_1 = 0
$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$y = 0$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = -\frac{p(x)}{2x - \frac{3\sqrt{p(x)}}{2}} \tag{3}$$

Inverting the above ode gives

$$\frac{d}{dp}x(p) = -\frac{2x(p) - \frac{3\sqrt{p}}{2}}{p} \tag{4}$$

This ODE is now solved for $x(p)$. The integrating factor is

$$\begin{aligned} \mu &= \mathrm{e}^{\int \frac{2}{p} dp} \\ \mu &= p^2 \\ \mu &= p^2 \end{aligned} \tag{5}$$

Integrating gives

$$\begin{aligned} x(p) &= \frac{1}{\mu}\left(\int \mu\left(\frac{3}{2\sqrt{p}}\right) dp + c_2\right) \\ &= \frac{1}{\mu}\left(\frac{3p^{5/2} + 5c_2}{5p^2} + c_2\right) \\ &= \frac{3p^{5/2} + 5c_2}{5p^2} \end{aligned} \tag{5}$$

Now we need to eliminate p between the above solution and (1A). The first method is to solve for p from Eq. (1A) and substitute the result into Eq. (5). The Second method is to solve for p from Eq. (5) and substitute the result into (1A).

Eliminating p from the following two equations

$$x = \frac{3p^{5/2} + 5c_2}{5p^2}$$

$$y = 2xp - p^{3/2}$$

results in

$$p = \mathrm{RootOf}\left(3\_Z^5 - 5x\_Z^4 + 5c_2\right)^2$$

Substituting the above into Eq (1A) and simplifying gives

$$y = 2x \operatorname{RootOf}\left(3\_Z^5 - 5x\_Z^4 + 5c_2\right)^2 - \left(\operatorname{RootOf}\left(3\_Z^5 - 5x\_Z^4 + 5c_2\right)^2\right)^{3/2}$$

The solution

$$y = 2x \operatorname{RootOf}\left(3\_Z^5 + 5x\_Z^4 - 5c_1\right)^2 + \left(\operatorname{RootOf}\left(3\_Z^5 + 5x\_Z^4 - 5c_1\right)^2\right)^{3/2}$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$y = 2x \operatorname{RootOf}\left(3\_Z^5 - 5x\_Z^4 + 5c_2\right)^2 - \left(\operatorname{RootOf}\left(3\_Z^5 - 5x\_Z^4 + 5c_2\right)^2\right)^{3/2}$$

was found not to satisfy the ode or the IC. Hence it is removed.

Maple step by step solution

Let's solve

$\left(y(x) - 2x\left(\frac{d}{dx}y(x)\right)\right)^2 = \left(\frac{d}{dx}y(x)\right)^3$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $$\left[\frac{d}{dx}y(x) = \frac{\left(-576x^3y(x)+108y(x)^2+512x^6+12\sqrt{-96x^3y(x)^3+81y(x)^4}\right)^{1/3}}{6} - \frac{6\left(\frac{4xy(x)}{3} - \frac{16x^4}{9}\right)}{\left(-576x^3y(x)+108y(x)^2+512x^6+12\sqrt{-96x^3y(x}\right.}\right.$$

- Solve the equation $\frac{d}{dx}y(x) = \frac{\left(-576x^3y(x)+108y(x)^2+512x^6+12\sqrt{-96x^3y(x)^3+81y(x)^4}\right)^{1/3}}{6} - \frac{}{\left(-576x^3y(x)+108y(x)^2\right.}$

- Solve the equation $\frac{d}{dx}y(x) = -\frac{\left(-576x^3y(x)+108y(x)^2+512x^6+12\sqrt{-96x^3y(x)^3+81y(x)^4}\right)^{1/3}}{12} + \frac{}{\left(-576x^3y(x)+108y(\right.}$

- Solve the equation $\frac{d}{dx}y(x) = -\frac{\left(-576x^3y(x)+108y(x)^2+512x^6+12\sqrt{-96x^3y(x)^3+81y(x)^4}\right)^{1/3}}{12} + \frac{}{\left(-576x^3y(x)+108y(\right.}$

- Set of solutions
  $\{workingODE, workingODE, workingODE\}$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying simple symmetries for implicit equations
Successful isolation of dy/dx: 3 solutions were found. Trying to solve each resulting
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying exact
   Looking for potential symmetries
   trying an equivalence to an Abel ODE
   trying 1st order ODE linearizable_by_differentiation
-> Solving 1st order ODE of high degree, Lie methods, 1st trial
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 2
-> Solving 1st order ODE of high degree, 2nd attempt. Trying parametric methods
trying dAlembert
<- dAlembert successful
<- dAlembert successful`
```

Maple dsolve solution

Solving time : 0.181 (sec)
Leaf size : 73

```
dsolve((-2*diff(y(x),x)*x+y(x))^2 = diff(y(x),x)^3,
       y(x),singsol=all)
```

$$y = 0$$
$$\left[x(\_T) = \frac{3\_T^{5/2} + 5c_1}{5\_T^2}, y(\_T) = \frac{\_T^{5/2} + 10c_1}{5\_T}\right]$$
$$\left[x(\_T) = \frac{-3\_T^{5/2} + 5c_1}{5\_T^2}, y(\_T) = \frac{-\_T^{5/2} + 10c_1}{5\_T}\right]$$

Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{(y[x]-2*x*D[y[x],x])^2== D[y[x],x]^3,{}},
       y[x],x,IncludeSingularSolutions->True]
```

Timed out

2.4.34 problem 31

Internal problem ID [8599]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 31
Date solved : `Thursday, December 12, 2024 at 09:31:57 AM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$x^2y'' + y = 0$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 0$$
$$q(x) = \frac{1}{x^2}$$

Table 2.195: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = \frac{1}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r}a_n(n+r)(n+r-1) + a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + a_0 x^r = 0$$

Or

$$(x^r r(-1+r) + x^r) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 - r + 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - r + 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$r_2 = \frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 - r + 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as $\left[\frac{1}{2} + \frac{i\sqrt{3}}{2}, \frac{1}{2} - \frac{i\sqrt{3}}{2}\right]$.

Since the roots are complex conjugates, then two linearly independent solutions can be constructed using

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}+\frac{i\sqrt{3}}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}-\frac{i\sqrt{3}}{2}}$$

$y_1(x)$ is found first. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $0 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = 0 \tag{4}$$

Which for the root $r = \frac{1}{2} + \frac{i\sqrt{3}}{2}$ becomes

$$a_n = 0 \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2} + \frac{i\sqrt{3}}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\
&= x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}\left(1 + O(x^6)\right)
\end{aligned}
$$

The second solution $y_2(x)$ is found by taking the complex conjugate of $y_1(x)$ which gives

$$
y_2(x) = x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\left(1 + O(x^6)\right)
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\, x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}\left(1 + O(x^6)\right) + c_2\, x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\left(1 + O(x^6)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\, x^{\frac{1}{2}+\frac{i\sqrt{3}}{2}}\left(1 + O(x^6)\right) + c_2\, x^{\frac{1}{2}-\frac{i\sqrt{3}}{2}}\left(1 + O(x^6)\right)
\end{aligned}
$$

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) + y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{y(x)}{x^2}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{y(x)}{x^2} = 0$
- Multiply by denominators of the ODE

  $x^2\left(\frac{d^2}{dx^2}y(x)\right) + y(x) = 0$
- Make a change of variables

$t = \ln(x)$

- □ Substitute the change of variables back into the ODE
  - ∘ Calculate the 1st derivative of y with respect to x , using the chain rule
    $\frac{d}{dx}y(x) = \left(\frac{d}{dt}y(t)\right)\left(\frac{d}{dx}t(x)\right)$
  - ∘ Compute derivative
    $\frac{d}{dx}y(x) = \frac{\frac{d}{dt}y(t)}{x}$
  - ∘ Calculate the 2nd derivative of y with respect to x , using the chain rule
    $\frac{d^2}{dx^2}y(x) = \left(\frac{d^2}{dt^2}y(t)\right)\left(\frac{d}{dx}t(x)\right)^2 + \left(\frac{d^2}{dx^2}t(x)\right)\left(\frac{d}{dt}y(t)\right)$
  - ∘ Compute derivative
    $\frac{d^2}{dx^2}y(x) = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE
  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + y(t) = 0$
- Simplify
  $\frac{d^2}{dt^2}y(t) - \frac{d}{dt}y(t) + y(t) = 0$
- Characteristic polynomial of ODE
  $r^2 - r + 1 = 0$
- Use quadratic formula to solve for r
  $r = \frac{1\pm(\sqrt{-3})}{2}$
- Roots of the characteristic polynomial
  $r = \left(\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, \frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)$
- 1st solution of the ODE
  $y_1(t) = \mathrm{e}^{\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right)$
- 2nd solution of the ODE
  $y_2(t) = \mathrm{e}^{\frac{t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)$
- General solution of the ODE
  $y(t) = C1y_1(t) + C2y_2(t)$
- Substitute in solutions
  $y(t) = C1\,\mathrm{e}^{\frac{t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right) + C2\,\mathrm{e}^{\frac{t}{2}}\sin\left(\frac{\sqrt{3}\,t}{2}\right)$
- Change variables back using $t = \ln(x)$
  $y(x) = C1\sqrt{x}\cos\left(\frac{\sqrt{3}\ln(x)}{2}\right) + C2\sqrt{x}\sin\left(\frac{\sqrt{3}\ln(x)}{2}\right)$

- Simplify

  $$y(x) = \sqrt{x}\left(C1\cos\left(\frac{\sqrt{3}\,\ln(x)}{2}\right) + C2\sin\left(\frac{\sqrt{3}\,\ln(x)}{2}\right)\right)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

Maple dsolve solution

Solving time : 0.018 (sec)
Leaf size : 32

```
dsolve(x^2*diff(diff(y(x),x),x)+y(x) = 0,y(x),
       series,x=0)
```

$$y = \sqrt{x}\left(c_1 x^{-\frac{i\sqrt{3}}{2}} + c_2 x^{\frac{i\sqrt{3}}{2}}\right) + O\left(x^6\right)$$

Mathematica DSolve solution

Solving time : 0.004 (sec)
Leaf size : 26

```
AsymptoticDSolveValue[{x^2*D[y[x],{x,2}] +y[x] == 0,{}},
       y[x],{x,0,5}]
```

$$y(x) \to c_1 x^{-(-1)^{2/3}} + c_2 x^{\sqrt[3]{-1}}$$

2.4.35 problem 32

Internal problem ID [8600]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 32
Date solved : `Thursday, December 12, 2024 at 09:31:58 AM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$xy'' + y' - y = 0$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + y' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{1}{x}$$

Table 2.197: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x=0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy''+y'-y=0$$

Let the solution be represented as Frobenius power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y'=\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right)+\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right)-\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right)=0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right)+\sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right)=0 \quad (2\text{A})$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right)=\sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n (n+r)(n+r-1) + (n+r) a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as $[0, 0]$.

Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln (x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-1}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_1 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{(r+1)^2}$	1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(r+1)^2 (r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{(r+1)^2}$	1
a_2	$\frac{1}{(r+1)^2(r+2)^2}$	$\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{(r+1)^2 (r+2)^2 (r+3)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{1}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{(r+1)^2}$	1
a_2	$\frac{1}{(r+1)^2(r+2)^2}$	$\frac{1}{4}$
a_3	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2}$	$\frac{1}{36}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r+1)^2 (r+2)^2 (r+3)^2 (r+4)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{576}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{(r+1)^2}$	1
a_2	$\frac{1}{(r+1)^2(r+2)^2}$	$\frac{1}{4}$
a_3	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2}$	$\frac{1}{36}$
a_4	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{1}{576}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{(r+1)^2 (r+2)^2 (r+3)^2 (r+4)^2 (r+5)^2}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{1}{14400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{(r+1)^2}$	1
a_2	$\frac{1}{(r+1)^2(r+2)^2}$	$\frac{1}{4}$
a_3	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2}$	$\frac{1}{36}$
a_4	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{1}{576}$
a_5	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$\frac{1}{14400}$

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned} y_1(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots \\ &= 1 + x + \frac{x^2}{4} + \frac{x^3}{36} + \frac{x^4}{576} + \frac{x^5}{14400} + O(x^6) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r=0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{1}{(r+1)^2}$	1	$-\frac{2}{(r+1)^3}$	-2
b_2	$\frac{1}{(r+1)^2(r+2)^2}$	$\frac{1}{4}$	$\frac{-4r-6}{(r+1)^3(r+2)^3}$	$-\frac{3}{4}$
b_3	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2}$	$\frac{1}{36}$	$\frac{-6r^2-24r-22}{(r+1)^3(r+2)^3(r+3)^3}$	$-\frac{11}{108}$
b_4	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{1}{576}$	$\frac{-8r^3-60r^2-140r-100}{(r+1)^3(r+2)^3(r+3)^3(r+4)^3}$	$-\frac{25}{3456}$
b_5	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$\frac{1}{14400}$	$\frac{-10r^4-120r^3-510r^2-900r-548}{(r+1)^3(r+2)^3(r+3)^3(r+4)^3(r+5)^3}$	$-\frac{137}{432000}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \ldots \\
&= \left(1 + x + \frac{x^2}{4} + \frac{x^3}{36} + \frac{x^4}{576} + \frac{x^5}{14400} + O(x^6)\right)\ln(x) \\
&\quad - 2x - \frac{3x^2}{4} - \frac{11x^3}{108} - \frac{25x^4}{3456} - \frac{137x^5}{432000} + O(x^6)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1 + x + \frac{x^2}{4} + \frac{x^3}{36} + \frac{x^4}{576} + \frac{x^5}{14400} + O(x^6)\right) \\
&\quad + c_2\left(\left(1 + x + \frac{x^2}{4} + \frac{x^3}{36} + \frac{x^4}{576} + \frac{x^5}{14400} + O(x^6)\right)\ln(x) - 2x - \frac{3x^2}{4} - \frac{11x^3}{108} \right. \\
&\quad \left. - \frac{25x^4}{3456} - \frac{137x^5}{432000} + O(x^6)\right)
\end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1+x+\frac{x^2}{4}+\frac{x^3}{36}+\frac{x^4}{576}+\frac{x^5}{14400}+O(x^6)\right) \\
&\quad + c_2\left(\left(1+x+\frac{x^2}{4}+\frac{x^3}{36}+\frac{x^4}{576}+\frac{x^5}{14400}+O(x^6)\right)\ln(x)-2x-\frac{3x^2}{4}-\frac{11x^3}{108}\right. \\
&\quad \left. -\frac{25x^4}{3456}-\frac{137x^5}{432000}+O(x^6)\right)
\end{aligned}
$$

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dx^2}y(x)\right)x+\frac{d}{dx}y(x)-y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{y(x)}{x}-\frac{\frac{d}{dx}y(x)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{\frac{d}{dx}y(x)}{x}-\frac{y(x)}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{1}{x}, P_3(x)=-\frac{1}{x}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  ◦ $x=0$is a regular singular point

    Check to see if $x_0=0$ is a regular singular point

    $x_0=0$

- Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right)x+\frac{d}{dx}y(x)-y(x)=0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - Convert $\frac{d}{dx}y(x)$ to series expansion

    $\frac{d}{dx}y(x) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$
  - Shift index using $k- >k+1$

    $\frac{d}{dx}y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$
  - Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$
  - Shift index using $k- >k+1$

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r^2 x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)^2 - a_k\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$
- Values of r that satisfy the indicial equation

  $r = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 - a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{(k+1)^2}$
- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{(k+1)^2}$
- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k}{(k+1)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

Maple dsolve solution

Solving time : `0.018` (sec)
Leaf size : 44

```
dsolve(x*diff(diff(y(x),x),x)+diff(y(x),x)-y(x) = 0,y(x),
       series,x=0)
```

$$
\begin{aligned}
y = (c_2 \ln (x) + c_1) \left(1 + x + \frac{1}{4}x^2 + \frac{1}{36}x^3 + \frac{1}{576}x^4 + \frac{1}{14400}x^5 + \mathrm{O}\left(x^6\right)\right) \\
+ \left((-2)\, x - \frac{3}{4}x^2 - \frac{11}{108}x^3 - \frac{25}{3456}x^4 - \frac{137}{432000}x^5 + \mathrm{O}\left(x^6\right)\right) c_2
\end{aligned}
$$

Mathematica DSolve solution

Solving time : 0.005 (sec)
Leaf size : 107

```
AsymptoticDSolveValue[{x*D[y[x],{x,2}] +D[y[x],x]-y[x] == 0,{}},
       y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^5}{14400}+\frac{x^4}{576}+\frac{x^3}{36}+\frac{x^2}{4}+x+1\right)+c_2\left(-\frac{137x^5}{432000}-\frac{25x^4}{3456}-\frac{11x^3}{108}-\frac{3x^2}{4}+\left(\frac{x^5}{14400}+\frac{x^4}{576}+\frac{x^3}{36}+\frac{x^2}{4}+x+1\right)\log(x)-2x\right)$$

2.4.36 problem 33

Internal problem ID [8601]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 33
Date solved : `Thursday, December 12, 2024 at 09:31:59 AM`
CAS classification :
`[[_Emden, _Fowler], [_2nd_order, _linear, '_with_symmetry_[0,F(x)]']]`

Solve

$$4xy'' + 2y' + y = 0$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$4xy'' + 2y' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{2x}$$
$$q(x) = \frac{1}{4x}$$

Table 2.199: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{4x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$4xy'' + 2y' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$4x\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + 2\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 4x^{n+r-1} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} 4x^{n+r-1}a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty} 2(n+r)\,a_n x^{n+r-1}\right)+\left(\sum_{n=1}^{\infty} a_{n-1}x^{n+r-1}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r-1}a_n(n+r)\,(n+r-1)+2(n+r)\,a_n x^{n+r-1}=0$$

When $n=0$ the above becomes

$$4x^{-1+r}a_0 r(-1+r)+2ra_0x^{-1+r}=0$$

Or

$$\left(4x^{-1+r}r(-1+r)+2r\,x^{-1+r}\right)a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(4r^2-2r\right)x^{-1+r}=0$$

Since the above is true for all x then the indicial equation becomes

$$4r^2-2r=0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= \frac{1}{2} \\ r_2 &= 0 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(4r^2-2r\right)x^{-1+r}=0$$

Solving for r gives the roots of the indicial equation as $\left[\frac{1}{2},0\right]$.

Since $r_1-r_2=\frac{1}{2}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^n$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$4a_n(n+r)(n+r-1) + 2a_n(n+r) + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{2(2n^2 + 4nr + 2r^2 - n - r)} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{a_{n-1}}{4n^2 + 2n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{4r^2 + 6r + 2}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4r^2+6r+2}$	$-\frac{1}{6}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{16r^4 + 80r^3 + 140r^2 + 100r + 24}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4r^2+6r+2}$	$-\frac{1}{6}$
a_2	$\frac{1}{16r^4+80r^3+140r^2+100r+24}$	$\frac{1}{120}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{64r^6 + 672r^5 + 2800r^4 + 5880r^3 + 6496r^2 + 3528r + 720}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = -\frac{1}{5040}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4r^2+6r+2}$	$-\frac{1}{6}$
a_2	$\frac{1}{16r^4+80r^3+140r^2+100r+24}$	$\frac{1}{120}$
a_3	$-\frac{1}{64r^6+672r^5+2800r^4+5880r^3+6496r^2+3528r+720}$	$-\frac{1}{5040}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16\left(8r^6 + 84r^5 + 350r^4 + 735r^3 + 812r^2 + 441r + 90\right)\left(2r^2 + 15r + 28\right)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{1}{362880}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4r^2+6r+2}$	$-\frac{1}{6}$
a_2	$\frac{1}{16r^4+80r^3+140r^2+100r+24}$	$\frac{1}{120}$
a_3	$-\frac{1}{64r^6+672r^5+2800r^4+5880r^3+6496r^2+3528r+720}$	$-\frac{1}{5040}$
a_4	$\frac{1}{16(8r^6+84r^5+350r^4+735r^3+812r^2+441r+90)(2r^2+15r+28)}$	$\frac{1}{362880}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{32\left(8r^6+84r^5+350r^4+735r^3+812r^2+441r+90\right)\left(2r^2+15r+28\right)\left(2r^2+19r+45\right)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = -\frac{1}{39916800}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{4r^2+6r+2}$	$-\frac{1}{6}$
a_2	$\frac{1}{16r^4+80r^3+140r^2+100r+24}$	$\frac{1}{120}$
a_3	$-\frac{1}{64r^6+672r^5+2800r^4+5880r^3+6496r^2+3528r+720}$	$-\frac{1}{5040}$
a_4	$\frac{1}{16(8r^6+84r^5+350r^4+735r^3+812r^2+441r+90)(2r^2+15r+28)}$	$\frac{1}{362880}$
a_5	$-\frac{1}{32(8r^6+84r^5+350r^4+735r^3+812r^2+441r+90)(2r^2+15r+28)(2r^2+19r+45)}$	$-\frac{1}{39916800}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\big(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\big) \\ &= \sqrt{x}\left(1 - \frac{x}{6} + \frac{x^2}{120} - \frac{x^3}{5040} + \frac{x^4}{362880} - \frac{x^5}{39916800} + O\left(x^6\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the

indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$4b_n(n+r)(n+r-1) + 2(n+r)\, b_n + b_{n-1} = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = -\frac{b_{n-1}}{2\left(2n^2 + 4nr + 2r^2 - n - r\right)} \tag{4}$$

Which for the root $r = 0$ becomes

$$b_n = -\frac{b_{n-1}}{4n^2 - 2n} \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{1}{4r^2 + 6r + 2}$$

Which for the root $r = 0$ becomes

$$b_1 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{4r^2+6r+2}$	$-\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{16r^4 + 80r^3 + 140r^2 + 100r + 24}$$

Which for the root $r = 0$ becomes

$$b_2 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{4r^2+6r+2}$	$-\frac{1}{2}$
b_2	$\frac{1}{16r^4+80r^3+140r^2+100r+24}$	$\frac{1}{24}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{1}{64r^6 + 672r^5 + 2800r^4 + 5880r^3 + 6496r^2 + 3528r + 720}$$

Which for the root $r = 0$ becomes

$$b_3 = -\frac{1}{720}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{4r^2+6r+2}$	$-\frac{1}{2}$
b_2	$\frac{1}{16r^4+80r^3+140r^2+100r+24}$	$\frac{1}{24}$
b_3	$-\frac{1}{64r^6+672r^5+2800r^4+5880r^3+6496r^2+3528r+720}$	$-\frac{1}{720}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{16\left(8r^6 + 84r^5 + 350r^4 + 735r^3 + 812r^2 + 441r + 90\right)\left(2r^2 + 15r + 28\right)}$$

Which for the root $r = 0$ becomes

$$b_4 = \frac{1}{40320}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{4r^2+6r+2}$	$-\frac{1}{2}$
b_2	$\frac{1}{16r^4+80r^3+140r^2+100r+24}$	$\frac{1}{24}$
b_3	$-\frac{1}{64r^6+672r^5+2800r^4+5880r^3+6496r^2+3528r+720}$	$-\frac{1}{720}$
b_4	$\frac{1}{16(8r^6+84r^5+350r^4+735r^3+812r^2+441r+90)(2r^2+15r+28)}$	$\frac{1}{40320}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{1}{32\left(8r^6+84r^5+350r^4+735r^3+812r^2+441r+90\right)\left(2r^2+15r+28\right)\left(2r^2+19r+45\right)}$$

Which for the root $r = 0$ becomes

$$b_5 = -\frac{1}{3628800}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{4r^2+6r+2}$	$-\frac{1}{2}$
b_2	$\frac{1}{16r^4+80r^3+140r^2+100r+24}$	$\frac{1}{24}$
b_3	$-\frac{1}{64r^6+672r^5+2800r^4+5880r^3+6496r^2+3528r+720}$	$-\frac{1}{720}$
b_4	$\frac{1}{16(8r^6+84r^5+350r^4+735r^3+812r^2+441r+90)(2r^2+15r+28)}$	$\frac{1}{40320}$
b_5	$-\frac{1}{32(8r^6+84r^5+350r^4+735r^3+812r^2+441r+90)(2r^2+15r+28)(2r^2+19r+45)}$	$-\frac{1}{3628800}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\ &= 1 - \frac{x}{2} + \frac{x^2}{24} - \frac{x^3}{720} + \frac{x^4}{40320} - \frac{x^5}{3628800} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \sqrt{x}\left(1 - \frac{x}{6} + \frac{x^2}{120} - \frac{x^3}{5040} + \frac{x^4}{362880} - \frac{x^5}{39916800} + O(x^6)\right) \\ &\quad + c_2\left(1 - \frac{x}{2} + \frac{x^2}{24} - \frac{x^3}{720} + \frac{x^4}{40320} - \frac{x^5}{3628800} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 \sqrt{x}\left(1 - \frac{x}{6} + \frac{x^2}{120} - \frac{x^3}{5040} + \frac{x^4}{362880} - \frac{x^5}{39916800} + O(x^6)\right) \\ &\quad + c_2\left(1 - \frac{x}{2} + \frac{x^2}{24} - \frac{x^3}{720} + \frac{x^4}{40320} - \frac{x^5}{3628800} + O(x^6)\right) \end{aligned}$$

Maple step by step solution

Let's solve

$4\left(\frac{d^2}{dx^2}y(x)\right)x + 2\frac{d}{dx}y(x) + y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{y(x)}{4x} - \frac{\frac{d}{dx}y(x)}{2x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{\frac{d}{dx}y(x)}{2x} + \frac{y(x)}{4x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{2x}, P_3(x) = \frac{1}{4x}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = \frac{1}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $4\left(\frac{d^2}{dx^2}y(x)\right)x + 2\frac{d}{dx}y(x) + y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $\frac{d}{dx}y(x)$ to series expansion

    $\frac{d}{dx}y(x) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1}$

  - Shift index using $k- >k+1$

    $\frac{d}{dx}y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,x^{k+r}$

- Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$
- Shift index using $k- >k+1$

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $2a_0r(-1+2r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} (2a_{k+1}(k+1+r)(2k+1+2r)+a_k)\, x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2r(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{0, \frac{1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $4(k+1+r)\left(k+\frac{1}{2}+r\right) a_{k+1} + a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k}{2(k+1+r)(2k+1+2r)}$
- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{a_k}{2(k+1)(2k+1)}$
- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k}{2(k+1)(2k+1)}\right]$
- Recursion relation for $r = \frac{1}{2}$

  $a_{k+1} = -\frac{a_k}{2\left(k+\frac{3}{2}\right)(2k+2)}$
- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+1} = -\frac{a_k}{2\left(k+\frac{3}{2}\right)(2k+2)}\right]$
- Combine solutions and rename parameters

  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+1} = -\frac{a_k}{2(k+1)(2k+1)}, b_{k+1} = -\frac{b_k}{2\left(k+\frac{3}{2}\right)(2k+2)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

Maple dsolve solution

Solving time : 0.020 (sec)
Leaf size : 44

```
dsolve(4*x*diff(diff(y(x),x),x)+2*diff(y(x),x)+y(x) = 0,y(x),
       series,x=0)
```

$$
\begin{aligned}
y = {} & c_1\sqrt{x}\left(1-\frac{1}{6}x+\frac{1}{120}x^2-\frac{1}{5040}x^3+\frac{1}{362880}x^4-\frac{1}{39916800}x^5+\mathrm{O}\left(x^6\right)\right) \\
& + c_2\left(1-\frac{1}{2}x+\frac{1}{24}x^2-\frac{1}{720}x^3+\frac{1}{40320}x^4-\frac{1}{3628800}x^5+\mathrm{O}\left(x^6\right)\right)
\end{aligned}
$$

Mathematica DSolve solution

Solving time : 0.004 (sec)
Leaf size : 85

```
AsymptoticDSolveValue[{4*x*D[y[x],{x,2}] +2*D[y[x],x]+y[x] == 0,{}},
      y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to {} & c_1\sqrt{x}\left(-\frac{x^5}{39916800}+\frac{x^4}{362880}-\frac{x^3}{5040}+\frac{x^2}{120}-\frac{x}{6}+1\right) \\
& + c_2\left(-\frac{x^5}{3628800}+\frac{x^4}{40320}-\frac{x^3}{720}+\frac{x^2}{24}-\frac{x}{2}+1\right)
\end{aligned}
$$

2.4.37 problem 34

Internal problem ID [8602]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 34
Date solved : `Thursday, December 12, 2024 at 09:32:00 AM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$xy'' + y' - y = 0$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + y' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{1}{x}$$

Table 2.201: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + y' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \quad (1)$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \quad (2\text{A})$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n(n+r)(n+r-1) + (n+r)\, a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as $[0, 0]$.

Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-1}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_1 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{(r+1)^2}$	1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(r+1)^2 (r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{(r+1)^2}$	1
a_2	$\frac{1}{(r+1)^2(r+2)^2}$	$\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{(r+1)^2 (r+2)^2 (r+3)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{1}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{(r+1)^2}$	1
a_2	$\frac{1}{(r+1)^2(r+2)^2}$	$\frac{1}{4}$
a_3	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2}$	$\frac{1}{36}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r+1)^2 (r+2)^2 (r+3)^2 (r+4)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{576}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{(r+1)^2}$	1
a_2	$\frac{1}{(r+1)^2(r+2)^2}$	$\frac{1}{4}$
a_3	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2}$	$\frac{1}{36}$
a_4	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{1}{576}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{(r+1)^2 (r+2)^2 (r+3)^2 (r+4)^2 (r+5)^2}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{1}{14400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{(r+1)^2}$	1
a_2	$\frac{1}{(r+1)^2(r+2)^2}$	$\frac{1}{4}$
a_3	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2}$	$\frac{1}{36}$
a_4	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{1}{576}$
a_5	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$\frac{1}{14400}$

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\ &= 1 + x + \frac{x^2}{4} + \frac{x^3}{36} + \frac{x^4}{576} + \frac{x^5}{14400} + O(x^6) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = 0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{1}{(r+1)^2}$	1	$-\frac{2}{(r+1)^3}$	-2
b_2	$\frac{1}{(r+1)^2(r+2)^2}$	$\frac{1}{4}$	$\frac{-4r-6}{(r+1)^3(r+2)^3}$	$-\frac{3}{4}$
b_3	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2}$	$\frac{1}{36}$	$\frac{-6r^2-24r-22}{(r+1)^3(r+2)^3(r+3)^3}$	$-\frac{11}{108}$
b_4	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{1}{576}$	$\frac{-8r^3-60r^2-140r-100}{(r+1)^3(r+2)^3(r+3)^3(r+4)^3}$	$-\frac{25}{3456}$
b_5	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$\frac{1}{14400}$	$\frac{-10r^4-120r^3-510r^2-900r-548}{(r+1)^3(r+2)^3(r+3)^3(r+4)^3(r+5)^3}$	$-\frac{137}{432000}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= \left(1 + x + \frac{x^2}{4} + \frac{x^3}{36} + \frac{x^4}{576} + \frac{x^5}{14400} + O(x^6)\right)\ln(x) \\
&\quad - 2x - \frac{3x^2}{4} - \frac{11x^3}{108} - \frac{25x^4}{3456} - \frac{137x^5}{432000} + O(x^6)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1 + x + \frac{x^2}{4} + \frac{x^3}{36} + \frac{x^4}{576} + \frac{x^5}{14400} + O(x^6)\right) \\
&\quad + c_2\left(\left(1 + x + \frac{x^2}{4} + \frac{x^3}{36} + \frac{x^4}{576} + \frac{x^5}{14400} + O(x^6)\right)\ln(x) - 2x - \frac{3x^2}{4} - \frac{11x^3}{108} \right. \\
&\quad \left. - \frac{25x^4}{3456} - \frac{137x^5}{432000} + O(x^6)\right)
\end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1+x+\frac{x^2}{4}+\frac{x^3}{36}+\frac{x^4}{576}+\frac{x^5}{14400}+O(x^6)\right) \\
&\quad + c_2\left(\left(1+x+\frac{x^2}{4}+\frac{x^3}{36}+\frac{x^4}{576}+\frac{x^5}{14400}+O(x^6)\right)\ln(x)-2x-\frac{3x^2}{4}-\frac{11x^3}{108}\right. \\
&\quad \left. -\frac{25x^4}{3456}-\frac{137x^5}{432000}+O(x^6)\right)
\end{aligned}
$$

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dx^2}y(x)\right)x+\frac{d}{dx}y(x)-y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{y(x)}{x}-\frac{\frac{d}{dx}y(x)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{\frac{d}{dx}y(x)}{x}-\frac{y(x)}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{1}{x},P_3(x)=-\frac{1}{x}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\Big|_{x=0}=1$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\Big|_{x=0}=0$

  - $x=0$is a regular singular point

    Check to see if $x_0=0$ is a regular singular point

    $x_0=0$

- Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right)x+\frac{d}{dx}y(x)-y(x)=0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - ○ Convert $\frac{d}{dx}y(x)$ to series expansion

    $\frac{d}{dx}y(x) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$
  - ○ Shift index using $k- > k+1$

    $\frac{d}{dx}y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$
  - ○ Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$
  - ○ Shift index using $k- > k+1$

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r^2 x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)^2 - a_k\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 - a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{(k+1)^2}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{(k+1)^2}$

- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k}{(k+1)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

Maple dsolve solution

Solving time : 0.020 (sec)
Leaf size : 44

```
dsolve(x*diff(diff(y(x),x),x)+diff(y(x),x)-y(x) = 0,y(x),
       series,x=0)
```

$$
\begin{aligned}
y = (c_2 \ln(x) + c_1)\left(1 + x + \frac{1}{4}x^2 + \frac{1}{36}x^3 + \frac{1}{576}x^4 + \frac{1}{14400}x^5 + \mathrm{O}\left(x^6\right)\right) \\
+ \left((-2)\,x - \frac{3}{4}x^2 - \frac{11}{108}x^3 - \frac{25}{3456}x^4 - \frac{137}{432000}x^5 + \mathrm{O}\left(x^6\right)\right) c_2
\end{aligned}
$$

Mathematica DSolve solution

Solving time : 0.005 (sec)
Leaf size : 107

```
AsymptoticDSolveValue[{x*D[y[x],{x,2}] +D[y[x],x]-y[x] == 0,{}},
       y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^5}{14400} + \frac{x^4}{576} + \frac{x^3}{36} + \frac{x^2}{4} + x + 1\right) + c_2\left(-\frac{137x^5}{432000} - \frac{25x^4}{3456} - \frac{11x^3}{108} - \frac{3x^2}{4} + \left(\frac{x^5}{14400} + \frac{x^4}{576} + \frac{x^3}{36} + \frac{x^2}{4} + x + 1\right)\log(x) - 2x\right)$$

2.4.38 problem 35

Internal problem ID [8603]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 35
Date solved : `Thursday, December 12, 2024 at 09:32:01 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' + (1 + x) y' + 2y = 0$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + (1 + x) y' + 2y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1 + x}{x}$$
$$q(x) = \frac{2}{x}$$

Table 2.203: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1+x}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{2}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (1 + x) y' + 2y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} & x\left(\sum_{n=0}^{\infty} (n + r) (n + r - 1) a_n x^{n+r-2}\right) \\ & + (1 + x) \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + 2\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\begin{aligned} & \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n + r) (n + r - 1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n + r)\right) \\ & + \left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r} a_n(n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} 2a_n x^{n+r} = \sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n(n+r)\,(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} 2a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n(n+r)\,(n+r-1) + (n+r)\, a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^{-1+r} r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^{-1+r} r^2 = 0$$

Solving for r gives the roots of the indicial equation as $[0, 0]$.

Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) + a_n(n+r) + 2a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}(n+r+1)}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{a_{n-1}(n+1)}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-2-r}{(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_1 = -2$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{(r+1)^2}$	-2

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{3+r}{(2+r)(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{3}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{(r+1)^2}$	-2
a_2	$\frac{3+r}{(2+r)(r+1)^2}$	$\frac{3}{2}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-4-r}{(3+r)(2+r)(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = -\frac{2}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{(r+1)^2}$	-2
a_2	$\frac{3+r}{(2+r)(r+1)^2}$	$\frac{3}{2}$
a_3	$\frac{-4-r}{(3+r)(2+r)(r+1)^2}$	$-\frac{2}{3}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{5+r}{(4+r)(3+r)(2+r)(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{5}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{(r+1)^2}$	-2
a_2	$\frac{3+r}{(2+r)(r+1)^2}$	$\frac{3}{2}$
a_3	$\frac{-4-r}{(3+r)(2+r)(r+1)^2}$	$-\frac{2}{3}$
a_4	$\frac{5+r}{(4+r)(3+r)(2+r)(r+1)^2}$	$\frac{5}{24}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-6-r}{(5+r)(4+r)(3+r)(2+r)(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{1}{20}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-2-r}{(r+1)^2}$	-2
a_2	$\frac{3+r}{(2+r)(r+1)^2}$	$\frac{3}{2}$
a_3	$\frac{-4-r}{(3+r)(2+r)(r+1)^2}$	$-\frac{2}{3}$
a_4	$\frac{5+r}{(4+r)(3+r)(2+r)(r+1)^2}$	$\frac{5}{24}$
a_5	$\frac{-6-r}{(5+r)(4+r)(3+r)(2+r)(r+1)^2}$	$-\frac{1}{20}$

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\ &= 1 - 2x + \frac{3x^2}{2} - \frac{2x^3}{3} + \frac{5x^4}{24} - \frac{x^5}{20} + O(x^6) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r=0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{-2-r}{(r+1)^2}$	-2	$\frac{3+r}{(r+1)^3}$	3
b_2	$\frac{3+r}{(2+r)(r+1)^2}$	$\frac{3}{2}$	$\frac{-2r^2-11r-13}{(2+r)^2(r+1)^3}$	$-\frac{13}{4}$
b_3	$\frac{-4-r}{(3+r)(2+r)(r+1)^2}$	$-\frac{2}{3}$	$\frac{3r^3+27r^2+74r+62}{(3+r)^2(2+r)^2(r+1)^3}$	$\frac{31}{18}$
b_4	$\frac{5+r}{(4+r)(3+r)(2+r)(r+1)^2}$	$\frac{5}{24}$	$\frac{-4r^4-54r^3-256r^2-504r-346}{(4+r)^2(3+r)^2(2+r)^2(r+1)^3}$	$-\frac{173}{288}$
b_5	$\frac{-6-r}{(5+r)(4+r)(3+r)(2+r)(r+1)^2}$	$-\frac{1}{20}$	$\frac{5r^5+95r^4+685r^3+2335r^2+3744r+2244}{(5+r)^2(4+r)^2(3+r)^2(2+r)^2(r+1)^3}$	$\frac{187}{1200}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots \\
&= \left(1 - 2x + \frac{3x^2}{2} - \frac{2x^3}{3} + \frac{5x^4}{24} - \frac{x^5}{20} + O(x^6)\right)\ln(x) \\
&\quad + 3x - \frac{13x^2}{4} + \frac{31x^3}{18} - \frac{173x^4}{288} + \frac{187x^5}{1200} + O(x^6)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1\left(1 - 2x + \frac{3x^2}{2} - \frac{2x^3}{3} + \frac{5x^4}{24} - \frac{x^5}{20} + O(x^6)\right) \\
&\quad + c_2\left(\left(1 - 2x + \frac{3x^2}{2} - \frac{2x^3}{3} + \frac{5x^4}{24} - \frac{x^5}{20} + O(x^6)\right)\ln(x) + 3x - \frac{13x^2}{4} + \frac{31x^3}{18} \right.\\
&\quad \left. - \frac{173x^4}{288} + \frac{187x^5}{1200} + O(x^6)\right)
\end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1-2x+\frac{3x^2}{2}-\frac{2x^3}{3}+\frac{5x^4}{24}-\frac{x^5}{20}+O(x^6)\right) \\
&\quad + c_2\left(\left(1-2x+\frac{3x^2}{2}-\frac{2x^3}{3}+\frac{5x^4}{24}-\frac{x^5}{20}+O(x^6)\right)\ln(x)+3x-\frac{13x^2}{4}+\frac{31x^3}{18}\right. \\
&\qquad \left. -\frac{173x^4}{288}+\frac{187x^5}{1200}+O(x^6)\right)
\end{aligned}
$$

Maple step by step solution

Let's solve

$\left(\frac{d^2}{dx^2}y(x)\right)x+(x+1)\left(\frac{d}{dx}y(x)\right)+2y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{2y(x)}{x}-\frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{(x+1)\left(\frac{d}{dx}y(x)\right)}{x}+\frac{2y(x)}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{x+1}{x}, P_3(x)=\frac{2}{x}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\bigg|_{x=0}=1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\bigg|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right)x+(x+1)\left(\frac{d}{dx}y(x)\right)+2y(x)=0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$

- ◦ Shift index using $k- >k+1-m$

  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$

- ◦ Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$

- ◦ Shift index using $k- >k+1$

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r^2 x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)^2 + a_k(k+r+2)\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 + a_k(k+2) = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = -\frac{a_k(k+2)}{(k+1)^2}$

- Recursion relation for $r = 0$

  $a_{k+1} = -\frac{a_k(k+2)}{(k+1)^2}$

- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k(k+2)}{(k+1)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.017` (sec)
Leaf size : 44

```
dsolve(x*diff(diff(y(x),x),x)+(x+1)*diff(y(x),x)+2*y(x) = 0,y(x),
       series,x=0)
```

$$y = (c_2 \ln(x) + c_1)\left(1 - 2x + \frac{3}{2}x^2 - \frac{2}{3}x^3 + \frac{5}{24}x^4 - \frac{1}{20}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(3x - \frac{13}{4}x^2 + \frac{31}{18}x^3 - \frac{173}{288}x^4 + \frac{187}{1200}x^5 + \mathrm{O}\left(x^6\right)\right)c_2$$

Mathematica DSolve solution

Solving time : 0.007 (sec)
Leaf size : 111

```
AsymptoticDSolveValue[{x*D[y[x],{x,2}] +(1+x)*D[y[x],x]+2*y[x] == 0,{}},
       y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to c_1\left(-\frac{x^5}{20}+\frac{5x^4}{24}-\frac{2x^3}{3}+\frac{3x^2}{2}-2x+1\right)+c_2\left(\frac{187x^5}{1200}-\frac{173x^4}{288}+\frac{31x^3}{18}-\frac{13x^2}{4}\right. \\
\left.+\left(-\frac{x^5}{20}+\frac{5x^4}{24}-\frac{2x^3}{3}+\frac{3x^2}{2}-2x+1\right)\log(x)+3x\right)
\end{aligned}
$$

2.4.39 problem 36

Internal problem ID [8604]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 36
Date solved : `Thursday, December 12, 2024 at 09:32:02 AM`
CAS classification : `[[_2nd_order, _exact, _linear, _homogeneous]]`

Solve

$$x(x-1)\,y''+3xy'+y=0$$

Using series expansion around $x=0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^2-x\right)y''+3xy'+y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{3}{x-1}$$
$$q(x)=\frac{1}{x\,(x-1)}$$

Table 2.205: Table $p(x), q(x)$ singularites.

$p(x)=\frac{3}{x-1}$	
singularity	type
$x=1$	"regular"

$q(x)=\frac{1}{x(x-1)}$	
singularity	type
$x=0$	"regular"
$x=1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[1, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x(x-1)\, y'' + 3xy' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} x(x-1) &\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\ &+ 3x\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\begin{aligned} &\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\,(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r-1} a_n (n+r)\,(n+r-1)\right) \\ &+ \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}$$
$$\sum_{n=0}^{\infty} 3x^{n+r} a_n(n+r) = \sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)\, x^{n+r-1}$$
$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-x^{n+r-1} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=1}^{\infty} 3a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$-x^{n+r-1} a_n(n+r)(n+r-1) = 0$$

When $n=0$ the above becomes

$$-x^{-1+r} a_0 r(-1+r) = 0$$

Or

$$-x^{-1+r} a_0 r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$-x^{-1+r} r(-1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$-r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$-x^{-1+r} r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as $[1, 0]$.

Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = x \left(\sum_{n=0}^{\infty} a_n x^n \right)$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n \right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_{n-1}(n+r-1)(n+r-2) - a_n(n+r)(n+r-1) + 3a_{n-1}(n+r-1) + a_{n-1} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{(n+r) a_{n-1}}{n+r-1} \quad (4)$$

Which for the root $r = 1$ becomes

$$a_n = \frac{(n+1)\, a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1+r}{r}$$

Which for the root $r = 1$ becomes

$$a_1 = 2$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{r}$	2

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{2+r}{r}$$

Which for the root $r = 1$ becomes

$$a_2 = 3$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{r}$	2
a_2	$\frac{2+r}{r}$	3

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{3+r}{r}$$

Which for the root $r = 1$ becomes

$$a_3 = 4$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{r}$	2
a_2	$\frac{2+r}{r}$	3
a_3	$\frac{3+r}{r}$	4

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{4+r}{r}$$

Which for the root $r = 1$ becomes

$$a_4 = 5$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{r}$	2
a_2	$\frac{2+r}{r}$	3
a_3	$\frac{3+r}{r}$	4
a_4	$\frac{4+r}{r}$	5

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{5+r}{r}$$

Which for the root $r = 1$ becomes

$$a_5 = 6$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1+r}{r}$	2
a_2	$\frac{2+r}{r}$	3
a_3	$\frac{3+r}{r}$	4
a_4	$\frac{4+r}{r}$	5
a_5	$\frac{5+r}{r}$	6

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x\left(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + O\left(x^6\right)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_1 \\
&= \frac{1+r}{r}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r \to r_2} \frac{1+r}{r} &= \lim_{r \to 0} \frac{1+r}{r} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)
$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $x(x-1)\,y'' + 3xy' + y = 0$ gives

$$
\begin{aligned}
&x(x-1)\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left. + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&+ 3x\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) \\
&+ Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(x(x-1)\,y_1''(x) + 3y_1'(x)\,x + y_1(x)\right)\ln(x) + x(x-1)\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)\right. \\
&\left. + 3y_1(x)\right)C + x(x-1)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&+ 3x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$x(x-1)\,y_1''(x)+3y_1'(x)\,x+y_1(x)=0$$

Eq (7) simplifes to

$$\begin{aligned}
&\left(x(x-1)\left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)+3y_1(x)\right)C\\
&+x(x-1)\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right)\\
&+3x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0
\end{aligned} \tag{8}$$

Substituting $y_1=\sum_{n=0}^{\infty}a_n x^{n+r_1}$ into the above gives

$$\begin{aligned}
&\frac{\left(2x(x-1)\left(\sum_{n=0}^{\infty}x^{-1+n+r_1}a_n(n+r_1)\right)+(2x+1)\left(\sum_{n=0}^{\infty}a_n x^{n+r_1}\right)\right)C}{x}\\
&+\frac{x^2(x-1)\left(\sum_{n=0}^{\infty}x^{-2+n+r_2}b_n(n+r_2)\,(-1+n+r_2)\right)+3\left(\sum_{n=0}^{\infty}x^{-1+n+r_2}b_n(n+r_2)\right)x^2+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)x}{x}\\
&=0
\end{aligned} \tag{9}$$

Since $r_1=1$ and $r_2=0$ then the above becomes

$$\begin{aligned}
&\frac{\left(2x(x-1)\left(\sum_{n=0}^{\infty}x^{n}a_n(n+1)\right)+(2x+1)\left(\sum_{n=0}^{\infty}a_n x^{n+1}\right)\right)C}{x}\\
&+\frac{x^2(x-1)\left(\sum_{n=0}^{\infty}x^{-2+n}b_n n(n-1)\right)+3\left(\sum_{n=0}^{\infty}x^{n-1}b_n n\right)x^2+\left(\sum_{n=0}^{\infty}b_n x^{n}\right)x}{x}=0
\end{aligned} \tag{10}$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty}2C\,x^{n+1}a_n(n+1)\right)+\sum_{n\,=0}^{\infty}\left(-2x^n a_n(n+1)\,C\right)\\
&+\left(\sum_{n=0}^{\infty}2C\,x^{n+1}a_n\right)+\left(\sum_{n=0}^{\infty}a_n x^n C\right)+\left(\sum_{n=0}^{\infty}x^n b_n n(n-1)\right)\\
&+\sum_{n\,=0}^{\infty}\left(-n\,x^{n-1}b_n(n-1)\right)+\left(\sum_{n=0}^{\infty}3x^n b_n n\right)+\left(\sum_{n=0}^{\infty}b_n x^n\right)=0
\end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\,x^{n+1} a_n(n+1) = \sum_{n=2}^{\infty} 2C a_{-2+n}(n-1)\,x^{n-1}$$
$$\sum_{n=0}^{\infty} \left(-2x^n a_n(n+1)\,C\right) = \sum_{n=1}^{\infty} \left(-2C a_{n-1} n\,x^{n-1}\right)$$
$$\sum_{n=0}^{\infty} 2C\,x^{n+1} a_n = \sum_{n=2}^{\infty} 2C a_{-2+n} x^{n-1}$$
$$\sum_{n=0}^{\infty} a_n x^n C = \sum_{n=1}^{\infty} C a_{n-1} x^{n-1}$$
$$\sum_{n=0}^{\infty} x^n b_n n(n-1) = \sum_{n=1}^{\infty} (n-1)\,b_{n-1}(-2+n)\,x^{n-1}$$
$$\sum_{n=0}^{\infty} 3x^n b_n n = \sum_{n=1}^{\infty} 3(n-1)\,b_{n-1} x^{n-1}$$
$$\sum_{n=0}^{\infty} b_n x^n = \sum_{n=1}^{\infty} b_{n-1} x^{n-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-1$.

$$\begin{aligned}
&\left(\sum_{n=2}^{\infty} 2C a_{-2+n}(n-1)\,x^{n-1}\right) + \sum_{n=1}^{\infty} \left(-2C a_{n-1} n\,x^{n-1}\right) \\
&+ \left(\sum_{n=2}^{\infty} 2C a_{-2+n} x^{n-1}\right) + \left(\sum_{n=1}^{\infty} C a_{n-1} x^{n-1}\right) \\
&+ \left(\sum_{n=1}^{\infty} (n-1)\,b_{n-1}(-2+n)\,x^{n-1}\right) + \sum_{n=0}^{\infty} \left(-n\,x^{n-1} b_n(n-1)\right) \\
&+ \left(\sum_{n=1}^{\infty} 3(n-1)\,b_{n-1} x^{n-1}\right) + \left(\sum_{n=1}^{\infty} b_{n-1} x^{n-1}\right) = 0
\end{aligned} \tag{2B}$$

For $n = 0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = N$, where $N = 1$ which is the difference between the two roots, we are free to choose $b_1 = 0$. Hence for $n = 1$, Eq (2B) gives

$$-C + 1 = 0$$

Which is solved for C. Solving for C gives

$$C = 1$$

For $n = 2$, Eq (2B) gives

$$(4a_0 - 3a_1) C + 4b_1 - 2b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-2 - 2b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = -1$$

For $n = 3$, Eq (2B) gives

$$(6a_1 - 5a_2) C + 9b_2 - 6b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-12 - 6b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = -2$$

For $n = 4$, Eq (2B) gives

$$(8a_2 - 7a_3) C + 16b_3 - 12b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-36 - 12b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = -3$$

For $n = 5$, Eq (2B) gives

$$(10a_3 - 9a_4) C + 25b_4 - 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-80 - 20b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = -4$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = 1$ and all b_n, then the second solution becomes

$$\begin{aligned} y_2(x) = {} & 1\big(x\big(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + O(x^6)\big)\big)\ln(x) \\ & + 1 - x^2 - 2x^3 - 3x^4 - 4x^5 + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\big(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + O(x^6)\big) \\ &\quad + c_2\big(1\big(x\big(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + O(x^6)\big)\big)\ln(x) + 1 - x^2 - 2x^3 \\ &\quad - 3x^4 - 4x^5 + O(x^6)\big) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x\big(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + O(x^6)\big) \\ &\quad + c_2\big(x\big(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + O(x^6)\big)\ln(x) + 1 - x^2 - 2x^3 - 3x^4 - 4x^5 \\ &\quad + O(x^6)\big) \end{aligned}$$

Maple step by step solution

Let's solve

$x(x-1)\left(\frac{d^2}{dx^2}y(x)\right) + 3x\left(\frac{d}{dx}y(x)\right) + y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x) = -\frac{y(x)}{x(x-1)} - \frac{3\left(\frac{d}{dx}y(x)\right)}{x-1}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{3\left(\frac{d}{dx}y(x)\right)}{x-1} + \frac{y(x)}{x(x-1)} = 0$

- □ Check to see if x_0 is a regular singular point
  - Define functions

    $\left[P_2(x) = \frac{3}{x-1}, P_3(x) = \frac{1}{x(x-1)}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $(x \cdot P_2(x))\Big|_{x=0} = 0$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $(x^2 \cdot P_3(x))\Big|_{x=0} = 0$
  - $x = 0$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $x(x-1)\left(\frac{d^2}{dx^2}y(x)\right) + 3x\left(\frac{d}{dx}y(x)\right) + y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$
  - Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 1..2$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-2+m}$
  - Shift index using $k- >k+2-m$

    $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)(k+1-m+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $-a_0 r(-1+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(-a_{k+1}(k+r+1)(k+r) + a_k(k+r+1)^2\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $-r(-1+r)=0$
- Values of r that satisfy the indicial equation
  $r\in\{0,1\}$
- Each term in the series must be 0, giving the recursion relation
  $(k+r+1)\left(-a_{k+1}(k+r)+a_k(k+r+1)\right)=0$
- Recursion relation that defines series solution to ODE
  $a_{k+1}=\frac{a_k(k+r+1)}{k+r}$
- Recursion relation for $r=0$
  $a_{k+1}=\frac{a_k(k+1)}{k}$
- Solution for $r=0$
  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^k, a_{k+1}=\frac{a_k(k+1)}{k}\right]$
- Recursion relation for $r=1$
  $a_{k+1}=\frac{a_k(k+2)}{k+1}$
- Solution for $r=1$
  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1}=\frac{a_k(k+2)}{k+1}\right]$
- Combine solutions and rename parameters
  $\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^k\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+1}=\frac{a_k(k+1)}{k}, b_{k+1}=\frac{b_k(k+2)}{k+1}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

Maple dsolve solution

Solving time : 0.025 (sec)
Leaf size : 60

```
dsolve(x*(x-1)*diff(diff(y(x),x),x)+3*diff(y(x),x)*x+y(x) = 0,y(x),
       series,x=0)
```

$$
\begin{aligned}
y = {} & \ln(x)\left(x+2x^2+3x^3+4x^4+5x^5+\mathrm{O}\left(x^6\right)\right)c_2 \\
& + c_1 x\left(1+2x+3x^2+4x^3+5x^4+6x^5+\mathrm{O}\left(x^6\right)\right) \\
& + \left(1+3x+5x^2+7x^3+9x^4+11x^5+\mathrm{O}\left(x^6\right)\right)c_2
\end{aligned}
$$

Mathematica DSolve solution

Solving time : 0.044 (sec)
Leaf size : 63

```
AsymptoticDSolveValue[{x*(x-1)*D[y[x],{x,2}] +3*x*D[y[x],x]+y[x] == 0,{}},
       y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to {} & c_1\left(x^4+x^3+x^2+\left(4x^3+3x^2+2x+1\right)x\log(x)+x+1\right) \\
& + c_2\left(5x^5+4x^4+3x^3+2x^2+x\right)
\end{aligned}
$$

2.4.40 problem 37

Internal problem ID [8605]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 37
Date solved : `Thursday, December 12, 2024 at 09:32:04 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2\left(x^2-2x+1\right)y''-x(3+x)\,y'+(4+x)\,y=0$$

Using series expansion around $x=0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^4-2x^3+x^2\right)y''+\left(-x^2-3x\right)y'+(4+x)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=-\frac{3+x}{x\,(x-1)^2}$$
$$q(x)=\frac{4+x}{x^2\,(x-1)^2}$$

Table 2.207: Table $p(x), q(x)$ singularites.

$p(x)=-\frac{3+x}{x(x-1)^2}$	
singularity	type
$x=0$	"regular"
$x=1$	"irregular"

$q(x)=\frac{4+x}{x^2(x-1)^2}$	
singularity	type
$x=0$	"regular"
$x=1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty]$

Irregular singular points : $[1]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2(x^2-2x+1)\,y''+(-x^2-3x)\,y'+(4+x)\,y=0$$

Let the solution be represented as Frobenius power series of the form

$$y=\sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y'=\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned}
&x^2(x^2-2x+1)\left(\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r-2}\right)\\
&+(-x^2-3x)\left(\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\right)+(4+x)\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right)=0
\end{aligned}\tag{1}$$

Which simplifies to

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty}x^{n+r+2}a_n(n+r)\,(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-2x^{1+n+r}a_n(n+r)\,(n+r-1)\right)\\
&+\left(\sum_{n=0}^{\infty}x^{n+r}a_n(n+r)\,(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-x^{1+n+r}a_n(n+r)\right)\\
&+\sum_{n=0}^{\infty}\left(-3x^{n+r}a_n(n+r)\right)+\left(\sum_{n=0}^{\infty}4a_n x^{n+r}\right)+\left(\sum_{n=0}^{\infty}x^{1+n+r}a_n\right)=0
\end{aligned}\tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n(n+r)(n+r-1) = \sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r)\,x^{n+r}$$

$$\sum_{n=0}^{\infty} \left(-2x^{1+n+r} a_n(n+r)(n+r-1)\right) = \sum_{n=1}^{\infty} \left(-2a_{n-1}(n+r-1)(n+r-2)\,x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n(n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\,x^{n+r}\right)$$

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\begin{aligned}&\left(\sum_{n=2}^{\infty} a_{n-2}(n+r-2)(n-3+r)\,x^{n+r}\right)\\ &+\sum_{n=1}^{\infty}\left(-2a_{n-1}(n+r-1)(n+r-2)\,x^{n+r}\right)\\ &+\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right)+\sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)\,x^{n+r}\right)\\ &+\sum_{n=0}^{\infty}\left(-3x^{n+r} a_n(n+r)\right)+\left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right)+\left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right)=0\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n(n+r)(n+r-1) - 3x^{n+r} a_n(n+r) + 4a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) - 3x^r a_0 r + 4a_0 x^r = 0$$

Or

$$\left(x^r r(-1+r) - 3x^r r + 4x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r-2)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r-2)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 2 \\ r_2 &= 2 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r-2)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as $[2, 2]$.

Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 2$, Eqs (1A,1B) become

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+2} \\ y_2(x) &= y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+2}\right) \end{aligned}$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = \frac{2r+1}{-1+r}$$

For $2 \leq n$ the recursive equation is

$$\begin{aligned} &a_{n-2}(n+r-2)(n-3+r) - 2a_{n-1}(n+r-1)(n+r-2) \\ &+ a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) - 3a_n(n+r) + 4a_n + a_{n-1} = 0 \end{aligned} \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{na_{n-2} - 2na_{n-1} + ra_{n-2} - 2ra_{n-1} - 3a_{n-2} + a_{n-1}}{n+r-2} \tag{4}$$

Which for the root $r = 2$ becomes

$$a_n = \frac{(-a_{n-2} + 2a_{n-1})\, n + a_{n-2} + 3a_{n-1}}{n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 2$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r+1}{-1+r}$	5

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{3r^2+10r+2}{r(-1+r)}$$

Which for the root $r = 2$ becomes

$$a_2 = 17$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r+1}{-1+r}$	5
a_2	$\frac{3r^2+10r+2}{r(-1+r)}$	17

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{4r^3 + 34r^2 + 54r + 10}{r^3 - r}$$

Which for the root $r = 2$ becomes

$$a_3 = \frac{143}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r+1}{-1+r}$	5
a_2	$\frac{3r^2+10r+2}{r(-1+r)}$	17
a_3	$\frac{4r^3+34r^2+54r+10}{r^3-r}$	$\frac{143}{3}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{5r^4 + 80r^3 + 321r^2 + 384r + 68}{(2 + r)\, r\, (r^2 - 1)}$$

Which for the root $r = 2$ becomes

$$a_4 = \frac{355}{3}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r+1}{-1+r}$	5
a_2	$\frac{3r^2+10r+2}{r(-1+r)}$	17
a_3	$\frac{4r^3+34r^2+54r+10}{r^3-r}$	$\frac{143}{3}$
a_4	$\frac{5r^4+80r^3+321r^2+384r+68}{(2+r)r(r^2-1)}$	$\frac{355}{3}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{6r^5 + 155r^4 + 1156r^3 + 3295r^2 + 3336r + 572}{r^5 + 5r^4 + 5r^3 - 5r^2 - 6r}$$

Which for the root $r = 2$ becomes

$$a_5 = \frac{4043}{15}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{2r+1}{-1+r}$	5
a_2	$\frac{3r^2+10r+2}{r(-1+r)}$	17
a_3	$\frac{4r^3+34r^2+54r+10}{r^3-r}$	$\frac{143}{3}$
a_4	$\frac{5r^4+80r^3+321r^2+384r+68}{(2+r)r(r^2-1)}$	$\frac{355}{3}$
a_5	$\frac{6r^5+155r^4+1156r^3+3295r^2+3336r+572}{r^5+5r^4+5r^3-5r^2-6r}$	$\frac{4043}{15}$

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x^2 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= x^2 \left(17x^2 + 5x + 1 + \frac{143x^3}{3} + \frac{355x^4}{3} + \frac{4043x^5}{15} + O(x^6)\right) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 2$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	$\frac{2r+1}{-1+r}$	5	$-\frac{3}{(-1+r)^2}$
b_2	$\frac{3r^2+10r+2}{r(-1+r)}$	17	$\frac{-13r^2-4r+2}{r^2(-1+r)^2}$
b_3	$\frac{4r^3+34r^2+54r+10}{r^3-r}$	$\frac{143}{3}$	$\frac{-34r^4-116r^3-64r^2+10}{r^2(r^2-1)^2}$
b_4	$\frac{5r^4+80r^3+321r^2+384r+68}{(2+r)r(r^2-1)}$	$\frac{355}{3}$	$\frac{-70r^6-652r^5-1904r^4-2128r^3-666r^2+136r+136}{(2+r)^2r^2(r^2-1)^2}$
b_5	$\frac{6r^5+155r^4+1156r^3+3295r^2+3336r+572}{r^5+5r^4+5r^3-5r^2-6r}$	$\frac{4043}{15}$	$\frac{-125r^8-2252r^7-14980r^6-47988r^5-77945r^4-58672r^3-11670r^2+5720r+3432}{r^2(r^4+5r^3+5r^2-5r-6)^2}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6\dots \\
&= x^2\left(17x^2 + 5x + 1 + \frac{143x^3}{3} + \frac{355x^4}{3} + \frac{4043x^5}{15} + O(x^6)\right)\ln(x) \\
&\quad + x^2\left(-3x - \frac{29x^2}{2} - \frac{859x^3}{18} - \frac{4693x^4}{36} - \frac{285181x^5}{900} + O(x^6)\right)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1\,x^2\left(17x^2 + 5x + 1 + \frac{143x^3}{3} + \frac{355x^4}{3} + \frac{4043x^5}{15} + O(x^6)\right) \\
&\quad + c_2\left(x^2\left(17x^2 + 5x + 1 + \frac{143x^3}{3} + \frac{355x^4}{3} + \frac{4043x^5}{15} + O(x^6)\right)\ln(x)\right. \\
&\qquad \left. + x^2\left(-3x - \frac{29x^2}{2} - \frac{859x^3}{18} - \frac{4693x^4}{36} - \frac{285181x^5}{900} + O(x^6)\right)\right)
\end{aligned}$$

Hence the final solution is

$$\begin{aligned}
y &= y_h \\
&= c_1\,x^2\left(17x^2 + 5x + 1 + \frac{143x^3}{3} + \frac{355x^4}{3} + \frac{4043x^5}{15} + O(x^6)\right) \\
&\quad + c_2\left(x^2\left(17x^2 + 5x + 1 + \frac{143x^3}{3} + \frac{355x^4}{3} + \frac{4043x^5}{15} + O(x^6)\right)\ln(x)\right. \\
&\qquad \left. + x^2\left(-3x - \frac{29x^2}{2} - \frac{859x^3}{18} - \frac{4693x^4}{36} - \frac{285181x^5}{900} + O(x^6)\right)\right)
\end{aligned}$$

Maple step by step solution

Let's solve

$x^2(x^2-2x+1)\left(\frac{d^2}{dx^2}y(x)\right)-x(x+3)\left(\frac{d}{dx}y(x)\right)+(x+4)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x)=-\frac{(x+4)y(x)}{x^2(x^2-2x+1)}+\frac{(x+3)\left(\frac{d}{dx}y(x)\right)}{x(x^2-2x+1)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x)-\frac{(x+3)\left(\frac{d}{dx}y(x)\right)}{x(x^2-2x+1)}+\frac{(x+4)y(x)}{x^2(x^2-2x+1)}=0$

□ Check to see if x_0 is a regular singular point

- ○ Define functions

$\left[P_2(x)=-\frac{x+3}{x(x^2-2x+1)},P_3(x)=\frac{x+4}{x^2(x^2-2x+1)}\right]$

- ○ $x\cdot P_2(x)$ is analytic at $x=0$

$\left(x\cdot P_2(x)\right)\bigg|_{x=0}=-3$

- ○ $x^2\cdot P_3(x)$ is analytic at $x=0$

$\left(x^2\cdot P_3(x)\right)\bigg|_{x=0}=4$

- ○ $x=0$is a regular singular point

Check to see if x_0 is a regular singular point

$x_0=0$

- Multiply by denominators

$x^2(x^2-2x+1)\left(\frac{d^2}{dx^2}y(x)\right)-x(x+3)\left(\frac{d}{dx}y(x)\right)+(x+4)\,y(x)=0$

- Assume series solution for $y(x)$

$y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

- ○ Convert $x^m\cdot y(x)$ to series expansion for $m=0..1$

$x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

- ○ Shift index using $k->k-m$

$$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 1..2$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- >k+1-m$

$$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion for $m = 2..4$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-2+m}$$

- Shift index using $k- >k+2-m$

$$x^m \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0(-2+r)^2 x^r + \left(a_1(-1+r)^2 - a_0(1+2r)\,(-1+r)\right) x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r-2)^2 - a_{k-1}(2k -\right.\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$(-2+r)^2 = 0$$

- Values of r that satisfy the indicial equation

$$r = 2$$

- Each term must be 0

$$a_1(-1+r)^2 - a_0(1+2r)\,(-1+r) = 0$$

- Solve for the dependent coefficient(s)

$$a_1 = \frac{a_0(1+2r)}{-1+r}$$

- Each term in the series must be 0, giving the recursion relation

$$(k+r-2)\left((a_k + a_{k-2} - 2a_{k-1})\,k + (a_k + a_{k-2} - 2a_{k-1})\,r - 2a_k - 3a_{k-2} + a_{k-1}\right) = 0$$

- Shift index using $k- >k+2$

$$(k+r)\left((a_{k+2} + a_k - 2a_{k+1})\,(k+2) + (a_{k+2} + a_k - 2a_{k+1})\,r - 2a_{k+2} - 3a_k + a_{k+1}\right) = 0$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{ka_k - 2ka_{k+1} + ra_k - 2ra_{k+1} - a_k - 3a_{k+1}}{k+r}$$

- Recursion relation for $r = 2$

$$a_{k+2} = -\frac{ka_k - 2ka_{k+1} + a_k - 7a_{k+1}}{k+2}$$

- Solution for $r = 2$

$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2}=-\frac{k a_k-2 k a_{k+1}+a_k-7 a_{k+1}}{k+2}, a_1=5 a_0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.020 (sec)
Leaf size : 48

```
dsolve(x^2*(x^2-2*x+1)*diff(diff(y(x),x),x)-x*(x+3)*diff(y(x),x)+(x+4)*y(x) = 0,y(x),
       series,x=0)
```

$$y=\left(\left(c_2 \ln (x)+c_1\right)\left(1+5 x+17 x^2+\frac{143}{3} x^3+\frac{355}{3} x^4+\frac{4043}{15} x^5+\mathrm{O}\left(x^6\right)\right)+\left((-3) x-\frac{29}{2} x^2-\frac{859}{18} x^3-\frac{4693}{36} x^4-\frac{285181}{900} x^5+\mathrm{O}\left(x^6\right)\right) c_2\right) x^2$$

Mathematica DSolve solution

Solving time : 0.024 (sec)
Leaf size : 118

```
AsymptoticDSolveValue[{x^2*(1-2*x+x^2)*D[y[x],{x,2}] -x*(3+x)*D[y[x],x]+(4+x)*y[x] == 0,{}},
       y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \rightarrow\ & c_1\left(\frac{4043x^5}{15}+\frac{355x^4}{3}+\frac{143x^3}{3}+17x^2+5x+1\right)x^2 \\
& + c_2\left(\left(-\frac{285181x^5}{900}-\frac{4693x^4}{36}-\frac{859x^3}{18}-\frac{29x^2}{2}-3x\right)x^2\right. \\
& \left.+\left(\frac{4043x^5}{15}+\frac{355x^4}{3}+\frac{143x^3}{3}+17x^2+5x+1\right)x^2\log(x)\right)
\end{aligned}
$$

2.4.41 problem 38

Internal problem ID [8606]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 38
Date solved : `Thursday, December 12, 2024 at 09:32:05 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2(2+x)\,y''+5x^2y'+(1+x)\,y=0$$

Using series expansion around $x=0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(2x^3+4x^2\right)y''+5x^2y'+(1+x)\,y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x)y'+q(x)y=0$$

Where

$$p(x)=\frac{5}{2\,(2+x)}$$
$$q(x)=\frac{1+x}{2x^2\,(2+x)}$$

Table 2.209: Table $p(x), q(x)$ singularites.

$p(x)=\frac{5}{2(2+x)}$	
singularity	type
$x=-2$	"regular"

$q(x)=\frac{1+x}{2x^2(2+x)}$	
singularity	type
$x=-2$	"regular"
$x=0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[-2, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2(2+x)\,y'' + 5x^2y' + (1+x)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &2x^2(2+x)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) \\ &+ 5x^2\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (1+x)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\begin{aligned} &\left(\sum_{n=0}^{\infty} 2x^{1+n+r}a_n(n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r}a_n(n+r)\,(n+r-1)\right) \\ &+ \left(\sum_{n=0}^{\infty} 5x^{1+n+r}a_n(n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r}a_n\right) = 0 \end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2x^{1+n+r}a_n(n+r)(n+r-1) = \sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)(n+r-2)\,x^{n+r}$$

$$\sum_{n=0}^{\infty} 5x^{1+n+r}a_n(n+r) = \sum_{n=1}^{\infty} 5a_{n-1}(n+r-1)\,x^{n+r}$$

$$\sum_{n=0}^{\infty} x^{1+n+r}a_n = \sum_{n=1}^{\infty} a_{n-1}x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=1}^{\infty} 2a_{n-1}(n+r-1)(n+r-2)\,x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r}a_n(n+r)(n+r-1)\right) \\ + \left(\sum_{n=1}^{\infty} 5a_{n-1}(n+r-1)\,x^{n+r}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) + \left(\sum_{n=1}^{\infty} a_{n-1}x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$4x^{n+r}a_n(n+r)(n+r-1) + a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$4x^r a_0 r(-1+r) + a_0 x^r = 0$$

Or

$$(4x^r r(-1+r) + x^r)\,a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(2r-1)^2\,x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(2r-1)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(2r-1)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as $\left[\frac{1}{2}, \frac{1}{2}\right]$.

Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = \frac{1}{2}$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+\frac{1}{2}}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_{n-1}(n+r-1)(n+r-2) + 4a_n(n+r)(n+r-1) + 5a_{n-1}(n+r-1) + a_n + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{(n+r)\, a_{n-1}}{-1+2n+2r} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{(2n+1)\, a_{n-1}}{4n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{-1-r}{1+2r}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_1 = -\frac{3}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{1+2r}$	$-\frac{3}{4}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{r^2+3r+2}{4r^2+8r+3}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = \frac{15}{32}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{1+2r}$	$-\frac{3}{4}$
a_2	$\frac{r^2+3r+2}{4r^2+8r+3}$	$\frac{15}{32}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{-r^3 - 6r^2 - 11r - 6}{8r^3 + 36r^2 + 46r + 15}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_3 = -\frac{35}{128}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{1+2r}$	$-\frac{3}{4}$
a_2	$\frac{r^2+3r+2}{4r^2+8r+3}$	$\frac{15}{32}$
a_3	$\frac{-r^3-6r^2-11r-6}{8r^3+36r^2+46r+15}$	$-\frac{35}{128}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{r^4 + 10r^3 + 35r^2 + 50r + 24}{16r^4 + 128r^3 + 344r^2 + 352r + 105}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{315}{2048}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{1+2r}$	$-\frac{3}{4}$
a_2	$\frac{r^2+3r+2}{4r^2+8r+3}$	$\frac{15}{32}$
a_3	$\frac{-r^3-6r^2-11r-6}{8r^3+36r^2+46r+15}$	$-\frac{35}{128}$
a_4	$\frac{r^4+10r^3+35r^2+50r+24}{16r^4+128r^3+344r^2+352r+105}$	$\frac{315}{2048}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{-r^5 - 15r^4 - 85r^3 - 225r^2 - 274r - 120}{32r^5 + 400r^4 + 1840r^3 + 3800r^2 + 3378r + 945}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_5 = -\frac{693}{8192}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{-1-r}{1+2r}$	$-\frac{3}{4}$
a_2	$\frac{r^2+3r+2}{4r^2+8r+3}$	$\frac{15}{32}$
a_3	$\frac{-r^3-6r^2-11r-6}{8r^3+36r^2+46r+15}$	$-\frac{35}{128}$
a_4	$\frac{r^4+10r^3+35r^2+50r+24}{16r^4+128r^3+344r^2+352r+105}$	$\frac{315}{2048}$
a_5	$\frac{-r^5-15r^4-85r^3-225r^2-274r-120}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	$-\frac{693}{8192}$

Using the above table, then the first solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= \sqrt{x}\left(1 - \frac{3x}{4} + \frac{15x^2}{32} - \frac{35x^3}{128} + \frac{315x^4}{2048} - \frac{693x^5}{8192} + O(x^6)\right)
\end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = \frac{1}{2}$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$
b_0	1	1	N/A since b_n starts from 1
b_1	$\frac{-1-r}{1+2r}$	$-\frac{3}{4}$	$\frac{1}{(1+2r)^2}$
b_2	$\frac{r^2+3r+2}{4r^2+8r+3}$	$\frac{15}{32}$	$\frac{-4r^2-10r-7}{(4r^2+8r+3)^2}$
b_3	$\frac{-r^3-6r^2-11r-6}{8r^3+36r^2+46r+15}$	$-\frac{35}{128}$	$\frac{12r^4+84r^3+219r^2+252r+111}{(8r^3+36r^2+46r+15)^2}$
b_4	$\frac{r^4+10r^3+35r^2+50r+24}{16r^4+128r^3+344r^2+352r+105}$	$\frac{315}{2048}$	$\frac{-32r^6-432r^5-2384r^4-6876r^3-10946r^2-9162r-3198}{(16r^4+128r^3+344r^2+352r+105)^2}$
b_5	$\frac{-r^5-15r^4-85r^3-225r^2-274r-120}{32r^5+400r^4+1840r^3+3800r^2+3378r+945}$	$-\frac{693}{8192}$	$\frac{80r^8+1760r^7+16600r^6+87560r^5+282265r^4+569360r^3+702575r^2+486750r+1}{(32r^5+400r^4+1840r^3+3800r^2+3378r+945)^2}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= \sqrt{x}\left(1 - \frac{3x}{4} + \frac{15x^2}{32} - \frac{35x^3}{128} + \frac{315x^4}{2048} - \frac{693x^5}{8192} + O(x^6)\right)\ln(x) \\
&\quad + \sqrt{x}\left(\frac{x}{4} - \frac{13x^2}{64} + \frac{101x^3}{768} - \frac{641x^4}{8192} + \frac{7303x^5}{163840} + O(x^6)\right)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\sqrt{x}\left(1 - \frac{3x}{4} + \frac{15x^2}{32} - \frac{35x^3}{128} + \frac{315x^4}{2048} - \frac{693x^5}{8192} + O(x^6)\right) \\
&\quad + c_2\left(\sqrt{x}\left(1 - \frac{3x}{4} + \frac{15x^2}{32} - \frac{35x^3}{128} + \frac{315x^4}{2048} - \frac{693x^5}{8192} + O(x^6)\right)\ln(x)\right. \\
&\quad \left. + \sqrt{x}\left(\frac{x}{4} - \frac{13x^2}{64} + \frac{101x^3}{768} - \frac{641x^4}{8192} + \frac{7303x^5}{163840} + O(x^6)\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\sqrt{x}\left(1 - \frac{3x}{4} + \frac{15x^2}{32} - \frac{35x^3}{128} + \frac{315x^4}{2048} - \frac{693x^5}{8192} + O(x^6)\right) \\
&\quad + c_2\left(\sqrt{x}\left(1 - \frac{3x}{4} + \frac{15x^2}{32} - \frac{35x^3}{128} + \frac{315x^4}{2048} - \frac{693x^5}{8192} + O(x^6)\right)\ln(x)\right. \\
&\quad \left. + \sqrt{x}\left(\frac{x}{4} - \frac{13x^2}{64} + \frac{101x^3}{768} - \frac{641x^4}{8192} + \frac{7303x^5}{163840} + O(x^6)\right)\right)
\end{aligned}
$$

Maple step by step solution

Let's solve

$2x^2(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+5x^2\left(\frac{d}{dx}y(x)\right)+(x+1)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(x+1)y(x)}{2(x+2)x^2}-\frac{5\left(\frac{d}{dx}y(x)\right)}{2(x+2)}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{5\left(\frac{d}{dx}y(x)\right)}{2(x+2)}+\frac{(x+1)y(x)}{2(x+2)x^2}=0$

□ Check to see if x_0 is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{5}{2(x+2)},P_3(x)=\frac{x+1}{2(x+2)x^2}\right]$

  ◦ $(x+2)\cdot P_2(x)$ is analytic at $x=-2$

  $\left((x+2)\cdot P_2(x)\right)\bigg|_{x=-2}=\frac{5}{2}$

  ◦ $(x+2)^2\cdot P_3(x)$ is analytic at $x=-2$

  $\left((x+2)^2\cdot P_3(x)\right)\bigg|_{x=-2}=0$

  ◦ $x=-2$is a regular singular point

  Check to see if x_0 is a regular singular point

  $x_0=-2$

- Multiply by denominators

  $2x^2(x+2)\left(\frac{d^2}{dx^2}y(x)\right)+5x^2\left(\frac{d}{dx}y(x)\right)+(x+1)\,y(x)=0$

- Change variables using $x=u-2$ so that the regular singular point is at $u=0$

  $(2u^3-8u^2+8u)\left(\frac{d^2}{du^2}y(u)\right)+(5u^2-20u+20)\left(\frac{d}{du}y(u)\right)+(u-1)\,y(u)=0$

- Assume series solution for $y(u)$

  $y(u)=\sum_{k=0}^{\infty}a_k u^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $u^m\cdot y(u)$ to series expansion for $m=0..1$

$$u^m \cdot y(u) = \sum_{k=0}^{\infty} a_k u^{k+r+m}$$

- Shift index using $k->k-m$

$$u^m \cdot y(u) = \sum_{k=m}^{\infty} a_{k-m} u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d}{du}y(u)\right)$ to series expansion for $m = 0..2$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, u^{k+r-1+m}$$

- Shift index using $k->k+1-m$

$$u^m \cdot \left(\frac{d}{du}y(u)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, u^{k+r}$$

- Convert $u^m \cdot \left(\frac{d^2}{du^2}y(u)\right)$ to series expansion for $m = 1..3$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, u^{k+r-2+m}$$

- Shift index using $k->k+2-m$

$$u^m \cdot \left(\frac{d^2}{du^2}y(u)\right) = \sum_{k=-2+m}^{\infty} a_{k+2-m}(k+2-m+r)\,(k+1-m+r)\, u^{k+r}$$

Rewrite ODE with series expansions

$$4a_0r(3+2r)\, u^{-1+r} + \left(4a_1(1+r)\,(5+2r) - a_0(8r^2+12r+1)\right) u^r + \left(\sum_{k=1}^{\infty} (4a_{k+1}(k+r+1)\,(2k\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation

$$4r(3+2r) = 0$$

- Values of r that satisfy the indicial equation

$$r \in \left\{0, -\frac{3}{2}\right\}$$

- Each term must be 0

$$4a_1(1+r)\,(5+2r) - a_0(8r^2+12r+1) = 0$$

- Each term in the series must be 0, giving the recursion relation

$$2(-4a_k + a_{k-1} + 4a_{k+1})\, k^2 + (4(-4a_k + a_{k-1} + 4a_{k+1})\, r - 12a_k - a_{k-1} + 28a_{k+1})\, k + 2(-4a_k +$$

- Shift index using $k->k+1$

$$2(-4a_{k+1} + a_k + 4a_{k+2})\,(k+1)^2 + (4(-4a_{k+1} + a_k + 4a_{k+2})\, r - 12a_{k+1} - a_k + 28a_{k+2})\,(k+1)$$

- Recursion relation that defines series solution to ODE

$$a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 4kra_k - 16kra_{k+1} + 2r^2a_k - 8r^2a_{k+1} + 3ka_k - 28ka_{k+1} + 3ra_k - 28ra_{k+1} + a_k - 21a_{k+1}}{4(2k^2+4kr+2r^2+11k+11r+14)}$$

- Recursion relation for $r = 0$

$$a_{k+2} = -\frac{2k^2a_k - 8k^2a_{k+1} + 3ka_k - 28ka_{k+1} + a_k - 21a_{k+1}}{4(2k^2+11k+14)}$$

- Solution for $r = 0$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^k, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 3k a_k - 28k a_{k+1} + a_k - 21 a_{k+1}}{4(2k^2+11k+14)}, 20a_1 - a_0 = 0\right]$$

- Revert the change of variables $u = x + 2$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k (x+2)^k, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 3k a_k - 28k a_{k+1} + a_k - 21 a_{k+1}}{4(2k^2+11k+14)}, 20a_1 - a_0 = 0\right]$$

- Recursion relation for $r = -\frac{3}{2}$

$$a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 3k a_k - 4k a_{k+1} + a_k + 3a_{k+1}}{4(2k^2+5k+2)}$$

- Solution for $r = -\frac{3}{2}$

$$\left[y(u) = \sum_{k=0}^{\infty} a_k u^{k-\frac{3}{2}}, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 3k a_k - 4k a_{k+1} + a_k + 3a_{k+1}}{4(2k^2+5k+2)}, -4a_1 - a_0 = 0\right]$$

- Revert the change of variables $u = x + 2$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k (x+2)^{k-\frac{3}{2}}, a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} - 3k a_k - 4k a_{k+1} + a_k + 3a_{k+1}}{4(2k^2+5k+2)}, -4a_1 - a_0 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k (x+2)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x+2)^{k-\frac{3}{2}}\right), a_{k+2} = -\frac{2k^2 a_k - 8k^2 a_{k+1} + 3k a_k - 28k a_{k+1} + a_k - 21 a_{k+1}}{4(2k^2+11k+14)}, 2$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.021 (sec)
Leaf size : 48

```
dsolve(2*x^2*(x+2)*diff(diff(y(x),x),x)+5*diff(y(x),x)*x^2+y(x)*(x+1) = 0,y(x),
       series,x=0)
```

$$y = \sqrt{x}\left((c_2 \ln(x) + c_1)\left(1 - \frac{3}{4}x + \frac{15}{32}x^2 - \frac{35}{128}x^3 + \frac{315}{2048}x^4 - \frac{693}{8192}x^5 + \mathrm{O}\left(x^6\right)\right) + \left(\frac{1}{4}x - \frac{13}{64}x^2 + \frac{101}{768}x^3 - \frac{641}{8192}x^4 + \frac{7303}{163840}x^5 + \mathrm{O}\left(x^6\right)\right)c_2\right)$$

Mathematica DSolve solution

Solving time : 0.022 (sec)
Leaf size : 134

```
AsymptoticDSolveValue[{2*x^2*(2+x)*D[y[x],{x,2}] +5*x^2*D[y[x],x]+(1+x)*y[x] == 0,{}},
       y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt{x}\left(-\frac{693x^5}{8192} + \frac{315x^4}{2048} - \frac{35x^3}{128} + \frac{15x^2}{32} - \frac{3x}{4} + 1\right) + c_2\left(\sqrt{x}\left(\frac{7303x^5}{163840} - \frac{641x^4}{8192} + \frac{101x^3}{768} - \frac{13x^2}{64} + \frac{x}{4}\right) + \sqrt{x}\left(-\frac{693x^5}{8192} + \frac{315x^4}{2048} - \frac{35x^3}{128} + \frac{15x^2}{32} - \frac{3x}{4} + 1\right)\log(x)\right)$$

2.4.42 problem 39

Internal problem ID [8607]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 39
Date solved : `Thursday, December 12, 2024 at 09:32:06 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$2x^2y'' + xy' + (x-5)\,y = 0$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' + xy' + (x-5)\,y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{2x}$$
$$q(x) = \frac{x-5}{2x^2}$$

Table 2.211: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x-5}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' + xy' + (x - 5)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &2x^2 \left(\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}\right) \\ &+ x\left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) + (x - 5)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\begin{aligned} &\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n + r)\,(n + r - 1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n + r)\right) \\ &+ \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) + \sum_{n\,=0}^{\infty} \left(-5a_n x^{n+r}\right) = 0 \end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-5a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n(n+r)(n+r-1) + x^{n+r} a_n(n+r) - 5a_n x^{n+r} = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) + x^r a_0 r - 5a_0 x^r = 0$$

Or

$$\left(2x^r r(-1+r) + x^r r - 5x^r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 - r - 5\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 - r - 5 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{4} + \frac{\sqrt{41}}{4}$$
$$r_2 = \frac{1}{4} - \frac{\sqrt{41}}{4}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 - r - 5\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as $\left[\frac{1}{4} + \frac{\sqrt{41}}{4}, \frac{1}{4} - \frac{\sqrt{41}}{4}\right]$.

Since $r_1 - r_2 = \frac{\sqrt{41}}{2}$ is not an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{4}+\frac{\sqrt{41}}{4}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{4}-\frac{\sqrt{41}}{4}}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-1} - 5a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{2n^2 + 4nr + 2r^2 - n - r - 5} \tag{4}$$

Which for the root $r = \frac{1}{4} + \frac{\sqrt{41}}{4}$ becomes

$$a_n = -\frac{a_{n-1}}{n\left(\sqrt{41} + 2n\right)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{4} + \frac{\sqrt{41}}{4}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{2r^2 + 3r - 4}$$

Which for the root $r = \frac{1}{4} + \frac{\sqrt{41}}{4}$ becomes

$$a_1 = \frac{1}{-2 - \sqrt{41}}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+3r-4}$	$\frac{1}{-2-\sqrt{41}}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{4r^4 + 20r^3 + 15r^2 - 25r - 4}$$

Which for the root $r = \frac{1}{4} + \frac{\sqrt{41}}{4}$ becomes

$$a_2 = \frac{1}{2\left(2 + \sqrt{41}\right)\left(4 + \sqrt{41}\right)}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+3r-4}$	$\frac{1}{-2-\sqrt{41}}$
a_2	$\frac{1}{4r^4+20r^3+15r^2-25r-4}$	$\frac{1}{2\left(2+\sqrt{41}\right)\left(4+\sqrt{41}\right)}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{8r^6 + 84r^5 + 290r^4 + 315r^3 - 133r^2 - 294r - 40}$$

Which for the root $r = \frac{1}{4} + \frac{\sqrt{41}}{4}$ becomes

$$a_3 = -\frac{1}{3240 + 510\sqrt{41}}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+3r-4}$	$\frac{1}{-2-\sqrt{41}}$
a_2	$\frac{1}{4r^4+20r^3+15r^2-25r-4}$	$\frac{1}{2\left(2+\sqrt{41}\right)\left(4+\sqrt{41}\right)}$
a_3	$-\frac{1}{8r^6+84r^5+290r^4+315r^3-133r^2-294r-40}$	$-\frac{1}{3240+510\sqrt{41}}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16r^8 + 288r^7 + 2024r^6 + 6912r^5 + 11129r^4 + 4662r^3 - 7549r^2 - 7362r - 920}$$

Which for the root $r = \frac{1}{4} + \frac{\sqrt{41}}{4}$ becomes

$$a_4 = \frac{1}{187320 + 29280\sqrt{41}}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+3r-4}$	$\frac{1}{-2-\sqrt{41}}$
a_2	$\frac{1}{4r^4+20r^3+15r^2-25r-4}$	$\frac{1}{2\left(2+\sqrt{41}\right)\left(4+\sqrt{41}\right)}$
a_3	$-\frac{1}{8r^6+84r^5+290r^4+315r^3-133r^2-294r-40}$	$-\frac{1}{3240+510\sqrt{41}}$
a_4	$\frac{1}{16r^8+288r^7+2024r^6+6912r^5+11129r^4+4662r^3-7549r^2-7362r-920}$	$\frac{1}{187320+29280\sqrt{41}}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{32r^{10} + 880r^9 + 10160r^8 + 63800r^7 + 234546r^6 + 497255r^5 + 518640r^4 + 28325r^3 - 443678r^2 - 3}$$

Which for the root $r = \frac{1}{4} + \frac{\sqrt{41}}{4}$ becomes

$$a_5 = -\frac{1}{600\left(1561 + 244\sqrt{41}\right)\left(10 + \sqrt{41}\right)}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{2r^2+3r-4}$	$\frac{1}{-2-\sqrt{41}}$
a_2	$\frac{1}{4r^4+20r^3+15r^2-25r-4}$	$\frac{1}{2(2+\sqrt{41})(4+\sqrt{41})}$
a_3	$-\frac{1}{8r^6+84r^5+290r^4+315r^3-133r^2-294r-40}$	$-\frac{1}{3240+510\sqrt{41}}$
a_4	$\frac{1}{16r^8+288r^7+2024r^6+6912r^5+11129r^4+4662r^3-7549r^2-7362r-920}$	$\frac{1}{187320+29280\sqrt{41}}$
a_5	$-\frac{1}{32r^{10}+880r^9+10160r^8+63800r^7+234546r^6+497255r^5+518640r^4+28325r^3-443678r^2-311960r-36800}$	$-\frac{1}{600(1561+244\sqrt{41})(}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^{\frac{1}{4}+\frac{\sqrt{41}}{4}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^{\frac{1}{4}+\frac{\sqrt{41}}{4}}\left(1 + \frac{x}{-2-\sqrt{41}} + \frac{x^2}{2\left(2+\sqrt{41}\right)\left(4+\sqrt{41}\right)} - \frac{x^3}{3240+510\sqrt{41}} + \frac{x^4}{187320+29280\sqrt{41}} - \frac{}{6}\right.
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$
2b_n(n+r)(n+r-1) + b_n(n+r) + b_{n-1} - 5b_n = 0 \tag{3}
$$

Solving for b_n from recursive equation (4) gives

$$
b_n = -\frac{b_{n-1}}{2n^2 + 4nr + 2r^2 - n - r - 5} \tag{4}
$$

Which for the root $r = \frac{1}{4} - \frac{\sqrt{41}}{4}$ becomes

$$
b_n = \frac{b_{n-1}}{n\left(\sqrt{41} - 2n\right)} \tag{5}
$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = \frac{1}{4} - \frac{\sqrt{41}}{4}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$
b_1 = -\frac{1}{2r^2 + 3r - 4}
$$

Which for the root $r = \frac{1}{4} - \frac{\sqrt{41}}{4}$ becomes

$$b_1 = \frac{1}{-2+\sqrt{41}}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+3r-4}$	$\frac{1}{-2+\sqrt{41}}$

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{4r^4+20r^3+15r^2-25r-4}$$

Which for the root $r = \frac{1}{4} - \frac{\sqrt{41}}{4}$ becomes

$$b_2 = \frac{1}{2\left(-2+\sqrt{41}\right)\left(-4+\sqrt{41}\right)}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+3r-4}$	$\frac{1}{-2+\sqrt{41}}$
b_2	$\frac{1}{4r^4+20r^3+15r^2-25r-4}$	$\frac{1}{2\left(-2+\sqrt{41}\right)\left(-4+\sqrt{41}\right)}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{1}{8r^6+84r^5+290r^4+315r^3-133r^2-294r-40}$$

Which for the root $r = \frac{1}{4} - \frac{\sqrt{41}}{4}$ becomes

$$b_3 = \frac{1}{-3240+510\sqrt{41}}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+3r-4}$	$\frac{1}{-2+\sqrt{41}}$
b_2	$\frac{1}{4r^4+20r^3+15r^2-25r-4}$	$\frac{1}{2\left(-2+\sqrt{41}\right)\left(-4+\sqrt{41}\right)}$
b_3	$-\frac{1}{8r^6+84r^5+290r^4+315r^3-133r^2-294r-40}$	$\frac{1}{-3240+510\sqrt{41}}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{16r^8 + 288r^7 + 2024r^6 + 6912r^5 + 11129r^4 + 4662r^3 - 7549r^2 - 7362r - 920}$$

Which for the root $r = \frac{1}{4} - \frac{\sqrt{41}}{4}$ becomes

$$b_4 = \frac{1}{187320 - 29280\sqrt{41}}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+3r-4}$	$\frac{1}{-2+\sqrt{41}}$
b_2	$\frac{1}{4r^4+20r^3+15r^2-25r-4}$	$\frac{1}{2\left(-2+\sqrt{41}\right)\left(-4+\sqrt{41}\right)}$
b_3	$-\frac{1}{8r^6+84r^5+290r^4+315r^3-133r^2-294r-40}$	$\frac{1}{-3240+510\sqrt{41}}$
b_4	$\frac{1}{16r^8+288r^7+2024r^6+6912r^5+11129r^4+4662r^3-7549r^2-7362r-920}$	$\frac{1}{187320-29280\sqrt{41}}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{1}{32r^{10} + 880r^9 + 10160r^8 + 63800r^7 + 234546r^6 + 497255r^5 + 518640r^4 + 28325r^3 - 443678r^2 - 3}$$

Which for the root $r = \frac{1}{4} - \frac{\sqrt{41}}{4}$ becomes

$$b_5 = -\frac{1}{600\left(-1561 + 244\sqrt{41}\right)\left(-10 + \sqrt{41}\right)}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{2r^2+3r-4}$	$\frac{1}{-2+\sqrt{41}}$
b_2	$\frac{1}{4r^4+20r^3+15r^2-25r-4}$	$\frac{1}{2\left(-2+\sqrt{41}\right)\left(-4+\sqrt{41}\right)}$
b_3	$-\frac{1}{8r^6+84r^5+290r^4+315r^3-133r^2-294r-40}$	$\frac{1}{-3240+510\sqrt{41}}$
b_4	$\frac{1}{16r^8+288r^7+2024r^6+6912r^5+11129r^4+4662r^3-7549r^2-7362r-920}$	$\frac{1}{187320-29280\sqrt{41}}$
b_5	$-\frac{1}{32r^{10}+880r^9+10160r^8+63800r^7+234546r^6+497255r^5+518640r^4+28325r^3-443678r^2-311960r-36800}$	$-\frac{1}{600\left(-1561+244\sqrt{41}\right)}$

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= x^{\frac{1}{4}+\frac{\sqrt{41}}{4}}\left(b_0+b_1x+b_2x^2+b_3x^3+b_4x^4+b_5x^5+b_6x^6\dots\right) \\
&= x^{\frac{1}{4}-\frac{\sqrt{41}}{4}}\left(1+\frac{x}{-2+\sqrt{41}}+\frac{x^2}{2\left(-2+\sqrt{41}\right)\left(-4+\sqrt{41}\right)}+\frac{x^3}{-3240+510\sqrt{41}}+\frac{x^4}{187320-29280\sqrt{41}}\right.
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1y_1(x)+c_2y_2(x) \\
&= c_1\,x^{\frac{1}{4}+\frac{\sqrt{41}}{4}}\left(1+\frac{x}{-2-\sqrt{41}}+\frac{x^2}{2\left(2+\sqrt{41}\right)\left(4+\sqrt{41}\right)}-\frac{x^3}{3240+510\sqrt{41}}\right. \\
&\quad\left.+\frac{x^4}{187320+29280\sqrt{41}}-\frac{x^5}{600\left(1561+244\sqrt{41}\right)\left(10+\sqrt{41}\right)}+O\left(x^6\right)\right) \\
&\quad+c_2\,x^{\frac{1}{4}-\frac{\sqrt{41}}{4}}\left(1+\frac{x}{-2+\sqrt{41}}+\frac{x^2}{2\left(-2+\sqrt{41}\right)\left(-4+\sqrt{41}\right)}+\frac{x^3}{-3240+510\sqrt{41}}\right. \\
&\quad\left.+\frac{x^4}{187320-29280\sqrt{41}}-\frac{x^5}{600\left(-1561+244\sqrt{41}\right)\left(-10+\sqrt{41}\right)}+O\left(x^6\right)\right)
\end{aligned}
$$

Hence the final solution is

$$
y = y_h
$$

$$
\begin{aligned}
&= c_1\, x^{\frac{1}{4}+\frac{\sqrt{41}}{4}} \left(1 + \frac{x}{-2-\sqrt{41}} + \frac{x^2}{2\left(2+\sqrt{41}\right)\left(4+\sqrt{41}\right)} - \frac{x^3}{3240+510\sqrt{41}}\right. \\
&\qquad \left. + \frac{x^4}{187320+29280\sqrt{41}} - \frac{x^5}{600\left(1561+244\sqrt{41}\right)\left(10+\sqrt{41}\right)} + O\left(x^6\right)\right) \\
&+ c_2\, x^{\frac{1}{4}-\frac{\sqrt{41}}{4}} \left(1 + \frac{x}{-2+\sqrt{41}} + \frac{x^2}{2\left(-2+\sqrt{41}\right)\left(-4+\sqrt{41}\right)} + \frac{x^3}{-3240+510\sqrt{41}}\right. \\
&\qquad \left. + \frac{x^4}{187320-29280\sqrt{41}} - \frac{x^5}{600\left(-1561+244\sqrt{41}\right)\left(-10+\sqrt{41}\right)} + O\left(x^6\right)\right)
\end{aligned}
$$

Maple step by step solution

Let's solve

$2x^2\left(\frac{d^2}{dx^2}y(x)\right) + x\left(\frac{d}{dx}y(x)\right) + (x-5)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{(x-5)y(x)}{2x^2} - \frac{\frac{d}{dx}y(x)}{2x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{\frac{d}{dx}y(x)}{2x} + \frac{(x-5)y(x)}{2x^2} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  - Define functions

    $\left[P_2(x) = \frac{1}{2x}, P_3(x) = \frac{x-5}{2x^2}\right]$

  - $x \cdot P_2(x)$ is analytic at $x = 0$

    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = \frac{1}{2}$

  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$

    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = -\frac{5}{2}$

  - $x = 0$is a regular singular point

    Check to see if $x_0 = 0$ is a regular singular point

    $x_0 = 0$

- Multiply by denominators

  $2x^2\left(\frac{d^2}{dx^2}y(x)\right) + x\left(\frac{d}{dx}y(x)\right) + (x-5)\,y(x) = 0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty} a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot y(x)$ to series expansion for $m=0..1$

    $x^m \cdot y(x)=\sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - ◦ Shift index using $k->k-m$

    $x^m \cdot y(x)=\sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - ◦ Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$
  - ◦ Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)=\sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2r^2-r-5)\, x^r+\left(\sum_{k=1}^{\infty} \left(a_k(2k^2+4kr+2r^2-k-r-5)+a_{k-1}\right) x^{k+r}\right)=0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $2r^2-r-5=0$
- Values of r that satisfy the indicial equation

  $r \in \left\{\frac{1}{4}-\frac{\sqrt{41}}{4}, \frac{1}{4}+\frac{\sqrt{41}}{4}\right\}$
- Each term in the series must be 0, giving the recursion relation

  $(2k^2+(4r-1)\,k+2r^2-r-5)\, a_k+a_{k-1}=0$
- Shift index using $k->k+1$

  $\left(2(k+1)^2+(4r-1)\,(k+1)+2r^2-r-5\right) a_{k+1}+a_k=0$
- Recursion relation that defines series solution to ODE

  $a_{k+1}=-\frac{a_k}{2k^2+4kr+2r^2+3k+3r-4}$
- Recursion relation for $r=\frac{1}{4}-\frac{\sqrt{41}}{4}$

  $a_{k+1}=-\frac{a_k}{2k^2+4k\left(\frac{1}{4}-\frac{\sqrt{41}}{4}\right)+2\left(\frac{1}{4}-\frac{\sqrt{41}}{4}\right)^2+3k-\frac{13}{4}-\frac{3\sqrt{41}}{4}}$
- Solution for $r=\frac{1}{4}-\frac{\sqrt{41}}{4}$

$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}-\frac{\sqrt{41}}{4}}, a_{k+1}=-\frac{a_k}{2k^2+4k\left(\frac{1}{4}-\frac{\sqrt{41}}{4}\right)+2\left(\frac{1}{4}-\frac{\sqrt{41}}{4}\right)^2+3k-\frac{13}{4}-\frac{3\sqrt{41}}{4}}\right]$$

- Recursion relation for $r=\frac{1}{4}+\frac{\sqrt{41}}{4}$

  $$a_{k+1}=-\frac{a_k}{2k^2+4k\left(\frac{1}{4}+\frac{\sqrt{41}}{4}\right)+2\left(\frac{1}{4}+\frac{\sqrt{41}}{4}\right)^2+3k-\frac{13}{4}+\frac{3\sqrt{41}}{4}}$$

- Solution for $r=\frac{1}{4}+\frac{\sqrt{41}}{4}$

  $$\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}+\frac{\sqrt{41}}{4}}, a_{k+1}=-\frac{a_k}{2k^2+4k\left(\frac{1}{4}+\frac{\sqrt{41}}{4}\right)+2\left(\frac{1}{4}+\frac{\sqrt{41}}{4}\right)^2+3k-\frac{13}{4}+\frac{3\sqrt{41}}{4}}\right]$$

- Combine solutions and rename parameters

  $$\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{4}-\frac{\sqrt{41}}{4}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{4}+\frac{\sqrt{41}}{4}}\right), a_{k+1}=-\frac{a_k}{2k^2+4k\left(\frac{1}{4}-\frac{\sqrt{41}}{4}\right)+2\left(\frac{1}{4}-\frac{\sqrt{41}}{4}\right)^2+3k-\frac{13}{4}-\frac{3\sqrt{41}}{4}},\right.$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

Maple dsolve solution

Solving time : 0.021 (sec)
Leaf size : 277

```
dsolve(2*x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x+(x-5)*y(x) = 0,y(x),
       series,x=0)
```

$$y = x^{1/4}\left(c_1 x^{-\frac{\sqrt{41}}{4}}\left(1+\frac{1}{-2+\sqrt{41}}x+\frac{1}{2}\frac{1}{\left(-2+\sqrt{41}\right)\left(-4+\sqrt{41}\right)}x^2+\frac{1}{6}\frac{1}{\left(-2+\sqrt{41}\right)\left(-4+\sqrt{41}\right)\left(-6+\sqrt{41}\right)} \right.\right.$$
$$+c_2 x^{\frac{\sqrt{41}}{4}}\left(1+\frac{1}{-2-\sqrt{41}}x+\frac{1}{2}\frac{1}{\left(2+\sqrt{41}\right)\left(4+\sqrt{41}\right)}x^2-\frac{1}{6}\frac{1}{\left(2+\sqrt{41}\right)\left(4+\sqrt{41}\right)\left(6+\sqrt{41}\right)}x^3+\frac{1}{24}\frac{}{(2+}\right.$$

Mathematica DSolve solution

Solving time : 0.006 (sec)
Leaf size : 1668

```
AsymptoticDSolveValue[{2*x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x-5)*y[x]==0,{}},
       y[x],{x,0,5}]
```

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2.4.43 problem 40

Internal problem ID [8608]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 40
Date solved : `Thursday, December 12, 2024 at 09:32:08 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$2x^2y'' + 2xy' - xy = \sin(x)$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' + 2xy' - xy = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{1}{2x}$$

Table 2.213: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' + 2xy' - xy = \sin(x)$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $2x^2y'' + 2xy' - xy = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + 2x \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) - x \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-x^{1+n+r}a_n\right)=\sum_{n=1}^{\infty}\left(-a_{n-1}x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty}2x^{n+r}a_n(n+r)\,(n+r-1)\right)+\left(\sum_{n=0}^{\infty}2x^{n+r}a_n(n+r)\right)+\sum_{n=1}^{\infty}\left(-a_{n-1}x^{n+r}\right)=0 \quad \text{(2B)}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r}a_n(n+r)\,(n+r-1)+2x^{n+r}a_n(n+r)=0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r)+2x^r a_0 r=0$$

Or

$$\left(2x^r r(-1+r)+2x^r r\right)a_0=0$$

Since $a_0\neq 0$ then the above simplifies to

$$2x^r r^2=0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2=0$$

Solving for r gives the roots of the indicial equation as

$$r_1=0$$
$$r_2=0$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$\left(2x^m m(-1+m)+2x^m m\right)c_0=\sin(x)$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$2x^r r^2 = 0$$

Solving for r gives the roots of the indicial equation as $[0, 0]$.

Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + 2a_n(n+r) - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{2n^2 + 4nr + 2r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-1}}{2n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{2(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_1 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2(r+1)^2}$	$\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{4(r+1)^2(r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{16}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2(r+1)^2}$	$\frac{1}{2}$
a_2	$\frac{1}{4(r+1)^2(r+2)^2}$	$\frac{1}{16}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{8(r+1)^2(r+2)^2(r+3)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{1}{288}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2(r+1)^2}$	$\frac{1}{2}$
a_2	$\frac{1}{4(r+1)^2(r+2)^2}$	$\frac{1}{16}$
a_3	$\frac{1}{8(r+1)^2(r+2)^2(r+3)^2}$	$\frac{1}{288}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16\left(r+1\right)^2\left(r+2\right)^2\left(r+3\right)^2\left(r+4\right)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{9216}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2(r+1)^2}$	$\frac{1}{2}$
a_2	$\frac{1}{4(r+1)^2(r+2)^2}$	$\frac{1}{16}$
a_3	$\frac{1}{8(r+1)^2(r+2)^2(r+3)^2}$	$\frac{1}{288}$
a_4	$\frac{1}{16(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{1}{9216}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{32\left(r+1\right)^2\left(r+2\right)^2\left(r+3\right)^2\left(r+4\right)^2\left(r+5\right)^2}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{1}{460800}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2(r+1)^2}$	$\frac{1}{2}$
a_2	$\frac{1}{4(r+1)^2(r+2)^2}$	$\frac{1}{16}$
a_3	$\frac{1}{8(r+1)^2(r+2)^2(r+3)^2}$	$\frac{1}{288}$
a_4	$\frac{1}{16(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{1}{9216}$
a_5	$\frac{1}{32(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$\frac{1}{460800}$

Using the above table, then the first solution $y_1(x)$ becomes

$$
\begin{aligned}
y_1(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots \\
&= 1 + \frac{x}{2} + \frac{x^2}{16} + \frac{x^3}{288} + \frac{x^4}{9216} + \frac{x^5}{460800} + O(x^6)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r=0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{1}{2(r+1)^2}$	$\frac{1}{2}$	$-\frac{1}{(r+1)^3}$	-1
b_2	$\frac{1}{4(r+1)^2(r+2)^2}$	$\frac{1}{16}$	$\frac{-2r-3}{2(r+1)^3(r+2)^3}$	$-\frac{3}{16}$
b_3	$\frac{1}{8(r+1)^2(r+2)^2(r+3)^2}$	$\frac{1}{288}$	$\frac{-3r^2-12r-11}{4(r+1)^3(r+2)^3(r+3)^3}$	$-\frac{11}{864}$
b_4	$\frac{1}{16(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{1}{9216}$	$\frac{-2r^3-15r^2-35r-25}{4(r+1)^3(r+2)^3(r+3)^3(r+4)^3}$	$-\frac{25}{55296}$
b_5	$\frac{1}{32(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$\frac{1}{460800}$	$\frac{-5r^4-60r^3-255r^2-450r-274}{16(r+1)^3(r+2)^3(r+3)^3(r+4)^3(r+5)^3}$	$-\frac{137}{13824000}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6\dots \\
&= \left(1 + \frac{x}{2} + \frac{x^2}{16} + \frac{x^3}{288} + \frac{x^4}{9216} + \frac{x^5}{460800} + O(x^6)\right)\ln(x) \\
&\quad - x - \frac{3x^2}{16} - \frac{11x^3}{864} - \frac{25x^4}{55296} - \frac{137x^5}{13824000} + O(x^6)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1y_1(x) + c_2y_2(x) \\
&= c_1\left(1 + \frac{x}{2} + \frac{x^2}{16} + \frac{x^3}{288} + \frac{x^4}{9216} + \frac{x^5}{460800} + O(x^6)\right) \\
&\quad + c_2\left(\left(1 + \frac{x}{2} + \frac{x^2}{16} + \frac{x^3}{288} + \frac{x^4}{9216} + \frac{x^5}{460800} + O(x^6)\right)\ln(x) - x - \frac{3x^2}{16} - \frac{11x^3}{864} \right. \\
&\quad \left. - \frac{25x^4}{55296} - \frac{137x^5}{13824000} + O(x^6)\right)
\end{aligned}$$

The particular solution is found by solving for c, m the balance equation

$$(2x^m m(-1+m) + 2x^m m)\, c_0 = F$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$a_1 = \frac{a_0}{2(r+1)^2}$
$a_2 = \frac{a_0}{4(r+1)^2(r+2)^2}$
$a_3 = \frac{a_0}{8(r+1)^2(r+2)^2(r+3)^2}$
$a_4 = \frac{a_0}{16(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$
$a_5 = \frac{a_0}{32(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$

Expanding the rhs of the ode $\sin(x)$ in series gives

$$\sin(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

Since the $F = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$ has more than one term then we find a particular solution for each term and add the result to find the particular solution to the ode.

Now we determine the particular solution y_p associated with $F = x$ by solving the balance equation

$$(2x^m m(-1+m) + 2x^m m)\, c_0 = x$$

For c_0 and x. This results in

$$\begin{aligned} c_0 &= \frac{1}{2} \\ m &= 1 \end{aligned}$$

The particular solution is therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= \sum_{n=0}^{\infty} c_n x^{n+1} \end{aligned}$$

Where in the above $c_0 = \frac{1}{2}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 1$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{2}$ and $r = m$ or $r = 1$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = \frac{1}{2}$
$c_1 = \frac{1}{16}$
$c_2 = \frac{1}{288}$
$c_3 = \frac{1}{9216}$
$c_4 = \frac{1}{460800}$
$c_5 = \frac{1}{33177600}$

The particular solution is now found using

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= x \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x\left(\frac{1}{2}+\frac{1}{16}x+\frac{1}{288}x^2+\frac{1}{9216}x^3+\frac{1}{460800}x^4+\frac{1}{33177600}x^5\right) \\
&= \frac{1}{2}x+\frac{1}{16}x^2+\frac{1}{288}x^3+\frac{1}{9216}x^4+\frac{1}{460800}x^5+\frac{1}{33177600}x^6
\end{aligned}
$$

Now we determine the particular solution y_p associated with $F=-\frac{x^3}{6}$ by solving the balance equation

$$
(2x^m m(-1+m)+2x^m m)\,c_0=-\frac{x^3}{6}
$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= -\frac{1}{108} \\
m &= 3
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= \sum_{n=0}^{\infty} c_n x^{n+3}
\end{aligned}
$$

Where in the above $c_0=-\frac{1}{108}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m=3$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0=c_0$ or $a_0=-\frac{1}{108}$ and $r=m$ or $r=3$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0=-\frac{1}{108}$
$c_1=-\frac{1}{3456}$
$c_2=-\frac{1}{172800}$
$c_3=-\frac{1}{12441600}$
$c_4=-\frac{1}{1219276800}$
$c_5=-\frac{1}{156067430400}$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
&= x^3 \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x^3 \left(-\frac{1}{108} - \frac{1}{3456}x - \frac{1}{172800}x^2 - \frac{1}{12441600}x^3 - \frac{1}{1219276800}x^4 - \frac{1}{156067430400}x^5\right) \\
&= -\frac{1}{108}x^3 - \frac{1}{3456}x^4 - \frac{1}{172800}x^5 - \frac{1}{12441600}x^6 - \frac{1}{1219276800}x^7 - \frac{1}{156067430400}x^8
\end{aligned}
$$

Now we determine the particular solution y_p associated with $F = \frac{x^5}{120}$ by solving the balance equation

$$
(2x^m m(-1+m) + 2x^m m)\, c_0 = \frac{x^5}{120}
$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= \frac{1}{6000} \\
m &= 5
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= \sum_{n=0}^{\infty} c_n x^{n+5}
\end{aligned}
$$

Where in the above $c_0 = \frac{1}{6000}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 5$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{6000}$ and $r = m$ or $r = 5$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$$\boxed{\begin{aligned} c_0 &= \tfrac{1}{6000} \\ c_1 &= \tfrac{1}{432000} \\ c_2 &= \tfrac{1}{42336000} \\ c_3 &= \tfrac{1}{5419008000} \\ c_4 &= \tfrac{1}{877879296000} \\ c_5 &= \tfrac{1}{175575859200000} \end{aligned}}$$

The particular solution is now found using

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= x^5 \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$\begin{aligned} y_p &= x^5 \left(\frac{1}{6000} + \frac{1}{432000} x + \frac{1}{42336000} x^2 + \frac{1}{5419008000} x^3 + \frac{1}{877879296000} x^4 \right. \\ &\quad \left. + \frac{1}{175575859200000} x^5 \right) \\ &= \frac{1}{6000} x^5 + \frac{1}{432000} x^6 + \frac{1}{42336000} x^7 + \frac{1}{5419008000} x^8 \\ &\quad + \frac{1}{877879296000} x^9 + \frac{1}{175575859200000} x^{10} \end{aligned}$$

Adding all the above particular solution(s) gives

$$\begin{aligned} y_p = {} & \frac{x}{2} + \frac{x^2}{16} - \frac{5x^3}{864} - \frac{5x^4}{27648} + \frac{1127x^5}{6912000} + \frac{1127x^6}{497664000} + \frac{139x^7}{6096384000} \\ & + \frac{139x^8}{780337152000} + \frac{x^9}{877879296000} + \frac{x^{10}}{175575859200000} + O(x^6) \end{aligned}$$

Truncating the particular solution to the order of series requested gives

$$y_p = \frac{x}{2} + \frac{x^2}{16} - \frac{5x^3}{864} - \frac{5x^4}{27648} + \frac{1127x^5}{6912000} + O(x^6)$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \frac{x}{2} + \frac{x^2}{16} - \frac{5x^3}{864} - \frac{5x^4}{27648} + \frac{1127x^5}{6912000} + O(x^6) \\
&\quad + c_1 \left(1 + \frac{x}{2} + \frac{x^2}{16} + \frac{x^3}{288} + \frac{x^4}{9216} + \frac{x^5}{460800} + O(x^6)\right) \\
&\quad + c_2 \left(\left(1 + \frac{x}{2} + \frac{x^2}{16} + \frac{x^3}{288} + \frac{x^4}{9216} + \frac{x^5}{460800} + O(x^6)\right) \ln(x) - x - \frac{3x^2}{16} - \frac{11x^3}{864} \right. \\
&\qquad \left. - \frac{25x^4}{55296} - \frac{137x^5}{13824000} + O(x^6)\right)
\end{aligned}
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : `0.019` (sec)
Leaf size : 62

```
dsolve(2*x^2*diff(diff(y(x),x),x)+2*diff(y(x),x)*x-x*y(x) = sin(x),y(x),
       series,x=0)
```

$$
\begin{aligned}
y = {} & (c_2 \ln(x) + c_1)\left(1 + \frac{1}{2}x + \frac{1}{16}x^2 + \frac{1}{288}x^3 + \frac{1}{9216}x^4 + \frac{1}{460800}x^5 + \mathrm{O}\left(x^6\right)\right) \\
& + x\left(\frac{1}{2} + \frac{1}{16}x - \frac{5}{864}x^2 - \frac{5}{27648}x^3 + \frac{1127}{6912000}x^4 + \frac{1127}{497664000}x^5 + \mathrm{O}\left(x^6\right)\right) \\
& + \left(-x - \frac{3}{16}x^2 - \frac{11}{864}x^3 - \frac{25}{55296}x^4 - \frac{137}{13824000}x^5 + \mathrm{O}\left(x^6\right)\right)c_2
\end{aligned}
$$

Mathematica DSolve solution

Solving time : `0.167` (sec)
Leaf size : 340

```
AsymptoticDSolveValue[{2*x^2*D[y[x],{x,2}]+2*x*D[y[x],x]-x*y[x]==Sin[x],{}},
       y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to {} & c_2\left(\frac{x^5}{460800} + \frac{x^4}{9216} + \frac{x^3}{288} + \frac{x^2}{16} + \frac{x}{2} + 1\right) \\
& + c_1\left(x^5\left(\frac{\log(x)}{460800} - \frac{107}{13824000}\right) + x^4\left(\frac{\log(x)}{9216} - \frac{19}{55296}\right) + x^3\left(\frac{\log(x)}{288} - \frac{1}{108}\right)\right. \\
& \left. + x^2\left(\frac{\log(x)}{16} - \frac{1}{8}\right) + x\left(\frac{\log(x)}{2} - \frac{1}{2}\right) + \log(x) + 1\right) + \left(\frac{4963x^6}{16588800} - \frac{91x^5}{460800}\right. \\
& \left. - \frac{23x^4}{2304} - \frac{5x^3}{288} + \frac{x^2}{8} + \frac{x}{2}\right)\left(x^5\left(\frac{\log(x)}{460800} - \frac{107}{13824000}\right) + x^4\left(\frac{\log(x)}{9216} - \frac{19}{55296}\right)\right. \\
& \left. + x^3\left(\frac{\log(x)}{288} - \frac{1}{108}\right) + x^2\left(\frac{\log(x)}{16} - \frac{1}{8}\right) + x\left(\frac{\log(x)}{2} - \frac{1}{2}\right) + \log(x) + 1\right) \\
& + \left(\frac{x^5}{460800} + \frac{x^4}{9216} + \frac{x^3}{288} + \frac{x^2}{16} + \frac{x}{2} + 1\right)\left(\frac{x^6(66968 - 74445\log(x))}{248832000}\right. \\
& + \frac{13x^5(210\log(x) - 3107)}{13824000} + \frac{x^4(276\log(x) - 325)}{27648} + \frac{1}{864}x^3(15\log(x) + 37) \\
& \left. + \frac{1}{16}x^2(3 - 2\log(x)) - \frac{1}{2}x\log(x)\right)
\end{aligned}
$$

2.4.44 problem 41

Internal problem ID [8609]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 41
Date solved : `Thursday, December 12, 2024 at 09:32:09 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$2x^2y'' + 2xy' - xy = x\sin(x)$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' + 2xy' - xy = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{1}{2x}$$

Table 2.214: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' + 2xy' - xy = x\sin(x)$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $2x^2y'' + 2xy' - xy = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} &2x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\ &+ 2x\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - x\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right) = 0 \quad \text{(2B)}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n(n+r)\,(n+r-1) + 2x^{n+r} a_n(n+r) = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) + 2x^r a_0 r = 0$$

Or

$$\left(2x^r r(-1+r) + 2x^r r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$2x^r r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$\left(2x^m m(-1+m) + 2x^m m\right) c_0 = x \sin(x)$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$2x^r r^2 = 0$$

Solving for r gives the roots of the indicial equation as $[0, 0]$.

Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + 2a_n(n+r) - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{2n^2 + 4nr + 2r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-1}}{2n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{2(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_1 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2(r+1)^2}$	$\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{4(r+1)^2(r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{16}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2(r+1)^2}$	$\frac{1}{2}$
a_2	$\frac{1}{4(r+1)^2(r+2)^2}$	$\frac{1}{16}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{8(r+1)^2(r+2)^2(r+3)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{1}{288}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2(r+1)^2}$	$\frac{1}{2}$
a_2	$\frac{1}{4(r+1)^2(r+2)^2}$	$\frac{1}{16}$
a_3	$\frac{1}{8(r+1)^2(r+2)^2(r+3)^2}$	$\frac{1}{288}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16\left(r+1\right)^2\left(r+2\right)^2\left(r+3\right)^2\left(r+4\right)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{9216}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2(r+1)^2}$	$\frac{1}{2}$
a_2	$\frac{1}{4(r+1)^2(r+2)^2}$	$\frac{1}{16}$
a_3	$\frac{1}{8(r+1)^2(r+2)^2(r+3)^2}$	$\frac{1}{288}$
a_4	$\frac{1}{16(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{1}{9216}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{32\left(r+1\right)^2\left(r+2\right)^2\left(r+3\right)^2\left(r+4\right)^2\left(r+5\right)^2}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{1}{460800}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2(r+1)^2}$	$\frac{1}{2}$
a_2	$\frac{1}{4(r+1)^2(r+2)^2}$	$\frac{1}{16}$
a_3	$\frac{1}{8(r+1)^2(r+2)^2(r+3)^2}$	$\frac{1}{288}$
a_4	$\frac{1}{16(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{1}{9216}$
a_5	$\frac{1}{32(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$\frac{1}{460800}$

Using the above table, then the first solution $y_1(x)$ becomes

$$
\begin{aligned}
y_1(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots \\
&= 1 + \frac{x}{2} + \frac{x^2}{16} + \frac{x^3}{288} + \frac{x^4}{9216} + \frac{x^5}{460800} + O(x^6)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x)\ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr}a_{n,r}$	$b_n(r=0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{1}{2(r+1)^2}$	$\frac{1}{2}$	$-\frac{1}{(r+1)^3}$	-1
b_2	$\frac{1}{4(r+1)^2(r+2)^2}$	$\frac{1}{16}$	$\frac{-2r-3}{2(r+1)^3(r+2)^3}$	$-\frac{3}{16}$
b_3	$\frac{1}{8(r+1)^2(r+2)^2(r+3)^2}$	$\frac{1}{288}$	$\frac{-3r^2-12r-11}{4(r+1)^3(r+2)^3(r+3)^3}$	$-\frac{11}{864}$
b_4	$\frac{1}{16(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{1}{9216}$	$\frac{-2r^3-15r^2-35r-25}{4(r+1)^3(r+2)^3(r+3)^3(r+4)^3}$	$-\frac{25}{55296}$
b_5	$\frac{1}{32(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$\frac{1}{460800}$	$\frac{-5r^4-60r^3-255r^2-450r-274}{16(r+1)^3(r+2)^3(r+3)^3(r+4)^3(r+5)^3}$	$-\frac{137}{13824000}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned} y_2(x) &= y_1(x)\ln(x) + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 \dots \\ &= \left(1 + \frac{x}{2} + \frac{x^2}{16} + \frac{x^3}{288} + \frac{x^4}{9216} + \frac{x^5}{460800} + O(x^6)\right)\ln(x) \\ &\quad - x - \frac{3x^2}{16} - \frac{11x^3}{864} - \frac{25x^4}{55296} - \frac{137x^5}{13824000} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1y_1(x) + c_2y_2(x) \\ &= c_1\left(1 + \frac{x}{2} + \frac{x^2}{16} + \frac{x^3}{288} + \frac{x^4}{9216} + \frac{x^5}{460800} + O(x^6)\right) \\ &\quad + c_2\left(\left(1 + \frac{x}{2} + \frac{x^2}{16} + \frac{x^3}{288} + \frac{x^4}{9216} + \frac{x^5}{460800} + O(x^6)\right)\ln(x) - x - \frac{3x^2}{16} - \frac{11x^3}{864} \right. \\ &\quad \left. - \frac{25x^4}{55296} - \frac{137x^5}{13824000} + O(x^6)\right) \end{aligned}$$

The particular solution is found by solving for c, m the balance equation

$$(2x^m m(-1+m) + 2x^m m)\, c_0 = F$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$a_1 =$	$\frac{a_0}{2(r+1)^2}$
$a_2 =$	$\frac{a_0}{4(r+1)^2(r+2)^2}$
$a_3 =$	$\frac{a_0}{8(r+1)^2(r+2)^2(r+3)^2}$
$a_4 =$	$\frac{a_0}{16(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$
$a_5 =$	$\frac{a_0}{32(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$

Expanding the rhs of the ode $x\sin(x)$ in series gives

$$x\sin(x) = x^2 - \frac{1}{6}x^4$$

Since the $F = x^2 - \frac{1}{6}x^4$ has more than one term then we find a particular solution for each term and add the result to find the particular solution to the ode.

Now we determine the particular solution y_p associated with $F = x^2$ by solving the balance equation

$$(2x^m m(-1+m) + 2x^m m)\, c_0 = x^2$$

For c_0 and x. This results in

$$\begin{aligned} c_0 &= \frac{1}{8} \\ m &= 2 \end{aligned}$$

The particular solution is therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= \sum_{n=0}^{\infty} c_n x^{n+2} \end{aligned}$$

Where in the above $c_0 = \frac{1}{8}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 2$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{8}$ and $r = m$ or $r = 2$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = \frac{1}{8}$
$c_1 = \frac{1}{144}$
$c_2 = \frac{1}{4608}$
$c_3 = \frac{1}{230400}$
$c_4 = \frac{1}{16588800}$
$c_5 = \frac{1}{1625702400}$

The particular solution is now found using

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= x^2 \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x^2\left(\frac{1}{8}+\frac{1}{144}x+\frac{1}{4608}x^2+\frac{1}{230400}x^3+\frac{1}{16588800}x^4+\frac{1}{1625702400}x^5\right)\\
&= \frac{1}{8}x^2+\frac{1}{144}x^3+\frac{1}{4608}x^4+\frac{1}{230400}x^5+\frac{1}{16588800}x^6+\frac{1}{1625702400}x^7
\end{aligned}
$$

Now we determine the particular solution y_p associated with $F = -\frac{x^4}{6}$ by solving the balance equation

$$
(2x^m m(-1+m) + 2x^m m)\, c_0 = -\frac{x^4}{6}
$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= -\frac{1}{192}\\
m &= 4
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m}\\
&= \sum_{n=0}^{\infty} c_n x^{n+4}
\end{aligned}
$$

Where in the above $c_0 = -\frac{1}{192}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 4$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = -\frac{1}{192}$ and $r = m$ or $r = 4$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = -\frac{1}{192}$
$c_1 = -\frac{1}{9600}$
$c_2 = -\frac{1}{691200}$
$c_3 = -\frac{1}{67737600}$
$c_4 = -\frac{1}{8670412800}$
$c_5 = -\frac{1}{1404606873600}$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
&= x^4 \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x^4 \left(-\frac{1}{192} - \frac{1}{9600}x - \frac{1}{691200}x^2 - \frac{1}{67737600}x^3 - \frac{1}{8670412800}x^4 - \frac{1}{1404606873600}x^5\right) \\
&= -\frac{1}{192}x^4 - \frac{1}{9600}x^5 - \frac{1}{691200}x^6 - \frac{1}{67737600}x^7 - \frac{1}{8670412800}x^8 - \frac{1}{1404606873600}x^9
\end{aligned}
$$

Adding all the above particular solution(s) gives

$$
\begin{aligned}
y_p = {} & \frac{x^2}{8} + \frac{x^3}{144} - \frac{23x^4}{4608} - \frac{23x^5}{230400} - \frac{23x^6}{16588800} - \frac{23x^7}{1625702400} \\
& - \frac{x^8}{8670412800} - \frac{x^9}{1404606873600} + O\left(x^6\right)
\end{aligned}
$$

Truncating the particular solution to the order of series requested gives

$$
y_p = \frac{x^2}{8} + \frac{x^3}{144} - \frac{23x^4}{4608} - \frac{23x^5}{230400} + O\left(x^6\right)
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \frac{x^2}{8} + \frac{x^3}{144} - \frac{23x^4}{4608} - \frac{23x^5}{230400} + O\left(x^6\right) + c_1\left(1 + \frac{x}{2} + \frac{x^2}{16} + \frac{x^3}{288} + \frac{x^4}{9216} + \frac{x^5}{460800} + O\left(x^6\right)\right) \\
&\quad + c_2\left(\left(1 + \frac{x}{2} + \frac{x^2}{16} + \frac{x^3}{288} + \frac{x^4}{9216} + \frac{x^5}{460800} + O\left(x^6\right)\right)\ln\left(x\right) - x - \frac{3x^2}{16} - \frac{11x^3}{864} \right. \\
&\quad \left. - \frac{25x^4}{55296} - \frac{137x^5}{13824000} + O\left(x^6\right)\right)
\end{aligned}
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.019 (sec)
Leaf size : 60

```
dsolve(2*x^2*diff(diff(y(x),x),x)+2*diff(y(x),x)*x-x*y(x) = x*sin(x),y(x),
       series,x=0)
```

$$
\begin{aligned}
y = {} & (c_2 \ln(x) + c_1) \left(1 + \frac{1}{2}x + \frac{1}{16}x^2 + \frac{1}{288}x^3 + \frac{1}{9216}x^4 + \frac{1}{460800}x^5 + \mathrm{O}\left(x^6\right)\right) \\
& + x^2 \left(\frac{1}{8} + \frac{1}{144}x - \frac{23}{4608}x^2 - \frac{23}{230400}x^3 + \mathrm{O}\left(x^4\right)\right) \\
& + \left(-x - \frac{3}{16}x^2 - \frac{11}{864}x^3 - \frac{25}{55296}x^4 - \frac{137}{13824000}x^5 + \mathrm{O}\left(x^6\right)\right) c_2
\end{aligned}
$$

Mathematica DSolve solution

Solving time : 0.167 (sec)
Leaf size : 328

```
AsymptoticDSolveValue[{2*x^2*D[y[x],{x,2}]+2*x*D[y[x],x]-x*y[x]==x*Sin[x],{}},
       y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to\, & c_2\left(\frac{x^5}{460800}+\frac{x^4}{9216}+\frac{x^3}{288}+\frac{x^2}{16}+\frac{x}{2}+1\right) \\
& + c_1\left(x^5\left(\frac{\log(x)}{460800}-\frac{107}{13824000}\right)+x^4\left(\frac{\log(x)}{9216}-\frac{19}{55296}\right)+x^3\left(\frac{\log(x)}{288}-\frac{1}{108}\right)\right. \\
& \left.\quad + x^2\left(\frac{\log(x)}{16}-\frac{1}{8}\right)+x\left(\frac{\log(x)}{2}-\frac{1}{2}\right)+\log(x)+1\right)+\left(-\frac{91x^6}{552960}-\frac{23x^5}{2880}\right. \\
& \left.\quad - \frac{5x^4}{384}+\frac{x^3}{12}+\frac{x^2}{4}\right)\left(x^5\left(\frac{\log(x)}{460800}-\frac{107}{13824000}\right)+x^4\left(\frac{\log(x)}{9216}-\frac{19}{55296}\right)\right. \\
& \left.\quad + x^3\left(\frac{\log(x)}{288}-\frac{1}{108}\right)+x^2\left(\frac{\log(x)}{16}-\frac{1}{8}\right)+x\left(\frac{\log(x)}{2}-\frac{1}{2}\right)+\log(x)+1\right) \\
& + \left(\frac{x^5}{460800}+\frac{x^4}{9216}+\frac{x^3}{288}+\frac{x^2}{16}+\frac{x}{2}+1\right)\left(\frac{13x^6(21\log(x)-310)}{1658880}\right. \\
& \quad + \frac{x^5(345\log(x)-389)}{43200}+\frac{x^4(20\log(x)+51)}{1536}+\frac{1}{36}x^3(4-3\log(x)) \\
& \left.\quad + \frac{1}{8}x^2(-2\log(x)-1)\right)
\end{aligned}
$$

2.4.45 problem 42

Internal problem ID [8610]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 42
Date solved : `Thursday, December 12, 2024 at 09:32:10 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$2x^2y'' + 2xy' - xy = \cos(x)\sin(x)$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' + 2xy' - xy = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{1}{2x}$$

Table 2.215: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' + 2xy' - xy = \frac{\sin(2x)}{2}$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $2x^2y'' + 2xy' - xy = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + 2x\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) - x\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-x^{1+n+r}a_n\right) = \sum_{n=1}^{\infty}\left(-a_{n-1}x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r}a_n(n+r)\left(n+r-1\right)\right)+\left(\sum_{n=0}^{\infty} 2x^{n+r}a_n(n+r)\right)+\sum_{n=1}^{\infty}\left(-a_{n-1}x^{n+r}\right)=0 \quad \text{(2B)}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r}a_n(n+r)\left(n+r-1\right)+2x^{n+r}a_n(n+r)=0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r)+2x^r a_0 r=0$$

Or

$$\left(2x^r r(-1+r)+2x^r r\right)a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$2x^r r^2=0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2=0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$\left(2x^m m(-1+m)+2x^m m\right)c_0=\frac{\sin\left(2x\right)}{2}$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$2x^r r^2 = 0$$

Solving for r gives the roots of the indicial equation as $[0, 0]$.

Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + 2a_n(n+r) - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{2n^2 + 4nr + 2r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-1}}{2n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{2(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_1 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2(r+1)^2}$	$\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{4(r+1)^2 (r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{16}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2(r+1)^2}$	$\frac{1}{2}$
a_2	$\frac{1}{4(r+1)^2(r+2)^2}$	$\frac{1}{16}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{8(r+1)^2 (r+2)^2 (r+3)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{1}{288}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2(r+1)^2}$	$\frac{1}{2}$
a_2	$\frac{1}{4(r+1)^2(r+2)^2}$	$\frac{1}{16}$
a_3	$\frac{1}{8(r+1)^2(r+2)^2(r+3)^2}$	$\frac{1}{288}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16\left(r+1\right)^2 \left(r+2\right)^2 \left(r+3\right)^2 \left(r+4\right)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{9216}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2(r+1)^2}$	$\frac{1}{2}$
a_2	$\frac{1}{4(r+1)^2(r+2)^2}$	$\frac{1}{16}$
a_3	$\frac{1}{8(r+1)^2(r+2)^2(r+3)^2}$	$\frac{1}{288}$
a_4	$\frac{1}{16(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{1}{9216}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{32\left(r+1\right)^2 \left(r+2\right)^2 \left(r+3\right)^2 \left(r+4\right)^2 \left(r+5\right)^2}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{1}{460800}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2(r+1)^2}$	$\frac{1}{2}$
a_2	$\frac{1}{4(r+1)^2(r+2)^2}$	$\frac{1}{16}$
a_3	$\frac{1}{8(r+1)^2(r+2)^2(r+3)^2}$	$\frac{1}{288}$
a_4	$\frac{1}{16(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{1}{9216}$
a_5	$\frac{1}{32(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$\frac{1}{460800}$

Using the above table, then the first solution $y_1(x)$ becomes

$$
\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\
&= 1 + \frac{x}{2} + \frac{x^2}{16} + \frac{x^3}{288} + \frac{x^4}{9216} + \frac{x^5}{460800} + O(x^6)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r} \right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = 0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{1}{2(r+1)^2}$	$\frac{1}{2}$	$-\frac{1}{(r+1)^3}$	-1
b_2	$\frac{1}{4(r+1)^2(r+2)^2}$	$\frac{1}{16}$	$\frac{-2r-3}{2(r+1)^3(r+2)^3}$	$-\frac{3}{16}$
b_3	$\frac{1}{8(r+1)^2(r+2)^2(r+3)^2}$	$\frac{1}{288}$	$\frac{-3r^2-12r-11}{4(r+1)^3(r+2)^3(r+3)^3}$	$-\frac{11}{864}$
b_4	$\frac{1}{16(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{1}{9216}$	$\frac{-2r^3-15r^2-35r-25}{4(r+1)^3(r+2)^3(r+3)^3(r+4)^3}$	$-\frac{25}{55296}$
b_5	$\frac{1}{32(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$\frac{1}{460800}$	$\frac{-5r^4-60r^3-255r^2-450r-274}{16(r+1)^3(r+2)^3(r+3)^3(r+4)^3(r+5)^3}$	$-\frac{137}{13824000}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= \left(1 + \frac{x}{2} + \frac{x^2}{16} + \frac{x^3}{288} + \frac{x^4}{9216} + \frac{x^5}{460800} + O(x^6)\right)\ln(x) \\
&\quad - x - \frac{3x^2}{16} - \frac{11x^3}{864} - \frac{25x^4}{55296} - \frac{137x^5}{13824000} + O(x^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1 + \frac{x}{2} + \frac{x^2}{16} + \frac{x^3}{288} + \frac{x^4}{9216} + \frac{x^5}{460800} + O(x^6)\right) \\
&\quad + c_2\left(\left(1 + \frac{x}{2} + \frac{x^2}{16} + \frac{x^3}{288} + \frac{x^4}{9216} + \frac{x^5}{460800} + O(x^6)\right)\ln(x) - x - \frac{3x^2}{16} - \frac{11x^3}{864} \right. \\
&\quad \left. - \frac{25x^4}{55296} - \frac{137x^5}{13824000} + O(x^6)\right)
\end{aligned}
$$

The particular solution is found by solving for c, m the balance equation

$$
(2x^m m(-1+m) + 2x^m m)\, c_0 = F
$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$
\begin{aligned}
a_1 &= \frac{a_0}{2(r+1)^2} \\
a_2 &= \frac{a_0}{4(r+1)^2(r+2)^2} \\
a_3 &= \frac{a_0}{8(r+1)^2(r+2)^2(r+3)^2} \\
a_4 &= \frac{a_0}{16(r+1)^2(r+2)^2(r+3)^2(r+4)^2} \\
a_5 &= \frac{a_0}{32(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}
\end{aligned}
$$

Expanding the rhs of the ode $\frac{\sin(2x)}{2}$ in series gives

$$
\frac{\sin(2x)}{2} = x - \frac{2}{3}x^3 + \frac{2}{15}x^5
$$

Since the $F = x - \frac{2}{3}x^3 + \frac{2}{15}x^5$ has more than one term then we find a particular solution for each term and add the result to find the particular solution to the ode.

Now we determine the particular solution y_p associated with $F = x$ by solving the balance equation

$$(2x^m m(-1+m) + 2x^m m)\, c_0 = x$$

For c_0 and x. This results in

$$\begin{aligned} c_0 &= \frac{1}{2} \\ m &= 1 \end{aligned}$$

The particular solution is therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= \sum_{n=0}^{\infty} c_n x^{n+1} \end{aligned}$$

Where in the above $c_0 = \frac{1}{2}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 1$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{2}$ and $r = m$ or $r = 1$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = \frac{1}{2}$
$c_1 = \frac{1}{16}$
$c_2 = \frac{1}{288}$
$c_3 = \frac{1}{9216}$
$c_4 = \frac{1}{460800}$
$c_5 = \frac{1}{33177600}$

The particular solution is now found using

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= x \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x\left(\frac{1}{2}+\frac{1}{16}x+\frac{1}{288}x^2+\frac{1}{9216}x^3+\frac{1}{460800}x^4+\frac{1}{33177600}x^5\right)\\
&= \frac{1}{2}x+\frac{1}{16}x^2+\frac{1}{288}x^3+\frac{1}{9216}x^4+\frac{1}{460800}x^5+\frac{1}{33177600}x^6
\end{aligned}
$$

Now we determine the particular solution y_p associated with $F=-\frac{2x^3}{3}$ by solving the balance equation

$$
(2x^m m(-1+m)+2x^m m)\, c_0 = -\frac{2x^3}{3}
$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= -\frac{1}{27}\\
m &= 3
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m}\\
&= \sum_{n=0}^{\infty} c_n x^{n+3}
\end{aligned}
$$

Where in the above $c_0=-\frac{1}{27}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m=3$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0=c_0$ or $a_0=-\frac{1}{27}$ and $r=m$ or $r=3$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0=-\frac{1}{27}$
$c_1=-\frac{1}{864}$
$c_2=-\frac{1}{43200}$
$c_3=-\frac{1}{3110400}$
$c_4=-\frac{1}{304819200}$
$c_5=-\frac{1}{39016857600}$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
&= x^3 \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x^3 \left(-\frac{1}{27} - \frac{1}{864}x - \frac{1}{43200}x^2 - \frac{1}{3110400}x^3 - \frac{1}{304819200}x^4 - \frac{1}{39016857600}x^5\right) \\
&= -\frac{1}{27}x^3 - \frac{1}{864}x^4 - \frac{1}{43200}x^5 - \frac{1}{3110400}x^6 - \frac{1}{304819200}x^7 - \frac{1}{39016857600}x^8
\end{aligned}
$$

Now we determine the particular solution y_p associated with $F = \frac{2x^5}{15}$ by solving the balance equation

$$
(2x^m m(-1+m) + 2x^m m)\, c_0 = \frac{2x^5}{15}
$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= \frac{1}{375} \\
m &= 5
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= \sum_{n=0}^{\infty} c_n x^{n+5}
\end{aligned}
$$

Where in the above $c_0 = \frac{1}{375}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 5$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{375}$ and $r = m$ or $r = 5$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$$\boxed{\begin{aligned} c_0 &= \frac{1}{375} \\ c_1 &= \frac{1}{27000} \\ c_2 &= \frac{1}{2646000} \\ c_3 &= \frac{1}{338688000} \\ c_4 &= \frac{1}{54867456000} \\ c_5 &= \frac{1}{10973491200000} \end{aligned}}$$

The particular solution is now found using

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= x^5 \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$\begin{aligned} y_p &= x^5 \left(\frac{1}{375} + \frac{1}{27000}x + \frac{1}{2646000}x^2 + \frac{1}{338688000}x^3 + \frac{1}{54867456000}x^4 \right. \\ &\qquad \left. + \frac{1}{10973491200000}x^5 \right) \\ &= \frac{1}{375}x^5 + \frac{1}{27000}x^6 + \frac{1}{2646000}x^7 + \frac{1}{338688000}x^8 + \frac{1}{54867456000}x^9 + \frac{1}{10973491200000}x^{10} \end{aligned}$$

Adding all the above particular solution(s) gives

$$\begin{aligned} y_p = {} & \frac{x}{2} + \frac{x^2}{16} - \frac{29x^3}{864} - \frac{29x^4}{27648} + \frac{18287x^5}{6912000} + \frac{18287x^6}{497664000} + \frac{571x^7}{1524096000} \\ & + \frac{571x^8}{195084288000} + \frac{x^9}{54867456000} + \frac{x^{10}}{10973491200000} + O(x^6) \end{aligned}$$

Truncating the particular solution to the order of series requested gives

$$y_p = \frac{x}{2} + \frac{x^2}{16} - \frac{29x^3}{864} - \frac{29x^4}{27648} + \frac{18287x^5}{6912000} + O(x^6)$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \frac{x}{2}+\frac{x^2}{16}-\frac{29x^3}{864}-\frac{29x^4}{27648}+\frac{18287x^5}{6912000}+O\left(x^6\right) \\
&\quad + c_1\left(1+\frac{x}{2}+\frac{x^2}{16}+\frac{x^3}{288}+\frac{x^4}{9216}+\frac{x^5}{460800}+O\left(x^6\right)\right) \\
&\quad + c_2\left(\left(1+\frac{x}{2}+\frac{x^2}{16}+\frac{x^3}{288}+\frac{x^4}{9216}+\frac{x^5}{460800}+O\left(x^6\right)\right)\ln\left(x\right)-x-\frac{3x^2}{16}-\frac{11x^3}{864}\right. \\
&\qquad \left. -\frac{25x^4}{55296}-\frac{137x^5}{13824000}+O\left(x^6\right)\right)
\end{aligned}
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : `0.019` (sec)
Leaf size : 62

```
dsolve(2*x^2*diff(diff(y(x),x),x)+2*diff(y(x),x)*x-x*y(x) = sin(x)*cos(x),y(x),
       series,x=0)
```

$$
\begin{aligned}
y = {} & (c_2 \ln(x) + c_1)\left(1 + \frac{1}{2}x + \frac{1}{16}x^2 + \frac{1}{288}x^3 + \frac{1}{9216}x^4 + \frac{1}{460800}x^5 + \mathrm{O}\left(x^6\right)\right) \\
& + x\left(\frac{1}{2} + \frac{1}{16}x - \frac{29}{864}x^2 - \frac{29}{27648}x^3 + \frac{18287}{6912000}x^4 + \frac{18287}{497664000}x^5 + \mathrm{O}\left(x^6\right)\right) \\
& + \left(-x - \frac{3}{16}x^2 - \frac{11}{864}x^3 - \frac{25}{55296}x^4 - \frac{137}{13824000}x^5 + \mathrm{O}\left(x^6\right)\right) c_2
\end{aligned}
$$

Mathematica DSolve solution

Solving time : `0.158` (sec)
Leaf size : 340

```
AsymptoticDSolveValue[{2*x^2*D[y[x],{x,2}]+2*x*D[y[x],x]-x*y[x]==Cos[x]*Sin[x],{}},
       y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to {} & c_2\left(\frac{x^5}{460800} + \frac{x^4}{9216} + \frac{x^3}{288} + \frac{x^2}{16} + \frac{x}{2} + 1\right) \\
& + c_1\left(x^5\left(\frac{\log(x)}{460800} - \frac{107}{13824000}\right) + x^4\left(\frac{\log(x)}{9216} - \frac{19}{55296}\right) + x^3\left(\frac{\log(x)}{288} - \frac{1}{108}\right)\right. \\
& \quad \left. + x^2\left(\frac{\log(x)}{16} - \frac{1}{8}\right) + x\left(\frac{\log(x)}{2} - \frac{1}{2}\right) + \log(x) + 1\right) + \left(\frac{88963x^6}{16588800} + \frac{4229x^5}{460800}\right. \\
& \left. - \frac{95x^4}{2304} - \frac{29x^3}{288} + \frac{x^2}{8} + \frac{x}{2}\right)\left(x^5\left(\frac{\log(x)}{460800} - \frac{107}{13824000}\right) + x^4\left(\frac{\log(x)}{9216} - \frac{19}{55296}\right)\right. \\
& \quad \left. + x^3\left(\frac{\log(x)}{288} - \frac{1}{108}\right) + x^2\left(\frac{\log(x)}{16} - \frac{1}{8}\right) + x\left(\frac{\log(x)}{2} - \frac{1}{2}\right) + \log(x) + 1\right) \\
& + \left(\frac{x^5}{460800} + \frac{x^4}{9216} + \frac{x^3}{288} + \frac{x^2}{16} + \frac{x}{2} + 1\right)\left(\frac{x^6(1476968 - 1334445\log(x))}{248832000}\right. \\
& \quad + \frac{x^5(-126870\log(x) - 273671)}{13824000} + \frac{5x^4(228\log(x) - 281)}{27648} \\
& \quad \left. + \frac{1}{864}x^3(87\log(x) + 85) + \frac{1}{16}x^2(3 - 2\log(x)) - \frac{1}{2}x\log(x)\right)
\end{aligned}
$$

2.4.46 problem 43

Internal problem ID [8611]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 43
Date solved : `Thursday, December 12, 2024 at 09:32:12 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$2x^2y'' + 2xy' - xy = x^3 + x\sin(x)$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' + 2xy' - xy = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{1}{2x}$$

Table 2.216: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{2x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' + 2xy' - xy = x^3 + x\sin(x)$$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $2x^2y'' + 2xy' - xy = 0$, and y_p is a particular solution to the inhomogeneous ode.which is found using the balance equation generated from indicial equation

First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} 2x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\ + 2x\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) - x\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r}a_n(n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r}a_n(n+r)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r}a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 2x^{n+r} a_n(n+r)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right) = 0 \quad \text{(2B)}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$2x^{n+r} a_n(n+r)(n+r-1) + 2x^{n+r} a_n(n+r) = 0$$

When $n=0$ the above becomes

$$2x^r a_0 r(-1+r) + 2x^r a_0 r = 0$$

Or

$$\left(2x^r r(-1+r) + 2x^r r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$2x^r r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$\left(2x^m m(-1+m) + 2x^m m\right) c_0 = x^3 + x \sin(x)$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$2x^r r^2 = 0$$

Solving for r gives the roots of the indicial equation as $[0, 0]$.

Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + 2a_n(n+r) - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{2n^2 + 4nr + 2r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-1}}{2n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{2(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_1 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2(r+1)^2}$	$\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{4(r+1)^2(r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{16}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2(r+1)^2}$	$\frac{1}{2}$
a_2	$\frac{1}{4(r+1)^2(r+2)^2}$	$\frac{1}{16}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{8(r+1)^2(r+2)^2(r+3)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{1}{288}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2(r+1)^2}$	$\frac{1}{2}$
a_2	$\frac{1}{4(r+1)^2(r+2)^2}$	$\frac{1}{16}$
a_3	$\frac{1}{8(r+1)^2(r+2)^2(r+3)^2}$	$\frac{1}{288}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{16(r+1)^2 (r+2)^2 (r+3)^2 (r+4)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{9216}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2(r+1)^2}$	$\frac{1}{2}$
a_2	$\frac{1}{4(r+1)^2(r+2)^2}$	$\frac{1}{16}$
a_3	$\frac{1}{8(r+1)^2(r+2)^2(r+3)^2}$	$\frac{1}{288}$
a_4	$\frac{1}{16(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{1}{9216}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{32(r+1)^2 (r+2)^2 (r+3)^2 (r+4)^2 (r+5)^2}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{1}{460800}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2(r+1)^2}$	$\frac{1}{2}$
a_2	$\frac{1}{4(r+1)^2(r+2)^2}$	$\frac{1}{16}$
a_3	$\frac{1}{8(r+1)^2(r+2)^2(r+3)^2}$	$\frac{1}{288}$
a_4	$\frac{1}{16(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{1}{9216}$
a_5	$\frac{1}{32(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$\frac{1}{460800}$

Using the above table, then the first solution $y_1(x)$ becomes

$$
\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\
&= 1 + \frac{x}{2} + \frac{x^2}{16} + \frac{x^3}{288} + \frac{x^4}{9216} + \frac{x^5}{460800} + O(x^6)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = 0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{1}{2(r+1)^2}$	$\frac{1}{2}$	$-\frac{1}{(r+1)^3}$	-1
b_2	$\frac{1}{4(r+1)^2(r+2)^2}$	$\frac{1}{16}$	$\frac{-2r-3}{2(r+1)^3(r+2)^3}$	$-\frac{3}{16}$
b_3	$\frac{1}{8(r+1)^2(r+2)^2(r+3)^2}$	$\frac{1}{288}$	$\frac{-3r^2-12r-11}{4(r+1)^3(r+2)^3(r+3)^3}$	$-\frac{11}{864}$
b_4	$\frac{1}{16(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{1}{9216}$	$\frac{-2r^3-15r^2-35r-25}{4(r+1)^3(r+2)^3(r+3)^3(r+4)^3}$	$-\frac{25}{55296}$
b_5	$\frac{1}{32(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$\frac{1}{460800}$	$\frac{-5r^4-60r^3-255r^2-450r-274}{16(r+1)^3(r+2)^3(r+3)^3(r+4)^3(r+5)^3}$	$-\frac{137}{13824000}$

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x)\ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= \left(1+\frac{x}{2}+\frac{x^2}{16}+\frac{x^3}{288}+\frac{x^4}{9216}+\frac{x^5}{460800}+O(x^6)\right)\ln(x) \\
&\quad - x - \frac{3x^2}{16} - \frac{11x^3}{864} - \frac{25x^4}{55296} - \frac{137x^5}{13824000} + O(x^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1+\frac{x}{2}+\frac{x^2}{16}+\frac{x^3}{288}+\frac{x^4}{9216}+\frac{x^5}{460800}+O(x^6)\right) \\
&\quad + c_2\left(\left(1+\frac{x}{2}+\frac{x^2}{16}+\frac{x^3}{288}+\frac{x^4}{9216}+\frac{x^5}{460800}+O(x^6)\right)\ln(x) - x - \frac{3x^2}{16} - \frac{11x^3}{864} \right. \\
&\qquad \left. - \frac{25x^4}{55296} - \frac{137x^5}{13824000} + O(x^6)\right)
\end{aligned}
$$

The particular solution is found by solving for c, m the balance equation

$$
(2x^m m(-1+m) + 2x^m m)\, c_0 = F
$$

Where $F(x)$ is the RHS of the ode. If $F(x)$ has more than one term, then this is done for each term one at a time and then all the particular solutions are added. The function $F(x)$ will be converted to series if needed. in order to solve for c_n, m for each term, the same recursive relation used to find $y_h(x)$ is used to find c_n, m which is used to find the particular solution $\sum_{n=0} c_n x^{n+m}$ by replacing a_n by c_n and r by m.

The following are the values of a_n found in terms of the indicial root r.

$$
\begin{aligned}
a_1 &= \frac{a_0}{2(r+1)^2} \\
a_2 &= \frac{a_0}{4(r+1)^2(r+2)^2} \\
a_3 &= \frac{a_0}{8(r+1)^2(r+2)^2(r+3)^2} \\
a_4 &= \frac{a_0}{16(r+1)^2(r+2)^2(r+3)^2(r+4)^2} \\
a_5 &= \frac{a_0}{32(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}
\end{aligned}
$$

Expanding the rhs of the ode $x^3 + x\sin(x)$ in series gives

$$
x^3 + x\sin(x) = x^2 + x^3 - \frac{1}{6}x^4
$$

Since the $F = x^2 + x^3 - \frac{1}{6}x^4$ has more than one term then we find a particular solution for each term and add the result to find the particular solution to the ode.

Now we determine the particular solution y_p associated with $F = x^2$ by solving the balance equation

$$(2x^m m(-1+m) + 2x^m m)\, c_0 = x^2$$

For c_0 and x. This results in

$$\begin{aligned} c_0 &= \frac{1}{8} \\ m &= 2 \end{aligned}$$

The particular solution is therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\ &= \sum_{n=0}^{\infty} c_n x^{n+2} \end{aligned}$$

Where in the above $c_0 = \frac{1}{8}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 2$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{8}$ and $r = m$ or $r = 2$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$$\boxed{\begin{aligned} c_0 &= \frac{1}{8} \\ c_1 &= \frac{1}{144} \\ c_2 &= \frac{1}{4608} \\ c_3 &= \frac{1}{230400} \\ c_4 &= \frac{1}{16588800} \\ c_5 &= \frac{1}{1625702400} \end{aligned}}$$

The particular solution is now found using

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= x^2 \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$\begin{aligned}
y_p &= x^2\left(\frac{1}{8}+\frac{1}{144}x+\frac{1}{4608}x^2+\frac{1}{230400}x^3+\frac{1}{16588800}x^4+\frac{1}{1625702400}x^5\right)\\
&= \frac{1}{8}x^2+\frac{1}{144}x^3+\frac{1}{4608}x^4+\frac{1}{230400}x^5+\frac{1}{16588800}x^6+\frac{1}{1625702400}x^7
\end{aligned}$$

Now we determine the particular solution y_p associated with $F = x^3$ by solving the balance equation

$$(2x^m m(-1+m)+2x^m m)\,c_0 = x^3$$

For c_0 and x. This results in

$$\begin{aligned}
c_0 &= \frac{1}{18}\\
m &= 3
\end{aligned}$$

The particular solution is therefore

$$\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m}\\
&= \sum_{n=0}^{\infty} c_n x^{n+3}
\end{aligned}$$

Where in the above $c_0 = \frac{1}{18}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 3$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = \frac{1}{18}$ and $r = m$ or $r = 3$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$c_0 = \frac{1}{18}$
$c_1 = \frac{1}{576}$
$c_2 = \frac{1}{28800}$
$c_3 = \frac{1}{2073600}$
$c_4 = \frac{1}{203212800}$
$c_5 = \frac{1}{26011238400}$

The particular solution is now found using

$$
\begin{aligned}
y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\
&= x^3 \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

Using the values found above for c_n into the above sum gives

$$
\begin{aligned}
y_p &= x^3 \left(\frac{1}{18} + \frac{1}{576}x + \frac{1}{28800}x^2 + \frac{1}{2073600}x^3 + \frac{1}{203212800}x^4 + \frac{1}{26011238400}x^5\right) \\
&= \frac{1}{18}x^3 + \frac{1}{576}x^4 + \frac{1}{28800}x^5 + \frac{1}{2073600}x^6 + \frac{1}{203212800}x^7 + \frac{1}{26011238400}x^8
\end{aligned}
$$

Now we determine the particular solution y_p associated with $F = -\frac{x^4}{6}$ by solving the balance equation

$$
\left(2x^m m(-1+m) + 2x^m m\right) c_0 = -\frac{x^4}{6}
$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= -\frac{1}{192} \\
m &= 4
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= \sum_{n=0}^{\infty} c_n x^{n+4}
\end{aligned}
$$

Where in the above $c_0 = -\frac{1}{192}$.

The remaining c_n values are found using the same recurrence relation given in the earlier table which was used to find the homogeneous solution but using c_0 in place of a_0 and using $m = 4$ in place of the root of the indicial equation used to find the homogeneous solution. By letting $a_0 = c_0$ or $a_0 = -\frac{1}{192}$ and $r = m$ or $r = 4$. The following table gives the resulting c_n values. These values will be used to find the particular solution. Values of c_n found not defined when doing the substitution will be discarded and not used

$$\boxed{\begin{aligned} c_0 &= -\frac{1}{192} \\ c_1 &= -\frac{1}{9600} \\ c_2 &= -\frac{1}{691200} \\ c_3 &= -\frac{1}{67737600} \\ c_4 &= -\frac{1}{8670412800} \\ c_5 &= -\frac{1}{1404606873600} \end{aligned}}$$

The particular solution is now found using

$$\begin{aligned} y_p &= x^m \sum_{n=0}^{\infty} c_n x^n \\ &= x^4 \sum_{n=0}^{\infty} c_n x^n \end{aligned}$$

Using the values found above for c_n into the above sum gives

$$\begin{aligned} y_p &= x^4 \left(-\frac{1}{192} - \frac{1}{9600}x - \frac{1}{691200}x^2 - \frac{1}{67737600}x^3 - \frac{1}{8670412800}x^4 - \frac{1}{1404606873600}x^5 \right) \\ &= -\frac{1}{192}x^4 - \frac{1}{9600}x^5 - \frac{1}{691200}x^6 - \frac{1}{67737600}x^7 - \frac{1}{8670412800}x^8 - \frac{1}{1404606873600}x^9 \end{aligned}$$

Adding all the above particular solution(s) gives

$$\begin{aligned} y_p = {} & \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{1536} - \frac{x^5}{15360} - \frac{x^6}{1105920} - \frac{x^7}{108380160} \\ & - \frac{x^8}{13005619200} - \frac{x^9}{1404606873600} + O(x^6) \end{aligned}$$

Truncating the particular solution to the order of series requested gives

$$y_p = \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{1536} - \frac{x^5}{15360} + O(x^6)$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{1536}-\frac{x^5}{15360}+O\left(x^6\right)+c_1\left(1+\frac{x}{2}+\frac{x^2}{16}+\frac{x^3}{288}+\frac{x^4}{9216}+\frac{x^5}{460800}+O\left(x^6\right)\right) \\
&\quad + c_2\left(\left(1+\frac{x}{2}+\frac{x^2}{16}+\frac{x^3}{288}+\frac{x^4}{9216}+\frac{x^5}{460800}+O\left(x^6\right)\right)\ln\left(x\right)-x-\frac{3x^2}{16}-\frac{11x^3}{864}\right. \\
&\quad \left. -\frac{25x^4}{55296}-\frac{137x^5}{13824000}+O\left(x^6\right)\right)
\end{aligned}
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      <- Bessel successful
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.028 (sec)
Leaf size : 60

```
dsolve(2*x^2*diff(diff(y(x),x),x)+2*diff(y(x),x)*x-x*y(x) = x^3+x*sin(x),y(x),
       series,x=0)
```

$$
\begin{aligned}
y = {} & (c_2 \ln(x) + c_1)\left(1 + \frac{1}{2}x + \frac{1}{16}x^2 + \frac{1}{288}x^3 + \frac{1}{9216}x^4 + \frac{1}{460800}x^5 + \mathrm{O}\left(x^6\right)\right) \\
& + x^2\left(\frac{1}{8} + \frac{1}{16}x - \frac{5}{1536}x^2 - \frac{1}{15360}x^3 + \mathrm{O}\left(x^4\right)\right) \\
& + \left(-x - \frac{3}{16}x^2 - \frac{11}{864}x^3 - \frac{25}{55296}x^4 - \frac{137}{13824000}x^5 + \mathrm{O}\left(x^6\right)\right)c_2
\end{aligned}
$$

Mathematica DSolve solution

Solving time : 0.278 (sec)
Leaf size : 268

```
AsymptoticDSolveValue[{2*x^2*D[y[x],{x,2}]+2*x*D[y[x],x]-x*y[x]==x^3*x*Sin[x],{}},
       y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to {} & c_2\left(\frac{x^5}{460800} + \frac{x^4}{9216} + \frac{x^3}{288} + \frac{x^2}{16} + \frac{x}{2} + 1\right) \\
& + c_1\left(x^5\left(\frac{\log(x)}{460800} - \frac{107}{13824000}\right) + x^4\left(\frac{\log(x)}{9216} - \frac{19}{55296}\right) + x^3\left(\frac{\log(x)}{288} - \frac{1}{108}\right)\right. \\
& \left. + x^2\left(\frac{\log(x)}{16} - \frac{1}{8}\right) + x\left(\frac{\log(x)}{2} - \frac{1}{2}\right) + \log(x) + 1\right) + \left(\frac{x^5}{460800} + \frac{x^4}{9216} + \frac{x^3}{288}\right. \\
& \left. + \frac{x^2}{16} + \frac{x}{2} + 1\right)\left(\frac{1}{144}x^6(7 - 6\log(x)) + \frac{1}{50}x^5(-5\log(x) - 4)\right) \\
& + \left(\frac{x^6}{24} + \frac{x^5}{10}\right)\left(x^5\left(\frac{\log(x)}{460800} - \frac{107}{13824000}\right) + x^4\left(\frac{\log(x)}{9216} - \frac{19}{55296}\right)\right. \\
& \left. + x^3\left(\frac{\log(x)}{288} - \frac{1}{108}\right) + x^2\left(\frac{\log(x)}{16} - \frac{1}{8}\right) + x\left(\frac{\log(x)}{2} - \frac{1}{2}\right) + \log(x) + 1\right)
\end{aligned}
$$

2.4.47 problem 44

Internal problem ID [8612]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 44
Date solved : `Thursday, December 12, 2024 at 09:32:13 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\cos(x)\, y'' + 2xy' - xy = 0$$

Using series expansion around $x = 0$

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y'_0$. Using Taylor series gives

$$\begin{aligned} y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\ &= y_0 + xy'_0 + \frac{x^2}{2} f|_{x_0,y_0,y'_0} + \frac{x^3}{3!} f'|_{x_0,y_0,y'_0} + \cdots \\ &= y_0 + xy'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y'_0} \end{aligned}$$

But

$$\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (2.9)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (2.10)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x, y, y') && (4) \\
F_1 &= \frac{df}{dx} \\
&= \frac{dF_0}{dx} \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' \\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f && (5) \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right) \\
&= \frac{d}{dx}(F_1) \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0 \\
&\vdots \\
F_n &= \frac{d}{dx}(F_{n-1}) \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y'' \\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 && (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n \big|_{x_0, y_0, y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{x(2y' - y)}{\cos(x)} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -2\left(\left(-2x^2 \sec(x) + x\tan(x) - \frac{x}{2} + 1\right) y' + y\left(x^2 \sec(x) - \frac{x\tan(x)}{2} - \frac{1}{2}\right)\right)\sec(x) \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= 4\Bigg(\Bigg(\frac{(1+x)\cos(x)^2}{2} + \left(\left(-1 + \frac{x}{2}\right)\sin(x) - x^2 + 3x\right)\cos(x) - 2x^3 + 3x^2\sin(x) - x\Bigg) y' + \Bigg(-\frac{x \mathrm{c}
\\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= -8\Bigg(\Bigg(\left(\left(\frac{3x}{8} - \frac{3}{4}\right)\cos(x)^3 + \left(\left(-\frac{3}{4} - \frac{x}{4}\right)\sin(x) + \frac{27x^2}{8} - \frac{3}{2} + \frac{9x}{4}\right)\cos(x)^2 + \Bigg(\left(\frac{9}{4}x^2 - 7x\right)\sin(
\\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= 16\sec(x)^5 \Bigg(\Bigg(\frac{\left(-3 - \frac{x}{2}\right)\cos(x)^4}{4} + \frac{\left(-3 + \left(-\frac{x}{2} + 1\right)\sin(x) - \frac{87x}{4} + 7x^2\right)\cos(x)^3}{2} + \Bigg(\Bigg(-7x + 5 -
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= 0 \\
F_1 &= -2y'(0) + y(0) \\
F_2 &= 2y'(0) \\
F_3 &= -3y(0) + 6y'(0) \\
F_4 &= -12y'(0) + 4y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$y = \left(1 + \frac{1}{6}x^3 - \frac{1}{40}x^5\right) y(0) + \left(x - \frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{20}x^5\right) y'(0) + O(x^6)$$

Since the expansion point $x = 0$ is an ordinary point, then this can also be solved using the standard power series method. Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} \end{aligned}$$

Substituting the above back into the ode gives

$$\cos(x) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + 2x \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - x \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Expanding $\cos(x)$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$\begin{aligned} \cos(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \dots \\ &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 \end{aligned}$$

Hence the ODE in Eq (1) becomes

$$\begin{aligned} &\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6\right) \left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) \\ &+ 2x \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) - x \left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \end{aligned}$$

Expanding the first term in (1) gives

$$\begin{aligned}&\left(1\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)\right)-\frac{x^2}{2}\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)\\&+\frac{x^4}{24}\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)-\frac{x^6}{720}\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)\\&+2x\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)-x\left(\sum_{n=0}^{\infty} a_n x^n\right)=0\end{aligned}$$

Which simplifies to

$$\begin{aligned}\sum_{n=2}^{\infty}\left(-\frac{n\, x^{n+4} a_n(n-1)}{720}\right)+\left(\sum_{n=2}^{\infty}\frac{n\, x^{n+2} a_n(n-1)}{24}\right)+\sum_{n=2}^{\infty}\left(-\frac{n a_n x^n(n-1)}{2}\right)\\+\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+\left(\sum_{n=1}^{\infty} 2n a_n x^n\right)+\sum_{n=0}^{\infty}\left(-x^{1+n} a_n\right)=0\end{aligned}\tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\begin{aligned}\sum_{n=2}^{\infty}\left(-\frac{n\, x^{n+4} a_n(n-1)}{720}\right)&=\sum_{n=6}^{\infty}\left(-\frac{(n-4)\, a_{n-4}(n-5)\, x^n}{720}\right)\\\sum_{n=2}^{\infty}\frac{n\, x^{n+2} a_n(n-1)}{24}&=\sum_{n=4}^{\infty}\frac{(n-2)\, a_{n-2}(n-3)\, x^n}{24}\\\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}&=\sum_{n=0}^{\infty}(n+2)\, a_{n+2}(1+n)\, x^n\\\sum_{n=0}^{\infty}\left(-x^{1+n} a_n\right)&=\sum_{n=1}^{\infty}\left(-a_{n-1} x^n\right)\end{aligned}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=6}^{\infty}\left(-\frac{(n-4)\, a_{n-4}(n-5)\, x^{n}}{720}\right)+\left(\sum_{n=4}^{\infty}\frac{(n-2)\, a_{n-2}(n-3)\, x^{n}}{24}\right) + \sum_{n=2}^{\infty}\left(-\frac{n a_{n} x^{n}(n-1)}{2}\right)+\left(\sum_{n=0}^{\infty}(n+2)\, a_{n+2}(1+n)\, x^{n}\right) + \left(\sum_{n=1}^{\infty}2n a_{n} x^{n}\right)+\sum_{n=1}^{\infty}(-a_{n-1}x^{n}) = 0 \tag{3}$$

$n = 1$ gives

$$6a_3 + 2a_1 - a_0 = 0$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_0}{6} - \frac{a_1}{3}$$

$n = 2$ gives

$$3a_2 + 12a_4 - a_1 = 0$$

Which after substituting earlier equations, simplifies to

$$a_4 = \frac{a_1}{12}$$

$n = 3$ gives

$$3a_3 + 20a_5 - a_2 = 0$$

Which after substituting earlier equations, simplifies to

$$\frac{a_0}{2} - a_1 + 20a_5 = 0$$

Or

$$a_5 = -\frac{a_0}{40} + \frac{a_1}{20}$$

$n = 4$ gives

$$\frac{a_2}{12} + 2a_4 + 30a_6 - a_3 = 0$$

Which after substituting earlier equations, simplifies to

$$\frac{a_1}{2}+30a_6-\frac{a_0}{6}=0$$

Or

$$a_6=\frac{a_0}{180}-\frac{a_1}{60}$$

$n=5$ gives

$$\frac{a_3}{4}+42a_7-a_4=0$$

Which after substituting earlier equations, simplifies to

$$\frac{a_0}{24}-\frac{a_1}{6}+42a_7=0$$

Or

$$a_7=-\frac{a_0}{1008}+\frac{a_1}{252}$$

For $6\leq n$, the recurrence equation is

$$\begin{aligned}&-\frac{(n-4)\,a_{n-4}(n-5)}{720}+\frac{(n-2)\,a_{n-2}(n-3)}{24}\\&-\frac{na_n(n-1)}{2}+(n+2)\,a_{n+2}(1+n)+2na_n-a_{n-1}=0\end{aligned} \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned}&a_{n+2}\\&=\frac{360n^2a_n+n^2a_{n-4}-30n^2a_{n-2}-1800na_n-9na_{n-4}+150na_{n-2}+20a_{n-4}-180a_{n-2}+720a_{n-1}}{720\,(n+2)\,(1+n)}\\&\quad=\frac{(360n^2-1800n)\,a_n}{720\,(n+2)\,(1+n)}+\frac{(n^2-9n+20)\,a_{n-4}}{720\,(n+2)\,(1+n)}\\&\qquad+\frac{(-30n^2+150n-180)\,a_{n-2}}{720\,(n+2)\,(1+n)}+\frac{a_{n-1}}{(n+2)\,(1+n)}\end{aligned} \tag{5}$$

And so on. Therefore the solution is

$$\begin{aligned}y&=\sum_{n=0}^{\infty}a_nx^n\\&=a_3x^3+a_2x^2+a_1x+a_0+\dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \left(\frac{a_0}{6} - \frac{a_1}{3}\right) x^3 + \frac{a_1 x^4}{12} + \left(-\frac{a_0}{40} + \frac{a_1}{20}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{6}x^3 - \frac{1}{40}x^5\right) a_0 + \left(x - \frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{20}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{6}x^3 - \frac{1}{40}x^5\right) c_1 + \left(x - \frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{20}x^5\right) c_2 + O(x^6)$$

Maple step by step solution

Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 44

```
dsolve(cos(x)*diff(diff(y(x),x),x)+2*diff(y(x),x)*x-x*y(x) = 0,y(x),
       series,x=0)
```

$$y = \left(1 + \frac{1}{6}x^3 - \frac{1}{40}x^5\right) y(0) + \left(x - \frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{20}x^5\right) y'(0) + O(x^6)$$

Mathematica DSolve solution

Solving time : 0.003 (sec)
Leaf size : 49

```
AsymptoticDSolveValue[{Cos[x]*D[y[x],{x,2}]+2*x*D[y[x],x]-x*y[x]==0,{}},
       y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(-\frac{x^5}{40} + \frac{x^3}{6} + 1\right) + c_2 \left(\frac{x^5}{20} + \frac{x^4}{12} - \frac{x^3}{3} + x\right)$$

2.4.48 problem 45

Internal problem ID [8613]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 45
Date solved : `Thursday, December 12, 2024 at 09:32:15 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + 4xy' + \left(x^2+2\right)y = 0$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + 4xy' + \left(x^2+2\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{4}{x}$$
$$q(x) = \frac{x^2+2}{x^2}$$

Table 2.217: Table $p(x), q(x)$ singularites.

$p(x) = \frac{4}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x^2+2}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x=0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + 4xy' + (x^2+2)\,y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) + 4x\left(\sum_{n=0}^{\infty} (n+r)\,a_n x^{n+r-1}\right) + (x^2+2)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 4x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \left(\sum_{n=0}^{\infty} 2a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + 4x^{n+r} a_n (n+r) + 2a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + 4x^r a_0 r + 2a_0 x^r = 0$$

Or

$$(x^r r(-1+r) + 4x^r r + 2x^r) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2 + 3r + 2\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 + 3r + 2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -1$$
$$r_2 = -2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2 + 3r + 2\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as $[-1, -2]$.

Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$

$$y_2(x) = Cy_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \frac{\sum_{n=0}^{\infty} a_n x^n}{x}$$

$$y_2(x) = Cy_1(x)\ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-1}$$

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n + r)(n + r - 1) + 4a_n(n + r) + a_{n-2} + 2a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2 + 3n + 3r + 2} \tag{4}$$

Which for the root $r = -1$ becomes

$$a_n = -\frac{a_{n-2}}{n(n + 1)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{r^2 + 7r + 12}$$

Which for the root $r = -1$ becomes

$$a_2 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+7r+12}$	$-\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+7r+12}$	$-\frac{1}{6}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r+4)(r+3)(r+6)(r+5)}$$

Which for the root $r = -1$ becomes

$$a_4 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+7r+12}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{1}{(r+4)(r+3)(r+6)(r+5)}$	$\frac{1}{120}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+7r+12}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{1}{(r+4)(r+3)(r+6)(r+5)}$	$\frac{1}{120}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \frac{1}{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \frac{1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)}{x} \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if

C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= 0 \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} 0 &= \lim_{r\to -2} 0 \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-2} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + 4b_n(n+r) + b_{n-2} + 2b_n = 0 \tag{4}$$

Which for for the root $r = -2$ becomes

$$b_n(n-2)(n-3) + 4b_n(n-2) + b_{n-2} + 2b_n = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{b_{n-2}}{n^2 + 2nr + r^2 + 3n + 3r + 2} \tag{5}$$

Which for the root $r = -2$ becomes

$$b_n = -\frac{b_{n-2}}{n^2 - n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -2$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{1}{r^2 + 7r + 12}$$

Which for the root $r = -2$ becomes

$$b_2 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+7r+12}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+7r+12}$	$-\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(r^2 + 7r + 12)(r^2 + 11r + 30)}$$

Which for the root $r = -2$ becomes

$$b_4 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+7r+12}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{(r+4)(r+3)(r+6)(r+5)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{1}{r^2+7r+12}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{1}{(r+4)(r+3)(r+6)(r+5)}$	$\frac{1}{24}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= \frac{1}{x}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)}{x^2} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= \frac{c_1\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right)}{x} + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)}{x^2} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= \frac{c_1\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right)}{x} + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)}{x^2} \end{aligned}$$

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)+4x\left(\frac{d}{dx}y(x)\right)+\left(x^2+2\right)y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(x^2+2)y(x)}{x^2}-\frac{4\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{4\left(\frac{d}{dx}y(x)\right)}{x}+\frac{(x^2+2)y(x)}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

- ◦ Define functions

  $\left[P_2(x)=\frac{4}{x}, P_3(x)=\frac{x^2+2}{x^2}\right]$

- ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left(x\cdot P_2(x)\right)\Big|_{x=0}=4$

- ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left(x^2\cdot P_3(x)\right)\Big|_{x=0}=2$

- ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right)+4x\left(\frac{d}{dx}y(x)\right)+\left(x^2+2\right)y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

- ◦ Shift index using $k->k-m$

$$x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$$

- Convert $x \cdot \left(\frac{d}{dx} y(x)\right)$ to series expansion

$$x \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right)$ to series expansion

$$x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 (2+r)\,(1+r)\, x^r + a_1 (3+r)\,(2+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k (k+r+2)\,(k+r+1) + a_{k-2}\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $(2+r)\,(1+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-2, -1\}$
- Each term must be 0
  $a_1 (3+r)\,(2+r) = 0$
- Solve for the dependent coefficient(s)
  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_k (k+r+2)\,(k+r+1) + a_{k-2} = 0$
- Shift index using $k- >k+2$
  $a_{k+2} (k+4+r)\,(k+3+r) + a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+2} = -\frac{a_k}{(k+4+r)(k+3+r)}$
- Recursion relation for $r = -2$
  $a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$
- Solution for $r = -2$
  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+2} = -\frac{a_k}{(k+2)(k+1)}, a_1 = 0\right]$$
- Recursion relation for $r = -1$
  $a_{k+2} = -\frac{a_k}{(k+3)(k+2)}$
- Solution for $r = -1$
  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+2} = -\frac{a_k}{(k+3)(k+2)}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k-2}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k-1}\right), a_{k+2}=-\frac{a_k}{(k+1)(k+2)}, a_1=0, b_{k+2}=-\frac{b_k}{(k+2)(k+3)}, b_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.029 (sec)
Leaf size : 34

```
dsolve(x^2*diff(diff(y(x),x),x)+4*diff(y(x),x)*x+(x^2+2)*y(x) = 0,y(x),
       series,x=0)
```

$$y=\frac{c_1\left(1-\frac{1}{6}x^2+\frac{1}{120}x^4+\mathrm{O}\left(x^6\right)\right)x+c_2\left(1-\frac{1}{2}x^2+\frac{1}{24}x^4+\mathrm{O}\left(x^6\right)\right)}{x^2}$$

Mathematica DSolve solution

Solving time : 0.013 (sec)
Leaf size : 40

```
AsymptoticDSolveValue[{x^2*D[y[x],{x,2}]+4*x*D[y[x],x]+(x^2+2)*y[x]==0,{}},
       y[x],{x,0,5}]
```

$$y(x)\to c_2\left(\frac{x^3}{120}-\frac{x}{6}+\frac{1}{x}\right)+c_1\left(\frac{x^2}{24}+\frac{1}{x^2}-\frac{1}{2}\right)$$

2.4.49 problem 46

Internal problem ID [8614]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 46
Date solved : `Thursday, December 12, 2024 at 09:32:17 AM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$x^2y'' + xy' - xy = 0$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + xy' - xy = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{1}{x}$$

Table 2.219: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + xy' - xy = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2} \right) + x \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \right) - x \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\,(n+r-1) \right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) \right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = 0 \quad \text{(2A)}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right) = 0 \quad \text{(2B)}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r} a_n(n+r)(n+r-1) + x^{n+r} a_n(n+r) = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r = 0$$

Or

$$\left(x^r r(-1+r) + x^r r\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^r r^2 = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^r r^2 = 0$$

Solving for r gives the roots of the indicial equation as $[0, 0]$.

Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \quad \text{(1A)}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right) \quad \text{(1B)}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{n^2 + 2nr + r^2} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-1}}{n^2} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{(r+1)^2}$$

Which for the root $r = 0$ becomes

$$a_1 = 1$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{(r+1)^2}$	1

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(r+1)^2 (r+2)^2}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{4}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{(r+1)^2}$	1
a_2	$\frac{1}{(r+1)^2(r+2)^2}$	$\frac{1}{4}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{(r+1)^2 (r+2)^2 (r+3)^2}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{1}{36}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{(r+1)^2}$	1
a_2	$\frac{1}{(r+1)^2(r+2)^2}$	$\frac{1}{4}$
a_3	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2}$	$\frac{1}{36}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r+1)^2 (r+2)^2 (r+3)^2 (r+4)^2}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{576}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{(r+1)^2}$	1
a_2	$\frac{1}{(r+1)^2(r+2)^2}$	$\frac{1}{4}$
a_3	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2}$	$\frac{1}{36}$
a_4	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{1}{576}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{(r+1)^2 (r+2)^2 (r+3)^2 (r+4)^2 (r+5)^2}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{1}{14400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{(r+1)^2}$	1
a_2	$\frac{1}{(r+1)^2(r+2)^2}$	$\frac{1}{4}$
a_3	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2}$	$\frac{1}{36}$
a_4	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2}$	$\frac{1}{576}$
a_5	$\frac{1}{(r+1)^2(r+2)^2(r+3)^2(r+4)^2(r+5)^2}$	$\frac{1}{14400}$

Using the above table, then the first solution $y_1(x)$ becomes

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \ldots \\ &= 1 + x + \frac{x^2}{4} + \frac{x^3}{36} + \frac{x^4}{576} + \frac{x^5}{14400} + O(x^6) \end{aligned}$$

Now the second solution is found. The second solution is given by

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r} \right)$$

Where b_n is found using

$$b_n = \frac{d}{dr} a_{n,r}$$

And the above is then evaluated at $r = 0$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = 0)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	$\frac{1}{(r+1)^2}$	1	$-\frac{2}{(r+1)^3}$	-2
b_2	$\frac{1}{(r+1)^2 (r+2)^2}$	$\frac{1}{4}$	$\frac{-4r-6}{(r+1)^3 (r+2)^3}$	$-\frac{3}{4}$
b_3	$\frac{1}{(r+1)^2 (r+2)^2 (r+3)^2}$	$\frac{1}{36}$	$\frac{-6r^2-24r-22}{(r+1)^3 (r+2)^3 (r+3)^3}$	$-\frac{11}{108}$
b_4	$\frac{1}{(r+1)^2 (r+2)^2 (r+3)^2 (r+4)^2}$	$\frac{1}{576}$	$\frac{-8r^3-60r^2-140r-100}{(r+1)^3 (r+2)^3 (r+3)^3 (r+4)^3}$	$-\frac{25}{3456}$
b_5	$\frac{1}{(r+1)^2 (r+2)^2 (r+3)^2 (r+4)^2 (r+5)^2}$	$\frac{1}{14400}$	$\frac{-10r^4-120r^3-510r^2-900r-548}{(r+1)^3 (r+2)^3 (r+3)^3 (r+4)^3 (r+5)^3}$	$-\frac{137}{432000}$

The above table gives all values of b_n needed. Hence the second solution is

$$\begin{aligned}
y_2(x) &= y_1(x) \ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= \left(1 + x + \frac{x^2}{4} + \frac{x^3}{36} + \frac{x^4}{576} + \frac{x^5}{14400} + O(x^6)\right) \ln(x) \\
&\quad - 2x - \frac{3x^2}{4} - \frac{11x^3}{108} - \frac{25x^4}{3456} - \frac{137x^5}{432000} + O(x^6)
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1 \left(1 + x + \frac{x^2}{4} + \frac{x^3}{36} + \frac{x^4}{576} + \frac{x^5}{14400} + O(x^6)\right) \\
&\quad + c_2 \left(\left(1 + x + \frac{x^2}{4} + \frac{x^3}{36} + \frac{x^4}{576} + \frac{x^5}{14400} + O(x^6)\right) \ln(x) - 2x - \frac{3x^2}{4} - \frac{11x^3}{108} \right. \\
&\quad \left. - \frac{25x^4}{3456} - \frac{137x^5}{432000} + O(x^6)\right)
\end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1+x+\frac{x^2}{4}+\frac{x^3}{36}+\frac{x^4}{576}+\frac{x^5}{14400}+O(x^6)\right) \\
&\quad + c_2\left(\left(1+x+\frac{x^2}{4}+\frac{x^3}{36}+\frac{x^4}{576}+\frac{x^5}{14400}+O(x^6)\right)\ln(x)-2x-\frac{3x^2}{4}-\frac{11x^3}{108}\right. \\
&\quad \left. -\frac{25x^4}{3456}-\frac{137x^5}{432000}+O(x^6)\right)
\end{aligned}
$$

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)+x\left(\frac{d}{dx}y(x)\right)-xy(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{y(x)}{x}-\frac{\frac{d}{dx}y(x)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{\frac{d}{dx}y(x)}{x}-\frac{y(x)}{x}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

    $\left[P_2(x)=\frac{1}{x}, P_3(x)=-\frac{1}{x}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=0$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right)x+\frac{d}{dx}y(x)-y(x)=0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

☐ Rewrite ODE with series expansions

  - Convert $\frac{d}{dx}y(x)$ to series expansion

    $\frac{d}{dx}y(x) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1}$

  - Shift index using $k->k+1$

    $\frac{d}{dx}y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\, x^{k+r}$

  - Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$

  - Shift index using $k->k+1$

    $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0 r^2 x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)^2 - a_k\right) x^{k+r}\right) = 0$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r^2 = 0$

- Values of r that satisfy the indicial equation

  $r = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_{k+1}(k+1)^2 - a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{(k+1)^2}$

- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{(k+1)^2}$

- Solution for $r = 0$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k}{(k+1)^2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

Maple dsolve solution

Solving time : 0.019 (sec)
Leaf size : 44

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x-x*y(x) = 0,y(x),
      series,x=0)
```

$$
\begin{aligned}
y = {} & (c_2 \ln(x) + c_1)\left(1 + x + \frac{1}{4}x^2 + \frac{1}{36}x^3 + \frac{1}{576}x^4 + \frac{1}{14400}x^5 + \mathrm{O}\left(x^6\right)\right) \\
& + \left((-2)\,x - \frac{3}{4}x^2 - \frac{11}{108}x^3 - \frac{25}{3456}x^4 - \frac{137}{432000}x^5 + \mathrm{O}\left(x^6\right)\right) c_2
\end{aligned}
$$

Mathematica DSolve solution

Solving time : 0.005 (sec)
Leaf size : 107

```
AsymptoticDSolveValue[{x^2*D[y[x],{x,2}]+x*D[y[x],x]-x*y[x]==0,{}},
       y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^5}{14400}+\frac{x^4}{576}+\frac{x^3}{36}+\frac{x^2}{4}+x+1\right)+c_2\left(-\frac{137x^5}{432000}-\frac{25x^4}{3456}-\frac{11x^3}{108}-\frac{3x^2}{4}+\left(\frac{x^5}{14400}+\frac{x^4}{576}+\frac{x^3}{36}+\frac{x^2}{4}+x+1\right)\log(x)-2x\right)$$

2.4.50 problem 47

Internal problem ID [8615]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 47
Date solved : `Thursday, December 12, 2024 at 09:32:18 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{4x^2 - 1}{4x^2}$$

Table 2.221: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{4x^2-1}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + x \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(x^2 - \frac{1}{4}\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-\frac{a_n x^{n+r}}{4}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-\frac{a_n x^{n+r}}{4}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) + x^{n+r} a_n (n+r) - \frac{a_n x^{n+r}}{4} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - \frac{a_0 x^r}{4} = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - \frac{x^r}{4}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\frac{(4r^2 - 1) x^r}{4} = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - \frac{1}{4} = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{(4r^2-1)\,x^r}{4} = 0$$

Solving for r gives the roots of the indicial equation as $\left[\frac{1}{2}, -\frac{1}{2}\right]$.

Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$

$$y_2(x) = C y_1(x)\ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sqrt{x}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$

$$y_2(x) = C y_1(x)\ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{\sqrt{x}}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$

$$y_2(x) = C y_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_n(n+r) + a_{n-2} - \frac{a_n}{4} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{4a_{n-2}}{4n^2+8nr+4r^2-1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{a_{n-2}}{n(n+1)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{4}{4r^2 + 16r + 15}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{6}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{(4r^2 + 16r + 15)(4r^2 + 32r + 63)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{120}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{120}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if

C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= 0 \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} 0 &= \lim_{r\to -\frac{1}{2}} 0 \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + b_n(n+r) + b_{n-2} - \frac{b_n}{4} = 0 \tag{4}$$

Which for for the root $r = -\frac{1}{2}$ becomes

$$b_n\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right) + b_n\left(n-\frac{1}{2}\right) + b_{n-2} - \frac{b_n}{4} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{4b_{n-2}}{4n^2 + 8nr + 4r^2 - 1} \tag{5}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_n = -\frac{4b_{n-2}}{4n^2 - 4n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{4}{4r^2 + 16r + 15}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16}{(4r^2 + 16r + 15)(4r^2 + 32r + 63)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{24}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= \sqrt{x}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)}{\sqrt{x}} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1\sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)}{\sqrt{x}} \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\sqrt{x}\left(1-\frac{x^2}{6}+\frac{x^4}{120}+O(x^6)\right)+\frac{c_2\left(1-\frac{x^2}{2}+\frac{x^4}{24}+O(x^6)\right)}{\sqrt{x}}
\end{aligned}
$$

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)+x\left(\frac{d}{dx}y(x)\right)+\left(x^2-\frac{1}{4}\right)y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(4x^2-1)y(x)}{4x^2}-\frac{\frac{d}{dx}y(x)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{\frac{d}{dx}y(x)}{x}+\frac{(4x^2-1)y(x)}{4x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{1}{x}, P_3(x)=\frac{4x^2-1}{4x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=-\frac{1}{4}$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right)+4x\left(\frac{d}{dx}y(x)\right)+(4x^2-1)y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

- □ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - ○ Shift index using $k- > k - m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - ○ Convert $x \cdot \left(\frac{d}{dx} y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$
  - ○ Convert $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-}\right.$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$
- Shift index using $k- > k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$
- Solution for $r = -\frac{1}{2}$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$$

- Recursion relation for $r = \frac{1}{2}$

$$a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$$

- Solution for $r = \frac{1}{2}$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0, b_{k+2} = -\frac{4b_k}{4k^2+20k+24}, b_1 = 0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.022` (sec)
Leaf size : 34

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x+(x^2-1/4)*y(x) = 0,y(x),
       series,x=0)
```

$$y = \frac{c_1\left(1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 + \mathrm{O}\left(x^6\right)\right)x + c_2\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \mathrm{O}\left(x^6\right)\right)}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.014 (sec)
Leaf size : 58

```
AsymptoticDSolveValue[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-1/4)*y[x]==0,{}},
       y[x],{x,0,5}]
```

$$y(x) \to c_1 \left(\frac{x^{7/2}}{24} - \frac{x^{3/2}}{2} + \frac{1}{\sqrt{x}}\right) + c_2 \left(\frac{x^{9/2}}{120} - \frac{x^{5/2}}{6} + \sqrt{x}\right)$$

2.4.51 problem 48

Internal problem ID [8616]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 48
Date solved : `Thursday, December 12, 2024 at 09:32:19 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\left(x^2-x\right) y''-x y'+y=0$$

Using series expansion around $x=0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$\left(x^2-x\right) y''-x y'+y=0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y''+p(x) y'+q(x) y=0$$

Where

$$\begin{aligned}
p(x) &= -\frac{1}{x-1} \\
q(x) &= \frac{1}{x\left(x-1\right)}
\end{aligned}$$

Table 2.223: Table $p(x), q(x)$ singularites.

$p(x)=-\frac{1}{x-1}$	
singularity	type
$x=1$	"regular"

$q(x)=\frac{1}{x(x-1)}$	
singularity	type
$x=0$	"regular"
$x=1$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[1, 0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x(x-1)\,y'' - xy' + y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x(x-1)\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) - x\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\,(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r-1} a_n (n+r)\,(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}$$

$$\sum_{n=0}^{\infty} \left(-x^{n+r} a_n(n+r)\right) = \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)(n+r-2)\, x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-x^{n+r-1} a_n(n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1}(n+r-1)\, x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$-x^{n+r-1} a_n(n+r)(n+r-1) = 0$$

When $n=0$ the above becomes

$$-x^{-1+r} a_0 r(-1+r) = 0$$

Or

$$-x^{-1+r} a_0 r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$-x^{-1+r} r(-1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$-r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$-x^{-1+r} r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as $[1, 0]$.

Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_{n-1}(n+r-1)(n+r-2) - a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) + a_{n-1} = 0 \quad (3)$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}(n^2 + 2nr + r^2 - 4n - 4r + 4)}{(n+r)(n+r-1)} \quad (4)$$

Which for the root $r = 1$ becomes

$$a_n = \frac{a_{n-1}(n-1)^2}{(n+1)\,n} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{(-1+r)^2}{(1+r)\,r}$$

Which for the root $r = 1$ becomes

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(-1+r)^2}{(1+r)r}$	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{(-1+r)^2\,r}{(1+r)^2\,(2+r)}$$

Which for the root $r = 1$ becomes

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(-1+r)^2}{(1+r)r}$	0
a_2	$\frac{(-1+r)^2r}{(1+r)^2(2+r)}$	0

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{(-1+r)^2 r}{(3+r)(2+r)^2}$$

Which for the root $r = 1$ becomes

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(-1+r)^2}{(1+r)r}$	0
a_2	$\frac{(-1+r)^2 r}{(1+r)^2(2+r)}$	0
a_3	$\frac{(-1+r)^2 r}{(3+r)(2+r)^2}$	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{(-1+r)^2 r}{(4+r)(3+r)^2}$$

Which for the root $r = 1$ becomes

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(-1+r)^2}{(1+r)r}$	0
a_2	$\frac{(-1+r)^2 r}{(1+r)^2(2+r)}$	0
a_3	$\frac{(-1+r)^2 r}{(3+r)(2+r)^2}$	0
a_4	$\frac{(-1+r)^2 r}{(4+r)(3+r)^2}$	0

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{(-1+r)^2 r}{(5+r)(4+r)^2}$$

Which for the root $r = 1$ becomes

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{(-1+r)^2}{(1+r)r}$	0
a_2	$\frac{(-1+r)^2 r}{(1+r)^2(2+r)}$	0
a_3	$\frac{(-1+r)^2 r}{(3+r)(2+r)^2}$	0
a_4	$\frac{(-1+r)^2 r}{(4+r)(3+r)^2}$	0
a_5	$\frac{(-1+r)^2 r}{(5+r)(4+r)^2}$	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\ &= x\left(1 + O\left(x^6\right)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= \frac{(-1+r)^2}{(1+r)\,r} \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} \frac{(-1+r)^2}{(1+r)\,r} &= \lim_{r\to 0} \frac{(-1+r)^2}{(1+r)\,r} \\ &= \textit{undefined} \end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $x(x-1)\,y'' - xy' + y = 0$ gives

$$\begin{aligned}
&x(x-1)\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\left. + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&- x\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) \\
&+ Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned}$$

Which can be written as

$$\left(\left(x(x-1)\, y_1''(x)-y_1'(x)\, x+y_1(x)\right) \ln\left(x\right)+x(x-1) \left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right) - y_1(x)\right) C+x(x-1) \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) - x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$x(x-1)\, y_1''(x)-y_1'(x)\, x+y_1(x)=0$$

Eq (7) simplifes to

$$\left(x(x-1) \left(\frac{2y_1'(x)}{x}-\frac{y_1(x)}{x^2}\right)-y_1(x)\right) C + x(x-1) \left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2}-\frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) - x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right)+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)=0 \tag{8}$$

Substituting $y_1=\sum_{n=0}^{\infty}a_n x^{n+r_1}$ into the above gives

$$\frac{\left(2x(x-1)\left(\sum_{n=0}^{\infty}x^{-1+n+r_1}a_n(n+r_1)\right)+(1-2x)\left(\sum_{n=0}^{\infty}a_n x^{n+r_1}\right)\right)C}{x} + \frac{x^2(x-1)\left(\sum_{n=0}^{\infty}x^{-2+n+r_2}b_n(n+r_2)\,(-1+n+r_2)\right)-\left(\sum_{n=0}^{\infty}x^{-1+n+r_2}b_n(n+r_2)\right)x^2+\left(\sum_{n=0}^{\infty}b_n x^{n+r_2}\right)x}{x} = 0 \tag{9}$$

Since $r_1=1$ and $r_2=0$ then the above becomes

$$\frac{\left(2x(x-1)\left(\sum_{n=0}^{\infty}x^{n}a_n(n+1)\right)+(1-2x)\left(\sum_{n=0}^{\infty}a_n x^{n+1}\right)\right)C}{x} + \frac{x^2(x-1)\left(\sum_{n=0}^{\infty}x^{-2+n}b_n n(n-1)\right)-\left(\sum_{n=0}^{\infty}x^{n-1}b_n n\right)x^2+\left(\sum_{n=0}^{\infty}b_n x^{n}\right)x}{x}=0 \tag{10}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{n+1} a_n (n+1)\right) + \sum_{n=0}^{\infty} \left(-2x^n a_n (n+1)\, C\right) \\
&+ \left(\sum_{n=0}^{\infty} a_n x^n C\right) + \sum_{n=0}^{\infty} \left(-2C\, x^{n+1} a_n\right) + \left(\sum_{n=0}^{\infty} x^n b_n n(n-1)\right) \\
&+ \sum_{n=0}^{\infty} \left(-n\, x^{n-1} b_n (n-1)\right) + \sum_{n=0}^{\infty} \left(-x^n b_n n\right) + \left(\sum_{n=0}^{\infty} b_n x^n\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-1} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 2C\, x^{n+1} a_n (n+1) &= \sum_{n=2}^{\infty} 2C a_{-2+n} (n-1)\, x^{n-1} \\
\sum_{n=0}^{\infty} \left(-2x^n a_n (n+1)\, C\right) &= \sum_{n=1}^{\infty} \left(-2C a_{n-1} n\, x^{n-1}\right) \\
\sum_{n=0}^{\infty} a_n x^n C &= \sum_{n=1}^{\infty} C a_{n-1} x^{n-1} \\
\sum_{n=0}^{\infty} \left(-2C\, x^{n+1} a_n\right) &= \sum_{n=2}^{\infty} \left(-2C a_{-2+n} x^{n-1}\right) \\
\sum_{n=0}^{\infty} x^n b_n n(n-1) &= \sum_{n=1}^{\infty} (n-1)\, b_{n-1} (-2+n)\, x^{n-1} \\
\sum_{n=0}^{\infty} \left(-x^n b_n n\right) &= \sum_{n=1}^{\infty} \left(-(n-1)\, b_{n-1} x^{n-1}\right) \\
\sum_{n=0}^{\infty} b_n x^n &= \sum_{n=1}^{\infty} b_{n-1} x^{n-1}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers

of x are the same and equal to $n-1$.

$$\begin{aligned}
&\left(\sum_{n=2}^{\infty} 2C a_{-2+n}(n-1)\, x^{n-1}\right)+\sum_{n\,=1}^{\infty}\left(-2C a_{n-1} n\, x^{n-1}\right) \\
&+\left(\sum_{n=1}^{\infty} C a_{n-1} x^{n-1}\right)+\sum_{n\,=2}^{\infty}\left(-2C a_{-2+n} x^{n-1}\right) \\
&+\left(\sum_{n=1}^{\infty}(n-1)\, b_{n-1}(-2+n)\, x^{n-1}\right)+\sum_{n\,=0}^{\infty}\left(-n\, x^{n-1} b_{n}(n-1)\right) \\
&+\sum_{n\,=1}^{\infty}\left(-(n-1)\, b_{n-1} x^{n-1}\right)+\left(\sum_{n=1}^{\infty} b_{n-1} x^{n-1}\right)=0
\end{aligned} \tag{2B}$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=N$, where $N=1$ which is the difference between the two roots, we are free to choose $b_1=0$. Hence for $n=1$, Eq (2B) gives

$$-C+1=0$$

Which is solved for C. Solving for C gives

$$C=1$$

For $n=2$, Eq (2B) gives

$$-3C a_1-2b_2=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-2b_2=0$$

Solving the above for b_2 gives

$$b_2=0$$

For $n=3$, Eq (2B) gives

$$(2a_1-5a_2)\, C+b_2-6b_3=0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-6b_3=0$$

Solving the above for b_3 gives

$$b_3=0$$

For $n = 4$, Eq (2B) gives

$$(4a_2 - 7a_3) C + 4b_3 - 12b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-12b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = 0$$

For $n = 5$, Eq (2B) gives

$$(6a_3 - 9a_4) C + 9b_4 - 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-20b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2} \right)$$

Using the above value found for $C = 1$ and all b_n, then the second solution becomes

$$y_2(x) = 1\big(x(1 + O(x^6))\big) \ln(x) + 1 + O(x^6)$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x(1 + O(x^6)) + c_2\Big(1\big(x(1 + O(x^6))\big) \ln(x) + 1 + O(x^6)\Big) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x(1 + O(x^6)) + c_2\big(x(1 + O(x^6)) \ln(x) + 1 + O(x^6)\big) \end{aligned}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.025 (sec)
Leaf size : 34

```
dsolve((x^2-x)*diff(diff(y(x),x),x)-diff(y(x),x)*x+y(x) = 0,y(x),
       series,x=0)
```

$$y = c_1 x \left(1 + \mathrm{O}\left(x^6\right)\right) + \left(x + \mathrm{O}\left(x^6\right)\right) \ln\left(x\right) c_2 + \left(1 - x + \mathrm{O}\left(x^6\right)\right) c_2$$

Mathematica DSolve solution

Solving time : 0.049 (sec)
Leaf size : 20

```
AsymptoticDSolveValue[{(x^2-x)*D[y[x],{x,2}]-x*D[y[x],x]+y[x]==0,{}},
       y[x],{x,0,5}]
```

$$y(x) \to c_2 x + c_1(-3x + x\log(x) + 1)$$

2.4.52 problem 49

Internal problem ID [8617]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 49
Date solved : `Thursday, December 12, 2024 at 09:32:20 AM`
CAS classification :
`[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, ‘_with_symmetry_[0,F(x)]`

Solve

$$x^2y'' + \left(x^2 + 6x\right) y' + xy = 0$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + \left(x^2 + 6x\right) y' + xy = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{x + 6}{x}$$
$$q(x) = \frac{1}{x}$$

Table 2.224: Table $p(x), q(x)$ singularites.

$p(x) = \frac{x+6}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + (x^2 + 6x)\, y' + xy = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + (x^2+6x) \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + x \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 6x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{1+n+r} a_n(n+r) = \sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}$$
$$\sum_{n=0}^{\infty} x^{1+n+r} a_n = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\begin{aligned}&\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\,(n+r-1)\right) + \left(\sum_{n=1}^{\infty} a_{n-1}(n+r-1)\, x^{n+r}\right) \\ &+ \left(\sum_{n=0}^{\infty} 6x^{n+r} a_n(n+r)\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r}\right) = 0\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n(n+r)\,(n+r-1) + 6x^{n+r} a_n(n+r) = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + 6x^r a_0 r = 0$$

Or

$$(x^r r(-1+r) + 6x^r r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^r r(5+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(5+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -5$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^r r(5+r) = 0$$

Solving for r gives the roots of the indicial equation as $[0, -5]$.

Since $r_1 - r_2 = 5$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^5}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-5}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + a_{n-1}(n+r-1) + 6a_n(n+r) + a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{a_{n-1}}{n+5+r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{a_{n-1}}{n+5} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = -\frac{1}{6+r}$$

Which for the root $r = 0$ becomes

$$a_1 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{6+r}$	$-\frac{1}{6}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(6+r)(7+r)}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{42}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{6+r}$	$-\frac{1}{6}$
a_2	$\frac{1}{(6+r)(7+r)}$	$\frac{1}{42}$

For $n = 3$, using the above recursive equation gives

$$a_3 = -\frac{1}{(6+r)(7+r)(8+r)}$$

Which for the root $r = 0$ becomes

$$a_3 = -\frac{1}{336}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{6+r}$	$-\frac{1}{6}$
a_2	$\frac{1}{(6+r)(7+r)}$	$\frac{1}{42}$
a_3	$-\frac{1}{(6+r)(7+r)(8+r)}$	$-\frac{1}{336}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(6+r)(7+r)(8+r)(9+r)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{3024}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{6+r}$	$-\frac{1}{6}$
a_2	$\frac{1}{(6+r)(7+r)}$	$\frac{1}{42}$
a_3	$-\frac{1}{(6+r)(7+r)(8+r)}$	$-\frac{1}{336}$
a_4	$\frac{1}{(6+r)(7+r)(8+r)(9+r)}$	$\frac{1}{3024}$

For $n = 5$, using the above recursive equation gives

$$a_5 = -\frac{1}{(6+r)(7+r)(8+r)(9+r)(10+r)}$$

Which for the root $r = 0$ becomes

$$a_5 = -\frac{1}{30240}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$-\frac{1}{6+r}$	$-\frac{1}{6}$
a_2	$\frac{1}{(6+r)(7+r)}$	$\frac{1}{42}$
a_3	$-\frac{1}{(6+r)(7+r)(8+r)}$	$-\frac{1}{336}$
a_4	$\frac{1}{(6+r)(7+r)(8+r)(9+r)}$	$\frac{1}{3024}$
a_5	$-\frac{1}{(6+r)(7+r)(8+r)(9+r)(10+r)}$	$-\frac{1}{30240}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\
&= 1 - \frac{x}{6} + \frac{x^2}{42} - \frac{x^3}{336} + \frac{x^4}{3024} - \frac{x^5}{30240} + O(x^6)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 5$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_5(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_5 \\
&= -\frac{1}{(6+r)(7+r)(8+r)(9+r)(10+r)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r \to r_2} -\frac{1}{(6+r)(7+r)(8+r)(9+r)(10+r)} &= \lim_{r \to -5} -\frac{1}{(6+r)(7+r)(8+r)(9+r)(10+r)} \\
&= -\frac{1}{120}
\end{aligned}
$$

The limit is $-\frac{1}{120}$. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-5}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + b_{n-1}(n+r-1) + 6b_n(n+r) + b_{n-1} = 0 \tag{4}$$

Which for for the root $r = -5$ becomes

$$b_n(n-5)(n-6) + b_{n-1}(n-6) + 6b_n(n-5) + b_{n-1} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{b_{n-1}}{n+5+r} \tag{5}$$

Which for the root $r = -5$ becomes

$$b_n = -\frac{b_{n-1}}{n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -5$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = -\frac{1}{6+r}$$

Which for the root $r = -5$ becomes

$$b_1 = -1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{6+r}$	-1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{(6+r)(7+r)}$$

Which for the root $r = -5$ becomes

$$b_2 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{6+r}$	-1
b_2	$\frac{1}{(6+r)(7+r)}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = -\frac{1}{(6+r)(7+r)(8+r)}$$

Which for the root $r = -5$ becomes

$$b_3 = -\frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{6+r}$	-1
b_2	$\frac{1}{(6+r)(7+r)}$	$\frac{1}{2}$
b_3	$-\frac{1}{(6+r)(7+r)(8+r)}$	$-\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(6+r)(7+r)(8+r)(9+r)}$$

Which for the root $r = -5$ becomes

$$b_4 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{6+r}$	-1
b_2	$\frac{1}{(6+r)(7+r)}$	$\frac{1}{2}$
b_3	$-\frac{1}{(6+r)(7+r)(8+r)}$	$-\frac{1}{6}$
b_4	$\frac{1}{(6+r)(7+r)(8+r)(9+r)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = -\frac{1}{(6+r)(7+r)(8+r)(9+r)(10+r)}$$

Which for the root $r = -5$ becomes

$$b_5 = -\frac{1}{120}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$-\frac{1}{6+r}$	-1
b_2	$\frac{1}{(6+r)(7+r)}$	$\frac{1}{2}$
b_3	$-\frac{1}{(6+r)(7+r)(8+r)}$	$-\frac{1}{6}$
b_4	$\frac{1}{(6+r)(7+r)(8+r)(9+r)}$	$\frac{1}{24}$
b_5	$-\frac{1}{(6+r)(7+r)(8+r)(9+r)(10+r)}$	$-\frac{1}{120}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned}
y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)}{x^5}
\end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1 - \frac{x}{6} + \frac{x^2}{42} - \frac{x^3}{336} + \frac{x^4}{3024} - \frac{x^5}{30240} + O(x^6)\right) \\
&\quad + \frac{c_2\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right)}{x^5}
\end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1-\frac{x}{6}+\frac{x^2}{42}-\frac{x^3}{336}+\frac{x^4}{3024}-\frac{x^5}{30240}+O(x^6)\right) \\
&\quad + \frac{c_2\left(1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\frac{x^5}{120}+O(x^6)\right)}{x^5}
\end{aligned}
$$

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)+(x^2+6x)\left(\frac{d}{dx}y(x)\right)+xy(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{y(x)}{x}-\frac{(6+x)\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{(6+x)\left(\frac{d}{dx}y(x)\right)}{x}+\frac{y(x)}{x}=0$

- ☐ Check to see if $x_0=0$ is a regular singular point
  - Define functions

    $\left[P_2(x)=\frac{6+x}{x}, P_3(x)=\frac{1}{x}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $(x\cdot P_2(x))\bigg|_{x=0}=6$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $(x^2\cdot P_3(x))\bigg|_{x=0}=0$

  - $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right)x+(6+x)\left(\frac{d}{dx}y(x)\right)+y(x)=0$

- Assume series solution for $y(x)$

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$

  $$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$

- Shift index using $k- > k+1-m$

  $$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$

- Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $$x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$

- Shift index using $k- > k+1$

  $$x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(5+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k+6+r) + a_k(k+1+r)\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(5+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-5, 0\}$
- Each term in the series must be 0, giving the recursion relation
  $(k+1+r)\,(a_{k+1}(k+6+r) + a_k) = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = -\frac{a_k}{k+6+r}$
- Recursion relation for $r = -5$
  $a_{k+1} = -\frac{a_k}{k+1}$
- Solution for $r = -5$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-5}, a_{k+1} = -\frac{a_k}{k+1}\right]$
- Recursion relation for $r = 0$
  $a_{k+1} = -\frac{a_k}{k+6}$
- Solution for $r = 0$

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = -\frac{a_k}{k+6}\right]$$

- Combine solutions and rename parameters

$$\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-5}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+1} = -\frac{a_k}{k+1}, b_{k+1} = -\frac{b_k}{k+6}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

Maple dsolve solution

Solving time : 0.024 (sec)
Leaf size : 44

```
dsolve(x^2*diff(diff(y(x),x),x)+(x^2+6*x)*diff(y(x),x)+x*y(x) = 0,y(x),
       series,x=0)
```

$$y = c_1\left(1 - \frac{1}{6}x + \frac{1}{42}x^2 - \frac{1}{336}x^3 + \frac{1}{3024}x^4 - \frac{1}{30240}x^5 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2(2880 - 2880x + 1440x^2 - 480x^3 + 120x^4 - 24x^5 + \mathrm{O}\left(x^6\right))}{x^5}$$

Mathematica DSolve solution

Solving time : 0.029 (sec)
Leaf size : 68

```
AsymptoticDSolveValue[{x^2*D[y[x],{x,2}]+(6*x+x^2)*D[y[x],x]+x*y[x]==0,{}},
       y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^4}{3024} - \frac{x^3}{336} + \frac{x^2}{42} - \frac{x}{6} + 1\right) + c_1\left(\frac{1}{x^5} - \frac{1}{x^4} + \frac{1}{2x^3} - \frac{1}{6x^2} + \frac{1}{24x}\right)$$

2.4.53 problem 50

Internal problem ID [8618]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 50
Date solved : `Thursday, December 12, 2024 at 09:32:22 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - xy' + (x^2 - 8)\, y = 0$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' - xy' + (x^2 - 8)\, y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{x^2 - 8}{x^2}$$

Table 2.226: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{x^2-8}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' - x y' + \left(x^2 - 8\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) - x \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(x^2 - 8\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-8 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\begin{aligned}\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-x^{n+r} a_n(n+r)\right) \\ +\left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-8 a_n x^{n+r}\right)=0\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n(n+r)(n+r-1)-x^{n+r} a_n(n+r)-8 a_n x^{n+r}=0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r)-x^r a_0 r-8 a_0 x^r=0$$

Or

$$\left(x^r r(-1+r)-x^r r-8 x^r\right) a_0=0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(r^2-2 r-8\right) x^r=0$$

Since the above is true for all x then the indicial equation becomes

$$r^2-2 r-8=0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 4 \\ r_2 &= -2 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(r^2-2 r-8\right) x^r=0$$

Solving for r gives the roots of the indicial equation as $[4, -2]$.

Since $r_1 - r_2 = 6$ is an integer, then we can construct two linearly independent solutions

$$
\begin{aligned}
y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= x^4\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2}
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+4} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$
a_1 = 0
$$

For $2 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) - a_n(n+r) + a_{n-2} - 8a_n = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = -\frac{a_{n-2}}{n^2 + 2nr + r^2 - 2n - 2r - 8} \tag{4}
$$

Which for the root $r = 4$ becomes

$$
a_n = -\frac{a_{n-2}}{n(n+6)} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 4$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{1}{r^2 + 2r - 8}$$

Which for the root $r = 4$ becomes

$$a_2 = -\frac{1}{16}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+2r-8}$	$-\frac{1}{16}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+2r-8}$	$-\frac{1}{16}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(r+4)(r-2)r(r+6)}$$

Which for the root $r = 4$ becomes

$$a_4 = \frac{1}{640}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+2r-8}$	$-\frac{1}{16}$
a_3	0	0
a_4	$\frac{1}{(r+4)(r-2)r(r+6)}$	$\frac{1}{640}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+2r-8}$	$-\frac{1}{16}$
a_3	0	0
a_4	$\frac{1}{(r+4)(r-2)r(r+6)}$	$\frac{1}{640}$
a_5	0	0

For $n = 6$, using the above recursive equation gives

$$a_6 = -\frac{1}{(r+4)(r-2)r(r+6)(r+8)(r+2)}$$

Which for the root $r = 4$ becomes

$$a_6 = -\frac{1}{46080}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{1}{r^2+2r-8}$	$-\frac{1}{16}$
a_3	0	0
a_4	$\frac{1}{(r+4)(r-2)r(r+6)}$	$\frac{1}{640}$
a_5	0	0
a_6	$-\frac{1}{(r+4)(r-2)r(r+6)(r+8)(r+2)}$	$-\frac{1}{46080}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^4\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 \dots\right) \\
&= x^4\left(1 - \frac{x^2}{16} + \frac{x^4}{640} - \frac{x^6}{46080} + O(x^7)\right)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 6$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_6(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_6 \\
&= -\frac{1}{(r+4)(r-2)r(r+6)(r+8)(r+2)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r \to r_2} -\frac{1}{(r+4)(r-2)r(r+6)(r+8)(r+2)} &= \lim_{r \to -2} -\frac{1}{(r+4)(r-2)r(r+6)(r+8)(r+2)} \\
&= \textit{undefined}
\end{aligned}
$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$
y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)
$$

Therefore

$$
\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n (n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n (n+r_2)(-1+n+r_2)\right)
\end{aligned}
$$

Substituting these back into the given ode $x^2y'' - xy' + (x^2-8)\,y = 0$ gives

$$
\begin{aligned}
&x^2\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2}\right. \\
&\quad \left. + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&\quad - x\left(Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right)\right) \\
&\quad + (x^2-8)\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
&\left(\left(x^2y_1''(x) - y_1'(x)\,x + (x^2-8)\,y_1(x)\right)\ln(x) + x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) - y_1(x)\right)C \\
&+ x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&- x\left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + (x^2-8)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{7}
$$

But since $y_1(x)$ is a solution to the ode, then

$$x^2 y_1''(x) - y_1'(x)\, x + \left(x^2 - 8\right) y_1(x) = 0$$

Eq (7) simplifes to

$$\begin{aligned}
&\left(x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right) - y_1(x)\right) C \\
&+ x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) \\
&- x\left(\sum_{n=0}^{\infty}\frac{b_n x^{n+r_2}(n+r_2)}{x}\right) + \left(x^2-8\right)\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\begin{aligned}
&\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n(n+r_1)\right) x - 2\left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right) C \\
&+ \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) x^2 + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)\left(-1+n+r_2\right)\right) x^2 \\
&- \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) x - 8\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0
\end{aligned} \tag{9}$$

Since $r_1 = 4$ and $r_2 = -2$ then the above becomes

$$\begin{aligned}
&\left(2\left(\sum_{n=0}^{\infty} x^{3+n} a_n(n+4)\right) x - 2\left(\sum_{n=0}^{\infty} a_n x^{n+4}\right)\right) C \\
&+ \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) x^2 + \left(\sum_{n=0}^{\infty} x^{-4+n} b_n(n-2)\left(-3+n\right)\right) x^2 \\
&- \left(\sum_{n=0}^{\infty} x^{-3+n} b_n(n-2)\right) x - 8\left(\sum_{n=0}^{\infty} b_n x^{n-2}\right) = 0
\end{aligned} \tag{10}$$

Which simplifies to

$$
\begin{aligned}
&\left(\sum_{n=0}^{\infty} 2C\, x^{n+4} a_n (n+4)\right) + \sum_{n=0}^{\infty} \left(-2a_n x^{n+4} C\right) \\
&+ \left(\sum_{n=0}^{\infty} b_n x^n\right) + \left(\sum_{n=0}^{\infty} x^{n-2} b_n \left(n^2-5n+6\right)\right) \\
&+ \sum_{n=0}^{\infty} \left(-x^{n-2} b_n (n-2)\right) + \sum_{n=0}^{\infty} \left(-8 b_n x^{n-2}\right) = 0
\end{aligned} \tag{2A}
$$

The next step is to make all powers of x be $n-2$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n-2} and adjusting the power and the corresponding index gives

$$
\begin{aligned}
\sum_{n=0}^{\infty} 2C\, x^{n+4} a_n (n+4) &= \sum_{n=6}^{\infty} 2C a_{n-6} (n-2)\, x^{n-2} \\
\sum_{n=0}^{\infty} \left(-2a_n x^{n+4} C\right) &= \sum_{n=6}^{\infty} \left(-2C a_{n-6} x^{n-2}\right) \\
\sum_{n=0}^{\infty} b_n x^n &= \sum_{n=2}^{\infty} b_{n-2} x^{n-2}
\end{aligned}
$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n-2$.

$$
\begin{aligned}
&\left(\sum_{n=6}^{\infty} 2C a_{n-6} (n-2)\, x^{n-2}\right) + \sum_{n=6}^{\infty} \left(-2C a_{n-6} x^{n-2}\right) \\
&+ \left(\sum_{n=2}^{\infty} b_{n-2} x^{n-2}\right) + \left(\sum_{n=0}^{\infty} x^{n-2} b_n \left(n^2-5n+6\right)\right) \\
&+ \sum_{n=0}^{\infty} \left(-x^{n-2} b_n (n-2)\right) + \sum_{n=0}^{\infty} \left(-8 b_n x^{n-2}\right) = 0
\end{aligned} \tag{2B}
$$

For $n=0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0=1$. For $n=1$, Eq (2B) gives

$$
-5b_1 = 0
$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-5b_1 = 0$$

Solving the above for b_1 gives

$$b_1 = 0$$

For $n = 2$, Eq (2B) gives

$$b_0 - 8b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$1 - 8b_2 = 0$$

Solving the above for b_2 gives

$$b_2 = \frac{1}{8}$$

For $n = 3$, Eq (2B) gives

$$b_1 - 9b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-9b_3 = 0$$

Solving the above for b_3 gives

$$b_3 = 0$$

For $n = 4$, Eq (2B) gives

$$b_2 - 8b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$\frac{1}{8} - 8b_4 = 0$$

Solving the above for b_4 gives

$$b_4 = \frac{1}{64}$$

For $n = 5$, Eq (2B) gives

$$b_3 - 5b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$-5b_5 = 0$$

Solving the above for b_5 gives

$$b_5 = 0$$

For $n = N$, where $N = 6$ which is the difference between the two roots, we are free to choose $b_6 = 0$. Hence for $n = 6$, Eq (2B) gives

$$6C + \frac{1}{64} = 0$$

Which is solved for C. Solving for C gives

$$C = -\frac{1}{384}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = -\frac{1}{384}$ and all b_n, then the second solution becomes

$$y_2(x) = -\frac{1}{384}\left(x^4\left(1 - \frac{x^2}{16} + \frac{x^4}{640} - \frac{x^6}{46080} + O(x^7)\right)\right)\ln(x) + \frac{1 + \frac{x^2}{8} + \frac{x^4}{64} + O(x^7)}{x^2}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x^4\left(1 - \frac{x^2}{16} + \frac{x^4}{640} - \frac{x^6}{46080} + O(x^7)\right) \\ &\quad + c_2\left(-\frac{1}{384}\left(x^4\left(1 - \frac{x^2}{16} + \frac{x^4}{640} - \frac{x^6}{46080} + O(x^7)\right)\right)\ln(x)\right. \\ &\quad \left. + \frac{1 + \frac{x^2}{8} + \frac{x^4}{64} + O(x^7)}{x^2}\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x^4\left(1 - \frac{x^2}{16} + \frac{x^4}{640} - \frac{x^6}{46080} + O(x^7)\right) \\ &\quad + c_2\left(-\frac{x^4\left(1 - \frac{x^2}{16} + \frac{x^4}{640} - \frac{x^6}{46080} + O(x^7)\right)\ln(x)}{384} + \frac{1 + \frac{x^2}{8} + \frac{x^4}{64} + O(x^7)}{x^2}\right) \end{aligned}$$

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)-x\left(\frac{d}{dx}y(x)\right)+(x^2-8)\,y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(x^2-8)y(x)}{x^2}+\frac{\frac{d}{dx}y(x)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)-\frac{\frac{d}{dx}y(x)}{x}+\frac{(x^2-8)y(x)}{x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{1}{x},P_3(x)=\frac{x^2-8}{x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $(x\cdot P_2(x))\bigg|_{x=0}=-1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $(x^2\cdot P_3(x))\bigg|_{x=0}=-8$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $x^2\left(\frac{d^2}{dx^2}y(x)\right)-x\left(\frac{d}{dx}y(x)\right)+(x^2-8)\,y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

  $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_k x^{k+r+m}$

  ◦ Shift index using $k->k-m$

  $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(2+r)\,(-4+r)\, x^r + a_1(3+r)\,(-3+r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(k+r+2)\,(k+r-4) + a_{k-2}\right) x^{k+r}\right)$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(2+r)\,(-4+r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \{-2, 4\}$

- Each term must be 0

  $a_1(3+r)\,(-3+r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(k+r+2)\,(k+r-4) + a_{k-2} = 0$

- Shift index using $k- >k+2$

  $a_{k+2}(k+4+r)\,(k-2+r) + a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{a_k}{(k+4+r)(k-2+r)}$

- Recursion relation for $r = -2$

  $a_{k+2} = -\frac{a_k}{(k+2)(k-4)}$

- Series not valid for $r = -2$, division by 0 in the recursion relation at $k = 4$

  $a_{k+2} = -\frac{a_k}{(k+2)(k-4)}$

- Recursion relation for $r = 4$

  $a_{k+2} = -\frac{a_k}{(k+8)(k+2)}$

- Solution for $r = 4$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+4}, a_{k+2} = -\frac{a_k}{(k+8)(k+2)}, a_1 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

Maple dsolve solution

Solving time : 0.023 (sec)
Leaf size : 35

```
dsolve(x^2*diff(diff(y(x),x),x)-diff(y(x),x)*x+(x^2-8)*y(x) = 0,y(x),
       series,x=0)
```

$$y = c_1 x^4 \left(1 - \frac{1}{16}x^2 + \frac{1}{640}x^4 + \mathrm{O}\left(x^6\right)\right) + \frac{c_2\left(-86400 - 10800x^2 - 1350x^4 + \mathrm{O}\left(x^6\right)\right)}{x^2}$$

Mathematica DSolve solution

Solving time : 0.013 (sec)
Leaf size : 42

```
AsymptoticDSolveValue[{x^2*D[y[x],{x,2}]-x*D[y[x],x]+(x^2-8)*y[x]==0,{}},
       y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^2}{64} + \frac{1}{x^2} + \frac{1}{8}\right) + c_2\left(\frac{x^8}{640} - \frac{x^6}{16} + x^4\right)$$

2.4.54 problem 51

Internal problem ID [8619]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 51
Date solved : `Thursday, December 12, 2024 at 09:32:23 AM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$x^2y'' - 9xy' + 25y = 0$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' - 9xy' + 25y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{9}{x}$$
$$q(x) = \frac{25}{x^2}$$

Table 2.228: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{9}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{25}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' - 9xy' + 25y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2\left(\sum_{n=0}^{\infty}(n+r)\,(n+r-1)\,a_n x^{n+r-2}\right) - 9x\left(\sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1}\right) + 25\left(\sum_{n=0}^{\infty}a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\,(n+r-1)\right) + \sum_{n=0}^{\infty}\left(-9x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 25 a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\,(n+r-1)\right) + \sum_{n=0}^{\infty}\left(-9x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 25 a_n x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) - 9x^{n+r} a_n (n+r) + 25 a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) - 9x^r a_0 r + 25 a_0 x^r = 0$$

Or

$$(x^r r(-1+r) - 9x^r r + 25x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$(r-5)^2 x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$(r-5)^2 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 5$$
$$r_2 = 5$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r-5)^2 x^r = 0$$

Solving for r gives the roots of the indicial equation as $[5, 5]$.

Since the root of the indicial equation is repeated, then we can construct two linearly independent solutions. The first solution has the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{1A}$$

Now the second solution y_2 is found using

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r} \right) \tag{1B}$$

Then the general solution will be

$$y = c_1 y_1(x) + c_2 y_2(x)$$

In Eq (1B) the sum starts from 1 and not zero. In Eq (1A), a_0 is never zero, and is arbitrary and is typically taken as $a_0 = 1$, and $\{c_1, c_2\}$ are two arbitray constants of integration which can be found from initial conditions. Using the value of the indicial root found earlier, $r = 5$, Eqs (1A,1B) become

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+5}$$

$$y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+5}\right)$$

We start by finding the first solution $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $0 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - 9a_n(n+r) + 25a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = 0 \tag{4}$$

Which for the root $r = 5$ becomes

$$a_n = 0 \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 5$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0
a_5	0	0

Using the above table, then the first solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= x^5 \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\
&= x^5 \left(1 + O(x^6)\right)
\end{aligned}
$$

Now the second solution is found. The second solution is given by

$$
y_2(x) = y_1(x) \ln(x) + \left(\sum_{n=1}^{\infty} b_n x^{n+r}\right)
$$

Where b_n is found using

$$
b_n = \frac{d}{dr} a_{n,r}
$$

And the above is then evaluated at $r = 5$. The above table for $a_{n,r}$ is used for this purpose. Computing the derivatives gives the following table

n	$b_{n,r}$	a_n	$b_{n,r} = \frac{d}{dr} a_{n,r}$	$b_n(r = 5)$
b_0	1	1	N/A since b_n starts from 1	N/A
b_1	0	0	0	0
b_2	0	0	0	0
b_3	0	0	0	0
b_4	0	0	0	0
b_5	0	0	0	0

The above table gives all values of b_n needed. Hence the second solution is

$$
\begin{aligned}
y_2(x) &= y_1(x) \ln(x) + b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots \\
&= x^5 \left(1 + O(x^6)\right) \ln(x) + x^5 O(x^6)
\end{aligned}
$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1\, x^5 \left(1+O\left(x^6\right)\right) + c_2 \left(x^5 \left(1+O\left(x^6\right)\right) \ln\left(x\right) + x^5 O\left(x^6\right)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1\, x^5 \left(1+O\left(x^6\right)\right) + c_2 \left(x^5 \left(1+O\left(x^6\right)\right) \ln\left(x\right) + x^5 O\left(x^6\right)\right) \end{aligned}$$

Maple step by step solution

Let's solve
$x^2 \left(\frac{d^2}{dx^2} y(x)\right) - 9x \left(\frac{d}{dx} y(x)\right) + 25y(x) = 0$

- Highest derivative means the order of the ODE is 2
$\frac{d^2}{dx^2} y(x)$
- Isolate 2nd derivative
$\frac{d^2}{dx^2} y(x) = -\frac{25y(x)}{x^2} + \frac{9\left(\frac{d}{dx} y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
$\frac{d^2}{dx^2} y(x) - \frac{9\left(\frac{d}{dx} y(x)\right)}{x} + \frac{25y(x)}{x^2} = 0$
- Multiply by denominators of the ODE
$x^2 \left(\frac{d^2}{dx^2} y(x)\right) - 9x \left(\frac{d}{dx} y(x)\right) + 25y(x) = 0$
- Make a change of variables
$t = \ln\left(x\right)$

□ Substitute the change of variables back into the ODE
  - Calculate the 1st derivative of y with respect to x , using the chain rule
  $\frac{d}{dx} y(x) = \left(\frac{d}{dt} y(t)\right) \left(\frac{d}{dx} t(x)\right)$
  - Compute derivative
  $\frac{d}{dx} y(x) = \frac{\frac{d}{dt} y(t)}{x}$
  - Calculate the 2nd derivative of y with respect to x , using the chain rule
  $\frac{d^2}{dx^2} y(x) = \left(\frac{d^2}{dt^2} y(t)\right) \left(\frac{d}{dx} t(x)\right)^2 + \left(\frac{d^2}{dx^2} t(x)\right) \left(\frac{d}{dt} y(t)\right)$
  - Compute derivative

$\frac{d^2}{dx^2}y(x) = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

Substitute the change of variables back into the ODE

$x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 9\frac{d}{dt}y(t) + 25y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - 10\frac{d}{dt}y(t) + 25y(t) = 0$
- Characteristic polynomial of ODE

  $r^2 - 10r + 25 = 0$
- Factor the characteristic polynomial

  $(r-5)^2 = 0$
- Root of the characteristic polynomial

  $r = 5$
- 1st solution of the ODE

  $y_1(t) = e^{5t}$
- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,e^{5t}$
- General solution of the ODE

  $y(t) = C1 y_1(t) + C2 y_2(t)$
- Substitute in solutions

  $y(t) = C1\,e^{5t} + C2 t\,e^{5t}$
- Change variables back using $t = \ln(x)$

  $y(x) = C2\ln(x)\,x^5 + C1\,x^5$
- Simplify

  $y(x) = x^5(C2\ln(x) + C1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

Maple dsolve solution

Solving time : 0.018 (sec)
Leaf size : 19

```
dsolve(x^2*diff(diff(y(x),x),x)-9*diff(y(x),x)*x+25*y(x) = 0,y(x),
       series,x=0)
```

$$y = x^5(c_2 \ln(x) + c_1) + O(x^6)$$

Mathematica DSolve solution

Solving time : 0.004 (sec)
Leaf size : 18

```
AsymptoticDSolveValue[{x^2*D[y[x],{x,2}]-9*x*D[y[x],x]+25*y[x]==0,{}},
       y[x],{x,0,5}]
```

$$y(x) \to c_1x^5 + c_2x^5 \log(x)$$

2.4.55 problem 52

Internal problem ID [8620]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 52
Date solved : `Thursday, December 12, 2024 at 09:32:24 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' - xy' - \left(x^2 + \frac{5}{4}\right) y = 0$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' - xy' + \left(-x^2 - \frac{5}{4}\right) y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = -\frac{4x^2 + 5}{4x^2}$$

Table 2.230: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{4x^2+5}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' - xy' + \left(-x^2 - \frac{5}{4}\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} x^2 &\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\ &- x\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(-x^2 - \frac{5}{4}\right)\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\begin{aligned} &\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\,(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n+r)\right) \\ &+ \sum_{n=0}^{\infty} \left(-x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-\frac{5 a_n x^{n+r}}{4}\right) = 0 \end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-x^{n+r+2}a_n\right)=\sum_{n=2}^{\infty}\left(-a_{n-2}x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r}a_n(n+r)\,(n+r-1)\right)+\sum_{n=0}^{\infty}\left(-x^{n+r}a_n(n+r)\right)+\sum_{n=2}^{\infty}\left(-a_{n-2}x^{n+r}\right)+\sum_{n=0}^{\infty}\left(-\frac{5a_n x^{n+r}}{4}\right)=0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r}a_n(n+r)\,(n+r-1)-x^{n+r}a_n(n+r)-\frac{5a_n x^{n+r}}{4}=0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r)-x^r a_0 r-\frac{5a_0 x^r}{4}=0$$

Or

$$\left(x^r r(-1+r)-x^r r-\frac{5x^r}{4}\right)a_0=0$$

Since $a_0\neq 0$ then the above simplifies to

$$\frac{(4r^2-8r-5)\,x^r}{4}=0$$

Since the above is true for all x then the indicial equation becomes

$$r^2-2r-\frac{5}{4}=0$$

Solving for r gives the roots of the indicial equation as

$$r_1=\frac{5}{2}$$
$$r_2=-\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{(4r^2 - 8r - 5)\, x^r}{4} = 0$$

Solving for r gives the roots of the indicial equation as $\left[\frac{5}{2}, -\frac{1}{2}\right]$.

Since $r_1 - r_2 = 3$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = x^{5/2} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{\sqrt{x}}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{5}{2}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_n(n+r) - a_{n-2} - \frac{5a_n}{4} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{4a_{n-2}}{4n^2 + 8nr + 4r^2 - 8n - 8r - 5} \tag{4}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_n = \frac{a_{n-2}}{n\,(n+3)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{5}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{4}{4r^2 + 8r - 5}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_2 = \frac{1}{10}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{4}{4r^2+8r-5}$	$\frac{1}{10}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{4}{4r^2+8r-5}$	$\frac{1}{10}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{(4r^2 + 8r - 5)\,(4r^2 + 24r + 27)}$$

Which for the root $r = \frac{5}{2}$ becomes

$$a_4 = \frac{1}{280}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{4}{4r^2+8r-5}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{16}{(4r^2+8r-5)(4r^2+24r+27)}$	$\frac{1}{280}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$\frac{4}{4r^2+8r-5}$	$\frac{1}{10}$
a_3	0	0
a_4	$\frac{16}{(4r^2+8r-5)(4r^2+24r+27)}$	$\frac{1}{280}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= x^{5/2}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right)\\ &= x^{5/2}\left(1 + \frac{x^2}{10} + \frac{x^4}{280} + O(x^6)\right)\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 3$. Now we need to determine if

C is zero or not. This is done by finding $\lim_{r\to r_2} a_3(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_3 \\ &= 0 \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} 0 &= \lim_{r\to -\frac{1}{2}} 0 \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) - b_n(n+r) - b_{n-2} - \frac{5b_n}{4} = 0 \tag{4}$$

Which for for the root $r = -\frac{1}{2}$ becomes

$$b_n\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right) - b_n\left(n-\frac{1}{2}\right) - b_{n-2} - \frac{5b_n}{4} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{4b_{n-2}}{4n^2+8nr+4r^2-8n-8r-5} \tag{5}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_n = \frac{4b_{n-2}}{4n^2-12n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{4}{4r^2 + 8r - 5}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{4}{4r^2+8r-5}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{4}{4r^2+8r-5}$	$-\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16}{(4r^2 + 8r - 5)(4r^2 + 24r + 27)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = -\frac{1}{8}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{4}{4r^2+8r-5}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{16}{(4r^2+8r-5)(4r^2+24r+27)}$	$-\frac{1}{8}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$\frac{4}{4r^2+8r-5}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{16}{(4r^2+8r-5)(4r^2+24r+27)}$	$-\frac{1}{8}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= x^{5/2}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - \frac{x^2}{2} - \frac{x^4}{8} + O(x^6)}{\sqrt{x}} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1\, x^{5/2}\left(1 + \frac{x^2}{10} + \frac{x^4}{280} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{2} - \frac{x^4}{8} + O(x^6)\right)}{\sqrt{x}} \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\, x^{5/2}\left(1+\frac{x^2}{10}+\frac{x^4}{280}+O\left(x^6\right)\right)+\frac{c_2\left(1-\frac{x^2}{2}-\frac{x^4}{8}+O(x^6)\right)}{\sqrt{x}}
\end{aligned}
$$

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)-x\left(\frac{d}{dx}y(x)\right)-\left(x^2+\frac{5}{4}\right)y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=\frac{(4x^2+5)y(x)}{4x^2}+\frac{\frac{d}{dx}y(x)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)-\frac{\frac{d}{dx}y(x)}{x}-\frac{(4x^2+5)y(x)}{4x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=-\frac{1}{x}, P_3(x)=-\frac{4x^2+5}{4x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=-1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=-\frac{5}{4}$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right)-4x\left(\frac{d}{dx}y(x)\right)+\left(-4x^2-5\right)y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

- Rewrite ODE with series expansions
  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - Shift index using $k- >k - m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$
  - Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-5+2r)\,x^r + a_1(3+2r)\,(-3+2r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-5) - 4a$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-5+2r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{5}{2}\right\}$
- Each term must be 0

  $a_1(3+2r)\,(-3+2r) = 0$
- Solve for the dependent coefficient(s)

  $a_1 = 0$
- Each term in the series must be 0, giving the recursion relation

  $4\left(k+r-\frac{5}{2}\right)\left(k+\frac{1}{2}+r\right)a_k - 4a_{k-2} = 0$
- Shift index using $k- >k + 2$

  $4\left(k-\frac{1}{2}+r\right)\left(k+\frac{5}{2}+r\right)a_{k+2} - 4a_k = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+2} = \frac{4a_k}{(2k-1+2r)(2k+5+2r)}$
- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = \frac{4a_k}{(2k-2)(2k+4)}$
- Solution for $r = -\frac{1}{2}$

$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2}=\frac{4a_k}{(2k-2)(2k+4)}, a_1=0\right]$$

- Recursion relation for $r=\frac{5}{2}$

$$a_{k+2}=\frac{4a_k}{(2k+4)(2k+10)}$$

- Solution for $r=\frac{5}{2}$

$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+\frac{5}{2}}, a_{k+2}=\frac{4a_k}{(2k+4)(2k+10)}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{5}{2}}\right), a_{k+2}=\frac{4a_k}{(2k-2)(2k+4)}, a_1=0, b_{k+2}=\frac{4b_k}{(2k+4)(2k+10)}, b_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Reducible group (found another exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.022 (sec)
Leaf size : 36

```
dsolve(x^2*diff(diff(y(x),x),x)-diff(y(x),x)*x-(x^2+5/4)*y(x) = 0,y(x),
       series,x=0)
```

$$y=\frac{c_1 x^3\left(1+\frac{1}{10}x^2+\frac{1}{280}x^4+\mathrm{O}\left(x^6\right)\right)+c_2\left(12-6x^2-\frac{3}{2}x^4+\mathrm{O}\left(x^6\right)\right)}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.014 (sec)
Leaf size : 58

```
AsymptoticDSolveValue[{x^2*D[y[x],{x,2}]-x*D[y[x],x]-(x^2+5/4)*y[x]==0,{}},
       y[x],{x,0,5}]
```

$$y(x) \to c_1\left(-\frac{x^{7/2}}{8} - \frac{x^{3/2}}{2} + \frac{1}{\sqrt{x}}\right) + c_2\left(\frac{x^{13/2}}{280} + \frac{x^{9/2}}{10} + x^{5/2}\right)$$

2.4.56 problem 53

Internal problem ID [8621]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 53
Date solved : `Thursday, December 12, 2024 at 09:32:25 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{4x^2 - 1}{4x^2}$$

Table 2.232: Table $p(x), q(x)$ singularites.

$p(x) = \frac{1}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{4x^2-1}{4x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + x \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(x^2 - \frac{1}{4}\right) \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} x^{n+r+2} a_n\right) + \sum_{n=0}^{\infty} \left(-\frac{a_n x^{n+r}}{4}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} x^{n+r+2} a_n = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} x^{n+r} a_n(n+r)\right) + \left(\sum_{n=2}^{\infty} a_{n-2} x^{n+r}\right) + \sum_{n=0}^{\infty} \left(-\frac{a_n x^{n+r}}{4}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r} a_n(n+r)(n+r-1) + x^{n+r} a_n(n+r) - \frac{a_n x^{n+r}}{4} = 0$$

When $n=0$ the above becomes

$$x^r a_0 r(-1+r) + x^r a_0 r - \frac{a_0 x^r}{4} = 0$$

Or

$$\left(x^r r(-1+r) + x^r r - \frac{x^r}{4}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\frac{(4r^2-1)\, x^r}{4} = 0$$

Since the above is true for all x then the indicial equation becomes

$$r^2 - \frac{1}{4} = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = \frac{1}{2}$$
$$r_2 = -\frac{1}{2}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\frac{(4r^2 - 1)\, x^r}{4} = 0$$

Solving for r gives the roots of the indicial equation as $\left[\frac{1}{2}, -\frac{1}{2}\right]$.

Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sqrt{x} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{\sqrt{x}}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

For $2 \leq n$ the recursive equation is

$$a_n(n + r)(n + r - 1) + a_n(n + r) + a_{n-2} - \frac{a_n}{4} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{4a_{n-2}}{4n^2 + 8nr + 4r^2 - 1} \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = -\frac{a_{n-2}}{n\,(n+1)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = -\frac{4}{4r^2 + 16r + 15}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_2 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{6}$
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{16}{(4r^2 + 16r + 15)\,(4r^2 + 32r + 63)}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_4 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{120}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{6}$
a_3	0	0
a_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{120}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if

C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_1 \\ &= 0 \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r\to r_2} 0 &= \lim_{r\to -\frac{1}{2}} 0 \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-\frac{1}{2}} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

For $2 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) + b_n(n+r) + b_{n-2} - \frac{b_n}{4} = 0 \tag{4}$$

Which for for the root $r = -\frac{1}{2}$ becomes

$$b_n\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right) + b_n\left(n-\frac{1}{2}\right) + b_{n-2} - \frac{b_n}{4} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{4b_{n-2}}{4n^2 + 8nr + 4r^2 - 1} \tag{5}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_n = -\frac{4b_{n-2}}{4n^2 - 4n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -\frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = -\frac{4}{4r^2 + 16r + 15}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_2 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{2}$
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{16}{(4r^2 + 16r + 15)(4r^2 + 32r + 63)}$$

Which for the root $r = -\frac{1}{2}$ becomes

$$b_4 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	$-\frac{4}{4r^2+16r+15}$	$-\frac{1}{2}$
b_3	0	0
b_4	$\frac{16}{(4r^2+16r+15)(4r^2+32r+63)}$	$\frac{1}{24}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= \sqrt{x}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)}{\sqrt{x}} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1\sqrt{x}\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + O(x^6)\right) + \frac{c_2\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)}{\sqrt{x}} \end{aligned}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\sqrt{x}\left(1-\frac{x^2}{6}+\frac{x^4}{120}+O(x^6)\right)+\frac{c_2\left(1-\frac{x^2}{2}+\frac{x^4}{24}+O(x^6)\right)}{\sqrt{x}}
\end{aligned}
$$

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)+x\left(\frac{d}{dx}y(x)\right)+\left(x^2-\frac{1}{4}\right)y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(4x^2-1)y(x)}{4x^2}-\frac{\frac{d}{dx}y(x)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{\frac{d}{dx}y(x)}{x}+\frac{(4x^2-1)y(x)}{4x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x)=\frac{1}{x}, P_3(x)=\frac{4x^2-1}{4x^2}\right]$

  ◦ $x\cdot P_2(x)$ is analytic at $x=0$

  $\left.(x\cdot P_2(x))\right|_{x=0}=1$

  ◦ $x^2\cdot P_3(x)$ is analytic at $x=0$

  $\left.(x^2\cdot P_3(x))\right|_{x=0}=-\frac{1}{4}$

  ◦ $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right)+4x\left(\frac{d}{dx}y(x)\right)+(4x^2-1)y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

- □ Rewrite ODE with series expansions
  - ○ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
  - ○ Shift index using $k- >k - m$

    $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
  - ○ Convert $x \cdot \left(\frac{d}{dx} y(x)\right)$ to series expansion

    $x \cdot \left(\frac{d}{dx} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r}$
  - ○ Convert $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right)$ to series expansion

    $x^2 \cdot \left(\frac{d^2}{dx^2} y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\,x^r + a_1(3+2r)\,(1+2r)\,x^{1+r} + \left(\sum_{k=2}^{\infty} \left(a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-}\right.\right.$

- • $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$
- • Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$
- • Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$
- • Solve for the dependent coefficient(s)

  $a_1 = 0$
- • Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$
- • Shift index using $k- >k + 2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\,r + 4r^2 - 1\right) + 4a_k = 0$
- • Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$
- • Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$
- • Solution for $r = -\frac{1}{2}$

$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2}=-\frac{4a_k}{4k^2+12k+8}, a_1=0\right]$$

- Recursion relation for $r=\frac{1}{2}$

$$a_{k+2}=-\frac{4a_k}{4k^2+20k+24}$$

- Solution for $r=\frac{1}{2}$

$$\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2}=-\frac{4a_k}{4k^2+20k+24}, a_1=0\right]$$

- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty} b_k x^{k+\frac{1}{2}}\right), a_{k+2}=-\frac{4a_k}{4k^2+12k+8}, a_1=0, b_{k+2}=-\frac{4b_k}{4k^2+20k+24}, b_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.023 (sec)
Leaf size : 34

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x+(x^2-1/4)*y(x) = 0,y(x),
       series,x=0)
```

$$y=\frac{c_1\left(1-\frac{1}{6}x^2+\frac{1}{120}x^4+\mathrm{O}\left(x^6\right)\right)x+c_2\left(1-\frac{1}{2}x^2+\frac{1}{24}x^4+\mathrm{O}\left(x^6\right)\right)}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.013 (sec)
Leaf size : 58

```
AsymptoticDSolveValue[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-1/4)*y[x]==0,{}},
       y[x],{x,0,5}]
```

$$y(x) \to c_1\left(\frac{x^{7/2}}{24} - \frac{x^{3/2}}{2} + \frac{1}{\sqrt{x}}\right) + c_2\left(\frac{x^{9/2}}{120} - \frac{x^{5/2}}{6} + \sqrt{x}\right)$$

2.4.57 problem 54

Internal problem ID [8622]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 54
Date solved : `Thursday, December 12, 2024 at 09:32:27 AM`
CAS classification : `[[_2nd_order, _exact, _linear, _homogeneous]]`

Solve

$$xy'' + (2-x)\,y' - y = 0$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$xy'' + (2-x)\,y' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = -\frac{x-2}{x}$$
$$q(x) = -\frac{1}{x}$$

Table 2.234: Table $p(x), q(x)$ singularites.

$p(x) = -\frac{x-2}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$xy'' + (2 - x)\, y' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} x\left(\sum_{n=0}^{\infty} (n + r)\, (n + r - 1)\, a_n x^{n+r-2}\right) \\ + (2 - x)\left(\sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\begin{aligned} \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n (n + r)\, (n + r - 1)\right) + \sum_{n=0}^{\infty} \left(-x^{n+r} a_n (n + r)\right) \\ + \left(\sum_{n=0}^{\infty} 2(n + r)\, a_n x^{n+r-1}\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \end{aligned} \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty}\left(-x^{n+r} a_n(n+r)\right) = \sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)\, x^{n+r-1}\right)$$

$$\sum_{n=0}^{\infty}\left(-a_n x^{n+r}\right) = \sum_{n=1}^{\infty}\left(-a_{n-1} x^{n+r-1}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\begin{aligned}\left(\sum_{n=0}^{\infty} x^{n+r-1} a_n(n+r)\,(n+r-1)\right) + \sum_{n=1}^{\infty}\left(-a_{n-1}(n+r-1)\, x^{n+r-1}\right) \\ + \left(\sum_{n=0}^{\infty} 2(n+r)\, a_n x^{n+r-1}\right) + \sum_{n=1}^{\infty}\left(-a_{n-1} x^{n+r-1}\right) = 0\end{aligned} \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2B) this gives

$$x^{n+r-1} a_n(n+r)\,(n+r-1) + 2(n+r)\, a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$x^{-1+r} a_0 r(-1+r) + 2r a_0 x^{-1+r} = 0$$

Or

$$\left(x^{-1+r} r(-1+r) + 2r\, x^{-1+r}\right) a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$r\, x^{-1+r}(1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= 0 \\ r_2 &= -1 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\,x^{-1+r}(1+r) = 0$$

Solving for r gives the roots of the indicial equation as $[0, -1]$.

Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x) \ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$
$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-1}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) - a_{n-1}(n+r-1) + 2a_n(n+r) - a_{n-1} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = \frac{a_{n-1}}{n+1+r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = \frac{a_{n-1}}{n+1} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = \frac{1}{2 + r}$$

Which for the root $r = 0$ becomes

$$a_1 = \frac{1}{2}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+r}$	$\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(2 + r)(3 + r)}$$

Which for the root $r = 0$ becomes

$$a_2 = \frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+r}$	$\frac{1}{2}$
a_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{6}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{(2 + r)(3 + r)(4 + r)}$$

Which for the root $r = 0$ becomes

$$a_3 = \frac{1}{24}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+r}$	$\frac{1}{2}$
a_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{6}$
a_3	$\frac{1}{(2+r)(3+r)(4+r)}$	$\frac{1}{24}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(2+r)(3+r)(5+r)(4+r)}$$

Which for the root $r = 0$ becomes

$$a_4 = \frac{1}{120}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+r}$	$\frac{1}{2}$
a_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{6}$
a_3	$\frac{1}{(2+r)(3+r)(4+r)}$	$\frac{1}{24}$
a_4	$\frac{1}{(2+r)(3+r)(5+r)(4+r)}$	$\frac{1}{120}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{(2+r)(3+r)(4+r)(6+r)(5+r)}$$

Which for the root $r = 0$ becomes

$$a_5 = \frac{1}{720}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{2+r}$	$\frac{1}{2}$
a_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{6}$
a_3	$\frac{1}{(2+r)(3+r)(4+r)}$	$\frac{1}{24}$
a_4	$\frac{1}{(2+r)(3+r)(5+r)(4+r)}$	$\frac{1}{120}$
a_5	$\frac{1}{(2+r)(3+r)(4+r)(6+r)(5+r)}$	$\frac{1}{720}$

Using the above table, then the solution $y_1(x)$ is

$$
\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\
&= 1 + \frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \frac{x^5}{720} + O(x^6)
\end{aligned}
$$

Now the second solution $y_2(x)$ is found. Let

$$
r_1 - r_2 = N
$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$
\begin{aligned}
a_N &= a_1 \\
&= \frac{1}{2+r}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lim_{r \to r_2} \frac{1}{2+r} &= \lim_{r \to -1} \frac{1}{2+r} \\
&= 1
\end{aligned}
$$

The limit is 1. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$
\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= \sum_{n=0}^{\infty} b_n x^{n-1}
\end{aligned}
$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $1 \leq n$ the recursive equation is

$$b_n(n+r)(n+r-1) - b_{n-1}(n+r-1) + 2(n+r)b_n - b_{n-1} = 0 \tag{4}$$

Which for for the root $r = -1$ becomes

$$b_n(n-1)(n-2) - b_{n-1}(n-2) + 2(n-1)b_n - b_{n-1} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = \frac{b_{n-1}}{n+1+r} \tag{5}$$

Which for the root $r = -1$ becomes

$$b_n = \frac{b_{n-1}}{n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = \frac{1}{2+r}$$

Which for the root $r = -1$ becomes

$$b_1 = 1$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+r}$	1

For $n = 2$, using the above recursive equation gives

$$b_2 = \frac{1}{(2+r)(3+r)}$$

Which for the root $r = -1$ becomes

$$b_2 = \frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+r}$	1
b_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$

For $n = 3$, using the above recursive equation gives

$$b_3 = \frac{1}{(2+r)(3+r)(4+r)}$$

Which for the root $r = -1$ becomes

$$b_3 = \frac{1}{6}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+r}$	1
b_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$
b_3	$\frac{1}{(2+r)(3+r)(4+r)}$	$\frac{1}{6}$

For $n = 4$, using the above recursive equation gives

$$b_4 = \frac{1}{(2+r)(3+r)(5+r)(4+r)}$$

Which for the root $r = -1$ becomes

$$b_4 = \frac{1}{24}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+r}$	1
b_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$
b_3	$\frac{1}{(2+r)(3+r)(4+r)}$	$\frac{1}{6}$
b_4	$\frac{1}{(2+r)(3+r)(5+r)(4+r)}$	$\frac{1}{24}$

For $n = 5$, using the above recursive equation gives

$$b_5 = \frac{1}{(2+r)(3+r)(4+r)(6+r)(5+r)}$$

Which for the root $r = -1$ becomes

$$b_5 = \frac{1}{120}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	$\frac{1}{2+r}$	1
b_2	$\frac{1}{(2+r)(3+r)}$	$\frac{1}{2}$
b_3	$\frac{1}{(2+r)(3+r)(4+r)}$	$\frac{1}{6}$
b_4	$\frac{1}{(2+r)(3+r)(5+r)(4+r)}$	$\frac{1}{24}$
b_5	$\frac{1}{(2+r)(3+r)(4+r)(6+r)(5+r)}$	$\frac{1}{120}$

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)}{x} \end{aligned}$$

Therefore the homogeneous solution is

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$= c_1 \left(1 + \frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{24} + \frac{x^4}{120} + \frac{x^5}{720} + O\left(x^6\right)\right) + \frac{c_2\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + O(x^6)\right)}{x}$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1+\frac{x}{2}+\frac{x^2}{6}+\frac{x^3}{24}+\frac{x^4}{120}+\frac{x^5}{720}+O(x^6)\right)+\frac{c_2\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+O(x^6)\right)}{x}
\end{aligned}
$$

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dx^2}y(x)\right)x+(-x+2)\left(\frac{d}{dx}y(x)\right)-y(x)=0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x)=\frac{y(x)}{x}+\frac{(x-2)\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x)-\frac{(x-2)\left(\frac{d}{dx}y(x)\right)}{x}-\frac{y(x)}{x}=0$
- □ Check to see if $x_0=0$ is a regular singular point
  - ∘ Define functions
    $\left[P_2(x)=-\frac{x-2}{x}, P_3(x)=-\frac{1}{x}\right]$
  - ∘ $x\cdot P_2(x)$ is analytic at $x=0$
    $(x\cdot P_2(x))\bigg|_{x=0}=2$
  - ∘ $x^2\cdot P_3(x)$ is analytic at $x=0$
    $(x^2\cdot P_3(x))\bigg|_{x=0}=0$
  - ∘ $x=0$is a regular singular point
    Check to see if $x_0=0$ is a regular singular point
    $x_0=0$
- Multiply by denominators
  $\left(\frac{d^2}{dx^2}y(x)\right)x+(-x+2)\left(\frac{d}{dx}y(x)\right)-y(x)=0$
- Assume series solution for $y(x)$
  $y(x)=\sum_{k=0}^{\infty}a_k x^{k+r}$

- ☐ Rewrite ODE with series expansions
  - ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$

    $$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r-1+m}$$
  - ◦ Shift index using $k- >k+1-m$

    $$x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$$
  - ◦ Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

    $$x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r-1}$$
  - ◦ Shift index using $k- >k+1$

    $$x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$$

  Rewrite ODE with series expansions

  $$a_0 r(1+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k+2+r) - a_k(k+1+r)\right) x^{k+r}\right) = 0$$
- $a_0$cannot be 0 by assumption, giving the indicial equation

  $r(1+r) = 0$
- Values of r that satisfy the indicial equation

  $r \in \{-1, 0\}$
- Each term in the series must be 0, giving the recursion relation

  $(k+1+r)\,(a_{k+1}(k+2+r) - a_k) = 0$
- Recursion relation that defines series solution to ODE

  $a_{k+1} = \frac{a_k}{k+2+r}$
- Recursion relation for $r = -1$

  $a_{k+1} = \frac{a_k}{k+1}$
- Solution for $r = -1$

  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-1}, a_{k+1} = \frac{a_k}{k+1}\right]$$
- Recursion relation for $r = 0$

  $a_{k+1} = \frac{a_k}{k+2}$
- Solution for $r = 0$

  $$\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k}{k+2}\right]$$
- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty} a_k x^{k-1}\right)+\left(\sum_{k=0}^{\infty} b_k x^k\right), a_{k+1}=\frac{a_k}{k+1}, b_{k+1}=\frac{b_k}{k+2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
<- linear_1 successful`
```

Maple dsolve solution

Solving time : 0.024 (sec)
Leaf size : 44

```
dsolve(x*diff(diff(y(x),x),x)+(-x+2)*diff(y(x),x)-y(x) = 0,y(x),
       series,x=0)
```

$$y = c_1\left(1+\frac{1}{2}x+\frac{1}{6}x^2+\frac{1}{24}x^3+\frac{1}{120}x^4+\frac{1}{720}x^5+\mathrm{O}\left(x^6\right)\right) + \frac{c_2\left(1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5+\mathrm{O}\left(x^6\right)\right)}{x}$$

Mathematica DSolve solution

Solving time : 0.024 (sec)
Leaf size : 62

```
AsymptoticDSolveValue[{x*D[y[x],{x,2}]+(2-x)*D[y[x],x]-y[x]==0,{}},
       y[x],{x,0,5}]
```

$$y(x)\to c_1\left(\frac{x^3}{24}+\frac{x^2}{6}+\frac{x}{2}+\frac{1}{x}+1\right)+c_2\left(\frac{x^4}{120}+\frac{x^3}{24}+\frac{x^2}{6}+\frac{x}{2}+1\right)$$

2.4.58 problem 55

Internal problem ID [8623]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 55
Date solved : `Thursday, December 12, 2024 at 09:32:28 AM`
CAS classification : `[[_2nd_order, _exact, _linear, _homogeneous]]`

Solve

$$2x^2y'' + 3xy' - y = 0$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' + 3xy' - y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{3}{2x}$$
$$q(x) = -\frac{1}{2x^2}$$

Table 2.236: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = -\frac{1}{2x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty]$

Irregular singular points : $[]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' + 3xy' - y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2\left(\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}\right) + 3x\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) - \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \quad \text{(2A)}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \sum_{n=0}^{\infty} \left(-a_n x^{n+r}\right) = 0 \quad \text{(2B)}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$2x^{n+r}a_n(n+r)(n+r-1) + 3x^{n+r}a_n(n+r) - a_n x^{n+r} = 0$$

When $n = 0$ the above becomes

$$2x^r a_0 r(-1+r) + 3x^r a_0 r - a_0 x^r = 0$$

Or

$$(2x^r r(-1+r) + 3x^r r - x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 + r - 1\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 + r - 1 = 0$$

Solving for r gives the roots of the indicial equation as

$$\begin{aligned} r_1 &= \frac{1}{2} \\ r_2 &= -1 \end{aligned}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 + r - 1\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as $\left[\frac{1}{2}, -1\right]$.

Since $r_1 - r_2 = \frac{3}{2}$ is not an integer, then we can construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right) \\ y_2(x) &= x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right) \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} \\ y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n-1} \end{aligned}$$

We start by finding $y_1(x)$. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $0 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + 3a_n(n+r) - a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = 0 \tag{4}$$

Which for the root $r = \frac{1}{2}$ becomes

$$a_n = 0 \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = \frac{1}{2}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= \sqrt{x}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right) \\ &= \sqrt{x}\left(1 + O(x^6)\right) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Eq (2B) derived above is now used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. For $0 \leq n$ the recursive equation is

$$2b_n(n+r)(n+r-1) + 3b_n(n+r) - b_n = 0 \tag{3}$$

Solving for b_n from recursive equation (4) gives

$$b_n = 0 \tag{4}$$

Which for the root $r = -1$ becomes

$$b_n = 0 \tag{5}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -1$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1

For $n = 1$, using the above recursive equation gives

$$b_1 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0

For $n = 2$, using the above recursive equation gives

$$b_2 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0

For $n = 3$, using the above recursive equation gives

$$b_3 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	0	0
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$\begin{aligned} y_2(x) &= \sqrt{x}\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\ &= \frac{1 + O(x^6)}{x} \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 \sqrt{x}\,\left(1 + O(x^6)\right) + \frac{c_2(1 + O(x^6))}{x} \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 \sqrt{x}\,\left(1 + O(x^6)\right) + \frac{c_2(1 + O(x^6))}{x} \end{aligned}$$

Maple step by step solution

Let's solve

$2x^2\left(\frac{d^2}{dx^2}y(x)\right) + 3x\left(\frac{d}{dx}y(x)\right) - y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{y(x)}{2x^2} - \frac{3\left(\frac{d}{dx}y(x)\right)}{2x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{3\left(\frac{d}{dx}y(x)\right)}{2x} - \frac{y(x)}{2x^2} = 0$

- Multiply by denominators of the ODE

  $2x^2\left(\frac{d^2}{dx^2}y(x)\right) + 3x\left(\frac{d}{dx}y(x)\right) - y(x) = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

  $\frac{d}{dx}y(x) = \left(\frac{d}{dt}y(t)\right)\left(\frac{d}{dx}t(x)\right)$

- Compute derivative

  $\frac{d}{dx}y(x) = \frac{\frac{d}{dt}y(t)}{x}$
- Calculate the 2nd derivative of y with respect to x , using the chain rule

  $\frac{d^2}{dx^2}y(x) = \left(\frac{d^2}{dt^2}y(t)\right)\left(\frac{d}{dx}t(x)\right)^2 + \left(\frac{d^2}{dx^2}t(x)\right)\left(\frac{d}{dt}y(t)\right)$
- Compute derivative

  $\frac{d^2}{dx^2}y(x) = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $2x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + 3\frac{d}{dt}y(t) - y(t) = 0$

- Simplify

  $2\frac{d^2}{dt^2}y(t) + \frac{d}{dt}y(t) - y(t) = 0$
- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = -\frac{\frac{d}{dt}y(t)}{2} + \frac{y(t)}{2}$
- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t) + \frac{\frac{d}{dt}y(t)}{2} - \frac{y(t)}{2} = 0$
- Characteristic polynomial of ODE

  $r^2 + \frac{1}{2}r - \frac{1}{2} = 0$
- Factor the characteristic polynomial

  $\frac{(r+1)(2r-1)}{2} = 0$
- Roots of the characteristic polynomial

  $r = \left(-1, \frac{1}{2}\right)$
- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{-t}$
- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{\frac{t}{2}}$
- General solution of the ODE

  $y(t) = C1y_1(t) + C2y_2(t)$
- Substitute in solutions

  $y(t) = C1\,\mathrm{e}^{-t} + C2\,\mathrm{e}^{\frac{t}{2}}$
- Change variables back using $t = \ln(x)$

  $y(x) = \frac{C1}{x} + C2\sqrt{x}$
- Simplify

$$y(x) = \frac{C1}{x} + C2\sqrt{x}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

Maple dsolve solution

Solving time : 0.018 (sec)
Leaf size : 20

```
dsolve(2*x^2*diff(diff(y(x),x),x)+3*diff(y(x),x)*x-y(x) = 0,y(x),
       series,x=0)
```

$$y = \frac{x^{3/2}c_2 + c_1}{x} + O(x^6)$$

Mathematica DSolve solution

Solving time : 0.004 (sec)
Leaf size : 18

```
AsymptoticDSolveValue[{2*x^2*D[y[x],{x,2}]+3*x*D[y[x],x]-y[x]==0,{}},
       y[x],{x,0,5}]
```

$$y(x) \to c_1\sqrt{x} + \frac{c_2}{x}$$

2.4.59 problem 56

Internal problem ID [8624]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 56
Date solved : `Thursday, December 12, 2024 at 09:32:29 AM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$2x^2y'' + 5xy' + 4y = 0$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$2x^2y'' + 5xy' + 4y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{5}{2x}$$
$$q(x) = \frac{2}{x^2}$$

Table 2.238: Table $p(x), q(x)$ singularites.

$p(x) = \frac{5}{2x}$	
singularity	type
$x = 0$	"regular"

$q(x) = \frac{2}{x^2}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty]$

Irregular singular points : []

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$2x^2y'' + 5xy' + 4y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$2x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) + 5x \left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 4\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0 \quad \text{(2A)}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} 2x^{n+r} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 5x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4a_n x^{n+r}\right) = 0 \quad \text{(2B)}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$2x^{n+r}a_n(n+r)(n+r-1) + 5x^{n+r}a_n(n+r) + 4a_nx^{n+r} = 0$$

When $n = 0$ the above becomes

$$2x^r a_0 r(-1+r) + 5x^r a_0 r + 4a_0 x^r = 0$$

Or

$$(2x^r r(-1+r) + 5x^r r + 4x^r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$\left(2r^2 + 3r + 4\right) x^r = 0$$

Since the above is true for all x then the indicial equation becomes

$$2r^2 + 3r + 4 = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = -\frac{3}{4} + \frac{i\sqrt{23}}{4}$$
$$r_2 = -\frac{3}{4} - \frac{i\sqrt{23}}{4}$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$\left(2r^2 + 3r + 4\right) x^r = 0$$

Solving for r gives the roots of the indicial equation as $\left[-\frac{3}{4} + \frac{i\sqrt{23}}{4}, -\frac{3}{4} - \frac{i\sqrt{23}}{4}\right]$.

Since the roots are complex conjugates, then two linearly independent solutions can be constructed using

$$y_1(x) = x^{r_1}\left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = x^{r_2}\left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n-\frac{3}{4}+\frac{i\sqrt{23}}{4}}$$
$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n-\frac{3}{4}-\frac{i\sqrt{23}}{4}}$$

$y_1(x)$ is found first. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $0 \leq n$ the recursive equation is

$$2a_n(n+r)(n+r-1) + 5a_n(n+r) + 4a_n = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = 0 \tag{4}$$

Which for the root $r = -\frac{3}{4} + \frac{i\sqrt{23}}{4}$ becomes

$$a_n = 0 \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = -\frac{3}{4} + \frac{i\sqrt{23}}{4}$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$a_1 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0

For $n = 2$, using the above recursive equation gives

$$a_2 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0

For $n = 3$, using the above recursive equation gives

$$a_3 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	0	0
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}y_1(x) &= x^{-\frac{3}{4}+\frac{i\sqrt{23}}{4}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots\right)\\ &= x^{-\frac{3}{4}+\frac{i\sqrt{23}}{4}}\left(1 + O(x^6)\right)\end{aligned}$$

The second solution $y_2(x)$ is found by taking the complex conjugate of $y_1(x)$ which gives

$$y_2(x) = x^{-\frac{3}{4}-\frac{i\sqrt{23}}{4}}\left(1 + O(x^6)\right)$$

Therefore the homogeneous solution is

$$\begin{aligned}y_h(x) &= c_1 y_1(x) + c_2 y_2(x)\\ &= c_1\, x^{-\frac{3}{4}+\frac{i\sqrt{23}}{4}}\left(1 + O(x^6)\right) + c_2\, x^{-\frac{3}{4}-\frac{i\sqrt{23}}{4}}\left(1 + O(x^6)\right)\end{aligned}$$

Hence the final solution is

$$\begin{aligned}y &= y_h\\ &= c_1\, x^{-\frac{3}{4}+\frac{i\sqrt{23}}{4}}\left(1 + O(x^6)\right) + c_2\, x^{-\frac{3}{4}-\frac{i\sqrt{23}}{4}}\left(1 + O(x^6)\right)\end{aligned}$$

Maple step by step solution

Let's solve

$2x^2\left(\frac{d^2}{dx^2}y(x)\right) + 5x\left(\frac{d}{dx}y(x)\right) + 4y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -\frac{2y(x)}{x^2} - \frac{5\left(\frac{d}{dx}y(x)\right)}{2x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) + \frac{5\left(\frac{d}{dx}y(x)\right)}{2x} + \frac{2y(x)}{x^2} = 0$
- Multiply by denominators of the ODE

  $2x^2\left(\frac{d^2}{dx^2}y(x)\right) + 5x\left(\frac{d}{dx}y(x)\right) + 4y(x) = 0$
- Make a change of variables

$t = \ln(x)$

- □ Substitute the change of variables back into the ODE
  - ○ Calculate the 1st derivative of y with respect to x , using the chain rule

    $\frac{d}{dx}y(x) = \left(\frac{d}{dt}y(t)\right)\left(\frac{d}{dx}t(x)\right)$
  - ○ Compute derivative

    $\frac{d}{dx}y(x) = \frac{\frac{d}{dt}y(t)}{x}$
  - ○ Calculate the 2nd derivative of y with respect to x , using the chain rule

    $\frac{d^2}{dx^2}y(x) = \left(\frac{d^2}{dt^2}y(t)\right)\left(\frac{d}{dx}t(x)\right)^2 + \left(\frac{d^2}{dx^2}t(x)\right)\left(\frac{d}{dt}y(t)\right)$
  - ○ Compute derivative

    $\frac{d^2}{dx^2}y(x) = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $2x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + 5\frac{d}{dt}y(t) + 4y(t) = 0$
- Simplify

  $2\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) + 4y(t) = 0$
- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = -\frac{3\frac{d}{dt}y(t)}{2} - 2y(t)$
- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t) + \frac{3\frac{d}{dt}y(t)}{2} + 2y(t) = 0$
- Characteristic polynomial of ODE

  $r^2 + \frac{3}{2}r + 2 = 0$
- Use quadratic formula to solve for r

  $r = \frac{\left(-\frac{3}{2}\right)\pm\left(\sqrt{-\frac{23}{4}}\right)}{2}$
- Roots of the characteristic polynomial

  $r = \left(-\frac{3}{4} - \frac{\mathrm{I}\sqrt{23}}{4}, -\frac{3}{4} + \frac{\mathrm{I}\sqrt{23}}{4}\right)$
- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{-\frac{3t}{4}}\cos\left(\frac{\sqrt{23}\,t}{4}\right)$
- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{-\frac{3t}{4}}\sin\left(\frac{\sqrt{23}\,t}{4}\right)$
- General solution of the ODE

  $y(t) = C1y_1(t) + C2y_2(t)$

- Substitute in solutions

$$y(t) = C1\,\mathrm{e}^{-\frac{3t}{4}} \cos\left(\frac{\sqrt{23}\,t}{4}\right) + C2\,\mathrm{e}^{-\frac{3t}{4}} \sin\left(\frac{\sqrt{23}\,t}{4}\right)$$

- Change variables back using $t = \ln(x)$

$$y(x) = \frac{C1 \cos\left(\frac{\sqrt{23}\,\ln(x)}{4}\right)}{x^{3/4}} + \frac{C2 \sin\left(\frac{\sqrt{23}\,\ln(x)}{4}\right)}{x^{3/4}}$$

- Simplify

$$y(x) = \frac{C1 \cos\left(\frac{\sqrt{23}\,\ln(x)}{4}\right)}{x^{3/4}} + \frac{C2 \sin\left(\frac{\sqrt{23}\,\ln(x)}{4}\right)}{x^{3/4}}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

Maple dsolve solution

Solving time : `0.021` (sec)
Leaf size : 32

```
dsolve(2*x^2*diff(diff(y(x),x),x)+5*diff(y(x),x)*x+4*y(x) = 0,y(x),
      series,x=0)
```

$$y = \frac{x^{-\frac{i\sqrt{23}}{4}} c_1 + x^{\frac{i\sqrt{23}}{4}} c_2}{x^{3/4}} + O\left(x^6\right)$$

Mathematica DSolve solution

Solving time : `0.004` (sec)
Leaf size : 44

```
AsymptoticDSolveValue[{2*x^2*D[y[x],{x,2}]+5*x*D[y[x],x]+4*y[x]==0,{}},
      y[x],{x,0,5}]
```

$$y(x) \to c_1 x^{\frac{1}{4}\left(-3+i\sqrt{23}\right)} + c_2 x^{\frac{1}{4}\left(-3-i\sqrt{23}\right)}$$

2.4.60 problem 57

Internal problem ID [8625]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 57
Date solved : `Thursday, December 12, 2024 at 09:32:30 AM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$x^2y'' + 3xy' + 4x^4y = 0$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' + 3xy' + 4x^4y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = \frac{3}{x}$$
$$q(x) = 4x^2$$

Table 2.240: Table $p(x), q(x)$ singularites.

$p(x) = \frac{3}{x}$	
singularity	type
$x = 0$	"regular"

$q(x) = 4x^2$	
singularity	type
$x = \infty$	"regular"
$x = -\infty$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[0, \infty, -\infty]$

Irregular singular points : $[\infty]$

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2y'' + 3xy' + 4x^4y = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$\begin{aligned} x^2 &\left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) \\ &+ 3x\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + 4x^4\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \end{aligned} \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\,(n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=0}^{\infty} 4x^{4+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 4x^{4+n+r} a_n = \sum_{n=4}^{\infty} 4a_{n-4} x^{n+r}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r) (n+r-1)\right) + \left(\sum_{n=0}^{\infty} 3x^{n+r} a_n (n+r)\right) + \left(\sum_{n=4}^{\infty} 4a_{n-4} x^{n+r}\right) = 0 \quad \text{(2B)}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r) (n+r-1) + 3x^{n+r} a_n (n+r) = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) + 3x^r a_0 r = 0$$

Or

$$(x^r r(-1+r) + 3x^r r)\, a_0 = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^r r(2+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(2+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 0$$
$$r_2 = -2$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^r r(2+r) = 0$$

Solving for r gives the roots of the indicial equation as $[0, -2]$.

Since $r_1 - r_2 = 2$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y_2(x) = Cy_1(x)\ln(x) + \frac{\sum_{n=0}^{\infty} b_n x^n}{x^2}$$

Or

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n-2}\right)$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. Substituting $n = 1$ in Eq. (2B) gives

$$a_1 = 0$$

Substituting $n = 2$ in Eq. (2B) gives

$$a_2 = 0$$

Substituting $n = 3$ in Eq. (2B) gives

$$a_3 = 0$$

For $4 \leq n$ the recursive equation is

$$a_n(n+r)(n+r-1) + 3a_n(n+r) + 4a_{n-4} = 0 \tag{3}$$

Solving for a_n from recursive equation (4) gives

$$a_n = -\frac{4a_{n-4}}{n^2 + 2nr + r^2 + 2n + 2r} \tag{4}$$

Which for the root $r = 0$ becomes

$$a_n = -\frac{4a_{n-4}}{n(n+2)} \tag{5}$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 0$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0

For $n = 4$, using the above recursive equation gives

$$a_4 = -\frac{4}{r^2 + 10r + 24}$$

Which for the root $r = 0$ becomes

$$a_4 = -\frac{1}{6}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	$-\frac{4}{r^2+10r+24}$	$-\frac{1}{6}$

For $n = 5$, using the above recursive equation gives

$$a_5 = 0$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	0	0
a_2	0	0
a_3	0	0
a_4	$-\frac{4}{r^2+10r+24}$	$-\frac{1}{6}$
a_5	0	0

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 \dots \\ &= 1 - \frac{x^4}{6} + O(x^6) \end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 2$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r \to r_2} a_2(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned} a_N &= a_2 \\ &= 0 \end{aligned}$$

Therefore

$$\begin{aligned} \lim_{r \to r_2} 0 &= \lim_{r \to -2} 0 \\ &= 0 \end{aligned}$$

The limit is 0. Since the limit exists then the log term is not needed and we can set $C = 0$. Therefore the second solution has the form

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-2} \end{aligned}$$

Eq (3) derived above is used to find all b_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. b_0 is arbitrary and taken as $b_0 = 1$. Substituting $n = 1$ in Eq(3) gives

$$b_1 = 0$$

Substituting $n = 2$ in Eq(3) gives

$$b_2 = 0$$

Substituting $n = 3$ in Eq(3) gives

$$b_3 = 0$$

For $4 \leq n$ the recursive equation is

$$b_n(n+r)\,(n+r-1) + 3b_n(n+r) + 4b_{n-4} = 0 \tag{4}$$

Which for for the root $r = -2$ becomes

$$b_n(n-2)\,(n-3) + 3b_n(n-2) + 4b_{n-4} = 0 \tag{4A}$$

Solving for b_n from the recursive equation (4) gives

$$b_n = -\frac{4b_{n-4}}{n^2 + 2nr + r^2 + 2n + 2r} \tag{5}$$

Which for the root $r = -2$ becomes

$$b_n = -\frac{4b_{n-4}}{n^2 - 2n} \tag{6}$$

At this point, it is a good idea to keep track of b_n in a table both before substituting $r = -2$ and after as more terms are found using the above recursive equation.

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0

For $n = 4$, using the above recursive equation gives

$$b_4 = -\frac{4}{r^2 + 10r + 24}$$

Which for the root $r = -2$ becomes

$$b_4 = -\frac{1}{2}$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	$-\frac{4}{r^2+10r+24}$	$-\frac{1}{2}$

For $n = 5$, using the above recursive equation gives

$$b_5 = 0$$

And the table now becomes

n	$b_{n,r}$	b_n
b_0	1	1
b_1	0	0
b_2	0	0
b_3	0	0
b_4	$-\frac{4}{r^2+10r+24}$	$-\frac{1}{2}$
b_5	0	0

Using the above table, then the solution $y_2(x)$ is

$$
\begin{aligned}
y_2(x) &= 1\left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 \dots\right) \\
&= \frac{1 - \frac{x^4}{2} + O(x^6)}{x^2}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1 - \frac{x^4}{6} + O\left(x^6\right)\right) + \frac{c_2\left(1 - \frac{x^4}{2} + O(x^6)\right)}{x^2}
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y &= y_h \\
&= c_1\left(1 - \frac{x^4}{6} + O\left(x^6\right)\right) + \frac{c_2\left(1 - \frac{x^4}{2} + O(x^6)\right)}{x^2}
\end{aligned}
$$

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) + 3x\left(\frac{d}{dx}y(x)\right) + 4x^4 y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = -4x^2 y(x) - \frac{3\left(\frac{d}{dx}y(x)\right)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$$\frac{d^2}{dx^2}y(x) + \frac{3\left(\frac{d}{dx}y(x)\right)}{x} + 4x^2y(x) = 0$$

□ Check to see if $x_0 = 0$ is a regular singular point

- ◦ Define functions

$$\left[P_2(x) = \frac{3}{x}, P_3(x) = 4x^2\right]$$

- ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

$$(x \cdot P_2(x))\bigg|_{x=0} = 3$$

- ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

$$(x^2 \cdot P_3(x))\bigg|_{x=0} = 0$$

- ◦ $x = 0$is a regular singular point

Check to see if $x_0 = 0$ is a regular singular point

$$x_0 = 0$$

- Multiply by denominators

$$4x^3y(x) + \left(\frac{d^2}{dx^2}y(x)\right)x + 3\frac{d}{dx}y(x) = 0$$

- Assume series solution for $y(x)$

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$$

□ Rewrite ODE with series expansions

- ◦ Convert $x^3 \cdot y(x)$ to series expansion

$$x^3 \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+3}$$

- ◦ Shift index using $k- > k - 3$

$$x^3 \cdot y(x) = \sum_{k=3}^{\infty} a_{k-3} x^{k+r}$$

- ◦ Convert $\frac{d}{dx}y(x)$ to series expansion

$$\frac{d}{dx}y(x) = \sum_{k=0}^{\infty} a_k(k+r)\,x^{k+r-1}$$

- ◦ Shift index using $k- > k + 1$

$$\frac{d}{dx}y(x) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,x^{k+r}$$

- ◦ Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

$$x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)(k+r-1)\, x^{k+r-1}$$

- Shift index using $k->k+1$

$$x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$$

Rewrite ODE with series expansions

$$a_0 r(2+r)\, x^{-1+r} + a_1(1+r)(3+r)\, x^r + a_2(2+r)(4+r)\, x^{1+r} + a_3(3+r)(5+r)\, x^{2+r} + \left(\sum_{k=3}^{\infty} (\right.$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{-2, 0\}$
- The coefficients of each power of x must be 0
  $[a_1(1+r)(3+r) = 0, a_2(2+r)(4+r) = 0, a_3(3+r)(5+r) = 0]$
- Solve for the dependent coefficient(s)
  $\{a_1 = 0, a_2 = 0, a_3 = 0\}$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)(k+r+3) + 4a_{k-3} = 0$
- Shift index using $k->k+3$
  $a_{k+4}(k+4+r)(k+6+r) + 4a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+4} = -\frac{4a_k}{(k+4+r)(k+6+r)}$
- Recursion relation for $r = -2$
  $a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}$
- Solution for $r = -2$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-2}, a_{k+4} = -\frac{4a_k}{(k+2)(k+4)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$
- Recursion relation for $r = 0$
  $a_{k+4} = -\frac{4a_k}{(k+4)(k+6)}$
- Solution for $r = 0$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{4a_k}{(k+4)(k+6)}, a_1 = 0, a_2 = 0, a_3 = 0\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^{k-2}\right) + \left(\sum_{k=0}^{\infty} b_k x^k\right), a_{4+k} = -\frac{4a_k}{(k+2)(4+k)}, a_1 = 0, a_2 = 0, a_3 = 0, b_{4+k} = -\frac{4b_k}{(4+k)(k\right.$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.023 (sec)
Leaf size : 28

```
dsolve(x^2*diff(diff(y(x),x),x)+3*diff(y(x),x)*x+4*y(x)*x^4 = 0,y(x),
       series,x=0)
```

$$y = c_1\left(1-\frac{1}{6}x^4+\mathrm{O}\left(x^6\right)\right)+\frac{c_2(-2+x^4+\mathrm{O}\left(x^6\right))}{x^2}$$

Mathematica DSolve solution

Solving time : 0.01 (sec)
Leaf size : 30

```
AsymptoticDSolveValue[{x^2*D[y[x],{x,2}]+3*x*D[y[x],x]+4*x^4*y[x]==0,{}},
       y[x],{x,0,5}]
```

$$y(x)\to c_2\left(1-\frac{x^4}{6}\right)+c_1\left(\frac{1}{x^2}-\frac{x^2}{2}\right)$$

2.4.61 problem 58

Internal problem ID [8626]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 58
Date solved : `Thursday, December 12, 2024 at 09:32:31 AM`
CAS classification : `[[_Emden, _Fowler]]`

Solve

$$x^2y'' - xy = 0$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$x^2y'' - xy = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 0$$
$$q(x) = -\frac{1}{x}$$

Table 2.242: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = -\frac{1}{x}$	
singularity	type
$x = 0$	"regular"

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : [0]

Irregular singular points : [∞]

Since $x = 0$ is regular singular point, then Frobenius power series is used. The ode is normalized to be

$$x^2 y'' - xy = 0$$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above back into the ode gives

$$x^2 \left(\sum_{n=0}^{\infty} (n+r)\,(n+r-1)\, a_n x^{n+r-2}\right) - x \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)\,(n+r-1)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n + r$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} \left(-x^{1+n+r} a_n\right) = \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r$.

$$\left(\sum_{n=0}^{\infty} x^{n+r} a_n (n+r)(n+r-1)\right) + \sum_{n=1}^{\infty} \left(-a_{n-1} x^{n+r}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2B) this gives

$$x^{n+r} a_n (n+r)(n+r-1) = 0$$

When $n = 0$ the above becomes

$$x^r a_0 r(-1+r) = 0$$

Or

$$x^r a_0 r(-1+r) = 0$$

Since $a_0 \neq 0$ then the above simplifies to

$$x^r r(-1+r) = 0$$

Since the above is true for all x then the indicial equation becomes

$$r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as

$$r_1 = 1$$
$$r_2 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$x^r r(-1+r) = 0$$

Solving for r gives the roots of the indicial equation as $[1, 0]$.

Since $r_1 - r_2 = 1$ is an integer, then we can construct two linearly independent solutions

$$y_1(x) = x^{r_1} \left(\sum_{n=0}^{\infty} a_n x^n\right)$$
$$y_2(x) = C y_1(x) \ln(x) + x^{r_2} \left(\sum_{n=0}^{\infty} b_n x^n\right)$$

Or

$$
\begin{aligned}
y_1(x) &= x\left(\sum_{n=0}^{\infty} a_n x^n\right) \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n x^{1+n} \\
y_2(x) &= C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^n\right)
\end{aligned}
$$

Where C above can be zero. We start by finding y_1. Eq (2B) derived above is now used to find all a_n coefficients. The case $n = 0$ is skipped since it was used to find the roots of the indicial equation. a_0 is arbitrary and taken as $a_0 = 1$. For $1 \leq n$ the recursive equation is

$$
a_n(n+r)(n+r-1) - a_{n-1} = 0 \tag{3}
$$

Solving for a_n from recursive equation (4) gives

$$
a_n = \frac{a_{n-1}}{(n+r)(n+r-1)} \tag{4}
$$

Which for the root $r = 1$ becomes

$$
a_n = \frac{a_{n-1}}{(1+n)\,n} \tag{5}
$$

At this point, it is a good idea to keep track of a_n in a table both before substituting $r = 1$ and after as more terms are found using the above recursive equation.

n	$a_{n,r}$	a_n
a_0	1	1

For $n = 1$, using the above recursive equation gives

$$
a_1 = \frac{1}{(1+r)\,r}
$$

Which for the root $r = 1$ becomes

$$
a_1 = \frac{1}{2}
$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{(1+r)r}$	$\frac{1}{2}$

For $n = 2$, using the above recursive equation gives

$$a_2 = \frac{1}{(1+r)^2 r (2+r)}$$

Which for the root $r = 1$ becomes

$$a_2 = \frac{1}{12}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{(1+r)r}$	$\frac{1}{2}$
a_2	$\frac{1}{(1+r)^2 r(2+r)}$	$\frac{1}{12}$

For $n = 3$, using the above recursive equation gives

$$a_3 = \frac{1}{(1+r)^2 r (2+r)^2 (3+r)}$$

Which for the root $r = 1$ becomes

$$a_3 = \frac{1}{144}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{(1+r)r}$	$\frac{1}{2}$
a_2	$\frac{1}{(1+r)^2 r(2+r)}$	$\frac{1}{12}$
a_3	$\frac{1}{(1+r)^2 r(2+r)^2 (3+r)}$	$\frac{1}{144}$

For $n = 4$, using the above recursive equation gives

$$a_4 = \frac{1}{(1+r)^2 r (2+r)^2 (3+r)^2 (4+r)}$$

Which for the root $r = 1$ becomes

$$a_4 = \frac{1}{2880}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{(1+r)r}$	$\frac{1}{2}$
a_2	$\frac{1}{(1+r)^2r(2+r)}$	$\frac{1}{12}$
a_3	$\frac{1}{(1+r)^2r(2+r)^2(3+r)}$	$\frac{1}{144}$
a_4	$\frac{1}{(1+r)^2r(2+r)^2(3+r)^2(4+r)}$	$\frac{1}{2880}$

For $n = 5$, using the above recursive equation gives

$$a_5 = \frac{1}{(1+r)^2 r (2+r)^2 (3+r)^2 (4+r)^2 (5+r)}$$

Which for the root $r = 1$ becomes

$$a_5 = \frac{1}{86400}$$

And the table now becomes

n	$a_{n,r}$	a_n
a_0	1	1
a_1	$\frac{1}{(1+r)r}$	$\frac{1}{2}$
a_2	$\frac{1}{(1+r)^2r(2+r)}$	$\frac{1}{12}$
a_3	$\frac{1}{(1+r)^2r(2+r)^2(3+r)}$	$\frac{1}{144}$
a_4	$\frac{1}{(1+r)^2r(2+r)^2(3+r)^2(4+r)}$	$\frac{1}{2880}$
a_5	$\frac{1}{(1+r)^2r(2+r)^2(3+r)^2(4+r)^2(5+r)}$	$\frac{1}{86400}$

Using the above table, then the solution $y_1(x)$ is

$$\begin{aligned}
y_1(x) &= x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 \dots\right) \\
&= x\left(1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{144} + \frac{x^4}{2880} + \frac{x^5}{86400} + O\left(x^6\right)\right)
\end{aligned}$$

Now the second solution $y_2(x)$ is found. Let

$$r_1 - r_2 = N$$

Where N is positive integer which is the difference between the two roots. r_1 is taken as the larger root. Hence for this problem we have $N = 1$. Now we need to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_1(r)$. If this limit exists, then $C = 0$, else we need to keep the log term and $C \neq 0$. The above table shows that

$$\begin{aligned}
a_N &= a_1 \\
&= \frac{1}{(1+r)\,r}
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{r\to r_2} \frac{1}{(1+r)\,r} &= \lim_{r\to 0} \frac{1}{(1+r)\,r} \\
&= \textit{undefined}
\end{aligned}$$

Since the limit does not exist then the log term is needed. Therefore the second solution has the form

$$y_2(x) = Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Therefore

$$\begin{aligned}
\frac{d}{dx}y_2(x) &= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} \frac{b_n x^{n+r_2}(n+r_2)}{x}\right) \\
&= Cy_1'(x)\ln(x) + \frac{Cy_1(x)}{x} + \left(\sum_{n=0}^{\infty} x^{-1+n+r_2} b_n(n+r_2)\right) \\
\frac{d^2}{dx^2}y_2(x) &= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} \\
&\quad + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right) \\
&= Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)\,(-1+n+r_2)\right)
\end{aligned}$$

Substituting these back into the given ode $x^2y'' - xy = 0$ gives

$$x^2\left(Cy_1''(x)\ln(x) + \frac{2Cy_1'(x)}{x} - \frac{Cy_1(x)}{x^2} + \sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) - x\left(Cy_1(x)\ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)\right) = 0$$

Which can be written as

$$\left(\left(y_1''(x)\,x^2 - y_1(x)\,x\right)\ln(x) + x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)\right)C + x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) - x\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{7}$$

But since $y_1(x)$ is a solution to the ode, then

$$y_1''(x)\,x^2 - y_1(x)\,x = 0$$

Eq (7) simplifes to

$$x^2\left(\frac{2y_1'(x)}{x} - \frac{y_1(x)}{x^2}\right)C + x^2\left(\sum_{n=0}^{\infty}\left(\frac{b_n x^{n+r_2}(n+r_2)^2}{x^2} - \frac{b_n x^{n+r_2}(n+r_2)}{x^2}\right)\right) - x\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{8}$$

Substituting $y_1 = \sum_{n=0}^{\infty} a_n x^{n+r_1}$ into the above gives

$$\left(2\left(\sum_{n=0}^{\infty} x^{-1+n+r_1} a_n(n+r_1)\right)x - \left(\sum_{n=0}^{\infty} a_n x^{n+r_1}\right)\right)C + \left(\sum_{n=0}^{\infty} x^{-2+n+r_2} b_n(n+r_2)(-1+n+r_2)\right)x^2 - x\left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right) = 0 \tag{9}$$

Since $r_1 = 1$ and $r_2 = 0$ then the above becomes

$$\left(2\left(\sum_{n=0}^{\infty} x^n a_n(1+n)\right)x - \left(\sum_{n=0}^{\infty} a_n x^{1+n}\right)\right)C \\ + \left(\sum_{n=0}^{\infty} x^{-2+n} b_n n(n-1)\right)x^2 - x\left(\sum_{n=0}^{\infty} b_n x^n\right) = 0 \tag{10}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} 2C\, x^{1+n} a_n(1+n)\right) + \sum_{n=0}^{\infty} \left(-a_n x^{1+n} C\right) + \left(\sum_{n=0}^{\infty} n\, x^n b_n(n-1)\right) + \sum_{n=0}^{\infty} \left(-x^{1+n} b_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} 2C\, x^{1+n} a_n(1+n) = \sum_{n=1}^{\infty} 2C a_{n-1} n\, x^n$$

$$\sum_{n=0}^{\infty} \left(-a_n x^{1+n} C\right) = \sum_{n=1}^{\infty} \left(-C a_{n-1} x^n\right)$$

$$\sum_{n=0}^{\infty} \left(-x^{1+n} b_n\right) = \sum_{n=1}^{\infty} \left(-b_{n-1} x^n\right)$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=1}^{\infty} 2C a_{n-1} n\, x^n\right) + \sum_{n=1}^{\infty} \left(-C a_{n-1} x^n\right) + \left(\sum_{n=0}^{\infty} n\, x^n b_n(n-1)\right) + \sum_{n=1}^{\infty} \left(-b_{n-1} x^n\right) = 0 \tag{2B}$$

For $n = 0$ in Eq. (2B), we choose arbitray value for b_0 as $b_0 = 1$. For $n = N$, where $N = 1$ which is the difference between the two roots, we are free to choose $b_1 = 0$. Hence for $n = 1$, Eq (2B) gives

$$C - 1 = 0$$

Which is solved for C. Solving for C gives

$$C = 1$$

For $n = 2$, Eq (2B) gives

$$3Ca_1 - b_1 + 2b_2 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$2b_2 + \frac{3}{2} = 0$$

Solving the above for b_2 gives

$$b_2 = -\frac{3}{4}$$

For $n = 3$, Eq (2B) gives

$$5Ca_2 - b_2 + 6b_3 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$6b_3 + \frac{7}{6} = 0$$

Solving the above for b_3 gives

$$b_3 = -\frac{7}{36}$$

For $n = 4$, Eq (2B) gives

$$7Ca_3 - b_3 + 12b_4 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$12b_4 + \frac{35}{144} = 0$$

Solving the above for b_4 gives

$$b_4 = -\frac{35}{1728}$$

For $n = 5$, Eq (2B) gives

$$9Ca_4 - b_4 + 20b_5 = 0$$

Which when replacing the above values found already for b_n and the values found earlier for a_n and for C, gives

$$20b_5 + \frac{101}{4320} = 0$$

Solving the above for b_5 gives

$$b_5 = -\frac{101}{86400}$$

Now that we found all b_n and C, we can calculate the second solution from

$$y_2(x) = C y_1(x) \ln(x) + \left(\sum_{n=0}^{\infty} b_n x^{n+r_2}\right)$$

Using the above value found for $C = 1$ and all b_n, then the second solution becomes

$$\begin{aligned} y_2(x) = 1\left(x\left(1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{144} + \frac{x^4}{2880} + \frac{x^5}{86400} + O(x^6)\right)\right)\ln(x) \\ + 1 - \frac{3x^2}{4} - \frac{7x^3}{36} - \frac{35x^4}{1728} - \frac{101x^5}{86400} + O(x^6) \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x\left(1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{144} + \frac{x^4}{2880} + \frac{x^5}{86400} + O(x^6)\right) \\ &\quad + c_2\left(1\left(x\left(1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{144} + \frac{x^4}{2880} + \frac{x^5}{86400} + O(x^6)\right)\right)\ln(x) + 1 - \frac{3x^2}{4} \right. \\ &\quad \left. - \frac{7x^3}{36} - \frac{35x^4}{1728} - \frac{101x^5}{86400} + O(x^6)\right) \end{aligned}$$

Hence the final solution is

$$\begin{aligned} y &= y_h \\ &= c_1 x\left(1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{144} + \frac{x^4}{2880} + \frac{x^5}{86400} + O(x^6)\right) \\ &\quad + c_2\left(x\left(1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{144} + \frac{x^4}{2880} + \frac{x^5}{86400} + O(x^6)\right)\ln(x) + 1 - \frac{3x^2}{4} - \frac{7x^3}{36} \right. \\ &\quad \left. - \frac{35x^4}{1728} - \frac{101x^5}{86400} + O(x^6)\right) \end{aligned}$$

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right) - xy(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x) = \frac{y(x)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x) - \frac{y(x)}{x} = 0$

□ Check to see if $x_0 = 0$ is a regular singular point

  ◦ Define functions

  $\left[P_2(x) = 0, P_3(x) = -\frac{1}{x}\right]$

  ◦ $x \cdot P_2(x)$ is analytic at $x = 0$

  $\left.(x \cdot P_2(x))\right|_{x=0} = 0$

  ◦ $x^2 \cdot P_3(x)$ is analytic at $x = 0$

  $\left.(x^2 \cdot P_3(x))\right|_{x=0} = 0$

  ◦ $x = 0$is a regular singular point

  Check to see if $x_0 = 0$ is a regular singular point

  $x_0 = 0$

- Multiply by denominators

  $\left(\frac{d^2}{dx^2}y(x)\right)x - y(x) = 0$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

  ◦ Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)(k+r-1)\, x^{k+r-1}$

  ◦ Shift index using $k- >k+1$

  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)(k+r)\, x^{k+r}$

Rewrite ODE with series expansions

$$a_0 r(-1+r)\, x^{-1+r} + \left(\sum_{k=0}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r) - a_k\right) x^{k+r}\right) = 0$$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-1+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 1\}$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)\,(k+r) - a_k = 0$
- Recursion relation that defines series solution to ODE
  $a_{k+1} = \frac{a_k}{(k+1+r)(k+r)}$
- Recursion relation for $r = 0$
  $a_{k+1} = \frac{a_k}{(k+1)k}$
- Solution for $r = 0$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+1} = \frac{a_k}{(k+1)k}\right]$
- Recursion relation for $r = 1$
  $a_{k+1} = \frac{a_k}{(k+2)(k+1)}$
- Solution for $r = 1$
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+1}, a_{k+1} = \frac{a_k}{(k+2)(k+1)}\right]$
- Combine solutions and rename parameters
  $\left[y(x) = \left(\sum_{k=0}^{\infty} a_k x^k\right) + \left(\sum_{k=0}^{\infty} b_k x^{k+1}\right), a_{k+1} = \frac{a_k}{(k+1)k}, b_{k+1} = \frac{b_k}{(k+2)(k+1)}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
```

```
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

Maple dsolve solution

Solving time : 0.023 (sec)
Leaf size : 58

```
dsolve(x^2*diff(diff(y(x),x),x)-x*y(x) = 0,y(x),
       series,x=0)
```

$$
\begin{aligned}
y = {} & c_1 x\left(1+\frac{1}{2}x+\frac{1}{12}x^2+\frac{1}{144}x^3+\frac{1}{2880}x^4+\frac{1}{86400}x^5+\operatorname{O}\left(x^6\right)\right) \\
& + c_2\left(\ln\left(x\right)\left(x+\frac{1}{2}x^2+\frac{1}{12}x^3+\frac{1}{144}x^4+\frac{1}{2880}x^5+\operatorname{O}\left(x^6\right)\right)\right. \\
& \left. + \left(1-\frac{3}{4}x^2-\frac{7}{36}x^3-\frac{35}{1728}x^4-\frac{101}{86400}x^5+\operatorname{O}\left(x^6\right)\right)\right)
\end{aligned}
$$

Mathematica DSolve solution

Solving time : 0.018 (sec)
Leaf size : 85

```
AsymptoticDSolveValue[{x^2*D[y[x],{x,2}]-x*y[x]==0,{}},
       y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to {} & c_1\left(\frac{1}{144}x\left(x^3+12x^2+72x+144\right)\log(x)\right. \\
& \left. + \frac{-47x^4-480x^3-2160x^2-1728x+1728}{1728}\right)+c_2\left(\frac{x^5}{2880}+\frac{x^4}{144}+\frac{x^3}{12}+\frac{x^2}{2}+x\right)
\end{aligned}
$$

2.4.62 problem 59

Internal problem ID [8627]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 59
Date solved : `Thursday, December 12, 2024 at 09:32:33 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$\left(-x^2+1\right) y''+y'+y=x\,\mathrm{e}^x$$

Using series expansion around $x=0$

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y''=f(x,y,y')$$

Assuming expansion is at $x_0=0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x,y,y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0)=y_0$ and $y'(x_0)=y'_0$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0)+(x-x_0)\,y'(x_0)+\frac{(x-x_0)^2}{2}y''(x_0)+\frac{(x-x_0)^3}{3!}y'''(x_0)+\cdots \\
&= y_0+xy'_0+\frac{x^2}{2}\,f|_{x_0,y_0,y'_0}+\frac{x^3}{3!}\,f'|_{x_0,y_0,y'_0}+\cdots \\
&= y_0+xy'_0+\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)\,!}\,\frac{d^nf}{dx^n}\bigg|_{x_0,y_0,y'_0}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (2.12)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (2.13)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{align*}
F_0 &= f(x,y,y') \tag{4}\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \tag{6}
\end{align*}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0,y_0,y_0'} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -\frac{x\,\mathrm{e}^{x} - y' - y}{x^2 - 1} \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= \frac{(x^2 - 2x)\, y' + (-x^3 + x^2 + 1)\, \mathrm{e}^{x} + (-2x + 1)\, y}{(x^2 - 1)^2} \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \frac{(-4x^3 + 8x^2 - 2x + 1)\, y' + (-x^5 + 2x^4 - 2x^3 + 5x^2 - 6x - 1)\, \mathrm{e}^{x} + y(7x^2 - 6x + 2)}{(x^2 - 1)^3} \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= \frac{(19x^4 - 42x^3 + 25x^2 - 18x + 1)\, y' + (-x^7 + 3x^6 - 5x^5 + 18x^4 - 40x^3 + 32x^2 + x + 7)\, \mathrm{e}^{x} - 32y\big(x^3}{(x^2 - 1)^4} \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \frac{(-108x^5 + 267x^4 - 264x^3 + 246x^2 - 48x + 12)\, y' + (-x^9 + 4x^8 - 10x^7 + 37x^6 - 144x^5 + 281x^4 -}{(x^2 - 1)^5}
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= -y'(0) - y(0) \\
F_1 &= 1 + y(0) \\
F_2 &= 1 - 2y(0) - y'(0) \\
F_3 &= 7 + 7y(0) + y'(0) \\
F_4 &= 8 - 29y(0) - 12y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1-\frac{1}{2}x^{2}+\frac{1}{6}x^{3}-\frac{1}{12}x^{4}+\frac{7}{120}x^{5}\right) y(0) \\
&+\left(x-\frac{1}{2}x^{2}-\frac{1}{24}x^{4}+\frac{1}{120}x^{5}\right) y'(0)+\frac{x^{3}}{6}+\frac{x^{4}}{24}+\frac{7x^{5}}{120}+O\left(x^{6}\right)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary point, then this can also be solved using the standard power series method. The ode is normalized to be

$$
\left(-x^{2}+1\right) y''+y'+y = x\,\mathrm{e}^{x}
$$

Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_{n} x^{n}
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_{n} x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_{n} x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(-x^{2}+1\right)\left(\sum_{n=2}^{\infty} n(n-1)\, a_{n} x^{n-2}\right)+\left(\sum_{n=1}^{\infty} n a_{n} x^{n-1}\right)+\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right) = x\,\mathrm{e}^{x} \tag{1}
$$

Expanding $x\,\mathrm{e}^{x}$ as Taylor series around $x = 0$ and keeping only the first 6 terms gives

$$
\begin{aligned}
x\,\mathrm{e}^{x} &= x+x^{2}+\frac{1}{2}x^{3}+\frac{1}{6}x^{4}+\frac{1}{24}x^{5}+\dots \\
&= x+x^{2}+\frac{1}{2}x^{3}+\frac{1}{6}x^{4}+\frac{1}{24}x^{5}
\end{aligned}
$$

Hence the ODE in Eq (1) becomes

$$
\begin{aligned}
&\left(-x^{2}+1\right)\left(\sum_{n=2}^{\infty} n(n-1)\, a_{n} x^{n-2}\right)+\left(\sum_{n=1}^{\infty} n a_{n} x^{n-1}\right)+\left(\sum_{n=0}^{\infty} a_{n} x^{n}\right) \\
&= x+x^{2}+\frac{1}{2}x^{3}+\frac{1}{6}x^{4}+\frac{1}{24}x^{5}
\end{aligned}
$$

Which simplifies to

$$\sum_{n=2}^{\infty} \left(-x^n a_n n(n-1)\right) + \left(\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}\right) + \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n$$

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\sum_{n=2}^{\infty} \left(-x^n a_n n(n-1)\right) + \left(\sum_{n=0}^{\infty} (n+2) a_{n+2}(n+1) x^n\right) + \left(\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5 \tag{3}$$

$n = 0$ gives

$$2a_2 + a_1 + a_0 = 0$$

$$a_2 = -\frac{a_0}{2} - \frac{a_1}{2}$$

$n = 1$ gives

$$(6a_3 + 2a_2 + a_1) x = x$$

$$6a_3 + 2a_2 + a_1 = 1$$

Which after substituting earlier equations, simplifies to

$$a_3 = \frac{a_0}{6} + \frac{1}{6}$$

For $2 \leq n$, the recurrence equation is

$$\left(-n a_n (n-1) + (n+2)\, a_{n+2}(n+1) + (n+1)\, a_{n+1} + a_n\right) x^n = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5 \tag{4}$$

For $n = 2$ the recurrence equation gives

$$\left(-a_2 + 12a_4 + 3a_3\right) x^2 = x^2$$
$$-a_2 + 12a_4 + 3a_3 = 1$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{1}{24} - \frac{a_0}{12} - \frac{a_1}{24}$$

For $n = 3$ the recurrence equation gives

$$\left(-5a_3 + 20a_5 + 4a_4\right) x^3 = \frac{x^3}{2}$$
$$-5a_3 + 20a_5 + 4a_4 = \frac{1}{2}$$

Which after substituting the earlier terms found becomes

$$a_5 = \frac{7}{120} + \frac{7a_0}{120} + \frac{a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$\left(-11a_4 + 30a_6 + 5a_5\right) x^4 = \frac{x^4}{6}$$
$$-11a_4 + 30a_6 + 5a_5 = \frac{1}{6}$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{1}{90} - \frac{29a_0}{720} - \frac{a_1}{60}$$

For $n = 5$ the recurrence equation gives

$$\begin{aligned}(-19a_5 + 42a_7 + 6a_6)\, x^5 &= \frac{x^5}{24} \\ -19a_5 + 42a_7 + 6a_6 &= \frac{1}{24}\end{aligned}$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{13}{504} + \frac{9a_0}{280} + \frac{31a_1}{5040}$$

And so on. Therefore the solution is

$$\begin{aligned}y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 + \dots\end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned}y = a_0 + a_1x + \left(-\frac{a_0}{2} - \frac{a_1}{2}\right)x^2 + \left(\frac{a_0}{6} + \frac{1}{6}\right)x^3 \\ + \left(\frac{1}{24} - \frac{a_0}{12} - \frac{a_1}{24}\right)x^4 + \left(\frac{7}{120} + \frac{7a_0}{120} + \frac{a_1}{120}\right)x^5 + \dots\end{aligned}$$

Collecting terms, the solution becomes

$$\begin{aligned}y = \left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{7}{120}x^5\right)a_0 \\ + \left(x - \frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{1}{120}x^5\right)a_1 + \frac{x^3}{6} + \frac{x^4}{24} + \frac{7x^5}{120} + O(x^6)\end{aligned} \tag{3}$$

At $x = 0$ the solution above becomes

$$\begin{aligned}y = \left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{7}{120}x^5\right)c_1 \\ + \left(x - \frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{1}{120}x^5\right)c_2 + \frac{x^3}{6} + \frac{x^4}{24} + \frac{7x^5}{120} + O(x^6)\end{aligned}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Group is reducible or imprimitive
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 69

```
dsolve((-x^2+1)*diff(diff(y(x),x),x)+diff(y(x),x)+y(x) = exp(x)*x,y(x),
       series,x=0)
```

$$
\begin{aligned}
y = &\left(1-\frac{1}{2}x^2+\frac{1}{6}x^3-\frac{1}{12}x^4+\frac{7}{120}x^5\right)y(0) \\
&+\left(x-\frac{1}{2}x^2-\frac{1}{24}x^4+\frac{1}{120}x^5\right)y'(0)+\frac{x^3}{6}+\frac{x^4}{24}+\frac{7x^5}{120}+O\left(x^6\right)
\end{aligned}
$$

Mathematica DSolve solution

Solving time : 0.02 (sec)
Leaf size : 63

```
AsymptoticDSolveValue[{(1-x^2)*D[y[x],{x,2}]+D[y[x],x]+y[x]==x*Exp[x],{}},
       y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{120} - \frac{x^4}{24} - \frac{x^2}{2} + x\right) + c_1\left(\frac{7x^5}{120} - \frac{x^4}{12} + \frac{x^3}{6} - \frac{x^2}{2} + 1\right)$$

2.4.63 problem 60

Internal problem ID [8628]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 60
Date solved : `Tuesday, December 17, 2024 at 12:56:43 PM`
CAS classification : `[_quadrature]`

Solve

$$y' = y\left(1-y^2\right)$$

Solved as first order autonomous ode

Time used: 0.428 (sec)

Integrating gives

$$\int -\frac{1}{y\left(y^2-1\right)}dy = dx$$
$$-\frac{\ln\left(y+1\right)}{2}-\frac{\ln\left(y-1\right)}{2}+\ln\left(y\right) = x+c_1$$

Singular solutions are found by solving

$$-y\left(y^2-1\right) = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = -1$$
$$y = 0$$
$$y = 1$$

The following diagram is the phase line diagram. It classifies each of the above equilibrium points as stable or not stable or semi-stable.

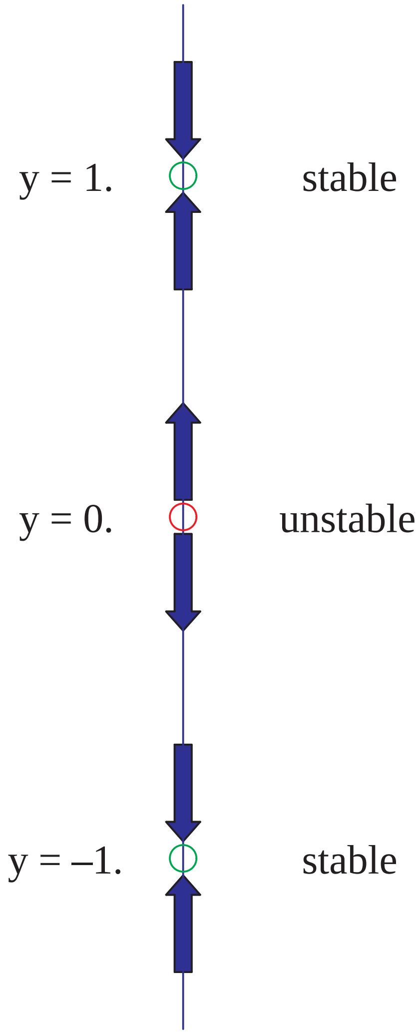


Figure 2.241: Phase line diagram

Solving for y gives

$$
\begin{aligned}
y &= -1 \\
y &= 0 \\
y &= 1 \\
y &= \frac{\sqrt{\left(\mathrm{e}^{2x+2c_1}-1\right)\mathrm{e}^{2x+2c_1}}}{\mathrm{e}^{2x+2c_1}-1} \\
y &= -\frac{\sqrt{\left(\mathrm{e}^{2x+2c_1}-1\right)\mathrm{e}^{2x+2c_1}}}{\mathrm{e}^{2x+2c_1}-1}
\end{aligned}
$$

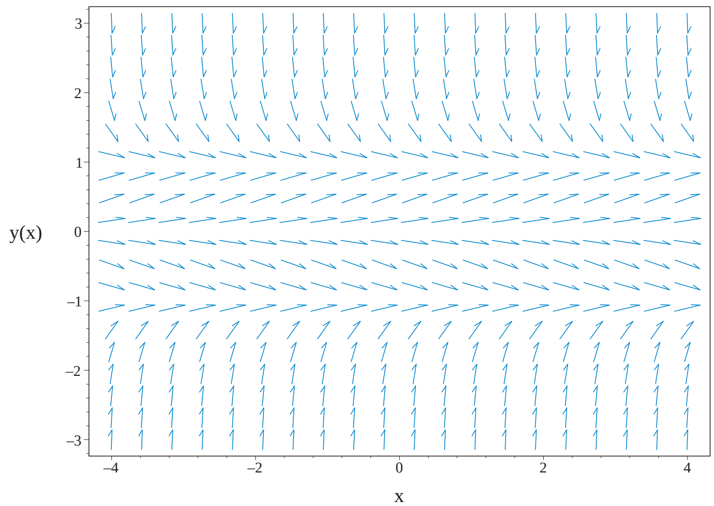


Figure 2.242: Slope field plot
$y' = y(1 - y^2)$

Summary of solutions found

$$y = -1$$
$$y = 0$$
$$y = 1$$
$$y = \frac{\sqrt{(e^{2x+2c_1} - 1)\, e^{2x+2c_1}}}{e^{2x+2c_1} - 1}$$
$$y = -\frac{\sqrt{(e^{2x+2c_1} - 1)\, e^{2x+2c_1}}}{e^{2x+2c_1} - 1}$$

Solved as first order Bernoulli ode

Time used: 1.106 (sec)

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -y(y^2 - 1) \end{aligned}$$

This is a Bernoulli ODE.

$$y' = (1)\, y + (-1)\, y^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

Comparing this to (1) shows that

$$
\begin{aligned}
f_0 &= 1 \\
f_1 &= -1
\end{aligned}
$$

The first step is to divide the above equation by y^n which gives

$$
\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}
$$

The next step is use the substitution $v = y^{1-n}$ in equation (3) which generates a new ODE in $v(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= 1 \\
f_1(x) &= -1 \\
n &= 3
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$
y'\frac{1}{y^3} = \frac{1}{y^2} - 1 \tag{4}
$$

Let

$$
\begin{aligned}
v &= y^{1-n} \\
&= \frac{1}{y^2}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
v' = -\frac{2}{y^3}y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-\frac{v'(x)}{2} &= v(x) - 1 \\
v' &= -2v + 2
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $v(x)$ which is now solved.

Integrating gives

$$\int \frac{1}{-2v+2}dv = dx$$
$$-\frac{\ln(v-1)}{2} = x + c_1$$

Applying the exponential to both sides gives

$$e^{\ln\left(\frac{1}{\sqrt{v-1}}\right)} = e^{x+c_1}$$
$$\frac{1}{\sqrt{v(x)-1}} = c_1 \,\mathrm{e}^{x}$$

Singular solutions are found by solving

$$-2v+2=0$$

for $v(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$v(x)=1$$

The substitution $v = y^{1-n}$ is now used to convert the above solution back to y which results in

$$\frac{1}{\sqrt{\frac{1}{y^2}-1}} = c_1 \,\mathrm{e}^{x}$$

$$\frac{1}{y^2} = 1$$

Solving for y gives

$$\frac{1}{\sqrt{\frac{1}{y^2}-1}} = c_1 \,\mathrm{e}^{x}$$
$$y = -1$$
$$y = 1$$

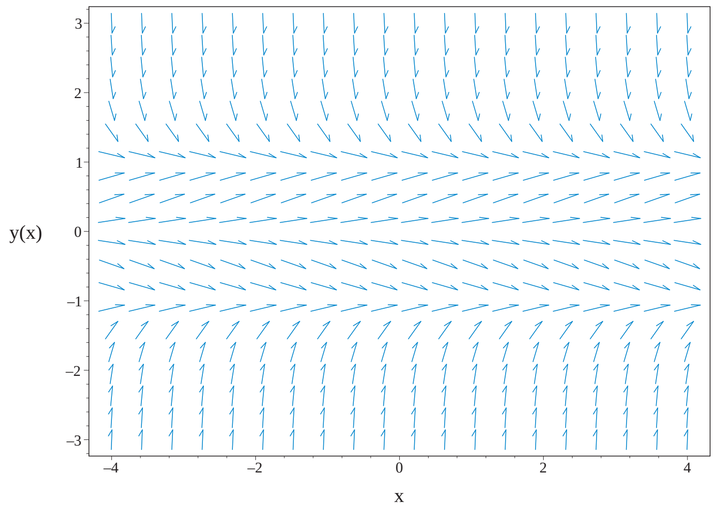


Figure 2.243: Slope field plot
$y' = y(1 - y^2)$

Summary of solutions found

$$\frac{1}{\sqrt{\frac{1}{y^2} - 1}} = c_1 \,\mathrm{e}^{x}$$

$$y = -1$$

$$y = 1$$

Solved as first order Exact ode

Time used: 0.259 (sec)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(y\left(-y^2 + 1\right)\right)\mathrm{d}x \\ \left(-y\left(-y^2 + 1\right)\right)\mathrm{d}x + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -y\left(-y^2 + 1\right) \\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-y\left(-y^2 + 1\right)\right) \\ &= 3y^2 - 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1\big((3y^2 - 1) - (0)\big) \\ &= 3y^2 - 1 \end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{1}{y^3 - y}\big((0) - (3y^2 - 1)\big) \\ &= \frac{-3y^2 + 1}{y^3 - y} \end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned} \mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int \frac{-3y^2+1}{y^3-y} \, \mathrm{d}y} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\ln(y(y^2-1))} \\ &= \frac{1}{y^3 - y} \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{y^3 - y}\big(-y(-y^2 + 1)\big) \\ &= 1 \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{y^3 - y}(1) \\ &= \frac{1}{y^3 - y} \end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$(1) + \left(\frac{1}{y^3 - y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int 1\,\mathrm{d}x$$
$$\phi = x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^3 - y}$. Therefore equation (4) becomes

$$\frac{1}{y^3 - y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y\,(y^2 - 1)}$$
$$= \frac{1}{y^3 - y}$$

Integrating the above w.r.t y results in

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y^3 - y}\right) \mathrm{d}y$$

$$f(y) = \frac{\ln(y+1)}{2} + \frac{\ln(y-1)}{2} - \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x + \frac{\ln(y+1)}{2} + \frac{\ln(y-1)}{2} - \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = x + \frac{\ln(y+1)}{2} + \frac{\ln(y-1)}{2} - \ln(y)$$

Solving for y gives

$$y = \frac{1}{\sqrt{1 - \mathrm{e}^{-2x+2c_1}}}$$

$$y = -\frac{1}{\sqrt{1 - \mathrm{e}^{-2x+2c_1}}}$$

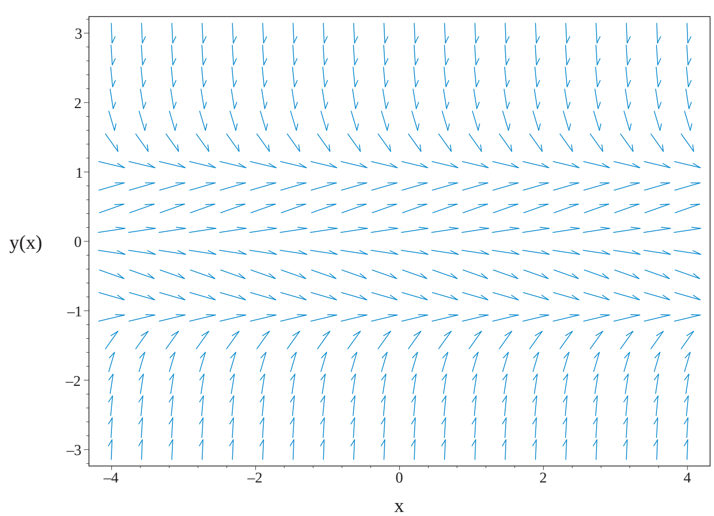


Figure 2.244: Slope field plot
$y' = y(1 - y^2)$

Summary of solutions found

$$y = \frac{1}{\sqrt{1 - \mathrm{e}^{-2x+2c_1}}}$$
$$y = -\frac{1}{\sqrt{1 - \mathrm{e}^{-2x+2c_1}}}$$

Solved using Lie symmetry for first order ode

Time used: 1.211 (sec)

Writing the ode as

$$y' = -y(y^2 - 1)$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - y(y^2 - 1)(b_3 - a_2) - y^2(y^2 - 1)^2 a_3 - (-3y^2 + 1)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-y^6 a_3 + 2y^4 a_3 + 3x y^2 b_2 + y^3 a_2 + 2y^3 b_3 - y^2 a_3 + 3y^2 b_1 - x b_2 - y a_2 - b_1 + b_2 = 0$$

Setting the numerator to zero gives

$$-y^6 a_3 + 2y^4 a_3 + 3x y^2 b_2 + y^3 a_2 + 2y^3 b_3 - y^2 a_3 + 3y^2 b_1 - x b_2 - y a_2 - b_1 + b_2 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_3v_2^6 + 2a_3v_2^4 + a_2v_2^3 + 3b_2v_1v_2^2 + 2b_3v_2^3 - a_3v_2^2 + 3b_1v_2^2 - a_2v_2 - b_2v_1 - b_1 + b_2 = 0 \quad (7E)$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$3b_2v_1v_2^2 - b_2v_1 - a_3v_2^6 + 2a_3v_2^4 + (a_2 + 2b_3)v_2^3 + (-a_3 + 3b_1)v_2^2 - a_2v_2 - b_1 + b_2 = 0 \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-a_2 &= 0 \\
-a_3 &= 0 \\
2a_3 &= 0 \\
-b_2 &= 0 \\
3b_2 &= 0 \\
a_2 + 2b_3 &= 0 \\
-a_3 + 3b_1 &= 0 \\
-b_1 + b_2 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= a_1 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 0
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = 1$$
$$\eta = 0$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= 0 - \left(-y(y^2 - 1)\right)(1) \\
&= y^3 - y \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{y^3 - y} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln (y + 1)}{2} + \frac{\ln (y - 1)}{2} - \ln (y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -y(y^2 - 1)$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= 0 \\
S_y &= \frac{1}{y^3 - y}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

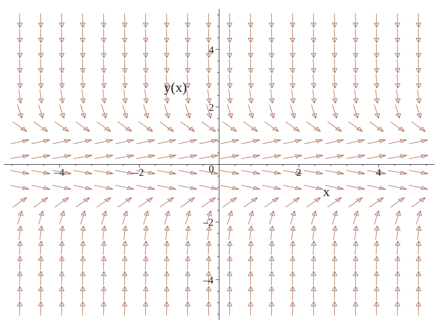
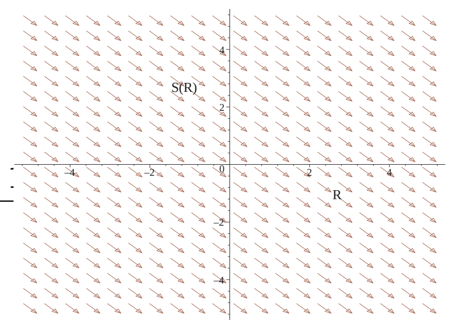
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned}
\int dS &= \int -1\, dR \\
S(R) &= -R + c_2
\end{aligned}$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\frac{\ln(y+1)}{2} + \frac{\ln(y-1)}{2} - \ln(y) = -x + c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -y(y^2 - 1)$ 	$R = x$ $S = \frac{\ln(y+1)}{2} + \frac{\ln(y-}{2}$	$\frac{dS}{dR} = -1$ 

Solving for y gives

$$y = \frac{1}{\sqrt{1 - e^{-2x+2c_2}}}$$

$$y = -\frac{1}{\sqrt{1 - e^{-2x+2c_2}}}$$

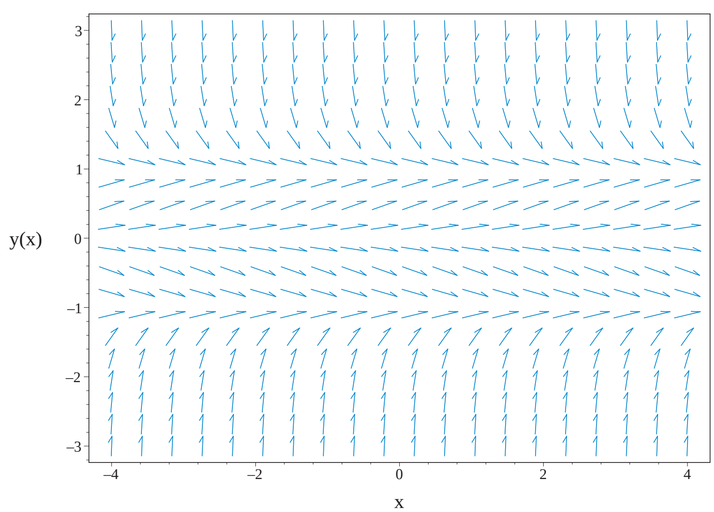


Figure 2.245: Slope field plot
$y' = y(1 - y^2)$

Summary of solutions found

$$y = \frac{1}{\sqrt{1 - e^{-2x+2c_2}}}$$

$$y = -\frac{1}{\sqrt{1 - e^{-2x+2c_2}}}$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) = y(x)\left(1 - y(x)^2\right)$

- Highest derivative means the order of the ODE is 1
$\frac{d}{dx}y(x)$
- Solve for the highest derivative
$\frac{d}{dx}y(x) = y(x)\left(1 - y(x)^2\right)$
- Separate variables
$\frac{\frac{d}{dx}y(x)}{y(x)\left(1-y(x)^2\right)} = 1$
- Integrate both sides with respect to x
$\int \frac{\frac{d}{dx}y(x)}{y(x)\left(1-y(x)^2\right)}dx = \int 1dx + C1$
- Evaluate integral
$-\frac{\ln(y(x)-1)}{2} + \ln\left(y(x)\right) - \frac{\ln(1+y(x))}{2} = x + C1$
- Solve for $y(x)$
$\left\{y(x) = \frac{\sqrt{\left(e^{2x+2C1}-1\right)e^{2x+2C1}}}{e^{2x+2C1}-1}, y(x) = -\frac{\sqrt{\left(e^{2x+2C1}-1\right)e^{2x+2C1}}}{e^{2x+2C1}-1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 29

```
dsolve(diff(y(x),x) = y(x)*(1-y(x)^2),
       y(x),singsol=all)
```

$$y = \frac{1}{\sqrt{c_1 \mathrm{e}^{-2x} + 1}}$$
$$y = -\frac{1}{\sqrt{c_1 \mathrm{e}^{-2x} + 1}}$$

Mathematica DSolve solution

Solving time : 0.716 (sec)
Leaf size : 100

```
DSolve[{D[y[x],x]==y[x]*(1-y[x]^2),{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{e^x}{\sqrt{e^{2x} + e^{2c_1}}}$$
$$y(x) \to \frac{e^x}{\sqrt{e^{2x} + e^{2c_1}}}$$
$$y(x) \to -1$$
$$y(x) \to 0$$
$$y(x) \to 1$$
$$y(x) \to -\frac{e^x}{\sqrt{e^{2x}}}$$
$$y(x) \to \frac{e^x}{\sqrt{e^{2x}}}$$

2.4.64 problem 61

Internal problem ID [8629]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 61
Date solved : `Wednesday, December 18, 2024 at 02:16:34 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$\frac{xy''}{1-x}+y=\frac{1}{1-x}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
```

```
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : `0.056` (sec)
Leaf size : 167

```
dsolve(x/(1-x)*diff(diff(y(x),x),x)+y(x) = 1/(1-x),
       y(x),singsol=all)
```

$$
\begin{aligned}
y = -x\Bigg(&(\operatorname{BesselK}(0,-x) \\
&-\operatorname{BesselK}(1,-x))\left(\int \frac{-\operatorname{BesselI}(0,-x)-\operatorname{BesselI}(1,-x)}{x\left(\operatorname{BesselI}(0,x)(x+1)\operatorname{BesselK}(1,-x)+1-(x+1)\operatorname{BesselK}(0,-x)\operatorname{BesselI}(\right.}\right. \\
&+(-\operatorname{BesselI}(0,-x) \\
&-\operatorname{BesselI}(1,-x))\left(\int \frac{-\operatorname{BesselK}(0,-x)+\operatorname{BesselK}(1,-x)}{(\operatorname{BesselI}(0,x)(x+1)\operatorname{BesselK}(1,-x)+1-(x+1)\operatorname{BesselK}(0,-x)\operatorname{BesselI}(1,}\right. \\
&-\operatorname{BesselK}(0,-x)c_1+\operatorname{BesselK}(1,-x)c_1-\operatorname{BesselI}(0,-x)c_2-\operatorname{BesselI}(1,-x)c_2\Bigg)
\end{aligned}
$$

Mathematica DSolve solution

Solving time : `0.248` (sec)
Leaf size : 136

```
DSolve[{x/(1-x)*D[y[x],{x,2}]+y[x]==1/(1-x),{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
y(x) \to e^{-x}x\Bigg(&e^{x}(\operatorname{BesselI}(0,x) \\
&-\operatorname{BesselI}(1,x))\int_1^x 2e^{-K[1]}\sqrt{\pi}\,\operatorname{HypergeometricU}\left(\frac{1}{2},2,2K[1]\right)dK[1] \\
&-2\sqrt{\pi}x\,\operatorname{HypergeometricU}\left(\frac{1}{2},2,2x\right){}_1F_2\left(\frac{1}{2};1,\frac{3}{2};\frac{x^2}{4}\right) \\
&+2\sqrt{\pi}\,\operatorname{HypergeometricU}\left(\frac{1}{2},2,2x\right)\operatorname{BesselI}(0,x) \\
&+c_1\operatorname{HypergeometricU}\left(\frac{1}{2},2,2x\right)+c_2e^{x}\operatorname{BesselI}(0,x)-c_2e^{x}\operatorname{BesselI}(1,x)\Bigg)
\end{aligned}
$$

2.4.65 problem 62

Internal problem ID [8630]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 62
Date solved : `Thursday, December 12, 2024 at 09:32:38 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\frac{xy''}{1-x} + xy = 0$$

Solved as second order ode adjoint method

Time used: 0.773 (sec)

In normal form the ode

$$\frac{xy''}{1-x} + xy = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 1-x \\ r(x) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (0)' + ((1-x)\,\xi(x)) &= 0 \\ \xi''(x) - (-1+x)\,\xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is Airy ODE. It has the general form

$$a\xi'' + b\xi' + c\xi x = F(x)$$

Where in this case

$$\begin{aligned}a &= 1\\ b &= 0\\ c &= \frac{1-x}{x}\\ F &= 0\end{aligned}$$

Therefore the solution to the homogeneous Airy ODE becomes

$$\xi = c_3 \operatorname{AiryAi}\left((1-x)\,(-1)^{1/3}\right) + c_4 \operatorname{AiryBi}\left((1-x)\,(-1)^{1/3}\right)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y &= \int \xi(x)\,r(x)\,dx\\ y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi\,(x)}\end{aligned}$$

Or

$$y' - \frac{y\left(-c_3(-1)^{1/3} \operatorname{AiryAi}\left(1, (1-x)\,(-1)^{1/3}\right) - c_4(-1)^{1/3} \operatorname{AiryBi}\left(1, (1-x)\,(-1)^{1/3}\right)\right)}{c_3 \operatorname{AiryAi}\left((1-x)\,(-1)^{1/3}\right) + c_4 \operatorname{AiryBi}\left((1-x)\,(-1)^{1/3}\right)} = 0$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned}q(x) &= \frac{(1+i\sqrt{3})\left(\operatorname{AiryAi}\left(1, -\frac{(-1+x)\left(1+i\sqrt{3}\right)}{2}\right) c_3 + \operatorname{AiryBi}\left(1, -\frac{(-1+x)\left(1+i\sqrt{3}\right)}{2}\right) c_4\right)}{2c_3 \operatorname{AiryAi}\left(-\frac{(-1+x)\left(1+i\sqrt{3}\right)}{2}\right) + 2c_4 \operatorname{AiryBi}\left(-\frac{(-1+x)\left(1+i\sqrt{3}\right)}{2}\right)}\\ p(x) &= 0\end{aligned}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= \mathrm{e}^{\int \frac{\left(1+i\sqrt{3}\right)\left(\mathrm{AiryAi}\left(1,-\frac{(-1+x)\left(1+i\sqrt{3}\right)}{2}\right)c_3+\mathrm{AiryBi}\left(1,-\frac{(-1+x)\left(1+i\sqrt{3}\right)}{2}\right)c_4\right)}{2c_3\,\mathrm{AiryAi}\left(-\frac{(-1+x)\left(1+i\sqrt{3}\right)}{2}\right)+2c_4\,\mathrm{AiryBi}\left(-\frac{(-1+x)\left(1+i\sqrt{3}\right)}{2}\right)}dx} \\
&= \frac{1}{c_3\,\mathrm{AiryAi}\left(-\frac{(-1+x)\left(1+i\sqrt{3}\right)}{2}\right)+c_4\,\mathrm{AiryBi}\left(-\frac{(-1+x)\left(1+i\sqrt{3}\right)}{2}\right)}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{c_3\,\mathrm{AiryAi}\left(-\frac{(-1+x)\left(1+i\sqrt{3}\right)}{2}\right)+c_4\,\mathrm{AiryBi}\left(-\frac{(-1+x)\left(1+i\sqrt{3}\right)}{2}\right)}\right) &= 0
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{c_3\,\mathrm{AiryAi}\left(-\frac{(-1+x)\left(1+i\sqrt{3}\right)}{2}\right)+c_4\,\mathrm{AiryBi}\left(-\frac{(-1+x)\left(1+i\sqrt{3}\right)}{2}\right)} &= \int 0\,dx + c_5 \\
&= c_5
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{1}{c_3\,\mathrm{AiryAi}\left(-\frac{(-1+x)\left(1+i\sqrt{3}\right)}{2}\right)+c_4\,\mathrm{AiryBi}\left(-\frac{(-1+x)\left(1+i\sqrt{3}\right)}{2}\right)}$ gives the final solution

$$
y = \left(c_3\,\mathrm{AiryAi}\left(-\frac{(-1+x)\left(1+i\sqrt{3}\right)}{2}\right)+c_4\,\mathrm{AiryBi}\left(-\frac{(-1+x)\left(1+i\sqrt{3}\right)}{2}\right)\right)c_5
$$

Hence, the solution found using Lagrange adjoint equation method is

$$
y = \left(c_3\,\mathrm{AiryAi}\left(-\frac{(-1+x)\left(1+i\sqrt{3}\right)}{2}\right)+c_4\,\mathrm{AiryBi}\left(-\frac{(-1+x)\left(1+i\sqrt{3}\right)}{2}\right)\right)c_5
$$

The constants can be merged to give

$$
y = c_3\,\mathrm{AiryAi}\left(-\frac{(-1+x)\left(1+i\sqrt{3}\right)}{2}\right)+c_4\,\mathrm{AiryBi}\left(-\frac{(-1+x)\left(1+i\sqrt{3}\right)}{2}\right)
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_3 \operatorname{AiryAi}\left(-\frac{(-1+x)\left(1+i\sqrt{3}\right)}{2}\right) + c_4 \operatorname{AiryBi}\left(-\frac{(-1+x)\left(1+i\sqrt{3}\right)}{2}\right)$$

Maple step by step solution

Let's solve

$\frac{x\left(\frac{d^2}{dx^2}y(x)\right)}{1-x} + xy(x) = 0$

- Highest derivative means the order of the ODE is 2

$\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

$\frac{d^2}{dx^2}y(x) = (x-1)\,y(x)$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

$\frac{d^2}{dx^2}y(x) + (1-x)\,y(x) = 0$

- Assume series solution for $y(x)$

$y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

  $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - Shift index using $k- >k-m$

  $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\,x^{k-2}$

  - Shift index using $k- >k+2$

  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\,x^k$

Rewrite ODE with series expansions

$$2a_2+a_0+\left(\sum_{k=1}^{\infty}\left(a_{k+2}(k+2)\left(k+1\right)+a_k-a_{k-1}\right)x^k\right)=0$$

- Each term must be 0

  $2a_2+a_0=0$
- Each term in the series must be 0, giving the recursion relation

  $(k^2+3k+2)\,a_{k+2}+a_k-a_{k-1}=0$
- Shift index using $k->k+1$

  $\left((k+1)^2+3k+5\right)a_{k+3}+a_{k+1}-a_k=0$
- Recursion relation that defines the series solution to the ODE

  $\left[y(x)=\sum_{k=0}^{\infty}a_k x^k, a_{k+3}=\frac{-a_{k+1}+a_k}{k^2+5k+6}, 2a_2+a_0=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

Maple dsolve solution

Solving time : 0.017 (sec)
Leaf size : 17

```
dsolve(x/(1-x)*diff(diff(y(x),x),x)+x*y(x) = 0,
       y(x),singsol=all)
```

$$y=c_1\,\mathrm{AiryAi}\left(x-1\right)+c_2\,\mathrm{AiryBi}\left(x-1\right)$$

Mathematica DSolve solution

Solving time : 0.015 (sec)
Leaf size : 20

```
DSolve[{x/(1-x)*D[y[x],{x,2}]+x*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 \operatorname{AiryAi}(x-1) + c_2 \operatorname{AiryBi}(x-1)$$

2.4.66 problem 63

Internal problem ID [8631]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 63
Date solved : Wednesday, December 18, 2024 at 02:16:35 AM
CAS classification : [[_2nd_order, _linear, _nonhomogeneous]]

Solve

$$\frac{xy''}{1-x}+y=\cos(x)$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
   <- No Liouvillian solutions exists
   -> Trying a solution in terms of special functions:
      -> Bessel
      -> elliptic
      -> Legendre
      <- Kummer successful
```

```
   <- special function solution successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.207 (sec)
Leaf size : 169

```
dsolve(x/(1-x)*diff(diff(y(x),x),x)+y(x) = cos(x),
       y(x),singsol=all)
```

$$
y = -x\left(\left(\operatorname{BesselI}(0,-x)+\operatorname{BesselI}(1,-x)\right)\left(\int -\frac{\cos(x)\left(\operatorname{BesselK}(0,-x)-\operatorname{BesselK}(1,-x)\right)(x-1)}{\left(\operatorname{BesselI}(0,x)(x+1)\operatorname{BesselK}(1,-x)+1-(x+1)\operatorname{BesselK}(0,-x)\operatorname{BesselI}(1,x)\right)x}dx\right) + \left(-\operatorname{BesselK}(0,-x)+\operatorname{BesselK}(1,-x)\right)\left(\int -\frac{\cos(x)\left(\operatorname{BesselI}(0,x)-\operatorname{BesselI}(1,x)\right)(x-1)}{\left(\operatorname{BesselI}(0,x)(x+1)\operatorname{BesselK}(1,-x)+1-(x+1)\operatorname{BesselK}(0,-x)\operatorname{BesselI}(1,x)\right)x}dx\right) + \operatorname{BesselK}(1,-x)c_1-\operatorname{BesselK}(0,-x)c_1-\operatorname{BesselI}(0,-x)c_2-\operatorname{BesselI}(1,-x)c_2\right)
$$

Mathematica DSolve solution

Solving time : 7.56 (sec)
Leaf size : 133

```
DSolve[{x/(1-x)*D[y[x],{x,2}]+y[x]==Cos[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
y(x) \to e^{-x}x\left(\text{HypergeometricU}\left(\frac{1}{2},2,2x\right)\int_1^x 2\sqrt{\pi}(\text{BesselI}(0,K[1]) - \text{BesselI}(1,K[1]))\cos(K[1])(K[1]-1)dK[1] + e^x(\text{BesselI}(0,x)-\text{BesselI}(1,x))\int_1^x -2e^{-K[2]}\sqrt{\pi}\cos(K[2])\,\text{HypergeometricU}\left(\frac{1}{2},2,2K[2]\right)(K[2]-1)dK[2] + c_1\,\text{HypergeometricU}\left(\frac{1}{2},2,2x\right)+c_2e^x\,\text{BesselI}(0,x)-c_2e^x\,\text{BesselI}(1,x)\right)
$$

2.4.67 problem 64

Internal problem ID [8632]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 64
Date solved : `Thursday, December 12, 2024 at 09:32:40 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$\frac{xy''}{-x^2+1}+y=0$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   -> elliptic
   -> Legendre
   -> Whittaker
      -> hyper3: Equivalence to 1F1 under a power @ Moebius
   -> hypergeometric
      -> heuristic approach
      -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
```

```
   -> Mathieu
      -> Equivalence to the rational form of Mathieu ODE under a power @ Moebius
trying a solution in terms of MeijerG functions
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
   trying differential order: 2; exact nonlinear
   trying symmetries linear in x and y(x)
   trying to convert to a linear ODE with constant coefficients
   trying to convert to an ODE of Bessel type
   -> trying reduction of order to Riccati
      trying Riccati sub-methods:
         -> trying a symmetry pattern of the form [F(x)*G(y), 0]
         -> trying a symmetry pattern of the form [0, F(x)*G(y)]
         -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
--- Trying Lie symmetry methods, 2nd order ---
`, `-> Computing symmetries using: way = 3`[0, y]
```

Maple dsolve solution

Solving time : 0.293 (sec)
Leaf size : maple_leaf_size

```
dsolve(x/(-x^2+1)*diff(diff(y(x),x),x)+y(x) = 0,
       y(x),singsol=all)
```

No solution found

Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{x/(1-x^2)*D[y[x],{x,2}]+y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

Not solved

2.4.68 problem 65

Internal problem ID [8633]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 65
Date solved : `Thursday, December 12, 2024 at 09:32:41 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' = \left(x^2+3\right) y$$

Solved as second order ode using Kovacic algorithm

Time used: 0.207 (sec)

Writing the ode as

$$y'' + \left(-x^2-3\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -x^2-3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 3}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 3 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(x^2 + 3\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.246: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 2 \\
&= -2
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$
v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1
$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\
&= \sum_{i=0}^{1} a_i x^i \qquad (8)
\end{aligned}
$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$
\sqrt{r} \approx x + \frac{3}{2x} - \frac{9}{8x^3} + \frac{27}{16x^5} - \frac{405}{128x^7} + \frac{1701}{256x^9} - \frac{15309}{1024x^{11}} + \frac{72171}{2048x^{13}} + \dots \qquad (9)
$$

Comparing Eq. (9) with Eq. (8) shows that

$$
a = 1
$$

From Eq. (9) the sum up to $v = 1$ gives

$$
\begin{aligned}
[\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\
&= x \qquad (10)
\end{aligned}
$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$
\left([\sqrt{r}]_\infty\right)^2 = x^2
$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2+3}{1} \\ &= Q + \frac{R}{1} \\ &= (x^2+3) + (0) \\ &= x^2+3 \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is 3. Now b can be found.

$$\begin{aligned} b &= (3) - (0) \\ &= 3 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= x \\ \alpha_\infty^{+} &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{3}{1} - 1\right) &&= 1 \\ \alpha_\infty^{-} &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{3}{1} - 1\right) &&= -2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = x^2 + 3$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
-2	x	1	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 1$, and since there are no poles, then

$$\begin{aligned} d &= \alpha_\infty^+ \\ &= 1 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

Substituting the above values in the above results in

$$\begin{aligned} \omega &= (+)[\sqrt{r}]_\infty \\ &= 0 + (x) \\ &= x \\ &= x \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}$$

Let

$$p(x) = x + a_0 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2(x)(1) + \left((1) + (x)^2 - \left(x^2+3\right)\right) &= 0 \\ -2a_0 &= 0 \end{aligned}$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$\{a_0 = 0\}$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$p(x) = x$$

Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= pe^{\int \omega \, dx} \\
&= (x) \, \mathrm{e}^{\int x dx} \\
&= (x) \, \mathrm{e}^{\frac{x^2}{2}} \\
&= x \, \mathrm{e}^{\frac{x^2}{2}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= x \, \mathrm{e}^{\frac{x^2}{2}}
\end{aligned}
$$

Which simplifies to

$$
y_1 = x \, \mathrm{e}^{\frac{x^2}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2} \, dx \\
&= x \, \mathrm{e}^{\frac{x^2}{2}} \int \frac{1}{x^2 \mathrm{e}^{x^2}} \, dx \\
&= x \, \mathrm{e}^{\frac{x^2}{2}} \left(-\frac{\mathrm{e}^{-x^2}}{x} - \sqrt{\pi} \, \mathrm{erf}(x) \right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x \, \mathrm{e}^{\frac{x^2}{2}} \right) + c_2 \left(x \, \mathrm{e}^{\frac{x^2}{2}} \left(-\frac{\mathrm{e}^{-x^2}}{x} - \sqrt{\pi} \, \mathrm{erf}(x) \right) \right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 x \,\mathrm{e}^{\frac{x^2}{2}} + c_2 \left(-\sqrt{\pi}\, \operatorname{erf}\left(x\right) x \,\mathrm{e}^{\frac{x^2}{2}} - \mathrm{e}^{-\frac{x^2}{2}}\right)$$

Solved as second order ode adjoint method

Time used: 0.900 (sec)

In normal form the ode

$$y'' = \left(x^2 + 3\right) y \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 0 \\ q(x) &= -x^2 - 3 \\ r(x) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (0)' + \left(\left(-x^2 - 3\right) \xi(x)\right) &= 0 \\ -\xi(x)\, x^2 + \xi''(x) - 3\xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. Writing the ode as

$$\xi'' + \left(-x^2 - 3\right) \xi = 0 \tag{1}$$

$$A\xi'' + B\xi' + C\xi = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -x^2 - 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = \xi e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{x^2 + 3}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= x^2 + 3 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = (x^2 + 3)\, z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then ξ is found using the inverse transformation

$$\xi = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.247: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 2 \\ &= -2 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is -2 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Attempting to find a solution using case $n = 1$.

Since the order of r at ∞ is $O_r(\infty) = -2$ then

$$v = \frac{-O_r(\infty)}{2} = \frac{2}{2} = 1$$

$[\sqrt{r}]_\infty$ is the sum of terms involving x^i for $0 \leq i \leq v$ in the Laurent series for $\sqrt{r}$ at ∞. Therefore

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{v} a_i x^i \\ &= \sum_{i=0}^{1} a_i x^i \end{aligned} \tag{8}$$

Let a be the coefficient of $x^v = x^1$ in the above sum. The Laurent series of $\sqrt{r}$ at ∞ is

$$\sqrt{r} \approx x + \frac{3}{2x} - \frac{9}{8x^3} + \frac{27}{16x^5} - \frac{405}{128x^7} + \frac{1701}{256x^9} - \frac{15309}{1024x^{11}} + \frac{72171}{2048x^{13}} + \dots \tag{9}$$

Comparing Eq. (9) with Eq. (8) shows that

$$a = 1$$

From Eq. (9) the sum up to $v = 1$ gives

$$\begin{aligned} [\sqrt{r}]_\infty &= \sum_{i=0}^{1} a_i x^i \\ &= x \end{aligned} \tag{10}$$

Now we need to find b, where b be the coefficient of $x^{v-1} = x^0 = 1$ in r minus the coefficient of same term but in $\left([\sqrt{r}]_\infty\right)^2$ where $[\sqrt{r}]_\infty$ was found above in Eq (10). Hence

$$\left([\sqrt{r}]_\infty\right)^2 = x^2$$

This shows that the coefficient of 1 in the above is 0. Now we need to find the coefficient of 1 in r. How this is done depends on if $v = 0$ or not. Since $v = 1$ which is not zero, then starting $r = \frac{s}{t}$, we do long division and write this in the form

$$r = Q + \frac{R}{t}$$

Where Q is the quotient and R is the remainder. Then the coefficient of 1 in r will be the coefficient this term in the quotient. Doing long division gives

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{x^2+3}{1} \\ &= Q + \frac{R}{1} \\ &= (x^2+3) + (0) \\ &= x^2 + 3 \end{aligned}$$

We see that the coefficient of the term $\frac{1}{x}$ in the quotient is 3. Now b can be found.

$$\begin{aligned} b &= (3) - (0) \\ &= 3 \end{aligned}$$

Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= x \\ \alpha_\infty^+ &= \frac{1}{2}\left(\frac{b}{a} - v\right) &&= \frac{1}{2}\left(\frac{3}{1} - 1\right) &&= 1 \\ \alpha_\infty^- &= \frac{1}{2}\left(-\frac{b}{a} - v\right) &&= \frac{1}{2}\left(-\frac{3}{1} - 1\right) &&= -2 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = x^2 + 3$$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
-2	x	1	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^+ = 1$, and since there are no poles, then

$$
\begin{aligned}
d &= \alpha_\infty^+ \\
&= 1
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty
$$

Substituting the above values in the above results in

$$
\begin{aligned}
\omega &= (+)[\sqrt{r}]_\infty \\
&= 0 + (x) \\
&= x \\
&= x
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 1$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = x + a_0 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2(x)\,(1) + \left((1) + (x)^2 - \left(x^2+3\right)\right) &= 0 \\
-2a_0 &= 0
\end{aligned}
$$

Solving for the coefficients a_i in the above using method of undetermined coefficients gives

$$
\{a_0 = 0\}
$$

Substituting these coefficients in $p(x)$ in eq. (2A) results in

$$
p(x) = x
$$

Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= pe^{\int \omega\,dx} \\
&= (x)\,\mathrm{e}^{\int x dx} \\
&= (x)\,\mathrm{e}^{\frac{x^2}{2}} \\
&= x\,\mathrm{e}^{\frac{x^2}{2}}
\end{aligned}$$

The first solution to the original ode in ξ is found from

$$\xi_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
\xi_1 &= z_1 \\
&= x\,\mathrm{e}^{\frac{x^2}{2}}
\end{aligned}$$

Which simplifies to

$$\xi_1 = x\,\mathrm{e}^{\frac{x^2}{2}}$$

The second solution ξ_2 to the original ode is found using reduction of order

$$\xi_2 = \xi_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{\xi_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
\xi_2 &= \xi_1 \int \frac{1}{\xi_1^2}\,dx \\
&= x\,\mathrm{e}^{\frac{x^2}{2}} \int \frac{1}{x^2 \mathrm{e}^{x^2}}\,dx \\
&= x\,\mathrm{e}^{\frac{x^2}{2}} \left(-\frac{\mathrm{e}^{-x^2}}{x} - \sqrt{\pi}\,\operatorname{erf}(x)\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
\xi &= c_1\xi_1 + c_2\xi_2 \\
&= c_1\left(x\,\mathrm{e}^{\frac{x^2}{2}}\right) + c_2\left(x\,\mathrm{e}^{\frac{x^2}{2}}\left(-\frac{\mathrm{e}^{-x^2}}{x} - \sqrt{\pi}\,\operatorname{erf}(x)\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$
\begin{aligned}
\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)}
\end{aligned}
$$

Or

$$
y' - \frac{y\left(c_1\, \mathrm{e}^{\frac{x^2}{2}} + c_1\, x^2 \mathrm{e}^{\frac{x^2}{2}} + c_2\left(-2\, \mathrm{e}^{-x^2} x\, \mathrm{e}^{\frac{x^2}{2}} - \sqrt{\pi}\, \operatorname{erf}(x)\, \mathrm{e}^{\frac{x^2}{2}} - \sqrt{\pi}\, \operatorname{erf}(x)\, x^2 \mathrm{e}^{\frac{x^2}{2}} + x\, \mathrm{e}^{-\frac{x^2}{2}}\right)\right)}{c_1 x\, \mathrm{e}^{\frac{x^2}{2}} + c_2\left(-\sqrt{\pi}\, \operatorname{erf}(x)\, x\, \mathrm{e}^{\frac{x^2}{2}} - \mathrm{e}^{-\frac{x^2}{2}}\right)} = 0
$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$
y' + q(x)y = p(x)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= -\frac{(x^2+1)\left(-c_2\sqrt{\pi}\, \operatorname{erf}(x) + c_1\right) \mathrm{e}^{x^2} - c_2 x}{x\left(-c_2\sqrt{\pi}\, \operatorname{erf}(x) + c_1\right) \mathrm{e}^{x^2} - c_2} \\
p(x) &= 0
\end{aligned}
$$

The integrating factor μ is

$$
\mu = \mathrm{e}^{\int -\frac{(x^2+1)(-c_2\sqrt{\pi}\, \operatorname{erf}(x)+c_1)\mathrm{e}^{x^2} - c_2 x}{x(-c_2\sqrt{\pi}\, \operatorname{erf}(x)+c_1)\mathrm{e}^{x^2} - c_2} dx}
$$

Therefore the solution is

$$
y = c_3\, \mathrm{e}^{-\int -\frac{(x^2+1)(-c_2\sqrt{\pi}\, \operatorname{erf}(x)+c_1)\mathrm{e}^{x^2} - c_2 x}{x(-c_2\sqrt{\pi}\, \operatorname{erf}(x)+c_1)\mathrm{e}^{x^2} - c_2} dx}
$$

Hence, the solution found using Lagrange adjoint equation method is

$$
y = c_3\, \mathrm{e}^{-\int -\frac{(x^2+1)(-c_2\sqrt{\pi}\, \operatorname{erf}(x)+c_1)\mathrm{e}^{x^2} - c_2 x}{x(-c_2\sqrt{\pi}\, \operatorname{erf}(x)+c_1)\mathrm{e}^{x^2} - c_2} dx}
$$

The constants can be merged to give

$$
y = \mathrm{e}^{-\int -\frac{(x^2+1)(-c_2\sqrt{\pi}\, \operatorname{erf}(x)+c_1)\mathrm{e}^{x^2} - c_2 x}{x(-c_2\sqrt{\pi}\, \operatorname{erf}(x)+c_1)\mathrm{e}^{x^2} - c_2} dx}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \mathrm{e}^{-\int -\frac{(x^2+1)\left(-c_2\sqrt{\pi}\,\operatorname{erf}(x)+c_1\right)\mathrm{e}^{x^2}-c_2 x}{x\left(-c_2\sqrt{\pi}\,\operatorname{erf}(x)+c_1\right)\mathrm{e}^{x^2}-c_2}dx}$$

Maple step by step solution

Let's solve
$\frac{d^2}{dx^2}y(x) = (x^2+3)\,y(x)$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) + (-x^2-3)\,y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$
  $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$
  ◦ Shift index using $k- >k-m$
  $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$
  ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion
  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\,x^{k-2}$
  ◦ Shift index using $k- >k+2$
  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\,x^k$

Rewrite ODE with series expansions
$2a_2 - 3a_0 + (6a_3 - 3a_1)\,x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - 3a_k - a_{k-2}\right) x^k\right) = 0$

- The coefficients of each power of x must be 0
  $[2a_2 - 3a_0 = 0, 6a_3 - 3a_1 = 0]$

- Solve for the dependent coefficient(s)
  $\left\{a_2 = \frac{3a_0}{2}, a_3 = \frac{a_1}{2}\right\}$
- Each term in the series must be 0, giving the recursion relation
  $(k^2 + 3k + 2)\, a_{k+2} - 3a_k - a_{k-2} = 0$
- Shift index using $k- >k + 2$
  $\left((k+2)^2 + 3k + 8\right) a_{k+4} - 3a_{k+2} - a_k = 0$
- Recursion relation that defines the series solution to the ODE
  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = \frac{3a_{k+2}+a_k}{k^2+7k+12}, a_2 = \frac{3a_0}{2}, a_3 = \frac{a_1}{2}\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
   Group is reducible, not completely reducible
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.010` (sec)
Leaf size : 30

```
dsolve(diff(diff(y(x),x),x) = (x^2+3)*y(x),
       y(x),singsol=all)
```

$$y = x\left(c_2\sqrt{\pi}\, \operatorname{erf}(x) + c_1\right) e^{\frac{x^2}{2}} + e^{-\frac{x^2}{2}} c_2$$

Mathematica DSolve solution

Solving time : 0.119 (sec)
Leaf size : 46

```
DSolve[{D[y[x],{x,2}]==(x^2+3)*y[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-\frac{x^2}{2}}\left(-\sqrt{\pi}c_2e^{x^2}x\operatorname{erf}(x)+c_1e^{x^2}x-c_2\right)$$

2.4.69 problem 66

Internal problem ID [8634]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 66
Date solved : `Thursday, December 12, 2024 at 09:32:43 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + (x - 1)\,y = 0$$

Using series expansion around $x = 0$

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\,y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \left.\frac{d^n f}{dx^n}\right|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$
\begin{align*}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} \tag{1}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' \tag{2.14}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \tag{2.15}\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f \tag{2}\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f \tag{3}\\
&\vdots
\end{align*}
$$

And so on. Hence if we name $F_0 = f(x, y, y')$ then the above can be written as

$$
\begin{align*}
F_0 &= f(x, y, y') \tag{4}\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} f \tag{5}\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx} f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_1 + \left(\frac{\partial F_1}{\partial y}\right) y' + \left(\frac{\partial F_1}{\partial y'}\right) F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) y''\\
&= \frac{\partial}{\partial x} F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right) y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right) F_0 \tag{6}
\end{align*}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + x y'_0 + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)!} F_n|_{x_0, y_0, y'_0} \tag{7}
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= -(x-1)\,y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -y - (x-1)\,y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= -2y' + (x-1)^2\,y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= ((x-1)\,y' + 4y)\,(x-1) \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= -(x-1)^3\,y + (6x-6)\,y' + 4y
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and $y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= y(0) \\
F_1 &= -y(0) + y'(0) \\
F_2 &= -2y'(0) + y(0) \\
F_3 &= y'(0) - 4y(0) \\
F_4 &= 5y(0) - 6y'(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
y = \left(1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{30}x^5\right) y(0) + \left(x + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{120}x^5\right) y'(0) + O(x^6)
$$

Since the expansion point $x = 0$ is an ordinary point, then this can also be solved using the standard power series method. Let the solution be represented as power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + (x-1)\left(\sum_{n=0}^{\infty} a_n x^n\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} x^{1+n} a_n\right) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\begin{aligned}
\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} &= \sum_{n=0}^{\infty} (n+2)\, a_{n+2} (1+n)\, x^n \\
\sum_{n=0}^{\infty} x^{1+n} a_n &= \sum_{n=1}^{\infty} a_{n-1} x^n
\end{aligned}$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty} (n+2)\, a_{n+2} (1+n)\, x^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^n\right) + \sum_{n=0}^{\infty} (-a_n x^n) = 0 \tag{3}$$

$n = 0$ gives

$$2a_2 - a_0 = 0$$

$$a_2 = \frac{a_0}{2}$$

For $1 \leq n$, the recurrence equation is

$$(n+2)\,a_{n+2}(1+n) + a_{n-1} - a_n = 0 \tag{4}$$

Solving for a_{n+2}, gives

$$\begin{aligned} a_{n+2} &= \frac{-a_{n-1} + a_n}{(n+2)\,(1+n)} \\ &= \frac{a_n}{(n+2)\,(1+n)} - \frac{a_{n-1}}{(n+2)\,(1+n)} \end{aligned} \tag{5}$$

For $n = 1$ the recurrence equation gives

$$6a_3 + a_0 - a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_0}{6} + \frac{a_1}{6}$$

For $n = 2$ the recurrence equation gives

$$12a_4 + a_1 - a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = -\frac{a_1}{12} + \frac{a_0}{24}$$

For $n = 3$ the recurrence equation gives

$$20a_5 + a_2 - a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_0}{30} + \frac{a_1}{120}$$

For $n = 4$ the recurrence equation gives

$$30a_6 + a_3 - a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_6 = \frac{a_0}{144} - \frac{a_1}{120}$$

For $n = 5$ the recurrence equation gives

$$42a_7 + a_4 - a_5 = 0$$

Which after substituting the earlier terms found becomes

$$a_7 = \frac{11a_1}{5040} - \frac{a_0}{560}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$y = a_0 + a_1 x + \frac{a_0 x^2}{2} + \left(-\frac{a_0}{6} + \frac{a_1}{6}\right) x^3 + \left(-\frac{a_1}{12} + \frac{a_0}{24}\right) x^4 + \left(-\frac{a_0}{30} + \frac{a_1}{120}\right) x^5 + \dots$$

Collecting terms, the solution becomes

$$y = \left(1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{30}x^5\right) a_0 + \left(x + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{120}x^5\right) a_1 + O(x^6) \tag{3}$$

At $x = 0$ the solution above becomes

$$y = \left(1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{30}x^5\right) c_1 + \left(x + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{120}x^5\right) c_2 + O(x^6)$$

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + (x-1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$

  $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  ◦ Shift index using $k->k-m$

  $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  ◦ Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

  $\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\,x^{k-2}$

  ◦ Shift index using $k->k+2$

  $\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\,x^k$

Rewrite ODE with series expansions

$2a_2 - a_0 + \left(\sum_{k=1}^{\infty} \left(a_{k+2}(k+2)\,(k+1) - a_k + a_{k-1}\right) x^k\right) = 0$

- Each term must be 0

  $2a_2 - a_0 = 0$

- Each term in the series must be 0, giving the recursion relation

  $(k^2 + 3k + 2)\,a_{k+2} + a_{k-1} - a_k = 0$

- Shift index using $k->k+1$

  $\left((k+1)^2 + 3k + 5\right) a_{k+3} + a_k - a_{k+1} = 0$

- Recursion relation that defines the series solution to the ODE

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+3} = -\frac{a_k - a_{k+1}}{k^2+5k+6}, 2a_2 - a_0 = 0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
<- No Liouvillian solutions exists
-> Trying a solution in terms of special functions:
   -> Bessel
   <- Bessel successful
<- special function solution successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 54

```
dsolve(diff(diff(y(x),x),x)+(x-1)*y(x) = 0,y(x),
       series,x=0)
```

$$y = \left(1+\frac{1}{2}x^2-\frac{1}{6}x^3+\frac{1}{24}x^4-\frac{1}{30}x^5\right)y(0)+\left(x+\frac{1}{6}x^3-\frac{1}{12}x^4+\frac{1}{120}x^5\right)y'(0)+O\left(x^6\right)$$

Mathematica DSolve solution

Solving time : 0.002 (sec)
Leaf size : 63

```
AsymptoticDSolveValue[{D[y[x],{x,2}]+(x-1)*y[x]==0,{}},
       y[x],{x,0,5}]
```

$$y(x) \to c_2\left(\frac{x^5}{120}-\frac{x^4}{12}+\frac{x^3}{6}+x\right)+c_1\left(-\frac{x^5}{30}+\frac{x^4}{24}-\frac{x^3}{6}+\frac{x^2}{2}+1\right)$$

2.4.70 problem 67

Internal problem ID [8635]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 67
Date solved : `Thursday, December 12, 2024 at 09:32:44 AM`
CAS classification : `system_of_ODEs`

$$
\begin{aligned}
x' &= x + 2y + 2t + 1 \\
y' &= 5x + y + 3t - 1
\end{aligned}
$$

Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2t+1 \\ 3t-1 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \left[\begin{array}{cc} \frac{e^{-\left(-1+\sqrt{10}\right)t}}{2}+\frac{e^{\left(1+\sqrt{10}\right)t}}{2} & -\frac{\left(-e^{\left(1+\sqrt{10}\right)t}+e^{-\left(-1+\sqrt{10}\right)t}\right)\sqrt{10}}{10} \\ -\frac{\left(-e^{\left(1+\sqrt{10}\right)t}+e^{-\left(-1+\sqrt{10}\right)t}\right)\sqrt{10}}{4} & \frac{e^{-\left(-1+\sqrt{10}\right)t}}{2}+\frac{e^{\left(1+\sqrt{10}\right)t}}{2} \end{array}\right]
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \left[\begin{array}{cc} \frac{e^{-\left(-1+\sqrt{10}\right)t}}{2}+\frac{e^{\left(1+\sqrt{10}\right)t}}{2} & -\frac{\left(-e^{\left(1+\sqrt{10}\right)t}+e^{-\left(-1+\sqrt{10}\right)t}\right)\sqrt{10}}{10} \\ -\frac{\left(-e^{\left(1+\sqrt{10}\right)t}+e^{-\left(-1+\sqrt{10}\right)t}\right)\sqrt{10}}{4} & \frac{e^{-\left(-1+\sqrt{10}\right)t}}{2}+\frac{e^{\left(1+\sqrt{10}\right)t}}{2} \end{array}\right] \left[\begin{array}{c} c_1 \\ c_2 \end{array}\right] \\
&= \left[\begin{array}{c} \left(\frac{e^{-\left(-1+\sqrt{10}\right)t}}{2}+\frac{e^{\left(1+\sqrt{10}\right)t}}{2}\right)c_1 - \frac{\left(-e^{\left(1+\sqrt{10}\right)t}+e^{-\left(-1+\sqrt{10}\right)t}\right)\sqrt{10}\,c_2}{10} \\ -\frac{\left(-e^{\left(1+\sqrt{10}\right)t}+e^{-\left(-1+\sqrt{10}\right)t}\right)\sqrt{10}\,c_1}{4} + \left(\frac{e^{-\left(-1+\sqrt{10}\right)t}}{2}+\frac{e^{\left(1+\sqrt{10}\right)t}}{2}\right)c_2 \end{array}\right] \\
&= \left[\begin{array}{c} \frac{\left(-c_2\sqrt{10}+5c_1\right)e^{-\left(-1+\sqrt{10}\right)t}}{10} + \frac{\left(\frac{c_2\sqrt{10}}{5}+c_1\right)e^{\left(1+\sqrt{10}\right)t}}{2} \\ \frac{\left(-c_1\sqrt{10}+2c_2\right)e^{-\left(-1+\sqrt{10}\right)t}}{4} + \frac{e^{\left(1+\sqrt{10}\right)t}\left(c_1\sqrt{10}+2c_2\right)}{4} \end{array}\right]
\end{aligned}
$$

The particular solution given by

$$
\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt
$$

But

$$
\begin{aligned}
e^{-At} &= \left(e^{At}\right)^{-1} \\
&= \left[\begin{array}{cc} \frac{e^{-2t}\left(e^{-\left(-1+\sqrt{10}\right)t}+e^{\left(1+\sqrt{10}\right)t}\right)}{2} & \frac{\sqrt{10}\,e^{-2t}\left(-e^{\left(1+\sqrt{10}\right)t}+e^{-\left(-1+\sqrt{10}\right)t}\right)}{10} \\ \frac{\sqrt{10}\,e^{-2t}\left(-e^{\left(1+\sqrt{10}\right)t}+e^{-\left(-1+\sqrt{10}\right)t}\right)}{4} & \frac{e^{-2t}\left(e^{-\left(-1+\sqrt{10}\right)t}+e^{\left(1+\sqrt{10}\right)t}\right)}{2} \end{array}\right]
\end{aligned}
$$

Hence

$$
\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \frac{e^{-\left(-1+\sqrt{10}\right)t}}{2}+\frac{e^{\left(1+\sqrt{10}\right)t}}{2} & -\frac{\left(-e^{\left(1+\sqrt{10}\right)t}+e^{-\left(-1+\sqrt{10}\right)t}\right)\sqrt{10}}{10} \\ -\frac{\left(-e^{\left(1+\sqrt{10}\right)t}+e^{-\left(-1+\sqrt{10}\right)t}\right)\sqrt{10}}{4} & \frac{e^{-\left(-1+\sqrt{10}\right)t}}{2}+\frac{e^{\left(1+\sqrt{10}\right)t}}{2} \end{bmatrix} \int \begin{bmatrix} \frac{e^{-2t}\left(e^{-\left(-1+\sqrt{10}\right)t}+e^{\left(1+\sqrt{10}\right)t}\right)}{2} \\ \frac{\sqrt{10}\,e^{-2t}\left(-e^{\left(1+\sqrt{10}\right)t}+e^{-\left(-1+\sqrt{10}\right)t}\right.}{4} \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{-\left(-1+\sqrt{10}\right)t}}{2}+\frac{e^{\left(1+\sqrt{10}\right)t}}{2} & -\frac{\left(-e^{\left(1+\sqrt{10}\right)t}+e^{-\left(-1+\sqrt{10}\right)t}\right)\sqrt{10}}{10} \\ -\frac{\left(-e^{\left(1+\sqrt{10}\right)t}+e^{-\left(-1+\sqrt{10}\right)t}\right)\sqrt{10}}{4} & \frac{e^{-\left(-1+\sqrt{10}\right)t}}{2}+\frac{e^{\left(1+\sqrt{10}\right)t}}{2} \end{bmatrix} \begin{bmatrix} \frac{\left((-63t-67)\sqrt{10}-180t+85\right)e^{-\left(1+\sqrt{10}\right)t}}{810} \\ \frac{\left((-36t+17)\sqrt{10}-126t-134\right)e^{-\left(1+\sqrt{10}\right)t}}{324} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{4t}{9}+\frac{17}{81} \\ -\frac{7t}{9}-\frac{67}{81} \end{bmatrix}
\end{aligned}
$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t)+\vec{x}_p(t) \\
&= \begin{bmatrix} \frac{17}{81}+\frac{\left(-c_2\sqrt{10}+5c_1\right)e^{-\left(-1+\sqrt{10}\right)t}}{10}+\frac{e^{\left(1+\sqrt{10}\right)t}\left(c_2\sqrt{10}+5c_1\right)}{10}-\frac{4t}{9} \\ \frac{\left(-c_1\sqrt{10}+2c_2\right)e^{-\left(-1+\sqrt{10}\right)t}}{4}+\frac{e^{\left(1+\sqrt{10}\right)t}\left(c_1\sqrt{10}+2c_2\right)}{4}-\frac{7t}{9}-\frac{67}{81} \end{bmatrix}
\end{aligned}
$$

Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)+\vec{G}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}+\begin{bmatrix} 2t+1 \\ 3t-1 \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$
\vec{x}(t) = \vec{x}_h(t)+\vec{x}_p(t)
$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t)+\vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A-\lambda I\right) = 0
$$

Expanding gives

$$\det\left(\begin{bmatrix} 1 & 2 \\ 5 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 1-\lambda & 2 \\ 5 & 1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 - 2\lambda - 9 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 1 + \sqrt{10}$$
$$\lambda_2 = 1 - \sqrt{10}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$1-\sqrt{10}$	1	real eigenvalue
$1+\sqrt{10}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 1 - \sqrt{10}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 \\ 5 & 1 \end{bmatrix} - \left(1-\sqrt{10}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} \sqrt{10} & 2 \\ 5 & \sqrt{10} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \sqrt{10} & 2 & 0 \\ 5 & \sqrt{10} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{\sqrt{10}\,R_1}{2} \Longrightarrow \left[\begin{array}{cc|c} \sqrt{10} & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \sqrt{10} & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{t\sqrt{10}}{5}\right\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{t\sqrt{10}}{5} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{\sqrt{10}}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{10}}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -\sqrt{10} \\ 5 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 1 + \sqrt{10}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 1 & 2 \\ 5 & 1 \end{bmatrix} - \left(1 + \sqrt{10}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\sqrt{10} & 2 \\ 5 & -\sqrt{10} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -\sqrt{10} & 2 & 0 \\ 5 & -\sqrt{10} & 0 \end{array}\right]$$

$$R_2 = R_2 + \frac{\sqrt{10}\,R_1}{2} \implies \left[\begin{array}{cc|c} -\sqrt{10} & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -\sqrt{10} & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{t\sqrt{10}}{5}\right\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} \frac{t\sqrt{10}}{5} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t \begin{bmatrix} \frac{\sqrt{10}}{5} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{10}}{5} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} \sqrt{10} \\ 5 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity: algebraic m	multiplicity: geometric k	defective?	eigenvectors
$1+\sqrt{10}$	1	1	No	$\left[\begin{array}{c}\frac{\sqrt{10}}{5} \\ 1\end{array}\right]$
$1-\sqrt{10}$	1	1	No	$\left[\begin{array}{c}-\frac{\sqrt{10}}{5} \\ 1\end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $1+\sqrt{10}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\left(1+\sqrt{10}\right)t} \\
&= \left[\begin{array}{c}\frac{\sqrt{10}}{5} \\ 1\end{array}\right] e^{\left(1+\sqrt{10}\right)t}
\end{aligned}
$$

Since eigenvalue $1-\sqrt{10}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{\left(1-\sqrt{10}\right)t} \\
&= \left[\begin{array}{c}-\frac{\sqrt{10}}{5} \\ 1\end{array}\right] e^{\left(1-\sqrt{10}\right)t}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)
$$

Which is written as

$$
\left[\begin{array}{c}x \\ y\end{array}\right] = c_1 \left[\begin{array}{c}\frac{\sqrt{10}\,\mathrm{e}^{\left(1+\sqrt{10}\right)t}}{5} \\ \mathrm{e}^{\left(1+\sqrt{10}\right)t}\end{array}\right] + c_2 \left[\begin{array}{c}-\frac{\mathrm{e}^{\left(1-\sqrt{10}\right)t}\sqrt{10}}{5} \\ \mathrm{e}^{\left(1-\sqrt{10}\right)t}\end{array}\right]
$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$
\Phi = \left[\vec{x}_1 \quad \vec{x}_2 \quad \cdots\right]
$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \left[\begin{array}{cc} \frac{\sqrt{10}\,e^{\left(1+\sqrt{10}\right)t}}{5} & -\frac{e^{\left(1-\sqrt{10}\right)t}\sqrt{10}}{5} \\ e^{\left(1+\sqrt{10}\right)t} & e^{\left(1-\sqrt{10}\right)t} \end{array}\right]$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\,dt$$

But

$$\Phi^{-1} = \left[\begin{array}{cc} \frac{\sqrt{10}\,e^{-\left(1+\sqrt{10}\right)t}}{4} & \frac{e^{-\left(1+\sqrt{10}\right)t}}{2} \\ -\frac{\sqrt{10}\,e^{\left(-1+\sqrt{10}\right)t}}{4} & \frac{e^{\left(-1+\sqrt{10}\right)t}}{2} \end{array}\right]$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \left[\begin{array}{cc} \frac{\sqrt{10}\,e^{\left(1+\sqrt{10}\right)t}}{5} & -\frac{e^{\left(1-\sqrt{10}\right)t}\sqrt{10}}{5} \\ e^{\left(1+\sqrt{10}\right)t} & e^{\left(1-\sqrt{10}\right)t} \end{array}\right] \int \left[\begin{array}{cc} \frac{\sqrt{10}\,e^{-\left(1+\sqrt{10}\right)t}}{4} & \frac{e^{-\left(1+\sqrt{10}\right)t}}{2} \\ -\frac{\sqrt{10}\,e^{\left(-1+\sqrt{10}\right)t}}{4} & \frac{e^{\left(-1+\sqrt{10}\right)t}}{2} \end{array}\right] \left[\begin{array}{c} 2t+1 \\ 3t-1 \end{array}\right] dt \\
&= \left[\begin{array}{cc} \frac{\sqrt{10}\,e^{\left(1+\sqrt{10}\right)t}}{5} & -\frac{e^{\left(1-\sqrt{10}\right)t}\sqrt{10}}{5} \\ e^{\left(1+\sqrt{10}\right)t} & e^{\left(1-\sqrt{10}\right)t} \end{array}\right] \int \left[\begin{array}{c} \frac{e^{-\left(1+\sqrt{10}\right)t}\left(2t\sqrt{10}+\sqrt{10}+6t-2\right)}{4} \\ -\frac{e^{\left(-1+\sqrt{10}\right)t}\left(2t\sqrt{10}+\sqrt{10}-6t+2\right)}{4} \end{array}\right] dt \\
&= \left[\begin{array}{cc} \frac{\sqrt{10}\,e^{\left(1+\sqrt{10}\right)t}}{5} & -\frac{e^{\left(1-\sqrt{10}\right)t}\sqrt{10}}{5} \\ e^{\left(1+\sqrt{10}\right)t} & e^{\left(1-\sqrt{10}\right)t} \end{array}\right] \left[\begin{array}{c} \frac{\left(18t\sqrt{10}+185\sqrt{10}-18t-572\right)e^{-\left(1+\sqrt{10}\right)t}\left(2t\sqrt{10}+\sqrt{10}+6t-2\right)}{-648t-5184+1620\sqrt{10}} \\ -\frac{\left(18t\sqrt{10}+185\sqrt{10}+18t+572\right)e^{\left(-1+\sqrt{10}\right)t}\left(2t\sqrt{10}+\sqrt{10}-6t+2\right)}{324\left(2t+16+5\sqrt{10}\right)} \end{array}\right] \\
&= \left[\begin{array}{c} \frac{-72t^3-1118t^2+436t+51}{162t^2+2592t+243} \\ \frac{-126t^3-2150t^2-2333t-201}{162t^2+2592t+243} \end{array}\right]
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
\left[\begin{array}{c} x \\ y \end{array}\right] &= \left[\begin{array}{c} \frac{c_1\sqrt{10}\,e^{\left(1+\sqrt{10}\right)t}}{5} \\ c_1\,e^{\left(1+\sqrt{10}\right)t} \end{array}\right] + \left[\begin{array}{c} -\frac{c_2\,e^{\left(1-\sqrt{10}\right)t}\sqrt{10}}{5} \\ c_2\,e^{\left(1-\sqrt{10}\right)t} \end{array}\right] + \left[\begin{array}{c} \frac{-72t^3-1118t^2+436t+51}{162t^2+2592t+243} \\ \frac{-126t^3-2150t^2-2333t-201}{162t^2+2592t+243} \end{array}\right]
\end{aligned}$$

Which becomes

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{c_1\sqrt{10}\,e^{\left(1+\sqrt{10}\right)t}}{5} - \frac{c_2\,e^{-\left(-1+\sqrt{10}\right)t}\sqrt{10}}{5} - \frac{4t}{9} + \frac{17}{81} \\ c_1\,e^{\left(1+\sqrt{10}\right)t} + c_2\,e^{-\left(-1+\sqrt{10}\right)t} - \frac{7t}{9} - \frac{67}{81} \end{bmatrix}$$

Maple step by step solution

Let's solve

$\left[\frac{d}{dt}x(t) = x(t) + 2y(t) + 2t + 1, \frac{d}{dt}y(t) = 5x(t) + y(t) + 3t - 1\right]$

- Define vector

  $\overrightarrow{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

- Convert system into a vector equation

  $\frac{d}{dt}\overrightarrow{x}(t) = \begin{bmatrix} 1 & 2 \\ 5 & 1 \end{bmatrix} \cdot \overrightarrow{x}(t) + \begin{bmatrix} 2t+1 \\ 3t-1 \end{bmatrix}$

- System to solve

  $\frac{d}{dt}\overrightarrow{x}(t) = \begin{bmatrix} 1 & 2 \\ 5 & 1 \end{bmatrix} \cdot \overrightarrow{x}(t) + \begin{bmatrix} 2t+1 \\ 3t-1 \end{bmatrix}$

- Define the forcing function

  $\overrightarrow{f}(t) = \begin{bmatrix} 2t+1 \\ 3t-1 \end{bmatrix}$

- Define the coefficient matrix

  $A = \begin{bmatrix} 1 & 2 \\ 5 & 1 \end{bmatrix}$

- Rewrite the system as

  $\frac{d}{dt}\overrightarrow{x}(t) = A \cdot \overrightarrow{x}(t) + \overrightarrow{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

  $\left[\left[1-\sqrt{10}, \begin{bmatrix} -\frac{\sqrt{10}}{5} \\ 1 \end{bmatrix}\right], \left[1+\sqrt{10}, \begin{bmatrix} \frac{\sqrt{10}}{5} \\ 1 \end{bmatrix}\right]\right]$

- Consider eigenpair

  $\left[1-\sqrt{10}, \begin{bmatrix} -\frac{\sqrt{10}}{5} \\ 1 \end{bmatrix}\right]$

- Solution to homogeneous system from eigenpair

$$\vec{x}_1 = \mathrm{e}^{\left(1-\sqrt{10}\right)t} \cdot \left[\begin{array}{c} -\frac{\sqrt{10}}{5} \\ 1 \end{array}\right]$$

- Consider eigenpair

$$\left[1+\sqrt{10}, \left[\begin{array}{c} \frac{\sqrt{10}}{5} \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_2 = \mathrm{e}^{\left(1+\sqrt{10}\right)t} \cdot \left[\begin{array}{c} \frac{\sqrt{10}}{5} \\ 1 \end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\vec{x}_p($

$$\vec{x}(t) = C1\vec{x}_1 + C2\vec{x}_2 + \vec{x}_p(t)$$

□ Fundamental matrix

◦ Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t) = \left[\begin{array}{cc} -\frac{\mathrm{e}^{\left(1-\sqrt{10}\right)t}\sqrt{10}}{5} & \frac{\mathrm{e}^{\left(1+\sqrt{10}\right)t}\sqrt{10}}{5} \\ \mathrm{e}^{\left(1-\sqrt{10}\right)t} & \mathrm{e}^{\left(1+\sqrt{10}\right)t} \end{array}\right]$$

◦ The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$$\Phi(t) = \phi(t) \cdot \phi(0)^{-1}$$

◦ Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \left[\begin{array}{cc} -\frac{\mathrm{e}^{\left(1-\sqrt{10}\right)t}\sqrt{10}}{5} & \frac{\mathrm{e}^{\left(1+\sqrt{10}\right)t}\sqrt{10}}{5} \\ \mathrm{e}^{\left(1-\sqrt{10}\right)t} & \mathrm{e}^{\left(1+\sqrt{10}\right)t} \end{array}\right] \cdot \left[\begin{array}{cc} -\frac{\sqrt{10}}{5} & \frac{\sqrt{10}}{5} \\ 1 & 1 \end{array}\right]^{-1}$$

◦ Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \left[\begin{array}{cc} \frac{\mathrm{e}^{-\left(-1+\sqrt{10}\right)t}}{2} + \frac{\mathrm{e}^{\left(1+\sqrt{10}\right)t}}{2} & -\frac{\left(\mathrm{e}^{-\left(-1+\sqrt{10}\right)t} - \mathrm{e}^{\left(1+\sqrt{10}\right)t}\right)\sqrt{10}}{10} \\ -\frac{\left(\mathrm{e}^{-\left(-1+\sqrt{10}\right)t} - \mathrm{e}^{\left(1+\sqrt{10}\right)t}\right)\sqrt{10}}{4} & \frac{\mathrm{e}^{-\left(-1+\sqrt{10}\right)t}}{2} + \frac{\mathrm{e}^{\left(1+\sqrt{10}\right)t}}{2} \end{array}\right]$$

□ Find a particular solution of the system of ODEs using variation of parameters

◦ Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

$$\vec{x}_p(t) = \Phi(t) \cdot \vec{v}(t)$$

◦ Take the derivative of the particular solution

$$\frac{d}{dt}\vec{x}_p(t) = \left(\frac{d}{dt}\Phi(t)\right) \cdot \vec{v}(t) + \Phi(t) \cdot \left(\frac{d}{dt}\vec{v}(t)\right)$$

- Substitute particular solution and its derivative into the system of ODEs
  $\left(\frac{d}{dt}\Phi(t)\right)\cdot \vec{v}(t)+\Phi(t)\cdot \left(\frac{d}{dt}\vec{v}(t)\right)=A\cdot \Phi(t)\cdot \vec{v}(t)+\vec{f}(t)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der
  $A\cdot \Phi(t)\cdot \vec{v}(t)+\Phi(t)\cdot \left(\frac{d}{dt}\vec{v}(t)\right)=A\cdot \Phi(t)\cdot \vec{v}(t)+\vec{f}(t)$
- Cancel like terms
  $\Phi(t)\cdot \left(\frac{d}{dt}\vec{v}(t)\right)=\vec{f}(t)$
- Multiply by the inverse of the fundamental matrix
  $\frac{d}{dt}\vec{v}(t)=\Phi(t)^{-1}\cdot \vec{f}(t)$
- Integrate to solve for $\vec{v}(t)$
  $\vec{v}(t)=\int_0^t \Phi(s)^{-1}\cdot \vec{f}(s)\,ds$
- Plug $\vec{v}(t)$ into the equation for the particular solution
  $\vec{x}_p(t)=\Phi(t)\cdot \left(\int_0^t \Phi(s)^{-1}\cdot \vec{f}(s)\,ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t)=\left[\begin{array}{c}\frac{\left(-67\sqrt{10}-85\right)\mathrm{e}^{-\left(-1+\sqrt{10}\right)t}}{810}+\frac{\left(67\sqrt{10}-85\right)\mathrm{e}^{\left(1+\sqrt{10}\right)t}}{810}-\frac{4t}{9}+\frac{17}{81}\\ \frac{\left(17\sqrt{10}+134\right)\mathrm{e}^{-\left(-1+\sqrt{10}\right)t}}{324}+\frac{\left(-17\sqrt{10}+134\right)\mathrm{e}^{\left(1+\sqrt{10}\right)t}}{324}-\frac{7t}{9}-\frac{67}{81}\end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{x}(t)=C1\vec{x}_1+C2\vec{x}_2+\left[\begin{array}{c}\frac{\left(-67\sqrt{10}-85\right)\mathrm{e}^{-\left(-1+\sqrt{10}\right)t}}{810}+\frac{\left(67\sqrt{10}-85\right)\mathrm{e}^{\left(1+\sqrt{10}\right)t}}{810}-\frac{4t}{9}+\frac{17}{81}\\ \frac{\left(17\sqrt{10}+134\right)\mathrm{e}^{-\left(-1+\sqrt{10}\right)t}}{324}+\frac{\left(-17\sqrt{10}+134\right)\mathrm{e}^{\left(1+\sqrt{10}\right)t}}{324}-\frac{7t}{9}-\frac{67}{81}\end{array}\right]$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c}x(t)\\ y(t)\end{array}\right]=\left[\begin{array}{c}\frac{\left(-85+(-162C1-67)\sqrt{10}\right)\mathrm{e}^{-\left(-1+\sqrt{10}\right)t}}{810}+\frac{\left(-85+(162C2+67)\sqrt{10}\right)\mathrm{e}^{\left(1+\sqrt{10}\right)t}}{810}-\frac{4t}{9}+\frac{17}{81}\\ \frac{\left(324C1+17\sqrt{10}+134\right)\mathrm{e}^{-\left(-1+\sqrt{10}\right)t}}{324}+\frac{\left(324C2-17\sqrt{10}+134\right)\mathrm{e}^{\left(1+\sqrt{10}\right)t}}{324}-\frac{7t}{9}-\frac{67}{81}\end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x(t)=\frac{\left(-85+(-162C1-67)\sqrt{10}\right)\mathrm{e}^{-\left(-1+\sqrt{10}\right)t}}{810}+\frac{\left(-85+(162C2+67)\sqrt{10}\right)\mathrm{e}^{\left(1+\sqrt{10}\right)t}}{810}-\frac{4t}{9}+\frac{17}{81}, y(t)=\frac{\left(324C1+17\right.}{}\right.$$

Maple dsolve solution

Solving time : 0.053 (sec)
Leaf size : 67

```
dsolve([diff(x(t),t) = x(t)+2*y(t)+2*t+1, diff(y(t),t) = 5*x(t)+y(t)+3*t-1]
       ,{op([x(t), y(t)])})
```

$$
\begin{aligned}
x(t) &= \mathrm{e}^{\left(1+\sqrt{10}\right)t} c_2 + \mathrm{e}^{-\left(-1+\sqrt{10}\right)t} c_1 - \frac{4t}{9} + \frac{17}{81} \\
y(t) &= \frac{\mathrm{e}^{\left(1+\sqrt{10}\right)t} c_2 \sqrt{10}}{2} - \frac{\mathrm{e}^{-\left(-1+\sqrt{10}\right)t} c_1 \sqrt{10}}{2} - \frac{7t}{9} - \frac{67}{81}
\end{aligned}
$$

Mathematica DSolve solution

Solving time : 11.175 (sec)
Leaf size : 158

```
DSolve[{{D[x[t],t]==x[t]+2*y[t]+2*t+1,D[y[t],t]==5*x[t]+y[t]+3*t-1},{}},
       {x[t],y[t]},t,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
x(t) &\to \frac{1}{810} e^{t-\sqrt{10}t} \left(e^{\left(\sqrt{10}-1\right)t} (170-360t) + 81 \left(5c_1 + \sqrt{10}c_2\right) e^{2\sqrt{10}t} + 81 \left(5c_1 - \sqrt{10}c_2\right) \right) \\
y(t) &\to \frac{1}{324} e^{t-\sqrt{10}t} \left(-4e^{\left(\sqrt{10}-1\right)t} (63t+67) + 81 \left(\sqrt{10}c_1 + 2c_2\right) e^{2\sqrt{10}t} - 81 \left(\sqrt{10}c_1 - 2c_2\right) \right)
\end{aligned}
$$

2.4.71 problem 68

Internal problem ID [8636]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 68
Date solved : `Thursday, December 12, 2024 at 09:32:45 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' + 20y' + 500y = 100000\cos(100x)$$

Solved as second order linear constant coeff ode

Time used: 0.168 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 20, C = 500, f(x) = 100000\cos(100x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 20y' + 500y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 20, C = 500$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + 20\lambda e^{x\lambda} + 500 e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 20\lambda + 500 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 20, C = 500$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-20}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{20^2 - (4)(1)(500)} \\ &= -10 \pm 20i\end{aligned}$$

Hence

$$\lambda_1 = -10 + 20i$$
$$\lambda_2 = -10 - 20i$$

Which simplifies to

$$\lambda_1 = -10 + 20i$$
$$\lambda_2 = -10 - 20i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -10$ and $\beta = 20$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-10x}(c_1 \cos(20x) + c_2 \sin(20x))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-10x}(c_1 \cos(20x) + c_2 \sin(20x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$100000 \cos(100x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(100x), \sin(100x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-10x}\cos(20x), e^{-10x}\sin(20x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\cos(100x) + A_2\sin(100x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&-9500A_1\cos(100x) - 9500A_2\sin(100x) - 2000A_1\sin(100x) + 2000A_2\cos(100x)\\&= 100000\cos(100x)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{3800}{377}, A_2 = \frac{800}{377}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{3800\cos(100x)}{377} + \frac{800\sin(100x)}{377}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p\\&= \left(e^{-10x}(c_1\cos(20x) + c_2\sin(20x))\right) + \left(-\frac{3800\cos(100x)}{377} + \frac{800\sin(100x)}{377}\right)\end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{3800\cos(100x)}{377} + \frac{800\sin(100x)}{377} + e^{-10x}(c_1\cos(20x) + c_2\sin(20x))$$

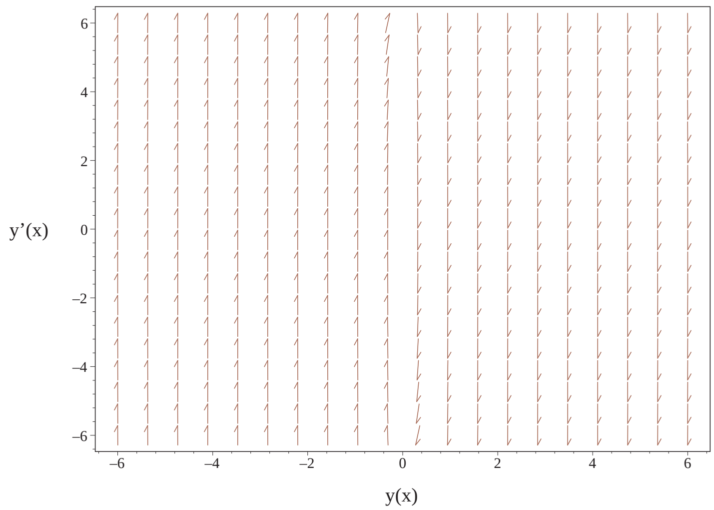


Figure 2.246: Slope field plot
$y'' + 20y' + 500y = 100000\cos(100x)$

Solved as second order ode using Kovacic algorithm

Time used: 0.190 (sec)

Writing the ode as

$$y'' + 20y' + 500y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 20 \\ C &= 500 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-400}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -400 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -400z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.251: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -400$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(20x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{20}{1}\,dx} \\
&= z_1 e^{-10x} \\
&= z_1\left(\mathrm{e}^{-10x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-10x}\cos(20x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{20}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-20x}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{\tan(20x)}{20}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-10x}\cos\left(20x\right)\right) + c_2\left(\mathrm{e}^{-10x}\cos\left(20x\right)\left(\frac{\tan\left(20x\right)}{20}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 20y' + 500y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1\,\mathrm{e}^{-10x}\cos\left(20x\right) + \frac{c_2\,\mathrm{e}^{-10x}\sin\left(20x\right)}{20}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
100000\cos\left(100x\right)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos\left(100x\right), \sin\left(100x\right)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\mathrm{e}^{-10x}\cos\left(20x\right), \frac{\mathrm{e}^{-10x}\sin\left(20x\right)}{20}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1\cos\left(100x\right) + A_2\sin\left(100x\right)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
\begin{aligned}
&-9500A_1\cos\left(100x\right) - 9500A_2\sin\left(100x\right) - 2000A_1\sin\left(100x\right) + 2000A_2\cos\left(100x\right) \\
&= 100000\cos\left(100x\right)
\end{aligned}
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{3800}{377}, A_2 = \frac{800}{377}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{3800\cos(100x)}{377} + \frac{800\sin(100x)}{377}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-10x}\cos(20x) + \frac{c_2 \mathrm{e}^{-10x}\sin(20x)}{20}\right) + \left(-\frac{3800\cos(100x)}{377} + \frac{800\sin(100x)}{377}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \mathrm{e}^{-10x}\cos(20x) + \frac{c_2 \mathrm{e}^{-10x}\sin(20x)}{20} - \frac{3800\cos(100x)}{377} + \frac{800\sin(100x)}{377}$$

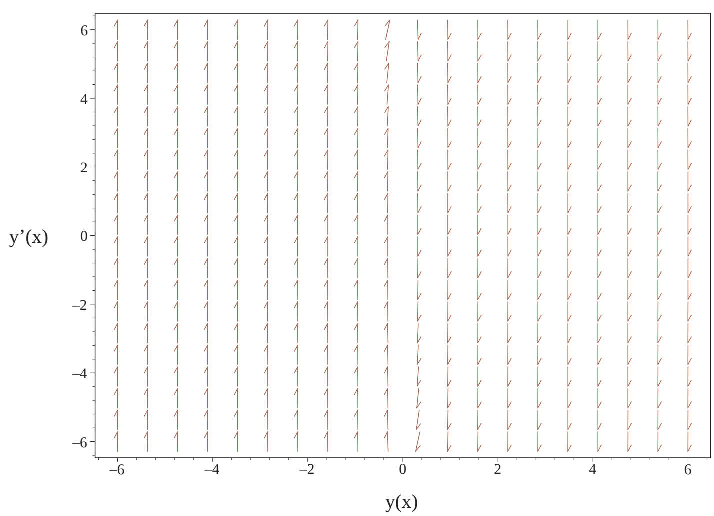


Figure 2.247: Slope field plot
$y'' + 20y' + 500y = 100000\cos(100x)$

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + 20\frac{d}{dx}y(x) + 500y(x) = 100000\cos(100x)$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Characteristic polynomial of homogeneous ODE

  $r^2 + 20r + 500 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-20)\pm(\sqrt{-1600})}{2}$

- Roots of the characteristic polynomial

  $r = (-10 - 20\,\mathrm{I}, -10 + 20\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-10x}\cos(20x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-10x}\sin(20x)$

- General solution of the ODE

  $y(x) = C1y_1(x) + C2y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y(x) = C1\,\mathrm{e}^{-10x}\cos(20x) + C2\,\mathrm{e}^{-10x}\sin(20x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 100000\cos(100x)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \mathrm{e}^{-10x}\cos(20x) & \mathrm{e}^{-10x}\sin(20x) \\ -10\,\mathrm{e}^{-10x}\cos(20x) - 20\,\mathrm{e}^{-10x}\sin(20x) & -10\,\mathrm{e}^{-10x}\sin(20x) + 20\,\mathrm{e}^{-10x}\cos \end{array}\right.$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 20\,\mathrm{e}^{-20x}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -5000\,\mathrm{e}^{-10x}\left(\cos(20x)\left(\int \sin(20x)\cos(100x)\,\mathrm{e}^{10x}dx\right) - \sin(20x)\left(\int \cos(20x)\cos(100x)\right.\right.$

- ○ Compute integrals

  $y_p(x) = -\frac{3800\cos(100x)}{377} + \frac{800\sin(100x)}{377}$

- Substitute particular solution into general solution to ODE

$$y(x) = C1\,\mathrm{e}^{-10x}\cos(20x) + C2\,\mathrm{e}^{-10x}\sin(20x) - \frac{3800\cos(100x)}{377} + \frac{800\sin(100x)}{377}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : `0.115` (sec)
Leaf size : 37

```
dsolve(diff(diff(y(x),x),x)+20*diff(y(x),x)+500*y(x) = 100000*cos(100*x),
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-10x}\sin(20x)\,c_2 + \mathrm{e}^{-10x}\cos(20x)\,c_1 - \frac{3800\cos(100x)}{377} + \frac{800\sin(100x)}{377}$$

Mathematica DSolve solution

Solving time : `0.026` (sec)
Leaf size : 47

```
DSolve[{D[y[x],{x,2}]+20*D[y[x],x]+500*y[x] == 100000*Cos[100*x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{200}{377}(19\cos(100x) - 4\sin(100x)) + c_2 e^{-10x}\cos(20x) + c_1 e^{-10x}\sin(20x)$$

2.4.72 problem 69

Internal problem ID [8637]
Book : Own collection of miscellaneous problems
Section : section 4.0
Problem number : 69
Date solved : `Thursday, December 12, 2024 at 09:34:22 AM`
CAS classification :
`[[_2nd_order, _with_linear_symmetries], [_2nd_order, _linear, '_with_symmetry_[0,F(x)]`

Solve

$$y'' \sin(2x)^2 + y' \sin(4x) - 4y = 0$$

Solved as second order ode using change of variable on x method 2

Time used: 0.581 (sec)

In normal form the ode

$$y'' \sin(2x)^2 + y' \sin(4x) - 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= 2\cot(2x) \\ q(x) &= -4\csc(2x)^2 \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int e^{-\int p(x)dx} dx \\
&= \int e^{-\int 2\cot(2x)dx} dx \\
&= \int e^{\frac{\ln\left(\csc(2x)^2\right)}{2}} dx \\
&= \int \csc(2x)\, dx \\
&= -\frac{\ln(\csc(2x) + \cot(2x))}{2}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-4\csc(2x)^2}{\csc(2x)^2} \\
&= -4
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) - 4y(\tau) = 0$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -4$. Let the solution be $y(\tau) = e^{\lambda \tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\tau \lambda} - 4\,\mathrm{e}^{\tau \lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda \tau}$ gives

$$\lambda^2 - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -4$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-4)} \\
&= \pm 2
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +2 \\
\lambda_2 &= -2
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 2 \\
\lambda_2 &= -2
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y(\tau) &= c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau} \\
y(\tau) &= c_1 e^{(2)\tau} + c_2 e^{(-2)\tau}
\end{aligned}$$

Or

$$y(\tau) = c_1\,\mathrm{e}^{2\tau} + c_2\,\mathrm{e}^{-2\tau}$$

Will add steps showing solving for IC soon.

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1}{\csc(2x) + \cot(2x)} + c_2(\csc(2x) + \cot(2x))$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{c_1}{\csc(2x) + \cot(2x)} + c_2(\csc(2x) + \cot(2x))$$

Solved as second order ode using change of variable on x method 1

Time used: 0.693 (sec)

In normal form the ode

$$y'' \sin(2x)^2 + y' \sin(4x) - 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= 2\cot(2x) \\ q(x) &= -4\csc(2x)^2 \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{-\csc(2x)^2}}{c} \\ \tau'' &= \frac{4\csc(2x)^2 \cot(2x)}{c\sqrt{-\csc(2x)^2}} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{\frac{4\csc(2x)^2\cot(2x)}{c\sqrt{-\csc(2x)^2}} + 2\cot(2x)\,\frac{2\sqrt{-\csc(2x)^2}}{c}}{\left(\frac{2\sqrt{-\csc(2x)^2}}{c}\right)^2} \\
&= 0
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = c_1\cos(c\tau) + c_2\sin(c\tau)
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int 2\sqrt{-\csc(2x)^2}\,dx}{c} \\
&= \frac{\sqrt{-\csc(2x)^2}\,\ln(\csc(2x) - \cot(2x))\sin(2x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
\begin{aligned}
y = {} & c_1\cos\left(\sqrt{-\csc(2x)^2}\,\ln(\csc(2x) - \cot(2x))\sin(2x)\right) \\
& + c_2\sin\left(\sqrt{-\csc(2x)^2}\,\ln(\csc(2x) - \cot(2x))\sin(2x)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
\begin{aligned}
y = {} & c_1\cos\left(\sqrt{-\csc(2x)^2}\,\ln(\csc(2x) - \cot(2x))\sin(2x)\right) \\
& + c_2\sin\left(\sqrt{-\csc(2x)^2}\,\ln(\csc(2x) - \cot(2x))\sin(2x)\right)
\end{aligned}
$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @ Moebiu
-> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), dx)) *
-> Trying changes of variables to rationalize or make the ODE simpler
   trying a quadrature
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   <- linear_1 successful
   Change of variables used:
      [x = 1/4*arccos(t)]
   Linear ODE actually solved:
      -u(t)+(3*t^2-2*t-1)*diff(u(t),t)+(2*t^3-2*t^2-2*t+2)*diff(diff(u(t),t),t) = 0
<- change of variables successful`
```

Maple dsolve solution

Solving time : 0.214 (sec)
Leaf size : 17

```
dsolve(diff(diff(y(x),x),x)*sin(2*x)^2+diff(y(x),x)*sin(4*x)-4*y(x) = 0,
       y(x),singsol=all)
```

$$y = c_1 \csc(2x) + \cot(2x) c_2$$

Mathematica DSolve solution

Solving time : 0.055 (sec)
Leaf size : 29

```
DSolve[{D[y[x],{x,2}]*Sin[2*x]^2+D[y[x],x]*Sin[4*x]-4*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c_1 - ic_2 \cos(2x)}{\sqrt{\sin^2(2x)}}$$

2.5 section 5.0

2.5.1 problem 1

Internal problem ID [8638]
Book : Own collection of miscellaneous problems
Section : section 5.0
Problem number : 1
Date solved : `Thursday, December 12, 2024 at 09:34:25 AM`
CAS classification :
`[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]`

Solve

$$
y'' = Ay^{2/3}
$$

Solved as second order missing x ode

Time used: 82.638 (sec)

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$
y' = p
$$

Then

$$
\begin{aligned}
y'' &= \frac{dp}{dx} \\
&= \frac{dp}{dy}\frac{dy}{dx} \\
&= p\frac{dp}{dy}
\end{aligned}
$$

Hence the ode becomes

$$
p(y) \left(\frac{d}{dy}p(y)\right) = A\, y^{2/3}
$$

Which is now solved as first order ode for $p(y)$.

The ode $p' = \frac{A\,y^{2/3}}{p}$ is separable as it can be written as

$$\begin{aligned} p' &= \frac{A\,y^{2/3}}{p} \\ &= f(y)g(p) \end{aligned}$$

Where

$$\begin{aligned} f(y) &= A\,y^{2/3} \\ g(p) &= \frac{1}{p} \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(p)}\,dp &= \int f(y)\,dy \\ \int p\,dp &= \int A\,y^{2/3}\,dy \\ \frac{p^2}{2} &= \frac{3y^{5/3}A}{5} + c_1 \end{aligned}$$

Solving for p gives

$$p = -\frac{\sqrt{30y^{5/3}A + 50c_1}}{5}$$

$$p = \frac{\sqrt{30y^{5/3}A + 50c_1}}{5}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\frac{\sqrt{30y^{5/3}A + 50c_1}}{5}$$

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} -\frac{5}{\sqrt{30\tau^{5/3}A + 50c_1}}d\tau = x + c_2$$

Singular solutions are found by solving

$$-\frac{\sqrt{30y^{5/3}A + 50c_1}}{5} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = \frac{5^{3/5} 3^{2/5} (-c_1 A^4)^{3/5}}{3A^3}$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \frac{\sqrt{30y^{5/3}A + 50c_1}}{5}$$

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^y \frac{5}{\sqrt{30\tau^{5/3}A + 50c_1}} d\tau = x + c_3$$

Singular solutions are found by solving

$$\frac{\sqrt{30y^{5/3}A + 50c_1}}{5} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = \frac{5^{3/5} 3^{2/5} (-c_1 A^4)^{3/5}}{3A^3}$$

Will add steps showing solving for IC soon.

The solution

$$y = \frac{5^{3/5} 3^{2/5} (-c_1 A^4)^{3/5}}{3A^3}$$

was found not to satisfy the ode or the IC. Hence it is removed.

Summary of solutions found

$$\int^y -\frac{5}{\sqrt{30\tau^{5/3}A + 50c_1}} d\tau = x + c_2$$

$$\int^y \frac{5}{\sqrt{30\tau^{5/3}A + 50c_1}} d\tau = x + c_3$$

Solved as second order can be made integrable

Time used: 81.053 (sec)

Multiplying the ode by y' gives

$$-y^{2/3}Ay' + y'y'' = 0$$

Integrating the above w.r.t x gives

$$\int \left(-y^{2/3}Ay' + y'y''\right) dx = 0$$
$$-\frac{3y^{5/3}A}{5} + \frac{y'^2}{2} = c_1$$

Which is now solved for y. Solving for the derivative gives these ODE's to solve

$$y' = \frac{\sqrt{30y^{5/3}A + 50c_1}}{5} \tag{1}$$
$$y' = -\frac{\sqrt{30y^{5/3}A + 50c_1}}{5} \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} \frac{5}{\sqrt{30\tau^{5/3}A + 50c_1}} d\tau = x + c_2$$

Singular solutions are found by solving

$$\frac{\sqrt{30y^{5/3}A + 50c_1}}{5} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = \frac{5^{3/5}3^{2/5}\left(-c_1\, A^4\right)^{3/5}}{3A^3}$$

Solving Eq. (2)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{y} -\frac{5}{\sqrt{30\tau^{5/3}A+50c_1}}d\tau = x+c_3$$

Singular solutions are found by solving

$$-\frac{\sqrt{30y^{5/3}A+50c_1}}{5} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = \frac{5^{3/5}3^{2/5}(-c_1\,A^4)^{3/5}}{3A^3}$$

Will add steps showing solving for IC soon.

The solution

$$y = \frac{5^{3/5}3^{2/5}(-c_1\,A^4)^{3/5}}{3A^3}$$

was found not to satisfy the ode or the IC. Hence it is removed.

Summary of solutions found

$$\int^{y} -\frac{5}{\sqrt{30\tau^{5/3}A+50c_1}}d\tau = x+c_3$$

$$\int^{y} \frac{5}{\sqrt{30\tau^{5/3}A+50c_1}}d\tau = x+c_2$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
```

```
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)-A*_a^(2/3) = 0, _b(_a), HI
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[_a, 5/6*_b]
```

Maple dsolve solution

Solving time : 0.086 (sec)
Leaf size : 61

```
dsolve(diff(diff(y(x),x),x) = A*y(x)^(2/3),
       y(x),singsol=all)
```

$$
\begin{aligned}
&y = 0\\
&-5\left(\int^{y}\frac{1}{\sqrt{30\_a^{5/3}A-5c_1}}d\_a\right)-x-c_2 = 0\\
&5\left(\int^{y}\frac{1}{\sqrt{30\_a^{5/3}A-5c_1}}d\_a\right)-x-c_2 = 0
\end{aligned}
$$

Mathematica DSolve solution

Solving time : 0.117 (sec)
Leaf size : 75

```
DSolve[{D[y[x],{x,2}]==A*y[x]^(2/3),{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
\text{Solve}\left[\frac{y(x)^2\left(1+\frac{6Ay(x)^{5/3}}{5c_1}\right)\text{Hypergeometric2F1}\left(\frac{1}{2},\frac{3}{5},\frac{8}{5},-\frac{6Ay(x)^{5/3}}{5c_1}\right)^2}{\frac{6}{5}Ay(x)^{5/3}+c_1}=(x+c_2)^2,y(x)\right]
$$

2.5.2 problem 2

Internal problem ID [8639]
Book : Own collection of miscellaneous problems
Section : section 5.0
Problem number : 2
Date solved : `Thursday, December 12, 2024 at 09:37:10 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 2xy' + \left(x^2+1\right) y = 0$$

Solved as second order solved by an integrating factor

Time used: 0.049 (sec)

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = 2x$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int 2x\, dx} \\ &= \mathrm{e}^{\frac{x^2}{2}} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = 0$$

$$\left(\mathrm{e}^{\frac{x^2}{2}} y\right)'' = 0$$

Integrating once gives

$$\left(e^{\frac{x^2}{2}} y\right)' = c_1$$

Integrating again gives

$$\left(e^{\frac{x^2}{2}} y\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{e^{\frac{x^2}{2}}}$$

Or

$$y = c_1 x\,e^{-\frac{x^2}{2}} + c_2\,e^{-\frac{x^2}{2}}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 x\,e^{-\frac{x^2}{2}} + c_2\,e^{-\frac{x^2}{2}}$$

Solved as second order ode using change of variable on y method 1

Time used: 0.344 (sec)

In normal form the given ode is written as

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= 2x \\ q(x) &= x^2 + 1 \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= x^2 + 1 - \frac{(2x)'}{2} - \frac{(2x)^2}{4} \\ &= x^2 + 1 - \frac{(2)}{2} - \frac{(4x^2)}{4} \\ &= x^2 + 1 - (1) - x^2 \\ &= 0 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\int \frac{p(x)}{2} dx} \\ &= e^{-\int \frac{2x}{2}} \\ &= \mathrm{e}^{-\frac{x^2}{2}} \end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x)\, \mathrm{e}^{-\frac{x^2}{2}} \tag{4}$$

Applying this change of variable to the original ode results in

$$v''(x)\, \mathrm{e}^{-\frac{x^2}{2}} = 0$$

Which is now solved for $v(x)$.

The above ode can be simplified to

$$v''(x) = 0$$

Integrating twice gives the solution

$$v(x) = c_1 x + c_2$$

Will add steps showing solving for IC soon.

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= (c_1 x + c_2)\, (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \mathrm{e}^{-\frac{x^2}{2}}$$

Hence (7) becomes

$$y = (c_1 x + c_2)\, \mathrm{e}^{-\frac{x^2}{2}}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = (c_1 x + c_2)\, \mathrm{e}^{-\frac{x^2}{2}}$$

Solved as second order ode using Kovacic algorithm

Time used: 0.049 (sec)

Writing the ode as

$$y'' + 2xy' + \left(x^2 + 1\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2x \\ C &= x^2 + 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.253: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x}{1}\, dx} \\
&= z_1 e^{-\frac{x^2}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x^2}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x^2}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-x^2}}{(y_1)^2}\, dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-\frac{x^2}{2}}\right) + c_2 \left(\mathrm{e}^{-\frac{x^2}{2}}(x)\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \,\mathrm{e}^{-\frac{x^2}{2}} + c_2 x \,\mathrm{e}^{-\frac{x^2}{2}}$$

Solved as second order ode adjoint method

Time used: 0.158 (sec)

In normal form the ode

$$y'' + 2xy' + \left(x^2 + 1\right) y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= 2x \\
q(x) &= x^2 + 1 \\
r(x) &= 0
\end{aligned}
$$

The Lagrange adjoint ode is given by

$$
\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - (2x\xi(x))' + \left(\left(x^2+1\right)\xi(x)\right) &= 0 \\
\xi(x)\, x^2 - 2x\xi'(x) + \xi''(x) - \xi(x) &= 0
\end{aligned}
$$

Which is solved for $\xi(x)$. The ode satisfies this form

$$
\xi'' + p(x)\,\xi' + \frac{\left(p(x)^2 + p'(x)\right)\xi}{2} = f(x)
$$

Where $p(x) = -2x$. Therefore, there is an integrating factor given by

$$
\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\,dx} \\
&= e^{\int -2x\,dx} \\
&= \mathrm{e}^{-\frac{x^2}{2}}
\end{aligned}
$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)\xi)'' &= 0 \\
\left(\mathrm{e}^{-\frac{x^2}{2}}\xi\right)'' &= 0
\end{aligned}
$$

Integrating once gives

$$
\left(\mathrm{e}^{-\frac{x^2}{2}}\xi\right)' = c_1
$$

Integrating again gives

$$
\left(\mathrm{e}^{-\frac{x^2}{2}}\xi\right) = c_1 x + c_2
$$

Hence the solution is

$$
\xi = \frac{c_1 x + c_2}{\mathrm{e}^{-\frac{x^2}{2}}}
$$

Or

$$
\xi = c_1 x\,\mathrm{e}^{\frac{x^2}{2}} + c_2\,\mathrm{e}^{\frac{x^2}{2}}
$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$
\begin{aligned}
\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi\,(x)}
\end{aligned}
$$

Or

$$
y' + y\left(2x - \frac{c_1\, e^{\frac{x^2}{2}} + c_1\, x^2 e^{\frac{x^2}{2}} + c_2\, e^{\frac{x^2}{2}}\, x}{c_1 x\, e^{\frac{x^2}{2}} + c_2\, e^{\frac{x^2}{2}}}\right) = 0
$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$
y' + q(x)y = p(x)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= -\frac{-c_1\, x^2 - c_2 x + c_1}{c_1 x + c_2} \\
p(x) &= 0
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\, dx} \\
&= e^{\int -\frac{-c_1\, x^2 - c_2 x + c_1}{c_1 x + c_2} dx} \\
&= \frac{e^{\frac{x^2}{2}}}{c_1 x + c_2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} \mu y &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y\, e^{\frac{x^2}{2}}}{c_1 x + c_2}\right) &= 0
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y\, e^{\frac{x^2}{2}}}{c_1 x + c_2} &= \int 0\, dx + c_3 \\
&= c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{e^{\frac{x^2}{2}}}{c_1x+c_2}$ gives the final solution

$$y = (c_1x + c_2)\,e^{-\frac{x^2}{2}}c_3$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = (c_1x + c_2)\,e^{-\frac{x^2}{2}}c_3$$

The constants can be merged to give

$$y = (c_1x + c_2)\,e^{-\frac{x^2}{2}}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = (c_1x + c_2)\,e^{-\frac{x^2}{2}}$$

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + 2x\left(\frac{d}{dx}y(x)\right) + (x^2+1)\,y(x) = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Assume series solution for $y(x)$

  $y(x) = \sum_{k=0}^{\infty} a_k x^k$

□ Rewrite ODE with series expansions

  - Convert $x^m \cdot y(x)$ to series expansion for $m = 0..2$

    $x^m \cdot y(x) = \sum_{k=\max(0,-m)}^{\infty} a_k x^{k+m}$

  - Shift index using $k- >k - m$

    $x^m \cdot y(x) = \sum_{k=\max(0,-m)+m}^{\infty} a_{k-m} x^k$

  - Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

$$x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k k\, x^k$$

- Convert $\frac{d^2}{dx^2}y(x)$ to series expansion

$$\frac{d^2}{dx^2}y(x) = \sum_{k=2}^{\infty} a_k k(k-1)\, x^{k-2}$$

- Shift index using $k->k+2$

$$\frac{d^2}{dx^2}y(x) = \sum_{k=0}^{\infty} a_{k+2}(k+2)\,(k+1)\, x^k$$

Rewrite ODE with series expansions

$$2a_2 + a_0 + (6a_3 + 3a_1)\, x + \left(\sum_{k=2}^{\infty} \left(a_{k+2}(k+2)\,(k+1) + a_k(2k+1) + a_{k-2}\right) x^k\right) = 0$$

- The coefficients of each power of x must be 0

$$[2a_2 + a_0 = 0, 6a_3 + 3a_1 = 0]$$

- Solve for the dependent coefficient(s)

$$\left\{a_2 = -\frac{a_0}{2}, a_3 = -\frac{a_1}{2}\right\}$$

- Each term in the series must be 0, giving the recursion relation

$$(k^2 + 3k + 2)\, a_{k+2} + 2a_k k + a_k + a_{k-2} = 0$$

- Shift index using $k->k+2$

$$\left((k+2)^2 + 3k + 8\right) a_{k+4} + 2a_{k+2}(k+2) + a_{k+2} + a_k = 0$$

- Recursion relation that defines the series solution to the ODE

$$\left[y(x) = \sum_{k=0}^{\infty} a_k x^k, a_{k+4} = -\frac{2ka_{k+2} + a_k + 5a_{k+2}}{k^2 + 7k + 12}, a_2 = -\frac{a_0}{2}, a_3 = -\frac{a_1}{2}\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 16

```
dsolve(diff(diff(y(x),x),x)+2*diff(y(x),x)*x+(x^2+1)*y(x) = 0,
       y(x),singsol=all)
```

$$y = \mathrm{e}^{-\frac{x^2}{2}}(c_2 x + c_1)$$

Mathematica DSolve solution

Solving time : 0.029 (sec)
Leaf size : 22

```
DSolve[{D[y[x],{x,2}]+2*x*D[y[x],x]+(x^2+1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{-\frac{x^2}{2}}(c_2 x + c_1)$$

2.5.3 problem 3

Internal problem ID [8640]
Book : Own collection of miscellaneous problems
Section : section 5.0
Problem number : 3
Date solved : `Thursday, December 12, 2024 at 09:37:11 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 2\cot(x)\, y' - y = 0$$

Solved as second order solved by an integrating factor

Time used: 0.040 (sec)

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = 2\cot(x)$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int 2\cot(x)\, dx} \\ &= \sin(x) \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = 0$$

$$(\sin(x)\, y)'' = 0$$

Integrating once gives

$$(\sin(x)\, y)' = c_1$$

Integrating again gives

$$(\sin(x)\, y) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\sin(x)}$$

Or

$$y = \frac{c_1 x}{\sin(x)} + \frac{c_2}{\sin(x)}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{c_1 x}{\sin(x)} + \frac{c_2}{\sin(x)}$$

Solved as second order ode using change of variable on y method 1

Time used: 0.368 (sec)

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= 2\cot(x) \\ q(x) &= -1 \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= -1 - \frac{(2\cot(x))'}{2} - \frac{(2\cot(x))^2}{4} \\ &= -1 - \frac{\left(-2 - 2\cot(x)^2\right)}{2} - \frac{\left(4\cot(x)^2\right)}{4} \\ &= -1 - \left(-1 - \cot(x)^2\right) - \cot(x)^2 \\ &= 0 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\int \frac{p(x)}{2} dx} \\ &= e^{-\int \frac{2\cot(x)}{2}} \\ &= \csc(x) \end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x) \csc(x) \tag{4}$$

Applying this change of variable to the original ode results in

$$v''(x) \csc(x) = 0$$

Which is now solved for $v(x)$.

The above ode can be simplified to

$$v''(x) = 0$$

Integrating twice gives the solution

$$v(x) = c_1 x + c_2$$

Will add steps showing solving for IC soon.

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= (c_1 x + c_2)\,(z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \csc(x)$$

Hence (7) becomes

$$y = (c_1 x + c_2) \csc(x)$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = (c_1 x + c_2) \csc(x)$$

Solved as second order ode using Kovacic algorithm

Time used: 0.065 (sec)

Writing the ode as

$$
\begin{align*}
y'' + 2\cot(x)\, y' - y &= 0 \tag{1}\\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= 1\\
B &= 2\cot(x) \tag{3}\\
C &= -1
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5}\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{0}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= 0\\
t &= 1
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(x) = 0 \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 2.255: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2\cot(x)}{1}\,dx} \\
&= z_1 e^{-\ln(\sin(x))} \\
&= z_1(\csc(x))
\end{aligned}$$

Which simplifies to

$$y_1 = \csc(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2\cot(x)}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{-2\ln(\sin(x))}}{(y_1)^2}\, dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\csc(x)) + c_2(\csc(x)(x))
\end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 \csc(x) + c_2 x \csc(x)$$

Solved as second order ode adjoint method

Time used: 0.484 (sec)

In normal form the ode

$$y'' + 2\cot(x)\, y' - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= 2\cot(x) \\
q(x) &= -1 \\
r(x) &= 0
\end{aligned}
$$

The Lagrange adjoint ode is given by

$$
\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - (2\cot(x)\, \xi(x))' + (-\xi(x)) &= 0 \\
2\cot(x)^2 \xi(x) - 2\cot(x)\, \xi'(x) + \xi''(x) + \xi(x) &= 0
\end{aligned}
$$

Which is solved for $\xi(x)$. The ode satisfies this form

$$
\xi'' + p(x)\, \xi' + \frac{\left(p(x)^2 + p'(x)\right) \xi}{2} = f(x)
$$

Where $p(x) = -2\cot(x)$. Therefore, there is an integrating factor given by

$$
\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\, dx} \\
&= e^{\int -2\cot(x)\, dx} \\
&= \csc(x)
\end{aligned}
$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)\xi)'' &= 0 \\
(\csc(x)\, \xi)'' &= 0
\end{aligned}
$$

Integrating once gives

$$
(\csc(x)\, \xi)' = c_1
$$

Integrating again gives

$$
(\csc(x)\, \xi) = c_1 x + c_2
$$

Hence the solution is

$$
\xi = \frac{c_1 x + c_2}{\csc(x)}
$$

Or

$$
\xi = \frac{c_1 x}{\csc(x)} + \frac{c_2}{\csc(x)}
$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$
\begin{aligned}
\xi(x)\,y' - y\xi'(x) + \xi(x)\,p(x)\,y &= \int \xi(x)\,r(x)\,dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi(x)}
\end{aligned}
$$

Or

$$
y' + y\left(2\cot(x) - \frac{\frac{c_1}{\csc(x)} + \frac{c_1 x\cot(x)}{\csc(x)} + \frac{c_2\cot(x)}{\csc(x)}}{\frac{c_1 x}{\csc(x)} + \frac{c_2}{\csc(x)}}\right) = 0
$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$
y' + q(x)y = p(x)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= -\frac{(-c_1x - c_2)\cot(x) + c_1}{c_1x + c_2} \\
p(x) &= 0
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= \mathrm{e}^{\int -\frac{(-c_1x - c_2)\cot(x) + c_1}{c_1x + c_2}dx} \\
&= \mathrm{e}^{-\frac{\ln\left(1+\tan(x)^2\right)}{2} - \ln(c_1x + c_2) + \ln(\tan(x))}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(y\,\mathrm{e}^{-\frac{\ln\left(1+\tan(x)^2\right)}{2} - \ln(c_1x + c_2) + \ln(\tan(x))}\right) &= 0
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
y\,\mathrm{e}^{-\frac{\ln\left(1+\tan(x)^2\right)}{2} - \ln(c_1x + c_2) + \ln(\tan(x))} &= \int 0\,dx + c_3 \\
&= c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $e^{-\frac{\ln\left(1+\tan(x)^2\right)}{2}-\ln(c_1x+c_2)+\ln(\tan(x))}$ gives the final solution

$$y = \frac{(c_1x + c_2)\sqrt{1 + \tan(x)^2}\, c_3}{\tan(x)}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{(c_1x + c_2)\sqrt{1 + \tan(x)^2}\, c_3}{\tan(x)}$$

The constants can be merged to give

$$y = \frac{(c_1x + c_2)\sqrt{1 + \tan(x)^2}}{\tan(x)}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{(c_1x + c_2)\sqrt{1 + \tan(x)^2}}{\tan(x)}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 12

```
dsolve(diff(diff(y(x),x),x)+2*cot(x)*diff(y(x),x)-y(x) = 0,
       y(x),singsol=all)
```

$$y = \csc(x)(c_2x + c_1)$$

Mathematica DSolve solution

Solving time : 0.034 (sec)
Leaf size : 15

```
DSolve[{D[y[x],{x,2}]+2*Cot[x]*D[y[x],x]-y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to (c_2x + c_1)\csc(x)$$

2.5.4 problem 4

Internal problem ID [8641]
Book : Own collection of miscellaneous problems
Section : section 5.0
Problem number : 4
Date solved : `Thursday, December 12, 2024 at 09:37:12 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$$

Solved as second order ode using change of variable on y method 1

Time used: 0.336 (sec)

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{x^2 - \frac{1}{4}}{x^2}$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{x^2 - \frac{1}{4}}{x^2} - \frac{\left(\frac{1}{x}\right)'}{2} - \frac{\left(\frac{1}{x}\right)^2}{4} \\
&= \frac{x^2 - \frac{1}{4}}{x^2} - \frac{\left(-\frac{1}{x^2}\right)}{2} - \frac{\left(\frac{1}{x^2}\right)}{4} \\
&= \frac{x^2 - \frac{1}{4}}{x^2} - \left(-\frac{1}{2x^2}\right) - \frac{1}{4x^2} \\
&= 1
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\int \frac{p(x)}{2} dx} \\
&= e^{-\int \frac{\frac{1}{x}}{2}} \\
&= \frac{1}{\sqrt{x}} \qquad (5)
\end{aligned}
$$

Hence (3) becomes

$$
y = \frac{v(x)}{\sqrt{x}} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
x^{3/2}(v''(x) + v(x)) = 0
$$

Which is now solved for $v(x)$.

The above ode can be simplified to

$$
v''(x) + v(x) = 0
$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Av''(x) + Bv'(x) + Cv(x) = 0
$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_3 \cos(\beta x) + c_4 \sin(\beta x))$$

Which becomes

$$v(x) = e^0(c_3 \cos(x) + c_4 \sin(x))$$

Or

$$v(x) = c_3 \cos(x) + c_4 \sin(x)$$

Will add steps showing solving for IC soon.

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= (c_3 \cos(x) + c_4 \sin(x))\,(z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \frac{1}{\sqrt{x}}$$

Hence (7) becomes

$$y = \frac{c_3 \cos(x) + c_4 \sin(x)}{\sqrt{x}}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{c_3 \cos(x) + c_4 \sin(x)}{\sqrt{x}}$$

Solved as second order Bessel ode

Time used: 0.053 (sec)

Writing the ode as

$$x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 0 \tag{1}$$

Bessel ode has the form

$$x^2 y'' + x y' + \left(-n^2 + x^2\right) y = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2 y'' + (1 - 2\alpha)\, x y' + \left(\beta^2 \gamma^2 x^{2\gamma} - n^2 \gamma^2 + \alpha^2\right) y = 0 \tag{3}$$

With the standard solution

$$y = x^{\alpha} (c_1 \operatorname{BesselJ}(n, \beta\, x^{\gamma}) + c_2 \operatorname{BesselY}(n, \beta\, x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$
\begin{aligned}
\alpha &= 0 \\
\beta &= 1 \\
n &= -\frac{1}{2} \\
\gamma &= 1
\end{aligned}
$$

Substituting all the above into (4) gives the solution as

$$
y = \frac{c_1 \sqrt{2}\, \cos\left(x\right)}{\sqrt{\pi}\, \sqrt{x}} + \frac{c_2 \sqrt{2}\, \sin\left(x\right)}{\sqrt{\pi}\, \sqrt{x}}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = \frac{c_1 \sqrt{2}\, \cos\left(x\right)}{\sqrt{\pi}\, \sqrt{x}} + \frac{c_2 \sqrt{2}\, \sin\left(x\right)}{\sqrt{\pi}\, \sqrt{x}}
$$

Solved as second order ode using Kovacic algorithm

Time used: 0.112 (sec)

Writing the ode as

$$
x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 0 \tag{1}
$$

$$
A y'' + B y' + C y = 0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= x^2 \\
B &= x \\
C &= x^2 - \frac{1}{4}
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = y e^{\int \frac{B}{2A}\, dx}
$$

Then (2) becomes

$$
z''(x) = r z(x) \tag{4}
$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.256: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx}\\ &= z_1 e^{-\frac{\ln(x)}{2}}\\ &= z_1\left(\frac{1}{\sqrt{x}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\cos(x)}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx\\ &= y_1(\tan(x))\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{\cos(x)}{\sqrt{x}}\right) + c_2 \left(\frac{\cos(x)}{\sqrt{x}} (\tan(x))\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = \frac{c_1 \cos(x)}{\sqrt{x}} + \frac{c_2 \sin(x)}{\sqrt{x}}
$$

Solved as second order ode adjoint method

Time used: 0.952 (sec)

In normal form the ode

$$
x^2 y'' + x y' + \left(x^2 - \frac{1}{4}\right) y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= \frac{x^2 - \frac{1}{4}}{x^2} \\
r(x) &= 0
\end{aligned}
$$

The Lagrange adjoint ode is given by

$$
\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - \left(\frac{\xi(x)}{x}\right)' + \left(\frac{\left(x^2 - \frac{1}{4}\right) \xi(x)}{x^2}\right) &= 0 \\
\frac{4\xi''(x)\, x^2 + 4\xi(x)\, x^2 - 4\xi'(x)\, x + 3\xi(x)}{4x^2} &= 0
\end{aligned}
$$

Which is solved for $\xi(x)$. In normal form the given ode is written as

$$\xi'' + p(x)\,\xi' + q(x)\,\xi = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= 1 + \frac{3}{4x^2} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= 1 + \frac{3}{4x^2} - \frac{\left(-\frac{1}{x}\right)'}{2} - \frac{\left(-\frac{1}{x}\right)^2}{4} \\ &= 1 + \frac{3}{4x^2} - \frac{\left(\frac{1}{x^2}\right)}{2} - \frac{\left(\frac{1}{x^2}\right)}{4} \\ &= 1 + \frac{3}{4x^2} - \left(\frac{1}{2x^2}\right) - \frac{1}{4x^2} \\ &= 1 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$\xi = v(x)\,z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\int \frac{p(x)}{2}dx} \\ &= e^{-\int \frac{-\frac{1}{x}}{2}} \\ &= \sqrt{x} \end{aligned} \tag{5}$$

Hence (3) becomes

$$\xi = v(x)\sqrt{x} \tag{4}$$

Applying this change of variable to the original ode results in

$$\sqrt{x}\,(v''(x) + v(x)) = 0$$

Which is now solved for $v(x)$.

The above ode can be simplified to

$$v''(x) + v(x) = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Av''(x) + Bv'(x) + Cv(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $v(x) = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$v(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$v(x) = e^{0}(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$v(x) = c_1 \cos(x) + c_2 \sin(x)$$

Will add steps showing solving for IC soon.

Now that $v(x)$ is known, then

$$\begin{aligned} \xi &= v(x)\, z(x) \\ &= (c_1 \cos(x) + c_2 \sin(x))\,(z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \sqrt{x}$$

Hence (7) becomes

$$\xi = (c_1 \cos(x) + c_2 \sin(x))\,\sqrt{x}$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned} \xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\ y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)} \end{aligned}$$

Or

$$y' + y\left(\frac{1}{x} - \frac{(-c_1 \sin(x) + c_2 \cos(x))\,\sqrt{x} + \frac{c_1 \cos(x) + c_2 \sin(x)}{2\sqrt{x}}}{(c_1 \cos(x) + c_2 \sin(x))\,\sqrt{x}}\right) = 0$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= -\frac{(-2c_1x - c_2)\sin(x) + 2\cos(x)\left(c_2x - \frac{c_1}{2}\right)}{2x\,(c_1 \cos(x) + c_2 \sin(x))} \\ p(x) &= 0 \end{aligned}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= \mathrm{e}^{\int -\frac{(-2c_1x-c_2)\sin(x)+2\cos(x)\left(c_2x-\frac{c_1}{2}\right)}{2x(c_1\cos(x)+c_2\sin(x))}dx} \\
&= \frac{\sqrt{x}\,\sec\left(\frac{x}{2}\right)^2}{c_1\tan\left(\frac{x}{2}\right)^2-2c_2\tan\left(\frac{x}{2}\right)-c_1}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y\sqrt{x}\,\sec\left(\frac{x}{2}\right)^2}{c_1\tan\left(\frac{x}{2}\right)^2-2c_2\tan\left(\frac{x}{2}\right)-c_1}\right) &= 0
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y\sqrt{x}\,\sec\left(\frac{x}{2}\right)^2}{c_1\tan\left(\frac{x}{2}\right)^2-2c_2\tan\left(\frac{x}{2}\right)-c_1} &= \int 0\,dx + c_3 \\
&= c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{\sqrt{x}\,\sec\left(\frac{x}{2}\right)^2}{c_1\tan\left(\frac{x}{2}\right)^2-2c_2\tan\left(\frac{x}{2}\right)-c_1}$ gives the final solution

$$
y = \frac{c_3(-c_2\sin(x)-c_1\cos(x))}{\sqrt{x}}
$$

Hence, the solution found using Lagrange adjoint equation method is

$$
y = \frac{c_3(-c_2\sin(x)-c_1\cos(x))}{\sqrt{x}}
$$

The constants can be merged to give

$$
y = \frac{-c_2\sin(x)-c_1\cos(x)}{\sqrt{x}}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = \frac{-c_2\sin(x)-c_1\cos(x)}{\sqrt{x}}
$$

Maple step by step solution

Let's solve

$x^2\left(\frac{d^2}{dx^2}y(x)\right)+x\left(\frac{d}{dx}y(x)\right)+\left(x^2-\frac{1}{4}\right)y(x)=0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=-\frac{(4x^2-1)y(x)}{4x^2}-\frac{\frac{d}{dx}y(x)}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+\frac{\frac{d}{dx}y(x)}{x}+\frac{(4x^2-1)y(x)}{4x^2}=0$

□ Check to see if $x_0=0$ is a regular singular point

  - Define functions

    $\left[P_2(x)=\frac{1}{x},P_3(x)=\frac{4x^2-1}{4x^2}\right]$

  - $x\cdot P_2(x)$ is analytic at $x=0$

    $\left.(x\cdot P_2(x))\right|_{x=0}=1$

  - $x^2\cdot P_3(x)$ is analytic at $x=0$

    $\left.(x^2\cdot P_3(x))\right|_{x=0}=-\frac{1}{4}$

  - $x=0$is a regular singular point

  Check to see if $x_0=0$ is a regular singular point

  $x_0=0$

- Multiply by denominators

  $4x^2\left(\frac{d^2}{dx^2}y(x)\right)+4x\left(\frac{d}{dx}y(x)\right)+(4x^2-1)y(x)=0$

- Assume series solution for $y(x)$

  $y(x)=\sum_{k=0}^{\infty}a_kx^{k+r}$

□ Rewrite ODE with series expansions

  - Convert $x^m\cdot y(x)$ to series expansion for $m=0..2$

    $x^m\cdot y(x)=\sum_{k=0}^{\infty}a_kx^{k+r+m}$

  - Shift index using $k->k-m$

    $x^m\cdot y(x)=\sum_{k=m}^{\infty}a_{k-m}x^{k+r}$

- Convert $x \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion

  $x \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\, x^{k+r}$

- Convert $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion

  $x^2 \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k(k+r)\,(k+r-1)\, x^{k+r}$

  Rewrite ODE with series expansions

  $a_0(1+2r)\,(-1+2r)\, x^r + a_1(3+2r)\,(1+2r)\, x^{1+r} + \left(\sum_{k=2}^{\infty} (a_k(2k+2r+1)\,(2k+2r-1) + 4a_{k-}\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation

  $(1+2r)\,(-1+2r) = 0$

- Values of r that satisfy the indicial equation

  $r \in \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

- Each term must be 0

  $a_1(3+2r)\,(1+2r) = 0$

- Solve for the dependent coefficient(s)

  $a_1 = 0$

- Each term in the series must be 0, giving the recursion relation

  $a_k(4k^2 + 8kr + 4r^2 - 1) + 4a_{k-2} = 0$

- Shift index using $k- >k+2$

  $a_{k+2}\left(4(k+2)^2 + 8(k+2)\, r + 4r^2 - 1\right) + 4a_k = 0$

- Recursion relation that defines series solution to ODE

  $a_{k+2} = -\frac{4a_k}{4k^2+8kr+4r^2+16k+16r+15}$

- Recursion relation for $r = -\frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+12k+8}$

- Solution for $r = -\frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k-\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+12k+8}, a_1 = 0\right]$

- Recursion relation for $r = \frac{1}{2}$

  $a_{k+2} = -\frac{4a_k}{4k^2+20k+24}$

- Solution for $r = \frac{1}{2}$

  $\left[y(x) = \sum_{k=0}^{\infty} a_k x^{k+\frac{1}{2}}, a_{k+2} = -\frac{4a_k}{4k^2+20k+24}, a_1 = 0\right]$

- Combine solutions and rename parameters

$$\left[y(x)=\left(\sum_{k=0}^{\infty}a_k x^{k-\frac{1}{2}}\right)+\left(\sum_{k=0}^{\infty}b_k x^{k+\frac{1}{2}}\right), a_{k+2}=-\frac{4a_k}{4k^2+12k+8}, a_1=0, b_{k+2}=-\frac{4b_k}{4k^2+20k+24}, b_1=0\right]$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Group is reducible or imprimitive
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : 0.050 (sec)
Leaf size : 17

```
dsolve(x^2*diff(diff(y(x),x),x)+diff(y(x),x)*x+(x^2-1/4)*y(x) = 0,
       y(x),singsol=all)
```

$$y=\frac{\sin(x)c_1+\cos(x)c_2}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.039 (sec)
Leaf size : 39

```
DSolve[{x^2*D[y[x],{x,2}]+x*D[y[x],x]+(x^2-1/4)*y[x] == 0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to\frac{e^{-ix}(2c_1-ic_2e^{2ix})}{2\sqrt{x}}$$

2.5.5 problem 5

Internal problem ID [8642]
Book : Own collection of miscellaneous problems
Section : section 5.0
Problem number : 5
Date solved : `Thursday, December 12, 2024 at 09:37:15 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$4x^2y'' + \left(-8x^2+4x\right)y' + \left(4x^2-4x-1\right)y = 4\sqrt{x}\,\mathrm{e}^x$$

Solved as second order solved by an integrating factor

Time used: 0.055 (sec)

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = \frac{-2x+1}{x}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int \frac{-2x+1}{x}\,dx} \\ &= \sqrt{x}\,\mathrm{e}^{-x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = \frac{\mathrm{e}^{-x}\mathrm{e}^{x}}{x}$$

$$\left(\sqrt{x}\,\mathrm{e}^{-x}y\right)'' = \frac{\mathrm{e}^{-x}\mathrm{e}^{x}}{x}$$

Integrating once gives

$$\left(\sqrt{x}\,\mathrm{e}^{-x}y\right)' = \ln(x) + c_1$$

Integrating again gives

$$\left(\sqrt{x}\,\mathrm{e}^{-x}y\right) = x(\ln(x) + c_1 - 1) + c_2$$

Hence the solution is

$$y = \frac{x(\ln(x) + c_1 - 1) + c_2}{\sqrt{x}\,\mathrm{e}^{-x}}$$

Or

$$y = c_1\sqrt{x}\,\mathrm{e}^{x} + \sqrt{x}\,\mathrm{e}^{x}\ln(x) + \frac{c_2\,\mathrm{e}^{x}}{\sqrt{x}} - \sqrt{x}\,\mathrm{e}^{x}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1\sqrt{x}\,\mathrm{e}^{x} + \sqrt{x}\,\mathrm{e}^{x}\ln(x) + \frac{c_2\,\mathrm{e}^{x}}{\sqrt{x}} - \sqrt{x}\,\mathrm{e}^{x}$$

Solved as second order ode using change of variable on y method 1

Time used: 0.445 (sec)

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$4x^2y'' + \left(-8x^2 + 4x\right)y' + \left(4x^2 - 4x - 1\right)y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{-8x^2 + 4x}{4x^2} \\ q(x) &= \frac{4x^2 - 4x - 1}{4x^2} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= \frac{4x^2-4x-1}{4x^2} - \frac{\left(\frac{-8x^2+4x}{4x^2}\right)'}{2} - \frac{\left(\frac{-8x^2+4x}{4x^2}\right)^2}{4} \\
&= \frac{4x^2-4x-1}{4x^2} - \frac{\left(\frac{-16x+4}{4x^2} - \frac{-8x^2+4x}{2x^3}\right)}{2} - \frac{\left(\frac{(-8x^2+4x)^2}{16x^4}\right)}{4} \\
&= \frac{4x^2-4x-1}{4x^2} - \left(\frac{-16x+4}{8x^2} - \frac{-8x^2+4x}{4x^3}\right) - \frac{(-8x^2+4x)^2}{64x^4} \\
&= 0
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
y = v(x)\, z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$
\begin{aligned}
z(x) &= \mathrm{e}^{-\int \frac{p(x)}{2} dx} \\
&= e^{-\int \frac{\frac{-8x^2+4x}{4x^2}}{2}} \\
&= \frac{\mathrm{e}^x}{\sqrt{x}}
\end{aligned} \tag{5}
$$

Hence (3) becomes

$$
y = \frac{v(x)\, \mathrm{e}^x}{\sqrt{x}} \tag{4}
$$

Applying this change of variable to the original ode results in

$$
4\,\mathrm{e}^x v''(x)\, x^{3/2} = 0
$$

Which is now solved for $v(x)$.

The above ode can be simplified to

$$
4v''(x) = 0
$$

The ODE simplifies to

$$
v''(x) = 0
$$

Integrating twice gives the solution

$$v(x) = c_1 x + c_2$$

Will add steps showing solving for IC soon.

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= (c_1 x + c_2)\,(z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \frac{e^x}{\sqrt{x}}$$

Hence (7) becomes

$$y = \frac{(c_1 x + c_2)\, e^x}{\sqrt{x}}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{(c_1 x + c_2)\, e^x}{\sqrt{x}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{e^x}{\sqrt{x}}$$

$$y_2 = \sqrt{x}\, e^x$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{e^x}{\sqrt{x}} & \sqrt{x}\, e^x \\ \frac{d}{dx}\left(\frac{e^x}{\sqrt{x}}\right) & \frac{d}{dx}\left(\sqrt{x}\, e^x\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{e^x}{\sqrt{x}} & \sqrt{x}\, e^x \\ -\frac{e^x}{2x^{3/2}} + \frac{e^x}{\sqrt{x}} & \frac{e^x}{2\sqrt{x}} + \sqrt{x}\, e^x \end{vmatrix}$$

Therefore

$$W = \left(\frac{e^x}{\sqrt{x}}\right)\left(\frac{e^x}{2\sqrt{x}} + \sqrt{x}\, e^x\right) - \left(\sqrt{x}\, e^x\right)\left(-\frac{e^x}{2x^{3/2}} + \frac{e^x}{\sqrt{x}}\right)$$

Which simplifies to

$$W = \frac{e^{2x}}{x}$$

Which simplifies to

$$W = \frac{e^{2x}}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4x\, e^{2x}}{4x\, e^{2x}}\, dx$$

Which simplifies to

$$u_1 = -\int 1 dx$$

Hence

$$u_1 = -x$$

And Eq. (3) becomes

$$u_2 = \int \frac{4\, e^{2x}}{4x\, e^{2x}}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x} dx$$

Hence

$$u_2 = \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\sqrt{x}\, e^x + \sqrt{x}\, e^x \ln(x)$$

Which simplifies to

$$y_p(x) = \sqrt{x}\, e^x (-1 + \ln(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{(c_1 x + c_2)\, e^x}{\sqrt{x}}\right) + \left(\sqrt{x}\, e^x (-1 + \ln(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{(c_1 x + c_2)\, e^x}{\sqrt{x}} + \sqrt{x}\, e^x (-1 + \ln(x))$$

Solved as second order ode using Kovacic algorithm

Time used: 0.214 (sec)

Writing the ode as

$$4x^2 y'' + \left(-8x^2 + 4x\right) y' + \left(4x^2 - 4x - 1\right) y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -8x^2 + 4x \\ C &= 4x^2 - 4x - 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.258: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - -\infty\\ &= \infty\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-8x^2+4x}{4x^2}\,dx}\\ &= z_1 e^{x-\frac{\ln(x)}{2}}\\ &= z_1\left(\frac{\mathrm{e}^x}{\sqrt{x}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \frac{\mathrm{e}^x}{\sqrt{x}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-8x^2+4x}{4x^2}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{2x-\ln(x)}}{(y_1)^2}\,dx\\ &= y_1(x)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{e^x}{\sqrt{x}}\right) + c_2\left(\frac{e^x}{\sqrt{x}}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
4x^2y'' + \left(-8x^2+4x\right)y' + \left(4x^2-4x-1\right)y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \frac{c_1 e^x}{\sqrt{x}} + c_2\sqrt{x}\, e^x
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \frac{e^x}{\sqrt{x}} \\
y_2 &= \sqrt{x}\, e^x
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{e^x}{\sqrt{x}} & \sqrt{x}\,e^x \\ \frac{d}{dx}\left(\frac{e^x}{\sqrt{x}}\right) & \frac{d}{dx}\left(\sqrt{x}\,e^x\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{e^x}{\sqrt{x}} & \sqrt{x}\,e^x \\ -\frac{e^x}{2x^{3/2}} + \frac{e^x}{\sqrt{x}} & \frac{e^x}{2\sqrt{x}} + \sqrt{x}\,e^x \end{vmatrix}$$

Therefore

$$W = \left(\frac{e^x}{\sqrt{x}}\right)\left(\frac{e^x}{2\sqrt{x}} + \sqrt{x}\,e^x\right) - \left(\sqrt{x}\,e^x\right)\left(-\frac{e^x}{2x^{3/2}} + \frac{e^x}{\sqrt{x}}\right)$$

Which simplifies to

$$W = \frac{e^{2x}}{x}$$

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$$W = \frac{e^{2x}}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{4x\,e^{2x}}{4x\,e^{2x}}\,dx$$

Which simplifies to

$$u_1 = -\int 1 dx$$

Hence

$$u_1 = -x$$

And Eq. (3) becomes

$$u_2 = \int \frac{4\,e^{2x}}{4x\,e^{2x}}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x} dx$$

Hence

$$u_2 = \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\sqrt{x}\, e^x + \sqrt{x}\, e^x \ln(x)$$

Which simplifies to

$$y_p(x) = \sqrt{x}\, e^x (-1 + \ln(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 e^x}{\sqrt{x}} + c_2 \sqrt{x}\, e^x\right) + \left(\sqrt{x}\, e^x (-1 + \ln(x))\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{c_1 e^x}{\sqrt{x}} + c_2 \sqrt{x}\, e^x + \sqrt{x}\, e^x (-1 + \ln(x))$$

Solved as second order ode adjoint method

Time used: 0.291 (sec)

In normal form the ode

$$4x^2 y'' + \left(-8x^2 + 4x\right) y' + \left(4x^2 - 4x - 1\right) y = 4\sqrt{x}\, e^x \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= \frac{-2x+1}{x} \\
q(x) &= \frac{4x^2-4x-1}{4x^2} \\
r(x) &= \frac{\mathrm{e}^x}{x^{3/2}}
\end{aligned}
$$

The Lagrange adjoint ode is given by

$$
\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - \left(\frac{(-2x+1)\,\xi(x)}{x}\right)' + \left(\frac{(4x^2-4x-1)\,\xi(x)}{4x^2}\right) &= 0 \\
\frac{\xi''(x)\,x^2 + (2x^2-x)\,\xi'(x) + \xi(x)\left(-x+\frac{3}{4}+x^2\right)}{x^2} &= 0
\end{aligned}
$$

Which is solved for $\xi(x)$. In normal form the given ode is written as

$$
\xi'' + p(x)\,\xi' + q(x)\,\xi = 0 \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= 2 - \frac{1}{x} \\
q(x) &= -\frac{1}{x} + \frac{3}{4x^2} + 1
\end{aligned}
$$

Calculating the Liouville ode invariant Q given by

$$
\begin{aligned}
Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\
&= -\frac{1}{x} + \frac{3}{4x^2} + 1 - \frac{\left(2-\frac{1}{x}\right)'}{2} - \frac{\left(2-\frac{1}{x}\right)^2}{4} \\
&= -\frac{1}{x} + \frac{3}{4x^2} + 1 - \frac{\left(\frac{1}{x^2}\right)}{2} - \frac{\left(\left(2-\frac{1}{x}\right)^2\right)}{4} \\
&= -\frac{1}{x} + \frac{3}{4x^2} + 1 - \left(\frac{1}{2x^2}\right) - \frac{\left(2-\frac{1}{x}\right)^2}{4} \\
&= 0
\end{aligned}
$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$
\xi = v(x)\,z(x) \tag{3}
$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\int \frac{p(x)}{2} dx} \\ &= e^{-\int \frac{2-\frac{1}{x}}{2}} \\ &= \sqrt{x}\, \mathrm{e}^{-x} \end{aligned} \tag{5}$$

Hence (3) becomes

$$\xi = v(x) \sqrt{x}\, \mathrm{e}^{-x} \tag{4}$$

Applying this change of variable to the original ode results in

$$v''(x) \sqrt{x}\, \mathrm{e}^{-x} = 0$$

Which is now solved for $v(x)$.

The above ode can be simplified to

$$v''(x) = 0$$

Integrating twice gives the solution

$$v(x) = c_1 x + c_2$$

Will add steps showing solving for IC soon.

Now that $v(x)$ is known, then

$$\begin{aligned} \xi &= v(x)\, z(x) \\ &= (c_1 x + c_2)\, (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \sqrt{x}\, \mathrm{e}^{-x}$$

Hence (7) becomes

$$\xi = (c_1 x + c_2) \sqrt{x}\, \mathrm{e}^{-x}$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned} \xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\ y' + y\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi(x)} \end{aligned}$$

Or

$$y' + y\left(\frac{-2x+1}{x} - \frac{\left(c_1\sqrt{x}\,\mathrm{e}^{-x} + \frac{(c_1x+c_2)\mathrm{e}^{-x}}{2\sqrt{x}} - (c_1x+c_2)\sqrt{x}\,\mathrm{e}^{-x}\right)\mathrm{e}^{x}}{(c_1x+c_2)\sqrt{x}}\right) = \frac{\mathrm{e}^{x}(c_1x + c_2\ln(x))}{(c_1x+c_2)\sqrt{x}}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= -\frac{2c_1\,x^2 + (c_1+2c_2)\,x - c_2}{2x\,(c_1x+c_2)} \\ p(x) &= \frac{\mathrm{e}^{x}(c_1x + c_2\ln(x))}{(c_1x+c_2)\sqrt{x}} \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= \mathrm{e}^{\int -\frac{2c_1\,x^2+(c_1+2c_2)x-c_2}{2x(c_1x+c_2)}dx} \\ &= \frac{\sqrt{x}\,\mathrm{e}^{-x}}{c_1x+c_2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{\mathrm{e}^{x}(c_1x + c_2\ln(x))}{(c_1x+c_2)\sqrt{x}}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y\sqrt{x}\,\mathrm{e}^{-x}}{c_1x+c_2}\right) &= \left(\frac{\sqrt{x}\,\mathrm{e}^{-x}}{c_1x+c_2}\right)\left(\frac{\mathrm{e}^{x}(c_1x + c_2\ln(x))}{(c_1x+c_2)\sqrt{x}}\right) \\ \mathrm{d}\left(\frac{y\sqrt{x}\,\mathrm{e}^{-x}}{c_1x+c_2}\right) &= \left(\frac{\mathrm{e}^{x}(c_1x + c_2\ln(x))\,\mathrm{e}^{-x}}{(c_1x+c_2)^2}\right)\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{y\sqrt{x}\,\mathrm{e}^{-x}}{c_1x+c_2} &= \int \frac{\mathrm{e}^{x}(c_1x + c_2\ln(x))\,\mathrm{e}^{-x}}{(c_1x+c_2)^2}\,dx \\ &= \frac{c_2}{c_1\,(c_1x+c_2)} + \frac{\ln(x)\,x}{c_1x+c_2} + c_3 \end{aligned}$$

Dividing throughout by the integrating factor $\frac{\sqrt{x}\,\mathrm{e}^{-x}}{c_1 x+c_2}$ gives the final solution

$$y = \frac{(c_1^2 c_3 x + \ln(x)\, x c_1 + c_1 c_2 c_3 + c_2)\,\mathrm{e}^{x}}{c_1 \sqrt{x}}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{(c_1^2 c_3 x + \ln(x)\, x c_1 + c_1 c_2 c_3 + c_2)\,\mathrm{e}^{x}}{c_1 \sqrt{x}}$$

The constants can be merged to give

$$y = \frac{(c_1^2 x + \ln(x)\, x c_1 + c_1 c_2 + c_2)\,\mathrm{e}^{x}}{c_1 \sqrt{x}}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{(c_1^2 x + \ln(x)\, x c_1 + c_1 c_2 + c_2)\,\mathrm{e}^{x}}{c_1 \sqrt{x}}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.038 (sec)
Leaf size : 21

```
dsolve(4*x^2*diff(diff(y(x),x),x)+(-8*x^2+4*x)*diff(y(x),x)+(4*x^2-4*x-1)*y(x) = 4*x^(
       y(x),singsol=all)
```

$$y = \frac{(x \ln (x) + (c_1 - 1)\, x + c_2)\, \mathrm{e}^x}{\sqrt{x}}$$

Mathematica DSolve solution

Solving time : 0.046 (sec)
Leaf size : 27

```
DSolve[{4*x^2*D[y[x],{x,2}]+(-8*x^2+4*x)*D[y[x],x]+(4*x^2-4*x-1)*y[x] == 4*x^(1/2)*Exp[x],{}
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{e^x (x \log(x) + (-1 + c_2) x + c_1)}{\sqrt{x}}$$

2.5.6 problem 6

Internal problem ID [8643]
Book : Own collection of miscellaneous problems
Section : section 5.0
Problem number : 6
Date solved : `Thursday, December 12, 2024 at 09:37:17 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$xy'' - (2x+2)\,y' + (2+x)\,y = 6x^3\mathrm{e}^x$$

Solved as second order ode using Kovacic algorithm

Time used: 0.196 (sec)

Writing the ode as

$$xy'' + (-2x-2)\,y' + (2+x)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2x-2 \\ C &= 2+x \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 2 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.259: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = 2 \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -1 \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + (-)(0) \\
&= -\frac{1}{x} \\
&= -\frac{1}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{1}{x} dx} \\ &= \frac{1}{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-2x-2}{x} \, dx} \\ &= z_1 e^{x + \ln(x)} \\ &= z_1 (x \, \mathrm{e}^x) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-2}{x} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{2x + 2\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{x \, \mathrm{e}^{2x + 2\ln(x)} \mathrm{e}^{-2x}}{3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2\left(\mathrm{e}^x\left(\frac{x\,\mathrm{e}^{2x+2\ln(x)}\mathrm{e}^{-2x}}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
xy'' + (-2x - 2)\,y' + (2 + x)\,y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1\,\mathrm{e}^x + \frac{c_2\,x^3\mathrm{e}^x}{3}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^x \\
y_2 &= \frac{x^3\mathrm{e}^x}{3}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^x & \frac{x^3 e^x}{3} \\ \frac{d}{dx}(e^x) & \frac{d}{dx}\left(\frac{x^3 e^x}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^x & \frac{x^3 e^x}{3} \\ e^x & x^2 e^x + \frac{x^3 e^x}{3} \end{vmatrix}$$

Therefore

$$W = (e^x)\left(x^2 e^x + \frac{x^3 e^x}{3}\right) - \left(\frac{x^3 e^x}{3}\right)(e^x)$$

Which simplifies to

$$W = x^2 e^{2x}$$

Which simplifies to

$$W = x^2 e^{2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{2x^6 e^{2x}}{x^3 e^{2x}}\, dx$$

Which simplifies to

$$u_1 = -\int 2x^3 dx$$

Hence

$$u_1 = -\frac{x^4}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{6x^3 e^{2x}}{x^3 e^{2x}}\, dx$$

Which simplifies to

$$u_2 = \int 6dx$$

Hence

$$u_2 = 6x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{3x^4 e^x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^x + \frac{c_2 x^3 e^x}{3}\right) + \left(\frac{3x^4 e^x}{2}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1 e^x + \frac{c_2 x^3 e^x}{3} + \frac{3x^4 e^x}{2}$$

Solved as second order ode adjoint method

Time used: 0.628 (sec)

In normal form the ode

$$xy'' - (2x+2)\, y' + (2+x)\, y = 6x^3 e^x \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{-2x-2}{x} \\ q(x) &= \frac{2+x}{x} \\ r(x) &= 6x^2 e^x \end{aligned}$$

The Lagrange adjoint ode is given by

$$
\begin{aligned}
\xi'' - (\xi\, p)' + \xi q &= 0 \\
\xi'' - \left(\frac{(-2x-2)\,\xi(x)}{x}\right)' + \left(\frac{(2+x)\,\xi(x)}{x}\right) &= 0 \\
\frac{\xi''(x)\,x^2 + (2x^2+2x)\,\xi'(x) + \xi(x)\,(x^2+2x-2)}{x^2} &= 0
\end{aligned}
$$

Which is solved for $\xi(x)$. Writing the ode as

$$
\begin{aligned}
\xi'' + \frac{(2x+2)\,\xi'}{x} + \frac{(x^2+2x-2)\,\xi}{x^2} &= 0 && (1)\\
A\xi'' + B\xi' + C\xi &= 0 && (2)
\end{aligned}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= \frac{2x+2}{x} \\
C &= \frac{x^2+2x-2}{x^2}
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = \xi e^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{2}{x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 2 \\
t &= x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{2}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then ξ is found using the inverse transformation

$$\xi = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.260: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{2}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{2}{x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 2$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 2 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -1
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{2}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	2	-1

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	2	-1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -1$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -1 - (-1) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{x} + (-)\,(0) \\
&= -\frac{1}{x} \\
&= -\frac{1}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(-\frac{1}{x}\right)(0) + \left(\left(\frac{1}{x^2}\right) + \left(-\frac{1}{x}\right)^2 - \left(\frac{2}{x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int -\frac{1}{x} dx} \\
&= \frac{1}{x}
\end{aligned}$$

The first solution to the original ode in ξ is found from

$$
\begin{aligned}
\xi_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{\frac{2x+2}{x}}{1}\,dx} \\
&= z_1 e^{-x-\ln(x)} \\
&= z_1\left(\frac{\mathrm{e}^{-x}}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
\xi_1 = \frac{\mathrm{e}^{-x}}{x^2}
$$

The second solution ξ_2 to the original ode is found using reduction of order

$$
\xi_2 = \xi_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{\xi_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
\xi_2 &= \xi_1 \int \frac{e^{\int -\frac{\frac{2x+2}{x}}{1}\,dx}}{(\xi_1)^2}\,dx \\
&= \xi_1 \int \frac{e^{-2x-2\ln(x)}}{(\xi_1)^2}\,dx \\
&= \xi_1\left(\frac{x^5\mathrm{e}^{-2x-2\ln(x)}\mathrm{e}^{2x}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
\xi &= c_1\xi_1 + c_2\xi_2 \\
&= c_1\left(\frac{\mathrm{e}^{-x}}{x^2}\right) + c_2\left(\frac{\mathrm{e}^{-x}}{x^2}\left(\frac{x^5\mathrm{e}^{-2x-2\ln(x)}\mathrm{e}^{2x}}{3}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$
\begin{aligned}
\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi\,(x)}
\end{aligned}
$$

Or

$$
y' + y\left(\frac{-2x-2}{x} - \frac{-\frac{2c_1\,\mathrm{e}^{-x}}{x^3} - \frac{c_1\,\mathrm{e}^{-x}}{x^2} + \frac{c_2\,\mathrm{e}^{-x}}{3} - \frac{c_2 x\,\mathrm{e}^{-x}}{3}}{\frac{c_1\,\mathrm{e}^{-x}}{x^2} + \frac{c_2 x\,\mathrm{e}^{-x}}{3}}\right) = \frac{x(c_2\,x^3 + 12c_1)}{\frac{2c_1\,\mathrm{e}^{-x}}{x^2} + \frac{2c_2 x\,\mathrm{e}^{-x}}{3}}
$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$
y' + q(x)y = p(x)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= -\frac{x^2(x+3)\,c_2 + 3c_1}{c_2\,x^3 + 3c_1} \\
p(x) &= \frac{3x^3(c_2\,x^3 + 12c_1)\,\mathrm{e}^x}{2c_2\,x^3 + 6c_1}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= \mathrm{e}^{\int -\frac{x^2(x+3)c_2+3c_1}{c_2\,x^3+3c_1}dx} \\
&= \frac{\mathrm{e}^{-x}}{c_2\,x^3 + 3c_1}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{3x^3(c_2\,x^3 + 12c_1)\,\mathrm{e}^x}{2c_2\,x^3 + 6c_1}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y\,\mathrm{e}^{-x}}{c_2\,x^3 + 3c_1}\right) &= \left(\frac{\mathrm{e}^{-x}}{c_2\,x^3 + 3c_1}\right)\left(\frac{3x^3(c_2\,x^3 + 12c_1)\,\mathrm{e}^x}{2c_2\,x^3 + 6c_1}\right) \\
\mathrm{d}\left(\frac{y\,\mathrm{e}^{-x}}{c_2\,x^3 + 3c_1}\right) &= \left(\frac{3x^3(c_2\,x^3 + 12c_1)\,\mathrm{e}^x\mathrm{e}^{-x}}{(2c_2\,x^3 + 6c_1)\,(c_2\,x^3 + 3c_1)}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$\begin{aligned}
\frac{y\,\mathrm{e}^{-x}}{c_2\,x^3+3c_1} &= \int \frac{3x^3(c_2\,x^3+12c_1)\,\mathrm{e}^{x}\mathrm{e}^{-x}}{(2c_2\,x^3+6c_1)\,(c_2\,x^3+3c_1)}\,dx \\
&= \frac{3x^4}{2\,(c_2\,x^3+3c_1)}+c_3
\end{aligned}$$

Dividing throughout by the integrating factor $\frac{\mathrm{e}^{-x}}{c_2\,x^3+3c_1}$ gives the final solution

$$y = \frac{((2c_2\,x^3+6c_1)\,c_3+3x^4)\,\mathrm{e}^{x}}{2}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \frac{((2c_2\,x^3+6c_1)\,c_3+3x^4)\,\mathrm{e}^{x}}{2}$$

The constants can be merged to give

$$y = \frac{(2c_2\,x^3+3x^4+6c_1)\,\mathrm{e}^{x}}{2}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \frac{(2c_2\,x^3+3x^4+6c_1)\,\mathrm{e}^{x}}{2}$$

Maple step by step solution

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
```

```
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.010 (sec)
Leaf size : 19

```
dsolve(x*diff(diff(y(x),x),x)-(2+2*x)*diff(y(x),x)+(x+2)*y(x) = 6*exp(x)*x^3,
       y(x),singsol=all)
```

$$y = e^{x}\left(c_2 + c_1 x^3 + \frac{3}{2}x^4\right)$$

Mathematica DSolve solution

Solving time : 0.038 (sec)
Leaf size : 29

```
DSolve[{x*D[y[x],{x,2}]-(2*x+2)*D[y[x],x]+(2+x)*y[x] == 6*x^3*Exp[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{6}e^{x}\left(9x^4 + 2c_2 x^3 + 6c_1\right)$$

2.5.7 problem 7

Internal problem ID [8644]
Book : Own collection of miscellaneous problems
Section : section 5.0
Problem number : 7
Date solved : `Tuesday, December 17, 2024 at 12:56:47 PM`
CAS classification : `[[_linear, ‘class A‘]]`

Solve

$$y' + y = \frac{1}{x}$$

Using series expansion around $x = 0$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $y' + y = 0$,and y_p is a particular solution to the inhomogeneous ode. First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2) this gives

$$(n+r)\, a_n x^{n+r-1} = 0$$

When $n=0$ the above becomes

$$r a_0 x^{-1+r} = 0$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$m c_0 x^{-1+m} = \frac{1}{x}$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r} = 0$$

Since the above is true for all x then the indicial equation simplifies to

$$r = 0$$

Solving for r gives the root of the indicial equation as

$$r = 0$$

For $1 \leq n$, the recurrence equation is

$$a_n(n+r) + a_{n-1} = 0 \tag{4}$$

For $n = 1$ the recurrence equation gives

$$a_1(1 + r) + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_1 = -\frac{a_0}{1 + r}$$

For $n = 2$ the recurrence equation gives

$$a_2(2 + r) + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = \frac{a_0}{(1 + r)(2 + r)}$$

For $n = 3$ the recurrence equation gives

$$a_3(3 + r) + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_0}{(1 + r)(2 + r)(3 + r)}$$

For $n = 4$ the recurrence equation gives

$$a_4(4 + r) + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{(1 + r)(2 + r)(3 + r)(4 + r)}$$

For $n = 5$ the recurrence equation gives

$$a_5(5 + r) + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_0}{(1+r)(2+r)(3+r)(4+r)(5+r)}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\ &= a_0 x^r + a_1 x^{1+r} + a_2 x^{2+r} + a_3 x^{3+r} + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y = a_0 x^r &- \frac{a_0 x^{1+r}}{1+r} + \frac{a_0 x^{2+r}}{(1+r)(2+r)} - \frac{a_0 x^{3+r}}{(1+r)(2+r)(3+r)} \\ &+ \frac{a_0 x^{4+r}}{(1+r)(2+r)(3+r)(4+r)} - \frac{a_0 x^{5+r}}{(1+r)(2+r)(3+r)(4+r)(5+r)} + \dots \end{aligned}$$

Which can be written as

$$\begin{aligned} y = x^r \Bigg(a_0 &- \frac{a_0 x}{1+r} + \frac{a_0 x^2}{(1+r)(2+r)} - \frac{a_0 x^3}{(1+r)(2+r)(3+r)} \\ &+ \frac{a_0 x^4}{(1+r)(2+r)(3+r)(4+r)} - \frac{a_0 x^5}{(1+r)(2+r)(3+r)(4+r)(5+r)} \\ &+ O(x^6)\, a_0 \Bigg) \end{aligned}$$

Collecting terms, the solution becomes

$$y = x^r \left(1 - \frac{x}{1+r} + \frac{x^2}{(1+r)(2+r)} - \frac{x^3}{(1+r)(2+r)(3+r)} + \frac{x^4}{(1+r)(2+r)(3+r)(4+r)} - \frac{}{(1+r)(} \right. \tag{3}$$

Finally, since $r = 0$, then the solution becomes

$$y = \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6)\right) a_0 \tag{3}$$

Unable to solve the balance equation $m c_0 x^{-1+m} = \frac{1}{x}$ for c_0 and x. No particular solution exists.

Unable to find the particular solution. No solution exist.

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) + y(x) = \frac{1}{x}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = -y(x) + \frac{1}{x}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $\frac{d}{dx}y(x) + y(x) = \frac{1}{x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(\frac{d}{dx}y(x) + y(x)\right) = \frac{\mu(x)}{x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\,\mu(x))$

  $\mu(x)\left(\frac{d}{dx}y(x) + y(x)\right) = \left(\frac{d}{dx}y(x)\right)\mu(x) + y(x)\left(\frac{d}{dx}\mu(x)\right)$

- Isolate $\frac{d}{dx}\mu(x)$

  $\frac{d}{dx}\mu(x) = \mu(x)$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(y(x)\,\mu(x))\right) dx = \int \frac{\mu(x)}{x} dx + C1$

- Evaluate the integral on the lhs

  $y(x)\,\mu(x) = \int \frac{\mu(x)}{x} dx + C1$

- Solve for $y(x)$

  $y(x) = \frac{\int \frac{\mu(x)}{x} dx + C1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{x}$

  $y(x) = \frac{\int \frac{\mathrm{e}^{x}}{x} dx + C1}{\mathrm{e}^{x}}$

- Evaluate the integrals on the rhs

  $y(x) = \frac{-\mathrm{Ei}_1(-x) + C1}{\mathrm{e}^{x}}$

- Simplify

  $y(x) = \mathrm{e}^{-x}(-\mathrm{Ei}_1(-x) + C1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.036 (sec)
Leaf size : maple_leaf_size

```
dsolve(diff(y(x),x)+y(x) = 1/x,y(x),
       series,x=0)
```

No solution found

Mathematica DSolve solution

Solving time : 0.017 (sec)
Leaf size : 113

```
AsymptoticDSolveValue[{D[y[x],x]+y[x]==1/x,{}},
       y[x],{x,0,5}]
```

$$y(x) \to \left(-\frac{x^5}{120}+\frac{x^4}{24}-\frac{x^3}{6}+\frac{x^2}{2}-x+1\right)\left(\frac{x^6}{2160}+\frac{x^5}{600}+\frac{x^4}{96}+\frac{x^3}{18}+\frac{x^2}{4}+x+\log(x)\right) + c_1\left(-\frac{x^5}{120}+\frac{x^4}{24}-\frac{x^3}{6}+\frac{x^2}{2}-x+1\right)$$

2.5.8 problem 8

Internal problem ID [8645]
Book : Own collection of miscellaneous problems
Section : section 5.0
Problem number : 8
Date solved : `Tuesday, December 17, 2024 at 12:56:47 PM`
CAS classification : `[[_linear, ‘class A‘]]`

Solve

$$y' + y = \frac{1}{x^2}$$

Using series expansion around $x = 0$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $y' + y = 0$,and y_p is a particular solution to the inhomogeneous ode. First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives

$$\sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}$$

Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=1}^{\infty} a_{n-1} x^{n+r-1}\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2) this gives

$$(n+r)\, a_n x^{n+r-1} = 0$$

When $n = 0$ the above becomes

$$r a_0 x^{-1+r} = 0$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$m c_0 x^{-1+m} = \frac{1}{x^2}$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r} = 0$$

Since the above is true for all x then the indicial equation simplifies to

$$r = 0$$

Solving for r gives the root of the indicial equation as

$$r = 0$$

For $1 \leq n$, the recurrence equation is

$$a_n(n+r) + a_{n-1} = 0 \tag{4}$$

For $n = 1$ the recurrence equation gives

$$a_1(1 + r) + a_0 = 0$$

Which after substituting the earlier terms found becomes

$$a_1 = -\frac{a_0}{1 + r}$$

For $n = 2$ the recurrence equation gives

$$a_2(2 + r) + a_1 = 0$$

Which after substituting the earlier terms found becomes

$$a_2 = \frac{a_0}{(1 + r)(2 + r)}$$

For $n = 3$ the recurrence equation gives

$$a_3(3 + r) + a_2 = 0$$

Which after substituting the earlier terms found becomes

$$a_3 = -\frac{a_0}{(1 + r)(2 + r)(3 + r)}$$

For $n = 4$ the recurrence equation gives

$$a_4(4 + r) + a_3 = 0$$

Which after substituting the earlier terms found becomes

$$a_4 = \frac{a_0}{(1 + r)(2 + r)(3 + r)(4 + r)}$$

For $n = 5$ the recurrence equation gives

$$a_5(5 + r) + a_4 = 0$$

Which after substituting the earlier terms found becomes

$$a_5 = -\frac{a_0}{(1+r)(2+r)(3+r)(4+r)(5+r)}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\ &= a_0 x^r + a_1 x^{1+r} + a_2 x^{2+r} + a_3 x^{3+r} + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y = a_0 x^r &- \frac{a_0 x^{1+r}}{1+r} + \frac{a_0 x^{2+r}}{(1+r)(2+r)} - \frac{a_0 x^{3+r}}{(1+r)(2+r)(3+r)} \\ &+ \frac{a_0 x^{4+r}}{(1+r)(2+r)(3+r)(4+r)} - \frac{a_0 x^{5+r}}{(1+r)(2+r)(3+r)(4+r)(5+r)} + \dots \end{aligned}$$

Which can be written as

$$\begin{aligned} y = x^r \Bigg(a_0 &- \frac{a_0 x}{1+r} + \frac{a_0 x^2}{(1+r)(2+r)} - \frac{a_0 x^3}{(1+r)(2+r)(3+r)} \\ &+ \frac{a_0 x^4}{(1+r)(2+r)(3+r)(4+r)} - \frac{a_0 x^5}{(1+r)(2+r)(3+r)(4+r)(5+r)} \\ &+ O(x^6)\, a_0 \Bigg) \end{aligned}$$

Collecting terms, the solution becomes

$$y = x^r \left(1 - \frac{x}{1+r} + \frac{x^2}{(1+r)(2+r)} - \frac{x^3}{(1+r)(2+r)(3+r)} + \frac{x^4}{(1+r)(2+r)(3+r)(4+r)} - \frac{}{(1+r)(} \right. \tag{3}$$

Finally, since $r = 0$, then the solution becomes

$$y = \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} + O(x^6) \right) a_0 \tag{3}$$

Now we determine the particular solution y_p by solving the balance equation

$$m c_0 x^{-1+m} = \frac{1}{x^2}$$

For c_0 and x. This results in

$$
\begin{aligned}
c_0 &= -1 \\
m &= -1
\end{aligned}
$$

The particular solution is therefore

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+m} \\
&= \sum_{n=0}^{\infty} c_n x^{n+-1}
\end{aligned}
$$

Where in the above $c_0 = -1$. The remaining c_n values are found using the same recurrence relation used to find the homogeneous solution but using c_0 in place of a_0 and using $m = -1$ in place of the root of the indicial equation used to find the homogeneous solution. The following are the values of a_n found in terms of the indicial root r. These will be now used to find find c_n by replacing $a_0 = -1$ and $r = -1$. The following table gives the a_n values found and the corresponding c_n values which will be used to find the particular solution

n	a_n	c_n
0	$a_0 = 1$	$c_0 = -1$

Unable to find particular solution .Unable to find the particular solution. No solution exist.

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) + y(x) = \frac{1}{x^2}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = -y(x) + \frac{1}{x^2}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $\frac{d}{dx}y(x) + y(x) = \frac{1}{x^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(\frac{d}{dx}y(x) + y(x)\right) = \frac{\mu(x)}{x^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\,\mu(x))$
  $\mu(x)\left(\frac{d}{dx}y(x)+y(x)\right)=\left(\frac{d}{dx}y(x)\right)\mu(x)+y(x)\left(\frac{d}{dx}\mu(x)\right)$
- Isolate $\frac{d}{dx}\mu(x)$
  $\frac{d}{dx}\mu(x)=\mu(x)$
- Solve to find the integrating factor
  $\mu(x)=\mathrm{e}^{x}$
- Integrate both sides with respect to x
  $\int\left(\frac{d}{dx}(y(x)\,\mu(x))\right)dx=\int\frac{\mu(x)}{x^2}dx+C1$
- Evaluate the integral on the lhs
  $y(x)\,\mu(x)=\int\frac{\mu(x)}{x^2}dx+C1$
- Solve for $y(x)$
  $y(x)=\frac{\int\frac{\mu(x)}{x^2}dx+C1}{\mu(x)}$
- Substitute $\mu(x)=\mathrm{e}^{x}$
  $y(x)=\frac{\int\frac{\mathrm{e}^{x}}{x^2}dx+C1}{\mathrm{e}^{x}}$
- Evaluate the integrals on the rhs
  $y(x)=\frac{-\frac{\mathrm{e}^{x}}{x}-\mathrm{Ei}_1(-x)+C1}{\mathrm{e}^{x}}$
- Simplify
  $y(x)=\frac{C1\,x\,\mathrm{e}^{-x}-\mathrm{e}^{-x}\mathrm{Ei}_1(-x)x-1}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.036 (sec)
Leaf size : maple_leaf_size

```
dsolve(diff(y(x),x)+y(x) = 1/x^2,y(x),
       series,x=0)
```

No solution found

Mathematica DSolve solution

Solving time : 0.015 (sec)
Leaf size : 122

```
AsymptoticDSolveValue[{D[y[x],x]+y[x]==1/x^2,{}},
       y[x],{x,0,5}]
```

$$y(x) \to \left(-\frac{x^5}{120}+\frac{x^4}{24}-\frac{x^3}{6}+\frac{x^2}{2}-x+1\right)\left(\frac{x^6}{2160}+\frac{x^5}{1800}+\frac{x^4}{480}+\frac{x^3}{72}+\frac{x^2}{12}+\frac{x}{2}-\frac{1}{x}+\log(x)\right) + c_1\left(-\frac{x^5}{120}+\frac{x^4}{24}-\frac{x^3}{6}+\frac{x^2}{2}-x+1\right)$$

2.5.9 problem 9

Internal problem ID [8646]
Book : Own collection of miscellaneous problems
Section : section 5.0
Problem number : 9
Date solved : `Tuesday, December 17, 2024 at 12:56:48 PM`
CAS classification : `[_separable]`

Solve

$$xy' + y = 0$$

Using series expansion around $x = 0$

Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

Substituting the above back into the ode gives

$$\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \frac{\sum_{n=0}^{\infty} a_n x^{n+r}}{x} = 0 \tag{1}$$

Which simplifies to

$$\left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n\right) = 0 \tag{2A}$$

The next step is to make all powers of x be $n + r - 1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\left(\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} x^{n+r-1} a_n\right) = 0 \tag{2B}$$

The indicial equation is obtained from $n=0$. From Eq (2) this gives

$$(n+r)\, a_n x^{n+r-1} + x^{n+r-1} a_n = 0$$

When $n=0$ the above becomes

$$r a_0 x^{-1+r} + x^{-1+r} a_0 = 0$$

Since $a_0 \neq 0$ then the indicial equation becomes

$$(r+1)\, x^{-1+r} = 0$$

Since the above is true for all x then the indicial equation simplifies to

$$r+1 = 0$$

Solving for r gives the root of the indicial equation as

$$r = -1$$

Replacing $r=-1$ found above results in

$$\left(\sum_{n=0}^{\infty} (n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} x^{n-2} a_n\right) = 0$$

From the above we see that there is no recurrence relation since there is only one summation term. Therefore all a_n terms are zero except for a_0. Hence

$$y_h = a_0 x^r$$

At $x=0$ the solution above becomes

$$y = c_1 \left(\frac{1}{x} + O(x^6)\right)$$

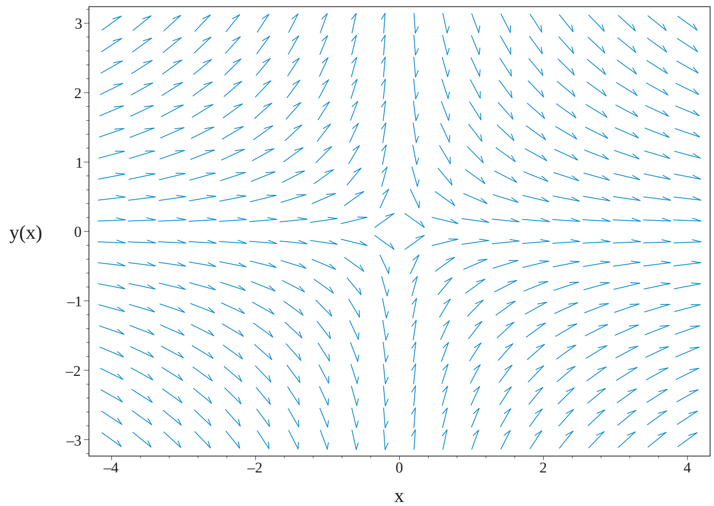


Figure 2.248: Slope field plot
$xy' + y = 0$

Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y(x)\right) + y(x) = 0$

- Highest derivative means the order of the ODE is 1

$\frac{d}{dx}y(x)$

- Separate variables

$\frac{\frac{d}{dx}y(x)}{y(x)} = -\frac{1}{x}$

- Integrate both sides with respect to x

$\int \frac{\frac{d}{dx}y(x)}{y(x)}dx = \int -\frac{1}{x}dx + C1$

- Evaluate integral

$\ln\left(y(x)\right) = -\ln\left(x\right) + C1$

- Solve for $y(x)$

$y(x) = \frac{e^{C1}}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.018 (sec)
Leaf size : 14

```
dsolve(diff(y(x),x)*x+y(x) = 0,y(x),
       series,x=0)
```

$$y = \frac{c_1}{x} + O(x^6)$$

Mathematica DSolve solution

Solving time : 0.002 (sec)
Leaf size : 9

```
AsymptoticDSolveValue[{x*D[y[x],x]+y[x]==0,{}},
       y[x],{x,0,5}]
```

$$y(x) \to \frac{c_1}{x}$$

2.5.10 problem 10

Internal problem ID [8647]
Book : Own collection of miscellaneous problems
Section : section 5.0
Problem number : 10
Date solved : `Tuesday, December 17, 2024 at 12:56:49 PM`
CAS classification : `[_quadrature]`

Solve

$$y' = \frac{1}{x}$$

Using series expansion around $x = 0$

Since this is an inhomogeneous, then let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ode $y' = 0$,and y_p is a particular solution to the inhomogeneous ode. First, we solve for y_h Let the solution be represented as Frobenius power series of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Then

$$y' = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$

Substituting the above back into the ode gives

$$\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} = 0 \tag{1}$$

Which simplifies to

$$\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} = 0 \tag{2A}$$

The next step is to make all powers of x be $n+r-1$ in each summation term. Going over each summation term above with power of x in it which is not already x^{n+r-1} and adjusting the power and the corresponding index gives Substituting all the above in Eq (2A) gives the following equation where now all powers of x are the same and equal to $n+r-1$.

$$\sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} = 0 \tag{2B}$$

The indicial equation is obtained from $n = 0$. From Eq (2) this gives

$$(n+r) a_n x^{n+r-1} = 0$$

When $n = 0$ the above becomes

$$r a_0 x^{-1+r} = 0$$

The corresponding balance equation is found by replacing r by m and a by c to avoid confusing terms between particular solution and the homogeneous solution. Hence the balance equation is

$$m c_0 x^{-1+m} = \frac{1}{x}$$

This equation will used later to find the particular solution.

Since $a_0 \neq 0$ then the indicial equation becomes

$$r\, x^{-1+r} = 0$$

Since the above is true for all x then the indicial equation simplifies to

$$r = 0$$

Solving for r gives the root of the indicial equation as

$$r = 0$$

From the above we see that there is no recurrence relation since there is only one summation term. Therefore all a_n terms are zero except for a_0. Hence

$$y_h = a_0 x^r$$

Unable to solve the balance equation $m c_0 x^{-1+m} = \frac{1}{x}$ for c_0 and x. No particular solution exists.

Unable to find the particular solution. No solution exist.

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) = \frac{1}{x}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}y(x)\right) dx = \int \frac{1}{x} dx + C1$

- Evaluate integral

  $y(x) = \ln(x) + C1$

- Solve for $y(x)$

  $y(x) = \ln(x) + C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : 0.034 (sec)
Leaf size : maple_leaf_size

```
dsolve(diff(y(x),x) = 1/x,y(x),
       series,x=0)
```

No solution found

Mathematica DSolve solution

Solving time : 0.01 (sec)
Leaf size : 8

```
AsymptoticDSolveValue[{D[y[x],x]==1/x,{}},
       y[x],{x,0,5}]
```

$$y(x) \to \log(x) + c_1$$

2.5.11 problem 11

Internal problem ID [8648]
Book : Own collection of miscellaneous problems
Section : section 5.0
Problem number : 11
Date solved : Thursday, December 12, 2024 at 09:37:21 AM
CAS classification : [[_2nd_order, _quadrature]]

Solve

$$y'' = \frac{1}{x}$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$y'' = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 0$$
$$q(x) = 0$$

Table 2.265: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = 0$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : $[\infty]$

Irregular singular points : []

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) = \frac{1}{x}$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Characteristic polynomial of homogeneous ODE

  $r^2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{0})}{2}$

- Roots of the characteristic polynomial

  $r = 0$

- 1st solution of the homogeneous ODE

  $y_1(x) = 1$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x$

- General solution of the ODE

  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y(x) = C1 + C2x + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \frac{1}{x}\right]$

- ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$

- ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$

- ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\left(\int 1dx\right) + \left(\int \frac{1}{x}dx\right)x$

- ◦ Compute integrals

  $y_p(x) = x(-1 + \ln(x))$

- Substitute particular solution into general solution to ODE

  $y(x) = C1 + C2x + x(-1 + \ln(x))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : 0.041 (sec)
Leaf size : maple_leaf_size

```
dsolve(diff(diff(y(x),x),x) = 1/x,y(x),
       series,x=0)
```

No solution found

Mathematica DSolve solution

Solving time : 0.031 (sec)
Leaf size : 17

```
AsymptoticDSolveValue[{D[y[x],{x,2}]==1/x,{}},
       y[x],{x,0,5}]
```

$$y(x) \rightarrow -x + x\log(x) + c_2x + c_1$$

2.5.12 problem 12

Internal problem ID [8649]
Book : Own collection of miscellaneous problems
Section : section 5.0
Problem number : 12
Date solved : `Thursday, December 12, 2024 at 09:37:21 AM`
CAS classification : `[[_2nd_order, _missing_y]]`

Solve

$$y'' + y' = \frac{1}{x}$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$y'' + y' = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 1$$
$$q(x) = 0$$

Table 2.267: Table $p(x), q(x)$ singularites.

$p(x) = 1$	
singularity	type

$q(x) = 0$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : []

Irregular singular points : [∞]

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + \frac{d}{dx}y(x) = \frac{1}{x}$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Characteristic polynomial of homogeneous ODE

  $r^2 + r = 0$

- Factor the characteristic polynomial

  $r(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 0)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y(x) = C1\,\mathrm{e}^{-x} + C2 + y_p(x)$

- □ Find a particular solution $y_p(x)$ of the ODE
  - ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \frac{1}{x}\right]$

  - ◦ Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-x} & 1 \\ -\mathrm{e}^{-x} & 0 \end{bmatrix}$

  - ◦ Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{-x}$

  - ◦ Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\mathrm{e}^{-x}\left(\int \frac{\mathrm{e}^{x}}{x}dx\right) + \int \frac{1}{x}dx$

  - ◦ Compute integrals

    $y_p(x) = \mathrm{e}^{-x}\mathrm{Ei}_1(-x) + \ln(x)$

- Substitute particular solution into general solution to ODE

  $y(x) = C1\,\mathrm{e}^{-x} + C2 + \mathrm{e}^{-x}\mathrm{Ei}_1(-x) + \ln(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
-> Calling odsolve with the ODE`, diff(_b(_a), _a) = -(_b(_a)*_a-1)/_a, _b(_a)`   ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   <- 1st order linear successful
<- high order exact linear fully integrable successful`
```

Maple dsolve solution

Solving time : 0.042 (sec)
Leaf size : maple_leaf_size

```
dsolve(diff(y(x),x)+diff(diff(y(x),x),x) = 1/x,y(x),
       series,x=0)
```

No solution found

Mathematica DSolve solution

Solving time : 0.024 (sec)
Leaf size : 159

```
AsymptoticDSolveValue[{D[y[x],{x,2}]+D[y[x],x]==1/x,{}},
       y[x],{x,0,5}]
```

$$y(x) \to -\frac{x^6}{4320} - \frac{x^5}{600} - \frac{x^4}{96} - \frac{x^3}{18} - \frac{x^2}{4} + c_2\left(-\frac{x^5}{720} + \frac{x^4}{120} - \frac{x^3}{24} + \frac{x^2}{6} - \frac{x}{2} + 1\right)x + \left(-\frac{x^5}{720} + \frac{x^4}{120} - \frac{x^3}{24} + \frac{x^2}{6} - \frac{x}{2} + 1\right)x\left(\frac{x^6}{2160} + \frac{x^5}{600} + \frac{x^4}{96} + \frac{x^3}{18} + \frac{x^2}{4} + x + \log(x)\right) - x + c_1$$

2.5.13 problem 13

Internal problem ID [8650]
Book : Own collection of miscellaneous problems
Section : section 5.0
Problem number : 13
Date solved : `Thursday, December 12, 2024 at 09:37:22 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' + y = \frac{1}{x}$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$y'' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 0$$
$$q(x) = 1$$

Table 2.269: Table $p(x), q(x)$ singularites.

$p(x) = 0$	
singularity	type

$q(x) = 1$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : []

Irregular singular points : [∞]

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + y(x) = \frac{1}{x}$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y(x) = C1 \cos(x) + C2 \sin(x) + y_p(x)$

- □ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \frac{1}{x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\cos(x)\left(\int \frac{\sin(x)}{x}dx\right) + \sin(x)\left(\int \frac{\cos(x)}{x}dx\right)$

  - Compute integrals

    $y_p(x) = -\cos(x)\,\mathrm{Si}(x) + \sin(x)\,\mathrm{Ci}(x)$

- Substitute particular solution into general solution to ODE

  $y(x) = C1 \cos(x) + C2 \sin(x) - \cos(x)\,\mathrm{Si}(x) + \sin(x)\,\mathrm{Ci}(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.041 (sec)
Leaf size : maple_leaf_size

```
dsolve(diff(diff(y(x),x),x)+y(x) = 1/x,y(x),
       series,x=0)
```

No solution found

Mathematica DSolve solution

Solving time : 0.021 (sec)
Leaf size : 148

```
AsymptoticDSolveValue[{D[y[x],{x,2}]+y[x]==1/x,{}},
       y[x],{x,0,5}]
```

$$\begin{aligned}y(x) \rightarrow\, & x\left(-\frac{x^6}{5040}+\frac{x^4}{120}-\frac{x^2}{6}+1\right)\left(-\frac{x^6}{4320}+\frac{x^4}{96}-\frac{x^2}{4}+\log(x)\right) \\ & +c_1\left(-\frac{x^6}{720}+\frac{x^4}{24}-\frac{x^2}{2}+1\right)+c_2x\left(-\frac{x^6}{5040}+\frac{x^4}{120}-\frac{x^2}{6}+1\right) \\ & +\left(-\frac{x^5}{600}+\frac{x^3}{18}-x\right)\left(-\frac{x^6}{720}+\frac{x^4}{24}-\frac{x^2}{2}+1\right)\end{aligned}$$

2.5.14 problem 14

Internal problem ID [8651]
Book : Own collection of miscellaneous problems
Section : section 5.0
Problem number : 14
Date solved : `Thursday, December 12, 2024 at 09:37:22 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' + y' + y = \frac{1}{x}$$

Using series expansion around $x = 0$

The type of the expansion point is first determined. This is done on the homogeneous part of the ODE.

$$y'' + y' + y = 0$$

The following is summary of singularities for the above ode. Writing the ode as

$$y'' + p(x)y' + q(x)y = 0$$

Where

$$p(x) = 1$$
$$q(x) = 1$$

Table 2.271: Table $p(x), q(x)$ singularites.

$p(x) = 1$	
singularity	type

$q(x) = 1$	
singularity	type

Combining everything together gives the following summary of singularities for the ode as

Regular singular points : []

Irregular singular points : $[\infty]$

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + \frac{d}{dx}y(x) + y(x) = \frac{1}{x}$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, -\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)$

- General solution of the ODE

  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y(x) = C1\,\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) + C2\,\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \frac{1}{x}\right]$

- ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right) & \mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right) \\ -\frac{\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} & -\frac{\mathrm{e}^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\sqrt{3}}{2} \end{bmatrix}$

- ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = \frac{\sqrt{3}\,\mathrm{e}^{-x}}{2}$

- ◦ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{2\sqrt{3}\,\mathrm{e}^{-\frac{x}{2}}\left(\cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int \frac{\mathrm{e}^{\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{x}dx\right)-\sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int \frac{\mathrm{e}^{\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{x}dx\right)\right)}{3}$$

- ◦ Compute integrals

$$y_p(x) = -\frac{\left(\left(\mathrm{I}\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\mathrm{Ei}_1\left(-\frac{\left(1+\mathrm{I}\sqrt{3}\right)x}{2}\right)-\left(\mathrm{I}\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\mathrm{Ei}_1\left(\frac{\left(\mathrm{I}\sqrt{3}-1\right)x}{2}\right)\right)\mathrm{e}^{-\frac{x}{2}}\sqrt{3}}{3}$$

- Substitute particular solution into general solution to ODE

$$y(x) = C1\,\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) + C2\,\mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right) - \frac{\left(\left(\mathrm{I}\cos\left(\frac{\sqrt{3}\,x}{2}\right)+\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\mathrm{Ei}_1\left(-\frac{\left(1+\mathrm{I}\sqrt{3}\right)x}{2}\right)-\left(\mathrm{I}\cos\left(\frac{\sqrt{3}\,x}{2}\right)-\right.}{3}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : `0.043` (sec)
Leaf size : maple_leaf_size

```
dsolve(diff(diff(y(x),x),x)+diff(y(x),x)+y(x) = 1/x,y(x),
       series,x=0)
```

No solution found

Mathematica DSolve solution

Solving time : 0.065 (sec)
Leaf size : 152

```
AsymptoticDSolveValue[{D[y[x],{x,2}]+D[y[x],x]+y[x]==1/x,{}},
       y[x],{x,0,5}]
```

$$
\begin{aligned}
y(x) \to\; & c_2 x\left(-\frac{x^4}{120}+\frac{x^3}{24}-\frac{x}{2}+1\right)+c_1\left(\frac{x^3}{6}-\frac{x^2}{2}+1\right) \\
& + x\left(-\frac{x^4}{120}+\frac{x^3}{24}-\frac{x}{2}+1\right)\left(\frac{41x^6}{4320}+\frac{x^5}{120}-\frac{x^4}{96}-\frac{x^3}{18}+x+\log(x)\right) \\
& + \left(\frac{x^3}{6}-\frac{x^2}{2}+1\right)\left(-\frac{x^6}{180}+\frac{x^5}{600}+\frac{x^4}{96}-\frac{x^2}{4}-x\right)
\end{aligned}
$$

2.5.15 problem 15

Internal problem ID [8652]
Book : Own collection of miscellaneous problems
Section : section 5.0
Problem number : 15
Date solved : `Tuesday, December 17, 2024 at 12:56:49 PM`
CAS classification : `[_quadrature]`

Solve

$$h^2 + \frac{2ah}{\sqrt{1+h'^2}} = b^2$$

Solving for the derivative gives these ODE's to solve

$$h' = -\frac{\sqrt{-h^4+4a^2h^2+2h^2b^2-b^4}}{(h+b)(h-b)} \tag{1}$$

$$h' = \frac{\sqrt{-h^4+4a^2h^2+2h^2b^2-b^4}}{(h+b)(h-b)} \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^h -\frac{(\tau+b)(\tau-b)}{\sqrt{4a^2\tau^2-b^4+2b^2\tau^2-\tau^4}}d\tau = u + c_3$$

Singular solutions are found by solving

$$-\frac{\sqrt{4a^2h^2-b^4+2h^2b^2-h^4}}{(h+b)(h-b)} = 0$$

for h. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$
\begin{aligned}
h &= -a-\sqrt{a^2+b^2} \\
h &= -a+\sqrt{a^2+b^2} \\
h &= a-\sqrt{a^2+b^2} \\
h &= a+\sqrt{a^2+b^2}
\end{aligned}
$$

We now need to find the singular solutions, these are found by finding for what values $\left(-\frac{\sqrt{4a^2h^2-b^4+2h^2b^2-h^4}}{(h+b)(h-b)}\right)$ is zero. These give

$$
h = -a-\sqrt{a^2+b^2}
$$

$$
h = -a+\sqrt{a^2+b^2}
$$

$$
h = a-\sqrt{a^2+b^2}
$$

$$
h = a+\sqrt{a^2+b^2}
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

The solution $h = -a-\sqrt{a^2+b^2}$ satisfies the ode and initial conditions.

The solution $h = -a+\sqrt{a^2+b^2}$ satisfies the ode and initial conditions.

The solution $h = a-\sqrt{a^2+b^2}$ does not satisfiy the ode and initial conditions. Hence will not be used.

The solution $h = a+\sqrt{a^2+b^2}$ does not satisfiy the ode and initial conditions. Hence will not be used.

Solving Eq. (2)

Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$
\int^{h}\frac{(\tau+b)(\tau-b)}{\sqrt{4a^2\tau^2-b^4+2b^2\tau^2-\tau^4}}d\tau = u+c_4
$$

Singular solutions are found by solving

$$
\frac{\sqrt{4a^2h^2-b^4+2h^2b^2-h^4}}{(h+b)(h-b)} = 0
$$

for h. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$h = -a - \sqrt{a^2+b^2}$$
$$h = -a + \sqrt{a^2+b^2}$$
$$h = a - \sqrt{a^2+b^2}$$
$$h = a + \sqrt{a^2+b^2}$$

We now need to find the singular solutions, these are found by finding for what values $\left(\frac{\sqrt{4a^2h^2-b^4+2h^2b^2-h^4}}{(h+b)(h-b)}\right)$ is zero. These give

$$h = -a - \sqrt{a^2+b^2}$$
$$h = -a + \sqrt{a^2+b^2}$$
$$h = a - \sqrt{a^2+b^2}$$
$$h = a + \sqrt{a^2+b^2}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

The solution $h = -a - \sqrt{a^2+b^2}$ satisfies the ode and initial conditions.

The solution $h = -a + \sqrt{a^2+b^2}$ satisfies the ode and initial conditions.

The solution $h = a - \sqrt{a^2+b^2}$ does not satisfiy the ode and initial conditions. Hence will not be used.

The solution $h = a + \sqrt{a^2+b^2}$ does not satisfiy the ode and initial conditions. Hence will not be used.

Maple step by step solution

Let's solve
$h(u)^2 + \frac{2ah(u)}{\sqrt{1+\left(\frac{d}{du}h(u)\right)^2}} = b^2$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{du}h(u)$
- Solve for the highest derivative

$$\left[\frac{d}{du}h(u)=\frac{\sqrt{-h(u)^4+4a^2h(u)^2+2h(u)^2b^2-b^4}}{(h(u)+b)(h(u)-b)}, \frac{d}{du}h(u)=-\frac{\sqrt{-h(u)^4+4a^2h(u)^2+2h(u)^2b^2-b^4}}{(h(u)+b)(h(u)-b)}\right]$$

- ☐ Solve the equation $\frac{d}{du}h(u)=\frac{\sqrt{-h(u)^4+4a^2h(u)^2+2h(u)^2b^2-b^4}}{(h(u)+b)(h(u)-b)}$
  - Separate variables
    $$\frac{\left(\frac{d}{du}h(u)\right)(h(u)+b)(h(u)-b)}{\sqrt{-h(u)^4+4a^2h(u)^2+2h(u)^2b^2-b^4}}=1$$
  - Integrate both sides with respect to u
    $$\int \frac{\left(\frac{d}{du}h(u)\right)(h(u)+b)(h(u)-b)}{\sqrt{-h(u)^4+4a^2h(u)^2+2h(u)^2b^2-b^4}}du=\int 1du+\_C1$$
  - Evaluate integral
    $$\frac{2b^4\sqrt{1+\frac{h(u)^2\left(2a\sqrt{a^2+b^2}-2a^2-b^2\right)}{b^4}}\sqrt{1-\frac{\left(2a\sqrt{a^2+b^2}+2a^2+b^2\right)h(u)^2}{b^4}}\left(EllipticF\left(h(u)\sqrt{-\frac{2a\sqrt{a^2+b^2}-2a^2-b^2}{b^4}},\sqrt{-1+\frac{\left(4a^2+2b^2\right)\left(2a\sqrt{a\right.}}{b^4}}\right.\right.}{\sqrt{-\frac{2a\sqrt{a^2+b^2}-2a^2-b^2}{b^4}}\sqrt{-h(u)^4+4a^2h(u)^2+2h(u)^2b^2-b}}$$
- ☐ Solve the equation $\frac{d}{du}h(u)=-\frac{\sqrt{-h(u)^4+4a^2h(u)^2+2h(u)^2b^2-b^4}}{(h(u)+b)(h(u)-b)}$
  - Separate variables
    $$\frac{\left(\frac{d}{du}h(u)\right)(h(u)+b)(h(u)-b)}{\sqrt{-h(u)^4+4a^2h(u)^2+2h(u)^2b^2-b^4}}=-1$$
  - Integrate both sides with respect to u
    $$\int \frac{\left(\frac{d}{du}h(u)\right)(h(u)+b)(h(u)-b)}{\sqrt{-h(u)^4+4a^2h(u)^2+2h(u)^2b^2-b^4}}du=\int (-1)\,du+\_C1$$
  - Evaluate integral
    $$\frac{2b^4\sqrt{1+\frac{h(u)^2\left(2a\sqrt{a^2+b^2}-2a^2-b^2\right)}{b^4}}\sqrt{1-\frac{\left(2a\sqrt{a^2+b^2}+2a^2+b^2\right)h(u)^2}{b^4}}\left(EllipticF\left(h(u)\sqrt{-\frac{2a\sqrt{a^2+b^2}-2a^2-b^2}{b^4}},\sqrt{-1+\frac{\left(4a^2+2b^2\right)\left(2a\sqrt{a\right.}}{b^4}}\right.\right.}{\sqrt{-\frac{2a\sqrt{a^2+b^2}-2a^2-b^2}{b^4}}\sqrt{-h(u)^4+4a^2h(u)^2+2h(u)^2b^2-b}}$$
- Set of solutions
  $$\left\{\frac{2b^4\sqrt{1+\frac{h(u)^2\left(2a\sqrt{a^2+b^2}-2a^2-b^2\right)}{b^4}}\sqrt{1-\frac{\left(2a\sqrt{a^2+b^2}+2a^2+b^2\right)h(u)^2}{b^4}}\left(EllipticF\left(h(u)\sqrt{-\frac{2a\sqrt{a^2+b^2}-2a^2-b^2}{b^4}},\sqrt{-1+\frac{\left(4a^2+2b^2\right)\left(2a\right.}{}}\right.\right.}{\sqrt{-\frac{2a\sqrt{a^2+b^2}-2a^2-b^2}{b^4}}\sqrt{-h(u)^4+4a^2h(u)^2+2h(u)^2b^2}}\right.$$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  x  successful`
```

Maple dsolve solution

Solving time : 3.867 (sec)
Leaf size : 103

```
dsolve(h(u)^2+2*a*h(u)/(1+diff(h(u),u)^2)^(1/2) = b^2,
       h(u),singsol=all)
```

$$u - \left(\int^{h} \frac{\_a^2 - b^2}{\sqrt{-\_a^4 + (4a^2 + 2b^2)\_a^2 - b^4}} d\_a\right) - c_1 = 0$$
$$u + \int^{h} \frac{\_a^2 - b^2}{\sqrt{-\_a^4 + (4a^2 + 2b^2)\_a^2 - b^4}} d\_a - c_1 = 0$$

Mathematica DSolve solution

Solving time : `24.227` (sec)
Leaf size : 913

```
DSolve[{h[u]^2 + 2*a*h[u]/Sqrt[1 + (D[ h[u],u])^2] == b^2,{}},
       h[u],u,IncludeSingularSolutions->True]
```

$$h(u) \to \text{InverseFunction}\left[-\frac{i\sqrt{\left(b^2-\#1^2\right)^2}\sqrt{1-\frac{\#1^2}{-2\sqrt{a^2(a^2+b^2)}+2a^2+b^2}}\sqrt{1-\frac{\#1^2}{2\sqrt{a^2(a^2+b^2)}+2a^2+b^2}}\left(\left(2\sqrt{a^2\left(a^2+b^2\right)}\right.\right.}{} + c_1\right]$$

$$h(u) \to \text{InverseFunction}\left[-\frac{i\sqrt{\left(b^2-\#1^2\right)^2}\sqrt{1-\frac{\#1^2}{-2\sqrt{a^2(a^2+b^2)}+2a^2+b^2}}\sqrt{1-\frac{\#1^2}{2\sqrt{a^2(a^2+b^2)}+2a^2+b^2}}\left(\left(2\sqrt{a^2\left(a^2+b^2\right)}\right.\right.}{} + c_1\right]$$

$$h(u) \to -\sqrt{a^2+b^2}-a$$
$$h(u) \to \sqrt{a^2+b^2}-a$$

2.5.16 problem 16

Internal problem ID [8653]
Book : Own collection of miscellaneous problems
Section : section 5.0
Problem number : 16
Date solved : `Thursday, December 12, 2024 at 09:37:45 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' + 2y' - 24y = 16 - (x + 2)\,e^{4x}$$

Solved as second order linear constant coeff ode

Time used: 0.158 (sec)

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = -24, f(x) = 16 + (-x - 2)\,e^{4x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' - 24y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = -24$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{x\lambda} + 2\lambda\,e^{x\lambda} - 24\,e^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda - 24 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = -24$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{2^2 - (4)(1)(-24)} \\ &= -1 \pm 5\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + 5 \\ \lambda_2 &= -1 - 5\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 4 \\ \lambda_2 &= -6\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(4)x} + c_2 e^{(-6)x}\end{aligned}$$

Or

$$y = c_1 \,\mathrm{e}^{4x} + c_2 \,\mathrm{e}^{-6x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \,\mathrm{e}^{4x} + c_2 \,\mathrm{e}^{-6x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$16 + (-x - 2)\,\mathrm{e}^{4x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{e^{4x}x, e^{4x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-6x}, e^{4x}\}$$

Since e^{4x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{1\}, \{x^2 e^{4x}, e^{4x}x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 + A_2 x^2 e^{4x} + A_3 e^{4x} x$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_2 e^{4x} + 20A_2 x\, e^{4x} + 10A_3 e^{4x} - 24A_1 = 16 - (x+2)\, e^{4x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{2}{3}, A_2 = -\frac{1}{20}, A_3 = -\frac{19}{100}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{2}{3} - \frac{x^2 e^{4x}}{20} - \frac{19\, e^{4x} x}{100}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\, e^{4x} + c_2\, e^{-6x}\right) + \left(-\frac{2}{3} - \frac{x^2 e^{4x}}{20} - \frac{19\, e^{4x} x}{100}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{2}{3} - \frac{x^2 e^{4x}}{20} - \frac{19\, e^{4x} x}{100} + c_1\, e^{4x} + c_2\, e^{-6x}$$

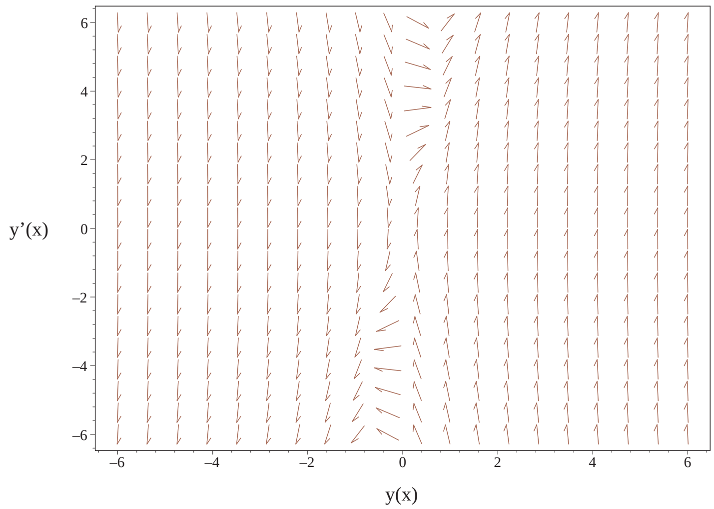


Figure 2.249: Slope field plot
$y'' + 2y' - 24y = 16 - (x + 2)\, e^{4x}$

Solved as second order ode using Kovacic algorithm

Time used: 0.270 (sec)

Writing the ode as

$$y'' + 2y' - 24y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= -24 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{25}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 25 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 25z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.274: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 25$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-5x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx}\\ &= z_1 e^{-x}\\ &= z_1\left(\mathrm{e}^{-x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-6x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx\\ &= y_1\left(\frac{\mathrm{e}^{-2x}\mathrm{e}^{12x}}{10}\right)\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-6x}\right) + c_2 \left(\mathrm{e}^{-6x} \left(\frac{\mathrm{e}^{-2x} \mathrm{e}^{12x}}{10}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 2y' - 24y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \,\mathrm{e}^{-6x} + \frac{c_2 \,\mathrm{e}^{4x}}{10}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \mathrm{e}^{-6x} \\
y_2 &= \frac{\mathrm{e}^{4x}}{10}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-6x} & \frac{e^{4x}}{10} \\ \frac{d}{dx}(e^{-6x}) & \frac{d}{dx}\left(\frac{e^{4x}}{10}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-6x} & \frac{e^{4x}}{10} \\ -6\,e^{-6x} & \frac{2\,e^{4x}}{5} \end{vmatrix}$$

Therefore

$$W = \left(e^{-6x}\right)\left(\frac{2\,e^{4x}}{5}\right) - \left(\frac{e^{4x}}{10}\right)\left(-6\,e^{-6x}\right)$$

Which simplifies to

$$W = e^{-6x} e^{4x}$$

Which simplifies to

$$W = e^{-2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{4x}(16+(-x-2)e^{4x})}{10}}{e^{-2x}}\,dx$$

Which simplifies to

$$u_1 = -\int -\frac{(-16+(x+2)\,e^{4x})\,e^{6x}}{10}dx$$

Hence

$$u_1 = \frac{e^{10x}x}{100} + \frac{19\,e^{10x}}{1000} - \frac{4\,e^{6x}}{15}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-6x}(16+(-x-2)\,e^{4x})}{e^{-2x}}\,dx$$

Which simplifies to

$$u_2 = \int \left(16\,\mathrm{e}^{-4x} - x - 2\right) dx$$

Hence

$$u_2 = -2x - \frac{x^2}{2} - 4\,\mathrm{e}^{-4x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{\mathrm{e}^{10x} x}{100} + \frac{19\,\mathrm{e}^{10x}}{1000} - \frac{4\,\mathrm{e}^{6x}}{15}\right) \mathrm{e}^{-6x} + \frac{\mathrm{e}^{4x}\left(-2x - \frac{x^2}{2} - 4\,\mathrm{e}^{-4x}\right)}{10}$$

Which simplifies to

$$y_p(x) = -\frac{2}{3} + \frac{(-50x^2 - 190x + 19)\,\mathrm{e}^{4x}}{1000}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\,\mathrm{e}^{-6x} + \frac{c_2\,\mathrm{e}^{4x}}{10}\right) + \left(-\frac{2}{3} + \frac{(-50x^2 - 190x + 19)\,\mathrm{e}^{4x}}{1000}\right) \end{aligned}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = c_1\,\mathrm{e}^{-6x} + \frac{c_2\,\mathrm{e}^{4x}}{10} - \frac{2}{3} + \frac{(-50x^2 - 190x + 19)\,\mathrm{e}^{4x}}{1000}$$

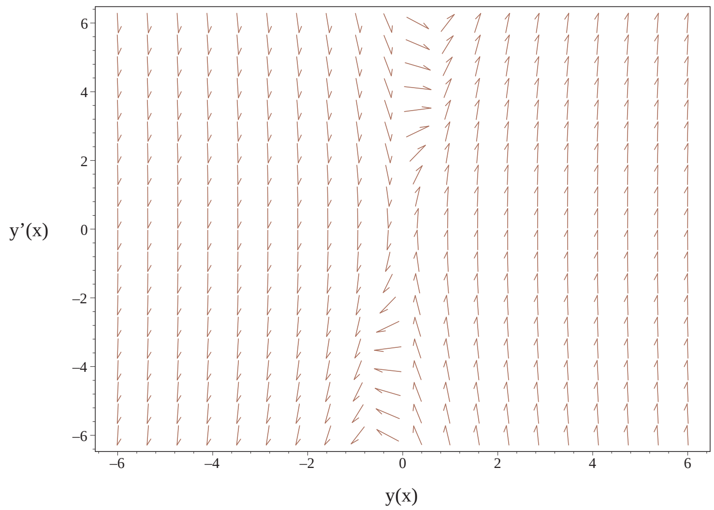


Figure 2.250: Slope field plot
$y'' + 2y' - 24y = 16 - (x + 2)\,e^{4x}$

Solved as second order ode adjoint method

Time used: 1.054 (sec)

In normal form the ode

$$y'' + 2y' - 24y = 16 - (x + 2)\,e^{4x} \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= 2 \\ q(x) &= -24 \\ r(x) &= 16 + (-x - 2)\,e^{4x} \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\, p)' + \xi q &= 0 \\ \xi'' - (2\xi(x))' + (-24\xi(x)) &= 0 \\ \xi''(x) - 2\xi'(x) - 24\xi(x) &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$A\xi''(x) + B\xi'(x) + C\xi(x) = 0$$

Where in the above $A = 1, B = -2, C = -24$. Let the solution be $\xi = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{x\lambda} - 2\lambda\, \mathrm{e}^{x\lambda} - 24\, \mathrm{e}^{x\lambda} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda - 24 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = -24$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-2^2 - (4)\,(1)\,(-24)} \\
&= 1 \pm 5
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= 1 + 5 \\
\lambda_2 &= 1 - 5
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 6 \\
\lambda_2 &= -4
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
\xi &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
\xi &= c_1 e^{(6)x} + c_2 e^{(-4)x}
\end{aligned}$$

Or

$$\xi = c_1\, \mathrm{e}^{6x} + c_2\, \mathrm{e}^{-4x}$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}
\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi\,(x)}
\end{aligned}$$

Or

$$y' + y\left(2 - \frac{6c_1\,\mathrm{e}^{6x} - 4c_2\,\mathrm{e}^{-4x}}{c_1\,\mathrm{e}^{6x} + c_2\,\mathrm{e}^{-4x}}\right) = \frac{-2c_2x + \frac{8c_1\,\mathrm{e}^{6x}}{3} - \frac{c_1\,\mathrm{e}^{10x}}{5} - \frac{c_2\,x^2}{2} - 4c_2\,\mathrm{e}^{-4x} - c_1\left(\frac{\mathrm{e}^{10x}x}{10} - \frac{\mathrm{e}^{10x}}{100}\right)}{c_1\,\mathrm{e}^{6x} + c_2\,\mathrm{e}^{-4x}}$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned}
q(x) &= -\frac{4c_1\,\mathrm{e}^{10x} - 6c_2}{c_1\,\mathrm{e}^{10x} + c_2} \\
p(x) &= \frac{800c_1\,\mathrm{e}^{10x} + (-30x - 57)\,c_1\,\mathrm{e}^{14x} - 150(8 + (x^2 + 4x)\,\mathrm{e}^{4x})\,c_2}{300c_1\,\mathrm{e}^{10x} + 300c_2}
\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= \mathrm{e}^{\int -\frac{4c_1\,\mathrm{e}^{10x} - 6c_2}{c_1\,\mathrm{e}^{10x} + c_2}dx} \\
&= \frac{(\mathrm{e}^{10x})^{3/5}}{c_1\,\mathrm{e}^{10x} + c_2}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{800c_1\,\mathrm{e}^{10x} + (-30x - 57)\,c_1\,\mathrm{e}^{14x} - 150(8 + (x^2 + 4x)\,\mathrm{e}^{4x})\,c_2}{300c_1\,\mathrm{e}^{10x} + 300c_2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y(\mathrm{e}^{10x})^{3/5}}{c_1\,\mathrm{e}^{10x} + c_2}\right) &= \left(\frac{(\mathrm{e}^{10x})^{3/5}}{c_1\,\mathrm{e}^{10x} + c_2}\right)\left(\frac{800c_1\,\mathrm{e}^{10x} + (-30x - 57)\,c_1\,\mathrm{e}^{14x} - 150(8 + (x^2 + 4x)\,\mathrm{e}^{4x})\,c_2}{300c_1\,\mathrm{e}^{10x} + 300c_2}\right) \\
\mathrm{d}\left(\frac{y(\mathrm{e}^{10x})^{3/5}}{c_1\,\mathrm{e}^{10x} + c_2}\right) &= \left(\frac{(800c_1\,\mathrm{e}^{10x} + (-30x - 57)\,c_1\,\mathrm{e}^{14x} - 150(8 + (x^2 + 4x)\,\mathrm{e}^{4x})\,c_2)\,(\mathrm{e}^{10x})^{3/5}}{(300c_1\,\mathrm{e}^{10x} + 300c_2)\,(c_1\,\mathrm{e}^{10x} + c_2)}\right)\,\mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{y(\mathrm{e}^{10x})^{3/5}}{c_1\,\mathrm{e}^{10x} + c_2} &= \int \frac{(800c_1\,\mathrm{e}^{10x} + (-30x - 57)\,c_1\,\mathrm{e}^{14x} - 150(8 + (x^2 + 4x)\,\mathrm{e}^{4x})\,c_2)\,(\mathrm{e}^{10x})^{3/5}}{(300c_1\,\mathrm{e}^{10x} + 300c_2)\,(c_1\,\mathrm{e}^{10x} + c_2)}\,dx \\
&= -\frac{(\mathrm{e}^{10x})^{3/5}\,\mathrm{e}^{-6x}(15x^2 + 57x)}{300c_1} + \frac{(\mathrm{e}^{10x})^{3/5}\,\mathrm{e}^{-6x}(-2000c_1\,\mathrm{e}^{6x} + 150c_2\,x^2 + 570c_2x - 57c_2)}{3000\,(c_1\,\mathrm{e}^{10x} + c_2)\,c_1} + c_3
\end{aligned}$$

Dividing throughout by the integrating factor $\frac{(e^{10x})^{3/5}}{c_1 e^{10x}+c_2}$ gives the final solution

$$y = -\frac{\left(\left(\frac{40c_1 e^{6x}}{3} + \frac{19c_2}{50}\right)(e^{10x})^{3/5} + \left(\left(x^2 + \frac{19}{5}x\right)(e^{10x})^{8/5} - 20c_3(c_1 e^{16x} + c_2 e^{6x})\right) c_1\right) e^{-6x}}{20\,(e^{10x})^{3/5}\, c_1}$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = -\frac{\left(\left(\frac{40c_1 e^{6x}}{3} + \frac{19c_2}{50}\right)(e^{10x})^{3/5} + \left(\left(x^2 + \frac{19}{5}x\right)(e^{10x})^{8/5} - 20c_3(c_1 e^{16x} + c_2 e^{6x})\right) c_1\right) e^{-6x}}{20\,(e^{10x})^{3/5}\, c_1}$$

The constants can be merged to give

$$y = -\frac{\left(\left(\frac{40c_1 e^{6x}}{3} + \frac{19c_2}{50}\right)(e^{10x})^{3/5} + \left(\left(x^2 + \frac{19}{5}x\right)(e^{10x})^{8/5} - 20c_1 e^{16x} - 20c_2 e^{6x}\right) c_1\right) e^{-6x}}{20\,(e^{10x})^{3/5}\, c_1}$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = -\frac{\left(\left(\frac{40c_1 e^{6x}}{3} + \frac{19c_2}{50}\right)(e^{10x})^{3/5} + \left(\left(x^2 + \frac{19}{5}x\right)(e^{10x})^{8/5} - 20c_1 e^{16x} - 20c_2 e^{6x}\right) c_1\right) e^{-6x}}{20\,(e^{10x})^{3/5}\, c_1}$$

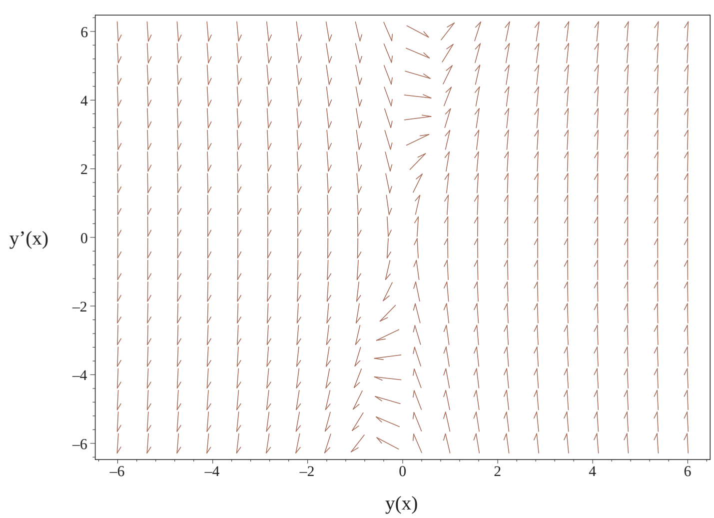


Figure 2.251: Slope field plot
$y'' + 2y' - 24y = 16 - (x + 2)\, e^{4x}$

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x)+2\frac{d}{dx}y(x)-24y(x)=16-(x+2)\,\mathrm{e}^{4x}$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Isolate 2nd derivative

  $\frac{d^2}{dx^2}y(x)=24y(x)-\mathrm{e}^{4x}x-2\,\mathrm{e}^{4x}-2\frac{d}{dx}y(x)+16$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dx^2}y(x)+2\frac{d}{dx}y(x)-24y(x)=-\mathrm{e}^{4x}x-2\,\mathrm{e}^{4x}+16$

- Characteristic polynomial of homogeneous ODE

  $r^2+2r-24=0$

- Factor the characteristic polynomial

  $(r+6)\,(r-4)=0$

- Roots of the characteristic polynomial

  $r=(-6,4)$

- 1st solution of the homogeneous ODE

  $y_1(x)=\mathrm{e}^{-6x}$

- 2nd solution of the homogeneous ODE

  $y_2(x)=\mathrm{e}^{4x}$

- General solution of the ODE

  $y(x)=C1y_1(x)+C2y_2(x)+y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y(x)=C1\,\mathrm{e}^{-6x}+C2\,\mathrm{e}^{4x}+y_p(x)$

- □ Find a particular solution $y_p(x)$ of the ODE
  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x)=-y_1(x)\left(\int\frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int\frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right),f(x)=-\mathrm{e}^{4x}x-2\,\mathrm{e}^{4x}+16\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x)\,,y_2(x))=\begin{bmatrix}\mathrm{e}^{-6x} & \mathrm{e}^{4x}\\ -6\,\mathrm{e}^{-6x} & 4\,\mathrm{e}^{4x}\end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x)\,,y_2(x))=10\,\mathrm{e}^{-2x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x)=\frac{\left(\mathrm{e}^{10x}\left(\int\left(-x-2+16\,\mathrm{e}^{-4x}\right)dx\right)+\int\left(-16+(x+2)\mathrm{e}^{4x}\right)\mathrm{e}^{6x}dx\right)\mathrm{e}^{-6x}}{10}$

- Compute integrals

  $y_p(x) = -\frac{2}{3} + \frac{(-50x^2-190x+19)\,e^{4x}}{1000}$

- Substitute particular solution into general solution to ODE

  $y(x) = \mathit{C1}\,e^{-6x} + \mathit{C2}\,e^{4x} - \frac{2}{3} + \frac{(-50x^2-190x+19)\,e^{4x}}{1000}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 36

```
dsolve(diff(diff(y(x),x),x)+2*diff(y(x),x)-24*y(x) = 16-(x+2)*exp(4*x),
       y(x),singsol=all)
```

$$y = -\frac{\left(\left(x^2+\frac{19}{5}x-20c_1-\frac{19}{50}\right)e^{10x}-20c_2+\frac{40\,e^{6x}}{3}\right)e^{-6x}}{20}$$

Mathematica DSolve solution

Solving time : 0.222 (sec)
Leaf size : 41

```
DSolve[{D[y[x],{x,2}]+2*D[y[x],x]-24*y[x]==16-(x+2)*Exp[4*x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to e^{4x}\left(-\frac{x^2}{20} - \frac{19x}{100} + \frac{19}{1000} + c_2\right) + c_1 e^{-6x} - \frac{2}{3}$$

2.5.17 problem 17

Internal problem ID [8654]
Book : Own collection of miscellaneous problems
Section : section 5.0
Problem number : 17
Date solved : `Thursday, December 12, 2024 at 09:37:47 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$y'' + 3y' - 4y = 6\,\mathrm{e}^{2t-2}$$

With initial conditions

$$\begin{aligned} y(1) &= 4 \\ y'(1) &= 5 \end{aligned}$$

Since both initial conditions are not at zero, then let

$$\begin{aligned} y(0) &= c_1 \\ y'(0) &= c_2 \end{aligned}$$

Solving using the Laplace transform method. Let

$$\mathcal{L}(y) = Y(s)$$

Taking the Laplace transform of the ode and using the relations that

$$\begin{aligned} \mathcal{L}(y') &= sY(s) - y(0) \\ \mathcal{L}(y'') &= s^2 Y(s) - y'(0) - sy(0) \end{aligned}$$

The given ode now becomes an algebraic equation in the Laplace domain.

$$s^2 Y(s) - y'(0) - sy(0) + 3sY(s) - 3y(0) - 4Y(s) = \frac{6\,\mathrm{e}^{-2}}{s-2} \tag{1}$$

Substituting the initial conditions in above in Eq (1) gives

$$s^2Y(s) - c_2 - sc_1 + 3sY(s) - 3c_1 - 4Y(s) = \frac{6\,e^{-2}}{s-2}$$

Solving the above equation for $Y(s)$ results in

$$Y(s) = \frac{s^2c_1 + sc_1 + c_2s + 6\,e^{-2} - 6c_1 - 2c_2}{(s-2)\,(s^2+3s-4)}$$

Applying partial fractions decomposition results in

$$Y(s) = \frac{e^{-2}}{s-2} + \frac{\frac{c_1}{5} - \frac{c_2}{5} + \frac{e^{-2}}{5}}{s+4} + \frac{\frac{4c_1}{5} + \frac{c_2}{5} - \frac{6\,e^{-2}}{5}}{s-1}$$

The inverse Laplace of each term above is now found, which gives

$$\mathcal{L}^{-1}\left(\frac{e^{-2}}{s-2}\right) = e^{2t-2}$$

$$\mathcal{L}^{-1}\left(\frac{\frac{c_1}{5} - \frac{c_2}{5} + \frac{e^{-2}}{5}}{s+4}\right) = \frac{(c_1 - c_2 + e^{-2})\,e^{-4t}}{5}$$

$$\mathcal{L}^{-1}\left(\frac{\frac{4c_1}{5} + \frac{c_2}{5} - \frac{6\,e^{-2}}{5}}{s-1}\right) = \frac{e^t(4c_1 + c_2 - 6\,e^{-2})}{5}$$

Adding the above results and simplifying gives

$$y = e^{2t-2} + \frac{e^t(4c_1 + c_2 - 6\,e^{-2})}{5} + \frac{(c_1 - c_2 + e^{-2})\,e^{-4t}}{5}$$

Solving for c_1 and c_2 from the given initial conditions results in

$$y = e^{2t-2} + \frac{e^t(4(e^{-1}+3)\,e^{-1} - 4\,e^{-2} + 3\,e^{-1})}{5} + \frac{((e^{-1}+3)\,e^{-1} - e^{-2} - 3\,e^{-1})\,e^{-4t}}{5}$$

Simplifying the solution gives

$$y = e^{2t-2} + 3\,e^{t-1}$$

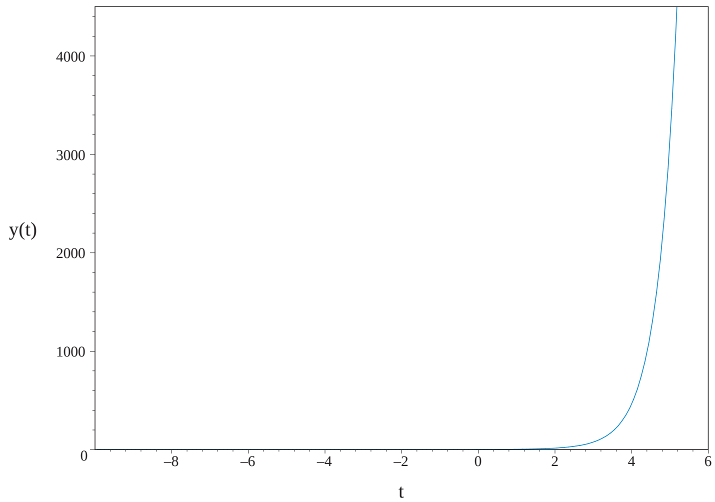


Figure 2.252: Solution plot
$y = e^{2t-2} + 3\,e^{t-1}$

Maple step by step solution

Let's solve

$\left[\frac{d^2}{dt^2}y(t) + 3\frac{d}{dt}y(t) - 4y(t) = 6\,e^{2t-2}, y(1) = 4, \left(\frac{d}{dt}y(t)\right)\Big|_{\{t=1\}} = 5\right]$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dt^2}y(t)$
- Characteristic polynomial of homogeneous ODE
  $r^2 + 3r - 4 = 0$
- Factor the characteristic polynomial
  $(r+4)(r-1) = 0$
- Roots of the characteristic polynomial
  $r = (-4, 1)$
- 1st solution of the homogeneous ODE
  $y_1(t) = e^{-4t}$
- 2nd solution of the homogeneous ODE
  $y_2(t) = e^t$
- General solution of the ODE
  $y(t) = C1 y_1(t) + C2 y_2(t) + y_p(t)$
- Substitute in solutions of the homogeneous ODE
  $y(t) = C1\,e^{-4t} + C2\,e^t + y_p(t)$

□ Find a particular solution $y_p(t)$ of the ODE

- Use variation of parameters to find y_p here $f(t)$ is the forcing function
  $\left[y_p(t) = -y_1(t)\left(\int \frac{y_2(t)f(t)}{W(y_1(t),y_2(t))}dt\right) + y_2(t)\left(\int \frac{y_1(t)f(t)}{W(y_1(t),y_2(t))}dt\right), f(t) = 6\,\mathrm{e}^{2t-2}\right]$
- Wronskian of solutions of the homogeneous equation
  $W(y_1(t), y_2(t)) = \begin{bmatrix} \mathrm{e}^{-4t} & \mathrm{e}^{t} \\ -4\,\mathrm{e}^{-4t} & \mathrm{e}^{t} \end{bmatrix}$
- Compute Wronskian
  $W(y_1(t), y_2(t)) = 5\,\mathrm{e}^{-3t}$
- Substitute functions into equation for $y_p(t)$
  $y_p(t) = -\frac{6\left(-\mathrm{e}^{5t}\left(\int \mathrm{e}^{t-2}dt\right)+\int \mathrm{e}^{6t-2}dt\right)\mathrm{e}^{-4t}}{5}$
- Compute integrals
  $y_p(t) = \mathrm{e}^{2t-2}$

- Substitute particular solution into general solution to ODE
  $y(t) = C1\,\mathrm{e}^{-4t} + C2\,\mathrm{e}^{t} + \mathrm{e}^{2t-2}$

☐ Check validity of solution $y(t) = \_C1\mathrm{e}^{-4t} + \_C2\mathrm{e}^{t} + \mathrm{e}^{2t-2}$

- Use initial condition $y(1) = 4$
  $4 = \_C1\,\mathrm{e}^{-4} + \_C2\,\mathrm{e} + 1$
- Compute derivative of the solution
  $\frac{d}{dt}y(t) = -4\_C1\,\mathrm{e}^{-4t} + \_C2\,\mathrm{e}^{t} + 2\,\mathrm{e}^{2t-2}$
- Use the initial condition $\left(\frac{d}{dt}y(t)\right)\Big|_{\{t=1\}} = 5$
  $5 = -4\_C1\,\mathrm{e}^{-4} + \_C2\,\mathrm{e} + 2$
- Solve for $\_C1$ and $\_C2$
  $\left\{\_C1 = 0, \_C2 = \frac{3}{\mathrm{e}}\right\}$
- Substitute constant values into general solution and simplify
  $y(t) = 3\,\mathrm{e}^{t-1} + \mathrm{e}^{2t-2}$

- Solution to the IVP
  $y(t) = 3\,\mathrm{e}^{t-1} + \mathrm{e}^{2t-2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 2.088 (sec)
Leaf size : 17

```
dsolve([diff(diff(y(t),t),t)+3*diff(y(t),t)-4*y(t) = 6*exp(2*t-2),
       op([y(1) = 4, D(y)(1) = 5])],
       y(t),method=laplace)
```

$$y = 3\,\mathrm{e}^{t-1} + \mathrm{e}^{2t-2}$$

Mathematica DSolve solution

Solving time : 0.075 (sec)
Leaf size : 18

```
DSolve[{D[y[t],{t,2}]+3*D[y[t],t]-4*y[t]==6*Exp[2*t-2],{y[1]==4,Derivative[1][y][1]==5}},
      y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to e^{t-2}\left(e^{t} + 3e\right)$$

2.5.18 problem 18

Internal problem ID [8655]
Book : Own collection of miscellaneous problems
Section : section 5.0
Problem number : 18
Date solved : `Thursday, December 12, 2024 at 09:37:48 AM`
CAS classification : `[[_2nd_order, _linear, _nonhomogeneous]]`

Solve

$$y'' + y = \mathrm{e}^{a\cos(x)}$$

Using series expansion around $x = 0$

Solving ode using Taylor series method. This gives review on how the Taylor series method works for solving second order ode.

Let

$$y'' = f(x, y, y')$$

Assuming expansion is at $x_0 = 0$ (we can always shift the actual expansion point to 0 by change of variables) and assuming $f(x, y, y')$ is analytic at x_0 which must be the case for an ordinary point. Let initial conditions be $y(x_0) = y_0$ and $y'(x_0) = y_0'$. Using Taylor series gives

$$\begin{aligned}
y(x) &= y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \cdots \\
&= y_0 + x y_0' + \frac{x^2}{2} f|_{x_0,y_0,y_0'} + \frac{x^3}{3!} f'|_{x_0,y_0,y_0'} + \cdots \\
&= y_0 + x y_0' + \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+2)\,!} \frac{d^n f}{dx^n}\bigg|_{x_0,y_0,y_0'}
\end{aligned}$$

But

$$
\begin{aligned}
\frac{df}{dx} &= \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} && (1)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y'' && (2.17)\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f && (2.18)\\
\frac{d^2 f}{dx^2} &= \frac{d}{dx}\left(\frac{df}{dx}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{df}{dx}\right) + \frac{\partial}{\partial y}\left(\frac{df}{dx}\right)y' + \frac{\partial}{\partial y'}\left(\frac{df}{dx}\right)f && (2)\\
\frac{d^3 f}{dx^3} &= \frac{d}{dx}\left(\frac{d^2 f}{dx^2}\right)\\
&= \frac{\partial}{\partial x}\left(\frac{d^2 f}{dx^2}\right) + \left(\frac{\partial}{\partial y}\frac{d^2 f}{dx^2}\right)y' + \frac{\partial}{\partial y'}\left(\frac{d^2 f}{dx^2}\right)f && (3)\\
&\vdots
\end{aligned}
$$

And so on. Hence if we name $F_0 = f(x,y,y')$ then the above can be written as

$$
\begin{aligned}
F_0 &= f(x,y,y') \qquad (4)\\
F_1 &= \frac{df}{dx}\\
&= \frac{dF_0}{dx}\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''\\
&= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}f \qquad (5)\\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y}y' + \frac{\partial F_0}{\partial y'}F_0\\
F_2 &= \frac{d}{dx}\left(\frac{d}{dx}f\right)\\
&= \frac{d}{dx}(F_1)\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_1 + \left(\frac{\partial F_1}{\partial y}\right)y' + \left(\frac{\partial F_1}{\partial y'}\right)F_0\\
&\vdots\\
F_n &= \frac{d}{dx}(F_{n-1})\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)y''\\
&= \frac{\partial}{\partial x}F_{n-1} + \left(\frac{\partial F_{n-1}}{\partial y}\right)y' + \left(\frac{\partial F_{n-1}}{\partial y'}\right)F_0 \qquad (6)
\end{aligned}
$$

Therefore (6) can be used from now on along with

$$
y(x) = y_0 + xy'_0 + \sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}F_n|_{x_0,y_0,y'_0} \qquad (7)
$$

To find $y(x)$ series solution around $x = 0$. Hence

$$
\begin{aligned}
F_0 &= \mathrm{e}^{a\cos(x)} - y \\
F_1 &= \frac{dF_0}{dx} \\
&= \frac{\partial F_0}{\partial x} + \frac{\partial F_0}{\partial y} y' + \frac{\partial F_0}{\partial y'} F_0 \\
&= -a \sin(x)\, \mathrm{e}^{a\cos(x)} - y' \\
F_2 &= \frac{dF_1}{dx} \\
&= \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} y' + \frac{\partial F_1}{\partial y'} F_1 \\
&= \mathrm{e}^{a\cos(x)} \left(a^2 \sin(x)^2 - a\cos(x) - 1\right) + y \\
F_3 &= \frac{dF_2}{dx} \\
&= \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y} y' + \frac{\partial F_2}{\partial y'} F_2 \\
&= -a\sin(x)\left(a^2 \sin(x)^2 - 3a\cos(x) - 2\right) \mathrm{e}^{a\cos(x)} + y' \\
F_4 &= \frac{dF_3}{dx} \\
&= \frac{\partial F_3}{\partial x} + \frac{\partial F_3}{\partial y} y' + \frac{\partial F_3}{\partial y'} F_3 \\
&= \left(8a^2\cos(x)^2 + \left(-6a^3\sin(x)^2 + 2a\right)\cos(x) + \sin(x)^4 a^4 - 5a^2 + 1\right) \mathrm{e}^{a\cos(x)} - y \\
F_5 &= \frac{dF_4}{dx} \\
&= \frac{\partial F_4}{\partial x} + \frac{\partial F_4}{\partial y} y' + \frac{\partial F_4}{\partial y'} F_4 \\
&= -a\sin(x)\left(\sin(x)^4 a^4 - 10\cos(x)\sin(x)^2 a^3 + 26a^2\cos(x)^2 + 18a\cos(x) - 11a^2 + 3\right) \mathrm{e}^{a\cos(x)} - y' \\
F_6 &= \frac{dF_5}{dx} \\
&= \frac{\partial F_5}{\partial x} + \frac{\partial F_5}{\partial y} y' + \frac{\partial F_5}{\partial y'} F_5 \\
&= \left(-96a^3\cos(x)^3 + \left(66\sin(x)^2 a^4 - 39a^2\right)\cos(x)^2 + \left(-15a^5\sin(x)^4 + 81a^3 - 3a\right)\cos(x) + \sin(x)^6 a
\end{aligned}
$$

And so on. Evaluating all the above at initial conditions $x = 0$ and $y(0) = y(0)$ and

$y'(0) = y'(0)$ gives

$$
\begin{aligned}
F_0 &= \mathrm{e}^a - y(0) \\
F_1 &= -y'(0) \\
F_2 &= -\mathrm{e}^a a - \mathrm{e}^a + y(0) \\
F_3 &= y'(0) \\
F_4 &= 3\,\mathrm{e}^a a^2 + 2\,\mathrm{e}^a a + \mathrm{e}^a - y(0) \\
F_5 &= -y'(0) \\
F_6 &= -15\,\mathrm{e}^a a^3 - 18\,\mathrm{e}^a a^2 - 3\,\mathrm{e}^a a - \mathrm{e}^a + y(0)
\end{aligned}
$$

Substituting all the above in (7) and simplifying gives the solution as

$$
\begin{aligned}
y = &\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6\right) y(0) + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7\right) y'(0) \\
&+ \frac{x^2 \mathrm{e}^a}{2} - \frac{x^4 \mathrm{e}^a a}{24} - \frac{x^4 \mathrm{e}^a}{24} + \frac{x^6 \mathrm{e}^a a^2}{240} + \frac{x^6 \mathrm{e}^a a}{360} + \frac{x^6 \mathrm{e}^a}{720} + O(x^8)
\end{aligned}
$$

Since the expansion point $x = 0$ is an ordinary point, then this can also be solved using the standard power series method. Let the solution be represented as power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^n
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right) = \mathrm{e}^{a\cos(x)} \tag{1}
$$

Expanding $\mathrm{e}^{a\cos(x)}$ as Taylor series around $x = 0$ and keeping only the first 8 terms gives

$$
\begin{aligned}
\mathrm{e}^{a\cos(x)} &= \mathrm{e}^a - \frac{\mathrm{e}^a a\, x^2}{2} + \mathrm{e}^a \left(\frac{1}{24}a + \frac{1}{8}a^2\right) x^4 + \mathrm{e}^a \left(-\frac{1}{720}a - \frac{1}{48}a^2 - \frac{1}{48}a^3\right) x^6 + \dots \\
&= \mathrm{e}^a - \frac{\mathrm{e}^a a\, x^2}{2} + \mathrm{e}^a \left(\frac{1}{24}a + \frac{1}{8}a^2\right) x^4 + \mathrm{e}^a \left(-\frac{1}{720}a - \frac{1}{48}a^2 - \frac{1}{48}a^3\right) x^6
\end{aligned}
$$

Hence the ODE in Eq (1) becomes

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+\left(\sum_{n=0}^{\infty} a_n x^n\right)=\mathrm{e}^a-\frac{\mathrm{e}^a a\, x^2}{2}+\mathrm{e}^a\left(\frac{1}{24}a+\frac{1}{8}a^2\right)x^4+\mathrm{e}^a\left(-\frac{1}{720}a-\frac{1}{48}a^2-\frac{1}{48}a^3\right)x^6$$

Which simplifies to

$$\left(\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}\right)+\left(\sum_{n=0}^{\infty} a_n x^n\right)=\mathrm{e}^a-\frac{\mathrm{e}^a a\, x^2}{2}+\mathrm{e}^a\left(\frac{1}{24}a+\frac{1}{8}a^2\right)x^4+\mathrm{e}^a\left(-\frac{1}{720}a-\frac{1}{48}a^2-\frac{1}{48}a^3\right)x^6 \tag{2}$$

The next step is to make all powers of x be n in each summation term. Going over each summation term above with power of x in it which is not already x^n and adjusting the power and the corresponding index gives

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}=\sum_{n=0}^{\infty}(n+2)\, a_{n+2}(n+1)\, x^n$$

Substituting all the above in Eq (2) gives the following equation where now all powers of x are the same and equal to n.

$$\left(\sum_{n=0}^{\infty}(n+2)\, a_{n+2}(n+1)\, x^n\right)+\left(\sum_{n=0}^{\infty} a_n x^n\right)=\mathrm{e}^a-\frac{\mathrm{e}^a a\, x^2}{2}+\mathrm{e}^a\left(\frac{1}{24}a+\frac{1}{8}a^2\right)x^4+\mathrm{e}^a\left(-\frac{1}{720}a-\frac{1}{48}a^2-\frac{1}{48}a^3\right)x^6 \tag{3}$$

For $0\leq n$, the recurrence equation is

$$((n+2)\, a_{n+2}(n+1)+a_n)\, x^n=\mathrm{e}^a-\frac{\mathrm{e}^a a\, x^2}{2}+\mathrm{e}^a\left(\frac{1}{24}a+\frac{1}{8}a^2\right)x^4+\mathrm{e}^a\left(-\frac{1}{720}a-\frac{1}{48}a^2-\frac{1}{48}a^3\right)x^6 \tag{4}$$

For $n = 0$ the recurrence equation gives

$$
\begin{aligned}
(2a_2 + a_0)\,1 &= \mathrm{e}^a \\
2a_2 + a_0 &= \mathrm{e}^a
\end{aligned}
$$

Which after substituting the earlier terms found becomes

$$
a_2 = \frac{\mathrm{e}^a}{2} - \frac{a_0}{2}
$$

For $n = 1$ the recurrence equation gives

$$
\begin{aligned}
(6a_3 + a_1)\,x &= 0 \\
6a_3 + a_1 &= 0
\end{aligned}
$$

Which after substituting the earlier terms found becomes

$$
a_3 = -\frac{a_1}{6}
$$

For $n = 2$ the recurrence equation gives

$$
\begin{aligned}
(12a_4 + a_2)\,x^2 &= -\frac{\mathrm{e}^a a\,x^2}{2} \\
12a_4 + a_2 &= -\frac{\mathrm{e}^a a}{2}
\end{aligned}
$$

Which after substituting the earlier terms found becomes

$$
a_4 = -\frac{\mathrm{e}^a a}{24} - \frac{\mathrm{e}^a}{24} + \frac{a_0}{24}
$$

For $n = 3$ the recurrence equation gives

$$
\begin{aligned}
(20a_5 + a_3)\,x^3 &= 0 \\
20a_5 + a_3 &= 0
\end{aligned}
$$

Which after substituting the earlier terms found becomes

$$
a_5 = \frac{a_1}{120}
$$

For $n = 4$ the recurrence equation gives

$$
\begin{aligned}
(30a_6 + a_4) x^4 &= \mathrm{e}^a \left(\frac{1}{24}a + \frac{1}{8}a^2\right) x^4 \\
30a_6 + a_4 &= \mathrm{e}^a \left(\frac{1}{24}a + \frac{1}{8}a^2\right)
\end{aligned}
$$

Which after substituting the earlier terms found becomes

$$
a_6 = \frac{\mathrm{e}^a a}{360} + \frac{\mathrm{e}^a a^2}{240} + \frac{\mathrm{e}^a}{720} - \frac{a_0}{720}
$$

For $n = 5$ the recurrence equation gives

$$
\begin{aligned}
(42a_7 + a_5) x^5 &= 0 \\
42a_7 + a_5 &= 0
\end{aligned}
$$

Which after substituting the earlier terms found becomes

$$
a_7 = -\frac{a_1}{5040}
$$

For $n = 6$ the recurrence equation gives

$$
\begin{aligned}
(56a_8 + a_6) x^6 &= \mathrm{e}^a \left(-\frac{1}{720}a - \frac{1}{48}a^2 - \frac{1}{48}a^3\right) x^6 \\
56a_8 + a_6 &= \mathrm{e}^a \left(-\frac{1}{720}a - \frac{1}{48}a^2 - \frac{1}{48}a^3\right)
\end{aligned}
$$

Which after substituting the earlier terms found becomes

$$
a_8 = -\frac{\mathrm{e}^a a}{13440} - \frac{\mathrm{e}^a a^2}{2240} - \frac{\mathrm{e}^a a^3}{2688} - \frac{\mathrm{e}^a}{40320} + \frac{a_0}{40320}
$$

For $n = 7$ the recurrence equation gives

$$
\begin{aligned}
(72a_9 + a_7) x^7 &= 0 \\
72a_9 + a_7 &= 0
\end{aligned}
$$

Which after substituting the earlier terms found becomes

$$a_9 = \frac{a_1}{362880}$$

And so on. Therefore the solution is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_3 x^3 + a_2 x^2 + a_1 x + a_0 + \dots \end{aligned}$$

Substituting the values for a_n found above, the solution becomes

$$\begin{aligned} y = a_0 + a_1 x + \left(\frac{e^a}{2} - \frac{a_0}{2}\right) x^2 - \frac{a_1 x^3}{6} + \left(-\frac{e^a a}{24} - \frac{e^a}{24} + \frac{a_0}{24}\right) x^4 \\ + \frac{a_1 x^5}{120} + \left(\frac{e^a a}{360} + \frac{e^a a^2}{240} + \frac{e^a}{720} - \frac{a_0}{720}\right) x^6 - \frac{a_1 x^7}{5040} + \dots \end{aligned}$$

Collecting terms, the solution becomes

$$\begin{aligned} y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6\right) a_0 + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7\right) a_1 \\ + \frac{x^2 e^a}{2} + \left(-\frac{e^a a}{24} - \frac{e^a}{24}\right) x^4 + \left(\frac{e^a a}{360} + \frac{e^a a^2}{240} + \frac{e^a}{720}\right) x^6 + O(x^8) \end{aligned} \tag{3}$$

At $x = 0$ the solution above becomes

$$\begin{aligned} y = \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6\right) c_1 + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7\right) c_2 \\ + \frac{x^2 e^a}{2} + \left(-\frac{e^a a}{24} - \frac{e^a}{24}\right) x^4 + \left(\frac{e^a a}{360} + \frac{e^a a^2}{240} + \frac{e^a}{720}\right) x^6 + O(x^8) \end{aligned}$$

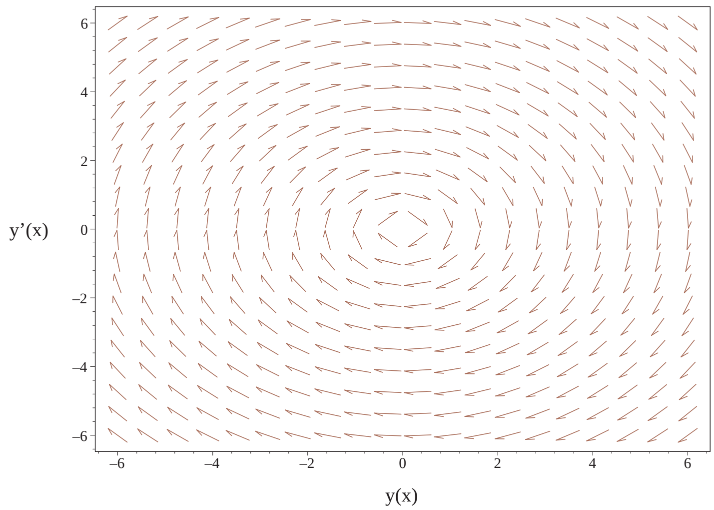


Figure 2.253: Slope field plot
$y'' + y = e^{a\cos(x)}$

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + y(x) = e^{a\cos(x)}$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y(x) = C1 y_1(x) + C2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y(x) = C1 \cos(x) + C2 \sin(x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function
  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = e^{a \cos(x)}\right]$
- Wronskian of solutions of the homogeneous equation
  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$
- Compute Wronskian
  $W(y_1(x), y_2(x)) = 1$
- Substitute functions into equation for $y_p(x)$
  $y_p(x) = -\cos(x) \left(\int \sin(x) e^{a \cos(x)} dx\right) + \sin(x) \left(\int \cos(x) e^{a \cos(x)} dx\right)$
- Compute integrals
  $y_p(x) = \frac{\sin(x)\left(\int \cos(x) e^{a \cos(x)} dx\right) a + \cos(x) e^{a \cos(x)}}{a}$

- Substitute particular solution into general solution to ODE
  $y(x) = C1 \cos(x) + C2 \sin(x) + \frac{\sin(x)\left(\int \cos(x) e^{a \cos(x)} dx\right) a + \cos(x) e^{a \cos(x)}}{a}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

Maple dsolve solution

Solving time : 0.007 (sec)
Leaf size : 85

```
dsolve(diff(diff(y(x),x),x)+y(x) = exp(a*cos(x)),y(x),
       series,x=0)
```

$$y = \left(-\frac{1}{720}x^6 + \frac{1}{24}x^4 + 1 - \frac{1}{2}x^2\right) y(0) + \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7\right) y'(0) + \frac{\mathrm{e}^a x^2}{2} + \frac{(-a-1)\,\mathrm{e}^a x^4}{24} + \frac{(3a^2+2a+1)\,\mathrm{e}^a x^6}{720} + O\left(x^8\right)$$

Mathematica DSolve solution

Solving time : 0.035 (sec)
Leaf size : 239

```
AsymptoticDSolveValue[{D[y[x],{x,2}]+y[x]==Exp[a*Cos[x]],{}},
       y[x],{x,0,7}]
```

$$\begin{aligned} y(x) \to & \left(-\frac{x^7}{5040} + \frac{x^5}{120} - \frac{x^3}{6} + x\right)\left(\frac{1}{120}\left(3a^2+7a+1\right)e^a x^5 - \frac{(15a^3+60a^2+31a+1)\,e^a x^7}{5040} - \frac{1}{6}(a+1)e^a x^3 + e^a x\right) \\ & + \left(-\frac{x^6}{720} + \frac{x^4}{24} - \frac{x^2}{2} + 1\right)\left(-\frac{1}{720}\left(15a^2+15a+1\right)e^a x^6 + \frac{(105a^3+210a^2+63a+1)\,e^a x^8}{40320} + \frac{1}{24}(3a+1)e^a x^4 - \frac{e^a x^2}{2}\right) \\ & + c_2\left(-\frac{x^7}{5040} + \frac{x^5}{120} - \frac{x^3}{6} + x\right) + c_1\left(-\frac{x^6}{720} + \frac{x^4}{24} - \frac{x^2}{2} + 1\right) \end{aligned}$$

2.5.19 problem 19

Internal problem ID [8656]
Book : Own collection of miscellaneous problems
Section : section 5.0
Problem number : 19
Date solved : `Tuesday, December 17, 2024 at 12:57:08 PM`
CAS classification : `[[_1st_order, _with_linear_symmetries]]`

Solve

$$y' = \frac{y}{2y\ln(y) + y - x}$$

Solved as first order Exact ode

Time used: 0.151 (sec)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(2y\ln(y)+y-x)\,\mathrm{d}y&=(y)\,\mathrm{d}x\\(-y)\,\mathrm{d}x+(2y\ln(y)+y-x)\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-y\\N(x,y)&=2y\ln(y)+y-x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-y)\\&=-1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(2y\ln(y)+y-x)\\&=-1\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \tag{1}$$

$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -y \mathrm{d}x \\
\phi &= -xy + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = -x + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2y \ln(y) + y - x$. Therefore equation (4) becomes

$$
2y \ln(y) + y - x = -x + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 2y \ln(y) + y
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \mathrm{d}y &= \int (y(2\ln(y) + 1)) \mathrm{d}y \\
f(y) &= y^2 \ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = -xy + y^2 \ln(y) + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$
c_1 = -xy + y^2 \ln(y)
$$

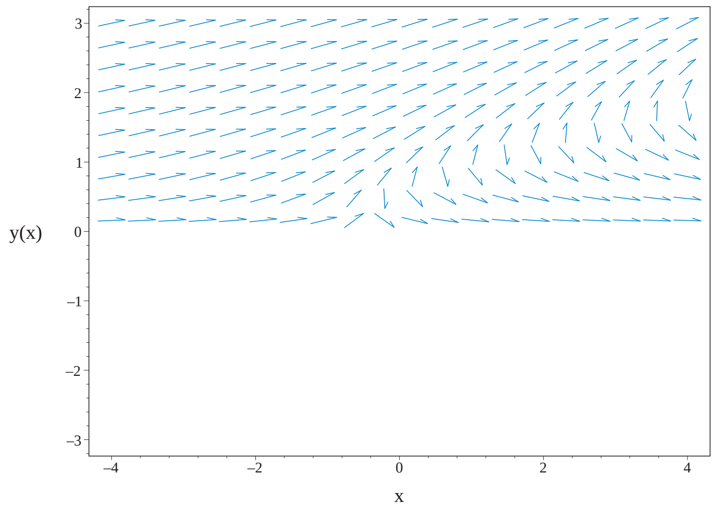


Figure 2.254: Slope field plot
$y' = \frac{y}{2y\ln(y)+y-x}$

Summary of solutions found

$$-xy + y^2 \ln(y) = c_1$$

Solved using Lie symmetry for first order ode

Time used: 0.657 (sec)

Writing the ode as

$$y' = \frac{y}{2y\ln(y) + y - x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2+\frac{y(b_3-a_2)}{2y\ln(y)+y-x}-\frac{y^2a_3}{(2y\ln(y)+y-x)^2}-\frac{y(xa_2+ya_3+a_1)}{(2y\ln(y)+y-x)^2} \\
& -\left(\frac{1}{2y\ln(y)+y-x}-\frac{y(2\ln(y)+3)}{(2y\ln(y)+y-x)^2}\right)(xb_2+yb_3+b_1)=0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$
\frac{4\ln(y)^2y^2b_2-4\ln(y)xyb_2-2\ln(y)y^2a_2+4\ln(y)y^2b_2+2\ln(y)y^2b_3+2x^2b_2-y^2a_2-2y^2a_3+y^2b_2+}{(2y\ln(y)+y-x)^2}
$$

$$
=0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
& 4\ln(y)^2y^2b_2-4\ln(y)xyb_2-2\ln(y)y^2a_2+4\ln(y)y^2b_2+2\ln(y)y^2b_3 \\
& +2x^2b_2-y^2a_2-2y^2a_3+y^2b_2+3y^2b_3+xb_1-ya_1+2yb_1=0
\end{aligned}
\tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$
\{x,y,\ln(y)\}
$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$
\{x=v_1,y=v_2,\ln(y)=v_3\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
& 4v_3^2v_2^2b_2-2v_3v_2^2a_2-4v_3v_1v_2b_2+4v_3v_2^2b_2+2v_3v_2^2b_3-v_2^2a_2 \\
& -2v_2^2a_3+2v_1^2b_2+v_2^2b_2+3v_2^2b_3-v_2a_1+v_1b_1+2v_2b_1=0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1,v_2,v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
& 2v_1^2b_2-4v_3v_1v_2b_2+v_1b_1+4v_3^2v_2^2b_2+(-2a_2+4b_2+2b_3)v_2^2v_3 \\
& +(-a_2-2a_3+b_2+3b_3)v_2^2+(-a_1+2b_1)v_2=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_1 &= 0\\
-4b_2 &= 0\\
2b_2 &= 0\\
4b_2 &= 0\\
-a_1 + 2b_1 &= 0\\
-2a_2 + 4b_2 + 2b_3 &= 0\\
-a_2 - 2a_3 + b_2 + 3b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0\\
a_2 &= b_3\\
a_3 &= b_3\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x + y\\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi\\
&= y - \left(\frac{y}{2y\ln(y) + y - x}\right)(x + y)\\
&= \frac{-2xy + 2y^2 \ln(y)}{2y\ln(y) + y - x}\\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-2xy+2y^2\ln(y)}{2y\ln(y)+y-x}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln\left(-xy + y^2\ln\left(y\right)\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y}{2y\ln\left(y\right) + y - x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{-2y\ln\left(y\right) + 2x} \\ S_y &= \frac{2y\ln\left(y\right) + y - x}{2y\left(y\ln\left(y\right) - x\right)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

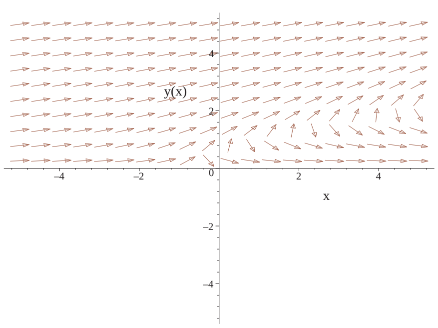
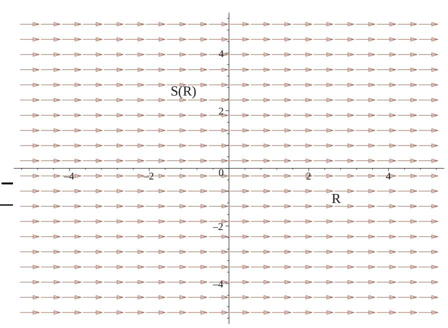
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int 0\, dR + c_2$$
$$S(R) = c_2$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\frac{\ln(y)}{2} + \frac{\ln(y\ln(y) - x)}{2} = c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{2y\ln(y)+y-x}$ 	$R = x$ $S = \frac{\ln(y)}{2} + \frac{\ln(y\ln(y) -}{2}$	$\frac{dS}{dR} = 0$ 

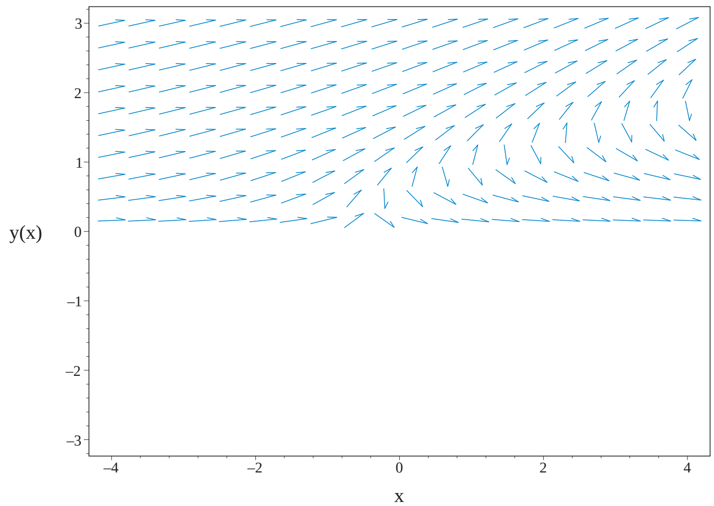


Figure 2.255: Slope field plot
$y' = \frac{y}{2y\ln(y)+y-x}$

Summary of solutions found

$$\frac{\ln(y)}{2} + \frac{\ln(y\ln(y) - x)}{2} = c_2$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) = \frac{y(x)}{2y(x)\ln(y(x))+y(x)-x}$

- Highest derivative means the order of the ODE is 1
$\frac{d}{dx}y(x)$
- Solve for the highest derivative
$\frac{d}{dx}y(x) = \frac{y(x)}{2y(x)\ln(y(x))+y(x)-x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
```

```
<- 1st order linear successful
<- inverse linear successful`
```

Maple dsolve solution

Solving time : 0.041 (sec)
Leaf size : 19

```
dsolve(diff(y(x),x) = y(x)/(2*y(x)*ln(y(x))+y(x)-x),
       y(x),singsol=all)
```

$$y = e^{\text{RootOf}\left(\_Z e^{2\_Z} - x e^{\_Z} + c_1\right)}$$

Mathematica DSolve solution

Solving time : 0.188 (sec)
Leaf size : 19

```
DSolve[{D[y[x],x]==y[x]/(2*y[x]*Log[y[x]]+y[x]-x),{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$\text{Solve}\left[x = y(x)\log(y(x)) + \frac{c_1}{y(x)}, y(x)\right]$$

2.5.20 problem 20

Internal problem ID [8657]
Book : Own collection of miscellaneous problems
Section : section 5.0
Problem number : 20
Date solved : `Thursday, December 12, 2024 at 09:37:51 AM`
CAS classification : `[[_2nd_order, _with_linear_symmetries]]`

Solve

$$xy'' - (2x+1)\,y' + (x+1)\,y = 0$$

Solved as second order ode using Kovacic algorithm

Time used: 0.125 (sec)

Writing the ode as

$$xy'' + (-2x-1)\,y' + (x+1)\,y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x \\ B &= -2x-1 \\ C &= x+1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.279: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)(0) \\ &= -\frac{1}{2x} \\ &= -\frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) &= 0\\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx}\\ &= e^{\int -\frac{1}{2x} dx}\\ &= \frac{1}{\sqrt{x}}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-2x-1}{x}\, dx}\\ &= z_1 e^{x + \frac{\ln(x)}{2}}\\ &= z_1\left(\sqrt{x}\, e^x\right)\end{aligned}$$

Which simplifies to

$$y_1 = e^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x-1}{x}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{2x+\ln(x)}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{x\, \mathrm{e}^{2x+\ln(x)} \mathrm{e}^{-2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{x}) + c_2\left(\mathrm{e}^{x}\left(\frac{x\, \mathrm{e}^{2x+\ln(x)} \mathrm{e}^{-2x}}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$
y = c_1\, \mathrm{e}^{x} + \frac{c_2\, x^2 \mathrm{e}^{x}}{2}
$$

Solved as second order ode adjoint method

Time used: 0.583 (sec)

In normal form the ode

$$
x y'' - (2x+1)\, y' + (x+1)\, y = 0 \tag{1}
$$

Becomes

$$
y'' + p(x)\, y' + q(x)\, y = r(x) \tag{2}
$$

Where

$$
\begin{aligned}
p(x) &= \frac{-2x-1}{x} \\
q(x) &= \frac{x+1}{x} \\
r(x) &= 0
\end{aligned}
$$

The Lagrange adjoint ode is given by

$$
\begin{gathered}
\xi''-(\xi\,p)'+\xi q=0\\
\xi''-\left(\frac{(-2x-1)\,\xi(x)}{x}\right)'+\left(\frac{(x+1)\,\xi(x)}{x}\right)=0\\
\frac{\xi''(x)\,x^2+(2x^2+x)\,\xi'(x)+\xi(x)\,(x^2+x-1)}{x^2}=0
\end{gathered}
$$

Which is solved for $\xi(x)$. Writing the ode as

$$
\xi''+\left(2+\frac{1}{x}\right)\xi'+\left(1+\frac{1}{x}-\frac{1}{x^2}\right)\xi=0 \tag{1}
$$

$$
A\xi''+B\xi'+C\xi=0 \tag{2}
$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A&=1\\
B&=2+\frac{1}{x}\\
C&=1+\frac{1}{x}-\frac{1}{x^2}
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x)=\xi e^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x)=rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r&=\frac{s}{t}\\
&=\frac{2AB'-2BA'+B^2-4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r=\frac{3}{4x^2} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s&=3\\
t&=4x^2
\end{aligned}
$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then ξ is found using the inverse transformation

$$\xi = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.280: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s=(s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^-=-\frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right)\\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right)\\
&= 0
\end{aligned}
$$

Since d an integer and $d\geq 0$ then it can be used to find ω using

$$
\omega=\sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c+\frac{\alpha_c^{s(c)}}{x-c}\right)+s(\infty)[\sqrt{r}]_\infty
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1}+\frac{\alpha_{c_1}^-}{x-c_1}\right)+(-)[\sqrt{r}]_\infty\\
&= -\frac{1}{2x}+(-)\,(0)\\
&= -\frac{1}{2x}\\
&= -\frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p''+2\omega p'+\left(\omega'+\omega^2-r\right)p=0 \tag{1A}
$$

Let

$$
p(x)=1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0)+2\left(-\frac{1}{2x}\right)(0)+\left(\left(\frac{1}{2x^2}\right)+\left(-\frac{1}{2x}\right)^2-\left(\frac{3}{4x^2}\right)\right) &= 0\\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$
\begin{aligned}
z_1(x) &= pe^{\int \omega\,dx}\\
&= \mathrm{e}^{\int -\frac{1}{2x}dx}\\
&= \frac{1}{\sqrt{x}}
\end{aligned}
$$

The first solution to the original ode in ξ is found from

$$
\begin{aligned}
\xi_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2+\frac{1}{x}}{1}\,dx} \\
&= z_1 e^{-x-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{\mathrm{e}^{-x}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
\xi_1 = \frac{\mathrm{e}^{-x}}{x}
$$

The second solution ξ_2 to the original ode is found using reduction of order

$$
\xi_2 = \xi_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{\xi_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
\xi_2 &= \xi_1 \int \frac{e^{\int -\frac{2+\frac{1}{x}}{1}\,dx}}{(\xi_1)^2}\,dx \\
&= \xi_1 \int \frac{e^{-2x-\ln(x)}}{(\xi_1)^2}\,dx \\
&= \xi_1 \left(\frac{x^3 \mathrm{e}^{-2x-\ln(x)}\mathrm{e}^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
\xi &= c_1\xi_1 + c_2\xi_2 \\
&= c_1\left(\frac{\mathrm{e}^{-x}}{x}\right) + c_2\left(\frac{\mathrm{e}^{-x}}{x}\left(\frac{x^3 \mathrm{e}^{-2x-\ln(x)}\mathrm{e}^{2x}}{2}\right)\right)
\end{aligned}
$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$
\begin{aligned}
\xi(x)\, y' - y\xi'(x) + \xi(x)\, p(x)\, y &= \int \xi(x)\, r(x)\, dx \\
y' + y\left(p(x) - \frac{\xi'(x)}{\xi\,(x)}\right) &= \frac{\int \xi(x)\, r(x)\, dx}{\xi\,(x)}
\end{aligned}
$$

Or

$$
y' + y\left(\frac{-2x-1}{x} - \frac{-\frac{c_1\,\mathrm{e}^{-x}}{x^2} - \frac{c_1\,\mathrm{e}^{-x}}{x} + \frac{c_2\,\mathrm{e}^{-x}}{2} - \frac{c_2 x\,\mathrm{e}^{-x}}{2}}{\frac{c_1\,\mathrm{e}^{-x}}{x} + \frac{c_2 x\,\mathrm{e}^{-x}}{2}}\right) = 0
$$

Which is now a first order ode. This is now solved for y. In canonical form a linear first order is

$$
y' + q(x)y = p(x)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= -\frac{(x^2+2x)\, c_2 + 2c_1}{c_2\, x^2 + 2c_1} \\
p(x) &= 0
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\, dx} \\
&= \mathrm{e}^{\int -\frac{(x^2+2x)c_2+2c_1}{c_2\, x^2+2c_1} dx} \\
&= \frac{\mathrm{e}^{-x}}{c_2\, x^2 + 2c_1}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y\,\mathrm{e}^{-x}}{c_2\, x^2 + 2c_1}\right) &= 0
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y\,\mathrm{e}^{-x}}{c_2\, x^2 + 2c_1} &= \int 0\, dx + c_3 \\
&= c_3
\end{aligned}
$$

Dividing throughout by the integrating factor $\frac{\mathrm{e}^{-x}}{c_2\, x^2+2c_1}$ gives the final solution

$$
y = (c_2\, x^2 + 2c_1)\, \mathrm{e}^{x} c_3
$$

Hence, the solution found using Lagrange adjoint equation method is

$$y = \left(c_2\, x^2 + 2c_1\right) \mathrm{e}^x c_3$$

The constants can be merged to give

$$y = \left(c_2\, x^2 + 2c_1\right) \mathrm{e}^x$$

Will add steps showing solving for IC soon.

Summary of solutions found

$$y = \left(c_2\, x^2 + 2c_1\right) \mathrm{e}^x$$

Maple step by step solution

Let's solve
$\left(\frac{d^2}{dx^2}y(x)\right) x - (2x+1)\left(\frac{d}{dx}y(x)\right) + (x+1)\, y(x) = 0$

- Highest derivative means the order of the ODE is 2
  $\frac{d^2}{dx^2}y(x)$
- Isolate 2nd derivative
  $\frac{d^2}{dx^2}y(x) = -\frac{(x+1)y(x)}{x} + \frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{x}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin
  $\frac{d^2}{dx^2}y(x) - \frac{(2x+1)\left(\frac{d}{dx}y(x)\right)}{x} + \frac{(x+1)y(x)}{x} = 0$

☐ Check to see if $x_0 = 0$ is a regular singular point
  - Define functions
    $\left[P_2(x) = -\frac{2x+1}{x}, P_3(x) = \frac{x+1}{x}\right]$
  - $x \cdot P_2(x)$ is analytic at $x = 0$
    $\left(x \cdot P_2(x)\right)\Big|_{x=0} = -1$
  - $x^2 \cdot P_3(x)$ is analytic at $x = 0$
    $\left(x^2 \cdot P_3(x)\right)\Big|_{x=0} = 0$
  - $x = 0$is a regular singular point
    Check to see if $x_0 = 0$ is a regular singular point
    $x_0 = 0$

- Multiply by denominators
  $\left(\frac{d^2}{dx^2}y(x)\right)x + (-2x-1)\left(\frac{d}{dx}y(x)\right) + (x+1)\,y(x) = 0$
- Assume series solution for $y(x)$
  $y(x) = \sum_{k=0}^{\infty} a_k x^{k+r}$

□ Rewrite ODE with series expansions

- ◦ Convert $x^m \cdot y(x)$ to series expansion for $m = 0..1$
  $x^m \cdot y(x) = \sum_{k=0}^{\infty} a_k x^{k+r+m}$
- ◦ Shift index using $k->k-m$
  $x^m \cdot y(x) = \sum_{k=m}^{\infty} a_{k-m} x^{k+r}$
- ◦ Convert $x^m \cdot \left(\frac{d}{dx}y(x)\right)$ to series expansion for $m = 0..1$
  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\, x^{k+r-1+m}$
- ◦ Shift index using $k->k+1-m$
  $x^m \cdot \left(\frac{d}{dx}y(x)\right) = \sum_{k=-1+m}^{\infty} a_{k+1-m}(k+1-m+r)\, x^{k+r}$
- ◦ Convert $x \cdot \left(\frac{d^2}{dx^2}y(x)\right)$ to series expansion
  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=0}^{\infty} a_k (k+r)\,(k+r-1)\, x^{k+r-1}$
- ◦ Shift index using $k->k+1$
  $x \cdot \left(\frac{d^2}{dx^2}y(x)\right) = \sum_{k=-1}^{\infty} a_{k+1}(k+1+r)\,(k+r)\, x^{k+r}$

  Rewrite ODE with series expansions
  $a_0 r(-2+r)\, x^{-1+r} + \left(a_1(1+r)\,(-1+r) - a_0(-1+2r)\right) x^r + \left(\sum_{k=1}^{\infty} \left(a_{k+1}(k+1+r)\,(k+r-1)\right.\right.$

- $a_0$cannot be 0 by assumption, giving the indicial equation
  $r(-2+r) = 0$
- Values of r that satisfy the indicial equation
  $r \in \{0, 2\}$
- Each term must be 0
  $a_1(1+r)\,(-1+r) - a_0(-1+2r) = 0$
- Each term in the series must be 0, giving the recursion relation
  $a_{k+1}(k+1+r)\,(k+r-1) + a_k(-2k-2r+1) + a_{k-1} = 0$

- Shift index using $k- >k+1$
  $a_{k+2}(k+2+r)(k+r)+a_{k+1}(-2k-1-2r)+a_k=0$
- Recursion relation that defines series solution to ODE
  $a_{k+2}=\frac{2ka_{k+1}+2ra_{k+1}-a_k+a_{k+1}}{(k+2+r)(k+r)}$
- Recursion relation for $r=0$
  $a_{k+2}=\frac{2ka_{k+1}-a_k+a_{k+1}}{(k+2)k}$
- Series not valid for $r=0$, division by 0 in the recursion relation at $k=0$
  $a_{k+2}=\frac{2ka_{k+1}-a_k+a_{k+1}}{(k+2)k}$
- Recursion relation for $r=2$
  $a_{k+2}=\frac{2ka_{k+1}-a_k+5a_{k+1}}{(k+4)(k+2)}$
- Solution for $r=2$
  $\left[y(x)=\sum_{k=0}^{\infty} a_k x^{k+2}, a_{k+2}=\frac{2ka_{k+1}-a_k+5a_{k+1}}{(k+4)(k+2)}, 3a_1-3a_0=0\right]$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
trying a symmetry of the form [xi=0, eta=F(x)]
checking if the LODE is missing y
-> Trying a Liouvillian solution using Kovacics algorithm
   A Liouvillian solution exists
   Reducible group (found an exponential solution)
<- Kovacics algorithm successful`
```

Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 14

```
dsolve(x*diff(diff(y(x),x),x)-(2*x+1)*diff(y(x),x)+y(x)*(x+1) = 0,
       y(x),singsol=all)
```

$$y=e^{x}\left(c_2x^2+c_1\right)$$

Mathematica DSolve solution

Solving time : 0.029 (sec)
Leaf size : 23

```
DSolve[{x*D[y[x],{x,2}]-(2*x+1)*D[y[x],x]+(x+1)*y[x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}e^x\left(c_2x^2 + 2c_1\right)$$

2.5.21 problem 21

Internal problem ID [8658]
Book : Own collection of miscellaneous problems
Section : section 5.0
Problem number : 21
Date solved : `Tuesday, December 17, 2024 at 12:57:10 PM`
CAS classification : `[_separable]`

Solve

$$x^2y' + \mathrm{e}^{-y} = 0$$

Solved as first order separable ode

Time used: 0.113 (sec)

The ode $y' = -\frac{\mathrm{e}^{-y}}{x^2}$ is separable as it can be written as

$$\begin{aligned} y' &= -\frac{\mathrm{e}^{-y}}{x^2} \\ &= f(x)g(y) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x^2} \\ g(y) &= \mathrm{e}^{-y} \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(y)}\, dy &= \int f(x)\, dx \\ \int \mathrm{e}^{y}\, dy &= \int -\frac{1}{x^2}\, dx \\ \mathrm{e}^{y} &= \frac{1}{x} + c_1 \end{aligned}$$

Solving for y gives

$$y = \ln\left(\frac{c_1 x + 1}{x}\right)$$

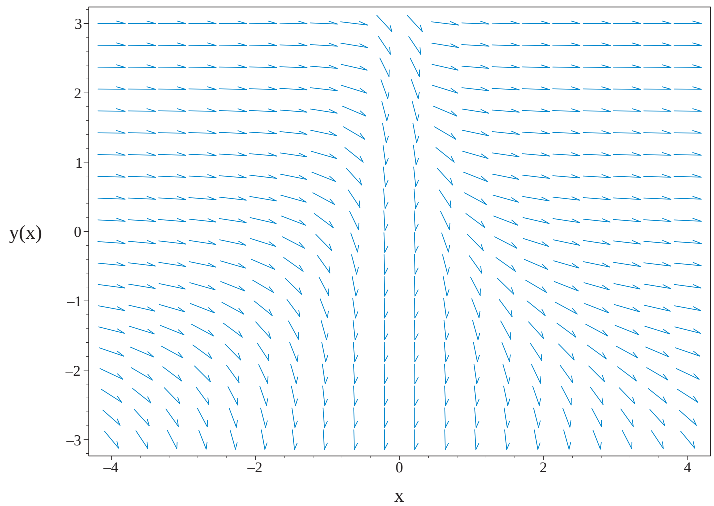


Figure 2.256: Slope field plot
$x^2 y' + \mathrm{e}^{-y} = 0$

Summary of solutions found

$$y = \ln\left(\frac{c_1 x + 1}{x}\right)$$

Solved using Lie symmetry for first order ode

Time used: 0.567 (sec)

Writing the ode as

$$y' = -\frac{\mathrm{e}^{-y}}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{e^{-y}(b_3 - a_2)}{x^2} - \frac{e^{-2y}a_3}{x^4} - \frac{2\,e^{-y}(xa_2 + ya_3 + a_1)}{x^3} - \frac{e^{-y}(xb_2 + yb_3 + b_1)}{x^2} = 0 \quad (5E)$$

Putting the above in normal form gives

$$-\frac{e^{-y}x^3b_2 + e^{-y}x^2yb_3 - b_2x^4 + e^{-y}x^2a_2 + e^{-y}x^2b_1 + e^{-y}x^2b_3 + 2\,e^{-y}xya_3 + e^{-2y}a_3 + 2\,e^{-y}xa_1}{x^4} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-e^{-y}x^3b_2 - e^{-y}x^2yb_3 + b_2x^4 - e^{-y}x^2a_2 - e^{-y}x^2b_1 \\ &- e^{-y}x^2b_3 - 2\,e^{-y}xya_3 - e^{-2y}a_3 - 2\,e^{-y}xa_1 = 0 \end{aligned} \quad (6E)$$

Simplifying the above gives

$$\begin{aligned} &-e^{-y}x^3b_2 - e^{-y}x^2yb_3 + b_2x^4 - e^{-y}x^2a_2 - e^{-y}x^2b_1 \\ &- e^{-y}x^2b_3 - 2\,e^{-y}xya_3 - e^{-2y}a_3 - 2\,e^{-y}xa_1 = 0 \end{aligned} \quad (6E)$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, e^{-2y}, e^{-y}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, e^{-2y} = v_3, e^{-y} = v_4\}$$

The above PDE (6E) now becomes

$$b_2v_1^4 - v_4v_1^3b_2 - v_4v_1^2v_2b_3 - v_4v_1^2a_2 - 2v_4v_1v_2a_3 - v_4v_1^2b_1 - v_4v_1^2b_3 - 2v_4v_1a_1 - v_3a_3 = 0 \quad (7E)$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$b_2v_1^4 - v_4v_1^3b_2 - v_4v_1^2v_2b_3 + (-a_2 - b_1 - b_3)\,v_1^2v_4 - 2v_4v_1v_2a_3 - 2v_4v_1a_1 - v_3a_3 = 0 \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_2 &= 0\\ -2a_1 &= 0\\ -2a_3 &= 0\\ -a_3 &= 0\\ -b_2 &= 0\\ -b_3 &= 0\\ -a_2 - b_1 - b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0\\ a_2 &= -b_1\\ a_3 &= 0\\ b_1 &= b_1\\ b_2 &= 0\\ b_3 &= 0 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= -x\\ \eta &= 1 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x,y)\,\xi\\ &= 1 - \left(-\frac{\mathrm{e}^{-y}}{x^2}\right)(-x)\\ &= \frac{-\mathrm{e}^{-y} + x}{x}\\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-e^{-y}+x}{x}} dy \end{aligned}$$

Which results in

$$S = \ln\left(x\,e^{y} - 1\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{e^{-y}}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{e^{y}}{x\,e^{y} - 1} \\ S_y &= \frac{x\,e^{y}}{x\,e^{y} - 1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int \frac{1}{R}\,dR$$
$$S(R) = \ln(R) + c_2$$

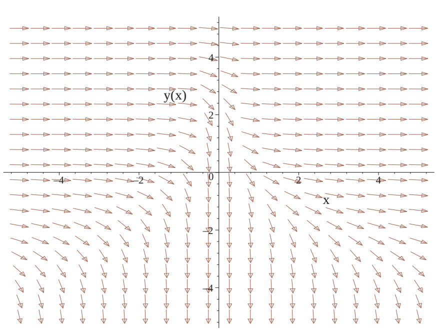
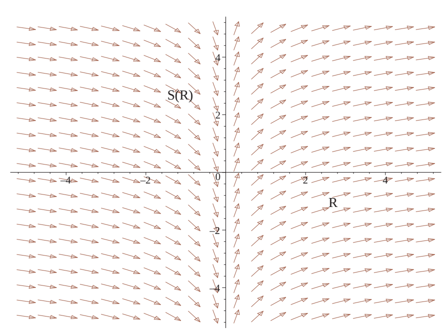
To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\ln(x\,e^{y} - 1) = \ln(x) + c_2$$

Which gives

$$y = \ln\left(\frac{1 + x\,e^{c_2}}{x}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{e^{-y}}{x^2}$ 	$R = x$ $S = \ln(x\,e^{y} - 1)$	$\frac{dS}{dR} = \frac{1}{R}$ 

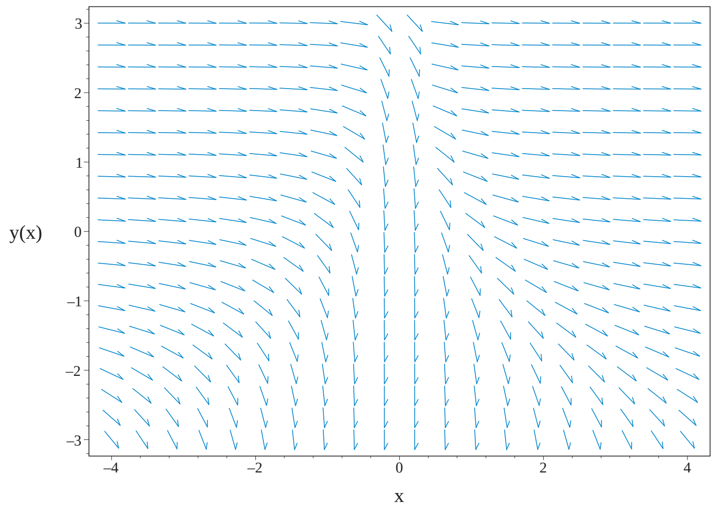


Figure 2.257: Slope field plot
$x^2y' + e^{-y} = 0$

Summary of solutions found

$$y = \ln\left(\frac{1 + x\,e^{c_2}}{x}\right)$$

Solved as first order ode of type ID 1

Time used: 0.053 (sec)

Writing the ode as

$$y' = -\frac{e^{-y}}{x^2} \tag{1}$$

And using the substitution $u = e^y$ then

$$u' = y' e^y$$

The above shows that

$$\begin{aligned} y' &= u'(x)\, e^{-y} \\ &= \frac{u'(x)}{u} \end{aligned}$$

Substituting this in (1) gives

$$\frac{u'(x)}{u} = -\frac{1}{x^2 u}$$

The above simplifies to

$$u'(x) = -\frac{1}{x^2} \tag{2}$$

Now ode (2) is solved for $u(x)$.

Since the ode has the form $u'(x) = f(x)$, then we only need to integrate $f(x)$.

$$\int du = \int -\frac{1}{x^2}\, dx$$
$$u(x) = \frac{1}{x} + c_1$$

Substituting the solution found for $u(x)$ in $u = e^y$ gives

$$\begin{aligned} y &= \ln(u(x)) \\ &= \ln\left(\ln\left(\frac{1}{x} + c_1\right)\right) \\ &= \ln\left(\frac{1}{x} + c_1\right) \end{aligned}$$

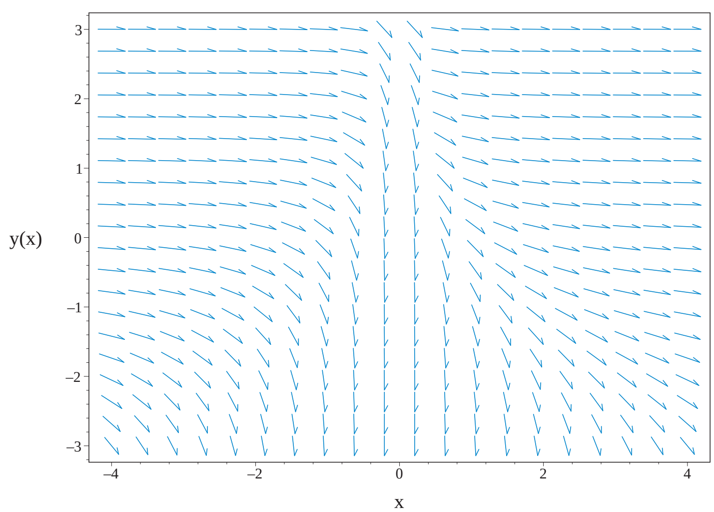


Figure 2.258: Slope field plot
$x^2y' + e^{-y} = 0$

Summary of solutions found

$$y = \ln\left(\frac{1}{x} + c_1\right)$$

Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y(x)\right)+\mathrm{e}^{-y(x)}=0$

- Highest derivative means the order of the ODE is 1

$\frac{d}{dx}y(x)$

- Solve for the highest derivative

$\frac{d}{dx}y(x)=-\frac{\mathrm{e}^{-y(x)}}{x^2}$

- Separate variables

$\frac{\frac{d}{dx}y(x)}{\mathrm{e}^{-y(x)}}=-\frac{1}{x^2}$

- Integrate both sides with respect to x

$\int\frac{\frac{d}{dx}y(x)}{\mathrm{e}^{-y(x)}}dx=\int-\frac{1}{x^2}dx+C1$

- Evaluate integral

$\frac{1}{\mathrm{e}^{-y(x)}}=\frac{1}{x}+C1$

- Solve for $y(x)$

$y(x)=-\ln\left(\frac{x}{C1x+1}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

Maple dsolve solution

Solving time : 0.008 (sec)
Leaf size : 15

```
dsolve(diff(y(x),x)*x^2+exp(-y(x)) = 0,
       y(x),singsol=all)
```

$$y=\ln\left(\frac{-c_1x+1}{x}\right)$$

Mathematica DSolve solution

Solving time : 0.481 (sec)
Leaf size : 12

```
DSolve[{x^2*D[y[x],x]+Exp[-y[x]]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \log \left(\frac{1}{x} + c_1\right)$$

2.5.22 problem 22

Internal problem ID [8659]
Book : Own collection of miscellaneous problems
Section : section 5.0
Problem number : 22
Date solved : `Thursday, December 12, 2024 at 09:37:53 AM`
CAS classification :
`[[_2nd_order, _missing_x], [_2nd_order, _reducible, _mu_x_y1]]`

Solve

$$y'' + e^{y} = 0$$

Solved as second order missing x ode

Time used: 7.970 (sec)

This is missing independent variable second order ode. Solved by reduction of order by using substitution which makes the dependent variable y an independent variable. Using

$$y' = p$$

Then

$$\begin{aligned} y'' &= \frac{dp}{dx} \\ &= \frac{dp}{dy}\frac{dy}{dx} \\ &= p\frac{dp}{dy} \end{aligned}$$

Hence the ode becomes

$$p(y)\left(\frac{d}{dy}p(y)\right) + e^{y} = 0$$

Which is now solved as first order ode for $p(y)$.

The ode $p' = -\frac{e^y}{p}$ is separable as it can be written as

$$\begin{aligned} p' &= -\frac{e^y}{p} \\ &= f(y)g(p) \end{aligned}$$

Where

$$\begin{aligned} f(y) &= -e^y \\ g(p) &= \frac{1}{p} \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(p)}\, dp &= \int f(y)\, dy \\ \int p\, dp &= \int -e^y\, dy \\ \frac{p^2}{2} &= -e^y + c_1 \end{aligned}$$

Solving for p gives

$$\begin{aligned} p &= \sqrt{-2\,e^y + 2c_1} \\ p &= -\sqrt{-2\,e^y + 2c_1} \end{aligned}$$

For solution (1) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = \sqrt{-2\,e^y + 2c_1}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{\sqrt{-2\,e^y + 2c_1}} dy &= dx \\ -\frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{-2\,e^y+2c_1}\,\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} &= x + c_2 \end{aligned}$$

Singular solutions are found by solving

$$\sqrt{-2\,e^y + 2c_1} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = \ln(c_1)$$

For solution (2) found earlier, since $p = y'$ then we now have a new first order ode to solve which is

$$y' = -\sqrt{-2\,e^{y} + 2c_1}$$

Integrating gives

$$\int -\frac{1}{\sqrt{-2\,e^{y} + 2c_1}} dy = dx$$

$$\frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{-2\,e^{y}+2c_1}\,\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} = x + c_3$$

Singular solutions are found by solving

$$-\sqrt{-2\,e^{y} + 2c_1} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = \ln(c_1)$$

Solving for y gives

$$y = \ln\left(-\tanh\left(\frac{\sqrt{c_1}\,(x + c_3)\sqrt{2}}{2}\right)^2 c_1 + c_1\right)$$

Will add steps showing solving for IC soon.

Solving for y from the above solution(s) gives (after possible removing of solutions that do not verify)

$$\begin{aligned}
y &= \ln(c_1) \\
y &= \ln\left(-\tanh\left(\frac{\sqrt{c_1}\,(x + c_2)\sqrt{2}}{2}\right)^2 c_1 + c_1\right) \\
y &= \ln\left(-\tanh\left(\frac{\sqrt{c_1}\,(x + c_3)\sqrt{2}}{2}\right)^2 c_1 + c_1\right)
\end{aligned}$$

The solution

$$y = \ln(c_1)$$

was found not to satisfy the ode or the IC. Hence it is removed.

Summary of solutions found

$$y = \ln\left(-\tanh\left(\frac{\sqrt{c_1}\,(x+c_2)\,\sqrt{2}}{2}\right)^2 c_1 + c_1\right)$$
$$y = \ln\left(-\tanh\left(\frac{\sqrt{c_1}\,(x+c_3)\,\sqrt{2}}{2}\right)^2 c_1 + c_1\right)$$

Solved as second order can be made integrable

Time used: 6.477 (sec)

Multiplying the ode by y' gives

$$y'y'' + y'\mathrm{e}^{y} = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + y'\mathrm{e}^{y})\, dx = 0$$
$$\frac{y'^2}{2} + \mathrm{e}^{y} = c_1$$

Which is now solved for y. Solving for the derivative gives these ODE's to solve

$$y' = \sqrt{-2\,\mathrm{e}^{y} + 2c_1} \tag{1}$$
$$y' = -\sqrt{-2\,\mathrm{e}^{y} + 2c_1} \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Integrating gives

$$\int \frac{1}{\sqrt{-2\,\mathrm{e}^{y} + 2c_1}} dy = dx$$
$$-\frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{-2\,\mathrm{e}^{y}+2c_1}\,\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} = x + c_2$$

Singular solutions are found by solving

$$\sqrt{-2\,e^{y}+2c_1} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = \ln(c_1)$$

Solving Eq. (2)

Integrating gives

$$\int -\frac{1}{\sqrt{-2\,e^{y}+2c_1}}dy = dx$$
$$\frac{\sqrt{2}\,\operatorname{arctanh}\left(\frac{\sqrt{-2\,e^{y}+2c_1}\,\sqrt{2}}{2\sqrt{c_1}}\right)}{\sqrt{c_1}} = x + c_3$$

Singular solutions are found by solving

$$-\sqrt{-2\,e^{y}+2c_1} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = \ln(c_1)$$

Solving for y gives

$$y = \ln\left(-\tanh\left(\frac{\sqrt{c_1}\,(x+c_3)\,\sqrt{2}}{2}\right)^2 c_1 + c_1\right)$$

Will add steps showing solving for IC soon.

Solving for y from the above solution(s) gives (after possible removing of solutions that do not verify)

$$y = \ln(c_1)$$
$$y = \ln\left(-\tanh\left(\frac{\sqrt{c_1}\,(x+c_2)\,\sqrt{2}}{2}\right)^2 c_1 + c_1\right)$$
$$y = \ln\left(-\tanh\left(\frac{\sqrt{c_1}\,(x+c_3)\,\sqrt{2}}{2}\right)^2 c_1 + c_1\right)$$

The solution

$$y = \ln(c_1)$$

was found not to satisfy the ode or the IC. Hence it is removed.

Summary of solutions found

$$y = \ln\left(-\tanh\left(\frac{\sqrt{c_1}\,(x+c_2)\,\sqrt{2}}{2}\right)^2 c_1 + c_1\right)$$

$$y = \ln\left(-\tanh\left(\frac{\sqrt{c_1}\,(x+c_3)\,\sqrt{2}}{2}\right)^2 c_1 + c_1\right)$$

Maple step by step solution

Let's solve

$\frac{d^2}{dx^2}y(x) + \mathrm{e}^{y(x)} = 0$

- Highest derivative means the order of the ODE is 2

  $\frac{d^2}{dx^2}y(x)$
- Define new dependent variable u

  $u(x) = \frac{d}{dx}y(x)$
- Compute $\frac{d^2}{dx^2}y(x)$

  $\frac{d}{dx}u(x) = \frac{d^2}{dx^2}y(x)$
- Use chain rule on the lhs

  $\left(\frac{d}{dx}y(x)\right)\left(\frac{d}{dy}u(y)\right) = \frac{d^2}{dx^2}y(x)$
- Substitute in the definition of u

  $u(y)\left(\frac{d}{dy}u(y)\right) = \frac{d^2}{dx^2}y(x)$
- Make substitutions $\frac{d}{dx}y(x) = u(y)\,, \frac{d^2}{dx^2}y(x) = u(y)\left(\frac{d}{dy}u(y)\right)$ to reduce order of ODE

  $u(y)\left(\frac{d}{dy}u(y)\right) + \mathrm{e}^{y} = 0$
- Separate variables

  $u(y)\left(\frac{d}{dy}u(y)\right) = -\mathrm{e}^{y}$
- Integrate both sides with respect to y

$$\int u(y)\left(\frac{d}{dy}u(y)\right)dy = \int -\mathrm{e}^{y}dy + C1$$

- Evaluate integral

$$\frac{u(y)^2}{2} = -\mathrm{e}^{y} + C1$$

- Solve for $u(y)$

$$\left\{u(y) = \sqrt{-2\,\mathrm{e}^{y} + 2C1}, u(y) = -\sqrt{-2\,\mathrm{e}^{y} + 2C1}\right\}$$

- Solve 1st ODE for $u(y)$

$$u(y) = \sqrt{-2\,\mathrm{e}^{y} + 2C1}$$

- Revert to original variables with substitution $u(y) = \frac{d}{dx}y(x)\,, y = y(x)$

$$\frac{d}{dx}y(x) = \sqrt{-2\,\mathrm{e}^{y(x)} + 2C1}$$

- Solve for the highest derivative

$$\frac{d}{dx}y(x) = \sqrt{-2\,\mathrm{e}^{y(x)} + 2C1}$$

- Separate variables

$$\frac{\frac{d}{dx}y(x)}{\sqrt{-2\,\mathrm{e}^{y(x)}+2C1}} = 1$$

- Integrate both sides with respect to x

$$\int \frac{\frac{d}{dx}y(x)}{\sqrt{-2\,\mathrm{e}^{y(x)}+2C1}}dx = \int 1dx + C2$$

- Evaluate integral

$$-\frac{\sqrt{2}\,\mathrm{arctanh}\left(\frac{\sqrt{-2\,\mathrm{e}^{y(x)}+2C1}\,\sqrt{2}}{2\sqrt{C1}}\right)}{\sqrt{C1}} = x + C2$$

- Solve for $y(x)$

$$y(x) = \ln\left(-\tanh\left(\frac{\sqrt{C1}\,(x+C2)\sqrt{2}}{2}\right)^2 C1 + C1\right)$$

- Solve 2nd ODE for $u(y)$

$$u(y) = -\sqrt{-2\,\mathrm{e}^{y} + 2C1}$$

- Revert to original variables with substitution $u(y) = \frac{d}{dx}y(x)\,, y = y(x)$

$$\frac{d}{dx}y(x) = -\sqrt{-2\,\mathrm{e}^{y(x)} + 2C1}$$

- Solve for the highest derivative

$$\frac{d}{dx}y(x) = -\sqrt{-2\,\mathrm{e}^{y(x)} + 2C1}$$

- Separate variables

$$\frac{\frac{d}{dx}y(x)}{\sqrt{-2\,\mathrm{e}^{y(x)}+2C1}} = -1$$

- Integrate both sides with respect to x

$$\int \frac{\frac{d}{dx}y(x)}{\sqrt{-2\,\mathrm{e}^{y(x)}+2C1}}dx = \int (-1)\,dx + C2$$

- Evaluate integral

$$-\frac{\sqrt{2}\,\mathrm{arctanh}\left(\frac{\sqrt{-2\,\mathrm{e}^{y(x)}+2C1}\,\sqrt{2}}{2\sqrt{C1}}\right)}{\sqrt{C1}} = -x + C2$$

- Solve for $y(x)$

$$y(x) = \ln\left(-\tanh\left(\frac{\sqrt{C1}\,(-x+C2)\sqrt{2}}{2}\right)^2 C1 + C1\right)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying 2nd order Liouville
trying 2nd order WeierstrassP
trying 2nd order JacobiSN
differential order: 2; trying a linearization to 3rd order
trying 2nd order ODE linearizable_by_differentiation
trying 2nd order, 2 integrating factors of the form mu(x,y)
trying differential order: 2; missing variables
`, `-> Computing symmetries using: way = 3
-> Calling odsolve with the ODE`, (diff(_b(_a), _a))*_b(_a)+exp(_a) = 0, _b(_a), HINT
   symmetry methods on request
`, `1st order, trying reduction of order with given symmetries:`[1, 1/2*_b]
```

Maple dsolve solution

Solving time : `0.087` (sec)
Leaf size : 25

```
dsolve(diff(diff(y(x),x),x)+exp(y(x)) = 0,
       y(x),singsol=all)
```

$$y = -\ln(2) + \ln\left(\frac{\operatorname{sech}\left(\frac{x+c_2}{2c_1}\right)^2}{c_1^2}\right)$$

Mathematica DSolve solution

Solving time : 21.487 (sec)
Leaf size : 60

```
DSolve[{D[y[x],{x,2}]+Exp[y[x]]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \log \left(\frac{1}{2}c_1 \operatorname{sech}^2\left(\frac{1}{2}\sqrt{c_1(x+c_2)^2}\right)\right)$$
$$y(x) \to \log \left(\frac{1}{2}c_1 \operatorname{sech}^2\left(\frac{\sqrt{c_1 x^2}}{2}\right)\right)$$

2.5.23 problem 23

Internal problem ID [8660]
Book : Own collection of miscellaneous problems
Section : section 5.0
Problem number : 23
Date solved : `Tuesday, December 17, 2024 at 12:57:12 PM`
CAS classification : `[_rational, [_Abel, '2nd type', 'class B']]`

Solve

$$y' = \frac{xy + 3x - 2y + 6}{xy - 3x - 2y + 6}$$

Unknown ode type.

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) = \frac{xy(x)+3x-2y(x)+6}{xy(x)-3x-2y(x)+6}$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = \frac{xy(x)+3x-2y(x)+6}{xy(x)-3x-2y(x)+6}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
```

```
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

Maple dsolve solution

Solving time : 0.031 (sec)
Leaf size : maple_leaf_size

```
dsolve(diff(y(x),x) = (x*y(x)+3*x-2*y(x)+6)/(x*y(x)-3*x-2*y(x)+6),
       y(x),singsol=all)
```

No solution found

Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{D[y[x],x]==(x*y[x]+3*x-2*y[x]+6)/(x*y[x]-3*x-2*y[x]+6),{}},
       y[x],x,IncludeSingularSolutions->True]
```

Not solved