A Solution Manual For

First order enumerated odes

Nasser M. Abbasi December 31, 2024 Compiled on December 31, 2024 at 7:27am

Contents

CHAPTER 1

Lookup tables for all problems in current book

1.1 section 1

Table 1.1: Lookup table for all problems in current section

ID	problem	ODE
8661	1	$y' = 0$
8662	2	$y' = a$
8663	3	$y' = x$
8664	4	$y' = 1$
8665	5	$y' = ax$
8666	6	$y' = axy$
8667	7	$y' = ax + y$
8668	8	$y' = ax + by$
8669	9	$y' = y$
8670	10	$y' = by$
8671	11	$y' = ax + by^2$
8672	12	$cy' = 0$
8673	13	$cy' = a$
8674	14	$cy' = ax$
8675	15	$cy' = ax + y$
8676	16	$cy' = ax + by$
8677	17	$cy' = y$
8678	18	$cy' = by$
8679	19	$cy' = ax + by^2$
8680	20	$cy' = \frac{ax+by^2}{r}$
8681	21	$cy' = \frac{ax+by^2}{rx}$
8682	22	$cy' = \frac{ax+by^2}{r\,x^2}$
8683	23	$cy' = \frac{ax+by^2}{y}$
8684	24	$a \sin(x)\, yxy' = 0$
8685	25	$f(x) \sin(x)\, yxy'\pi = 0$
8686	26	$y' = \sin(x) + y$
8687	27	$y' = \sin(x) + y^2$
8688	28	$y' = \cos(x) + \frac{y}{x}$
8689	29	$y' = \cos(x) + \frac{y^2}{x}$
8690	30	$y' = x + y + by^2$
8691	31	$xy' = 0$
8692	32	$5y' = 0$
8693	33	$\mathrm{e}y' = 0$

Continued on next page

Table 1.1 Lookup table
Continued from previous page

ID	problem	ODE
8694	34	$\pi y' = 0$
8695	35	$\sin(x) y' = 0$
8696	36	$f(x) y' = 0$
8697	37	$xy' = 1$
8698	38	$xy' = \sin(x)$
8699	39	$(x-1) y' = 0$
8700	40	$yy' = 0$
8701	41	$xyy' = 0$
8702	42	$xy \sin(x) y' = 0$
8703	43	$\pi y \sin(x) y' = 0$
8704	44	$x \sin(x) y' = 0$
8705	45	$x \sin(x) {y'}^2 = 0$
8706	46	$y{y'}^2 = 0$
8707	47	${y'}^n = 0$
8708	48	$x{y'}^n = 0$
8709	49	${y'}^2 = x$
8710	50	${y'}^2 = x + y$
8711	51	${y'}^2 = \frac{y}{x}$
8712	52	${y'}^2 = \frac{y^2}{x}$
8713	53	${y'}^2 = \frac{y^3}{x}$
8714	54	${y'}^3 = \frac{y^2}{x}$
8715	55	${y'}^2 = \frac{1}{yx}$
8716	56	${y'}^2 = \frac{1}{xy^3}$
8717	57	${y'}^2 = \frac{1}{x^2y^3}$
8718	58	${y'}^4 = \frac{1}{xy^3}$
8719	59	${y'}^2 = \frac{1}{x^3y^4}$
8720	60	$y' = \sqrt{1 + 6x + y}$
8721	61	$y' = (1 + 6x + y)^{1/3}$
8722	62	$y' = (1 + 6x + y)^{1/4}$
8723	63	$y' = (a + bx + y)^4$
8724	64	$y' = (\pi + x + 7y)^{7/2}$
8725	65	$y' = (a + bx + cy)^6$
8726	66	$y' = e^{x+y}$
8727	67	$y' = 10 + e^{x+y}$
8728	68	$y' = 10\, e^{x+y} + x^2$

Continued on next page

Table 1.1 Lookup table
Continued from previous page

ID	problem	ODE
8729	69	$y' = x\,\mathrm{e}^{x+y} + \sin(x)$
8730	70	$y' = 5\,\mathrm{e}^{x^2+20y} + \sin(x)$

1.2 section 2 (system of first order odes)

Table 1.2: Lookup table for all problems in current section

ID	problem	ODE
8731	1	$[x'(t) + y'(t) - x(t) = y(t) + t, x'(t) + y'(t) = 2x(t) + 3y(t) + \mathrm{e}^t]$
8732	2	$[2x'(t) + y'(t) - x(t) = y(t) + t, x'(t) + y'(t) = 2x(t) + 3y(t) + \mathrm{e}^t]$
8733	3	$[x'(t) + y'(t) - x(t) = y(t) + t + \sin(t) + \cos(t), x'(t) + y'(t) = 2x(t) + 3y(t) + \mathrm{e}^t]$

1.3 section 3. First order odes solved using Laplace method

Table 1.3: Lookup table for all problems in current section

ID	problem	ODE
8734	1	$y't + y = t$
8735	2	$y' - yt = 0$
8736	3	$y't + y = 0$
8737	4	$y't + y = 0$
8738	5	$y't + y = 0$
8739	6	$y't + y = 0$
8740	7	$y't + y = 0$
8741	8	$y't + y = \sin(t)$
8742	9	$y't + y = t$
8743	10	$y't + y = t$
8744	11	$y' + t^2y = 0$
8745	12	$(at + 1)\,y' + y = t$
8746	13	$y' + (at + bt)\,y = 0$
8747	14	$y' + (at + bt)\,y = 0$

CHAPTER 2

Book Solved Problems

2.1 section 1

2.1.1 problem 1

Internal problem ID [8661]
Book : First order enumerated odes
Section : section 1
Problem number : 1
Date solved : `Tuesday, December 17, 2024 at 12:57:13 PM`
CAS classification : `[_quadrature]`

Solve

$$y' = 0$$

Solved as first order quadrature ode

Time used: 0.025 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\,dx + c_1$$
$$y = c_1$$

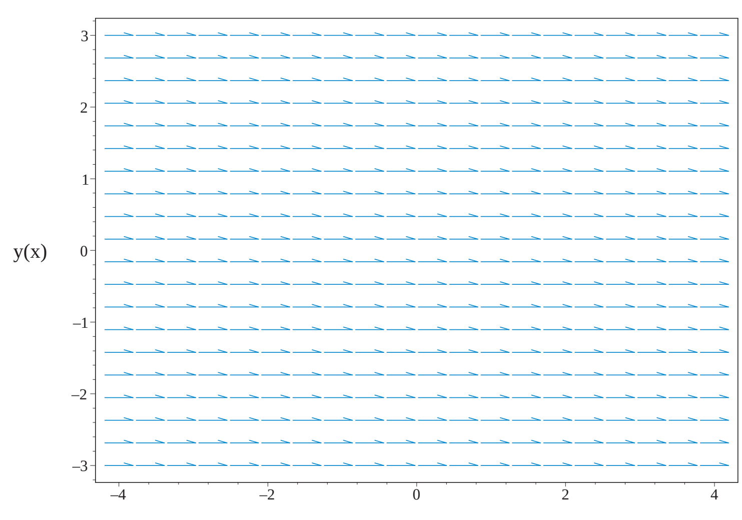


Figure 2.1: Slope field plot
$y' = 0$

Summary of solutions found

$$y = c_1$$

Solved as first order homogeneous class D2 ode

Time used: 0.155 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$u'(x)\,x + u(x) = 0$$

Which is now solved The ode $u'(x) = -\frac{u(x)}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(u) &= u \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\ \ln\left(u(x)\right) &= \ln\left(\frac{1}{x}\right) + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} \ln\left(u(x)\right) &= \ln\left(\frac{1}{x}\right) + c_1 \\ u(x) &= 0 \end{aligned}$$

Solving for $u(x)$ gives

$$\begin{aligned} u(x) &= 0 \\ u(x) &= \frac{e^{c_1}}{x} \end{aligned}$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = \frac{e^{c_1}}{x}$ back to y gives

$$y = e^{c_1}$$

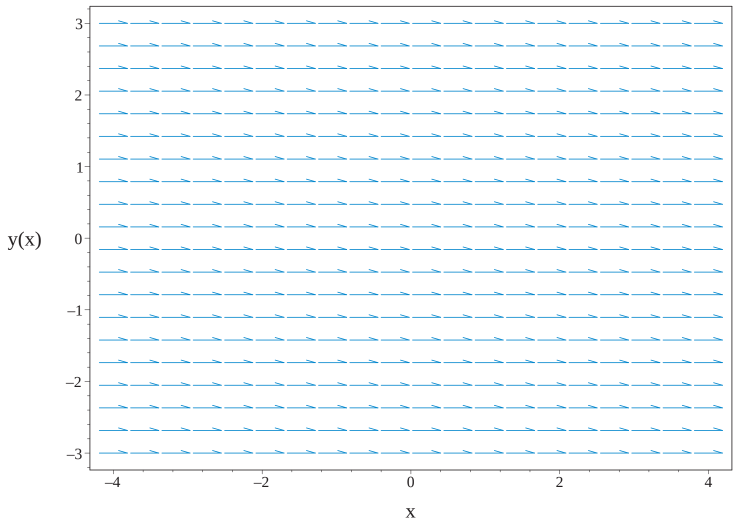


Figure 2.2: Slope field plot
$y' = 0$

Summary of solutions found

$$y = 0$$
$$y = e^{c_1}$$

Solved as first order ode of type differential

Time used: 0.010 (sec)

Writing the ode as

$$y' = 0 \tag{1}$$

Which becomes

$$(1)\, dy = (0)\, dx \tag{2}$$

But the RHS is complete differential because

$$(0)\, dx = d(0)$$

Hence (2) becomes

$$(1)\, dy = d(0)$$

Integrating gives

$$y = c_1$$

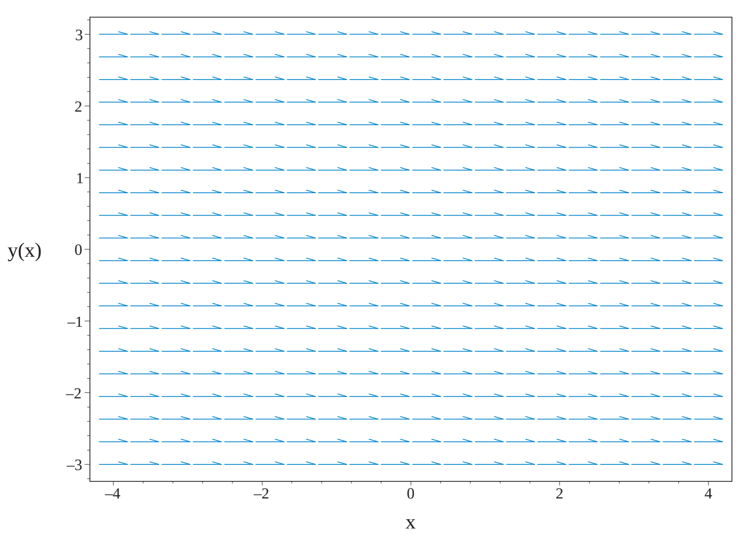


Figure 2.3: Slope field plot
$y' = 0$

Summary of solutions found

$$y = c_1$$

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) = 0$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}y(x)\right) dx = \int 0 dx + C1$

- Evaluate integral

  $y(x) = C1$

- Solve for $y(x)$

  $y(x) = C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : `0.001` (sec)
Leaf size : 5

```
dsolve(diff(y(x),x) = 0,
       y(x),singsol=all)
```

$$y = c_1$$

Mathematica DSolve solution

Solving time : `0.002` (sec)
Leaf size : 7

```
DSolve[{D[y[x],x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1$$

2.1.2 problem 2

Internal problem ID [8662]
Book : First order enumerated odes
Section : section 1
Problem number : 2
Date solved : `Tuesday, December 17, 2024 at 12:57:14 PM`
CAS classification : `[_quadrature]`

Solve

$$y' = a$$

Solved as first order quadrature ode

Time used: 0.036 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int a\,dx$$
$$y = ax + c_1$$

Summary of solutions found

$$y = ax + c_1$$

Solved as first order homogeneous class D2 ode

Time used: 0.186 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$u'(x)\,x + u(x) = a$$

Which is now solved The ode $u'(x) = -\frac{u(x)-a}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x) - a}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{1}{x} \\ g(u) &= -u + a \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\ \int \frac{1}{-u+a}\,du &= \int \frac{1}{x}\,dx \\ -\ln(-u(x) + a) &= \ln(x) + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $-u + a = 0$ for $u(x)$ gives

$$u(x) = a$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$-\ln(-u(x) + a) = \ln(x) + c_1$$
$$u(x) = a$$

Solving for $u(x)$ gives

$$u(x) = a$$
$$u(x) = \frac{(x\,e^{c_1}a - 1)\,e^{-c_1}}{x}$$

Converting $u(x) = a$ back to y gives

$$y = ax$$

Converting $u(x) = \frac{(x\,e^{c_1}a-1)e^{-c_1}}{x}$ back to y gives

$$y = (x\,e^{c_1}a - 1)\,e^{-c_1}$$

Summary of solutions found

$$y = ax$$
$$y = (x\,e^{c_1}a - 1)\,e^{-c_1}$$

Solved as first order Exact ode

Time used: 0.060 (sec)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} \mathrm{d}y &= (a)\,\mathrm{d}x \\ (-a)\,\mathrm{d}x + \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -a \\ N(x,y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-a) \\ &= 0 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -a\,\mathrm{d}x \end{aligned}$$

$$\phi = -ax + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 1$. Therefore equation (4) becomes

$$1 = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 1$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (1)\,\mathrm{d}y$$
$$f(y) = y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -ax + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -ax + y$$

Solving for y gives

$$y = ax + c_1$$

Summary of solutions found

$$y = ax + c_1$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) = a$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Integrate both sides with respect to x
  $\int \left(\frac{d}{dx}y(x)\right) dx = \int a dx + C1$
- Evaluate integral
  $y(x) = xa + C1$
- Solve for $y(x)$
  $y(x) = xa + C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : `0.002` (sec)
Leaf size : 9

```
dsolve(diff(y(x),x) = a,
       y(x),singsol=all)
```

$$y = ax + c_1$$

Mathematica DSolve solution

Solving time : 0.002 (sec)
Leaf size : 11

```
DSolve[{D[y[x],x]==a,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to ax + c_1$$

2.1.3 problem 3

Internal problem ID [8663]
Book : First order enumerated odes
Section : section 1
Problem number : 3
Date solved : `Tuesday, December 17, 2024 at 12:57:15 PM`
CAS classification : `[_quadrature]`

Solve

$$y' = x$$

Solved as first order quadrature ode

Time used: 0.038 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int x\,dx$$
$$y = \frac{x^2}{2} + c_1$$

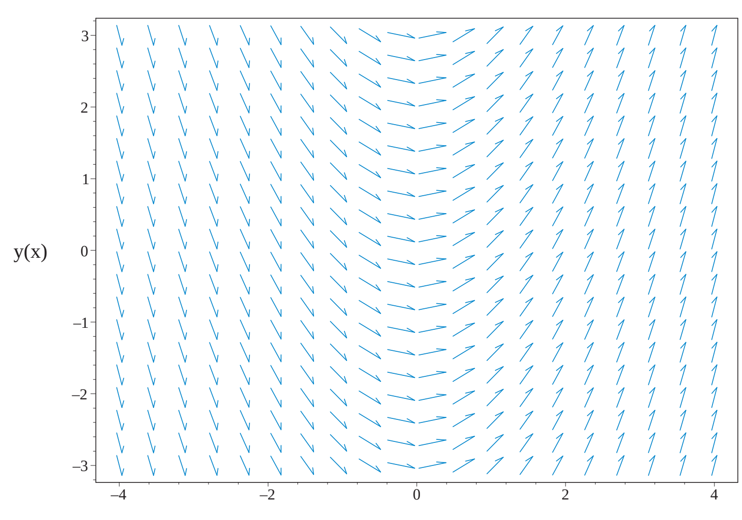


Figure 2.4: Slope field plot
$y' = x$

Summary of solutions found

$$y = \frac{x^2}{2} + c_1$$

Solved as first order Exact ode

Time used: 0.056 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y&=(x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x+\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x\\ N(x,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-x)\\ &=0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(1)\\ &=0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -x \, \mathrm{d}x$$

$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 1$. Therefore equation (4) becomes

$$1 = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 1$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (1) \, \mathrm{d}y$$

$$f(y) = y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + y$$

Solving for y gives

$$y = \frac{x^2}{2} + c_1$$

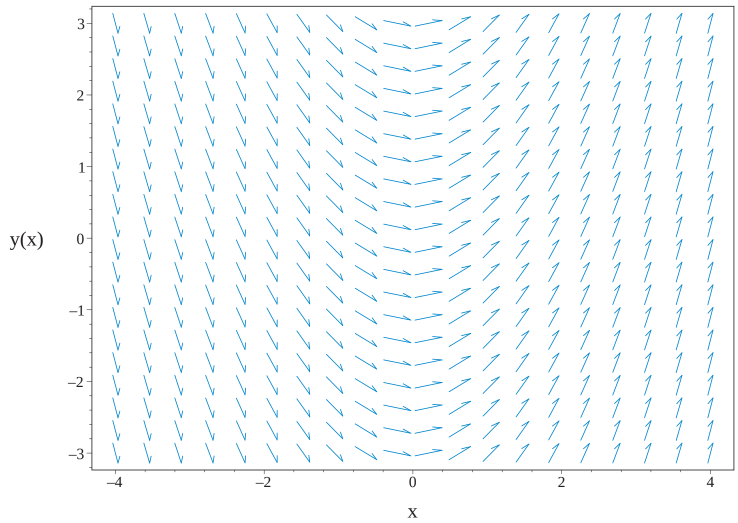


Figure 2.5: Slope field plot
$y' = x$

Summary of solutions found

$$y = \frac{x^2}{2} + c_1$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) = x$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Integrate both sides with respect to x
  $\int \left(\frac{d}{dx}y(x)\right) dx = \int x dx + C1$
- Evaluate integral
  $y(x) = \frac{x^2}{2} + C1$
- Solve for $y(x)$
  $y(x) = \frac{x^2}{2} + C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : `0.002` (sec)
Leaf size : 11

```
dsolve(diff(y(x),x) = x,
       y(x),singsol=all)
```

$$y = \frac{x^2}{2} + c_1$$

Mathematica DSolve solution

Solving time : 0.002 (sec)
Leaf size : 15

```
DSolve[{D[y[x],x]==x,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{x^2}{2} + c_1$$

2.1.4 problem 4

Internal problem ID [8664]
Book : First order enumerated odes
Section : section 1
Problem number : 4
Date solved : `Tuesday, December 17, 2024 at 12:57:15 PM`
CAS classification : `[_quadrature]`

Solve

$$y' = 1$$

Solved as first order quadrature ode

Time used: 0.032 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 1\, dx$$
$$y = x + c_1$$

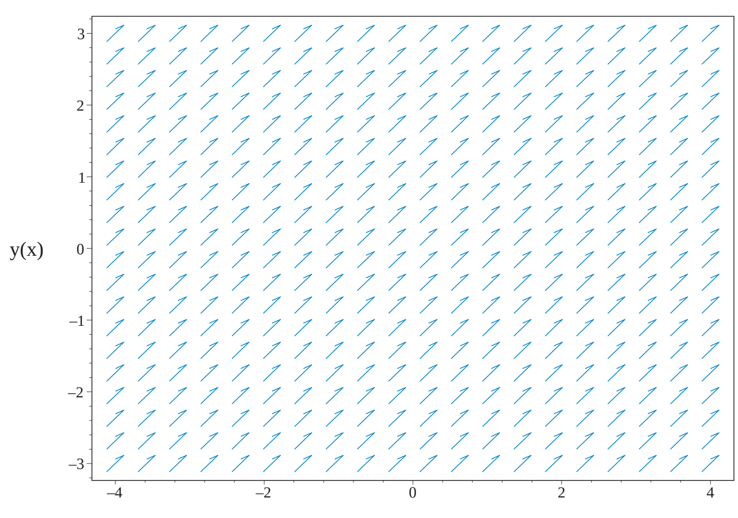


Figure 2.6: Slope field plot
$y' = 1$

Summary of solutions found

$$y = x + c_1$$

Solved as first order homogeneous class D2 ode

Time used: 0.173 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$u'(x)\,x + u(x) = 1$$

Which is now solved The ode $u'(x) = -\frac{u(x)-1}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)-1}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{1}{x} \\ g(u) &= -u+1 \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\ \int \frac{1}{-u+1}\,du &= \int \frac{1}{x}\,dx \\ -\ln\left(u(x)-1\right) &= \ln\left(x\right)+c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $-u+1 = 0$ for $u(x)$ gives

$$u(x) = 1$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} -\ln\left(u(x)-1\right) &= \ln\left(x\right)+c_1 \\ u(x) &= 1 \end{aligned}$$

Solving for $u(x)$ gives

$$\begin{aligned} u(x) &= 1 \\ u(x) &= \frac{\left(x\,\mathrm{e}^{c_1}+1\right)\mathrm{e}^{-c_1}}{x} \end{aligned}$$

Converting $u(x) = 1$ back to y gives

$$y = x$$

Converting $u(x) = \frac{\left(x\,\mathrm{e}^{c_1}+1\right)\mathrm{e}^{-c_1}}{x}$ back to y gives

$$y = \left(x\,\mathrm{e}^{c_1}+1\right)\mathrm{e}^{-c_1}$$

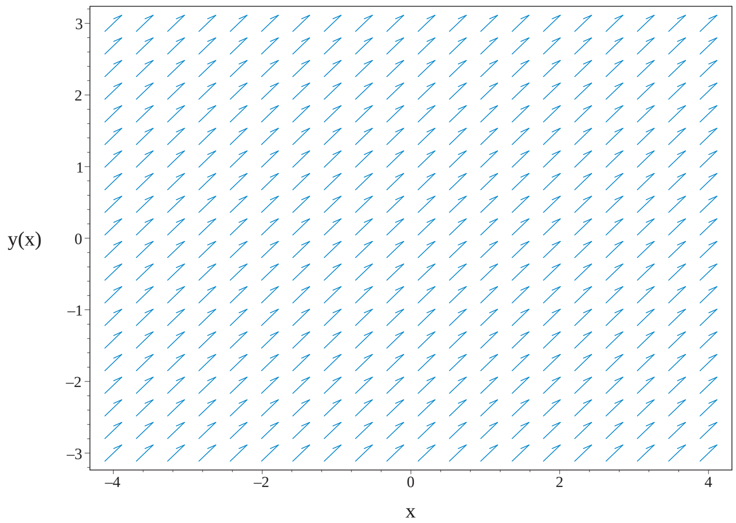


Figure 2.7: Slope field plot
$y' = 1$

Summary of solutions found

$$
\begin{aligned}
y &= x \\
y &= (x\,\mathrm{e}^{c_1} + 1)\,\mathrm{e}^{-c_1}
\end{aligned}
$$

Solved as first order Exact ode

Time used: 0.056 (sec)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$
\begin{aligned}
(1)\,\mathrm{d}y &= \mathrm{d}x \\
-\,\mathrm{d}x + (1)\,\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -1 \\ N(x,y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-1) \\ &= 0 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -1 \, \mathrm{d}x \end{aligned}$$

$$\phi = -x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 1$. Therefore equation (4) becomes

$$1 = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 1$$

Integrating the above w.r.t y gives

$$\begin{aligned} \int f'(y) \, \mathrm{d}y &= \int (1) \, \mathrm{d}y \\ f(y) &= y + c_1 \end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -x + y$$

Solving for y gives

$$y = x + c_1$$

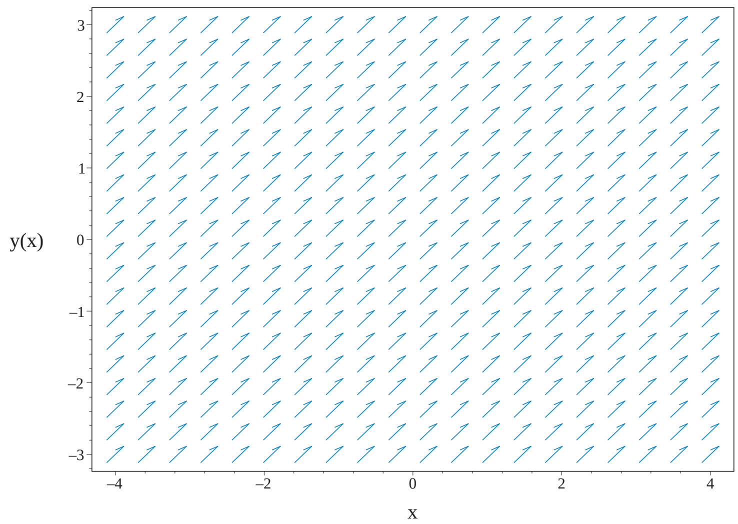


Figure 2.8: Slope field plot
$y' = 1$

Summary of solutions found

$$y = x + c_1$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) = 1$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Integrate both sides with respect to x
  $\int \left(\frac{d}{dx}y(x)\right) dx = \int 1 dx + C1$
- Evaluate integral
  $y(x) = x + C1$
- Solve for $y(x)$
  $y(x) = x + C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 7

```
dsolve(diff(y(x),x) = 1,
       y(x),singsol=all)
```

$$y = x + c_1$$

Mathematica DSolve solution

Solving time : 0.002 (sec)
Leaf size : 9

```
DSolve[{D[y[x],x]==1,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x + c_1$$

2.1.5 problem 5

Internal problem ID [8665]
Book : First order enumerated odes
Section : section 1
Problem number : 5
Date solved : `Tuesday, December 17, 2024 at 12:57:16 PM`
CAS classification : `[_quadrature]`

Solve

$$y' = ax$$

Solved as first order quadrature ode

Time used: 0.040 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int ax\, dx$$
$$y = \frac{a\,x^2}{2} + c_1$$

Summary of solutions found

$$y = \frac{a\,x^2}{2} + c_1$$

Solved as first order Exact ode

Time used: 0.061 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y&=(ax)\,\mathrm{d}x\\(-ax)\,\mathrm{d}x+\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-ax\\N(x,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-ax)\\&=0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(1)\\&=0\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \tag{1}$$

$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x&=\int M\,\mathrm{d}x\\\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x&=\int -ax\,\mathrm{d}x\end{aligned}$$

$$\phi=-\frac{a\,x^2}{2}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y}=0+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y}=1$. Therefore equation (4) becomes

$$1=0+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 1$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (1)\,\mathrm{d}y$$

$$f(y) = y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{a\,x^2}{2} + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -\frac{a\,x^2}{2} + y$$

Solving for y gives

$$y = \frac{a\,x^2}{2} + c_1$$

Summary of solutions found

$$y = \frac{a\,x^2}{2} + c_1$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) = xa$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Integrate both sides with respect to x
  $\int \left(\frac{d}{dx}y(x)\right) dx = \int xadx + C1$
- Evaluate integral
  $y(x) = \frac{x^2 a}{2} + C1$
- Solve for $y(x)$
  $y(x) = \frac{x^2 a}{2} + C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 12

```
dsolve(diff(y(x),x) = a*x,
       y(x),singsol=all)
```

$$y = \frac{a\,x^2}{2} + c_1$$

Mathematica DSolve solution

Solving time : 0.002 (sec)
Leaf size : 16

```
DSolve[{D[y[x],x]==a*x,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{ax^2}{2} + c_1$$

2.1.6 problem 6

Internal problem ID [8666]
Book : First order enumerated odes
Section : section 1
Problem number : 6
Date solved : `Tuesday, December 17, 2024 at 12:57:17 PM`
CAS classification : `[_separable]`

Solve

$$y' = axy$$

Solved as first order linear ode

Time used: 0.086 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = -ax$$
$$p(x) = 0$$

The integrating factor μ is

$$\mu = e^{\int q\,dx}$$
$$= e^{\int -ax dx}$$
$$= e^{-\frac{a x^2}{2}}$$

The ode becomes

$$\frac{d}{dx}\mu y = 0$$
$$\frac{d}{dx}\left(y\,e^{-\frac{a x^2}{2}}\right) = 0$$

Integrating gives

$$y\,e^{-\frac{a x^2}{2}} = \int 0\,dx + c_1$$
$$= c_1$$

Dividing throughout by the integrating factor $e^{-\frac{a x^2}{2}}$ gives the final solution

$$y = e^{\frac{a x^2}{2}} c_1$$

Summary of solutions found

$$y = e^{\frac{a x^2}{2}} c_1$$

Solved as first order separable ode

Time used: 0.109 (sec)

The ode $y' = axy$ is separable as it can be written as

$$\begin{aligned} y' &= axy \\ &= f(x)g(y) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= ax \\ g(y) &= y \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(y)}\, dy &= \int f(x)\, dx \\ \int \frac{1}{y}\, dy &= \int ax\, dx \\ \ln(y) &= \frac{a\,x^2}{2} + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(y)$ is zero, since we had to divide by this above. Solving $g(y) = 0$ or $y = 0$ for y gives

$$y = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} \ln(y) &= \frac{a\,x^2}{2} + c_1 \\ y &= 0 \end{aligned}$$

Solving for y gives

$$\begin{aligned} y &= 0 \\ y &= \mathrm{e}^{\frac{a\,x^2}{2} + c_1} \end{aligned}$$

Summary of solutions found

$$\begin{aligned} y &= 0 \\ y &= \mathrm{e}^{\frac{a\,x^2}{2} + c_1} \end{aligned}$$

Solved as first order homogeneous class D2 ode

Time used: 0.126 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$u'(x)\,x + u(x) = a\,x^2 u(x)$$

Which is now solved The ode $u'(x) = \frac{u(x)(a\,x^2-1)}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= \frac{u(x)\,(a\,x^2 - 1)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$f(x) = \frac{a\,x^2 - 1}{x}$$
$$g(u) = u$$

Integrating gives

$$\int \frac{1}{g(u)}\,du = \int f(x)\,dx$$
$$\int \frac{1}{u}\,du = \int \frac{a\,x^2 - 1}{x}\,dx$$
$$\ln(u(x)) = \frac{a\,x^2}{2} + \ln\left(\frac{1}{x}\right) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\ln(u(x)) = \frac{a\,x^2}{2} + \ln\left(\frac{1}{x}\right) + c_1$$
$$u(x) = 0$$

Solving for $u(x)$ gives

$$u(x) = 0$$
$$u(x) = \frac{e^{\frac{a\,x^2}{2} + c_1}}{x}$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = \frac{e^{\frac{a\,x^2}{2} + c_1}}{x}$ back to y gives

$$y = e^{\frac{a\,x^2}{2} + c_1}$$

Summary of solutions found

$$y = 0$$
$$y = e^{\frac{a\,x^2}{2} + c_1}$$

Solved as first order Exact ode

Time used: 0.173 (sec)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= (axy)\,\mathrm{d}x\\ (-axy)\,\mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -axy\\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-axy)\\ &= -ax\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1((-ax) - (0))\\ &= -ax\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -ax\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\mu = e^{-\frac{a x^2}{2}}$$
$$= \mathrm{e}^{-\frac{a x^2}{2}}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\overline{M} = \mu M$$
$$= \mathrm{e}^{-\frac{a x^2}{2}}(-axy)$$
$$= -axy\,\mathrm{e}^{-\frac{a x^2}{2}}$$

And

$$\overline{N} = \mu N$$
$$= \mathrm{e}^{-\frac{a x^2}{2}}(1)$$
$$= \mathrm{e}^{-\frac{a x^2}{2}}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(-axy\,\mathrm{e}^{-\frac{a x^2}{2}}\right) + \left(\mathrm{e}^{-\frac{a x^2}{2}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\int \frac{\partial \phi}{\partial y}\,\mathrm{d}y = \int \overline{N}\,\mathrm{d}y$$
$$\int \frac{\partial \phi}{\partial y}\,\mathrm{d}y = \int \mathrm{e}^{-\frac{a x^2}{2}}\,\mathrm{d}y$$
$$\phi = y\,\mathrm{e}^{-\frac{a x^2}{2}} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = -axy\,\mathrm{e}^{-\frac{a x^2}{2}} + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = -axy\,\mathrm{e}^{-\frac{a x^2}{2}}$. Therefore equation (4) becomes

$$-axy\,\mathrm{e}^{-\frac{a x^2}{2}} = -axy\,\mathrm{e}^{-\frac{a x^2}{2}} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 0$$

Therefore

$$f(x) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = y\,\mathrm{e}^{-\frac{a\,x^2}{2}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = y\,\mathrm{e}^{-\frac{a\,x^2}{2}}$$

Solving for y gives

$$y = \mathrm{e}^{\frac{a\,x^2}{2}} c_1$$

Summary of solutions found

$$y = \mathrm{e}^{\frac{a\,x^2}{2}} c_1$$

Solved using Lie symmetry for first order ode

Time used: 0.212 (sec)

Writing the ode as

$$\begin{aligned} y' &= axy \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + axy(b_3 - a_2) - a^2x^2y^2a_3 - ay(xa_2 + ya_3 + a_1) - ax(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-a^2x^2y^2a_3 - a\,x^2b_2 - 2axya_2 - a\,y^2a_3 - axb_1 - aya_1 + b_2 = 0$$

Setting the numerator to zero gives

$$-a^2x^2y^2a_3 - a\,x^2b_2 - 2axya_2 - a\,y^2a_3 - axb_1 - aya_1 + b_2 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a^2 a_3 v_1^2 v_2^2 - 2aa_2 v_1 v_2 - aa_3 v_2^2 - ab_2 v_1^2 - aa_1 v_2 - ab_1 v_1 + b_2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-a^2 a_3 v_1^2 v_2^2 - 2aa_2 v_1 v_2 - aa_3 v_2^2 - ab_2 v_1^2 - aa_1 v_2 - ab_1 v_1 + b_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} b_2 &= 0 \\ -aa_1 &= 0 \\ -2aa_2 &= 0 \\ -aa_3 &= 0 \\ -ab_1 &= 0 \\ -ab_2 &= 0 \\ -a^2 a_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 0 \\ \eta &= y \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y} dy \end{aligned}$$

Which results in

$$S = \ln(y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = axy$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 0 \\ S_y &= \frac{1}{y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = ax \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = aR$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned} \int dS &= \int aR\, dR \\ S(R) &= \frac{a\,R^2}{2} + c_2 \end{aligned}$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\ln(y) = \frac{a\,x^2}{2} + c_2$$

Which gives

$$y = \mathrm{e}^{\frac{a\,x^2}{2} + c_2}$$

Summary of solutions found

$$y = \mathrm{e}^{\frac{a\,x^2}{2} + c_2}$$

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) = xay(x)$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = xay(x)$

- Separate variables

  $\frac{\frac{d}{dx}y(x)}{y(x)} = xa$

- Integrate both sides with respect to x

  $\int \frac{\frac{d}{dx}y(x)}{y(x)}dx = \int xadx + C1$

- Evaluate integral

  $\ln(y(x)) = \frac{x^2a}{2} + C1$

- Solve for $y(x)$

  $y(x) = e^{\frac{x^2a}{2}+C1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 13

```
dsolve(diff(y(x),x) = a*x*y(x),
       y(x),singsol=all)
```

$$y = c_1 e^{\frac{a x^2}{2}}$$

Mathematica DSolve solution

Solving time : 0.027 (sec)
Leaf size : 23

```
DSolve[{D[y[x],x]==a*x*y[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{\frac{ax^2}{2}}$$
$$y(x) \to 0$$

2.1.7 problem 7

Internal problem ID [8667]
Book : First order enumerated odes
Section : section 1
Problem number : 7
Date solved : `Tuesday, December 17, 2024 at 12:57:18 PM`
CAS classification : `[[_linear, ‘class A‘]]`

Solve

$$y' = ax + y$$

Solved as first order linear ode

Time used: 0.105 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = -1$$
$$p(x) = ax$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dx} \\ &= e^{\int (-1)dx} \\ &= e^{-x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= \mu p \\ \frac{d}{dx}(\mu y) &= (\mu)(ax) \\ \frac{d}{dx}\left(y\,e^{-x}\right) &= \left(e^{-x}\right)(ax) \\ d\left(y\,e^{-x}\right) &= \left(ax\,e^{-x}\right)\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}y\,e^{-x} &= \int ax\,e^{-x}\,dx \\ &= -(x+1)\,a\,e^{-x} + c_1\end{aligned}$$

Dividing throughout by the integrating factor e^{-x} gives the final solution

$$y = c_1\,e^{x} - a(x+1)$$

Summary of solutions found

$$y = c_1\,e^{x} - a(x+1)$$

Solved as first order Exact ode

Time used: 0.105 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y&=(ax+y)\,\mathrm{d}x\\ (-ax-y)\,\mathrm{d}x+\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-ax-y\\ N(x,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-ax-y)\\ &=-1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(1)\\ &=0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((-1) - (0)) \\ &= -1 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -1 \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-x} \\ &= \mathrm{e}^{-x} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{-x}(-ax - y) \\ &= -(ax + y)\,\mathrm{e}^{-x} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{-x}(1) \\ &= \mathrm{e}^{-x} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-(ax + y)\,\mathrm{e}^{-x}\right) + \left(\mathrm{e}^{-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial y} \, \mathrm{d}y &= \int \overline{N} \, \mathrm{d}y \\ \int \frac{\partial \phi}{\partial y} \, \mathrm{d}y &= \int \mathrm{e}^{-x} \, \mathrm{d}y \end{aligned}$$

$$\phi = y\,\mathrm{e}^{-x} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = -y\,\mathrm{e}^{-x} + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = -(ax+y)\,\mathrm{e}^{-x}$. Therefore equation (4) becomes

$$-(ax+y)\,\mathrm{e}^{-x} = -y\,\mathrm{e}^{-x} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -ax\,\mathrm{e}^{-x}$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int \left(-ax\,\mathrm{e}^{-x}\right) \mathrm{d}x$$
$$f(x) = (x+1)\,a\,\mathrm{e}^{-x} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = y\,\mathrm{e}^{-x} + (x+1)\,a\,\mathrm{e}^{-x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = y\,\mathrm{e}^{-x} + (x+1)\,a\,\mathrm{e}^{-x}$$

Solving for y gives

$$y = -\left(ax\,\mathrm{e}^{-x} + a\,\mathrm{e}^{-x} - c_1\right)\mathrm{e}^{x}$$

Summary of solutions found

$$y = -\left(ax\,\mathrm{e}^{-x} + a\,\mathrm{e}^{-x} - c_1\right)\mathrm{e}^{x}$$

Solved using Lie symmetry for first order ode

Time used: 0.347 (sec)

Writing the ode as

$$y' = ax + y$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + (ax+y)(b_3 - a_2) - (ax+y)^2 a_3 - a(xa_2 + ya_3 + a_1) - xb_2 - yb_3 - b_1 = 0 \tag{5E}$$

Putting the above in normal form gives

$$-a^2x^2a_3 - 2axya_3 - 2axa_2 + axb_3 - aya_3 - y^2a_3 - aa_1 - xb_2 - ya_2 - b_1 + b_2 = 0$$

Setting the numerator to zero gives

$$-a^2x^2a_3 - 2axya_3 - 2axa_2 + axb_3 - aya_3 - y^2a_3 - aa_1 - xb_2 - ya_2 - b_1 + b_2 = 0 \quad (6E)$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a^2a_3v_1^2 - 2aa_3v_1v_2 - 2aa_2v_1 - aa_3v_2 + ab_3v_1 - a_3v_2^2 - aa_1 - a_2v_2 - b_2v_1 - b_1 + b_2 = 0 \quad (7E)$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-a^2a_3v_1^2 - 2aa_3v_1v_2 + (-2aa_2 + ab_3 - b_2)\,v_1 - a_3v_2^2 + (-aa_3 - a_2)\,v_2 - aa_1 - b_1 + b_2 = 0 \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -a_3 &= 0 \\ -2aa_3 &= 0 \\ -a^2a_3 &= 0 \\ -aa_3 - a_2 &= 0 \\ -aa_1 - b_1 + b_2 &= 0 \\ -2aa_2 + ab_3 - b_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= a_1 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= -aa_1 + ab_3 \\ b_2 &= ab_3 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 0 \\ \eta &= ax + a + y \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{ax + a + y} dy \end{aligned}$$

Which results in

$$S = \ln(ax + a + y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = ax + y$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{a}{ax + a + y} \\ S_y &= \frac{1}{ax + a + y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned} \int dS &= \int 1 \, dR \\ S(R) &= R + c_2 \end{aligned}$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\ln(ax + a + y) = x + c_2$$

Which gives

$$y = e^{x+c_2} - ax - a$$

Summary of solutions found

$$y = e^{x+c_2} - ax - a$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) = xa + y(x)$

- Highest derivative means the order of the ODE is 1
$\frac{d}{dx}y(x)$
- Solve for the highest derivative
$\frac{d}{dx}y(x) = xa + y(x)$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE
$\frac{d}{dx}y(x) - y(x) = xa$
- The ODE is linear; multiply by an integrating factor $\mu(x)$
$\mu(x)\left(\frac{d}{dx}y(x) - y(x)\right) = \mu(x)\, xa$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\,\mu(x))$
$\mu(x)\left(\frac{d}{dx}y(x) - y(x)\right) = \left(\frac{d}{dx}y(x)\right)\mu(x) + y(x)\left(\frac{d}{dx}\mu(x)\right)$
- Isolate $\frac{d}{dx}\mu(x)$
$\frac{d}{dx}\mu(x) = -\mu(x)$
- Solve to find the integrating factor
$\mu(x) = e^{-x}$
- Integrate both sides with respect to x
$\int \left(\frac{d}{dx}(y(x)\,\mu(x))\right) dx = \int \mu(x)\, xadx + C1$
- Evaluate the integral on the lhs
$y(x)\,\mu(x) = \int \mu(x)\, xadx + C1$
- Solve for $y(x)$
$y(x) = \frac{\int \mu(x) xadx + C1}{\mu(x)}$
- Substitute $\mu(x) = e^{-x}$
$y(x) = \frac{\int e^{-x} xadx + C1}{e^{-x}}$
- Evaluate the integrals on the rhs
$y(x) = \frac{-(x+1)e^{-x}a + C1}{e^{-x}}$
- Simplify
$y(x) = C1\, e^{x} - a(x+1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 15

```
dsolve(diff(y(x),x) = a*x+y(x),
       y(x),singsol=all)
```

$$y = e^x c_1 - a(x+1)$$

Mathematica DSolve solution

Solving time : 0.027 (sec)
Leaf size : 18

```
DSolve[{D[y[x],x]==a*x+y[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -a(x+1) + c_1 e^x$$

2.1.8 problem 8

Internal problem ID [8668]
Book : First order enumerated odes
Section : section 1
Problem number : 8
Date solved : `Tuesday, December 17, 2024 at 12:57:19 PM`
CAS classification : `[[_linear, 'class A']]`

Solve

$$y' = ax + by$$

Solved as first order linear ode

Time used: 0.124 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= -b \\ p(x) &= ax \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= \mathrm{e}^{\int -b dx} \\ &= \mathrm{e}^{-bx} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\,(ax) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(y\,\mathrm{e}^{-bx}\right) &= \left(\mathrm{e}^{-bx}\right)(ax) \\ \mathrm{d}\left(y\,\mathrm{e}^{-bx}\right) &= \left(ax\,\mathrm{e}^{-bx}\right)\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} y\,\mathrm{e}^{-bx} &= \int ax\,\mathrm{e}^{-bx}\,dx \\ &= -\frac{(bx+1)\,a\,\mathrm{e}^{-bx}}{b^2} + c_1 \end{aligned}$$

Dividing throughout by the integrating factor e^{-bx} gives the final solution

$$y = \frac{c_1\,\mathrm{e}^{bx} b^2 - abx - a}{b^2}$$

Summary of solutions found

$$y = \frac{c_1 \,\mathrm{e}^{bx} b^2 - abx - a}{b^2}$$

Solved as first order Exact ode

Time used: 0.123 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x \partial y} = \frac{\partial^2\phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x \partial y} = \frac{\partial^2\phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = (ax + by)\,\mathrm{d}x$$
$$(-ax - by)\,\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -ax - by$$
$$N(x,y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(-ax - by)$$
$$= -b$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1((-b) - (0))\\ &= -b\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}x}\\ &= e^{\int -b \,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-bx}\\ &= \mathrm{e}^{-bx}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{-bx}(-ax - by)\\ &= -(ax + by)\,\mathrm{e}^{-bx}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{-bx}(1)\\ &= \mathrm{e}^{-bx}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(-(ax + by)\,\mathrm{e}^{-bx}\right) + \left(\mathrm{e}^{-bx}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial y}\,\mathrm{d}y &= \int \overline{N}\,\mathrm{d}y\\ \int \frac{\partial \phi}{\partial y}\,\mathrm{d}y &= \int \mathrm{e}^{-bx}\,\mathrm{d}y\end{aligned}$$

$$\phi = y\,\mathrm{e}^{-bx} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = -y b\, \mathrm{e}^{-bx} + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = -(ax + by)\, \mathrm{e}^{-bx}$. Therefore equation (4) becomes

$$-(ax + by)\, \mathrm{e}^{-bx} = -y b\, \mathrm{e}^{-bx} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -ax\, \mathrm{e}^{-bx}$$

Integrating the above w.r.t x gives

$$\int f'(x)\, \mathrm{d}x = \int \left(-ax\, \mathrm{e}^{-bx}\right) \mathrm{d}x$$

$$f(x) = \frac{(bx + 1)\, a\, \mathrm{e}^{-bx}}{b^2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = y\, \mathrm{e}^{-bx} + \frac{(bx + 1)\, a\, \mathrm{e}^{-bx}}{b^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = y\, \mathrm{e}^{-bx} + \frac{(bx + 1)\, a\, \mathrm{e}^{-bx}}{b^2}$$

Solving for y gives

$$y = -\frac{\left(axb\, \mathrm{e}^{-bx} - c_1 b^2 + a\, \mathrm{e}^{-bx}\right) \mathrm{e}^{bx}}{b^2}$$

Summary of solutions found

$$y = -\frac{\left(axb\, \mathrm{e}^{-bx} - c_1 b^2 + a\, \mathrm{e}^{-bx}\right) \mathrm{e}^{bx}}{b^2}$$

Solved using Lie symmetry for first order ode

Time used: 0.415 (sec)

Writing the ode as

$$y' = ax + by$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + (ax + by)(b_3 - a_2) - (ax + by)^2 a_3 - a(xa_2 + ya_3 + a_1) - b(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-a^2x^2a_3 - 2abxya_3 - b^2y^2a_3 - 2axa_2 + axb_3 - aya_3 - bxb_2 - bya_2 - aa_1 - bb_1 + b_2 = 0$$

Setting the numerator to zero gives

$$-a^2x^2a_3 - 2abxya_3 - b^2y^2a_3 - 2axa_2 + axb_3 - aya_3 - bxb_2 - bya_2 - aa_1 - bb_1 + b_2 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a^2a_3v_1^2 - 2aba_3v_1v_2 - b^2a_3v_2^2 - 2aa_2v_1 - aa_3v_2 + ab_3v_1 - ba_2v_2 - bb_2v_1 - aa_1 - bb_1 + b_2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &-a^2a_3v_1^2 - 2aba_3v_1v_2 + (-2aa_2 + ab_3 - bb_2)v_1 \\ &- b^2a_3v_2^2 + (-aa_3 - ba_2)v_2 - aa_1 - bb_1 + b_2 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -a^2a_3 &= 0 \\ -b^2a_3 &= 0 \\ -2aba_3 &= 0 \\ -aa_3 - ba_2 &= 0 \\ -aa_1 - bb_1 + b_2 &= 0 \\ -2aa_2 + ab_3 - bb_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= a_1 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= b_1 \\
b_2 &= aa_1 + bb_1 \\
b_3 &= \frac{b(aa_1 + bb_1)}{a}
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 0 \\
\eta &= \frac{abx + b^2y + a}{a}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{abx+b^2y+a}{a}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{a \ln \left(abx + b^2y + a\right)}{b^2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = ax + by
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{a^2}{b\left(abx + b^2y + a\right)} \\
S_y &= \frac{a}{abx + b^2y + a}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{a}{b} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{a}{b}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int \frac{a}{b}\, dR$$
$$S(R) = \frac{aR}{b} + c_2$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\frac{a \ln\left(abx + b^2 y + a\right)}{b^2} = \frac{ax}{b} + c_2$$

Which gives

$$y = \frac{\mathrm{e}^{\frac{b(c_2 b + ax)}{a}} - abx - a}{b^2}$$

Summary of solutions found

$$y = \frac{\mathrm{e}^{\frac{b(c_2 b + ax)}{a}} - abx - a}{b^2}$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) = xa + by(x)$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = xa + by(x)$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE
  $\frac{d}{dx}y(x) - by(x) = xa$
- The ODE is linear; multiply by an integrating factor $\mu(x)$
  $\mu(x)\left(\frac{d}{dx}y(x) - by(x)\right) = \mu(x)\, xa$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\,\mu(x))$
  $\mu(x)\left(\frac{d}{dx}y(x) - by(x)\right) = \left(\frac{d}{dx}y(x)\right)\mu(x) + y(x)\left(\frac{d}{dx}\mu(x)\right)$
- Isolate $\frac{d}{dx}\mu(x)$
  $\frac{d}{dx}\mu(x) = -\mu(x)\, b$
- Solve to find the integrating factor
  $\mu(x) = \mathrm{e}^{-bx}$
- Integrate both sides with respect to x

$\int \left(\frac{d}{dx}(y(x)\,\mu(x))\right) dx = \int \mu(x)\, xadx + C1$

- Evaluate the integral on the lhs

  $y(x)\,\mu(x) = \int \mu(x)\, xadx + C1$

- Solve for $y(x)$

  $y(x) = \frac{\int \mu(x)xadx + C1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{-bx}$

  $y(x) = \frac{\int \mathrm{e}^{-bx}xadx + C1}{\mathrm{e}^{-bx}}$

- Evaluate the integrals on the rhs

  $y(x) = \frac{-\frac{(bx+1)\mathrm{e}^{-bx}a}{b^2} + C1}{\mathrm{e}^{-bx}}$

- Simplify

  $y(x) = \frac{C1\,\mathrm{e}^{bx}b^2 - bxa - a}{b^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 26

```
dsolve(diff(y(x),x) = a*x+b*y(x),
       y(x),singsol=all)
```

$$y = \frac{\mathrm{e}^{bx}c_1 b^2 - axb - a}{b^2}$$

Mathematica DSolve solution

Solving time : 0.056 (sec)
Leaf size : 25

```
DSolve[{D[y[x],x]==a*x+b*y[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{abx + a}{b^2} + c_1 e^{bx}$$

2.1.9 problem 9

Internal problem ID [8669]
Book : First order enumerated odes
Section : section 1
Problem number : 9
Date solved : `Tuesday, December 17, 2024 at 12:57:20 PM`
CAS classification : `[_quadrature]`

Solve

$$y' = y$$

Solved as first order autonomous ode

Time used: 0.083 (sec)

Integrating gives

$$\int \frac{1}{y} dy = 1$$
$$\ln(y) = x + c_1$$
$$e^{\ln(y)} = e^{x+c_1}$$
$$y = c_1 e^x$$

The following diagram is the phase line diagram. It classifies each of the above equilibrium points as stable or not stable or semi-stable.

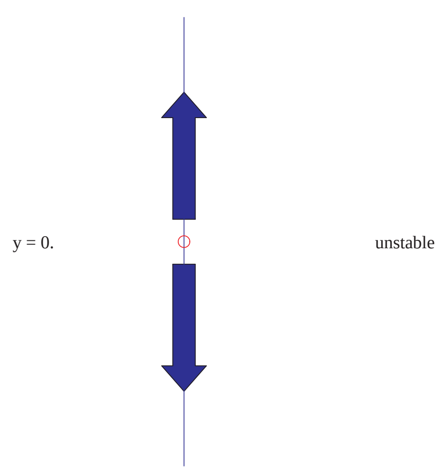


Figure 2.9: Phase line diagram

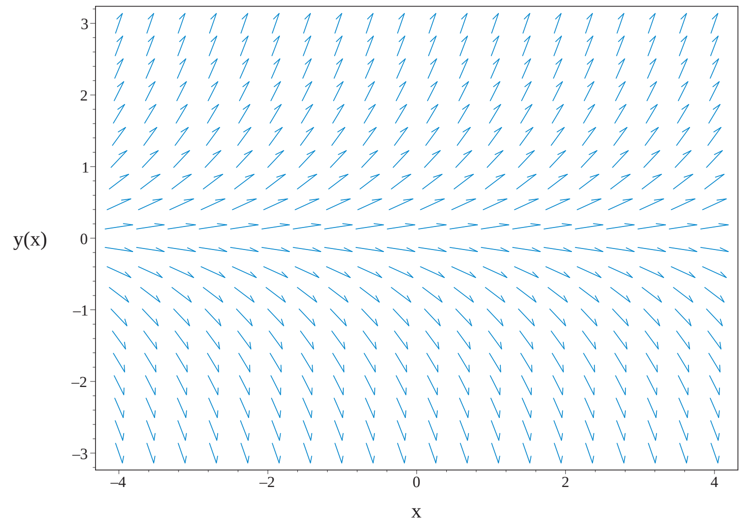


Figure 2.10: Slope field plot
$y' = y$

Summary of solutions found

$$y = 0$$
$$y = c_1 e^x$$

Solved as first order homogeneous class D2 ode

Time used: 0.139 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$u'(x)\,x + u(x) = u(x)\,x$$

Which is now solved The ode $u'(x) = \frac{u(x)(x-1)}{x}$ is separable as it can be written as

$$\begin{aligned}u'(x) &= \frac{u(x)\,(x-1)}{x}\\ &= f(x)g(u)\end{aligned}$$

Where

$$\begin{aligned}f(x) &= \frac{x-1}{x}\\ g(u) &= u\end{aligned}$$

Integrating gives

$$\begin{aligned}\int \frac{1}{g(u)}\,du &= \int f(x)\,dx\\ \int \frac{1}{u}\,du &= \int \frac{x-1}{x}\,dx\\ \ln(u(x)) &= x + \ln\left(\frac{1}{x}\right) + c_1\end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\ln(u(x)) = x + \ln\left(\frac{1}{x}\right) + c_1$$
$$u(x) = 0$$

Solving for $u(x)$ gives

$$u(x) = 0$$

$$u(x) = \frac{e^{x+c_1}}{x}$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = \frac{e^{x+c_1}}{x}$ back to y gives

$$y = e^{x+c_1}$$

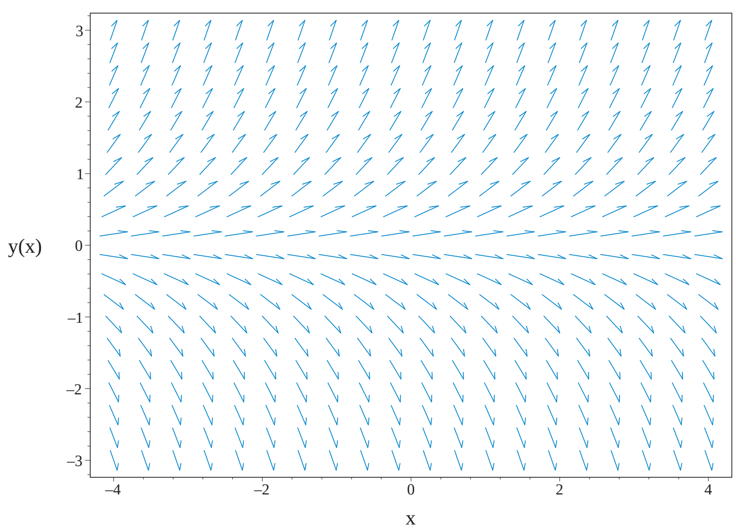


Figure 2.11: Slope field plot
$y' = y$

<u>Summary of solutions found</u>

$$y = 0$$

$$y = e^{x+c_1}$$

Solved as first order Exact ode

Time used: 0.096 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$

$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= (y)\,\mathrm{d}x\\ (-y)\,\mathrm{d}x + \mathrm{d}y &= 0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -y\\ N(x,y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y)\\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1((-1) - (0))\\ &= -1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -1\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-x}\\ &= \mathrm{e}^{-x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{-x}(-y) \\ &= -y\,\mathrm{e}^{-x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{-x}(1) \\ &= \mathrm{e}^{-x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-y\,\mathrm{e}^{-x}\right) + \left(\mathrm{e}^{-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial y}\,\mathrm{d}y &= \int \overline{N}\,\mathrm{d}y \\ \int \frac{\partial \phi}{\partial y}\,\mathrm{d}y &= \int \mathrm{e}^{-x}\,\mathrm{d}y \\ \phi &= y\,\mathrm{e}^{-x} + f(x) \qquad (3)\end{aligned}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = -y\,\mathrm{e}^{-x} + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = -y\,\mathrm{e}^{-x}$. Therefore equation (4) becomes

$$-y\,\mathrm{e}^{-x} = -y\,\mathrm{e}^{-x} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 0$$

Therefore

$$f(x) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = y\,\mathrm{e}^{-x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = y\,\mathrm{e}^{-x}$$

Solving for y gives

$$y = c_1\,\mathrm{e}^{x}$$

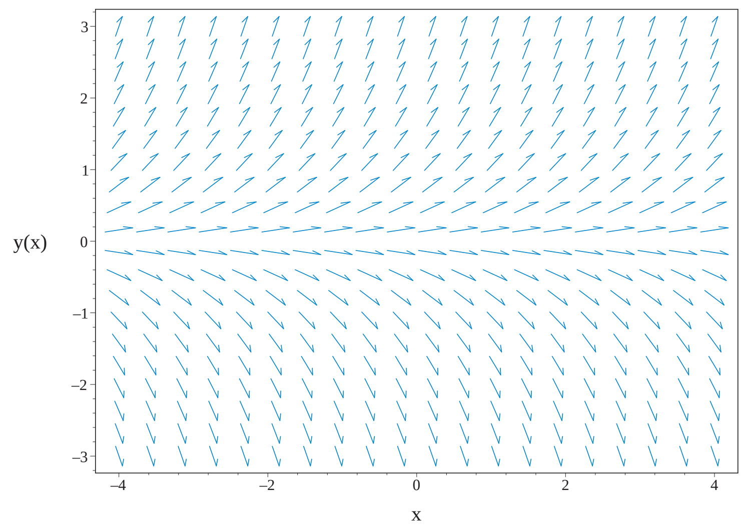


Figure 2.12: Slope field plot
$y' = y$

Summary of solutions found

$$y = c_1\,\mathrm{e}^{x}$$

Solved using Lie symmetry for first order ode

Time used: 0.389 (sec)

Writing the ode as

$$y' = y$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + y(b_3 - a_2) - y^2a_3 - xb_2 - yb_3 - b_1 = 0 \tag{5E}$$

Putting the above in normal form gives

$$-y^2a_3 - xb_2 - ya_2 - b_1 + b_2 = 0$$

Setting the numerator to zero gives

$$-y^2a_3 - xb_2 - ya_2 - b_1 + b_2 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_3v_2^2 - a_2v_2 - b_2v_1 - b_1 + b_2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-a_3v_2^2 - a_2v_2 - b_2v_1 - b_1 + b_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -a_2 &= 0 \\ -a_3 &= 0 \\ -b_2 &= 0 \\ -b_1 + b_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= a_1 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 0 \\ \eta &= y \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y} dy \end{aligned}$$

Which results in

$$S = \ln(y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = y$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 0 \\ S_y &= \frac{1}{y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned} \int dS &= \int 1\, dR \\ S(R) &= R + c_2 \end{aligned}$$

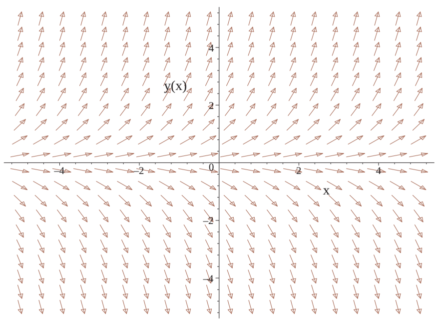
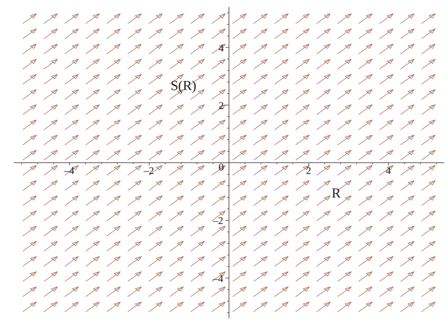
To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\ln(y) = x + c_2$$

Which gives

$$y = \mathrm{e}^{x+c_2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = y$ 	$R = x$ $S = \ln(y)$	$\frac{dS}{dR} = 1$ 

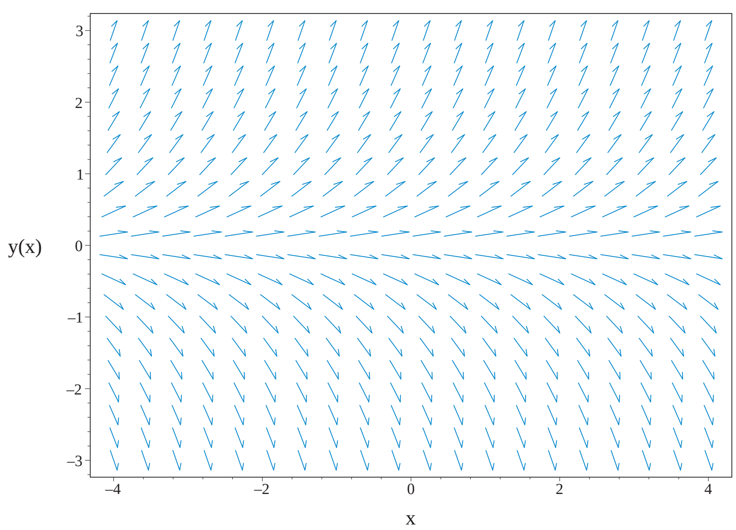


Figure 2.13: Slope field plot
$y' = y$

Summary of solutions found

$$y = \mathrm{e}^{x+c_2}$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) = y(x)$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = y(x)$
- Separate variables
  $\frac{\frac{d}{dx}y(x)}{y(x)} = 1$
- Integrate both sides with respect to x
  $\int \frac{\frac{d}{dx}y(x)}{y(x)}dx = \int 1dx + C1$
- Evaluate integral

$$\ln(y(x)) = x + C1$$

- Solve for $y(x)$

$$y(x) = e^{x+C1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 8

```
dsolve(diff(y(x),x) = y(x),
       y(x),singsol=all)
```

$$y = e^{x} c_1$$

Mathematica DSolve solution

Solving time : `0.021` (sec)
Leaf size : 16

```
DSolve[{D[y[x],x]==y[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{x}$$
$$y(x) \to 0$$

2.1.10 problem 10

Internal problem ID [8670]
Book : First order enumerated odes
Section : section 1
Problem number : 10
Date solved : `Tuesday, December 17, 2024 at 12:57:21 PM`
CAS classification : `[_quadrature]`

Solve

$$y' = by$$

Solved as first order autonomous ode

Time used: 0.155 (sec)

Integrating gives

$$\int \frac{1}{by} dy = dx$$
$$\frac{\ln(y)}{b} = x + c_1$$

Singular solutions are found by solving

$$by = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = 0$$

The following diagram is the phase line diagram. It classifies each of the above equilibrium points as stable or not stable or semi-stable.

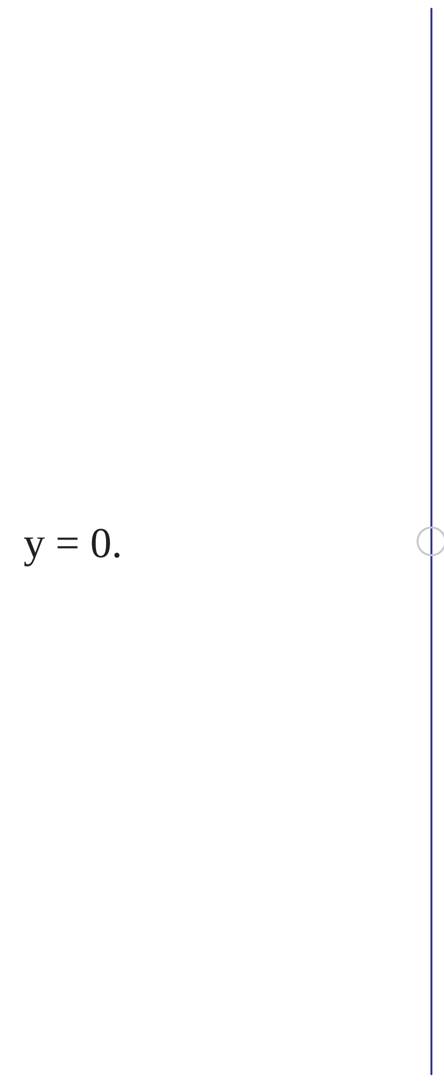


Figure 2.14: Phase line diagram

Solving for y gives

$$y = 0$$
$$y = e^{c_1 b + x b}$$

Summary of solutions found

$$y = 0$$
$$y = e^{c_1 b + x b}$$

Solved as first order homogeneous class D2 ode

Time used: 0.115 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$u'(x)\,x + u(x) = bu(x)\,x$$

Which is now solved The ode $u'(x) = \frac{u(x)(xb-1)}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= \frac{u(x)\,(xb-1)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{xb-1}{x} \\ g(u) &= u \end{aligned}$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\
\int \frac{1}{u}\,du &= \int \frac{xb-1}{x}\,dx \\
\ln\left(u(x)\right) &= xb + \ln\left(\frac{1}{x}\right) + c_1
\end{aligned}
$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$
u(x) = 0
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
\ln\left(u(x)\right) &= xb + \ln\left(\frac{1}{x}\right) + c_1 \\
u(x) &= 0
\end{aligned}
$$

Solving for $u(x)$ gives

$$
\begin{aligned}
u(x) &= 0 \\
u(x) &= \frac{\mathrm{e}^{xb+c_1}}{x}
\end{aligned}
$$

Converting $u(x) = 0$ back to y gives

$$
y = 0
$$

Converting $u(x) = \frac{\mathrm{e}^{xb+c_1}}{x}$ back to y gives

$$
y = \mathrm{e}^{xb+c_1}
$$

Summary of solutions found

$$
\begin{aligned}
y &= 0 \\
y &= \mathrm{e}^{xb+c_1}
\end{aligned}
$$

Solved as first order Exact ode

Time used: 0.104 (sec)

To solve an ode of the form

$$
M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y&=(by)\,\mathrm{d}x\\(-by)\,\mathrm{d}x+\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-by\\N(x,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-by)\\&=-b\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(1)\\&=0\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\&=1((-b)-(0))\\&=-b\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu&=e^{\int A\,\mathrm{d}x}\\&=e^{\int -b\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu&=e^{-xb}\\&=\mathrm{e}^{-xb}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{-xb}(-by)\\ &= -by\,\mathrm{e}^{-xb}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{-xb}(1)\\ &= \mathrm{e}^{-xb}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(-by\,\mathrm{e}^{-xb}\right) + \left(\mathrm{e}^{-xb}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial y}\,\mathrm{d}y &= \int \overline{N}\,\mathrm{d}y\\ \int \frac{\partial \phi}{\partial y}\,\mathrm{d}y &= \int \mathrm{e}^{-xb}\,\mathrm{d}y\end{aligned}$$

$$\phi = \mathrm{e}^{-xb}y + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = -by\,\mathrm{e}^{-xb} + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = -by\,\mathrm{e}^{-xb}$. Therefore equation (4) becomes

$$-by\,\mathrm{e}^{-xb} = -by\,\mathrm{e}^{-xb} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 0$$

Therefore

$$f(x) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{-xb}y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = e^{-xb}y$$

Solving for y gives

$$y = c_1 e^{xb}$$

Summary of solutions found

$$y = c_1 e^{xb}$$

Solved using Lie symmetry for first order ode

Time used: 0.244 (sec)

Writing the ode as

$$y' = by$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + by(b_3 - a_2) - b^2 y^2 a_3 - b(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-b^2 y^2 a_3 - bxb_2 - bya_2 - bb_1 + b_2 = 0$$

Setting the numerator to zero gives

$$-b^2 y^2 a_3 - bxb_2 - bya_2 - bb_1 + b_2 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-b^2 a_3 v_2^2 - ba_2 v_2 - bb_2 v_1 - bb_1 + b_2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-b^2 a_3 v_2^2 - b a_2 v_2 - b b_2 v_1 - b b_1 + b_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -b a_2 &= 0 \\ -b b_2 &= 0 \\ -b^2 a_3 &= 0 \\ -b b_1 + b_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= a_1 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 0 \\ \eta &= y \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y} dy \end{aligned}$$

Which results in

$$S = \ln(y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = by$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1\\ R_y &= 0\\ S_x &= 0\\ S_y &= \frac{1}{y}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = b \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = b$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned}\int dS &= \int b\,dR\\ S(R) &= bR + c_2\end{aligned}$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\ln(y) = xb + c_2$$

Which gives

$$y = e^{xb+c_2}$$

Summary of solutions found

$$y = e^{xb+c_2}$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) = by(x)$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = by(x)$
- Separate variables
  $\frac{\frac{d}{dx}y(x)}{y(x)} = b$
- Integrate both sides with respect to x

$$\int \frac{\frac{d}{dx}y(x)}{y(x)}dx = \int b dx + C1$$

- Evaluate integral

  $$\ln(y(x)) = bx + C1$$

- Solve for $y(x)$

  $$y(x) = e^{bx+C1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 10

```
dsolve(diff(y(x),x) = b*y(x),
       y(x),singsol=all)
```

$$y = e^{bx}c_1$$

Mathematica DSolve solution

Solving time : 0.024 (sec)
Leaf size : 18

```
DSolve[{D[y[x],x]==b*y[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{bx}$$
$$y(x) \to 0$$

2.1.11 problem 11

Internal problem ID [8671]
Book : First order enumerated odes
Section : section 1
Problem number : 11
Date solved : `Tuesday, December 17, 2024 at 12:57:23 PM`
CAS classification : `[[_Riccati, _special]]`

Solve

$$y' = ax + by^2$$

Solved as first order ode of type reduced Riccati

Time used: 0.135 (sec)

This is reduced Riccati ode of the form

$$y' = a\,x^n + by^2$$

Comparing the given ode to the above shows that

$$a = a$$
$$b = b$$
$$n = 1$$

Since $n \neq -2$ then the solution of the reduced Riccati ode is given by

$$w = \sqrt{x}\begin{cases} c_1 \operatorname{BesselJ}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{ab}x^k\right) + c_2 \operatorname{BesselY}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{ab}x^k\right) & ab > 0 \\ c_1 \operatorname{BesselI}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{-ab}x^k\right) + c_2 \operatorname{BesselK}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{-ab}x^k\right) & ab < 0 \end{cases} \tag{1}$$
$$y = -\frac{1}{b}\frac{w'}{w}$$
$$k = 1 + \frac{n}{2}$$

EQ(1) gives

$$k = \frac{3}{2}$$
$$w = \sqrt{x}\left(c_1 \operatorname{BesselJ}\left(\frac{1}{3}, \frac{2\sqrt{ab}\,x^{3/2}}{3}\right) + c_2 \operatorname{BesselY}\left(\frac{1}{3}, \frac{2\sqrt{ab}\,x^{3/2}}{3}\right)\right)$$

Therefore the solution becomes

$$y = -\frac{1}{b}\frac{w'}{w}$$

Substituting the value of b, w found above and simplyfing gives

$$y = \frac{\left(-\operatorname{BesselY}\left(-\frac{2}{3}, \frac{2\sqrt{ab}\,x^{3/2}}{3}\right)c_2 - \operatorname{BesselJ}\left(-\frac{2}{3}, \frac{2\sqrt{ab}\,x^{3/2}}{3}\right)c_1\right)\sqrt{ab}\,\sqrt{x}}{b\left(c_1 \operatorname{BesselJ}\left(\frac{1}{3}, \frac{2\sqrt{ab}\,x^{3/2}}{3}\right) + c_2 \operatorname{BesselY}\left(\frac{1}{3}, \frac{2\sqrt{ab}\,x^{3/2}}{3}\right)\right)}$$

Letting $c_2 = 1$ the above becomes

$$y = \frac{\left(-\operatorname{BesselY}\left(-\frac{2}{3}, \frac{2\sqrt{ab}\,x^{3/2}}{3}\right) - \operatorname{BesselJ}\left(-\frac{2}{3}, \frac{2\sqrt{ab}\,x^{3/2}}{3}\right) c_1\right) \sqrt{ab}\,\sqrt{x}}{b\left(c_1 \operatorname{BesselJ}\left(\frac{1}{3}, \frac{2\sqrt{ab}\,x^{3/2}}{3}\right) + \operatorname{BesselY}\left(\frac{1}{3}, \frac{2\sqrt{ab}\,x^{3/2}}{3}\right)\right)}$$

Summary of solutions found

$$y = \frac{\left(-\operatorname{BesselY}\left(-\frac{2}{3}, \frac{2\sqrt{ab}\,x^{3/2}}{3}\right) - \operatorname{BesselJ}\left(-\frac{2}{3}, \frac{2\sqrt{ab}\,x^{3/2}}{3}\right) c_1\right) \sqrt{ab}\,\sqrt{x}}{b\left(c_1 \operatorname{BesselJ}\left(\frac{1}{3}, \frac{2\sqrt{ab}\,x^{3/2}}{3}\right) + \operatorname{BesselY}\left(\frac{1}{3}, \frac{2\sqrt{ab}\,x^{3/2}}{3}\right)\right)}$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) = xa + by(x)^2$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = xa + by(x)^2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
<- Riccati Special successful`
```

Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 59

```
dsolve(diff(y(x),x) = a*x+b*y(x)^2,
       y(x),singsol=all)
```

$$y = \frac{(ba)^{1/3}\left(\operatorname{AiryAi}\left(1, -(ba)^{1/3} x\right) c_1 + \operatorname{AiryBi}\left(1, -(ba)^{1/3} x\right)\right)}{b\left(c_1 \operatorname{AiryAi}\left(-(ba)^{1/3} x\right) + \operatorname{AiryBi}\left(-(ba)^{1/3} x\right)\right)}$$

Mathematica DSolve solution

Solving time : `0.156` (sec)
Leaf size : 331

```
DSolve[{D[y[x],x]==a*x+b*y[x]^2,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \rightarrow \frac{\sqrt{a}\sqrt{b}x^{3/2}\left(-2\,\mathrm{BesselJ}\left(-\frac{2}{3},\frac{2}{3}\sqrt{a}\sqrt{b}x^{3/2}\right)+c_1\left(\mathrm{BesselJ}\left(\frac{2}{3},\frac{2}{3}\sqrt{a}\sqrt{b}x^{3/2}\right)-\mathrm{BesselJ}\left(-\frac{4}{3},\frac{2}{3}\sqrt{a}\sqrt{b}x^{3/2}\right)\right)\right)}{2bx\left(\mathrm{BesselJ}\left(\frac{1}{3},\frac{2}{3}\sqrt{a}\sqrt{b}x^{3/2}\right)+c_1\,\mathrm{BesselJ}\left(-\frac{1}{3},\frac{2}{3}\sqrt{a}\sqrt{b}x^{3/2}\right)\right)}$$

$$y(x) \rightarrow -\frac{\sqrt{a}\sqrt{b}x^{3/2}\,\mathrm{BesselJ}\left(-\frac{4}{3},\frac{2}{3}\sqrt{a}\sqrt{b}x^{3/2}\right)-\sqrt{a}\sqrt{b}x^{3/2}\,\mathrm{BesselJ}\left(\frac{2}{3},\frac{2}{3}\sqrt{a}\sqrt{b}x^{3/2}\right)+\mathrm{BesselJ}\left(-\frac{1}{3},\frac{2}{3}\sqrt{a}\sqrt{b}x^{3/2}\right)}{2bx\,\mathrm{BesselJ}\left(-\frac{1}{3},\frac{2}{3}\sqrt{a}\sqrt{b}x^{3/2}\right)}$$

2.1.12 problem 12

Internal problem ID [8672]
Book : First order enumerated odes
Section : section 1
Problem number : 12
Date solved : `Tuesday, December 17, 2024 at 12:57:24 PM`
CAS classification : `[_quadrature]`

Solve

$$cy' = 0$$

Solved as first order quadrature ode

Time used: 0.024 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\, dx + c_1$$
$$y = c_1$$

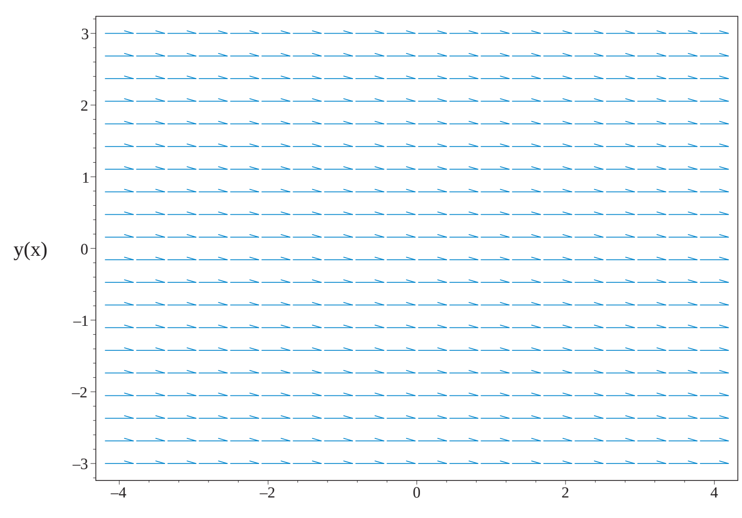


Figure 2.15: Slope field plot
$cy' = 0$

Summary of solutions found

$$y = c_1$$

Solved as first order homogeneous class D2 ode

Time used: 0.129 (sec)

Applying change of variables $y = u(x)\, x$, then the ode becomes

$$c(u'(x)\, x + u(x)) = 0$$

Which is now solved The ode $u'(x) = -\frac{u(x)}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(u) &= u \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\, du &= \int f(x)\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{1}{x}\, dx \\ \ln\left(u(x)\right) &= \ln\left(\frac{1}{x}\right) + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} \ln\left(u(x)\right) &= \ln\left(\frac{1}{x}\right) + c_1 \\ u(x) &= 0 \end{aligned}$$

Solving for $u(x)$ gives

$$\begin{aligned} u(x) &= 0 \\ u(x) &= \frac{e^{c_1}}{x} \end{aligned}$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = \frac{e^{c_1}}{x}$ back to y gives

$$y = e^{c_1}$$

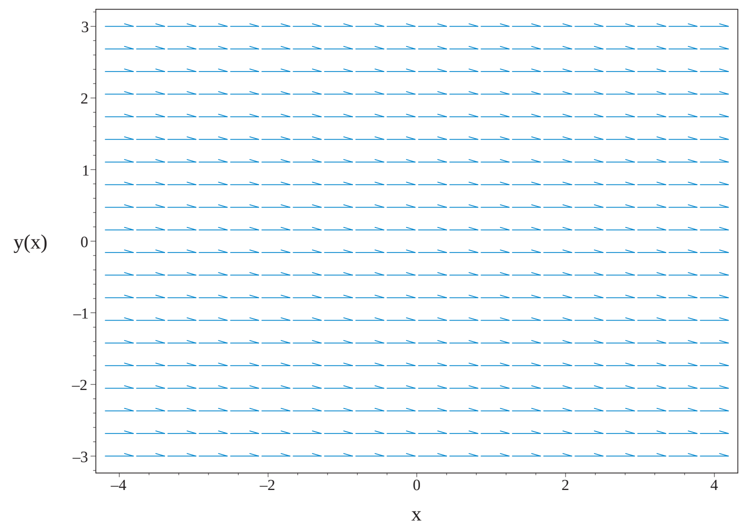


Figure 2.16: Slope field plot
$cy' = 0$

Summary of solutions found

$$y = 0$$

$$y = e^{c_1}$$

Solved as first order ode of type differential

Time used: 0.011 (sec)

Writing the ode as

$$y' = 0 \tag{1}$$

Which becomes

$$(1)\, dy = (0)\, dx \tag{2}$$

But the RHS is complete differential because

$$(0)\, dx = d(0)$$

Hence (2) becomes

$$(1)\, dy = d(0)$$

Integrating gives

$$y = c_1$$

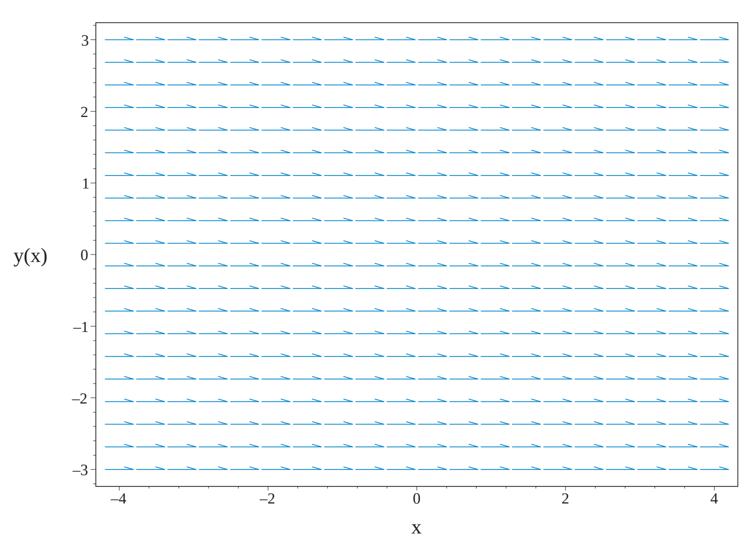


Figure 2.17: Slope field plot
$cy' = 0$

Summary of solutions found

$$y = c_1$$

Maple step by step solution

Let's solve

$c\left(\frac{d}{dx}y(x)\right) = 0$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$
- Separate variables

  $\frac{d}{dx}y(x) = 0$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}y(x)\right) dx = \int 0dx + C1$
- Evaluate integral

  $y(x) = C1$
- Solve for $y(x)$

  $y(x) = C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 5

```
dsolve(c*diff(y(x),x) = 0,
       y(x),singsol=all)
```

$$y = c_1$$

Mathematica DSolve solution

Solving time : `0.002` (sec)
Leaf size : 7

```
DSolve[{c*D[y[x],x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1$$

2.1.13 problem 13

Internal problem ID [8673]
Book : First order enumerated odes
Section : section 1
Problem number : 13
Date solved : `Tuesday, December 17, 2024 at 12:57:25 PM`
CAS classification : `[_quadrature]`

Solve

$$cy' = a$$

Solved as first order quadrature ode

Time used: 0.038 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int \frac{a}{c}\, dx$$
$$y = \frac{ax}{c} + c_1$$

Summary of solutions found

$$y = \frac{ax}{c} + c_1$$

Solved as first order homogeneous class D2 ode

Time used: 0.174 (sec)

Applying change of variables $y = u(x)\, x$, then the ode becomes

$$c(u'(x)\, x + u(x)) = a$$

Which is now solved The ode $u'(x) = -\frac{cu(x)-a}{cx}$ is separable as it can be written as

$$u'(x) = -\frac{cu(x) - a}{cx}$$
$$= f(x)g(u)$$

Where

$$f(x) = -\frac{1}{xc}$$
$$g(u) = cu - a$$

Integrating gives

$$\int \frac{1}{g(u)}\, du = \int f(x)\, dx$$
$$\int \frac{1}{cu - a}\, du = \int -\frac{1}{xc}\, dx$$
$$\frac{\ln(-cu(x) + a)}{c} = \frac{\ln\left(\frac{1}{x}\right)}{c} + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $cu - a = 0$ for $u(x)$ gives

$$u(x) = \frac{a}{c}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\frac{\ln(-cu(x) + a)}{c} = \frac{\ln\left(\frac{1}{x}\right)}{c} + c_1$$
$$u(x) = \frac{a}{c}$$

Solving for $u(x)$ gives

$$u(x) = \frac{a}{c}$$
$$u(x) = \frac{(\mathrm{e}^{-c_1 c} a x - 1)\,\mathrm{e}^{c_1 c}}{cx}$$

Converting $u(x) = \frac{a}{c}$ back to y gives

$$y = \frac{ax}{c}$$

Converting $u(x) = \frac{(\mathrm{e}^{-c_1 c} a x - 1)\mathrm{e}^{c_1 c}}{cx}$ back to y gives

$$y = \frac{(\mathrm{e}^{-c_1 c} a x - 1)\,\mathrm{e}^{c_1 c}}{c}$$

Summary of solutions found

$$y = \frac{ax}{c}$$
$$y = \frac{(\mathrm{e}^{-c_1 c} a x - 1)\,\mathrm{e}^{c_1 c}}{c}$$

Solved as first order Exact ode

Time used: 0.066 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(c)\,\mathrm{d}y &= (a)\,\mathrm{d}x\\ (-a)\,\mathrm{d}x + (c)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -a\\ N(x, y) &= c\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-a)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(c)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int -a\,\mathrm{d}x\end{aligned}$$

$$\phi = -ax + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = c$. Therefore equation (4) becomes

$$c = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = c$$

Integrating the above w.r.t y gives

$$\int f'(y) \, dy = \int (c) \, dy$$
$$f(y) = cy + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -ax + cy + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -ax + cy$$

Solving for y gives

$$y = \frac{ax + c_1}{c}$$

Summary of solutions found

$$y = \frac{ax + c_1}{c}$$

Maple step by step solution

Let's solve
$c\left(\frac{d}{dx}y(x)\right) = a$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Separate variables
  $\frac{d}{dx}y(x) = \frac{a}{c}$
- Integrate both sides with respect to x
  $\int \left(\frac{d}{dx}y(x)\right) dx = \int \frac{a}{c} dx + C1$
- Evaluate integral
  $y(x) = \frac{ax}{c} + C1$
- Solve for $y(x)$
  $y(x) = \frac{C1c + xa}{c}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 12

```
dsolve(c*diff(y(x),x) = a,
       y(x),singsol=all)
```

$$y = \frac{ax}{c} + c_1$$

Mathematica DSolve solution

Solving time : 0.002 (sec)
Leaf size : 14

```
DSolve[{c*D[y[x],x]==a,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{ax}{c} + c_1$$

2.1.14 problem 14

Internal problem ID [8674]
Book : First order enumerated odes
Section : section 1
Problem number : 14
Date solved : `Tuesday, December 17, 2024 at 12:57:25 PM`
CAS classification : `[_quadrature]`

Solve

$$cy' = ax$$

Solved as first order quadrature ode

Time used: 0.042 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int \frac{ax}{c}\, dx$$
$$y = \frac{a\, x^2}{2c} + c_1$$

Summary of solutions found

$$y = \frac{a\, x^2}{2c} + c_1$$

Solved as first order Exact ode

Time used: 0.061 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(c)\,\mathrm{d}y &= (ax)\,\mathrm{d}x\\ (-ax)\,\mathrm{d}x + (c)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -ax\\ N(x,y) &= c\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-ax)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(c)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -ax\,\mathrm{d}x\end{aligned}$$

$$\phi = -\frac{a\,x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = c$. Therefore equation (4) becomes

$$c = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = c$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (c)\,\mathrm{d}y$$
$$f(y) = cy + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{a\,x^2}{2} + cy + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -\frac{a\,x^2}{2} + cy$$

Solving for y gives

$$y = \frac{a\,x^2 + 2c_1}{2c}$$

Summary of solutions found

$$y = \frac{a\,x^2 + 2c_1}{2c}$$

Maple step by step solution

Let's solve
$c\left(\frac{d}{dx}y(x)\right) = xa$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Separate variables
  $\frac{d}{dx}y(x) = \frac{ax}{c}$
- Integrate both sides with respect to x
  $\int \left(\frac{d}{dx}y(x)\right) dx = \int \frac{ax}{c} dx + C1$
- Evaluate integral
  $y(x) = \frac{a\,x^2}{2c} + C1$
- Solve for $y(x)$
  $y(x) = \frac{x^2 a + 2C1c}{2c}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 15

```
dsolve(c*diff(y(x),x) = a*x,
       y(x),singsol=all)
```

$$y = \frac{a\,x^2}{2c} + c_1$$

Mathematica DSolve solution

Solving time : 0.002 (sec)
Leaf size : 19

```
DSolve[{c*D[y[x],x]==a*x,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{ax^2}{2c} + c_1$$

2.1.15 problem 15

Internal problem ID [8675]
Book : First order enumerated odes
Section : section 1
Problem number : 15
Date solved : `Tuesday, December 17, 2024 at 12:57:26 PM`
CAS classification : `[[_linear, ‘class A‘]]`

Solve

$$cy' = ax + y$$

Solved as first order linear ode

Time used: 0.125 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= -\frac{1}{c} \\ p(x) &= \frac{ax}{c} \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= \mathrm{e}^{\int -\frac{1}{c}dx} \\ &= \mathrm{e}^{-\frac{x}{c}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{ax}{c}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(y\,\mathrm{e}^{-\frac{x}{c}}\right) &= \left(\mathrm{e}^{-\frac{x}{c}}\right)\left(\frac{ax}{c}\right) \\ \mathrm{d}\left(y\,\mathrm{e}^{-\frac{x}{c}}\right) &= \left(\frac{ax\,\mathrm{e}^{-\frac{x}{c}}}{c}\right)\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} y\,\mathrm{e}^{-\frac{x}{c}} &= \int \frac{ax\,\mathrm{e}^{-\frac{x}{c}}}{c}\,dx \\ &= -(c+x)\,a\,\mathrm{e}^{-\frac{x}{c}} + c_1 \end{aligned}$$

Dividing throughout by the integrating factor $e^{-\frac{x}{c}}$ gives the final solution

$$y = c_1 e^{\frac{x}{c}} - a(c + x)$$

Summary of solutions found

$$y = c_1 e^{\frac{x}{c}} - a(c + x)$$

Solved as first order Exact ode

Time used: 0.128 (sec)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} (c)\,\mathrm{d}y &= (ax + y)\,\mathrm{d}x \\ (-ax - y)\,\mathrm{d}x + (c)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -ax - y \\ N(x, y) &= c \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-ax - y) \\ &= -1 \end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(c)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{c}((-1) - (0))\\ &= -\frac{1}{c}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x}\\ &= e^{\int -\frac{1}{c} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{x}{c}}\\ &= \mathrm{e}^{-\frac{x}{c}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{-\frac{x}{c}}(-ax - y)\\ &= -(ax + y)\,\mathrm{e}^{-\frac{x}{c}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{-\frac{x}{c}}(c)\\ &= c\,\mathrm{e}^{-\frac{x}{c}}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(-(ax + y)\,\mathrm{e}^{-\frac{x}{c}}\right) + \left(c\,\mathrm{e}^{-\frac{x}{c}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial y} \, \mathrm{d}y &= \int \overline{N} \, \mathrm{d}y\\ \int \frac{\partial \phi}{\partial y} \, \mathrm{d}y &= \int c\,\mathrm{e}^{-\frac{x}{c}} \, \mathrm{d}y\end{aligned}$$

$$\phi = c\,\mathrm{e}^{-\frac{x}{c}} y + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = -y\,\mathrm{e}^{-\frac{x}{c}} + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = -(ax+y)\,\mathrm{e}^{-\frac{x}{c}}$. Therefore equation (4) becomes

$$-(ax+y)\,\mathrm{e}^{-\frac{x}{c}} = -y\,\mathrm{e}^{-\frac{x}{c}} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -ax\,\mathrm{e}^{-\frac{x}{c}}$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int \left(-ax\,\mathrm{e}^{-\frac{x}{c}}\right)\mathrm{d}x$$

$$f(x) = c(c+x)\,a\,\mathrm{e}^{-\frac{x}{c}} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = c\,\mathrm{e}^{-\frac{x}{c}}y + c(c+x)\,a\,\mathrm{e}^{-\frac{x}{c}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = c\,\mathrm{e}^{-\frac{x}{c}}y + c(c+x)\,a\,\mathrm{e}^{-\frac{x}{c}}$$

Solving for y gives

$$y = -\frac{\left(a\,\mathrm{e}^{-\frac{x}{c}}c^2 + cax\,\mathrm{e}^{-\frac{x}{c}} - c_1\right)\mathrm{e}^{\frac{x}{c}}}{c}$$

Summary of solutions found

$$y = -\frac{\left(a\,\mathrm{e}^{-\frac{x}{c}}c^2 + cax\,\mathrm{e}^{-\frac{x}{c}} - c_1\right)\mathrm{e}^{\frac{x}{c}}}{c}$$

Solved using Lie symmetry for first order ode

Time used: 0.365 (sec)

Writing the ode as

$$y' = \frac{ax+y}{c}$$

$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{(ax+y)(b_3-a_2)}{c} - \frac{(ax+y)^2 a_3}{c^2} - \frac{a(xa_2+ya_3+a_1)}{c} - \frac{xb_2+yb_3+b_1}{c} = 0 \quad \text{(5E)}$$

Putting the above in normal form gives

$$-\frac{a^2x^2a_3 + 2acxa_2 - acxb_3 + acya_3 + 2axya_3 + aca_1 - b_2c^2 + cxb_2 + cya_2 + y^2a_3 + cb_1}{c^2} = 0$$

Setting the numerator to zero gives

$$-a^2x^2a_3 - 2acxa_2 + acxb_3 - acya_3 - 2axya_3 - aca_1 + b_2c^2 - cxb_2 - cya_2 - y^2a_3 - cb_1 = 0 \quad \text{(6E)}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-a^2a_3v_1^2 - 2aca_2v_1 - aca_3v_2 + acb_3v_1 - 2aa_3v_1v_2 \\ &- aca_1 + b_2c^2 - ca_2v_2 - cb_2v_1 - a_3v_2^2 - cb_1 = 0 \end{aligned} \quad \text{(7E)}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &-a^2a_3v_1^2 - 2aa_3v_1v_2 + (-2aca_2 + acb_3 - cb_2)\, v_1 \\ &- a_3v_2^2 + (-aca_3 - ca_2)\, v_2 - aca_1 + b_2c^2 - cb_1 = 0 \end{aligned} \quad \text{(8E)}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -a_3 &= 0 \\ -2aa_3 &= 0 \\ -a^2a_3 &= 0 \\ -aca_3 - ca_2 &= 0 \\ -aca_1 + b_2c^2 - cb_1 &= 0 \\ -2aca_2 + acb_3 - cb_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= a_1 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= acb_3 - aa_1 \\
b_2 &= ab_3 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= 0 \\
\eta &= ac + ax + y
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{ac + ax + y} dy
\end{aligned}$$

Which results in

$$S = \ln(ac + ax + y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{ax + y}{c}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{a}{a(c + x) + y} \\
S_y &= \frac{1}{a(c + x) + y}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{c} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{c}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int \frac{1}{c}\, dR$$
$$S(R) = \frac{R}{c} + c_2$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\ln\left(a(c+x)+y\right) = \frac{x}{c} + c_2$$

Which gives

$$y = -ac - ax + \mathrm{e}^{\frac{c_2 c + x}{c}}$$

Summary of solutions found

$$y = -ac - ax + \mathrm{e}^{\frac{c_2 c + x}{c}}$$

Maple step by step solution

Let's solve
$c\left(\frac{d}{dx}y(x)\right) = xa + y(x)$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = \frac{xa+y(x)}{c}$
- Collect w.r.t. $y(x)$ and simplify
  $\frac{d}{dx}y(x) = \frac{y(x)}{c} + \frac{ax}{c}$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE
  $\frac{d}{dx}y(x) - \frac{y(x)}{c} = \frac{ax}{c}$
- The ODE is linear; multiply by an integrating factor $\mu(x)$
  $\mu(x)\left(\frac{d}{dx}y(x) - \frac{y(x)}{c}\right) = \frac{\mu(x)ax}{c}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\,\mu(x))$
  $\mu(x)\left(\frac{d}{dx}y(x) - \frac{y(x)}{c}\right) = \left(\frac{d}{dx}y(x)\right)\mu(x) + y(x)\left(\frac{d}{dx}\mu(x)\right)$
- Isolate $\frac{d}{dx}\mu(x)$
  $\frac{d}{dx}\mu(x) = -\frac{\mu(x)}{c}$
- Solve to find the integrating factor

$$\mu(x) = \mathrm{e}^{-\frac{x}{c}}$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(y(x)\,\mu(x))\right) dx = \int \frac{\mu(x) a x}{c} dx + C1$$

- Evaluate the integral on the lhs

$$y(x)\,\mu(x) = \int \frac{\mu(x) a x}{c} dx + C1$$

- Solve for $y(x)$

$$y(x) = \frac{\int \frac{\mu(x) a x}{c} dx + C1}{\mu(x)}$$

- Substitute $\mu(x) = \mathrm{e}^{-\frac{x}{c}}$

$$y(x) = \frac{\int \frac{\mathrm{e}^{-\frac{x}{c}} a x}{c} dx + C1}{\mathrm{e}^{-\frac{x}{c}}}$$

- Evaluate the integrals on the rhs

$$y(x) = \frac{-(x+c)\mathrm{e}^{-\frac{x}{c}} a + C1}{\mathrm{e}^{-\frac{x}{c}}}$$

- Simplify

$$y(x) = C1\,\mathrm{e}^{\frac{x}{c}} - a(x+c)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 19

```
dsolve(c*diff(y(x),x) = a*x+y(x),
       y(x),singsol=all)
```

$$y = \mathrm{e}^{\frac{x}{c}} c_1 - a(c+x)$$

Mathematica DSolve solution

Solving time : 0.052 (sec)
Leaf size : 22

```
DSolve[{c*D[y[x],x]==a*x+y[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -a(c+x) + c_1 e^{\frac{x}{c}}$$

2.1.16 problem 16

Internal problem ID [8676]
Book : First order enumerated odes
Section : section 1
Problem number : 16
Date solved : `Tuesday, December 17, 2024 at 12:57:27 PM`
CAS classification : `[[_linear, 'class A']]`

Solve

$$cy' = ax + by$$

Solved as first order linear ode

Time used: 0.124 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= -\frac{b}{c} \\ p(x) &= \frac{ax}{c} \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= e^{\int -\frac{b}{c}dx} \\ &= e^{-\frac{bx}{c}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= \mu p \\ \frac{d}{dx}(\mu y) &= (\mu)\left(\frac{ax}{c}\right) \\ \frac{d}{dx}\left(y\,e^{-\frac{bx}{c}}\right) &= \left(e^{-\frac{bx}{c}}\right)\left(\frac{ax}{c}\right) \\ d\left(y\,e^{-\frac{bx}{c}}\right) &= \left(\frac{ax\,e^{-\frac{bx}{c}}}{c}\right)dx \end{aligned}$$

Integrating gives

$$\begin{aligned} y\,e^{-\frac{bx}{c}} &= \int \frac{ax\,e^{-\frac{bx}{c}}}{c}\,dx \\ &= -\frac{(bx+c)\,a\,e^{-\frac{bx}{c}}}{b^2} + c_1 \end{aligned}$$

Dividing throughout by the integrating factor $e^{-\frac{bx}{c}}$ gives the final solution

$$y = \frac{c_1 \, e^{\frac{bx}{c}} b^2 - a(bx + c)}{b^2}$$

Summary of solutions found

$$y = \frac{c_1 \, e^{\frac{bx}{c}} b^2 - a(bx + c)}{b^2}$$

Solved as first order Exact ode

Time used: 0.140 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\, \mathrm{d}x + N(x,y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(c)\, \mathrm{d}y &= (ax + by)\, \mathrm{d}x\\ (-ax - by)\, \mathrm{d}x + (c)\, \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -ax - by\\ N(x,y) &= c\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-ax - by) \\ &= -b\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(c) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{c}((-b) - (0)) \\ &= -\frac{b}{c}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\frac{b}{c} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{bx}{c}} \\ &= \mathrm{e}^{-\frac{bx}{c}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{-\frac{bx}{c}}(-ax - by) \\ &= -(ax + by)\,\mathrm{e}^{-\frac{bx}{c}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{-\frac{bx}{c}}(c) \\ &= c\,\mathrm{e}^{-\frac{bx}{c}}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-(ax + by)\,\mathrm{e}^{-\frac{bx}{c}}\right) + \left(c\,\mathrm{e}^{-\frac{bx}{c}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\int \frac{\partial \phi}{\partial y} \,\mathrm{d}y = \int \overline{N} \,\mathrm{d}y$$
$$\int \frac{\partial \phi}{\partial y} \,\mathrm{d}y = \int c\,\mathrm{e}^{-\frac{bx}{c}} \,\mathrm{d}y$$
$$\phi = c\,\mathrm{e}^{-\frac{bx}{c}} y + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = -b\,\mathrm{e}^{-\frac{bx}{c}} y + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = -(ax+by)\,\mathrm{e}^{-\frac{bx}{c}}$. Therefore equation (4) becomes

$$-(ax+by)\,\mathrm{e}^{-\frac{bx}{c}} = -b\,\mathrm{e}^{-\frac{bx}{c}} y + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -ax\,\mathrm{e}^{-\frac{bx}{c}}$$

Integrating the above w.r.t x gives

$$\int f'(x) \,\mathrm{d}x = \int \left(-ax\,\mathrm{e}^{-\frac{bx}{c}}\right) \mathrm{d}x$$
$$f(x) = \frac{c(bx+c)\,a\,\mathrm{e}^{-\frac{bx}{c}}}{b^2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = c\,\mathrm{e}^{-\frac{bx}{c}} y + \frac{c(bx+c)\,a\,\mathrm{e}^{-\frac{bx}{c}}}{b^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = c\,\mathrm{e}^{-\frac{bx}{c}} y + \frac{c(bx+c)\,a\,\mathrm{e}^{-\frac{bx}{c}}}{b^2}$$

Solving for y gives

$$y = -\frac{\left(\mathrm{e}^{-\frac{bx}{c}} abcx + \mathrm{e}^{-\frac{bx}{c}} a\,c^2 - c_1\,b^2\right) \mathrm{e}^{\frac{bx}{c}}}{c\,b^2}$$

Summary of solutions found

$$y = -\frac{\left(\mathrm{e}^{-\frac{bx}{c}} abcx + \mathrm{e}^{-\frac{bx}{c}} a\,c^2 - c_1\,b^2\right) \mathrm{e}^{\frac{bx}{c}}}{c\,b^2}$$

Solved using Lie symmetry for first order ode

Time used: 0.383 (sec)

Writing the ode as

$$
\begin{aligned}
y' &= \frac{ax+by}{c} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = xa_2 + ya_3 + a_1 \tag{1E}
$$

$$
\eta = xb_2 + yb_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
b_2 + \frac{(ax+by)(b_3-a_2)}{c} - \frac{(ax+by)^2 a_3}{c^2} - \frac{a(xa_2+ya_3+a_1)}{c} - \frac{b(xb_2+yb_3+b_1)}{c} = 0 \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{a^2x^2a_3 + 2abxya_3 + b^2y^2a_3 + 2acxa_2 - acxb_3 + acya_3 + bcxb_2 + bcya_2 + aca_1 + bcb_1 - b_2c^2}{c^2} = 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-a^2x^2a_3 - 2abxya_3 - b^2y^2a_3 - 2acxa_2 + acxb_3 \\
&- acya_3 - bcxb_2 - bcya_2 - aca_1 - bcb_1 + b_2c^2 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x = v_1, y = v_2\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-a^2a_3v_1^2 - 2aba_3v_1v_2 - b^2a_3v_2^2 - 2aca_2v_1 - aca_3v_2 \\
&+ acb_3v_1 - bca_2v_2 - bcb_2v_1 - aca_1 - bcb_1 + b_2c^2 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&-a^2a_3v_1^2 - 2aba_3v_1v_2 + (-2aca_2 + acb_3 - bcb_2)\, v_1 \\
&- b^2a_3v_2^2 + (-aca_3 - bca_2)\, v_2 - aca_1 - bcb_1 + b_2c^2 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-a^2a_3 &= 0 \\
-b^2a_3 &= 0 \\
-2aba_3 &= 0 \\
-aca_3 - bca_2 &= 0 \\
-aca_1 - bcb_1 + b_2c^2 &= 0 \\
-2aca_2 + acb_3 - bcb_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= a_1 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= -\frac{a(ba_1 - cb_3)}{b^2} \\
b_2 &= \frac{ab_3}{b} \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 0 \\
\eta &= \frac{abx + b^2y + ac}{b^2}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{abx + b^2y + ac}{b^2}} dy
\end{aligned}
$$

Which results in

$$
S = \ln\left(abx + b^2y + ac\right)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{ax+by}{c}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{ab}{a\,(bx+c)+b^2y} \\ S_y &= \frac{b^2}{abx+b^2y+ac} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{b}{c} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{b}{c}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned} \int dS &= \int \frac{b}{c}\,dR \\ S(R) &= \frac{bR}{c} + c_2 \end{aligned}$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\ln\left(a(bx+c)+b^2y\right) = \frac{bx}{c} + c_2$$

Which gives

$$y = \frac{-abx - ac + e^{\frac{c_2c+bx}{c}}}{b^2}$$

<u>Summary of solutions found</u>

$$y = \frac{-abx - ac + e^{\frac{c_2c+bx}{c}}}{b^2}$$

Maple step by step solution

Let's solve

$c\left(\frac{d}{dx}y(x)\right) = xa + by(x)$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = \frac{xa+by(x)}{c}$

- Collect w.r.t. $y(x)$ and simplify

  $\frac{d}{dx}y(x) = \frac{by(x)}{c} + \frac{ax}{c}$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

  $\frac{d}{dx}y(x) - \frac{by(x)}{c} = \frac{ax}{c}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(\frac{d}{dx}y(x) - \frac{by(x)}{c}\right) = \frac{\mu(x)ax}{c}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\,\mu(x))$

  $\mu(x)\left(\frac{d}{dx}y(x) - \frac{by(x)}{c}\right) = \left(\frac{d}{dx}y(x)\right)\mu(x) + y(x)\left(\frac{d}{dx}\mu(x)\right)$

- Isolate $\frac{d}{dx}\mu(x)$

  $\frac{d}{dx}\mu(x) = -\frac{\mu(x)b}{c}$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{-\frac{xb}{c}}$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(y(x)\,\mu(x))\right)dx = \int\frac{\mu(x)ax}{c}dx + C1$

- Evaluate the integral on the lhs

  $y(x)\,\mu(x) = \int\frac{\mu(x)ax}{c}dx + C1$

- Solve for $y(x)$

  $y(x) = \frac{\int\frac{\mu(x)ax}{c}dx+C1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{-\frac{xb}{c}}$

  $y(x) = \frac{\int\frac{\mathrm{e}^{-\frac{xb}{c}}ax}{c}dx+C1}{\mathrm{e}^{-\frac{xb}{c}}}$

- Evaluate the integrals on the rhs

  $y(x) = \frac{-\frac{(bx+c)\mathrm{e}^{-\frac{xb}{c}}a}{b^2}+C1}{\mathrm{e}^{-\frac{xb}{c}}}$

- Simplify

  $y(x) = \frac{C1\,\mathrm{e}^{\frac{xb}{c}}b^2-a(bx+c)}{b^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 29

```
dsolve(c*diff(y(x),x) = a*x+b*y(x),
       y(x),singsol=all)
```

$$y = \frac{e^{\frac{bx}{c}} c_1 b^2 - a(bx + c)}{b^2}$$

Mathematica DSolve solution

Solving time : 0.061 (sec)
Leaf size : 28

```
DSolve[{c*D[y[x],x]==a*x+b*y[x],{}},
      y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{a(bx + c)}{b^2} + c_1 e^{\frac{bx}{c}}$$

2.1.17 problem 17

Internal problem ID [8677]
Book : First order enumerated odes
Section : section 1
Problem number : 17
Date solved : `Tuesday, December 17, 2024 at 12:57:28 PM`
CAS classification : `[_quadrature]`

Solve

$$cy' = y$$

Solved as first order autonomous ode

Time used: 0.161 (sec)

Integrating gives

$$\int \frac{c}{y} dy = dx$$
$$c \ln (y) = x + c_1$$

Singular solutions are found by solving

$$\frac{y}{c} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = 0$$

The following diagram is the phase line diagram. It classifies each of the above equilibrium points as stable or not stable or semi-stable.

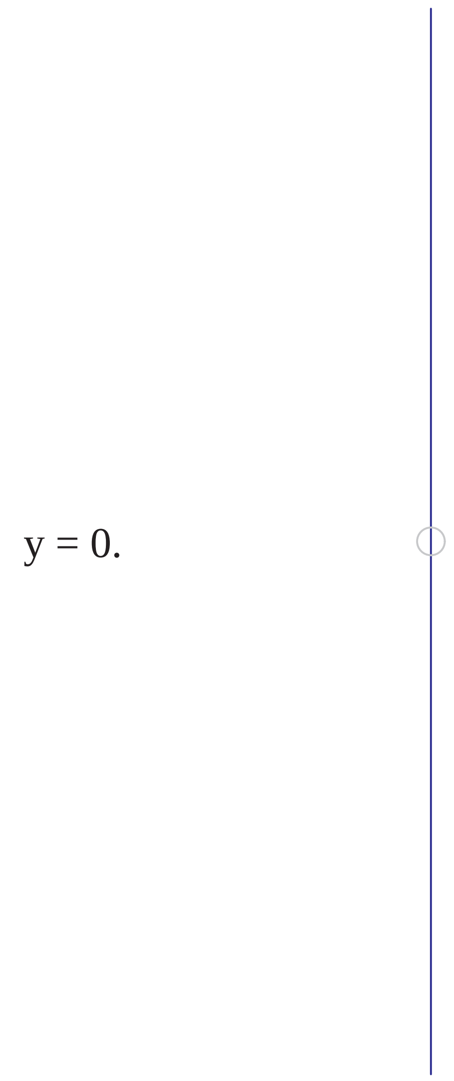


Figure 2.18: Phase line diagram

Solving for y gives

$$y = 0$$
$$y = e^{\frac{x+c_1}{c}}$$

Summary of solutions found

$$y = 0$$
$$y = e^{\frac{x+c_1}{c}}$$

Solved as first order homogeneous class D2 ode

Time used: 0.184 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$c(u'(x)\,x + u(x)) = u(x)\,x$$

Which is now solved The ode $u'(x) = -\frac{u(x)(c-x)}{cx}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)\,(c-x)}{cx} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{c-x}{xc} \\ g(u) &= u \end{aligned}$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(u)} du &= \int f(x) dx \\
\int \frac{1}{u} du &= \int -\frac{c-x}{xc} dx \\
\ln (u(x)) &= \ln \left(\frac{1}{x}\right) + \frac{x}{c} + c_1
\end{aligned}
$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$
u(x) = 0
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
\ln (u(x)) = \ln \left(\frac{1}{x}\right) + \frac{x}{c} + c_1 \\
u(x) = 0
\end{aligned}
$$

Solving for $u(x)$ gives

$$
\begin{aligned}
u(x) &= 0 \\
u(x) &= \mathrm{e}^{\frac{\ln\left(\frac{1}{x}\right)c + c_1 c + x}{c}}
\end{aligned}
$$

Converting $u(x) = 0$ back to y gives

$$
y = 0
$$

Converting $u(x) = \mathrm{e}^{\frac{\ln\left(\frac{1}{x}\right)c + c_1 c + x}{c}}$ back to y gives

$$
y = \mathrm{e}^{\frac{\ln\left(\frac{1}{x}\right)c + c_1 c + x}{c}} x
$$

Summary of solutions found

$$
\begin{aligned}
y &= 0 \\
y &= \mathrm{e}^{\frac{\ln\left(\frac{1}{x}\right)c + c_1 c + x}{c}} x
\end{aligned}
$$

Solved as first order Exact ode

Time used: 0.170 (sec)

To solve an ode of the form

$$
M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx} \phi(x, y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(c)\,\mathrm{d}y = (y)\,\mathrm{d}x$$
$$(-y)\,\mathrm{d}x + (c)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -y$$
$$N(x,y) = c$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y) \\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(c) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{c}((-1) - (0)) \\ &= -\frac{1}{c}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x} \\ &= e^{\int -\frac{1}{c}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{x}{c}}\\ &= \mathrm{e}^{-\frac{x}{c}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{-\frac{x}{c}}(-y)\\ &= -y\,\mathrm{e}^{-\frac{x}{c}}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \mathrm{e}^{-\frac{x}{c}}(c)\\ &= c\,\mathrm{e}^{-\frac{x}{c}}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(-y\,\mathrm{e}^{-\frac{x}{c}}\right) + \left(c\,\mathrm{e}^{-\frac{x}{c}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\begin{aligned}\int\frac{\partial\phi}{\partial y}\,\mathrm{d}y &= \int\overline{N}\,\mathrm{d}y\\ \int\frac{\partial\phi}{\partial y}\,\mathrm{d}y &= \int c\,\mathrm{e}^{-\frac{x}{c}}\,\mathrm{d}y\end{aligned}$$

$$\phi = c\,\mathrm{e}^{-\frac{x}{c}}y + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial\phi}{\partial x} = -y\,\mathrm{e}^{-\frac{x}{c}} + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial\phi}{\partial x} = -y\,\mathrm{e}^{-\frac{x}{c}}$. Therefore equation (4) becomes

$$-y\,\mathrm{e}^{-\frac{x}{c}} = -y\,\mathrm{e}^{-\frac{x}{c}} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 0$$

Therefore

$$f(x) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = c\,\mathrm{e}^{-\frac{x}{c}}y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = c\,\mathrm{e}^{-\frac{x}{c}}y$$

Solving for y gives

$$y = \frac{c_1\,\mathrm{e}^{\frac{x}{c}}}{c}$$

Summary of solutions found

$$y = \frac{c_1\,\mathrm{e}^{\frac{x}{c}}}{c}$$

Solved using Lie symmetry for first order ode

Time used: 0.201 (sec)

Writing the ode as

$$y' = \frac{y}{c}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{y(b_3 - a_2)}{c} - \frac{y^2a_3}{c^2} - \frac{xb_2 + yb_3 + b_1}{c} = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{b_2c^2 - cxb_2 - yca_2 - y^2a_3 - cb_1}{c^2} = 0$$

Setting the numerator to zero gives

$$b_2c^2 - cxb_2 - yca_2 - y^2a_3 - cb_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$b_2c^2 - ca_2v_2 - cb_2v_1 - a_3v_2^2 - cb_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$b_2c^2 - ca_2v_2 - cb_2v_1 - a_3v_2^2 - cb_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -a_3 &= 0 \\ -ca_2 &= 0 \\ -cb_2 &= 0 \\ b_2c^2 - cb_1 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= a_1 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 0 \\ \eta &= y \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y} dy \end{aligned}$$

Which results in

$$S = \ln(y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y}{c}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 0 \\ S_y &= \frac{1}{y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{c} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{c}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned} \int dS &= \int \frac{1}{c}\, dR \\ S(R) &= \frac{R}{c} + c_2 \end{aligned}$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\ln(y) = \frac{x}{c} + c_2$$

Which gives

$$y = \mathrm{e}^{\frac{c_2 c + x}{c}}$$

Summary of solutions found

$$y = \mathrm{e}^{\frac{c_2 c + x}{c}}$$

Maple step by step solution

Let's solve

$c\left(\frac{d}{dx}y(x)\right) = y(x)$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = \frac{y(x)}{c}$

- Separate variables

  $\frac{\frac{d}{dx}y(x)}{y(x)} = \frac{1}{c}$

- Integrate both sides with respect to x

  $\int \frac{\frac{d}{dx}y(x)}{y(x)}dx = \int \frac{1}{c}dx + C1$

- Evaluate integral

  $\ln(y(x)) = \frac{x}{c} + C1$

- Solve for $y(x)$

  $y(x) = e^{\frac{C1c+x}{c}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 12

```
dsolve(c*diff(y(x),x) = y(x),
       y(x),singsol=all)
```

$$y = e^{\frac{x}{c}} c_1$$

Mathematica DSolve solution

Solving time : 0.024 (sec)
Leaf size : 20

```
DSolve[{c*D[y[x],x]==y[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{\frac{x}{c}}$$
$$y(x) \to 0$$

2.1.18 problem 18

Internal problem ID [8678]
Book : First order enumerated odes
Section : section 1
Problem number : 18
Date solved : `Tuesday, December 17, 2024 at 12:57:29 PM`
CAS classification : `[_quadrature]`

Solve

$$cy' = by$$

Solved as first order autonomous ode

Time used: 0.173 (sec)

Integrating gives

$$\int \frac{c}{by} dy = dx$$
$$\frac{c \ln (y)}{b} = x + c_1$$

Singular solutions are found by solving

$$\frac{by}{c} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = 0$$

The following diagram is the phase line diagram. It classifies each of the above equilibrium points as stable or not stable or semi-stable.

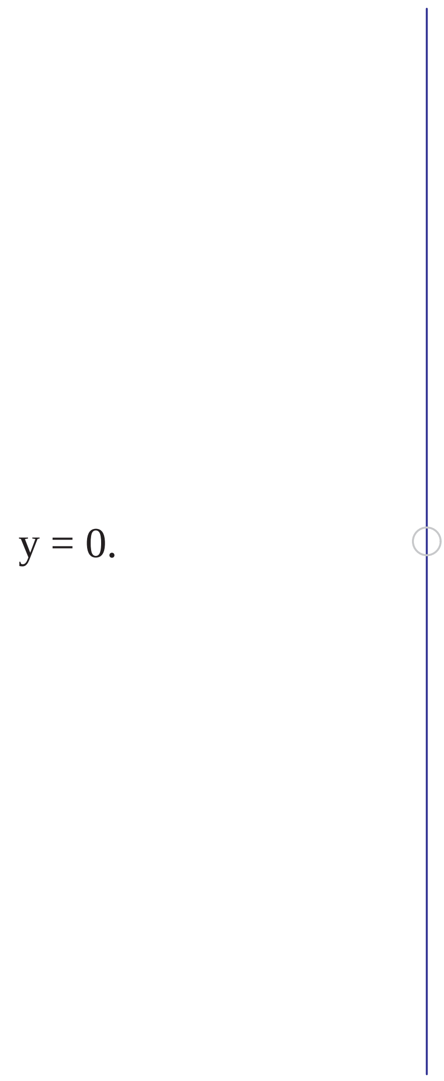


Figure 2.19: Phase line diagram

Solving for y gives

$$y = 0$$
$$y = e^{\frac{b(x+c_1)}{c}}$$

Summary of solutions found

$$y = 0$$
$$y = e^{\frac{b(x+c_1)}{c}}$$

Solved as first order homogeneous class D2 ode

Time used: 0.202 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$c(u'(x)\,x + u(x)) = bu(x)\,x$$

Which is now solved The ode $u'(x) = \frac{u(x)(xb-c)}{cx}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= \frac{u(x)\,(xb-c)}{cx} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{xb-c}{xc} \\ g(u) &= u \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(u)}\, du = \int f(x)\, dx$$
$$\int \frac{1}{u}\, du = \int \frac{xb-c}{xc}\, dx$$
$$\ln(u(x)) = \ln\left(\frac{1}{x}\right) + \frac{xb}{c} + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\ln(u(x)) = \ln\left(\frac{1}{x}\right) + \frac{xb}{c} + c_1$$
$$u(x) = 0$$

Solving for $u(x)$ gives

$$u(x) = 0$$
$$u(x) = e^{\frac{\ln\left(\frac{1}{x}\right)c + c_1 c + xb}{c}}$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = e^{\frac{\ln\left(\frac{1}{x}\right)c + c_1 c + xb}{c}}$ back to y gives

$$y = x\, e^{\frac{\ln\left(\frac{1}{x}\right)c + c_1 c + xb}{c}}$$

Summary of solutions found

$$y = 0$$
$$y = x\, e^{\frac{\ln\left(\frac{1}{x}\right)c + c_1 c + xb}{c}}$$

Solved as first order Exact ode

Time used: 0.170 (sec)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(c)\,\mathrm{d}y = (by)\,\mathrm{d}x$$
$$(-by)\,\mathrm{d}x + (c)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -by$$
$$N(x, y) = c$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-by)\\ &= -b\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(c)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{c}((-b) - (0))\\ &= -\frac{b}{c}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\frac{b}{c}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\mu = e^{-\frac{xb}{c}}$$
$$= \mathrm{e}^{-\frac{xb}{c}}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\overline{M} = \mu M$$
$$= \mathrm{e}^{-\frac{xb}{c}}(-by)$$
$$= -by\,\mathrm{e}^{-\frac{xb}{c}}$$

And

$$\overline{N} = \mu N$$
$$= \mathrm{e}^{-\frac{xb}{c}}(c)$$
$$= c\,\mathrm{e}^{-\frac{xb}{c}}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\left(-by\,\mathrm{e}^{-\frac{xb}{c}}\right) + \left(c\,\mathrm{e}^{-\frac{xb}{c}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\int \frac{\partial \phi}{\partial y}\,\mathrm{d}y = \int \overline{N}\,\mathrm{d}y$$
$$\int \frac{\partial \phi}{\partial y}\,\mathrm{d}y = \int c\,\mathrm{e}^{-\frac{xb}{c}}\,\mathrm{d}y$$
$$\phi = c\,\mathrm{e}^{-\frac{xb}{c}}y + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = -by\,\mathrm{e}^{-\frac{xb}{c}} + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = -by\,\mathrm{e}^{-\frac{xb}{c}}$. Therefore equation (4) becomes

$$-by\,\mathrm{e}^{-\frac{xb}{c}} = -by\,\mathrm{e}^{-\frac{xb}{c}} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = 0$$

Therefore

$$f(x) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(x)$ into equation (3) gives ϕ

$$\phi = c\,\mathrm{e}^{-\frac{xb}{c}} y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = c\,\mathrm{e}^{-\frac{xb}{c}} y$$

Solving for y gives

$$y = \frac{c_1\,\mathrm{e}^{\frac{xb}{c}}}{c}$$

Summary of solutions found

$$y = \frac{c_1\,\mathrm{e}^{\frac{xb}{c}}}{c}$$

Solved using Lie symmetry for first order ode

Time used: 0.231 (sec)

Writing the ode as

$$y' = \frac{by}{c}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{by(b_3 - a_2)}{c} - \frac{b^2 y^2 a_3}{c^2} - \frac{b(xb_2 + yb_3 + b_1)}{c} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{b^2 y^2 a_3 + bcxb_2 + byca_2 + bcb_1 - b_2 c^2}{c^2} = 0$$

Setting the numerator to zero gives

$$-b^2 y^2 a_3 - bcxb_2 - byca_2 - bcb_1 + b_2 c^2 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-b^2 a_3 v_2^2 - bca_2 v_2 - bcb_2 v_1 - bcb_1 + b_2 c^2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-b^2 a_3 v_2^2 - bca_2 v_2 - bcb_2 v_1 - bcb_1 + b_2 c^2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -b^2 a_3 &= 0 \\ -bca_2 &= 0 \\ -bcb_2 &= 0 \\ -bcb_1 + b_2 c^2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= a_1 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 0 \\ \eta &= y \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y} dy \end{aligned}$$

Which results in

$$S = \ln(y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{by}{c}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 0 \\ S_y &= \frac{1}{y} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{b}{c} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{b}{c}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned} \int dS &= \int \frac{b}{c}\, dR \\ S(R) &= \frac{bR}{c} + c_2 \end{aligned}$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\ln(y) = \frac{xb}{c} + c_2$$

Which gives

$$y = e^{\frac{c_2 c + xb}{c}}$$

Summary of solutions found

$$y = e^{\frac{c_2 c + xb}{c}}$$

Maple step by step solution

Let's solve

$c\left(\frac{d}{dx}y(x)\right) = by(x)$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = \frac{by(x)}{c}$

- Separate variables

  $\frac{\frac{d}{dx}y(x)}{y(x)} = \frac{b}{c}$

- Integrate both sides with respect to x

  $\int \frac{\frac{d}{dx}y(x)}{y(x)}dx = \int \frac{b}{c}dx + C1$

- Evaluate integral

  $\ln(y(x)) = \frac{xb}{c} + C1$

- Solve for $y(x)$

  $y(x) = e^{\frac{C1\,c+bx}{c}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 13

```
dsolve(c*diff(y(x),x) = b*y(x),
       y(x),singsol=all)
```

$$y = e^{\frac{bx}{c}} c_1$$

Mathematica DSolve solution

Solving time : 0.025 (sec)
Leaf size : 21

```
DSolve[{c*D[y[x],x]==b*y[x],{}},
      y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1 e^{\frac{bx}{c}}$$
$$y(x) \to 0$$

2.1.19 problem 19

Internal problem ID [8679]
Book : First order enumerated odes
Section : section 1
Problem number : 19
Date solved : `Tuesday, December 17, 2024 at 12:57:31 PM`
CAS classification : `[[_Riccati, _special]]`

Solve

$$cy' = ax + by^2$$

Solved as first order ode of type reduced Riccati

Time used: 0.151 (sec)

This is reduced Riccati ode of the form

$$y' = a\,x^n + by^2$$

Comparing the given ode to the above shows that

$$\begin{aligned} a &= \frac{a}{c} \\ b &= \frac{b}{c} \\ n &= 1 \end{aligned}$$

Since $n \neq -2$ then the solution of the reduced Riccati ode is given by

$$\begin{aligned} w &= \sqrt{x}\begin{cases} c_1 \operatorname{BesselJ}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{ab}x^k\right) + c_2 \operatorname{BesselY}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{ab}x^k\right) & ab > 0 \\ c_1 \operatorname{BesselI}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{-ab}x^k\right) + c_2 \operatorname{BesselK}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{-ab}x^k\right) & ab < 0 \end{cases} \\ y &= -\frac{1}{b}\frac{w'}{w} \\ k &= 1 + \frac{n}{2} \end{aligned} \tag{1}$$

EQ(1) gives

$$\begin{aligned} k &= \frac{3}{2} \\ w &= \sqrt{x}\left(c_1 \operatorname{BesselJ}\left(\frac{1}{3}, \frac{2\sqrt{\frac{ab}{c^2}}\,x^{3/2}}{3}\right) + c_2 \operatorname{BesselY}\left(\frac{1}{3}, \frac{2\sqrt{\frac{ab}{c^2}}\,x^{3/2}}{3}\right)\right) \end{aligned}$$

Therefore the solution becomes

$$y = -\frac{1}{b}\frac{w'}{w}$$

Substituting the value of b, w found above and simplyfing gives

$$y = \frac{\left(-\operatorname{BesselY}\left(-\frac{2}{3}, \frac{2\sqrt{\frac{ab}{c^2}}\,x^{3/2}}{3}\right) c_2 - \operatorname{BesselJ}\left(-\frac{2}{3}, \frac{2\sqrt{\frac{ab}{c^2}}\,x^{3/2}}{3}\right) c_1\right) c\sqrt{\frac{ab}{c^2}}\,\sqrt{x}}{b\left(c_1 \operatorname{BesselJ}\left(\frac{1}{3}, \frac{2\sqrt{\frac{ab}{c^2}}\,x^{3/2}}{3}\right) + c_2 \operatorname{BesselY}\left(\frac{1}{3}, \frac{2\sqrt{\frac{ab}{c^2}}\,x^{3/2}}{3}\right)\right)}$$

Letting $c_2 = 1$ the above becomes

$$y = \frac{\left(-\operatorname{BesselY}\left(-\frac{2}{3}, \frac{2\sqrt{\frac{ab}{c^2}}\,x^{3/2}}{3}\right) - \operatorname{BesselJ}\left(-\frac{2}{3}, \frac{2\sqrt{\frac{ab}{c^2}}\,x^{3/2}}{3}\right) c_1\right) c\sqrt{\frac{ab}{c^2}}\,\sqrt{x}}{b\left(c_1 \operatorname{BesselJ}\left(\frac{1}{3}, \frac{2\sqrt{\frac{ab}{c^2}}\,x^{3/2}}{3}\right) + \operatorname{BesselY}\left(\frac{1}{3}, \frac{2\sqrt{\frac{ab}{c^2}}\,x^{3/2}}{3}\right)\right)}$$

Summary of solutions found

$$y = \frac{\left(-\operatorname{BesselY}\left(-\frac{2}{3}, \frac{2\sqrt{\frac{ab}{c^2}}\,x^{3/2}}{3}\right) - \operatorname{BesselJ}\left(-\frac{2}{3}, \frac{2\sqrt{\frac{ab}{c^2}}\,x^{3/2}}{3}\right) c_1\right) c\sqrt{\frac{ab}{c^2}}\,\sqrt{x}}{b\left(c_1 \operatorname{BesselJ}\left(\frac{1}{3}, \frac{2\sqrt{\frac{ab}{c^2}}\,x^{3/2}}{3}\right) + \operatorname{BesselY}\left(\frac{1}{3}, \frac{2\sqrt{\frac{ab}{c^2}}\,x^{3/2}}{3}\right)\right)}$$

Maple step by step solution

Let's solve

$c\left(\frac{d}{dx}y(x)\right) = xa + by(x)^2$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = \frac{xa+by(x)^2}{c}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
<- Riccati Special successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 75

```
dsolve(c*diff(y(x),x) = a*x+b*y(x)^2,
       y(x),singsol=all)
```

$$y = \frac{\left(\frac{ba}{c^2}\right)^{1/3}\left(\operatorname{AiryAi}\left(1, -\left(\frac{ba}{c^2}\right)^{1/3} x\right) c_1 + \operatorname{AiryBi}\left(1, -\left(\frac{ba}{c^2}\right)^{1/3} x\right)\right) c}{b\left(c_1 \operatorname{AiryAi}\left(-\left(\frac{ba}{c^2}\right)^{1/3} x\right) + \operatorname{AiryBi}\left(-\left(\frac{ba}{c^2}\right)^{1/3} x\right)\right)}$$

Mathematica DSolve solution

Solving time : `0.186` (sec)
Leaf size : 437

```
DSolve[{c*D[y[x],x]==a*x+b*y[x]^2,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{c\left(x^{3/2}\sqrt{\frac{a}{c}}\sqrt{\frac{b}{c}}\left(-2\,\text{BesselJ}\left(-\frac{2}{3},\frac{2}{3}\sqrt{\frac{a}{c}}\sqrt{\frac{b}{c}}x^{3/2}\right)+c_1\left(\text{BesselJ}\left(\frac{2}{3},\frac{2}{3}\sqrt{\frac{a}{c}}\sqrt{\frac{b}{c}}x^{3/2}\right)-\text{BesselJ}\left(-\frac{4}{3},\frac{2}{3}\sqrt{\frac{a}{c}}\sqrt{\right.\right.\right.\right.}{2bx\left(\text{BesselJ}\left(\frac{1}{3},\frac{2}{3}\sqrt{\frac{a}{c}}\sqrt{\frac{b}{c}}x^{3/2}\right)+c_1\,\text{BesselJ}\left(-\frac{1}{3},\frac{2}{3}\sqrt{\frac{a}{c}}\sqrt{\frac{b}{c}}x^{3/\right.\right.}$$

$$y(x) \to -\frac{c\left(x^{3/2}\sqrt{\frac{a}{c}}\sqrt{\frac{b}{c}}\,\text{BesselJ}\left(-\frac{4}{3},\frac{2}{3}\sqrt{\frac{a}{c}}\sqrt{\frac{b}{c}}x^{3/2}\right)-x^{3/2}\sqrt{\frac{a}{c}}\sqrt{\frac{b}{c}}\,\text{BesselJ}\left(\frac{2}{3},\frac{2}{3}\sqrt{\frac{a}{c}}\sqrt{\frac{b}{c}}x^{3/2}\right)+\text{BesselJ}\left(-\frac{1}{3},\frac{2}{3}\right.\right.}{2bx\,\text{BesselJ}\left(-\frac{1}{3},\frac{2}{3}\sqrt{\frac{a}{c}}\sqrt{\frac{b}{c}}x^{3/2}\right)}$$

2.1.20 problem 20

Internal problem ID [8680]
Book : First order enumerated odes
Section : section 1
Problem number : 20
Date solved : `Tuesday, December 17, 2024 at 12:57:32 PM`
CAS classification : `[[_Riccati, _special]]`

Solve

$$cy' = \frac{ax + by^2}{r}$$

Solved as first order ode of type reduced Riccati

Time used: 0.155 (sec)

This is reduced Riccati ode of the form

$$y' = a\,x^n + by^2$$

Comparing the given ode to the above shows that

$$\begin{aligned} a &= \frac{a}{rc} \\ b &= \frac{b}{cr} \\ n &= 1 \end{aligned}$$

Since $n \neq -2$ then the solution of the reduced Riccati ode is given by

$$w = \sqrt{x}\left\{\begin{array}{ll} c_1 \operatorname{BesselJ}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{ab}x^k\right) + c_2 \operatorname{BesselY}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{ab}x^k\right) & ab > 0 \\ c_1 \operatorname{BesselI}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{-ab}x^k\right) + c_2 \operatorname{BesselK}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{-ab}x^k\right) & ab < 0 \end{array}\right. \tag{1}$$

$$y = -\frac{1}{b}\frac{w'}{w}$$

$$k = 1 + \frac{n}{2}$$

EQ(1) gives

$$k = \frac{3}{2}$$

$$w = \sqrt{x}\left(c_1 \operatorname{BesselJ}\left(\frac{1}{3}, \frac{2\sqrt{\frac{ab}{r^2c^2}}\,x^{3/2}}{3}\right) + c_2 \operatorname{BesselY}\left(\frac{1}{3}, \frac{2\sqrt{\frac{ab}{r^2c^2}}\,x^{3/2}}{3}\right)\right)$$

Therefore the solution becomes

$$y = -\frac{1}{b}\frac{w'}{w}$$

Substituting the value of b, w found above and simplyfing gives

$$y = \frac{\left(-\operatorname{BesselY}\left(-\frac{2}{3}, \frac{2\sqrt{\frac{ab}{r^2c^2}}\,x^{3/2}}{3}\right)c_2 - \operatorname{BesselJ}\left(-\frac{2}{3}, \frac{2\sqrt{\frac{ab}{r^2c^2}}\,x^{3/2}}{3}\right)c_1\right)cr\sqrt{\frac{ab}{r^2c^2}}\,\sqrt{x}}{b\left(c_1 \operatorname{BesselJ}\left(\frac{1}{3}, \frac{2\sqrt{\frac{ab}{r^2c^2}}\,x^{3/2}}{3}\right) + c_2 \operatorname{BesselY}\left(\frac{1}{3}, \frac{2\sqrt{\frac{ab}{r^2c^2}}\,x^{3/2}}{3}\right)\right)}$$

Letting $c_2 = 1$ the above becomes

$$y = \frac{\left(-\operatorname{BesselY}\left(-\frac{2}{3}, \frac{2\sqrt{\frac{ab}{r^2c^2}}\, x^{3/2}}{3}\right) - \operatorname{BesselJ}\left(-\frac{2}{3}, \frac{2\sqrt{\frac{ab}{r^2c^2}}\, x^{3/2}}{3}\right) c_1\right) cr\sqrt{\frac{ab}{r^2c^2}}\, \sqrt{x}}{b\left(c_1 \operatorname{BesselJ}\left(\frac{1}{3}, \frac{2\sqrt{\frac{ab}{r^2c^2}}\, x^{3/2}}{3}\right) + \operatorname{BesselY}\left(\frac{1}{3}, \frac{2\sqrt{\frac{ab}{r^2c^2}}\, x^{3/2}}{3}\right)\right)}$$

Summary of solutions found

$$y = \frac{\left(-\operatorname{BesselY}\left(-\frac{2}{3}, \frac{2\sqrt{\frac{ab}{r^2c^2}}\, x^{3/2}}{3}\right) - \operatorname{BesselJ}\left(-\frac{2}{3}, \frac{2\sqrt{\frac{ab}{r^2c^2}}\, x^{3/2}}{3}\right) c_1\right) cr\sqrt{\frac{ab}{r^2c^2}}\, \sqrt{x}}{b\left(c_1 \operatorname{BesselJ}\left(\frac{1}{3}, \frac{2\sqrt{\frac{ab}{r^2c^2}}\, x^{3/2}}{3}\right) + \operatorname{BesselY}\left(\frac{1}{3}, \frac{2\sqrt{\frac{ab}{r^2c^2}}\, x^{3/2}}{3}\right)\right)}$$

Maple step by step solution

Let's solve

$c\left(\frac{d}{dx}y(x)\right) = \frac{xa+by(x)^2}{r}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = \frac{xa+by(x)^2}{rc}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
<- Riccati Special successful`
```

Maple dsolve solution

Solving time : `0.004` (sec)
Leaf size : 91

```
dsolve(c*diff(y(x),x) = (a*x+b*y(x)^2)/r,
       y(x),singsol=all)
```

$$y = \frac{\left(\frac{ba}{r^2c^2}\right)^{1/3}\left(\operatorname{AiryAi}\left(1, -\left(\frac{ba}{r^2c^2}\right)^{1/3} x\right) c_1 + \operatorname{AiryBi}\left(1, -\left(\frac{ba}{r^2c^2}\right)^{1/3} x\right)\right) rc}{b\left(c_1 \operatorname{AiryAi}\left(-\left(\frac{ba}{r^2c^2}\right)^{1/3} x\right) + \operatorname{AiryBi}\left(-\left(\frac{ba}{r^2c^2}\right)^{1/3} x\right)\right)}$$

Mathematica DSolve solution

Solving time : 0.21 (sec)
Leaf size : 517

```
DSolve[{c*D[y[x],x]==(a*x+b*y[x]^2)/r,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{cr\left(x^{3/2}\sqrt{\frac{a}{cr}}\sqrt{\frac{b}{cr}}\left(-2\,\text{BesselJ}\left(-\frac{2}{3},\frac{2}{3}\sqrt{\frac{a}{cr}}\sqrt{\frac{b}{cr}}x^{3/2}\right)+c_1\left(\text{BesselJ}\left(\frac{2}{3},\frac{2}{3}\sqrt{\frac{a}{cr}}\sqrt{\frac{b}{cr}}x^{3/2}\right)-\text{BesselJ}\left(-\frac{4}{3},\frac{2}{3}\sqrt{\frac{a}{cr}}\right.\right.\right.\right.}{2bx\left(\text{BesselJ}\left(\frac{1}{3},\frac{2}{3}\sqrt{\frac{a}{cr}}\sqrt{\frac{b}{cr}}x^{3/2}\right)+c_1\,\text{BesselJ}\left(-\frac{1}{3},\frac{2}{3}\sqrt{\frac{a}{cr}}\sqrt{\frac{b}{cr}}\right.\right.}$$

$$y(x) \to -\frac{cr\left(x^{3/2}\sqrt{\frac{a}{cr}}\sqrt{\frac{b}{cr}}\,\text{BesselJ}\left(-\frac{4}{3},\frac{2}{3}\sqrt{\frac{a}{cr}}\sqrt{\frac{b}{cr}}x^{3/2}\right)-x^{3/2}\sqrt{\frac{a}{cr}}\sqrt{\frac{b}{cr}}\,\text{BesselJ}\left(\frac{2}{3},\frac{2}{3}\sqrt{\frac{a}{cr}}\sqrt{\frac{b}{cr}}x^{3/2}\right)+\text{BesselJ}\left(-\right.\right.}{2bx\,\text{BesselJ}\left(-\frac{1}{3},\frac{2}{3}\sqrt{\frac{a}{cr}}\sqrt{\frac{b}{cr}}x^{3/2}\right)}$$

2.1.21 problem 21

Internal problem ID [8681]
Book : First order enumerated odes
Section : section 1
Problem number : 21
Date solved : `Tuesday, December 17, 2024 at 12:57:34 PM`
CAS classification : `[_rational, _Riccati]`

Solve

$$c y' = \frac{a x + b y^2}{r x}$$

Solved as first order ode of type Riccati

Time used: 2.946 (sec)

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{b\,y^2 + a x}{r x c} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{a}{r c} + \frac{b\,y^2}{c r x}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x) y + f_2(x) y^2$$

Shows that $f_0(x) = \frac{a}{rc}$, $f_1(x) = 0$ and $f_2(x) = \frac{b}{crx}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{ub}{crx}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{b}{c r\, x^2} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \frac{b^2 a}{c^3 r^3 x^2} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{b u''(x)}{c r x} + \frac{b u'(x)}{c r\, x^2} + \frac{b^2 a u(x)}{c^3 r^3 x^2} = 0$$

In normal form the ode

$$\frac{b\left(\frac{d^2u}{dx^2}\right)}{crx} + \frac{b\left(\frac{du}{dx}\right)}{cr\,x^2} + \frac{b^2au}{c^3r^3x^2} = 0 \tag{1}$$

Becomes

$$\frac{d^2u}{dx^2} + p(x)\left(\frac{du}{dx}\right) + q(x)\,u = r(x) \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{ab}{r^2c^2x} \\ r(x) &= 0 \end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned} \xi'' - (\xi\,p)' + \xi q &= 0 \\ \xi'' - \left(\frac{\xi(x)}{x}\right)' + \left(\frac{ab\xi(x)}{r^2c^2x}\right) &= 0 \\ \frac{d^2}{dx^2}\xi(x) - \frac{\frac{d}{dx}\xi(x)}{x} + \frac{(c^2r^2 + abx)\,\xi(x)}{r^2x^2c^2} &= 0 \end{aligned}$$

Which is solved for $\xi(x)$. Writing the ode as

$$x^2\xi'' - x\xi' + \left(1 + \frac{abx}{r^2c^2}\right)\xi = 0 \tag{1}$$

Bessel ode has the form

$$x^2\xi'' + x\xi' + \left(-n^2 + x^2\right)\xi = 0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2\xi'' + (1 - 2\alpha)\,x\xi' + \left(\beta^2\gamma^2x^{2\gamma} - n^2\gamma^2 + \alpha^2\right)\xi = 0 \tag{3}$$

With the standard solution

$$\xi = x^{\alpha}(c_5 \operatorname{BesselJ}(n, \beta\,x^{\gamma}) + c_6 \operatorname{BesselY}(n, \beta\,x^{\gamma})) \tag{4}$$

Comparing (3) to (1) and solving for α, β, n, γ gives

$$\begin{aligned} \alpha &= 1 \\ \beta &= \frac{2\sqrt{ab}}{rc} \\ n &= 0 \\ \gamma &= \frac{1}{2} \end{aligned}$$

Substituting all the above into (4) gives the solution as

$$\xi = c_5x \operatorname{BesselJ}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right) + c_6x \operatorname{BesselY}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned} \xi(x)\,u' - u\xi'(x) + \xi(x)\,p(x)\,u &= \int \xi(x)\,r(x)\,dx \\ u' + u\left(p(x) - \frac{\xi'(x)}{\xi(x)}\right) &= \frac{\int \xi(x)\,r(x)\,dx}{\xi(x)} \end{aligned}$$

Or

$$u' + u\left(\frac{1}{x} - \frac{c_5 \operatorname{BesselJ}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right) - \frac{c_5\sqrt{x}\operatorname{BesselJ}\left(1, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right)\sqrt{ab}}{rc} + c_6 \operatorname{BesselY}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right) - \frac{c_6\sqrt{x}\operatorname{BesselY}\left(1, \frac{2\sqrt{}}{}\right.}{rc}}{c_5 x \operatorname{BesselJ}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right) + c_6 x \operatorname{BesselY}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right)}\right.$$

Which is now a first order ode. This is now solved for u. In canonical form a linear first order is

$$u' + q(x)u = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= \frac{\sqrt{ab}\left(\operatorname{BesselJ}\left(1, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right) c_5 + \operatorname{BesselY}\left(1, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right) c_6\right)}{\sqrt{x}\, rc\left(c_5 \operatorname{BesselJ}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right) + c_6 \operatorname{BesselY}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right)\right)} \\ p(x) &= 0 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= e^{\int \frac{\sqrt{ab}\left(\operatorname{BesselJ}\left(1, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right) c_5 + \operatorname{BesselY}\left(1, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right) c_6\right)}{\sqrt{x}\, rc\left(c_5 \operatorname{BesselJ}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right) + c_6 \operatorname{BesselY}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right)\right)} dx} \\ &= \frac{1}{c_5 \operatorname{BesselJ}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right) + c_6 \operatorname{BesselY}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right)} \end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\mu u = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{c_5 \operatorname{BesselJ}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right) + c_6 \operatorname{BesselY}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right)}\right) = 0$$

Integrating gives

$$\begin{aligned} \frac{u}{c_5 \operatorname{BesselJ}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right) + c_6 \operatorname{BesselY}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right)} &= \int 0\,dx + c_7 \\ &= c_7 \end{aligned}$$

Dividing throughout by the integrating factor $\frac{1}{c_5 \operatorname{BesselJ}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right) + c_6 \operatorname{BesselY}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right)}$ gives the final solution

$$u = \left(c_5 \operatorname{BesselJ}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right) + c_6 \operatorname{BesselY}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right)\right) c_7$$

Hence, the solution found using Lagrange adjoint equation method is

$$u = \left(c_5 \operatorname{BesselJ}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right) + c_6 \operatorname{BesselY}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right)\right) c_7$$

The constants can be merged to give

$$u = c_5 \operatorname{BesselJ}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right) + c_6 \operatorname{BesselY}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right)$$

Will add steps showing solving for IC soon.

Taking derivative gives

$$u'(x) = -\frac{c_5 \operatorname{BesselJ}\left(1, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right)\sqrt{ab}}{rc\sqrt{x}} - \frac{c_6 \operatorname{BesselY}\left(1, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right)\sqrt{ab}}{rc\sqrt{x}}$$

Doing change of constants, the solution becomes

$$y = -\frac{\left(-\frac{c_8 \operatorname{BesselJ}\left(1, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right)\sqrt{ab}}{rc\sqrt{x}} - \frac{\operatorname{BesselY}\left(1, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right)\sqrt{ab}}{rc\sqrt{x}}\right) crx}{b\left(c_8 \operatorname{BesselJ}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right) + \operatorname{BesselY}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right)\right)}$$

Summary of solutions found

$$y = -\frac{\left(-\frac{c_8 \operatorname{BesselJ}\left(1, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right)\sqrt{ab}}{rc\sqrt{x}} - \frac{\operatorname{BesselY}\left(1, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right)\sqrt{ab}}{rc\sqrt{x}}\right) crx}{b\left(c_8 \operatorname{BesselJ}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right) + \operatorname{BesselY}\left(0, \frac{2\sqrt{ab}\,\sqrt{x}}{rc}\right)\right)}$$

Maple step by step solution

Let's solve

$c\left(\frac{d}{dx}y(x)\right) = \frac{xa+by(x)^2}{rx}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = \frac{xa+by(x)^2}{rxc}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
      -> Trying a Liouvillian solution using Kovacics algorithm
      <- No Liouvillian solutions exists
   <- Abel AIR successful: ODE belongs to the 0F1 1-parameter (Bessel type) class`
```

Maple dsolve solution

Solving time : 0.016 (sec)
Leaf size : 98

```
dsolve(c*diff(y(x),x) = (a*x+b*y(x)^2)/r/x,
       y(x),singsol=all)
```

$$y = \frac{\sqrt{\frac{xba}{r^2c^2}}\, cr\left(\text{BesselY}\left(1, 2\sqrt{\frac{xba}{r^2c^2}}\right) c_1 cr + \text{BesselJ}\left(1, 2\sqrt{\frac{xba}{r^2c^2}}\right)\right)}{b\left(c_1 cr \,\text{BesselY}\left(0, 2\sqrt{\frac{xba}{r^2c^2}}\right) + \text{BesselJ}\left(0, 2\sqrt{\frac{xba}{r^2c^2}}\right)\right)}$$

Mathematica DSolve solution

Solving time : 0.288 (sec)
Leaf size : 207

```
DSolve[{c*D[y[x],x]==(a*x+b*y[x]^2)/(r*x),{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\sqrt{a}\sqrt{x}\left(2\,\text{BesselY}\left(1, \frac{2\sqrt{a}\sqrt{b}\sqrt{x}}{cr}\right) + c_1\,\text{BesselJ}\left(1, \frac{2\sqrt{a}\sqrt{b}\sqrt{x}}{cr}\right)\right)}{\sqrt{b}\left(2\,\text{BesselY}\left(0, \frac{2\sqrt{a}\sqrt{b}\sqrt{x}}{cr}\right) + c_1\,\text{BesselJ}\left(0, \frac{2\sqrt{a}\sqrt{b}\sqrt{x}}{cr}\right)\right)}$$

$$y(x) \to \frac{\sqrt{a}\sqrt{x}\,\text{BesselJ}\left(1, \frac{2\sqrt{a}\sqrt{b}\sqrt{x}}{cr}\right)}{\sqrt{b}\,\text{BesselJ}\left(0, \frac{2\sqrt{a}\sqrt{b}\sqrt{x}}{cr}\right)}$$

2.1.22 problem 22

Internal problem ID [8682]
Book : First order enumerated odes
Section : section 1
Problem number : 22
Date solved : `Tuesday, December 17, 2024 at 12:57:38 PM`
CAS classification : `[_rational, _Riccati]`

Solve

$$cy' = \frac{ax + by^2}{r\,x^2}$$

Solved as first order ode of type Riccati

Time used: 4.780 (sec)

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{b\,y^2 + ax}{r\,x^2 c} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{a}{xcr} + \frac{b\,y^2}{r\,x^2 c}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{a}{xcr}$, $f_1(x) = 0$ and $f_2(x) = \frac{b}{cr\,x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{ub}{cr\,x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2b}{cr\,x^3} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \frac{b^2 a}{c^3 r^3 x^5} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{bu''(x)}{cr\,x^2} + \frac{2bu'(x)}{cr\,x^3} + \frac{b^2 au(x)}{c^3 r^3 x^5} = 0$$

In normal form the ode

$$\frac{b\left(\frac{d^2u}{dx^2}\right)}{cr\,x^2}+\frac{2b\left(\frac{du}{dx}\right)}{cr\,x^3}+\frac{b^2au}{c^3r^3x^5}=0 \tag{1}$$

Becomes

$$\frac{d^2u}{dx^2}+p(x)\left(\frac{du}{dx}\right)+q(x)\,u=r(x) \tag{2}$$

Where

$$\begin{aligned}p(x)&=\frac{2}{x}\\ q(x)&=\frac{ba}{x^3c^2r^2}\\ r(x)&=0\end{aligned}$$

The Lagrange adjoint ode is given by

$$\begin{aligned}\xi''-(\xi\,p)'+\xi q&=0\\ \xi''-\left(\frac{2\xi(x)}{x}\right)'+\left(\frac{ba\xi(x)}{x^3c^2r^2}\right)&=0\\ \frac{d^2}{dx^2}\xi(x)-\frac{2\left(\frac{d}{dx}\xi(x)\right)}{x}+\frac{(2c^2r^2x+ba)\,\xi(x)}{x^3c^2r^2}&=0\end{aligned}$$

Which is solved for $\xi(x)$. Writing the ode as

$$x^2\xi''-2x\xi'+\left(2+\frac{ba}{x\,c^2r^2}\right)\xi=0 \tag{1}$$

Bessel ode has the form

$$x^2\xi''+x\xi'+\left(-n^2+x^2\right)\xi=0 \tag{2}$$

The generalized form of Bessel ode is given by Bowman (1958) as the following

$$x^2\xi''+(1-2\alpha)\,x\xi'+\left(\beta^2\gamma^2x^{2\gamma}-n^2\gamma^2+\alpha^2\right)\xi=0 \tag{3}$$

With the standard solution

$$\xi=x^\alpha(c_5\,\mathrm{BesselJ}\,(n,\beta\,x^\gamma)+c_6\,\mathrm{BesselY}\,(n,\beta\,x^\gamma)) \tag{4}$$

Comparing (3) to (1) and solving for α,β,n,γ gives

$$\begin{aligned}\alpha&=\frac{3}{2}\\ \beta&=\frac{2\sqrt{ba}}{rc}\\ n&=-1\\ \gamma&=-\frac{1}{2}\end{aligned}$$

Substituting all the above into (4) gives the solution as

$$\xi=-c_5\,x^{3/2}\,\mathrm{BesselJ}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)-c_6\,x^{3/2}\,\mathrm{BesselY}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)$$

Will add steps showing solving for IC soon.

The original ode now reduces to first order ode

$$\begin{aligned}\xi(x)\,u'-u\xi'(x)+\xi(x)\,p(x)\,u&=\int\xi(x)\,r(x)\,dx\\ u'+u\left(p(x)-\frac{\xi'(x)}{\xi\,(x)}\right)&=\frac{\int\xi(x)\,r(x)\,dx}{\xi\,(x)}\end{aligned}$$

Or

$$u' + u\left(\frac{2}{x} - \frac{-\frac{3c_5\sqrt{x}\,\mathrm{BesselJ}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)}{2} + \frac{c_5\left(\mathrm{BesselJ}\left(0,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right) - \frac{rc\sqrt{x}\,\mathrm{BesselJ}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)}{2\sqrt{ba}}\right)\sqrt{ba}}{rc} - \frac{3c_6\sqrt{x}\,\mathrm{BesselY}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)}{2} + \frac{c_6\left(\mathrm{BesselY}\right.}{}}{-c_5\,x^{3/2}\,\mathrm{BesselJ}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right) - c_6\,x^{3/2}\,\mathrm{BesselY}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)}\right.$$

Which is now a first order ode. This is now solved for u. In canonical form a linear first order is

$$u' + q(x)u = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = \frac{\sqrt{ba}\left(\mathrm{BesselJ}\left(0,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)c_5 + \mathrm{BesselY}\left(0,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)c_6\right)}{x^{3/2}rc\left(\mathrm{BesselJ}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)c_5 + \mathrm{BesselY}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)c_6\right)}$$
$$p(x) = 0$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= \mathrm{e}^{\int \frac{\sqrt{ba}\left(\mathrm{BesselJ}\left(0,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)c_5+\mathrm{BesselY}\left(0,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)c_6\right)}{x^{3/2}rc\left(\mathrm{BesselJ}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)c_5+\mathrm{BesselY}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)c_6\right)}dx} \\
&= \frac{rc\sqrt{x}}{\sqrt{ba}\left(2\,\mathrm{BesselJ}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)c_5 + 2\,\mathrm{BesselY}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)c_6\right)}
\end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\mu u = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{urc\sqrt{x}}{\sqrt{ba}\left(2\,\mathrm{BesselJ}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)c_5 + 2\,\mathrm{BesselY}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)c_6\right)}\right) = 0$$

Integrating gives

$$\begin{aligned}
\frac{urc\sqrt{x}}{\sqrt{ba}\left(2\,\mathrm{BesselJ}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)c_5 + 2\,\mathrm{BesselY}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)c_6\right)} &= \int 0\,dx + c_7 \\
&= c_7
\end{aligned}$$

Dividing throughout by the integrating factor $\frac{rc\sqrt{x}}{\sqrt{ba}\left(2\,\mathrm{BesselJ}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)c_5+2\,\mathrm{BesselY}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)c_6\right)}$ gives the final solution

$$u = \frac{2\sqrt{ba}\left(\mathrm{BesselJ}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)c_5 + \mathrm{BesselY}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)c_6\right)c_7}{rc\sqrt{x}}$$

Hence, the solution found using Lagrange adjoint equation method is

$$u = \frac{2\sqrt{ba}\left(\mathrm{BesselJ}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)c_5 + \mathrm{BesselY}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)c_6\right)c_7}{rc\sqrt{x}}$$

The constants can be merged to give

$$u = \frac{2\sqrt{ba}\left(\mathrm{BesselJ}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)c_5 + \mathrm{BesselY}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)c_6\right)}{rc\sqrt{x}}$$

Will add steps showing solving for IC soon.

Taking derivative gives

$$u'(x) = \frac{2\sqrt{ba}\left(-\frac{\left(\mathrm{BesselJ}\left(0,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)-\frac{rc\sqrt{x}\,\mathrm{BesselJ}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)}{2\sqrt{ba}}\right)\sqrt{ba}\,c_5}{rc\,x^{3/2}}-\frac{\left(\mathrm{BesselY}\left(0,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)-\frac{rc\sqrt{x}\,\mathrm{BesselY}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)}{2\sqrt{ba}}\right)\sqrt{ba}\,c_6}{rc\,x^{3/2}}\right)}{rc\sqrt{x}} - \frac{\sqrt{ba}\left(\mathrm{BesselJ}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)c_5+\mathrm{BesselY}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)c_6\right)}{rc\,x^{3/2}}$$

Doing change of constants, the solution becomes

$$y = -\frac{\left(\frac{2\sqrt{ba}\left(-\frac{\left(\mathrm{BesselJ}\left(0,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)-\frac{rc\sqrt{x}\,\mathrm{BesselJ}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)}{2\sqrt{ba}}\right)\sqrt{ba}\,c_8}{rc\,x^{3/2}}-\frac{\left(\mathrm{BesselY}\left(0,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)-\frac{rc\sqrt{x}\,\mathrm{BesselY}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)}{2\sqrt{ba}}\right)\sqrt{ba}}{rc\,x^{3/2}}\right)}{rc\sqrt{x}} - \frac{\sqrt{ba}\left(\mathrm{BesselJ}\left(1,\cdots\right.\right.}{\cdots}\cdots\right)}{2b\sqrt{ba}\left(\mathrm{BesselJ}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)c_8+\mathrm{BesselY}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)\right)\cdots}$$

Summary of solutions found

$$y = -\frac{\left(\frac{2\sqrt{ba}\left(-\frac{\left(\mathrm{BesselJ}\left(0,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)-\frac{rc\sqrt{x}\,\mathrm{BesselJ}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)}{2\sqrt{ba}}\right)\sqrt{ba}\,c_8}{rc\,x^{3/2}}-\frac{\left(\mathrm{BesselY}\left(0,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)-\frac{rc\sqrt{x}\,\mathrm{BesselY}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)}{2\sqrt{ba}}\right)\sqrt{ba}}{rc\,x^{3/2}}\right)}{rc\sqrt{x}} - \frac{\sqrt{ba}\left(\mathrm{BesselJ}\left(1,\cdots\right.\right.}{\cdots}\cdots\right)}{2b\sqrt{ba}\left(\mathrm{BesselJ}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)c_8+\mathrm{BesselY}\left(1,\frac{2\sqrt{ba}}{rc\sqrt{x}}\right)\right)\cdots}$$

Maple step by step solution

Let's solve

$c\left(\frac{d}{dx}y(x)\right) = \frac{xa+by(x)^2}{r\,x^2}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = \frac{xa+by(x)^2}{r\,x^2c}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
      -> Trying a Liouvillian solution using Kovacics algorithm
      <- No Liouvillian solutions exists
   <- Abel AIR successful: ODE belongs to the 0F1 1-parameter (Bessel type) class`
```

Maple dsolve solution

Solving time : `0.020` (sec)
Leaf size : 110

```
dsolve(c*diff(y(x),x) = (a*x+b*y(x)^2)/r/x^2,
       y(x),singsol=all)
```

$$y = \frac{a\left(\operatorname{BesselY}\left(0, 2\sqrt{\frac{ba}{c^2r^2x}}\right) c_1 cr + \operatorname{BesselJ}\left(0, 2\sqrt{\frac{ba}{c^2r^2x}}\right)\right)}{cr\sqrt{\frac{ba}{c^2r^2x}}\left(c_1 cr \operatorname{BesselY}\left(1, 2\sqrt{\frac{ba}{c^2r^2x}}\right) + \operatorname{BesselJ}\left(1, 2\sqrt{\frac{ba}{c^2r^2x}}\right)\right)}$$

Mathematica DSolve solution

Solving time : `0.341` (sec)
Leaf size : 492

```
DSolve[{c*D[y[x],x]==(a*x+b*y[x]^2)/(r*x^2),{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{2\sqrt{a}\sqrt{b}\operatorname{BesselY}\left(0, \frac{2\sqrt{a}\sqrt{b}\sqrt{\frac{1}{x}}}{cr}\right) + \frac{2cr\operatorname{BesselY}\left(1, \frac{2\sqrt{a}\sqrt{b}\sqrt{\frac{1}{x}}}{cr}\right)}{\sqrt{\frac{1}{x}}} - 2\sqrt{a}\sqrt{b}\operatorname{BesselY}\left(2, \frac{2\sqrt{a}\sqrt{b}\sqrt{\frac{1}{x}}}{cr}\right) - i\sqrt{a}\sqrt{b}c_1 \operatorname{Besse}}{2b\sqrt{\frac{1}{x}}\left(2\operatorname{BesselY}\left(1, \frac{2\sqrt{a}\sqrt{b}\sqrt{\frac{1}{x}}}{cr}\right) - ic_1 \operatorname{BesselJ}\right.}$$

$$y(x) \to \frac{x\left(\sqrt{a}\sqrt{b}\sqrt{\frac{1}{x}}\operatorname{BesselJ}\left(0, \frac{2\sqrt{a}\sqrt{b}\sqrt{\frac{1}{x}}}{cr}\right) + cr\operatorname{BesselJ}\left(1, \frac{2\sqrt{a}\sqrt{b}\sqrt{\frac{1}{x}}}{cr}\right) - \sqrt{a}\sqrt{b}\sqrt{\frac{1}{x}}\operatorname{BesselJ}\left(2, \frac{2\sqrt{a}\sqrt{b}\sqrt{\frac{1}{x}}}{cr}\right)\right)}{2b\operatorname{BesselJ}\left(1, \frac{2\sqrt{a}\sqrt{b}\sqrt{\frac{1}{x}}}{cr}\right)}$$

2.1.23 problem 23

Internal problem ID [8683]
Book : First order enumerated odes
Section : section 1
Problem number : 23
Date solved : `Tuesday, December 17, 2024 at 12:57:44 PM`
CAS classification : `[_rational, _Bernoulli]`

Solve

$$cy' = \frac{ax + by^2}{y}$$

Solved as first order Bernoulli ode

Time used: 0.301 (sec)

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{b\,y^2 + ax}{yc} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \left(\frac{b}{c}\right) y + \left(\frac{ax}{c}\right) \frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

Comparing this to (1) shows that

$$\begin{aligned} f_0 &= \frac{b}{c} \\ f_1 &= \frac{ax}{c} \end{aligned}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $v = y^{1-n}$ in equation (3) which generates a new ODE in $v(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{b}{c} \\ f_1(x) &= \frac{ax}{c} \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = \frac{b\,y^2}{c} + \frac{ax}{c} \tag{4}$$

Let

$$\begin{aligned} v &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$v' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{v'(x)}{2} &= \frac{bv(x)}{c} + \frac{ax}{c} \\ v' &= \frac{2bv}{c} + \frac{2ax}{c} \end{aligned} \tag{7}$$

The above now is a linear ODE in $v(x)$ which is now solved.

In canonical form a linear first order is

$$v'(x) + q(x)v(x) = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= -\frac{2b}{c} \\ p(x) &= \frac{2ax}{c} \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= \mathrm{e}^{\int -\frac{2b}{c}dx} \\ &= \mathrm{e}^{-\frac{2bx}{c}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu v) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu v) &= (\mu)\left(\frac{2ax}{c}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(v\,\mathrm{e}^{-\frac{2bx}{c}}\right) &= \left(\mathrm{e}^{-\frac{2bx}{c}}\right)\left(\frac{2ax}{c}\right) \\ \mathrm{d}\left(v\,\mathrm{e}^{-\frac{2bx}{c}}\right) &= \left(\frac{2ax\,\mathrm{e}^{-\frac{2bx}{c}}}{c}\right)\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} v\,\mathrm{e}^{-\frac{2bx}{c}} &= \int \frac{2ax\,\mathrm{e}^{-\frac{2bx}{c}}}{c}\,dx \\ &= -\frac{(2bx+c)\,a\,\mathrm{e}^{-\frac{2bx}{c}}}{2b^2} + c_1 \end{aligned}$$

Dividing throughout by the integrating factor $\mathrm{e}^{-\frac{2bx}{c}}$ gives the final solution

$$v(x) = \frac{c_1\,\mathrm{e}^{\frac{2bx}{c}}b^2 - \left(bx + \frac{c}{2}\right)a}{b^2}$$

The substitution $v = y^{1-n}$ is now used to convert the above solution back to y which results in

$$y^2 = \frac{c_1 \, \mathrm{e}^{\frac{2bx}{c}} b^2 - \left(bx + \frac{c}{2}\right) a}{b^2}$$

Solving for y gives

$$y = -\frac{\sqrt{4c_1 \, \mathrm{e}^{\frac{2bx}{c}} b^2 - 4axb - 2ac}}{2b}$$
$$y = \frac{\sqrt{4c_1 \, \mathrm{e}^{\frac{2bx}{c}} b^2 - 4axb - 2ac}}{2b}$$

Summary of solutions found

$$y = -\frac{\sqrt{4c_1 \, \mathrm{e}^{\frac{2bx}{c}} b^2 - 4axb - 2ac}}{2b}$$
$$y = \frac{\sqrt{4c_1 \, \mathrm{e}^{\frac{2bx}{c}} b^2 - 4axb - 2ac}}{2b}$$

Solved as first order Exact ode

Time used: 0.322 (sec)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(yc) \, \mathrm{d}y = \left(b \, y^2 + ax\right) \mathrm{d}x$$
$$\left(-b \, y^2 - ax\right) \mathrm{d}x + (yc) \, \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -b\,y^2 - ax \\
N(x,y) &= yc
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-b\,y^2 - ax\right) \\
&= -2by
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(yc) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{yc}((-2by) - (0)) \\
&= -\frac{2b}{c}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A\,\mathrm{d}x} \\
&= e^{\int -\frac{2b}{c}\,\mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{2bx}{c}} \\
&= \mathrm{e}^{-\frac{2bx}{c}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-\frac{2bx}{c}}\left(-b\,y^2 - ax\right) \\
&= -\left(b\,y^2 + ax\right)\mathrm{e}^{-\frac{2bx}{c}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-\frac{2bx}{c}}(yc) \\
&= yc\,\mathrm{e}^{-\frac{2bx}{c}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\left(b\,y^2 + ax\right)\mathrm{e}^{-\frac{2bx}{c}}\right) + \left(yc\,\mathrm{e}^{-\frac{2bx}{c}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int \overline{M}\, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int -\left(b\,y^2 + ax\right) \mathrm{e}^{-\frac{2bx}{c}}\, \mathrm{d}x$$

$$\phi = \frac{c(2b^2y^2 + 2axb + ac)\,\mathrm{e}^{-\frac{2bx}{c}}}{4b^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = yc\,\mathrm{e}^{-\frac{2bx}{c}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = yc\,\mathrm{e}^{-\frac{2bx}{c}}$. Therefore equation (4) becomes

$$yc\,\mathrm{e}^{-\frac{2bx}{c}} = yc\,\mathrm{e}^{-\frac{2bx}{c}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{c(2b^2y^2 + 2axb + ac)\,\mathrm{e}^{-\frac{2bx}{c}}}{4b^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \frac{c(2b^2y^2 + 2axb + ac)\,\mathrm{e}^{-\frac{2bx}{c}}}{4b^2}$$

Solving for y gives

$$y = -\frac{\mathrm{e}^{\frac{2bx}{c}}\sqrt{-2\,\mathrm{e}^{-\frac{2bx}{c}} c\left(2\,\mathrm{e}^{-\frac{2bx}{c}} abcx + \mathrm{e}^{-\frac{2bx}{c}} a\,c^2 - 4c_1\,b^2\right)}}{2cb}$$

$$y = \frac{\mathrm{e}^{\frac{2bx}{c}}\sqrt{-2\,\mathrm{e}^{-\frac{2bx}{c}} c\left(2\,\mathrm{e}^{-\frac{2bx}{c}} abcx + \mathrm{e}^{-\frac{2bx}{c}} a\,c^2 - 4c_1\,b^2\right)}}{2cb}$$

Summary of solutions found

$$y = -\frac{\mathrm{e}^{\frac{2bx}{c}}\sqrt{-2\,\mathrm{e}^{-\frac{2bx}{c}} c\left(2\,\mathrm{e}^{-\frac{2bx}{c}} abcx + \mathrm{e}^{-\frac{2bx}{c}} a\,c^2 - 4c_1\,b^2\right)}}{2cb}$$

$$y = \frac{\mathrm{e}^{\frac{2bx}{c}}\sqrt{-2\,\mathrm{e}^{-\frac{2bx}{c}} c\left(2\,\mathrm{e}^{-\frac{2bx}{c}} abcx + \mathrm{e}^{-\frac{2bx}{c}} a\,c^2 - 4c_1\,b^2\right)}}{2cb}$$

Maple step by step solution

Let's solve

$c\left(\frac{d}{dx}y(x)\right) = \frac{xa+by(x)^2}{y(x)}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = \frac{xa+by(x)^2}{y(x)c}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

Maple dsolve solution

Solving time : `0.017` (sec)
Leaf size : 69

```
dsolve(c*diff(y(x),x) = (a*x+b*y(x)^2)/y(x),
       y(x),singsol=all)
```

$$y = -\frac{\sqrt{4\,\mathrm{e}^{\frac{2bx}{c}} c_1 b^2 - 4axb - 2ac}}{2b}$$
$$y = \frac{\sqrt{4\,\mathrm{e}^{\frac{2bx}{c}} c_1 b^2 - 4axb - 2ac}}{2b}$$

Mathematica DSolve solution

Solving time : `5.76` (sec)
Leaf size : 85

```
DSolve[{c*D[y[x],x]==(a*x+b*y[x]^2)/y[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{i\sqrt{abx + \frac{ac}{2} + b^2 c_1 \left(-e^{\frac{2bx}{c}}\right)}}{b}$$
$$y(x) \to \frac{i\sqrt{abx + \frac{ac}{2} + b^2 c_1 \left(-e^{\frac{2bx}{c}}\right)}}{b}$$

2.1.24 problem 24

Internal problem ID [8684]
Book : First order enumerated odes
Section : section 1
Problem number : 24
Date solved : `Tuesday, December 17, 2024 at 12:57:46 PM`
CAS classification : `[_quadrature]`

Solve

$$a\sin(x)\, yxy' = 0$$

Factoring the ode gives these factors

$$y = 0 \tag{1}$$

$$y' = 0 \tag{2}$$

Now each of the above equations is solved in turn.

Solving equation (1)

Solving for y from

$$y = 0$$

Solving gives $y = 0$

Solving equation (2)

Solved as first order quadrature ode

Time used: 0.013 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\, dx + c_1$$

$$y = c_1$$

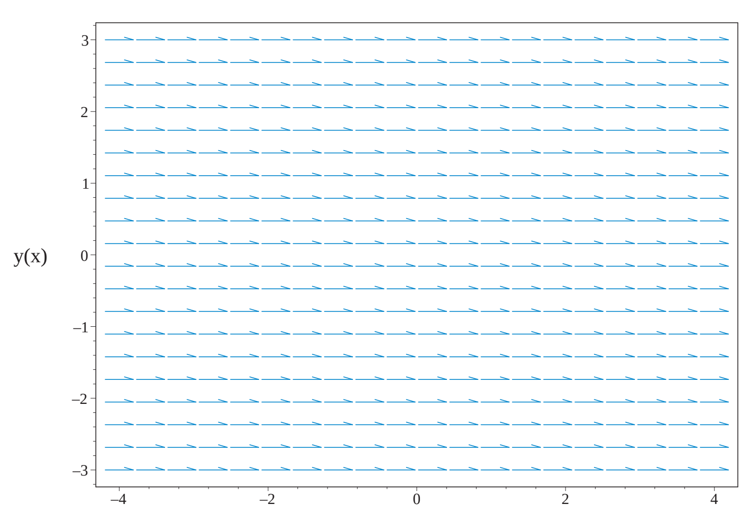


Figure 2.20: Slope field plot
$y' = 0$

Summary of solutions found

$$y = c_1$$

Solved as first order homogeneous class D2 ode

Time used: 0.152 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$u'(x)\,x + u(x) = 0$$

Which is now solved The ode $u'(x) = -\frac{u(x)}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(u) &= u \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\ \ln(u(x)) &= \ln\left(\frac{1}{x}\right) + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} \ln(u(x)) &= \ln\left(\frac{1}{x}\right) + c_1 \\ u(x) &= 0 \end{aligned}$$

Solving for $u(x)$ gives

$$\begin{aligned} u(x) &= 0 \\ u(x) &= \frac{\mathrm{e}^{c_1}}{x} \end{aligned}$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = \frac{\mathrm{e}^{c_1}}{x}$ back to y gives

$$y = \mathrm{e}^{c_1}$$

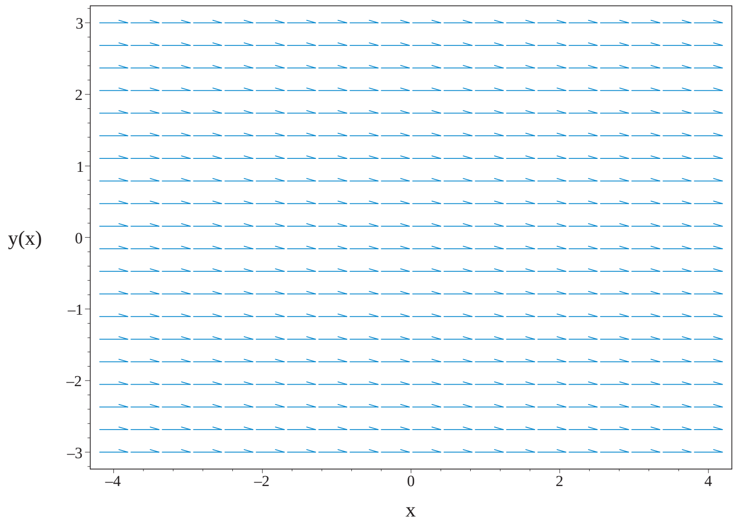


Figure 2.21: Slope field plot
$y' = 0$

Summary of solutions found

$$y = 0$$
$$y = e^{c_1}$$

Solved as first order ode of type differential

Time used: 0.012 (sec)

Writing the ode as

$$y' = 0 \tag{1}$$

Which becomes

$$(1)\, dy = (0)\, dx \tag{2}$$

But the RHS is complete differential because

$$(0)\, dx = d(0)$$

Hence (2) becomes

$$(1)\, dy = d(0)$$

Integrating gives

$$y = c_1$$

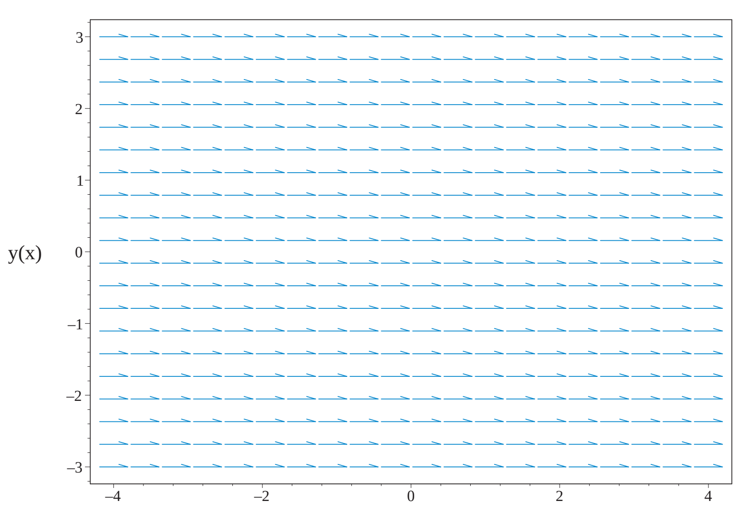


Figure 2.22: Slope field plot
$y' = 0$

Summary of solutions found

$$y = c_1$$

Maple step by step solution

Let's solve

$a\sin(x)\, y(x)\, x\left(\frac{d}{dx}y(x)\right) = 0$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$
- Solve for the highest derivative

  $\frac{d}{dx}y(x) = 0$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}y(x)\right) dx = \int 0 dx + C1$
- Evaluate integral

  $y(x) = C1$
- Solve for $y(x)$

  $y(x) = C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : `0.002` (sec)
Leaf size : 9

```
dsolve(a*sin(x)*y(x)*x*diff(y(x),x) = 0,
       y(x),singsol=all)
```

$$y = 0$$
$$y = c_1$$

Mathematica DSolve solution

Solving time : `0.003` (sec)
Leaf size : 12

```
DSolve[{a*Sin[x]*y[x]*x*D[y[x],x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to 0$$
$$y(x) \to c_1$$

2.1.25 problem 25

Internal problem ID [8685]
Book : First order enumerated odes
Section : section 1
Problem number : 25
Date solved : `Tuesday, December 17, 2024 at 12:57:47 PM`
CAS classification : `[_quadrature]`

Solve

$$f(x)\sin(x)\,yxy'\pi = 0$$

Factoring the ode gives these factors

$$y = 0 \tag{1}$$

$$y' = 0 \tag{2}$$

Now each of the above equations is solved in turn.

Solving equation (1)

Solving for y from

$$y = 0$$

Solving gives $y = 0$

Solving equation (2)

Solved as first order quadrature ode

Time used: 0.013 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\,dx + c_1$$

$$y = c_1$$

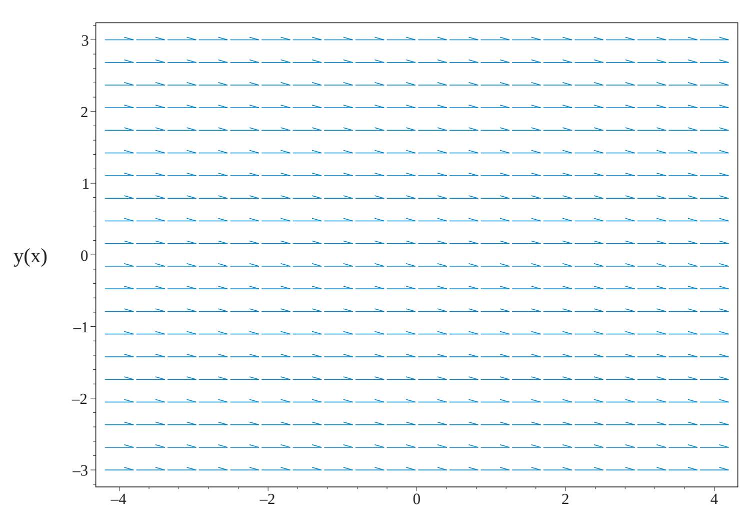


Figure 2.23: Slope field plot
$y' = 0$

Summary of solutions found

$$y = c_1$$

Solved as first order homogeneous class D2 ode

Time used: 0.158 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$u'(x)\,x + u(x) = 0$$

Which is now solved The ode $u'(x) = -\frac{u(x)}{x}$ is separable as it can be written as

$$\begin{aligned}
u'(x) &= -\frac{u(x)}{x} \\
&= f(x)g(u)
\end{aligned}$$

Where

$$\begin{aligned}
f(x) &= -\frac{1}{x} \\
g(u) &= u
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\
\ln\left(u(x)\right) &= \ln\left(\frac{1}{x}\right) + c_1
\end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned}
\ln\left(u(x)\right) &= \ln\left(\frac{1}{x}\right) + c_1 \\
u(x) &= 0
\end{aligned}$$

Solving for $u(x)$ gives

$$\begin{aligned}
u(x) &= 0 \\
u(x) &= \frac{e^{c_1}}{x}
\end{aligned}$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = \frac{e^{c_1}}{x}$ back to y gives

$$y = e^{c_1}$$

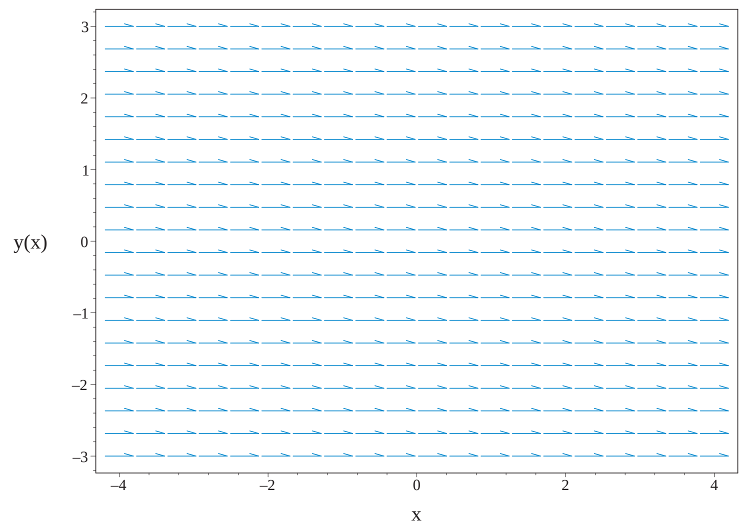


Figure 2.24: Slope field plot
$y' = 0$

Summary of solutions found

$$y = 0$$

$$y = e^{c_1}$$

Solved as first order ode of type differential

Time used: 0.012 (sec)

Writing the ode as

$$y' = 0 \tag{1}$$

Which becomes

$$(1)\, dy = (0)\, dx \tag{2}$$

But the RHS is complete differential because

$$(0)\, dx = d(0)$$

Hence (2) becomes

$$(1)\, dy = d(0)$$

Integrating gives

$$y = c_1$$

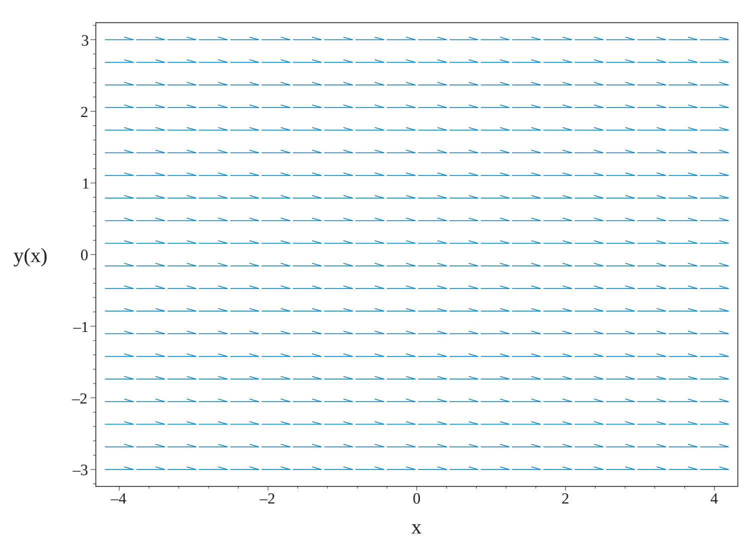


Figure 2.25: Slope field plot
$y' = 0$

Summary of solutions found

$$y = c_1$$

Maple step by step solution

Let's solve

$f(x)\sin(x)\, y(x)\, x\left(\frac{d}{dx}y(x)\right)\pi = 0$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = 0$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}y(x)\right)dx = \int 0dx + C1$

- Evaluate integral

  $y(x) = C1$

- Solve for $y(x)$

  $y(x) = C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 9

```
dsolve(f(x)*sin(x)*y(x)*x*diff(y(x),x)*Pi = 0,
       y(x),singsol=all)
```

$$y = 0$$
$$y = c_1$$

Mathematica DSolve solution

Solving time : 0.003 (sec)
Leaf size : 12

```
DSolve[{f(x)*Sin[x]*y[x]*x*D[y[x],x]*Pi==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to 0$$
$$y(x) \to c_1$$

2.1.26 problem 26

Internal problem ID [8686]
Book : First order enumerated odes
Section : section 1
Problem number : 26
Date solved : `Tuesday, December 17, 2024 at 12:57:48 PM`
CAS classification : `[[_linear, ‘class A‘]]`

Solve

$$y' = \sin(x) + y$$

Solved as first order linear ode

Time used: 0.148 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= -1 \\ p(x) &= \sin(x) \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= \mathrm{e}^{\int (-1)dx} \\ &= \mathrm{e}^{-x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)(\sin(x)) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(y\,\mathrm{e}^{-x}\right) &= \left(\mathrm{e}^{-x}\right)(\sin(x)) \\ \mathrm{d}\left(y\,\mathrm{e}^{-x}\right) &= \left(\sin(x)\,\mathrm{e}^{-x}\right)\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} y\,\mathrm{e}^{-x} &= \int \sin(x)\,\mathrm{e}^{-x}\,dx \\ &= -\frac{\cos(x)\,\mathrm{e}^{-x}}{2} - \frac{\sin(x)\,\mathrm{e}^{-x}}{2} + c_1 \end{aligned}$$

Dividing throughout by the integrating factor e^{-x} gives the final solution

$$y = c_1\,\mathrm{e}^{x} - \frac{\cos(x)}{2} - \frac{\sin(x)}{2}$$

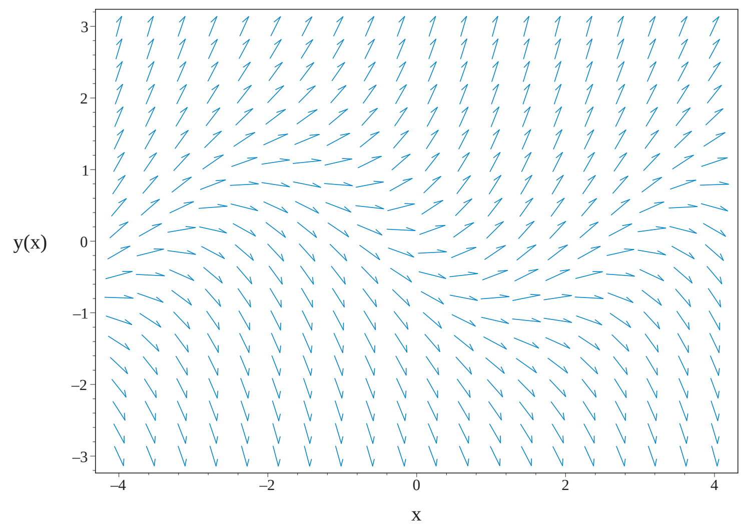


Figure 2.26: Slope field plot
$y' = \sin(x) + y$

Summary of solutions found

$$y = c_1 e^x - \frac{\cos(x)}{2} - \frac{\sin(x)}{2}$$

Solved as first order Exact ode

Time used: 0.120 (sec)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = (\sin(x) + y)\,\mathrm{d}x$$
$$(-\sin(x) - y)\,\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -\sin(x) - y \\ N(x,y) &= 1 \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\sin(x) - y) \\ &= -1 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((-1) - (0)) \\ &= -1 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -1 \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-x} \\ &= \mathrm{e}^{-x} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{-x}(-\sin(x) - y) \\ &= -(\sin(x) + y)\,\mathrm{e}^{-x} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{-x}(1) \\ &= \mathrm{e}^{-x} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-(\sin(x) + y)\,\mathrm{e}^{-x}\right) + \left(\mathrm{e}^{-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\int \frac{\partial \phi}{\partial y} \, dy = \int \overline{N} \, dy$$

$$\int \frac{\partial \phi}{\partial y} \, dy = \int e^{-x} \, dy$$

$$\phi = y\, e^{-x} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = -y\, e^{-x} + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = -(\sin(x) + y)\, e^{-x}$. Therefore equation (4) becomes

$$-(\sin(x) + y)\, e^{-x} = -y\, e^{-x} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -\sin(x)\, e^{-x}$$

Integrating the above w.r.t x gives

$$\int f'(x) \, dx = \int \left(-\sin(x)\, e^{-x}\right) dx$$

$$f(x) = \frac{\cos(x)\, e^{-x}}{2} + \frac{\sin(x)\, e^{-x}}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = y\, e^{-x} + \frac{\cos(x)\, e^{-x}}{2} + \frac{\sin(x)\, e^{-x}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = y\, e^{-x} + \frac{\cos(x)\, e^{-x}}{2} + \frac{\sin(x)\, e^{-x}}{2}$$

Solving for y gives

$$y = -\frac{(\sin(x)\, e^{-x} + \cos(x)\, e^{-x} - 2c_1)\, e^{x}}{2}$$

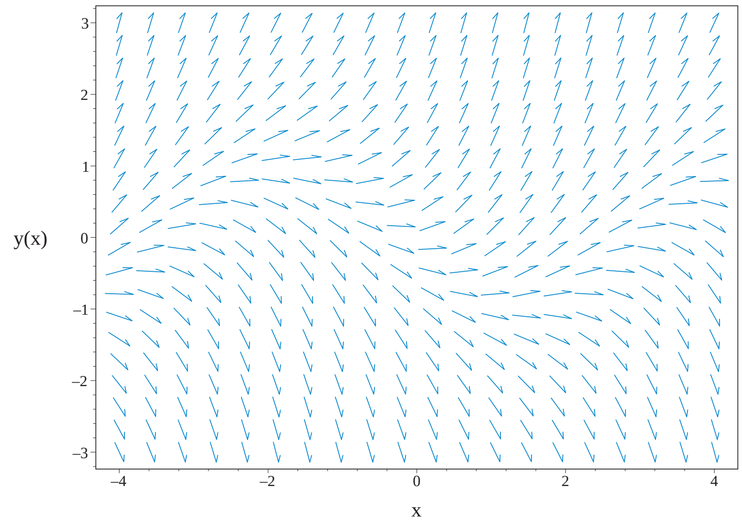


Figure 2.27: Slope field plot
$y' = \sin(x) + y$

Summary of solutions found

$$y = -\frac{(\sin(x)\,e^{-x} + \cos(x)\,e^{-x} - 2c_1)\,e^{x}}{2}$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) = y(x) + \sin(x)$

- Highest derivative means the order of the ODE is 1
$\frac{d}{dx}y(x)$
- Solve for the highest derivative
$\frac{d}{dx}y(x) = y(x) + \sin(x)$
- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE
$\frac{d}{dx}y(x) - y(x) = \sin(x)$
- The ODE is linear; multiply by an integrating factor $\mu(x)$
$\mu(x)\left(\frac{d}{dx}y(x) - y(x)\right) = \mu(x)\sin(x)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\,\mu(x))$
$\mu(x)\left(\frac{d}{dx}y(x) - y(x)\right) = \left(\frac{d}{dx}y(x)\right)\mu(x) + y(x)\left(\frac{d}{dx}\mu(x)\right)$
- Isolate $\frac{d}{dx}\mu(x)$
$\frac{d}{dx}\mu(x) = -\mu(x)$
- Solve to find the integrating factor
$\mu(x) = e^{-x}$
- Integrate both sides with respect to x
$\int\left(\frac{d}{dx}(y(x)\,\mu(x))\right)dx = \int \mu(x)\sin(x)\,dx + C1$
- Evaluate the integral on the lhs
$y(x)\,\mu(x) = \int \mu(x)\sin(x)\,dx + C1$
- Solve for $y(x)$
$y(x) = \frac{\int \mu(x)\sin(x)dx + C1}{\mu(x)}$
- Substitute $\mu(x) = e^{-x}$
$y(x) = \frac{\int e^{-x}\sin(x)dx + C1}{e^{-x}}$
- Evaluate the integrals on the rhs
$y(x) = \frac{-\frac{e^{-x}\cos(x)}{2} - \frac{e^{-x}\sin(x)}{2} + C1}{e^{-x}}$

- Simplify
  $y(x) = C1\ \mathrm{e}^{x} - \frac{\cos(x)}{2} - \frac{\sin(x)}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 17

```
dsolve(diff(y(x),x) = sin(x)+y(x),
       y(x),singsol=all)
```

$$y = -\frac{\cos(x)}{2} - \frac{\sin(x)}{2} + \mathrm{e}^{x} c_1$$

Mathematica DSolve solution

Solving time : 0.037 (sec)
Leaf size : 24

```
DSolve[{D[y[x],x]==Sin[x]+y[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{\sin(x)}{2} - \frac{\cos(x)}{2} + c_1 e^{x}$$

2.1.27 problem 27

Internal problem ID [8687]
Book : First order enumerated odes
Section : section 1
Problem number : 27
Date solved : `Tuesday, December 17, 2024 at 12:57:49 PM`
CAS classification : `[_Riccati]`

Solve

$$y' = \sin(x) + y^2$$

Solved as first order ode of type Riccati

Time used: 0.641 (sec)

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \sin(x) + y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \sin(x) + y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \sin(x)$, $f_1(x) = 0$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= \sin(x) \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + \sin(x) u(x) = 0$$

Unable to solve. Will ask Maple to solve this ode now.

Solution obtained is

$$u(x) = c_1 \operatorname{MathieuC}\left(0, -2, -\frac{\pi}{4} + \frac{x}{2}\right) + c_2 \operatorname{MathieuS}\left(0, -2, -\frac{\pi}{4} + \frac{x}{2}\right)$$

Taking derivative gives

$$u'(x) = \frac{c_1 \operatorname{MathieuCPrime}\left(0, -2, -\frac{\pi}{4} + \frac{x}{2}\right)}{2} + \frac{c_2 \operatorname{MathieuSPrime}\left(0, -2, -\frac{\pi}{4} + \frac{x}{2}\right)}{2}$$

Doing change of constants, the solution becomes

$$y = -\frac{\frac{c_1 \operatorname{MathieuCPrime}\left(0,-2,-\frac{\pi}{4}+\frac{x}{2}\right)}{2} + \frac{\operatorname{MathieuSPrime}\left(0,-2,-\frac{\pi}{4}+\frac{x}{2}\right)}{2}}{c_1 \operatorname{MathieuC}\left(0, -2, -\frac{\pi}{4} + \frac{x}{2}\right) + \operatorname{MathieuS}\left(0, -2, -\frac{\pi}{4} + \frac{x}{2}\right)}$$

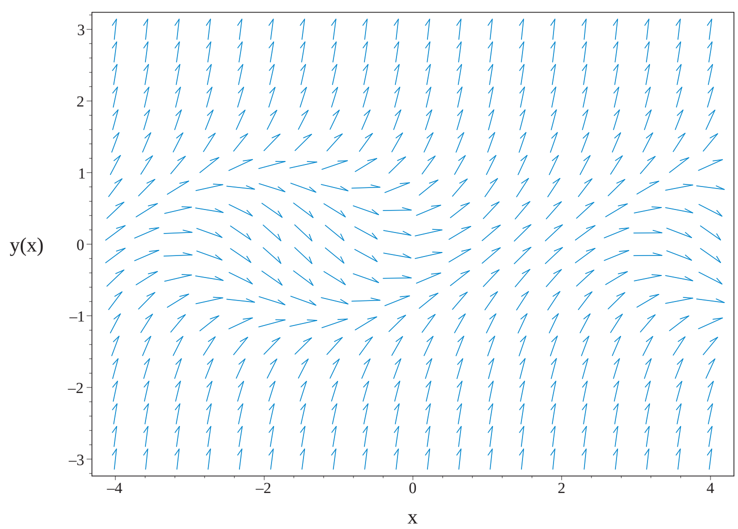


Figure 2.28: Slope field plot
$y' = \sin(x) + y^2$

Summary of solutions found

$$y = -\frac{\frac{c_1 \operatorname{MathieuCPrime}\left(0,-2,-\frac{\pi}{4}+\frac{x}{2}\right)}{2} + \frac{\operatorname{MathieuSPrime}\left(0,-2,-\frac{\pi}{4}+\frac{x}{2}\right)}{2}}{c_1 \operatorname{MathieuC}\left(0, -2, -\frac{\pi}{4} + \frac{x}{2}\right) + \operatorname{MathieuS}\left(0, -2, -\frac{\pi}{4} + \frac{x}{2}\right)}$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) = \sin(x) + y(x)^2$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = \sin(x) + y(x)^2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -y(x)*sin(x), y(x)`
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), d
      -> Trying changes of variables to rationalize or make the ODE simpler
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Trying an equivalence, under non-integer power transformations,
            to LODEs admitting Liouvillian solutions.
            -> Trying a Liouvillian solution using Kovacics algorithm
            <- No Liouvillian solutions exists
         -> Trying a solution in terms of special functions:
            -> Bessel
            -> elliptic
            -> Legendre
            -> Whittaker
               -> hyper3: Equivalence to 1F1 under a power @ Moebius
            -> hypergeometric
               -> heuristic approach
               -> hyper3: Equivalence to 2F1, 1F1 or 0F1 under a power @ Moebius
            -> Mathieu
               -> Equivalence to the rational form of Mathieu ODE under a power @ Moeb
               Equivalence transformation and function parameters: {t = 1/2*t+1/2}, {k
               <- Equivalence to the rational form of Mathieu ODE successful
            <- Mathieu successful
         <- special function solution successful
         Change of variables used:
            [x = arccos(t)]
         Linear ODE actually solved:
            (-t^2+1)^(1/2)*u(t)-t*diff(u(t),t)+(-t^2+1)*diff(diff(u(t),t),t) = 0
      <- change of variables successful
   <- Riccati to 2nd Order successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 59

```
dsolve(diff(y(x),x) = sin(x)+y(x)^2,
       y(x),singsol=all)
```

$$y = \frac{-c_1 \operatorname{MathieuSPrime}\left(0, -2, -\frac{\pi}{4}+\frac{x}{2}\right) - \operatorname{MathieuCPrime}\left(0, -2, -\frac{\pi}{4}+\frac{x}{2}\right)}{2c_1 \operatorname{MathieuS}\left(0, -2, -\frac{\pi}{4}+\frac{x}{2}\right) + 2 \operatorname{MathieuC}\left(0, -2, -\frac{\pi}{4}+\frac{x}{2}\right)}$$

Mathematica DSolve solution

Solving time : 0.174 (sec)
Leaf size : 105

```
DSolve[{D[y[x],x]==Sin[x]+y[x]^2,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \rightarrow \frac{-\text{MathieuSPrime}\left[0,-2,\frac{1}{4}(\pi-2x)\right] + c_1\text{MathieuCPrime}\left[0,-2,\frac{1}{4}(\pi-2x)\right]}{2\left(\text{MathieuS}\left[0,-2,\frac{1}{4}(2x-\pi)\right] + c_1\text{MathieuC}\left[0,-2,\frac{1}{4}(\pi-2x)\right]\right)}$$

$$y(x) \rightarrow \frac{\text{MathieuCPrime}\left[0,-2,\frac{1}{4}(\pi-2x)\right]}{2\text{MathieuC}\left[0,-2,\frac{1}{4}(\pi-2x)\right]}$$

2.1.28 problem 28

Internal problem ID [8688]
Book : First order enumerated odes
Section : section 1
Problem number : 28
Date solved : `Tuesday, December 17, 2024 at 12:57:52 PM`
CAS classification : `[_linear]`

Solve

$$y' = \cos(x) + \frac{y}{x}$$

Solved as first order linear ode

Time used: 0.105 (sec)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= -\frac{1}{x} \\ p(x) &= \cos(x) \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= \mu p \\ \frac{d}{dx}(\mu y) &= (\mu)(\cos(x)) \\ \frac{d}{dx}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)(\cos(x)) \\ d\left(\frac{y}{x}\right) &= \left(\frac{\cos(x)}{x}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{y}{x} &= \int \frac{\cos(x)}{x}\,dx \\ &= \operatorname{Ci}(x) + c_1 \end{aligned}$$

Dividing throughout by the integrating factor $\frac{1}{x}$ gives the final solution

$$y = x(\mathrm{Ci}(x) + c_1)$$

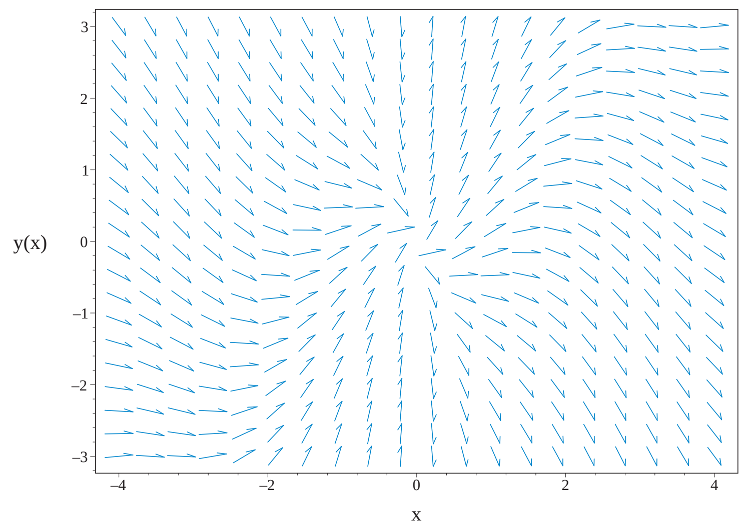


Figure 2.29: Slope field plot
$y' = \cos(x) + \frac{y}{x}$

Summary of solutions found

$$y = x(\mathrm{Ci}(x) + c_1)$$

Solved as first order homogeneous class D2 ode

Time used: 0.030 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$u'(x)\,x + u(x) = \cos(x) + u(x)$$

Which is now solved Since the ode has the form $u'(x) = f(x)$, then we only need to integrate $f(x)$.

$$\int du = \int \frac{\cos(x)}{x}\,dx$$

$$u(x) = \mathrm{Ci}(x) + c_1$$

Converting $u(x) = \mathrm{Ci}(x) + c_1$ back to y gives

$$y = x(\mathrm{Ci}(x) + c_1)$$

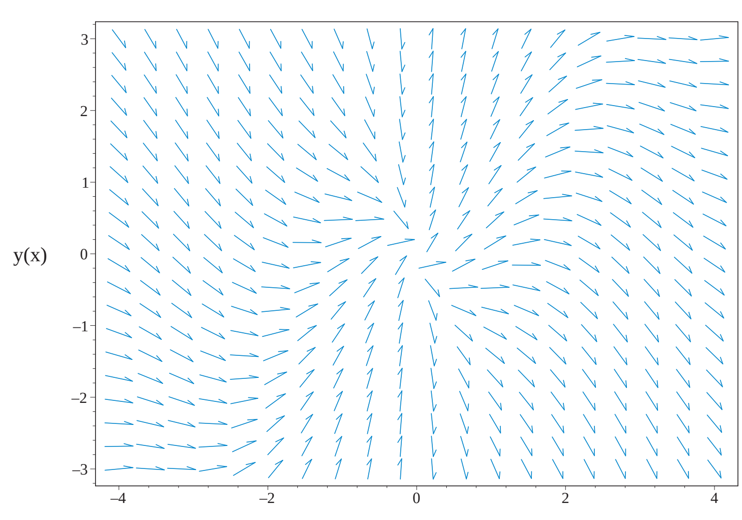


Figure 2.30: Slope field plot
$y' = \cos(x) + \frac{y}{x}$

Summary of solutions found

$$y = x(\mathrm{Ci}(x) + c_1)$$

Solved as first order Exact ode

Time used: 0.109 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y&=\left(\frac{y}{x}+\cos(x)\right)\mathrm{d}x\\ \left(-\cos(x)-\frac{y}{x}\right)\mathrm{d}x+\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\cos(x)-\frac{y}{x}\\ N(x,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\cos(x)-\frac{y}{x}\right)\\ &=-\frac{1}{x}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(1)\\ &=0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1\left(\left(-\frac{1}{x}\right) - (0)\right) \\ &= -\frac{1}{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\frac{1}{x} \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\ln(x)} \\ &= \frac{1}{x} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{x}\left(-\cos(x) - \frac{y}{x}\right) \\ &= \frac{-\cos(x)\,x - y}{x^2} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{x}(1) \\ &= \frac{1}{x} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{-\cos(x)\,x - y}{x^2}\right) + \left(\frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (2) w.r.t. y gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial y} \, \mathrm{d}y &= \int \overline{N} \, \mathrm{d}y \\ \int \frac{\partial \phi}{\partial y} \, \mathrm{d}y &= \int \frac{1}{x} \, \mathrm{d}y \end{aligned}$$

$$\phi = \frac{y}{x} + f(x) \tag{3}$$

Where $f(x)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t x gives

$$\frac{\partial \phi}{\partial x} = -\frac{y}{x^2} + f'(x) \tag{4}$$

But equation (1) says that $\frac{\partial \phi}{\partial x} = \frac{-\cos(x)x-y}{x^2}$. Therefore equation (4) becomes

$$\frac{-\cos(x)\,x - y}{x^2} = -\frac{y}{x^2} + f'(x) \tag{5}$$

Solving equation (5) for $f'(x)$ gives

$$f'(x) = -\frac{\cos(x)}{x}$$

Integrating the above w.r.t x gives

$$\int f'(x)\,\mathrm{d}x = \int \left(-\frac{\cos(x)}{x}\right)\mathrm{d}x$$
$$f(x) = -\operatorname{Ci}(x) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(x)$ into equation (3) gives ϕ

$$\phi = \frac{y}{x} - \operatorname{Ci}(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \frac{y}{x} - \operatorname{Ci}(x)$$

Solving for y gives

$$y = x(\operatorname{Ci}(x) + c_1)$$

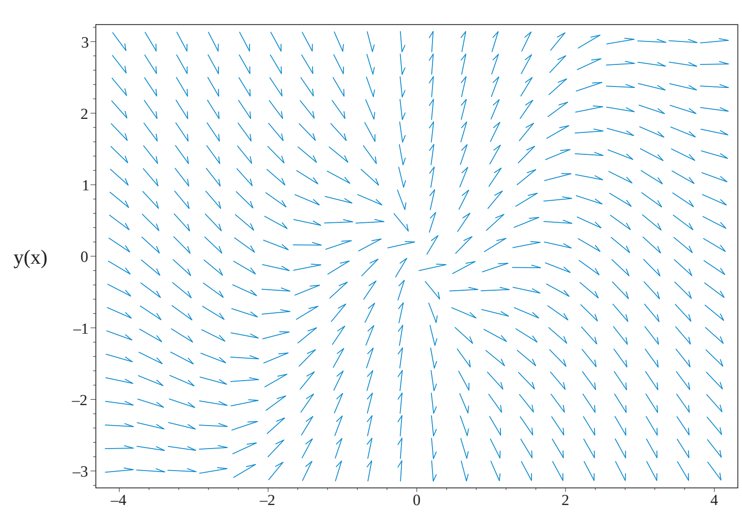


Figure 2.31: Slope field plot
$y' = \cos(x) + \frac{y}{x}$

Summary of solutions found

$$y = x(\operatorname{Ci}(x) + c_1)$$

Solved using Lie symmetry for first order ode

Time used: 0.398 (sec)

Writing the ode as

$$\begin{aligned} y' &= \frac{\cos(x)\, x + y}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 + \frac{(\cos(x)\, x + y)(b_3 - a_2)}{x} - \frac{(\cos(x)\, x + y)^2 a_3}{x^2} \\ & - \left(\frac{-\sin(x)\, x + \cos(x)}{x} - \frac{\cos(x)\, x + y}{x^2} \right) (x a_2 + y a_3 + a_1) - \frac{x b_2 + y b_3 + b_1}{x} = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{\cos(x)^2 x^2 a_3 - \sin(x)\, x^3 a_2 - \sin(x)\, x^2 y a_3 + \cos(x)\, x^2 a_2 - \cos(x)\, x^2 b_3 + 2\cos(x)\, x y a_3 - \sin(x)\, x^2 a_1 + xb}{x^2}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} & -\cos(x)^2 x^2 a_3 + \sin(x)\, x^3 a_2 + \sin(x)\, x^2 y a_3 - \cos(x)\, x^2 a_2 \\ & + \cos(x)\, x^2 b_3 - 2\cos(x)\, x y a_3 + \sin(x)\, x^2 a_1 - x b_1 + y a_1 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} & -x b_1 + y a_1 - \frac{x^2 a_3}{2} - \frac{x^2 a_3 \cos(2x)}{2} + \sin(x)\, x^3 a_2 + \sin(x)\, x^2 y a_3 \\ & - \cos(x)\, x^2 a_2 + \cos(x)\, x^2 b_3 - 2\cos(x)\, x y a_3 + \sin(x)\, x^2 a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \cos(x), \cos(2x), \sin(x)\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \cos(x) = v_3, \cos(2x) = v_4, \sin(x) = v_5\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
-v_1 b_1 + v_2 a_1 - \frac{1}{2} v_1^2 a_3 - \frac{1}{2} v_1^2 a_3 v_4 + v_5 v_1^3 a_2 + v_5 v_1^2 v_2 a_3 \\
- v_3 v_1^2 a_2 + v_3 v_1^2 b_3 - 2 v_3 v_1 v_2 a_3 + v_5 v_1^2 a_1 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3, v_4, v_5\}
$$

Equation (7E) now becomes

$$
-\frac{v_1^2 a_3 v_4}{2} + v_5 v_1^3 a_2 + (b_3 - a_2) v_1^2 v_3 + v_5 v_1^2 a_1 - \frac{v_1^2 a_3}{2} - v_1 b_1 + v_2 a_1 + v_5 v_1^2 v_2 a_3 - 2 v_3 v_1 v_2 a_3 = 0 \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 0 \\
a_3 &= 0 \\
-2a_3 &= 0 \\
-\frac{a_3}{2} &= 0 \\
-b_1 &= 0 \\
b_3 - a_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= b_2 \\
b_3 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 0 \\
\eta &= x
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{\cos(x) x + y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\cos(x)}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\cos(R)}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR} S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned} \int dS &= \int \frac{\cos(R)}{R} dR \\ S(R) &= \mathrm{Ci}(R) + c_2 \end{aligned}$$

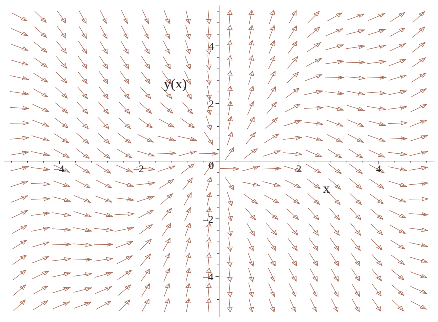
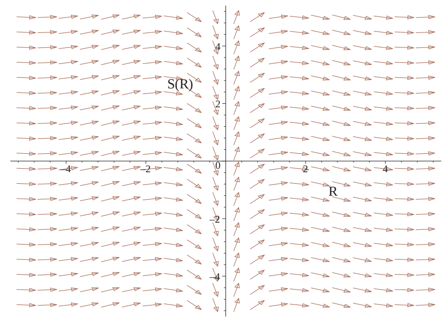
To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\frac{y}{x} = \mathrm{Ci}(x) + c_2$$

Which gives

$$y = (\mathrm{Ci}(x) + c_2) x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\cos(x)x+y}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = \frac{\cos(R)}{R}$ 

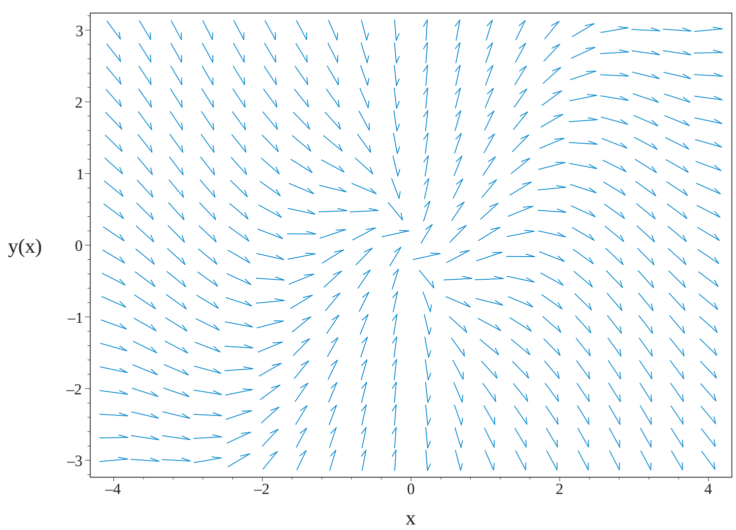


Figure 2.32: Slope field plot
$y' = \cos(x) + \frac{y}{x}$

Summary of solutions found

$$y = (\mathrm{Ci}(x) + c_2)\, x$$

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) = \frac{y(x)}{x} + \cos(x)$

- Highest derivative means the order of the ODE is 1

$\frac{d}{dx}y(x)$

- Solve for the highest derivative

$\frac{d}{dx}y(x) = \frac{y(x)}{x} + \cos(x)$

- Group terms with $y(x)$ on the lhs of the ODE and the rest on the rhs of the ODE

$\frac{d}{dx}y(x) - \frac{y(x)}{x} = \cos(x)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x)\left(\frac{d}{dx}y(x) - \frac{y(x)}{x}\right) = \mu(x)\cos(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(y(x)\,\mu(x))$

$\mu(x)\left(\frac{d}{dx}y(x) - \frac{y(x)}{x}\right) = \left(\frac{d}{dx}y(x)\right)\mu(x) + y(x)\left(\frac{d}{dx}\mu(x)\right)$

- Isolate $\frac{d}{dx}\mu(x)$

$\frac{d}{dx}\mu(x) = -\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(y(x)\,\mu(x))\right) dx = \int \mu(x)\cos(x)\,dx + C1$

- Evaluate the integral on the lhs

  $y(x)\,\mu(x) = \int \mu(x)\cos(x)\,dx + C1$

- Solve for $y(x)$

  $y(x) = \frac{\int \mu(x)\cos(x)dx + C1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x}$

  $y(x) = x\left(\int \frac{\cos(x)}{x}dx + C1\right)$

- Evaluate the integrals on the rhs

  $y(x) = x(\mathrm{Ci}(x) + C1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 10

```
dsolve(diff(y(x),x) = cos(x)+y(x)/x,
       y(x),singsol=all)
```

$$y = (\mathrm{Ci}(x) + c_1)\,x$$

Mathematica DSolve solution

Solving time : 0.034 (sec)
Leaf size : 12

```
DSolve[{D[y[x],x]==Cos[x]+y[x]/x,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to x(\mathrm{CosIntegral}(x) + c_1)$$

2.1.29 problem 29

Internal problem ID [8689]
Book : First order enumerated odes
Section : section 1
Problem number : 29
Date solved : `Tuesday, December 17, 2024 at 12:57:54 PM`
CAS classification : `[_Riccati]`

Solve

$$y' = \cos(x) + \frac{y^2}{x}$$

Unknown ode type.

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) = \cos(x) + \frac{y(x)^2}{x}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = \cos(x) + \frac{y(x)^2}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   trying Riccati_symmetries
   trying Riccati to 2nd Order
   -> Calling odsolve with the ODE`, diff(diff(y(x), x), x) = -(diff(y(x), x))/x-cos(x
      Methods for second order ODEs:
      --- Trying classification methods ---
      trying a symmetry of the form [xi=0, eta=F(x)]
      checking if the LODE is missing y
      -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @
      -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x), d
      -> Trying changes of variables to rationalize or make the ODE simpler
```

```
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x),
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
            -> trying with_periodic_functions in the coefficients
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x),
            trying a symmetry of the form [xi=0, eta=F(x)]
            trying 2nd order exact linear
            trying symmetries linear in x and y(x)
            trying to convert to a linear ODE with constant coefficients
      <- unable to find a useful change of variables
         trying a symmetry of the form [xi=0, eta=F(x)]
         trying 2nd order exact linear
         trying symmetries linear in x and y(x)
         trying to convert to a linear ODE with constant coefficients
         trying 2nd order, integrating factor of the form mu(x,y)
         trying a symmetry of the form [xi=0, eta=F(x)]
         checking if the LODE is missing y
         -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a power @
         -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x),
         -> Trying changes of variables to rationalize or make the ODE simpler
            trying a symmetry of the form [xi=0, eta=F(x)]
            checking if the LODE is missing y
            -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a powe
            -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x
               trying a symmetry of the form [xi=0, eta=F(x)]
               trying 2nd order exact linear
               trying symmetries linear in x and y(x)
               trying to convert to a linear ODE with constant coefficients
               -> trying with_periodic_functions in the coefficients
            trying a symmetry of the form [xi=0, eta=F(x)]
            checking if the LODE is missing y
            -> Heun: Equivalence to the GHE or one of its 4 confluent cases under a powe
            -> trying a solution of the form r0(x) * Y + r1(x) * Y where Y = exp(int(r(x
               trying a symmetry of the form [xi=0, eta=F(x)]
               trying 2nd order exact linear
               trying symmetries linear in x and y(x)
               trying to convert to a linear ODE with constant coefficients
         <- unable to find a useful change of variables
            trying a symmetry of the form [xi=0, eta=F(x)]
         trying to convert to an ODE of Bessel type
         -> trying with_periodic_functions in the coefficients
   -> Trying a change of variables to reduce to Bernoulli
   -> Calling odsolve with the ODE`, diff(y(x), x)-(y(x)^2/x+y(x)+x^2*cos(x))/x, y(x), e
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
```

```
      trying 1st order linear
      trying Bernoulli
      trying separable
      trying inverse linear
      trying homogeneous types:
      trying Chini
      differential order: 1; looking for linear symmetries
      trying exact
      Looking for potential symmetries
      trying Riccati
      trying Riccati sub-methods:
         trying Riccati_symmetries
      trying inverse_Riccati
      trying 1st order ODE linearizable_by_differentiation
   -> trying a symmetry pattern of the form [F(x)*G(y), 0]
   -> trying a symmetry pattern of the form [0, F(x)*G(y)]
   -> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
trying inverse_Riccati
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
`, `-> Computing symmetries using: way = 6`
```

Maple dsolve solution

Solving time : 0.543 (sec)
Leaf size : maple_leaf_size

```
dsolve(diff(y(x),x) = cos(x)+y(x)^2/x,
       y(x),singsol=all)
```

No solution found

Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{D[y[x],x]==Cos[x]+y[x]^2/x,{}},
       y[x],x,IncludeSingularSolutions->True]
```

Not solved

2.1.30 problem 30

Internal problem ID [8690]
Book : First order enumerated odes
Section : section 1
Problem number : 30
Date solved : `Tuesday, December 17, 2024 at 12:57:58 PM`
CAS classification : `[_Riccati]`

Solve

$$y' = x + y + by^2$$

Solved as first order ode of type Riccati

Time used: 0.185 (sec)

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= b\,y^2 + x + y \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = b\,y^2 + x + y$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x$, $f_1(x) = 1$ and $f_2(x) = b$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{ub} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= b \\ f_2^2 f_0 &= b^2 x \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$bu''(x) - bu'(x) + b^2 x u(x) = 0$$

This is Airy ODE. It has the general form

$$a\frac{d^2u}{dx^2} + b\frac{du}{dx} + cux = F(x)$$

Where in this case

$$
\begin{aligned}
a &= b \\
b &= -b \\
c &= b^2 \\
F &= 0
\end{aligned}
$$

Therefore the solution to the homogeneous Airy ODE becomes

$$
u = c_1 \,\mathrm{e}^{\frac{x}{2}} \operatorname{AiryAi}\left(-\frac{b^3 x - \frac{1}{4} b^2}{b^{8/3}}\right) + c_2 \,\mathrm{e}^{\frac{x}{2}} \operatorname{AiryBi}\left(-\frac{b^3 x - \frac{1}{4} b^2}{b^{8/3}}\right)
$$

Will add steps showing solving for IC soon.

Taking derivative gives

$$
u'(x) = \frac{c_1 \,\mathrm{e}^{\frac{x}{2}} \operatorname{AiryAi}\left(-\frac{b^3x-\frac{1}{4}b^2}{b^{8/3}}\right)}{2} - c_1 \,\mathrm{e}^{\frac{x}{2}} b^{1/3} \operatorname{AiryAi}\left(1, -\frac{b^3 x - \frac{1}{4} b^2}{b^{8/3}}\right) + \frac{c_2 \,\mathrm{e}^{\frac{x}{2}} \operatorname{AiryBi}\left(-\frac{b^3x-\frac{1}{4}b^2}{b^{8/3}}\right)}{2} - c_2 \,\mathrm{e}^{\frac{x}{2}} b^{1/3} \operatorname{AiryBi}\left(1, -\frac{b^3 x - \frac{1}{4} b^2}{b^{8/3}}\right)
$$

Doing change of constants, the solution becomes

$$
y = -\frac{\frac{c_3 \,\mathrm{e}^{\frac{x}{2}} \operatorname{AiryAi}\left(-\frac{b^3x-\frac{1}{4}b^2}{b^{8/3}}\right)}{2} - c_3 \,\mathrm{e}^{\frac{x}{2}} b^{1/3} \operatorname{AiryAi}\left(1, -\frac{b^3x-\frac{1}{4}b^2}{b^{8/3}}\right) + \frac{\mathrm{e}^{\frac{x}{2}} \operatorname{AiryBi}\left(-\frac{b^3x-\frac{1}{4}b^2}{b^{8/3}}\right)}{2} - \mathrm{e}^{\frac{x}{2}} b^{1/3} \operatorname{AiryBi}\left(1, -\frac{b^3x-}{b^8}\right.}{b\left(c_3 \,\mathrm{e}^{\frac{x}{2}} \operatorname{AiryAi}\left(-\frac{b^3x-\frac{1}{4}b^2}{b^{8/3}}\right) + \mathrm{e}^{\frac{x}{2}} \operatorname{AiryBi}\left(-\frac{b^3x-\frac{1}{4}b^2}{b^{8/3}}\right)\right)}
$$

Summary of solutions found

$$
y = -\frac{\frac{c_3 \,\mathrm{e}^{\frac{x}{2}} \operatorname{AiryAi}\left(-\frac{b^3x-\frac{1}{4}b^2}{b^{8/3}}\right)}{2} - c_3 \,\mathrm{e}^{\frac{x}{2}} b^{1/3} \operatorname{AiryAi}\left(1, -\frac{b^3x-\frac{1}{4}b^2}{b^{8/3}}\right) + \frac{\mathrm{e}^{\frac{x}{2}} \operatorname{AiryBi}\left(-\frac{b^3x-\frac{1}{4}b^2}{b^{8/3}}\right)}{2} - \mathrm{e}^{\frac{x}{2}} b^{1/3} \operatorname{AiryBi}\left(1, -\frac{b^3x-}{b^8}\right.}{b\left(c_3 \,\mathrm{e}^{\frac{x}{2}} \operatorname{AiryAi}\left(-\frac{b^3x-\frac{1}{4}b^2}{b^{8/3}}\right) + \mathrm{e}^{\frac{x}{2}} \operatorname{AiryBi}\left(-\frac{b^3x-\frac{1}{4}b^2}{b^{8/3}}\right)\right)}
$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) = x + y(x) + b y(x)^2$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = x + y(x) + b y(x)^2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati sub-methods:
   <- Abel AIR successful: ODE belongs to the 0F1 0-parameter (Airy type) class`
```

Maple dsolve solution

Solving time : 0.022 (sec)
Leaf size : 105

```
dsolve(diff(y(x),x) = x+y(x)+b*y(x)^2,
       y(x),singsol=all)
```

$$y = \frac{2\,\mathrm{AiryAi}\left(1, -\frac{4bx-1}{4b^{2/3}}\right) b^{1/3} c_1 - \mathrm{AiryAi}\left(-\frac{4bx-1}{4b^{2/3}}\right) c_1 + 2\,\mathrm{AiryBi}\left(1, -\frac{4bx-1}{4b^{2/3}}\right) b^{1/3} - \mathrm{AiryBi}\left(-\frac{4bx-1}{4b^{2/3}}\right)}{2b\left(\mathrm{AiryAi}\left(-\frac{4bx-1}{4b^{2/3}}\right) c_1 + \mathrm{AiryBi}\left(-\frac{4bx-1}{4b^{2/3}}\right)\right)}$$

Mathematica DSolve solution

Solving time : 0.201 (sec)
Leaf size : 211

```
DSolve[{D[y[x],x]==x+y[x]+b*y[x]^2,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \rightarrow -\frac{-(-b)^{2/3}\,\mathrm{AiryBi}\left(\frac{\frac{1}{4}-bx}{(-b)^{2/3}}\right) + 2b\,\mathrm{AiryBiPrime}\left(\frac{\frac{1}{4}-bx}{(-b)^{2/3}}\right) + c_1\left(2b\,\mathrm{AiryAiPrime}\left(\frac{\frac{1}{4}-bx}{(-b)^{2/3}}\right) - (-b)^{2/3}\,\mathrm{AiryAi}\right.}{2(-b)^{5/3}\left(\mathrm{AiryBi}\left(\frac{\frac{1}{4}-bx}{(-b)^{2/3}}\right) + c_1\,\mathrm{AiryAi}\left(\frac{\frac{1}{4}-bx}{(-b)^{2/3}}\right)\right)}$$

$$y(x) \rightarrow -\frac{\frac{2\sqrt[3]{-b}\,\mathrm{AiryAiPrime}\left(\frac{\frac{1}{4}-bx}{(-b)^{2/3}}\right)}{\mathrm{AiryAi}\left(\frac{\frac{1}{4}-bx}{(-b)^{2/3}}\right)} + 1}{2b}$$

2.1.31 problem 31

Internal problem ID [8691]
Book : First order enumerated odes
Section : section 1
Problem number : 31
Date solved : `Tuesday, December 17, 2024 at 12:58:00 PM`
CAS classification : `[_quadrature]`

Solve

$$xy' = 0$$

Solved as first order quadrature ode

Time used: 0.026 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\, dx + c_1$$
$$y = c_1$$

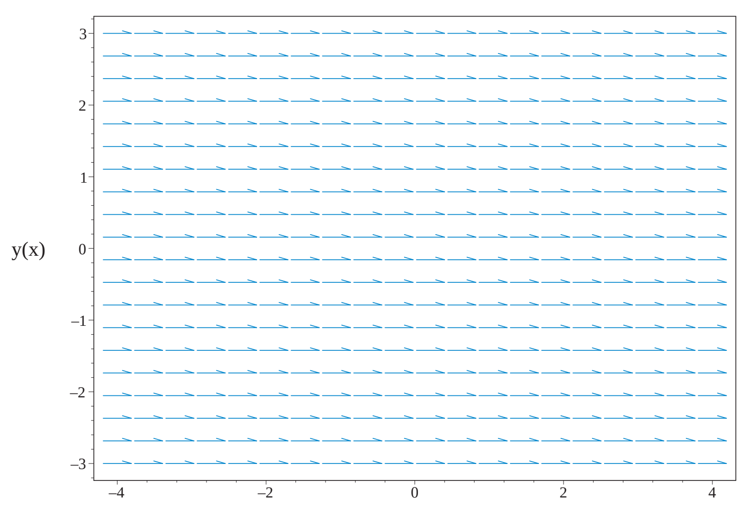


Figure 2.33: Slope field plot
$xy' = 0$

Summary of solutions found

$$y = c_1$$

Solved as first order homogeneous class D2 ode

Time used: 0.132 (sec)

Applying change of variables $y = u(x)\, x$, then the ode becomes

$$x(u'(x)\, x + u(x)) = 0$$

Which is now solved The ode $u'(x) = -\frac{u(x)}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(u) &= u \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\, du &= \int f(x)\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{1}{x}\, dx \\ \ln(u(x)) &= \ln\left(\frac{1}{x}\right) + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} \ln(u(x)) &= \ln\left(\frac{1}{x}\right) + c_1 \\ u(x) &= 0 \end{aligned}$$

Solving for $u(x)$ gives

$$\begin{aligned} u(x) &= 0 \\ u(x) &= \frac{e^{c_1}}{x} \end{aligned}$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = \frac{e^{c_1}}{x}$ back to y gives

$$y = e^{c_1}$$

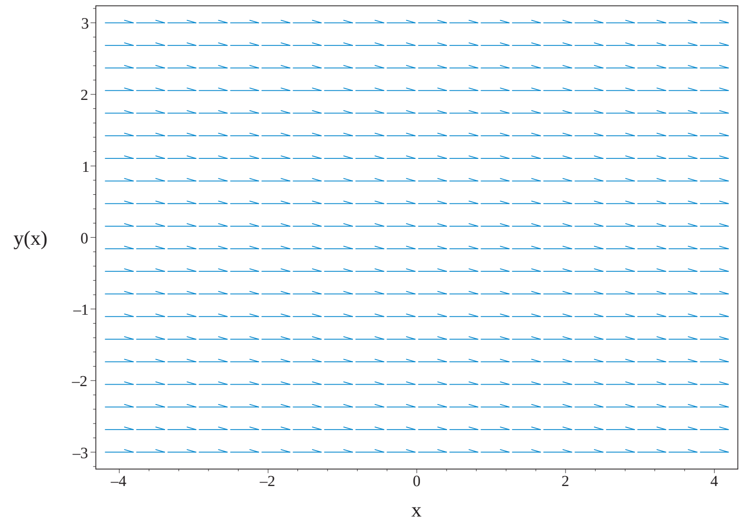


Figure 2.34: Slope field plot
$xy' = 0$

Summary of solutions found

$$y = 0$$
$$y = e^{c_1}$$

Solved as first order ode of type differential

Time used: 0.010 (sec)

Writing the ode as

$$y' = 0 \tag{1}$$

Which becomes

$$(1)\,dy = (0)\,dx \tag{2}$$

But the RHS is complete differential because

$$(0)\,dx = d(0)$$

Hence (2) becomes

$$(1)\,dy = d(0)$$

Integrating gives

$$y = c_1$$

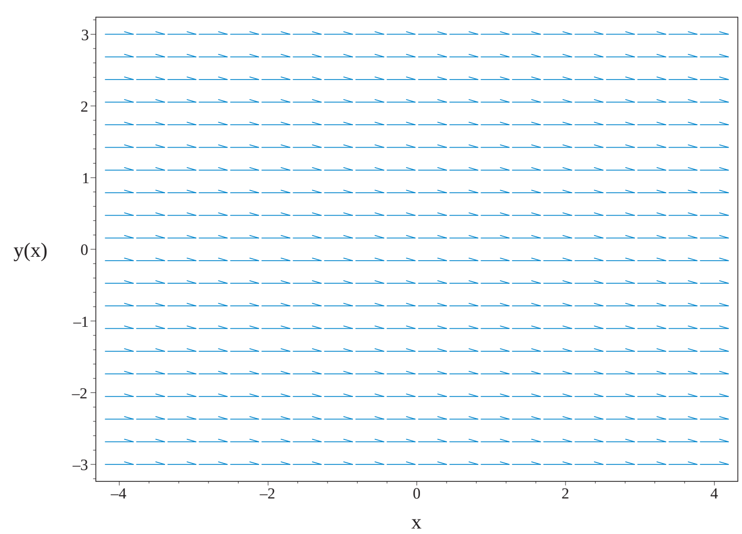


Figure 2.35: Slope field plot
$xy' = 0$

Summary of solutions found

$$y = c_1$$

Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y(x)\right) = 0$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = 0$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}y(x)\right) dx = \int 0 dx + C1$

- Evaluate integral

  $y(x) = C1$

- Solve for $y(x)$

  $y(x) = C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 5

```
dsolve(diff(y(x),x)*x = 0,
       y(x),singsol=all)
```

$$y = c_1$$

Mathematica DSolve solution

Solving time : `0.002` (sec)
Leaf size : 7

```
DSolve[{x*D[y[x],x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1$$

2.1.32 problem 32

Internal problem ID [8692]
Book : First order enumerated odes
Section : section 1
Problem number : 32
Date solved : `Tuesday, December 17, 2024 at 12:58:01 PM`
CAS classification : `[_quadrature]`

Solve

$$5y' = 0$$

Solved as first order quadrature ode

Time used: 0.025 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\, dx + c_1$$
$$y = c_1$$

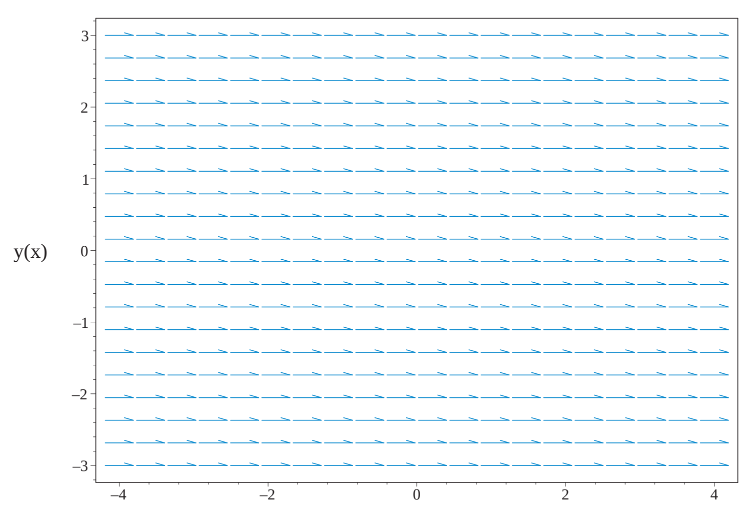


Figure 2.36: Slope field plot
$5y' = 0$

Summary of solutions found

$$y = c_1$$

Solved as first order homogeneous class D2 ode

Time used: 0.130 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$5u'(x)\,x + 5u(x) = 0$$

Which is now solved The ode $u'(x) = -\frac{u(x)}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(u) &= u \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\ \ln(u(x)) &= \ln\left(\frac{1}{x}\right) + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} \ln(u(x)) &= \ln\left(\frac{1}{x}\right) + c_1 \\ u(x) &= 0 \end{aligned}$$

Solving for $u(x)$ gives

$$\begin{aligned} u(x) &= 0 \\ u(x) &= \frac{e^{c_1}}{x} \end{aligned}$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = \frac{e^{c_1}}{x}$ back to y gives

$$y = e^{c_1}$$

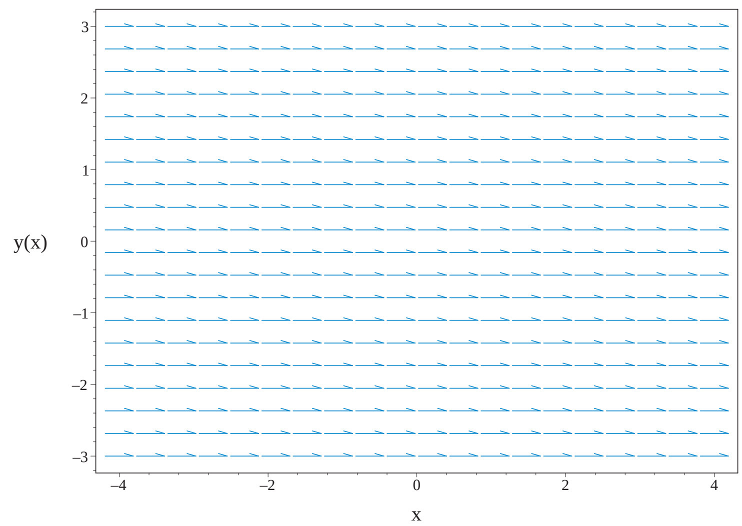


Figure 2.37: Slope field plot
$5y' = 0$

Summary of solutions found

$$y = 0$$

$$y = e^{c_1}$$

Solved as first order ode of type differential

Time used: 0.010 (sec)

Writing the ode as

$$y' = 0 \tag{1}$$

Which becomes

$$(1)\,dy = (0)\,dx \tag{2}$$

But the RHS is complete differential because

$$(0)\,dx = d(0)$$

Hence (2) becomes

$$(1)\,dy = d(0)$$

Integrating gives

$$y = c_1$$

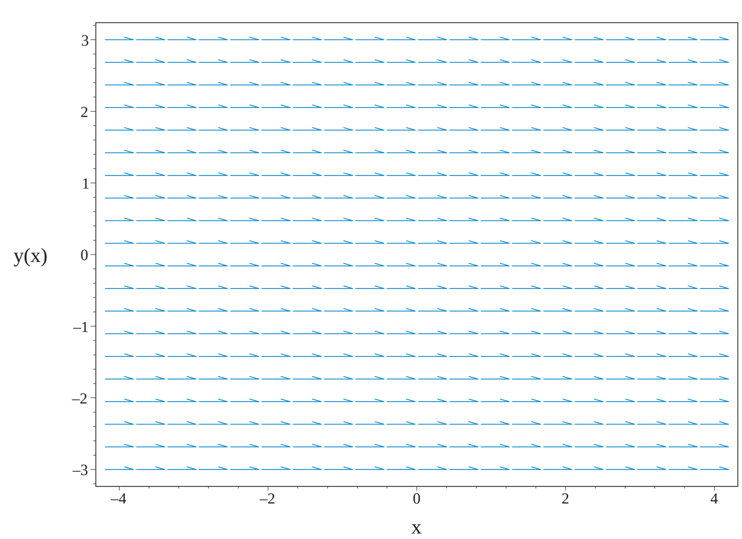


Figure 2.38: Slope field plot
$5y' = 0$

Summary of solutions found

$$y = c_1$$

Maple step by step solution

Let's solve

$5\frac{d}{dx}y(x) = 0$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$
- Separate variables

  $\frac{d}{dx}y(x) = 0$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}y(x)\right) dx = \int 0dx + C1$
- Evaluate integral

  $y(x) = C1$
- Solve for $y(x)$

  $y(x) = C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 5

```
dsolve(5*diff(y(x),x) = 0,
       y(x),singsol=all)
```

$$y = c_1$$

Mathematica DSolve solution

Solving time : 0.002 (sec)
Leaf size : 7

```
DSolve[{5*D[y[x],x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1$$

2.1.33 problem 33

Internal problem ID [8693]
Book : First order enumerated odes
Section : section 1
Problem number : 33
Date solved : `Tuesday, December 17, 2024 at 12:58:02 PM`
CAS classification : `[_quadrature]`

Solve

$$ey' = 0$$

Solved as first order quadrature ode

Time used: 0.023 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\,dx + c_1$$

$$y = c_1$$

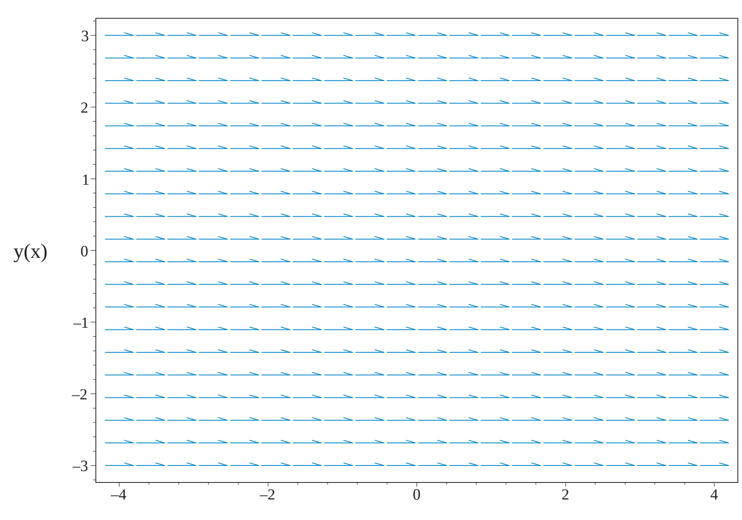


Figure 2.39: Slope field plot
$ey' = 0$

Summary of solutions found

$$y = c_1$$

Solved as first order homogeneous class D2 ode

Time used: 0.132 (sec)

Applying change of variables $y = u(x)\, x$, then the ode becomes

$$\mathrm{e}(u'(x)\, x + u(x)) = 0$$

Which is now solved The ode $u'(x) = -\frac{u(x)}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(u) &= u \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\, du &= \int f(x)\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{1}{x}\, dx \\ \ln\left(u(x)\right) &= \ln\left(\frac{1}{x}\right) + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} \ln\left(u(x)\right) &= \ln\left(\frac{1}{x}\right) + c_1 \\ u(x) &= 0 \end{aligned}$$

Solving for $u(x)$ gives

$$\begin{aligned} u(x) &= 0 \\ u(x) &= \frac{\mathrm{e}^{c_1}}{x} \end{aligned}$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = \frac{\mathrm{e}^{c_1}}{x}$ back to y gives

$$y = \mathrm{e}^{c_1}$$

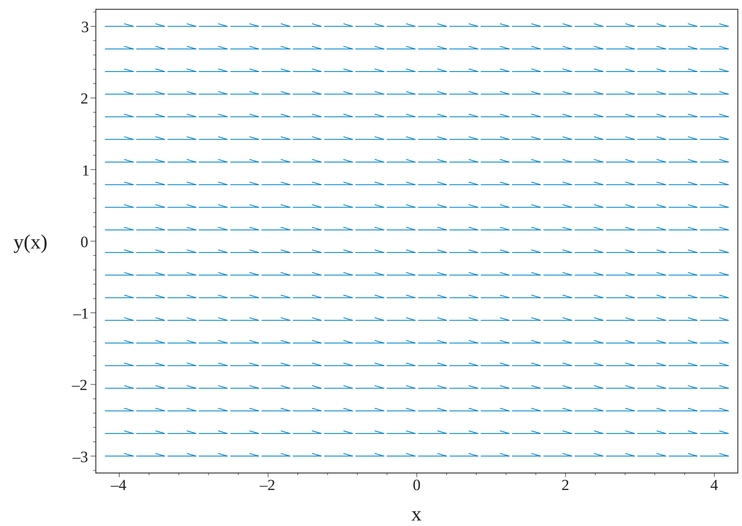


Figure 2.40: Slope field plot
$ey' = 0$

Summary of solutions found

$$y = 0$$
$$y = e^{c_1}$$

Solved as first order ode of type differential

Time used: 0.010 (sec)

Writing the ode as

$$y' = 0 \tag{1}$$

Which becomes

$$(1)\,dy = (0)\,dx \tag{2}$$

But the RHS is complete differential because

$$(0)\,dx = d(0)$$

Hence (2) becomes

$$(1)\,dy = d(0)$$

Integrating gives

$$y = c_1$$

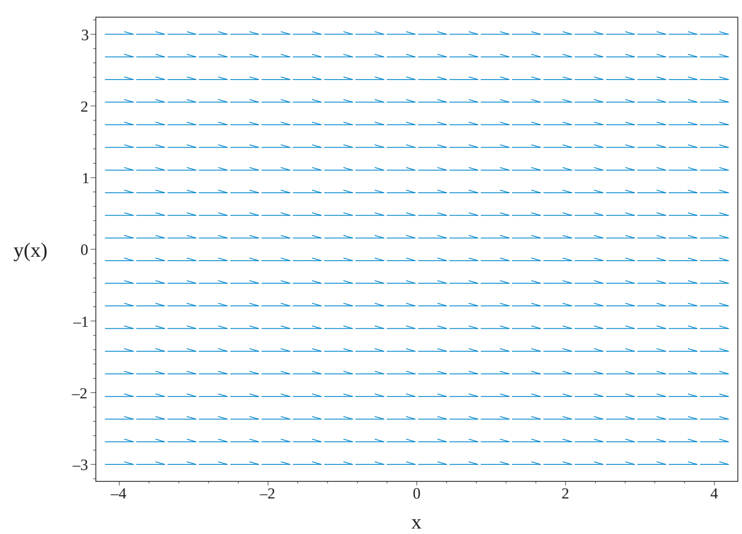


Figure 2.41: Slope field plot
$ey' = 0$

Summary of solutions found

$$y = c_1$$

Maple step by step solution

Let's solve

$e\left(\frac{d}{dx}y(x)\right) = 0$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Separate variables

  $\frac{d}{dx}y(x) = 0$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}y(x)\right) dx = \int 0 dx + C1$

- Evaluate integral

  $y(x) = C1$

- Solve for $y(x)$

  $y(x) = C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 5

```
dsolve(exp(1)*diff(y(x),x) = 0,
       y(x),singsol=all)
```

$$y = c_1$$

Mathematica DSolve solution

Solving time : 0.002 (sec)
Leaf size : 7

```
DSolve[{Exp[1]*D[y[x],x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1$$

2.1.34 problem 34

Internal problem ID [8694]
Book : First order enumerated odes
Section : section 1
Problem number : 34
Date solved : `Tuesday, December 17, 2024 at 12:58:02 PM`
CAS classification : `[_quadrature]`

Solve

$$\pi y' = 0$$

Solved as first order quadrature ode

Time used: 0.023 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\,dx + c_1$$
$$y = c_1$$

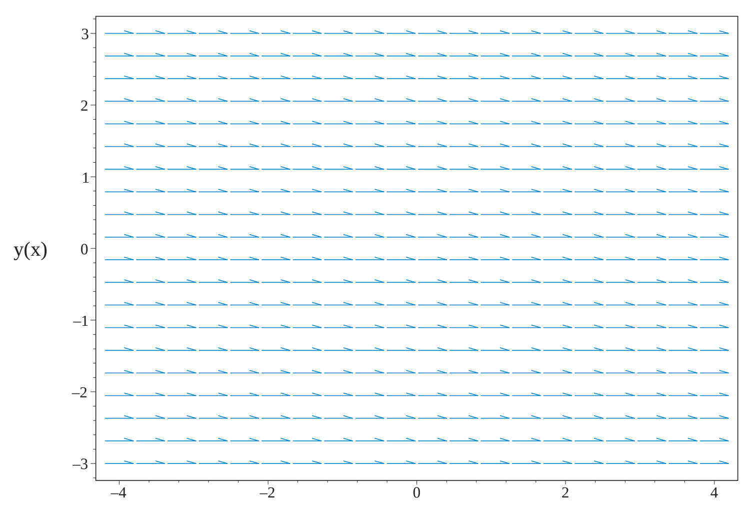


Figure 2.42: Slope field plot
$\pi y' = 0$

Summary of solutions found

$$y = c_1$$

Solved as first order homogeneous class D2 ode

Time used: 0.132 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$\pi(u'(x)\,x + u(x)) = 0$$

Which is now solved The ode $u'(x) = -\frac{u(x)}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(u) &= u \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\, du &= \int f(x)\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{1}{x}\, dx \\ \ln(u(x)) &= \ln\left(\frac{1}{x}\right) + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} \ln(u(x)) &= \ln\left(\frac{1}{x}\right) + c_1 \\ u(x) &= 0 \end{aligned}$$

Solving for $u(x)$ gives

$$\begin{aligned} u(x) &= 0 \\ u(x) &= \frac{e^{c_1}}{x} \end{aligned}$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = \frac{e^{c_1}}{x}$ back to y gives

$$y = e^{c_1}$$

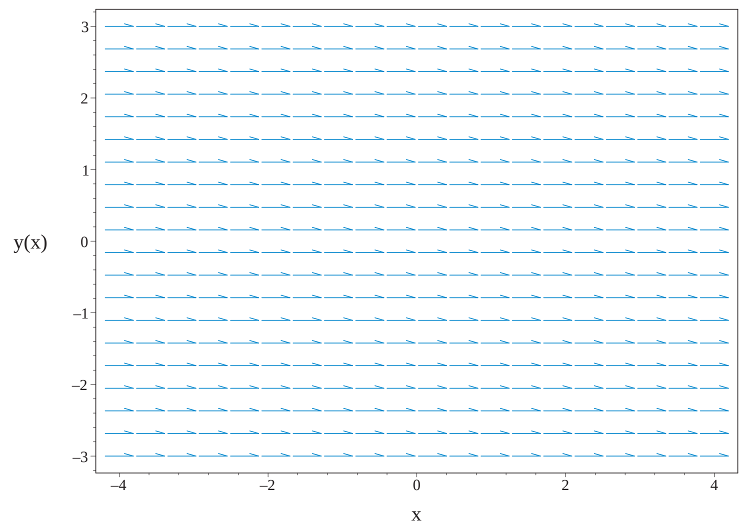


Figure 2.43: Slope field plot
$\pi y' = 0$

Summary of solutions found

$$y = 0$$
$$y = e^{c_1}$$

Solved as first order ode of type differential

Time used: 0.010 (sec)

Writing the ode as

$$y' = 0 \tag{1}$$

Which becomes

$$(1)\, dy = (0)\, dx \tag{2}$$

But the RHS is complete differential because

$$(0)\, dx = d(0)$$

Hence (2) becomes

$$(1)\, dy = d(0)$$

Integrating gives

$$y = c_1$$

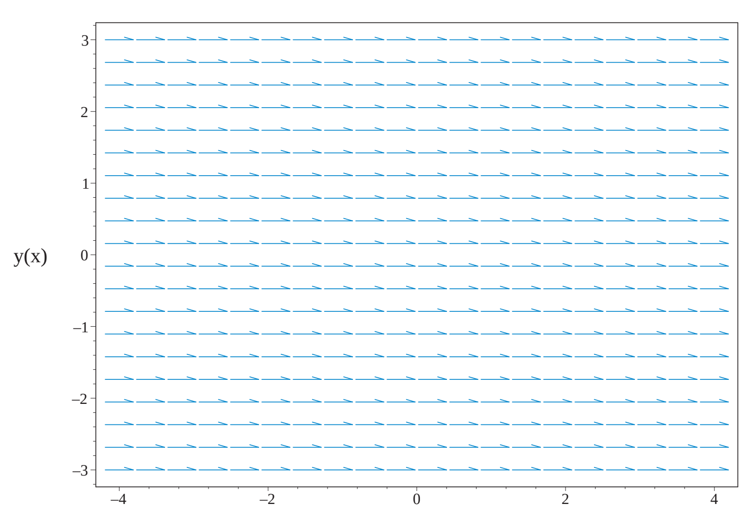


Figure 2.44: Slope field plot
$\pi y' = 0$

Summary of solutions found

$$y = c_1$$

Maple step by step solution

Let's solve
$\pi\left(\frac{d}{dx}y(x)\right) = 0$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Separate variables
  $\frac{d}{dx}y(x) = 0$
- Integrate both sides with respect to x
  $\int \left(\frac{d}{dx}y(x)\right) dx = \int 0 dx + C1$
- Evaluate integral
  $y(x) = C1$
- Solve for $y(x)$
  $y(x) = C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 5

```
dsolve(Pi*diff(y(x),x) = 0,
       y(x),singsol=all)
```

$$y = c_1$$

Mathematica DSolve solution

Solving time : 0.002 (sec)
Leaf size : 7

```
DSolve[{Pi*D[y[x],x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1$$

2.1.35 problem 35

Internal problem ID [8695]
Book : First order enumerated odes
Section : section 1
Problem number : 35
Date solved : Tuesday, December 17, 2024 at 12:58:03 PM
CAS classification : [_quadrature]

Solve

$$\sin(x)\, y' = 0$$

Solved as first order quadrature ode

Time used: 0.026 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\, dx + c_1$$

$$y = c_1$$

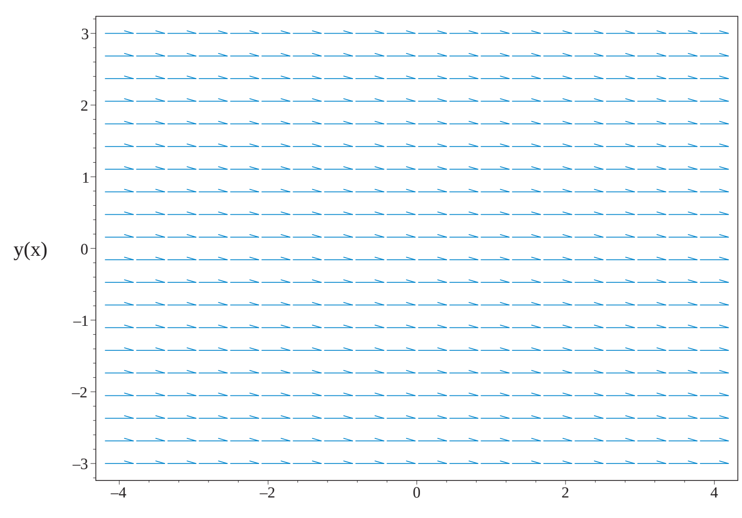


Figure 2.45: Slope field plot
$\sin(x)\, y' = 0$

Summary of solutions found

$$y = c_1$$

Solved as first order homogeneous class D2 ode

Time used: 0.156 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$\sin(x)\,(u'(x)\,x + u(x)) = 0$$

Which is now solved The ode $u'(x) = -\frac{u(x)}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(u) &= u \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\ \ln(u(x)) &= \ln\left(\frac{1}{x}\right) + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} \ln(u(x)) &= \ln\left(\frac{1}{x}\right) + c_1 \\ u(x) &= 0 \end{aligned}$$

Solving for $u(x)$ gives

$$\begin{aligned} u(x) &= 0 \\ u(x) &= \frac{e^{c_1}}{x} \end{aligned}$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = \frac{e^{c_1}}{x}$ back to y gives

$$y = e^{c_1}$$

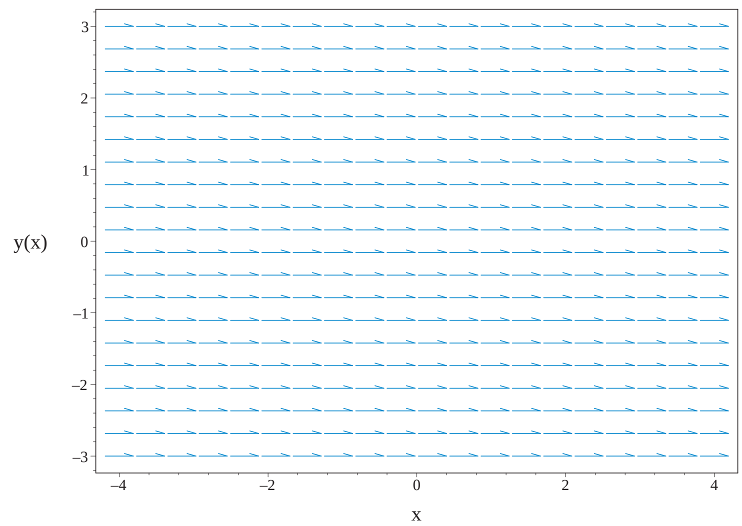


Figure 2.46: Slope field plot
$\sin(x)\, y' = 0$

Summary of solutions found

$$y = 0$$

$$y = e^{c_1}$$

Solved as first order ode of type differential

Time used: 0.012 (sec)

Writing the ode as

$$y' = 0 \tag{1}$$

Which becomes

$$(1)\, dy = (0)\, dx \tag{2}$$

But the RHS is complete differential because

$$(0)\, dx = d(0)$$

Hence (2) becomes

$$(1)\, dy = d(0)$$

Integrating gives

$$y = c_1$$

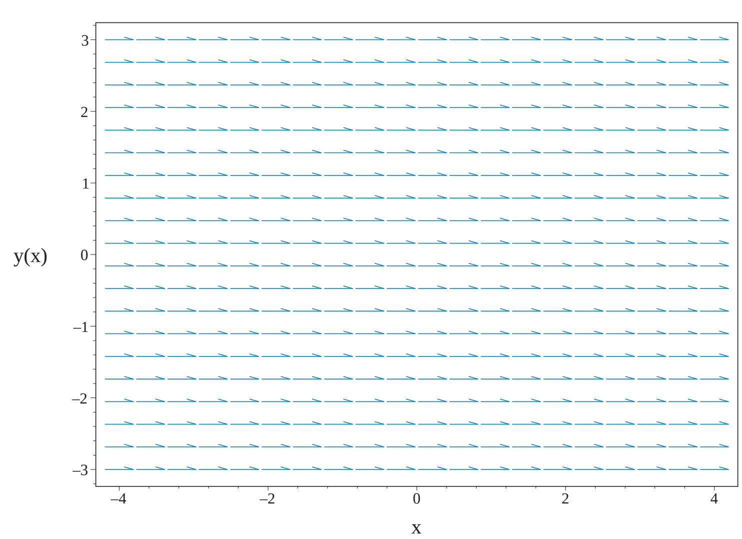


Figure 2.47: Slope field plot
$\sin(x)\, y' = 0$

Summary of solutions found

$$y = c_1$$

Maple step by step solution

Let's solve

$\sin\left(x\right)\left(\frac{d}{dx}y(x)\right)=0$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x)=0$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}y(x)\right)dx=\int 0dx+C1$

- Evaluate integral

  $y(x)=C1$

- Solve for $y(x)$

  $y(x)=C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 5

```
dsolve(sin(x)*diff(y(x),x) = 0,
       y(x),singsol=all)
```

$$y=c_1$$

Mathematica DSolve solution

Solving time : 0.002 (sec)
Leaf size : 7

```
DSolve[{Sin[x]*D[y[x],x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to c_1$$

2.1.36 problem 36

Internal problem ID [8696]
Book : First order enumerated odes
Section : section 1
Problem number : 36
Date solved : `Tuesday, December 17, 2024 at 12:58:04 PM`
CAS classification : `[_quadrature]`

Solve

$$f(x)\, y' = 0$$

Solved as first order quadrature ode

Time used: 0.026 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\, dx + c_1$$
$$y = c_1$$

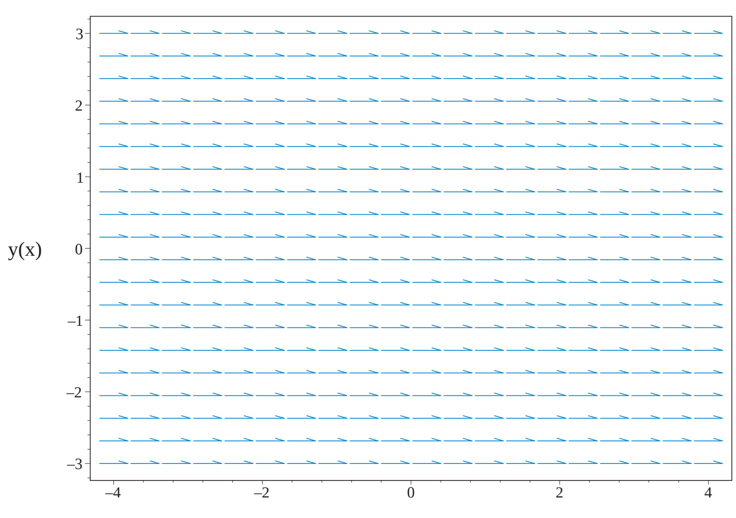


Figure 2.48: Slope field plot
$f(x)\, y' = 0$

Summary of solutions found

$$y = c_1$$

Solved as first order homogeneous class D2 ode

Time used: 0.145 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$f(x)\,(u'(x)\,x + u(x)) = 0$$

Which is now solved The ode $u'(x) = -\frac{u(x)}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(u) &= u \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\ \ln(u(x)) &= \ln\left(\frac{1}{x}\right) + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} \ln(u(x)) &= \ln\left(\frac{1}{x}\right) + c_1 \\ u(x) &= 0 \end{aligned}$$

Solving for $u(x)$ gives

$$\begin{aligned} u(x) &= 0 \\ u(x) &= \frac{e^{c_1}}{x} \end{aligned}$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = \frac{e^{c_1}}{x}$ back to y gives

$$y = e^{c_1}$$

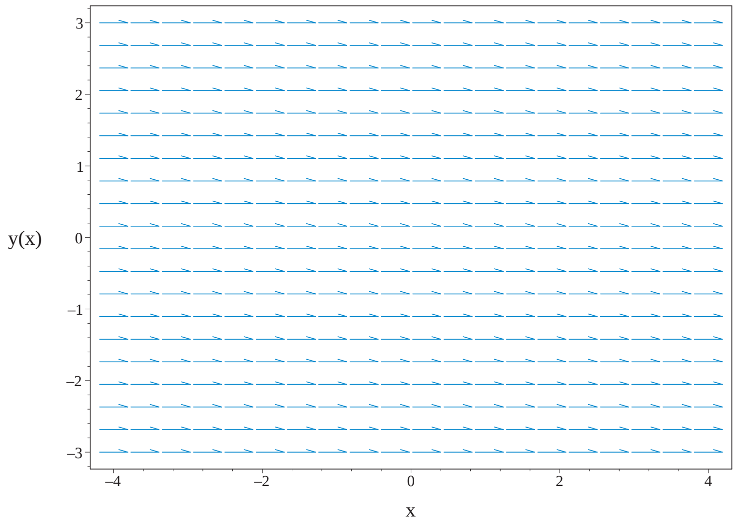


Figure 2.49: Slope field plot
$f(x)\, y' = 0$

Summary of solutions found

$$y = 0$$

$$y = e^{c_1}$$

Solved as first order ode of type differential

Time used: 0.012 (sec)

Writing the ode as

$$y' = 0 \tag{1}$$

Which becomes

$$(1)\, dy = (0)\, dx \tag{2}$$

But the RHS is complete differential because

$$(0)\, dx = d(0)$$

Hence (2) becomes

$$(1)\, dy = d(0)$$

Integrating gives

$$y = c_1$$

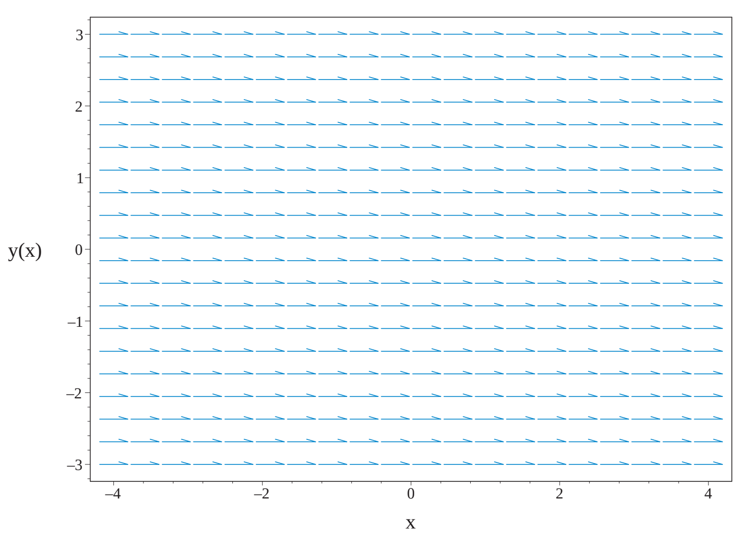


Figure 2.50: Slope field plot
$f(x)\, y' = 0$

Summary of solutions found

$$y = c_1$$

Maple step by step solution

Let's solve

$f(x) \left(\frac{d}{dx}y(x)\right) = 0$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = 0$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}y(x)\right) dx = \int 0 dx + C1$

- Evaluate integral

  $y(x) = C1$

- Solve for $y(x)$

  $y(x) = C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 5

```
dsolve(f(x)*diff(y(x),x) = 0,
       y(x),singsol=all)
```

$$y = c_1$$

Mathematica DSolve solution

Solving time : 0.002 (sec)
Leaf size : 7

```
DSolve[{f[x]*D[y[x],x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1$$

2.1.37 problem 37

Internal problem ID [8697]
Book : First order enumerated odes
Section : section 1
Problem number : 37
Date solved : `Tuesday, December 17, 2024 at 12:58:05 PM`
CAS classification : `[_quadrature]`

Solve

$$xy' = 1$$

Solved as first order quadrature ode

Time used: 0.036 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int \frac{1}{x}\, dx$$
$$y = \ln(x) + c_1$$

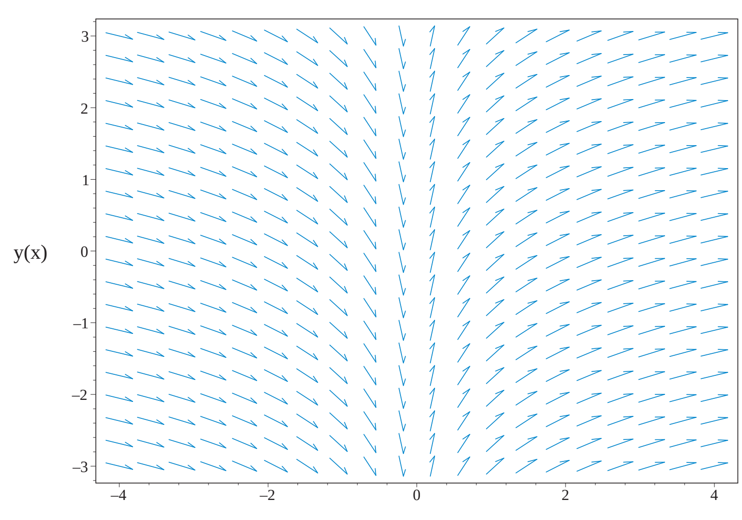


Figure 2.51: Slope field plot
$xy' = 1$

Summary of solutions found

$$y = \ln(x) + c_1$$

Solved as first order Exact ode

Time used: 0.075 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y &= \mathrm{d}x\\ -\,\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -1\\ N(x,y) &= x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-1)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x}((0) - (1)) \\ &= -\frac{1}{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\frac{1}{x} \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\ln(x)} \\ &= \frac{1}{x} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{x}(-1) \\ &= -\frac{1}{x} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{x}(x) \\ &= 1 \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-\frac{1}{x}\right) + (1)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned} \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{x} \, \mathrm{d}x \end{aligned}$$

$$\phi = -\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 1$. Therefore equation (4) becomes

$$1 = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 1$$

Integrating the above w.r.t y gives

$$\int f'(y)\, \mathrm{d}y = \int (1)\, \mathrm{d}y$$
$$f(y) = y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -\ln(x) + y$$

Solving for y gives

$$y = \ln(x) + c_1$$

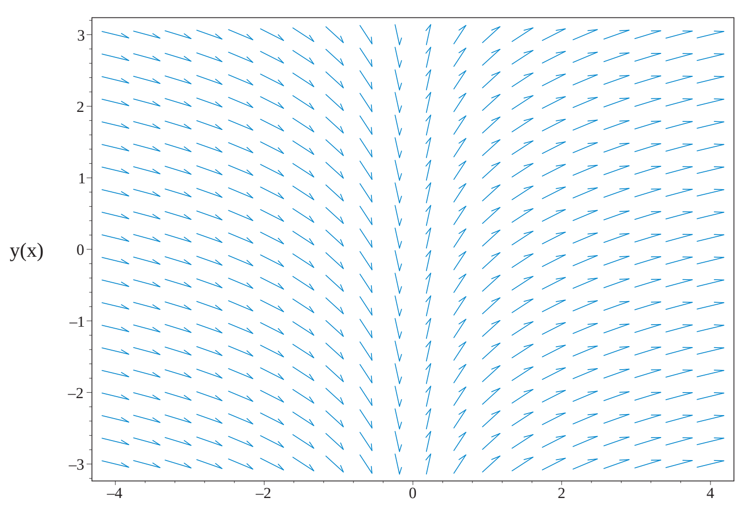


Figure 2.52: Slope field plot
$xy' = 1$

Summary of solutions found

$$y = \ln(x) + c_1$$

Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y(x)\right) = 1$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$
- Solve for the highest derivative

  $\frac{d}{dx}y(x) = \frac{1}{x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}y(x)\right) dx = \int \frac{1}{x}dx + C1$
- Evaluate integral

  $y(x) = \ln(x) + C1$
- Solve for $y(x)$

  $y(x) = \ln(x) + C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : 0.000 (sec)
Leaf size : 8

```
dsolve(diff(y(x),x)*x = 1,
       y(x),singsol=all)
```

$$y = \ln(x) + c_1$$

Mathematica DSolve solution

Solving time : 0.003 (sec)
Leaf size : 10

```
DSolve[{x*D[y[x],x]==1,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \log(x) + c_1$$

2.1.38 problem 38

Internal problem ID [8698]
Book : First order enumerated odes
Section : section 1
Problem number : 38
Date solved : `Tuesday, December 17, 2024 at 12:58:05 PM`
CAS classification : `[_quadrature]`

Solve

$$xy' = \sin(x)$$

Solved as first order quadrature ode

Time used: 0.086 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int \frac{\sin(x)}{x} dx$$
$$y = \mathrm{Si}(x) + c_1$$

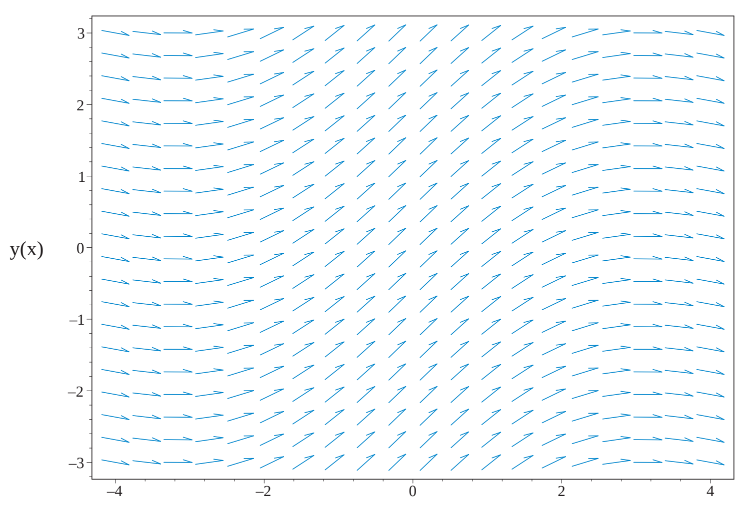


Figure 2.53: Slope field plot
$xy' = \sin(x)$

Summary of solutions found

$$y = \mathrm{Si}(x) + c_1$$

Solved as first order Exact ode

Time used: 0.085 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y&=(\sin(x))\,\mathrm{d}x\\ (-\sin(x))\,\mathrm{d}x+(x)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\sin(x)\\ N(x,y)&=x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-\sin(x))\\ &=0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(x)\\ &=1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x}((0) - (1))\\ &= -\frac{1}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x}\\ &= e^{\int -\frac{1}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\ln(x)}\\ &= \frac{1}{x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x}(-\sin(x))\\ &= -\frac{\sin(x)}{x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x}(x)\\ &= 1\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(-\frac{\sin(x)}{x}\right) + (1)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x\\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{\sin(x)}{x} \, \mathrm{d}x\end{aligned}$$

$$\phi = -\operatorname{Si}(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 1$. Therefore equation (4) becomes

$$1 = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 1$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (1) \, \mathrm{d}y$$
$$f(y) = y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\operatorname{Si}(x) + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -\operatorname{Si}(x) + y$$

Solving for y gives

$$y = \operatorname{Si}(x) + c_1$$

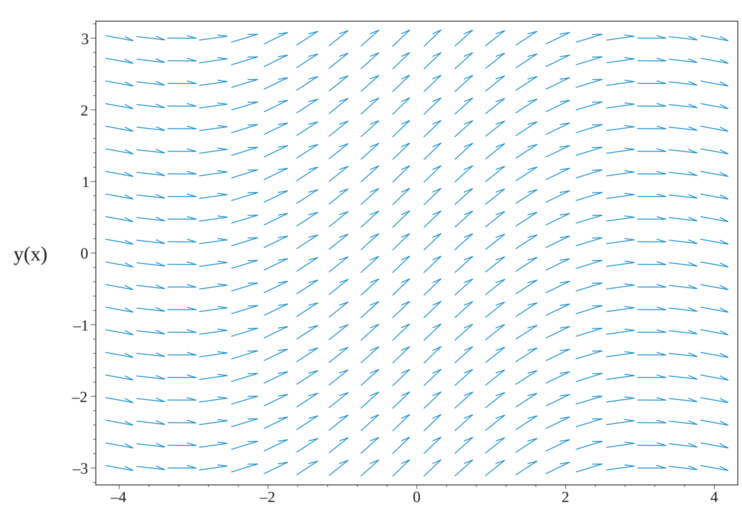


Figure 2.54: Slope field plot
$xy' = \sin(x)$

Summary of solutions found

$$y = \operatorname{Si}(x) + c_1$$

Maple step by step solution

Let's solve

$x\left(\frac{d}{dx}y(x)\right) = \sin(x)$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$
- Solve for the highest derivative

  $\frac{d}{dx}y(x) = \frac{\sin(x)}{x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}y(x)\right) dx = \int \frac{\sin(x)}{x} dx + C1$
- Evaluate integral

  $y(x) = \mathrm{Si}(x) + C1$
- Solve for $y(x)$

  $y(x) = \mathrm{Si}(x) + C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 8

```
dsolve(diff(y(x),x)*x = sin(x),
       y(x),singsol=all)
```

$$y = \mathrm{Si}(x) + c_1$$

Mathematica DSolve solution

Solving time : 0.006 (sec)
Leaf size : 10

```
DSolve[{x*D[y[x],x]==Sin[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \mathrm{Si}(x) + c_1$$

2.1.39 problem 39

Internal problem ID [8699]
Book : First order enumerated odes
Section : section 1
Problem number : 39
Date solved : `Tuesday, December 17, 2024 at 12:58:06 PM`
CAS classification : `[_quadrature]`

Solve

$$(x-1)\,y' = 0$$

Solved as first order quadrature ode

Time used: 0.024 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\,dx + c_1$$
$$y = c_1$$

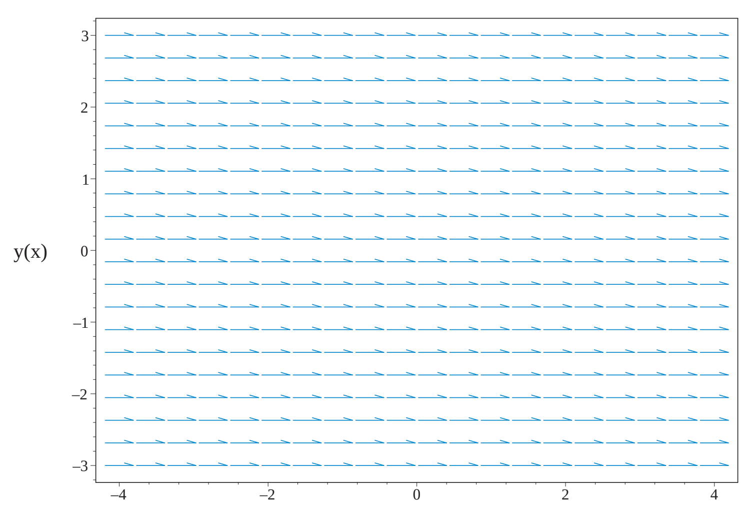


Figure 2.55: Slope field plot
$(x-1)\,y' = 0$

Summary of solutions found

$$y = c_1$$

Solved as first order homogeneous class D2 ode

Time used: 0.130 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$(x-1)\left(u'(x)\,x+u(x)\right)=0$$

Which is now solved The ode $u'(x) = -\frac{u(x)}{x}$ is separable as it can be written as

$$\begin{aligned}
u'(x) &= -\frac{u(x)}{x} \\
&= f(x)g(u)
\end{aligned}$$

Where

$$\begin{aligned}
f(x) &= -\frac{1}{x} \\
g(u) &= u
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\
\ln\left(u(x)\right) &= \ln\left(\frac{1}{x}\right)+c_1
\end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned}
\ln\left(u(x)\right) &= \ln\left(\frac{1}{x}\right)+c_1 \\
u(x) &= 0
\end{aligned}$$

Solving for $u(x)$ gives

$$\begin{aligned}
u(x) &= 0 \\
u(x) &= \frac{e^{c_1}}{x}
\end{aligned}$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = \frac{e^{c_1}}{x}$ back to y gives

$$y = e^{c_1}$$

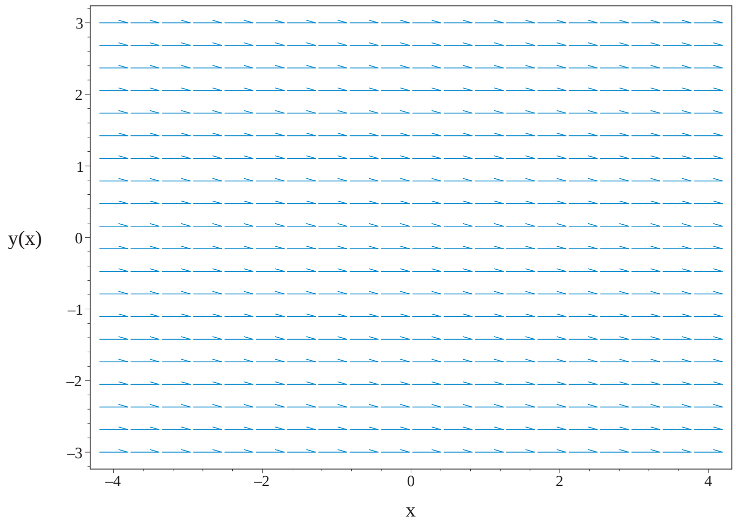


Figure 2.56: Slope field plot
$(x-1)\,y'=0$

Summary of solutions found

$$y = 0$$
$$y = e^{c_1}$$

Solved as first order ode of type differential

Time used: 0.010 (sec)

Writing the ode as

$$y' = 0 \tag{1}$$

Which becomes

$$(1)\,dy = (0)\,dx \tag{2}$$

But the RHS is complete differential because

$$(0)\,dx = d(0)$$

Hence (2) becomes

$$(1)\,dy = d(0)$$

Integrating gives

$$y = c_1$$

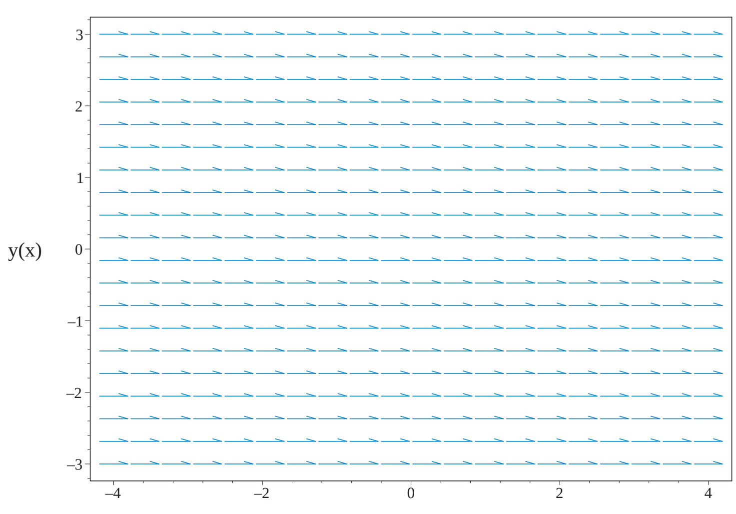


Figure 2.57: Slope field plot
$(x-1)\,y'=0$

Summary of solutions found

$$y = c_1$$

Maple step by step solution

Let's solve

$(x-1)\left(\frac{d}{dx}y(x)\right)=0$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x)=0$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}y(x)\right)dx=\int 0dx+C1$

- Evaluate integral

  $y(x)=C1$

- Solve for $y(x)$

  $y(x)=C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.001 (sec)
Leaf size : 5

```
dsolve((x-1)*diff(y(x),x) = 0,
       y(x),singsol=all)
```

$$y=c_1$$

Mathematica DSolve solution

Solving time : 0.002 (sec)
Leaf size : 7

```
DSolve[{(x-1)*D[y[x],x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to c_1$$

2.1.40 problem 40

Internal problem ID [8700]
Book : First order enumerated odes
Section : section 1
Problem number : 40
Date solved : `Tuesday, December 17, 2024 at 12:58:07 PM`
CAS classification : `[_quadrature]`

Solve

$$yy' = 0$$

Factoring the ode gives these factors

$$y = 0 \tag{1}$$

$$y' = 0 \tag{2}$$

Now each of the above equations is solved in turn.

Solving equation (1)

Solving for y from

$$y = 0$$

Solving gives $y = 0$

Solving equation (2)

Solved as first order quadrature ode

Time used: 0.013 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\, dx + c_1$$

$$y = c_1$$

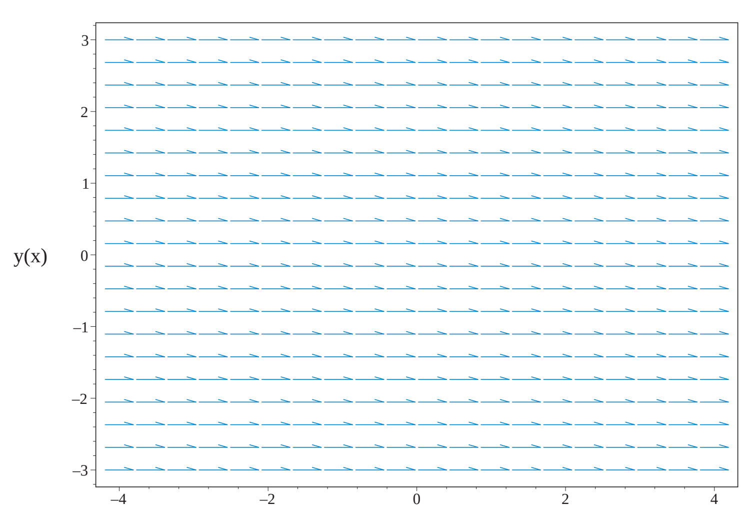


Figure 2.58: Slope field plot
$y' = 0$

Summary of solutions found

$$y = c_1$$

Solved as first order homogeneous class D2 ode

Time used: 0.154 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$u'(x)\,x + u(x) = 0$$

Which is now solved The ode $u'(x) = -\frac{u(x)}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(u) &= u \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\ \ln(u(x)) &= \ln\left(\frac{1}{x}\right) + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} \ln(u(x)) &= \ln\left(\frac{1}{x}\right) + c_1 \\ u(x) &= 0 \end{aligned}$$

Solving for $u(x)$ gives

$$\begin{aligned} u(x) &= 0 \\ u(x) &= \frac{e^{c_1}}{x} \end{aligned}$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = \frac{e^{c_1}}{x}$ back to y gives

$$y = e^{c_1}$$

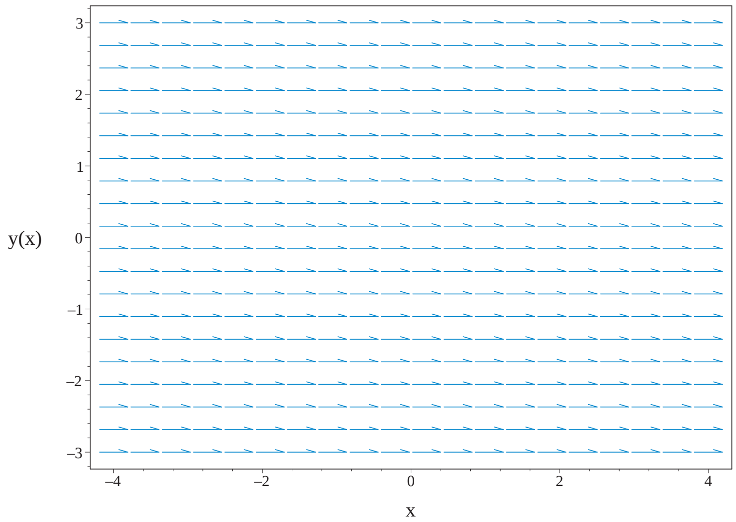


Figure 2.59: Slope field plot
$y' = 0$

Summary of solutions found

$$y = 0$$

$$y = e^{c_1}$$

Solved as first order ode of type differential

Time used: 0.009 (sec)

Writing the ode as

$$y' = 0 \tag{1}$$

Which becomes

$$(1)\, dy = (0)\, dx \tag{2}$$

But the RHS is complete differential because

$$(0)\, dx = d(0)$$

Hence (2) becomes

$$(1)\, dy = d(0)$$

Integrating gives

$$y = c_1$$

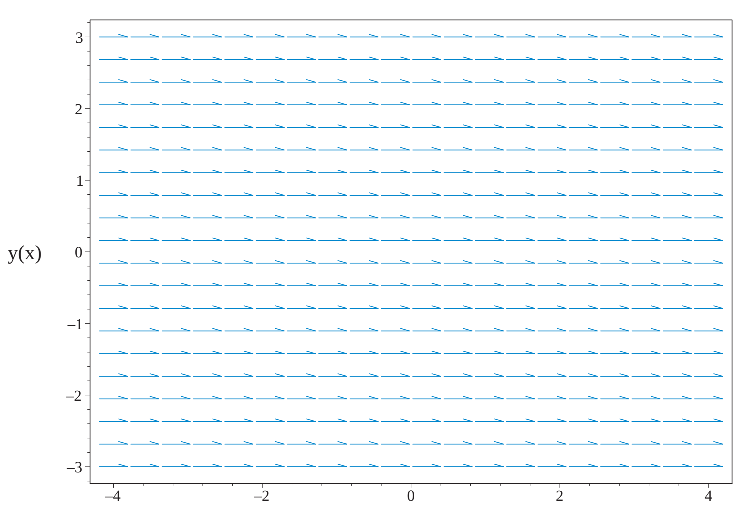


Figure 2.60: Slope field plot
$y' = 0$

Summary of solutions found

$$y = c_1$$

Maple step by step solution

Let's solve

$y(x)\left(\frac{d}{dx}y(x)\right)=0$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Separate variables

  $\frac{d}{dx}y(x)=0$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}y(x)\right)dx=\int 0dx+C1$

- Evaluate integral

  $y(x)=C1$

- Solve for $y(x)$

  $y(x)=C1$

Maple trace

```
`Classification methods on request
Methods to be used are: [exact]
----------------------------
* Tackling ODE using method: exact
--- Trying classification methods ---
trying exact
<- exact successful`
```

Maple dsolve solution

Solving time : 0.003 (sec)
Leaf size : 11

```
dsolve(y(x)*diff(y(x),x) = 0,
       y(x),singsol=all)
```

$$y=0$$
$$y=-c_1$$

Mathematica DSolve solution

Solving time : 0.002 (sec)
Leaf size : 12

```
DSolve[{y[x]*D[y[x],x]==0,{}},
      y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to 0$$
$$y(x)\to c_1$$

2.1.41 problem 41

Internal problem ID [8701]
Book : First order enumerated odes
Section : section 1
Problem number : 41
Date solved : `Tuesday, December 17, 2024 at 12:58:07 PM`
CAS classification : `[_quadrature]`

Solve

$$xyy' = 0$$

Factoring the ode gives these factors

$$y = 0 \tag{1}$$

$$y' = 0 \tag{2}$$

Now each of the above equations is solved in turn.

Solving equation (1)

Solving for y from

$$y = 0$$

Solving gives $y = 0$

Solving equation (2)

Solved as first order quadrature ode

Time used: 0.013 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\, dx + c_1$$

$$y = c_1$$

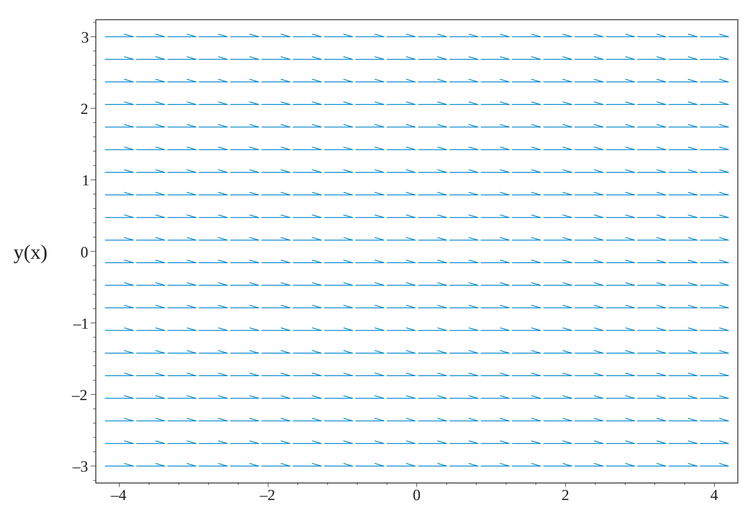


Figure 2.61: Slope field plot
$y' = 0$

Summary of solutions found

$$y = c_1$$

Solved as first order homogeneous class D2 ode

Time used: 0.152 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$u'(x)\,x + u(x) = 0$$

Which is now solved The ode $u'(x) = -\frac{u(x)}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(u) &= u \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\ \ln(u(x)) &= \ln\left(\frac{1}{x}\right) + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} \ln(u(x)) &= \ln\left(\frac{1}{x}\right) + c_1 \\ u(x) &= 0 \end{aligned}$$

Solving for $u(x)$ gives

$$\begin{aligned} u(x) &= 0 \\ u(x) &= \frac{e^{c_1}}{x} \end{aligned}$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = \frac{e^{c_1}}{x}$ back to y gives

$$y = e^{c_1}$$

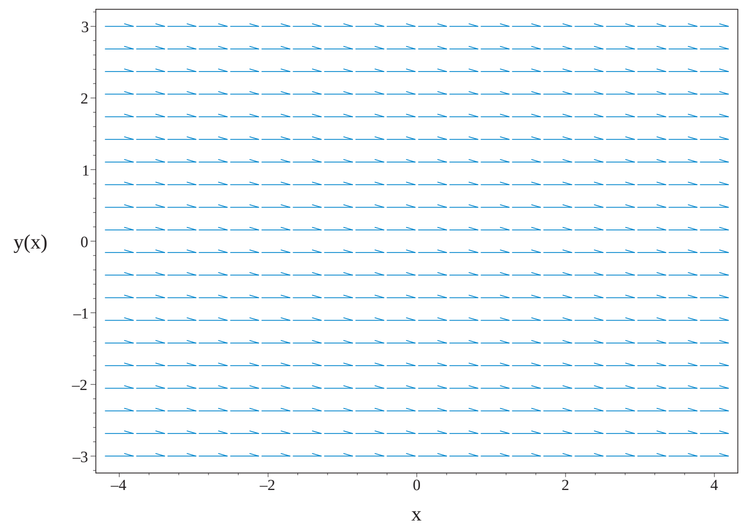


Figure 2.62: Slope field plot
$y' = 0$

Summary of solutions found

$$y = 0$$

$$y = e^{c_1}$$

Solved as first order ode of type differential

Time used: 0.009 (sec)

Writing the ode as

$$y' = 0 \tag{1}$$

Which becomes

$$(1)\,dy = (0)\,dx \tag{2}$$

But the RHS is complete differential because

$$(0)\,dx = d(0)$$

Hence (2) becomes

$$(1)\,dy = d(0)$$

Integrating gives

$$y = c_1$$

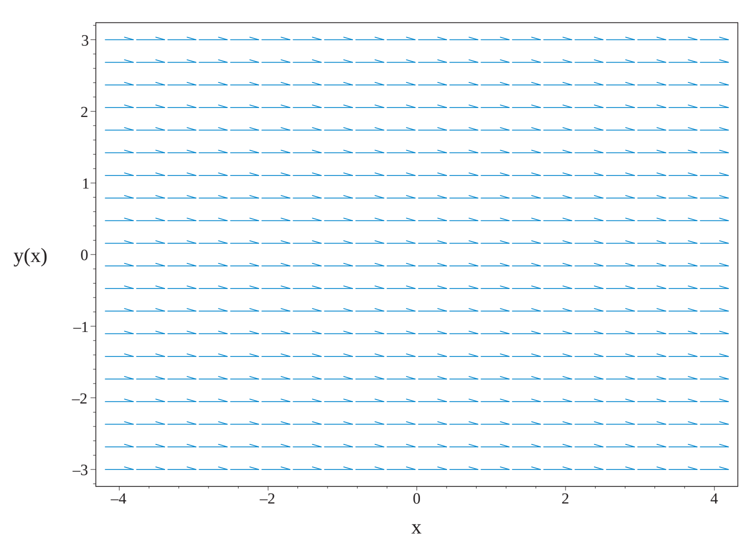


Figure 2.63: Slope field plot
$y' = 0$

Summary of solutions found

$$y = c_1$$

Maple step by step solution

Let's solve

$xy(x)\left(\frac{d}{dx}y(x)\right) = 0$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = 0$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}y(x)\right) dx = \int 0dx + C1$

- Evaluate integral

  $y(x) = C1$

- Solve for $y(x)$

  $y(x) = C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.002 (sec)
Leaf size : 9

```
dsolve(x*y(x)*diff(y(x),x) = 0,
       y(x),singsol=all)
```

$$y = 0$$
$$y = c_1$$

Mathematica DSolve solution

Solving time : 0.002 (sec)
Leaf size : 12

```
DSolve[{x*y[x]*D[y[x],x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to 0$$
$$y(x) \to c_1$$

2.1.42 problem 42

Internal problem ID [8702]
Book : First order enumerated odes
Section : section 1
Problem number : 42
Date solved : `Tuesday, December 17, 2024 at 12:58:08 PM`
CAS classification : `[_quadrature]`

Solve

$$xy\sin(x)\,y' = 0$$

Factoring the ode gives these factors

$$y = 0 \tag{1}$$

$$y' = 0 \tag{2}$$

Now each of the above equations is solved in turn.

Solving equation (1)

Solving for y from

$$y = 0$$

Solving gives $y = 0$

Solving equation (2)

Solved as first order quadrature ode

Time used: 0.013 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\,dx + c_1$$

$$y = c_1$$

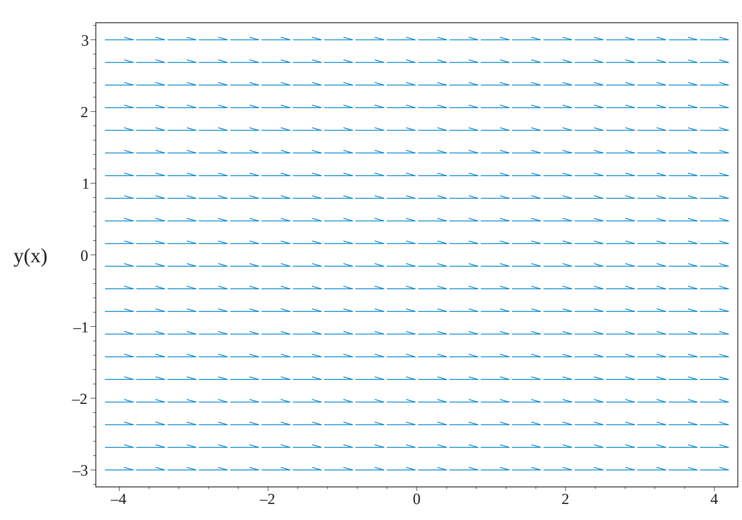


Figure 2.64: Slope field plot
$y' = 0$

Summary of solutions found

$$y = c_1$$

Solved as first order homogeneous class D2 ode

Time used: 0.153 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$u'(x)\,x + u(x) = 0$$

Which is now solved The ode $u'(x) = -\frac{u(x)}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(u) &= u \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\ \ln(u(x)) &= \ln\left(\frac{1}{x}\right) + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} \ln(u(x)) &= \ln\left(\frac{1}{x}\right) + c_1 \\ u(x) &= 0 \end{aligned}$$

Solving for $u(x)$ gives

$$\begin{aligned} u(x) &= 0 \\ u(x) &= \frac{e^{c_1}}{x} \end{aligned}$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = \frac{e^{c_1}}{x}$ back to y gives

$$y = e^{c_1}$$

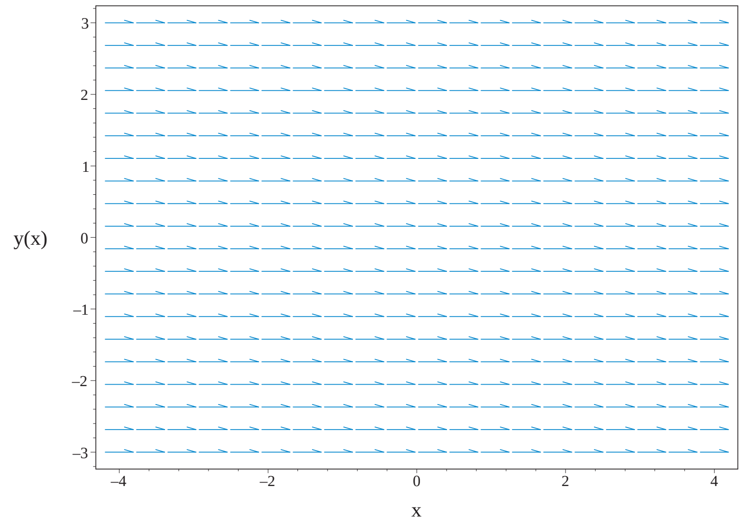


Figure 2.65: Slope field plot

$y' = 0$

Summary of solutions found

$$y = 0$$

$$y = e^{c_1}$$

Solved as first order ode of type differential

Time used: 0.009 (sec)

Writing the ode as

$$y' = 0 \tag{1}$$

Which becomes

$$(1)\, dy = (0)\, dx \tag{2}$$

But the RHS is complete differential because

$$(0)\, dx = d(0)$$

Hence (2) becomes

$$(1)\, dy = d(0)$$

Integrating gives

$$y = c_1$$

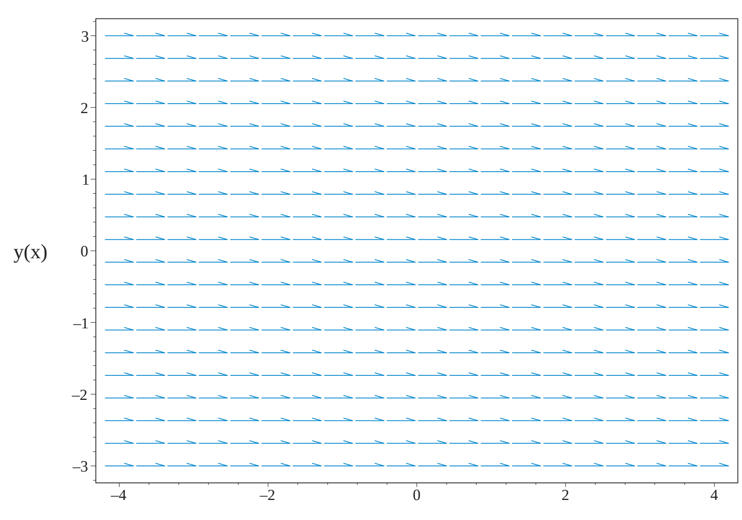


Figure 2.66: Slope field plot

$y' = 0$

Summary of solutions found

$$y = c_1$$

Maple step by step solution

Let's solve

$xy(x)\sin(x)\left(\frac{d}{dx}y(x)\right)=0$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x)=0$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}y(x)\right)dx=\int 0dx+C1$

- Evaluate integral

  $y(x)=C1$

- Solve for $y(x)$

  $y(x)=C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : `0.003` (sec)
Leaf size : 9

```
dsolve(x*y(x)*sin(x)*diff(y(x),x) = 0,
       y(x),singsol=all)
```

$$y=0$$
$$y=c_1$$

Mathematica DSolve solution

Solving time : `0.003` (sec)
Leaf size : 12

```
DSolve[{x*y[x]*Sin[x]*D[y[x],x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to 0$$
$$y(x)\to c_1$$

2.1.43 problem 43

Internal problem ID [8703]
Book : First order enumerated odes
Section : section 1
Problem number : 43
Date solved : `Tuesday, December 17, 2024 at 12:58:09 PM`
CAS classification : `[_quadrature]`

Solve

$$\pi y \sin(x) y' = 0$$

Factoring the ode gives these factors

$$y = 0 \tag{1}$$

$$y' = 0 \tag{2}$$

Now each of the above equations is solved in turn.

Solving equation (1)

Solving for y from

$$y = 0$$

Solving gives $y = 0$

Solving equation (2)

Solved as first order quadrature ode

Time used: 0.014 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\, dx + c_1$$

$$y = c_1$$

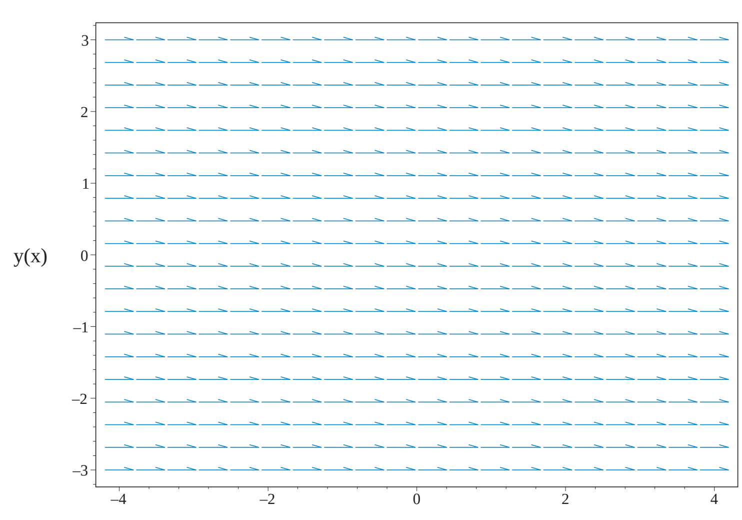


Figure 2.67: Slope field plot
$y' = 0$

Summary of solutions found

$$y = c_1$$

Solved as first order homogeneous class D2 ode

Time used: 0.153 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$u'(x)\,x + u(x) = 0$$

Which is now solved The ode $u'(x) = -\frac{u(x)}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(u) &= u \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\ \ln\left(u(x)\right) &= \ln\left(\frac{1}{x}\right) + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} \ln\left(u(x)\right) &= \ln\left(\frac{1}{x}\right) + c_1 \\ u(x) &= 0 \end{aligned}$$

Solving for $u(x)$ gives

$$u(x) = 0$$

$$u(x) = \frac{e^{c_1}}{x}$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = \frac{e^{c_1}}{x}$ back to y gives

$$y = e^{c_1}$$

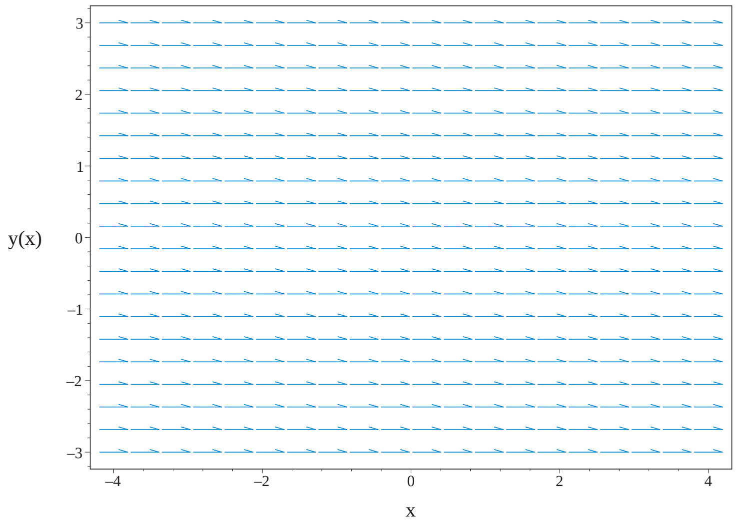


Figure 2.68: Slope field plot
$y' = 0$

Summary of solutions found

$$y = 0$$

$$y = e^{c_1}$$

Solved as first order ode of type differential

Time used: 0.010 (sec)

Writing the ode as

$$y' = 0 \tag{1}$$

Which becomes

$$(1)\, dy = (0)\, dx \tag{2}$$

But the RHS is complete differential because

$$(0)\, dx = d(0)$$

Hence (2) becomes

$$(1)\, dy = d(0)$$

Integrating gives

$$y = c_1$$

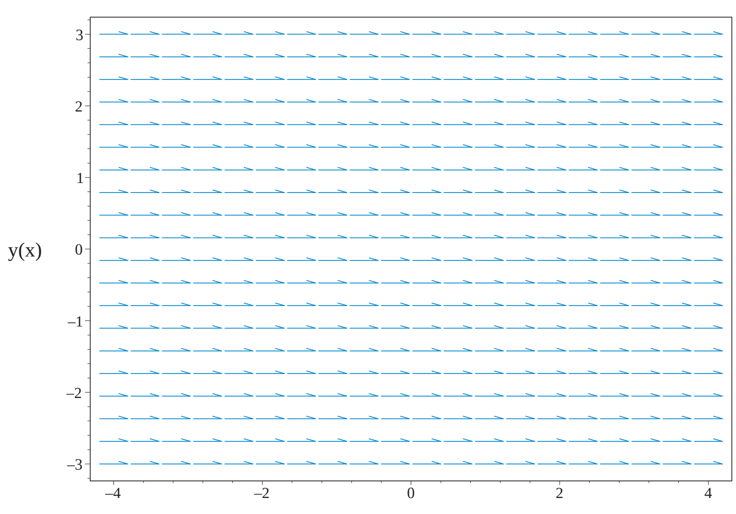


Figure 2.69: Slope field plot
$y' = 0$

Summary of solutions found

$$y = c_1$$

Maple step by step solution

Let's solve

$\pi y(x) \sin(x) \left(\frac{d}{dx}y(x)\right) = 0$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$
- Solve for the highest derivative

  $\frac{d}{dx}y(x) = 0$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}y(x)\right) dx = \int 0dx + C1$
- Evaluate integral

  $y(x) = C1$
- Solve for $y(x)$

  $y(x) = C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : `0.002` (sec)
Leaf size : 9

```
dsolve(Pi*y(x)*sin(x)*diff(y(x),x) = 0,
       y(x),singsol=all)
```

$$y = 0$$
$$y = c_1$$

Mathematica DSolve solution

Solving time : `0.003` (sec)
Leaf size : 12

```
DSolve[{Pi*y[x]*Sin[x]*D[y[x],x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to 0$$
$$y(x) \to c_1$$

2.1.44 problem 44

Internal problem ID [8704]
Book : First order enumerated odes
Section : section 1
Problem number : 44
Date solved : Tuesday, December 17, 2024 at 12:58:10 PM
CAS classification : [_quadrature]

Solve

$$x \sin(x) y' = 0$$

Solved as first order quadrature ode

Time used: 0.026 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\,dx + c_1$$
$$y = c_1$$

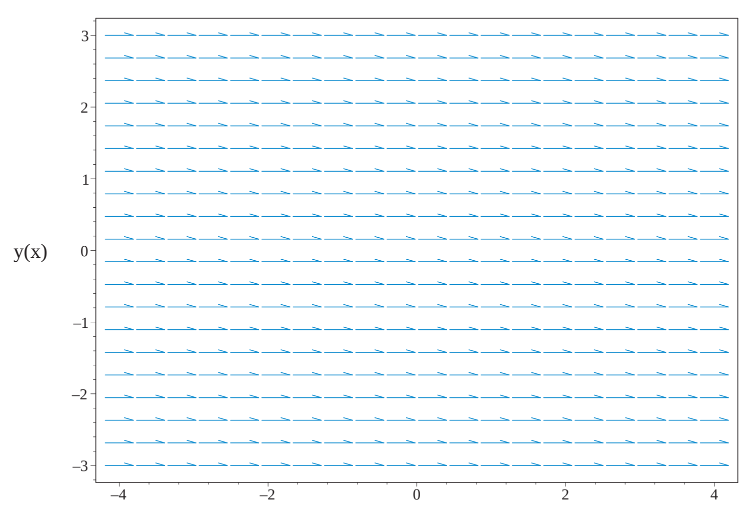


Figure 2.70: Slope field plot
$x \sin(x) y' = 0$

Summary of solutions found

$$y = c_1$$

Solved as first order homogeneous class D2 ode

Time used: 0.158 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$x \sin(x) \left(u'(x)\,x + u(x)\right) = 0$$

Which is now solved The ode $u'(x) = -\frac{u(x)}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(u) &= u \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\ \ln(u(x)) &= \ln\left(\frac{1}{x}\right) + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} \ln(u(x)) &= \ln\left(\frac{1}{x}\right) + c_1 \\ u(x) &= 0 \end{aligned}$$

Solving for $u(x)$ gives

$$\begin{aligned} u(x) &= 0 \\ u(x) &= \frac{e^{c_1}}{x} \end{aligned}$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = \frac{e^{c_1}}{x}$ back to y gives

$$y = e^{c_1}$$

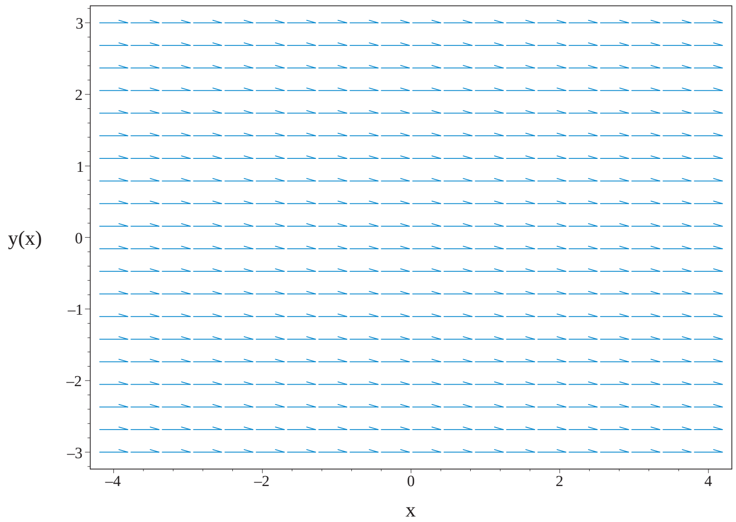


Figure 2.71: Slope field plot
$x\sin(x)\, y' = 0$

Summary of solutions found

$$y = 0$$
$$y = e^{c_1}$$

Solved as first order ode of type differential

Time used: 0.012 (sec)

Writing the ode as

$$y' = 0 \tag{1}$$

Which becomes

$$(1)\, dy = (0)\, dx \tag{2}$$

But the RHS is complete differential because

$$(0)\, dx = d(0)$$

Hence (2) becomes

$$(1)\, dy = d(0)$$

Integrating gives

$$y = c_1$$

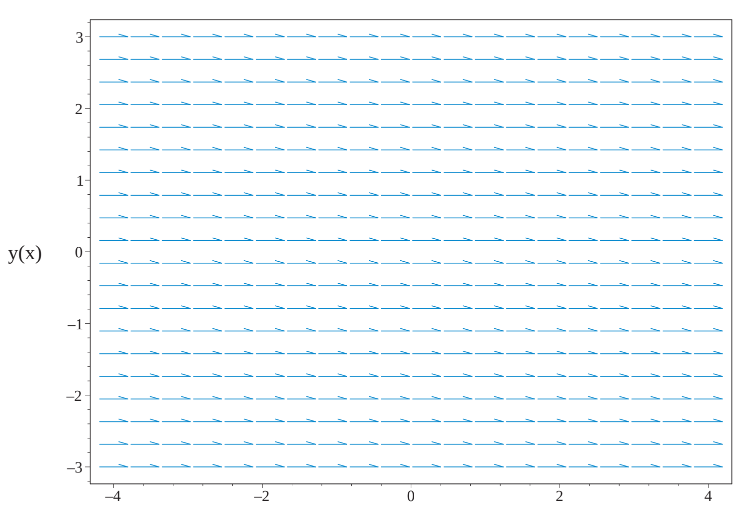


Figure 2.72: Slope field plot
$x\sin(x)\, y' = 0$

Summary of solutions found

$$y = c_1$$

Maple step by step solution

Let's solve

$x \sin(x) \left(\frac{d}{dx}y(x)\right) = 0$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = 0$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}y(x)\right) dx = \int 0 dx + C1$

- Evaluate integral

  $y(x) = C1$

- Solve for $y(x)$

  $y(x) = C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : `0.001` (sec)
Leaf size : 5

```
dsolve(x*sin(x)*diff(y(x),x) = 0,
       y(x),singsol=all)
```

$$y = c_1$$

Mathematica DSolve solution

Solving time : `0.002` (sec)
Leaf size : 7

```
DSolve[{x*Sin[x]*D[y[x],x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1$$

2.1.45 problem 45

Internal problem ID [8705]
Book : First order enumerated odes
Section : section 1
Problem number : 45
Date solved : `Tuesday, December 17, 2024 at 12:58:10 PM`
CAS classification : `[_quadrature]`

Solve

$$x \sin(x) {y'}^2 = 0$$

Solving for the derivative gives these ODE's to solve

$$y' = 0 \tag{1}$$
$$y' = 0 \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\, dx + c_1$$
$$y = c_1$$

Solving Eq. (2)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\, dx + c_2$$
$$y = c_2$$

Maple step by step solution

Let's solve
$x \sin(x) \left(\frac{d}{dx}y(x)\right)^2 = 0$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = 0$
- Integrate both sides with respect to x
  $\int \left(\frac{d}{dx}y(x)\right) dx = \int 0 dx + C1$
- Evaluate integral
  $y(x) = C1$
- Solve for $y(x)$
  $y(x) = C1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

Maple dsolve solution

Solving time : 0.029 (sec)
Leaf size : 5

```
dsolve(x*sin(x)*diff(y(x),x)^2 = 0,
       y(x),singsol=all)
```

$$y = c_1$$

Mathematica DSolve solution

Solving time : 0.003 (sec)
Leaf size : 7

```
DSolve[{x*Sin[x]*D[y[x],x]^2==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to c_1$$

2.1.46 problem 46

Internal problem ID [8706]
Book : First order enumerated odes
Section : section 1
Problem number : 46
Date solved : `Tuesday, December 17, 2024 at 12:58:11 PM`
CAS classification : `[_quadrature]`

Solve

$$yy'^2 = 0$$

Factoring the ode gives these factors

$$y = 0 \tag{1}$$

$$y'^2 = 0 \tag{2}$$

Now each of the above equations is solved in turn.

Solving equation (1)

Solving for y from

$$y = 0$$

Solving gives $y = 0$

Solving equation (2)

Solving for the derivative gives these ODE's to solve

$$y' = 0 \tag{1}$$

$$y' = 0 \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\, dx + c_1$$

$$y = c_1$$

Solving Eq. (2)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\, dx + c_2$$

$$y = c_2$$

Maple step by step solution

Let's solve

$y(x)\left(\frac{d}{dx}y(x)\right)^2=0$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x)=0$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}y(x)\right)dx=\int 0dx+C1$

- Evaluate integral

  $y(x)=C1$

- Solve for $y(x)$

  $y(x)=C1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : `0.002` (sec)
Leaf size : 9

```
dsolve(y(x)*diff(y(x),x)^2 = 0,
       y(x),singsol=all)
```

$$y=0$$
$$y=c_1$$

Mathematica DSolve solution

Solving time : `0.002` (sec)
Leaf size : 12

```
DSolve[{y[x]*(D[y[x],x])^2==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\rightarrow 0$$
$$y(x)\rightarrow c_1$$

2.1.47 problem 47

Internal problem ID [8707]
Book : First order enumerated odes
Section : section 1
Problem number : 47
Date solved : `Tuesday, December 17, 2024 at 12:58:11 PM`
CAS classification : `[_quadrature]`

Solve

$$y'^n = 0$$

Solved as first order quadrature ode

Time used: 0.043 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\,dx + c_1$$
$$y = c_1$$

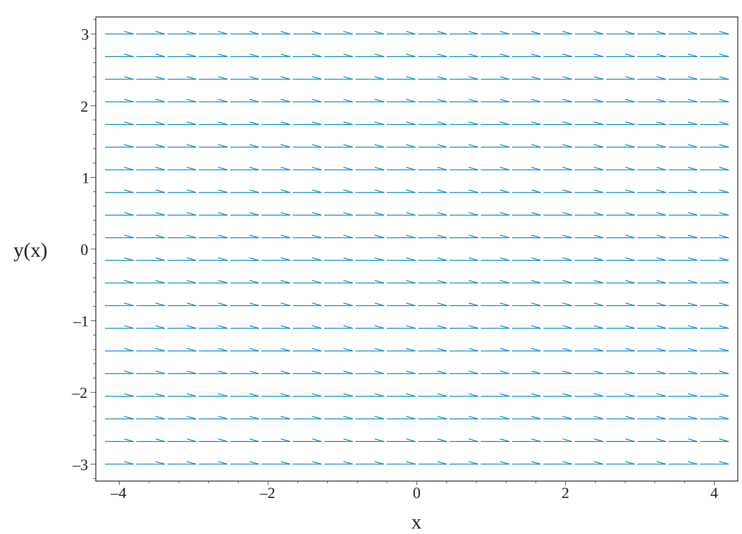


Figure 2.73: Slope field plot
$y'^n = 0$

Summary of solutions found

$$y = c_1$$

Solved as first order homogeneous class D2 ode

Time used: 0.228 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$(u'(x)\,x + u(x))^n = 0$$

Which is now solved The ode $u'(x) = -\frac{u(x)}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(u) &= u \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\ \ln(u(x)) &= \ln\left(\frac{1}{x}\right) + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Solving for $u(x)$ gives

$$u(x) = \frac{e^{c_1}}{x}$$

Converting $u(x) = \frac{e^{c_1}}{x}$ back to y gives

$$y = e^{c_1}$$

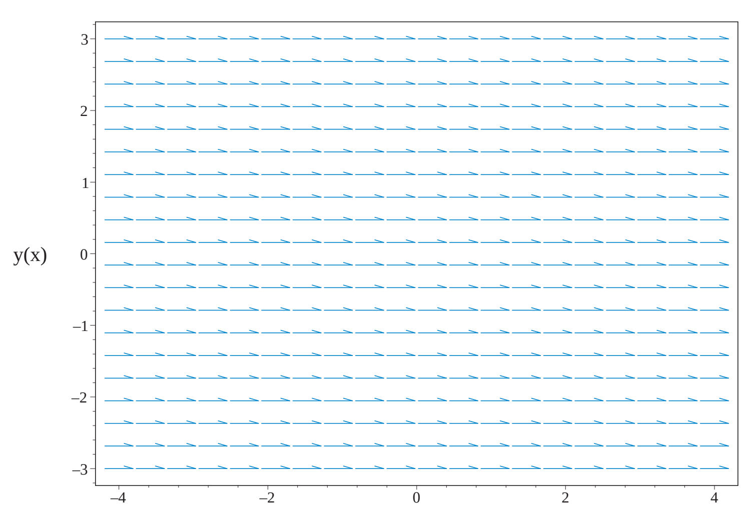


Figure 2.74: Slope field plot
$y'^n = 0$

<u>Summary of solutions found</u>

$$y = e^{c_1}$$

Maple step by step solution

Let's solve
$\left(\frac{d}{dx}y(x)\right)^{n}=0$

- Highest derivative means the order of the ODE is 1
$\frac{d}{dx}y(x)$
- Solve for the highest derivative
$\frac{d}{dx}y(x)=0$
- Integrate both sides with respect to x
$\int \left(\frac{d}{dx}y(x)\right)dx=\int 0dx+C1$
- Evaluate integral
$y(x)=C1$
- Solve for $y(x)$
$y(x)=C1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 5

```
dsolve(diff(y(x),x)^n = 0,
       y(x),singsol=all)
```

$$y=c_1$$

Mathematica DSolve solution

Solving time : 0.003 (sec)
Leaf size : 15

```
DSolve[{(D[y[x],x])^n==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to 0^{\frac{1}{n}}x+c_1$$

2.1.48 problem 48

Internal problem ID [8708]
Book : First order enumerated odes
Section : section 1
Problem number : 48
Date solved : `Tuesday, December 17, 2024 at 12:58:12 PM`
CAS classification : `[_quadrature]`

Solve

$$xy'^n = 0$$

Solved as first order quadrature ode

Time used: 0.026 (sec)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int 0\,dx + c_1$$
$$y = c_1$$

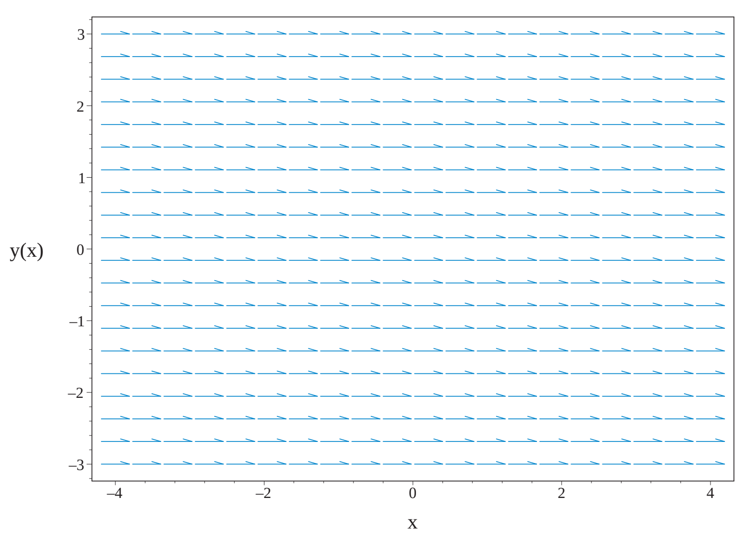


Figure 2.75: Slope field plot
$xy'^n = 0$

Summary of solutions found

$$y = c_1$$

Solved as first order homogeneous class D2 ode

Time used: 0.155 (sec)

Applying change of variables $y = u(x)\,x$, then the ode becomes

$$x(u'(x)\,x + u(x))^n = 0$$

Which is now solved The ode $u'(x) = -\frac{u(x)}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= -\frac{u(x)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{1}{x} \\ g(u) &= u \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\,du &= \int f(x)\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\ \ln(u(x)) &= \ln\left(\frac{1}{x}\right) + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} \ln(u(x)) &= \ln\left(\frac{1}{x}\right) + c_1 \\ u(x) &= 0 \end{aligned}$$

Solving for $u(x)$ gives

$$\begin{aligned} u(x) &= 0 \\ u(x) &= \frac{e^{c_1}}{x} \end{aligned}$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = \frac{e^{c_1}}{x}$ back to y gives

$$y = e^{c_1}$$

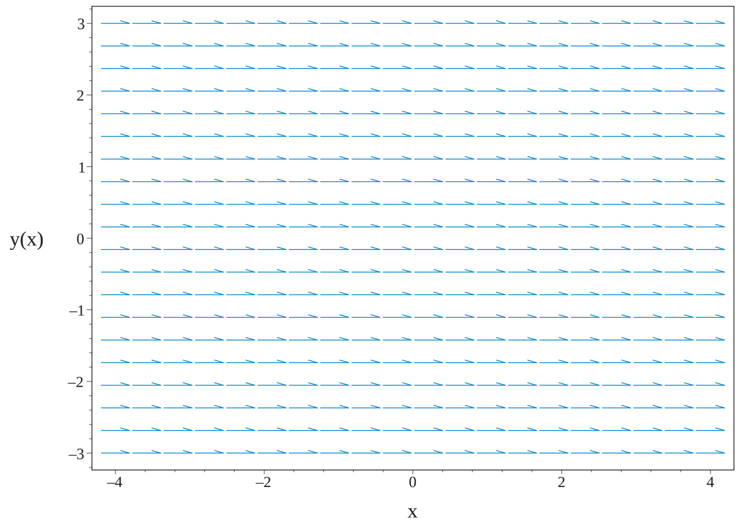


Figure 2.76: Slope field plot
$xy'^n = 0$

Summary of solutions found

$$y = 0$$

$$y = e^{c_1}$$

Maple step by step solution

Let's solve
$x\left(\frac{d}{dx}y(x)\right)^n = 0$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = 0$
- Integrate both sides with respect to x
  $\int \left(\frac{d}{dx}y(x)\right) dx = \int 0dx + C1$
- Evaluate integral
  $y(x) = C1$
- Solve for $y(x)$
  $y(x) = C1$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

Maple dsolve solution

Solving time : 0.005 (sec)
Leaf size : 5

```
dsolve(x*diff(y(x),x)^n = 0,
       y(x),singsol=all)
```

$$y = c_1$$

Mathematica DSolve solution

Solving time : 0.003 (sec)
Leaf size : 15

```
DSolve[{x*(D[y[x],x])^n==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to 0^{\frac{1}{n}} x + c_1$$

2.1.49 problem 49

Internal problem ID [8709]
Book : First order enumerated odes
Section : section 1
Problem number : 49
Date solved : `Tuesday, December 17, 2024 at 12:58:13 PM`
CAS classification : `[_quadrature]`

Solve

$$y'^2 = x$$

Solving for the derivative gives these ODE's to solve

$$y' = \sqrt{x} \tag{1}$$
$$y' = -\sqrt{x} \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int \sqrt{x}\, dx$$
$$y = \frac{2x^{3/2}}{3} + c_1$$

Solving Eq. (2)

Since the ode has the form $y' = f(x)$, then we only need to integrate $f(x)$.

$$\int dy = \int -\sqrt{x}\, dx$$
$$y = -\frac{2x^{3/2}}{3} + c_2$$

Maple step by step solution

Let's solve
$\left(\frac{d}{dx}y(x)\right)^2 = x$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\left[\frac{d}{dx}y(x) = \sqrt{x}, \frac{d}{dx}y(x) = -\sqrt{x}\right]$

☐ Solve the equation $\frac{d}{dx}y(x) = \sqrt{x}$
  ◦ Integrate both sides with respect to x
    $\int \left(\frac{d}{dx}y(x)\right) dx = \int \sqrt{x} dx + \_C1$
  ◦ Evaluate integral
    $y(x) = \frac{2x^{3/2}}{3} + \_C1$

- Solve for $y(x)$

  $y(x) = \frac{2x^{3/2}}{3} + \_C1$

☐ Solve the equation $\frac{d}{dx}y(x) = -\sqrt{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}y(x)\right) dx = \int -\sqrt{x} dx + \_C1$

- Evaluate integral

  $y(x) = -\frac{2x^{3/2}}{3} + \_C1$

- Solve for $y(x)$

  $y(x) = -\frac{2x^{3/2}}{3} + \_C1$

- Set of solutions

  $\left\{y(x) = -\frac{2x^{3/2}}{3} + C1, y(x) = \frac{2x^{3/2}}{3} + C1\right\}$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  y(x)  successful`
```

Maple dsolve solution

Solving time : 0.040 (sec)
Leaf size : 21

```
dsolve(diff(y(x),x)^2 = x,
       y(x),singsol=all)
```

$$y = \frac{2x^{3/2}}{3} + c_1$$
$$y = -\frac{2x^{3/2}}{3} + c_1$$

Mathematica DSolve solution

Solving time : 0.004 (sec)
Leaf size : 33

```
DSolve[{(D[y[x],x])^2==x,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{2x^{3/2}}{3} + c_1$$
$$y(x) \to \frac{2x^{3/2}}{3} + c_1$$

2.1.50 problem 50

Internal problem ID [8710]
Book : First order enumerated odes
Section : section 1
Problem number : 50
Date solved : `Tuesday, December 17, 2024 at 12:58:14 PM`
CAS classification : `[[_homogeneous, 'class C'], _dAlembert]`

Solve

$$y'^2 = x + y$$

Solved as first order ode of type dAlembert

Time used: 0.226 (sec)

Let $p = y'$ the ode becomes

$$p^2 = x + y$$

Solving for y from the above results in

$$y = p^2 - x \tag{1}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = -1$$
$$g = p^2$$

Hence (2) becomes

$$p + 1 = 2pp'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p + 1 = 0$$

Solving the above for p results in

$$p_1 = -1$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$y = 1 - x$$

The general solution is found when $\frac{\mathrm{dp}}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) + 1}{2p(x)} \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed. Integrating gives

$$\int \frac{2p}{p+1} dp = dx$$
$$2p - 2\ln(p+1) = x + c_1$$

Singular solutions are found by solving

$$\frac{p+1}{2p} = 0$$

for $p(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$p(x) = -1$$

Solving for $p(x)$ gives

$$p(x) = -1$$
$$p(x) = -\operatorname{LambertW}\left(-\mathrm{e}^{-1-\frac{x}{2}-\frac{c_1}{2}}\right) - 1$$

Substituing the above solution for p in (2A) gives

$$y = 1 - x$$
$$y = \left(-\operatorname{LambertW}\left(-\mathrm{e}^{-1-\frac{x}{2}-\frac{c_1}{2}}\right) - 1\right)^2 - x$$

Summary of solutions found

$$y = 1 - x$$
$$y = \left(-\operatorname{LambertW}\left(-\mathrm{e}^{-1-\frac{x}{2}-\frac{c_1}{2}}\right) - 1\right)^2 - x$$

Maple step by step solution

Let's solve
$\left(\frac{d}{dx}y(x)\right)^2 = x + y(x)$
- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\left[\frac{d}{dx}y(x) = \sqrt{x + y(x)}, \frac{d}{dx}y(x) = -\sqrt{x + y(x)}\right]$
- Solve the equation $\frac{d}{dx}y(x) = \sqrt{x + y(x)}$
- Solve the equation $\frac{d}{dx}y(x) = -\sqrt{x + y(x)}$
- Set of solutions

$$\{workingODE, workingODE\}$$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying dAlembert
<- dAlembert successful`
```

Maple dsolve solution

Solving time : 0.048 (sec)
Leaf size : 33

```
dsolve(diff(y(x),x)^2 = x+y(x),
       y(x),singsol=all)
```

$$y = \operatorname{LambertW}\left(-c_1 \mathrm{e}^{-\frac{x}{2}-1}\right)^2 + 2\operatorname{LambertW}\left(-c_1 \mathrm{e}^{-\frac{x}{2}-1}\right) - x + 1$$

Mathematica DSolve solution

Solving time : 14.92 (sec)
Leaf size : 100

```
DSolve[{(D[y[x],x])^2==x+y[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to W\left(-e^{-\frac{x}{2}-1-\frac{c_1}{2}}\right)^2 + 2W\left(-e^{-\frac{x}{2}-1-\frac{c_1}{2}}\right) - x + 1$$
$$y(x) \to W\left(e^{\frac{1}{2}(-x-2+c_1)}\right)^2 + 2W\left(e^{\frac{1}{2}(-x-2+c_1)}\right) - x + 1$$
$$y(x) \to 1 - x$$

2.1.51 problem 51

Internal problem ID [8711]
Book : First order enumerated odes
Section : section 1
Problem number : 51
Date solved : `Tuesday, December 17, 2024 at 12:58:14 PM`
CAS classification : `[[_homogeneous, 'class A'], _rational, _dAlembert]`

Solve

$$y'^2 = \frac{y}{x}$$

Solved as first order homogeneous class A ode

Time used: 0.832 (sec)

Solving for y' gives

$$y' = \frac{\sqrt{xy}}{x} \tag{1}$$
$$y' = -\frac{\sqrt{xy}}{x} \tag{2}$$

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{\sqrt{xy}}{x} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = \sqrt{xy}$ and $N = x$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\frac{\mathrm{d}u}{\mathrm{d}x}x + u = \sqrt{u}$$
$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\sqrt{u(x)} - u(x)}{x}$$

Or

$$u'(x) - \frac{\sqrt{u(x)} - u(x)}{x} = 0$$

Or

$$u'(x)\,x - \sqrt{u(x)} + u(x) = 0$$

Which is now solved as separable in $u(x)$.

The ode $u'(x) = \frac{\sqrt{u(x)}-u(x)}{x}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= \frac{\sqrt{u(x)}-u(x)}{x} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{1}{x} \\ g(u) &= \sqrt{u}-u \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(u)}\, du &= \int f(x)\, dx \\ \int \frac{1}{\sqrt{u}-u}\, du &= \int \frac{1}{x}\, dx \\ -2\ln\left(\sqrt{u(x)}-1\right) &= \ln(x)+c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $\sqrt{u}-u = 0$ for $u(x)$ gives

$$\begin{aligned} u(x) &= 0 \\ u(x) &= 1 \end{aligned}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} -2\ln\left(\sqrt{u(x)}-1\right) &= \ln(x)+c_1 \\ u(x) &= 0 \\ u(x) &= 1 \end{aligned}$$

Converting $-2\ln\left(\sqrt{u(x)}-1\right) = \ln(x)+c_1$ back to y gives

$$-2\ln\left(\sqrt{\frac{y}{x}}-1\right) = \ln(x)+c_1$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Converting $u(x) = 1$ back to y gives

$$y = x$$

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{\sqrt{xy}}{x} \end{aligned} \tag{1}$$

An ode of the form $y' = \frac{M(x,y)}{N(x,y)}$ is called homogeneous if the functions $M(x,y)$ and $N(x,y)$ are both homogeneous functions and of the same order. Recall that a function $f(x,y)$ is homogeneous of order n if

$$f(t^n x, t^n y) = t^n f(x,y)$$

In this case, it can be seen that both $M = -\sqrt{xy}$ and $N = x$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{y}{x}$, or $y = ux$. Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}x + u$$

Applying the transformation $y = ux$ to the above ODE in (1) gives

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}x}x + u &= -\sqrt{u}\\ \frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{-\sqrt{u(x)} - u(x)}{x}\end{aligned}$$

Or

$$u'(x) - \frac{-\sqrt{u(x)} - u(x)}{x} = 0$$

Or

$$u'(x)\,x + \sqrt{u(x)} + u(x) = 0$$

Which is now solved as separable in $u(x)$.

The ode $u'(x) = -\frac{\sqrt{u(x)}+u(x)}{x}$ is separable as it can be written as

$$\begin{aligned}u'(x) &= -\frac{\sqrt{u(x)} + u(x)}{x}\\ &= f(x)g(u)\end{aligned}$$

Where

$$\begin{aligned}f(x) &= \frac{1}{x}\\ g(u) &= -\sqrt{u} - u\end{aligned}$$

Integrating gives

$$\begin{aligned}\int \frac{1}{g(u)}\,du &= \int f(x)\,dx\\ \int \frac{1}{-\sqrt{u} - u}\,du &= \int \frac{1}{x}\,dx\\ \ln\left(\frac{1}{\left(\sqrt{u(x)}+1\right)^2}\right) &= \ln(x) + c_2\end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $-\sqrt{u} - u = 0$ for $u(x)$ gives

$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned}\ln\left(\frac{1}{\left(\sqrt{u(x)}+1\right)^2}\right) &= \ln(x) + c_2\\ u(x) &= 0\end{aligned}$$

Converting $\ln\left(\frac{1}{\left(\sqrt{u(x)}+1\right)^2}\right) = \ln(x) + c_2$ back to y gives

$$\ln\left(\frac{1}{\left(\sqrt{\frac{y}{x}}+1\right)^2}\right) = \ln(x) + c_2$$

Converting $u(x) = 0$ back to y gives

$$y = 0$$

Solving for y gives

$$\begin{aligned}
y &= 0 \\
y &= x \\
y &= \left(\frac{2x\,\mathrm{e}^{c_1}\left(\sqrt{x\,\mathrm{e}^{c_1}}-1\right)}{\sqrt{x\,\mathrm{e}^{c_1}}} - x\,\mathrm{e}^{c_1} + 1\right)\mathrm{e}^{-c_1} \\
y &= \left(\frac{2x\,\mathrm{e}^{c_1}\left(\sqrt{x\,\mathrm{e}^{c_1}}+1\right)}{\sqrt{x\,\mathrm{e}^{c_1}}} - x\,\mathrm{e}^{c_1} + 1\right)\mathrm{e}^{-c_1} \\
y &= -\left(-\frac{2x\,\mathrm{e}^{c_2}\left(\sqrt{x\,\mathrm{e}^{c_2}}-1\right)}{\sqrt{x\,\mathrm{e}^{c_2}}} + x\,\mathrm{e}^{c_2} - 1\right)\mathrm{e}^{-c_2} \\
y &= -\left(-\frac{2x\,\mathrm{e}^{c_2}\left(\sqrt{x\,\mathrm{e}^{c_2}}+1\right)}{\sqrt{x\,\mathrm{e}^{c_2}}} + x\,\mathrm{e}^{c_2} - 1\right)\mathrm{e}^{-c_2}
\end{aligned}$$

Summary of solutions found

$$\begin{aligned}
y &= 0 \\
y &= x \\
y &= \left(\frac{2x\,\mathrm{e}^{c_1}\left(\sqrt{x\,\mathrm{e}^{c_1}}-1\right)}{\sqrt{x\,\mathrm{e}^{c_1}}} - x\,\mathrm{e}^{c_1} + 1\right)\mathrm{e}^{-c_1} \\
y &= \left(\frac{2x\,\mathrm{e}^{c_1}\left(\sqrt{x\,\mathrm{e}^{c_1}}+1\right)}{\sqrt{x\,\mathrm{e}^{c_1}}} - x\,\mathrm{e}^{c_1} + 1\right)\mathrm{e}^{-c_1} \\
y &= -\left(-\frac{2x\,\mathrm{e}^{c_2}\left(\sqrt{x\,\mathrm{e}^{c_2}}-1\right)}{\sqrt{x\,\mathrm{e}^{c_2}}} + x\,\mathrm{e}^{c_2} - 1\right)\mathrm{e}^{-c_2} \\
y &= -\left(-\frac{2x\,\mathrm{e}^{c_2}\left(\sqrt{x\,\mathrm{e}^{c_2}}+1\right)}{\sqrt{x\,\mathrm{e}^{c_2}}} + x\,\mathrm{e}^{c_2} - 1\right)\mathrm{e}^{-c_2}
\end{aligned}$$

Solved as first order ode of type nonlinear p but separable

Time used: 0.242 (sec)

The ode has the form

$$(y')^{\frac{n}{m}} = f(x)g(y) \tag{1}$$

Where $n = 2, m = 1, f = \frac{1}{x}, g = y$. Hence the ode is

$$(y')^2 = \frac{y}{x}$$

Solving for y' from (1) gives

$$
\begin{aligned}
y' &= \sqrt{fg} \\
y' &= -\sqrt{fg}
\end{aligned}
$$

To be able to solve as separable ode, we have to now assume that $f > 0, g > 0$.

$$
\begin{aligned}
\frac{1}{x} &> 0 \\
y &> 0
\end{aligned}
$$

Under the above assumption the differential equations become separable and can be written as

$$
\begin{aligned}
y' &= \sqrt{f}\,\sqrt{g} \\
y' &= -\sqrt{f}\,\sqrt{g}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{1}{\sqrt{g}}\,dy &= \left(\sqrt{f}\right)\,dx \\
-\frac{1}{\sqrt{g}}\,dy &= \left(\sqrt{f}\right)\,dx
\end{aligned}
$$

Replacing $f(x), g(y)$ by their values gives

$$
\begin{aligned}
\frac{1}{\sqrt{y}}\,dy &= \left(\sqrt{\frac{1}{x}}\right)\,dx \\
-\frac{1}{\sqrt{y}}\,dy &= \left(\sqrt{\frac{1}{x}}\right)\,dx
\end{aligned}
$$

Integrating now gives the following solutions

$$
\begin{aligned}
\int \frac{1}{\sqrt{y}}dy &= \int \sqrt{\frac{1}{x}}dx + c_1 \\
2\sqrt{y} &= 2x\sqrt{\frac{1}{x}} \\
\int -\frac{1}{\sqrt{y}}dy &= \int \sqrt{\frac{1}{x}}dx + c_1 \\
-2\sqrt{y} &= 2x\sqrt{\frac{1}{x}}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
y &= x\sqrt{\frac{1}{x}}\,c_1 + \frac{c_1^2}{4} + x \\
y &= x\sqrt{\frac{1}{x}}\,c_1 + \frac{c_1^2}{4} + x
\end{aligned}
$$

Summary of solutions found

$$
y = x\sqrt{\frac{1}{x}}\,c_1 + \frac{c_1^2}{4} + x
$$

Solved as first order ode of type dAlembert

Time used: 0.071 (sec)

Let $p = y'$ the ode becomes

$$p^2 = \frac{y}{x}$$

Solving for y from the above results in

$$y = p^2 x \tag{1}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= p^2 \\ g &= 0 \end{aligned}$$

Hence (2) becomes

$$-p^2 + p = 2xpp'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$-p^2 + p = 0$$

Solving the above for p results in

$$\begin{aligned} p_1 &= 0 \\ p_2 &= 1 \end{aligned}$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$\begin{aligned} y &= 0 \\ y &= x \end{aligned}$$

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{-p(x)^2 + p(x)}{2xp(x)} \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed. In canonical form a linear first order is

$$p'(x) + q(x)p(x) = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= \frac{1}{2x} \\ p(x) &= \frac{1}{2x} \end{aligned}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dx}\\ &= \mathrm{e}^{\int \frac{1}{2x}dx}\\ &= \sqrt{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu p) &= \mu p\\ \frac{\mathrm{d}}{\mathrm{d}x}(\mu p) &= (\mu)\left(\frac{1}{2x}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(p\sqrt{x}\right) &= \left(\sqrt{x}\right)\left(\frac{1}{2x}\right)\\ \mathrm{d}\left(p\sqrt{x}\right) &= \left(\frac{1}{2\sqrt{x}}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}p\sqrt{x} &= \int \frac{1}{2\sqrt{x}}\,dx\\ &= \sqrt{x} + c_1\end{aligned}$$

Dividing throughout by the integrating factor $\sqrt{x}$ gives the final solution

$$p(x) = \frac{\sqrt{x} + c_1}{\sqrt{x}}$$

Substituing the above solution for p in (2A) gives

$$y = \left(\sqrt{x} + c_1\right)^2$$

<u>Summary of solutions found</u>

$$\begin{aligned}y &= 0\\ y &= x\\ y &= \left(\sqrt{x} + c_1\right)^2\end{aligned}$$

Maple step by step solution

Let's solve
$\left(\frac{d}{dx}y(x)\right)^2 = \frac{y(x)}{x}$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\left[\frac{d}{dx}y(x) = \frac{\sqrt{xy(x)}}{x}, \frac{d}{dx}y(x) = -\frac{\sqrt{xy(x)}}{x}\right]$
- Solve the equation $\frac{d}{dx}y(x) = \frac{\sqrt{xy(x)}}{x}$
- Solve the equation $\frac{d}{dx}y(x) = -\frac{\sqrt{xy(x)}}{x}$
- Set of solutions
  $\{workingODE, workingODE\}$

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

Maple dsolve solution

Solving time : 0.048 (sec)
Leaf size : 39

```
dsolve(diff(y(x),x)^2 = y(x)/x,
       y(x),singsol=all)
```

$$
\begin{aligned}
y &= 0 \\
y &= \frac{\left(x+\sqrt{c_1 x}\right)^2}{x} \\
y &= \frac{\left(-x+\sqrt{c_1 x}\right)^2}{x}
\end{aligned}
$$

Mathematica DSolve solution

Solving time : 0.047 (sec)
Leaf size : 46

```
DSolve[{(D[y[x],x])^2==y[x]/x,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
y(x) &\to \frac{1}{4}\left(-2\sqrt{x}+c_1\right)^2 \\
y(x) &\to \frac{1}{4}\left(2\sqrt{x}+c_1\right)^2 \\
y(x) &\to 0
\end{aligned}
$$

2.1.52 problem 52

Internal problem ID [8712]
Book : First order enumerated odes
Section : section 1
Problem number : 52
Date solved : `Tuesday, December 17, 2024 at 12:58:16 PM`
CAS classification : `[_separable]`

Solve

$$y'^2 = \frac{y^2}{x}$$

Solving for the derivative gives these ODE's to solve

$$y' = \frac{y}{\sqrt{x}} \tag{1}$$

$$y' = -\frac{y}{\sqrt{x}} \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$q(x) = -\frac{1}{\sqrt{x}}$$
$$p(x) = 0$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,dx} \\ &= e^{\int -\frac{1}{\sqrt{x}}dx} \\ &= e^{-2\sqrt{x}}\end{aligned}$$

The ode becomes

$$\frac{d}{dx}\mu y = 0$$
$$\frac{d}{dx}\left(y\,e^{-2\sqrt{x}}\right) = 0$$

Integrating gives

$$\begin{aligned}y\,e^{-2\sqrt{x}} &= \int 0\,dx + c_2 \\ &= c_2\end{aligned}$$

Dividing throughout by the integrating factor $e^{-2\sqrt{x}}$ gives the final solution

$$y = e^{2\sqrt{x}}c_2$$

We now need to find the singular solutions, these are found by finding for what values $\left(\frac{y}{\sqrt{x}}\right)$ is zero. These give

$$y = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

The solution $y = 0$ satisfies the ode and initial conditions.

Solving Eq. (2)

In canonical form a linear first order is

$$y' + q(x)y = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= \frac{1}{\sqrt{x}} \\ p(x) &= 0 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= \mathrm{e}^{\int \frac{1}{\sqrt{x}}dx} \\ &= \mathrm{e}^{2\sqrt{x}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(y\,\mathrm{e}^{2\sqrt{x}}\right) &= 0 \end{aligned}$$

Integrating gives

$$\begin{aligned} y\,\mathrm{e}^{2\sqrt{x}} &= \int 0\,dx + c_3 \\ &= c_3 \end{aligned}$$

Dividing throughout by the integrating factor $\mathrm{e}^{2\sqrt{x}}$ gives the final solution

$$y = \mathrm{e}^{-2\sqrt{x}}c_3$$

We now need to find the singular solutions, these are found by finding for what values $\left(-\frac{y}{\sqrt{x}}\right)$ is zero. These give

$$y = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

The solution $y = 0$ satisfies the ode and initial conditions.

Maple step by step solution

Let's solve

$\left(\frac{d}{dx}y(x)\right)^2 = \frac{y(x)^2}{x}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\left[\frac{d}{dx}y(x) = \frac{y(x)}{\sqrt{x}}, \frac{d}{dx}y(x) = -\frac{y(x)}{\sqrt{x}}\right]$

☐ Solve the equation $\frac{d}{dx}y(x) = \frac{y(x)}{\sqrt{x}}$

- Separate variables

  $\frac{\frac{d}{dx}y(x)}{y(x)} = \frac{1}{\sqrt{x}}$

- Integrate both sides with respect to x

  $\int \frac{\frac{d}{dx}y(x)}{y(x)}dx = \int \frac{1}{\sqrt{x}}dx + \_C1$

- Evaluate integral

  $\ln(y(x)) = 2\sqrt{x} + \_C1$

- Solve for $y(x)$

  $y(x) = e^{2\sqrt{x}+\_C1}$

☐ Solve the equation $\frac{d}{dx}y(x) = -\frac{y(x)}{\sqrt{x}}$

- Separate variables

  $\frac{\frac{d}{dx}y(x)}{y(x)} = -\frac{1}{\sqrt{x}}$

- Integrate both sides with respect to x

  $\int \frac{\frac{d}{dx}y(x)}{y(x)}dx = \int -\frac{1}{\sqrt{x}}dx + \_C1$

- Evaluate integral

  $\ln(y(x)) = -2\sqrt{x} + \_C1$

- Solve for $y(x)$

  $y(x) = e^{-2\sqrt{x}+\_C1}$

- Set of solutions

  $\left\{y(x) = e^{-2\sqrt{x}+C1}, y(x) = e^{2\sqrt{x}+C1}\right\}$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying simple symmetries for implicit equations
<- symmetries for implicit equations successful`
```

Maple dsolve solution

Solving time : 0.061 (sec)
Leaf size : 27

```
dsolve(diff(y(x),x)^2 = y(x)^2/x,
       y(x),singsol=all)
```

$$
\begin{aligned}
y &= 0 \\
y &= c_1 \mathrm{e}^{-2\sqrt{x}} \\
y &= c_1 \mathrm{e}^{2\sqrt{x}}
\end{aligned}
$$

Mathematica DSolve solution

Solving time : 0.067 (sec)
Leaf size : 38

```
DSolve[{(D[y[x],x])^2==y[x]^2/x,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
y(x) &\to c_1 e^{-2\sqrt{x}} \\
y(x) &\to c_1 e^{2\sqrt{x}} \\
y(x) &\to 0
\end{aligned}
$$

2.1.53 problem 53

Internal problem ID [8713]
Book : First order enumerated odes
Section : section 1
Problem number : 53
Date solved : `Tuesday, December 17, 2024 at 12:58:18 PM`
CAS classification : `[[_homogeneous, 'class G']]`

Solve

$$y'^2 = \frac{y^3}{x}$$

Solved as first order ode of type nonlinear p but separable

Time used: 0.425 (sec)

The ode has the form

$$(y')^{\frac{n}{m}} = f(x)g(y) \tag{1}$$

Where $n = 2, m = 1, f = \frac{1}{x}, g = y^3$. Hence the ode is

$$(y')^2 = \frac{y^3}{x}$$

Solving for y' from (1) gives

$$y' = \sqrt{fg}$$
$$y' = -\sqrt{fg}$$

To be able to solve as separable ode, we have to now assume that $f > 0, g > 0$.

$$\frac{1}{x} > 0$$
$$y^3 > 0$$

Under the above assumption the differential equations become separable and can be written as

$$y' = \sqrt{f}\,\sqrt{g}$$
$$y' = -\sqrt{f}\,\sqrt{g}$$

Therefore

$$\frac{1}{\sqrt{g}}\,dy = \left(\sqrt{f}\right)\,dx$$
$$-\frac{1}{\sqrt{g}}\,dy = \left(\sqrt{f}\right)\,dx$$

Replacing $f(x), g(y)$ by their values gives

$$\frac{1}{\sqrt{y^3}}\,dy = \left(\sqrt{\frac{1}{x}}\right)\,dx$$
$$-\frac{1}{\sqrt{y^3}}\,dy = \left(\sqrt{\frac{1}{x}}\right)\,dx$$

Integrating now gives the following solutions

$$\int \frac{1}{\sqrt{y^3}}dy = \int \sqrt{\frac{1}{x}}dx + c_1$$
$$-\frac{2\sqrt{y^3}}{y^2} = 2x\sqrt{\frac{1}{x}}$$
$$\int -\frac{1}{\sqrt{y^3}}dy = \int \sqrt{\frac{1}{x}}dx + c_1$$
$$\frac{2\sqrt{y^3}}{y^2} = 2x\sqrt{\frac{1}{x}}$$

Therefore

$$y = \frac{4}{4x\sqrt{\frac{1}{x}}\, c_1 + c_1^2 + 4x}$$
$$y = \frac{4}{4x\sqrt{\frac{1}{x}}\, c_1 + c_1^2 + 4x}$$

Summary of solutions found

$$y = \frac{4}{4x\sqrt{\frac{1}{x}}\, c_1 + c_1^2 + 4x}$$

Maple step by step solution

Let's solve
$\left(\frac{d}{dx}y(x)\right)^2 = \frac{y(x)^3}{x}$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\left[\frac{d}{dx}y(x) = \frac{\sqrt{xy(x)}\, y(x)}{x}, \frac{d}{dx}y(x) = -\frac{\sqrt{xy(x)}\, y(x)}{x}\right]$
- Solve the equation $\frac{d}{dx}y(x) = \frac{\sqrt{xy(x)}\, y(x)}{x}$
- Solve the equation $\frac{d}{dx}y(x) = -\frac{\sqrt{xy(x)}\, y(x)}{x}$
- Set of solutions
  $\{workingODE, workingODE\}$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
<- 1st_order WeierstrassP successful`
```

Maple dsolve solution

Solving time : 0.045 (sec)
Leaf size : 27

```
dsolve(diff(y(x),x)^2 = y(x)^3/x,
       y(x),singsol=all)
```

$$y = 0$$
$$y = \frac{\text{WeierstrassP}(1, 0, 0)\, 2^{2/3}}{\left(\sqrt{x}\, 2^{1/3} + c_1\right)^2}$$

Mathematica DSolve solution

Solving time : `0.071` (sec)
Leaf size : 42

```
DSolve[{(D[y[x],x])^2==y[x]^3/x,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{4}{\left(-2\sqrt{x} + c_1\right)^2}$$
$$y(x) \to \frac{4}{\left(2\sqrt{x} + c_1\right)^2}$$
$$y(x) \to 0$$

2.1.54 problem 54

Internal problem ID [8714]
Book : First order enumerated odes
Section : section 1
Problem number : 54
Date solved : `Tuesday, December 17, 2024 at 12:58:19 PM`
CAS classification : `[[_homogeneous, 'class G'], _rational]`

Solve

$$y'^3 = \frac{y^2}{x}$$

Solved as first order ode of type nonlinear p but separable

Time used: 0.947 (sec)

The ode has the form

$$(y')^{\frac{n}{m}} = f(x)g(y) \tag{1}$$

Where $n = 3, m = 1, f = \frac{1}{x}, g = y^2$. Hence the ode is

$$(y')^3 = \frac{y^2}{x}$$

Solving for y' from (1) gives

$$\begin{aligned} y' &= (fg)^{1/3} \\ y' &= -\frac{(fg)^{1/3}}{2} + \frac{i\sqrt{3}\,(fg)^{1/3}}{2} \\ y' &= -\frac{(fg)^{1/3}}{2} - \frac{i\sqrt{3}\,(fg)^{1/3}}{2} \end{aligned}$$

To be able to solve as separable ode, we have to now assume that $f > 0, g > 0$.

$$\begin{aligned} \frac{1}{x} &> 0 \\ y^2 &> 0 \end{aligned}$$

Under the above assumption the differential equations become separable and can be written as

$$\begin{aligned} y' &= f^{1/3} g^{1/3} \\ y' &= \frac{f^{1/3} g^{1/3} \left(-1 + i\sqrt{3}\right)}{2} \\ y' &= -\frac{f^{1/3} g^{1/3} \left(1 + i\sqrt{3}\right)}{2} \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{g^{1/3}}\, dy &= \left(f^{1/3}\right)\, dx \\ \frac{2}{g^{1/3}\left(-1 + i\sqrt{3}\right)}\, dy &= \left(f^{1/3}\right)\, dx \\ -\frac{2}{g^{1/3}\left(1 + i\sqrt{3}\right)}\, dy &= \left(f^{1/3}\right)\, dx \end{aligned}$$

Replacing $f(x), g(y)$ by their values gives

$$\frac{1}{(y^2)^{1/3}}\, dy = \left(\left(\frac{1}{x}\right)^{1/3}\right) dx$$
$$\frac{2}{(y^2)^{1/3}\left(-1+i\sqrt{3}\right)}\, dy = \left(\left(\frac{1}{x}\right)^{1/3}\right) dx$$
$$-\frac{2}{(y^2)^{1/3}\left(1+i\sqrt{3}\right)}\, dy = \left(\left(\frac{1}{x}\right)^{1/3}\right) dx$$

Integrating now gives the following solutions

$$\int \frac{1}{(y^2)^{1/3}} dy = \int \left(\frac{1}{x}\right)^{1/3} dx + c_1$$
$$\frac{3(y^2)^{2/3}}{y} = \frac{3x\left(\frac{1}{x}\right)^{1/3}}{2}$$
$$\int \frac{2}{(y^2)^{1/3}\left(-1+i\sqrt{3}\right)} dy = \int \left(\frac{1}{x}\right)^{1/3} dx + c_1$$
$$-\frac{3(y^2)^{2/3}\left(1+i\sqrt{3}\right)}{2y} = \frac{3x\left(\frac{1}{x}\right)^{1/3}}{2}$$
$$\int -\frac{2}{(y^2)^{1/3}\left(1+i\sqrt{3}\right)} dy = \int \left(\frac{1}{x}\right)^{1/3} dx + c_1$$
$$\frac{3(y^2)^{2/3}\left(-1+i\sqrt{3}\right)}{2y} = \frac{3x\left(\frac{1}{x}\right)^{1/3}}{2}$$

Therefore

$$\frac{3(y^2)^{2/3}}{y} = \frac{3x\left(\frac{1}{x}\right)^{1/3}}{2} + c_1$$
$$y = \frac{x^2}{8} + \frac{\left(\frac{1}{x}\right)^{2/3} c_1\, x^2}{4} + \frac{\left(\frac{1}{x}\right)^{1/3} c_1^2 x}{6} + \frac{c_1^3}{27}$$
$$y = \frac{x^2}{8} + \frac{\left(\frac{1}{x}\right)^{2/3} c_1\, x^2}{4} + \frac{\left(\frac{1}{x}\right)^{1/3} c_1^2 x}{6} + \frac{c_1^3}{27}$$

Summary of solutions found

$$\frac{3(y^2)^{2/3}}{y} = \frac{3x\left(\frac{1}{x}\right)^{1/3}}{2} + c_1$$
$$y = \frac{x^2}{8} + \frac{\left(\frac{1}{x}\right)^{2/3} c_1\, x^2}{4} + \frac{\left(\frac{1}{x}\right)^{1/3} c_1^2 x}{6} + \frac{c_1^3}{27}$$

Maple step by step solution

Let's solve

$\left(\frac{d}{dx}y(x)\right)^3 = \frac{y(x)^2}{x}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\left[\frac{d}{dx}y(x) = \frac{\left(x^2 y(x)^2\right)^{1/3}}{x}, \frac{d}{dx}y(x) = -\frac{\left(x^2 y(x)^2\right)^{1/3}}{2x} - \frac{I\sqrt{3}\left(x^2 y(x)^2\right)^{1/3}}{2x}, \frac{d}{dx}y(x) = -\frac{\left(x^2 y(x)^2\right)^{1/3}}{2x} + \frac{I\sqrt{3}\left(x^2 y(x)^2\right.}{2x}\right.$

- Solve the equation $\frac{d}{dx}y(x) = \frac{\left(x^2 y(x)^2\right)^{1/3}}{x}$
- Solve the equation $\frac{d}{dx}y(x) = -\frac{\left(x^2 y(x)^2\right)^{1/3}}{2x} - \frac{I\sqrt{3}\left(x^2 y(x)^2\right)^{1/3}}{2x}$
- Solve the equation $\frac{d}{dx}y(x) = -\frac{\left(x^2 y(x)^2\right)^{1/3}}{2x} + \frac{I\sqrt{3}\left(x^2 y(x)^2\right)^{1/3}}{2x}$
- Set of solutions
  $\{workingODE, workingODE, workingODE\}$

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying simple symmetries for implicit equations
   Successful isolation of dy/dx: 3 solutions were found. Trying to solve each resulting
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      trying an integrating factor from the invariance group
      <- integrating factor successful
      <- homogeneous successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      trying an integrating factor from the invariance group
      <- integrating factor successful
      <- homogeneous successful
   -------------------
   * Tackling next ODE.
      *** Sublevel 3 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying homogeneous types:
      trying homogeneous G
      trying an integrating factor from the invariance group
      <- integrating factor successful
      <- homogeneous successful`
```

Maple dsolve solution

Solving time : `0.104` (sec)
Leaf size : 341

```
dsolve(diff(y(x),x)^3 = y(x)^2/x,
       y(x),singsol=all)
```

$$
\begin{aligned}
y &= 0 \\
y &= -\frac{3x^{4/3}c_1}{8} + \frac{3x^{2/3}c_1^2}{8} - \frac{c_1^3}{8} + \frac{x^2}{8} \\
y &= \frac{3\left(-i\sqrt{3}-1\right)c_1^2x^{2/3}}{16} + \frac{3c_1\left(1-i\sqrt{3}\right)x^{4/3}}{16} - \frac{c_1^3}{8} + \frac{x^2}{8} \\
y &= \frac{3c_1^2\left(i\sqrt{3}-1\right)x^{2/3}}{16} + \frac{3c_1\left(1+i\sqrt{3}\right)x^{4/3}}{16} - \frac{c_1^3}{8} + \frac{x^2}{8} \\
y &= \frac{3x^{4/3}c_1}{16} + \frac{3x^{2/3}c_1^2}{32} + \frac{c_1^3}{64} + \frac{x^2}{8} \\
y &= \frac{3\left(-i\sqrt{3}-1\right)c_1^2x^{2/3}}{64} + \frac{3c_1\left(i\sqrt{3}-1\right)x^{4/3}}{32} + \frac{c_1^3}{64} + \frac{x^2}{8} \\
y &= \frac{3c_1^2\left(i\sqrt{3}-1\right)x^{2/3}}{64} + \frac{3c_1\left(-i\sqrt{3}-1\right)x^{4/3}}{32} + \frac{c_1^3}{64} + \frac{x^2}{8} \\
y &= -\frac{3x^{4/3}c_1}{16} + \frac{3x^{2/3}c_1^2}{32} - \frac{c_1^3}{64} + \frac{x^2}{8} \\
y &= \frac{3\left(-i\sqrt{3}-1\right)c_1^2x^{2/3}}{64} + \frac{3c_1\left(1-i\sqrt{3}\right)x^{4/3}}{32} - \frac{c_1^3}{64} + \frac{x^2}{8} \\
y &= \frac{3c_1^2\left(i\sqrt{3}-1\right)x^{2/3}}{64} + \frac{3c_1\left(1+i\sqrt{3}\right)x^{4/3}}{32} - \frac{c_1^3}{64} + \frac{x^2}{8}
\end{aligned}
$$

Mathematica DSolve solution

Solving time : `0.084` (sec)
Leaf size : 152

```
DSolve[{(D[y[x],x])^3==y[x]^2/x,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
y(x) &\to \frac{1}{216}\left(3x^{2/3}+2c_1\right)^3 \\
y(x) &\to \frac{1}{216}\left(18i\left(\sqrt{3}+i\right)c_1{}^2x^{2/3} - 27i\left(\sqrt{3}-i\right)c_1x^{4/3} + 27x^2 + 8c_1{}^3\right) \\
y(x) &\to \frac{1}{216}\left(-18i\left(\sqrt{3}-i\right)c_1{}^2x^{2/3} + 27i\left(\sqrt{3}+i\right)c_1x^{4/3} + 27x^2 + 8c_1{}^3\right) \\
y(x) &\to 0
\end{aligned}
$$

2.1.55 problem 55

Internal problem ID [8715]
Book : First order enumerated odes
Section : section 1
Problem number : 55
Date solved : `Tuesday, December 17, 2024 at 12:58:21 PM`
CAS classification : `[[_homogeneous, 'class G']]`

Solve

$$y'^2 = \frac{1}{yx}$$

Solved as first order ode of type nonlinear p but separable

Time used: 0.305 (sec)

The ode has the form

$$(y')^{\frac{n}{m}} = f(x)g(y) \tag{1}$$

Where $n = 2, m = 1, f = \frac{1}{x}, g = \frac{1}{y}$. Hence the ode is

$$(y')^2 = \frac{1}{yx}$$

Solving for y' from (1) gives

$$y' = \sqrt{fg}$$
$$y' = -\sqrt{fg}$$

To be able to solve as separable ode, we have to now assume that $f > 0, g > 0$.

$$\frac{1}{x} > 0$$
$$\frac{1}{y} > 0$$

Under the above assumption the differential equations become separable and can be written as

$$y' = \sqrt{f}\,\sqrt{g}$$
$$y' = -\sqrt{f}\,\sqrt{g}$$

Therefore

$$\frac{1}{\sqrt{g}}\, dy = \left(\sqrt{f}\right)\, dx$$
$$-\frac{1}{\sqrt{g}}\, dy = \left(\sqrt{f}\right)\, dx$$

Replacing $f(x), g(y)$ by their values gives

$$\frac{1}{\sqrt{\frac{1}{y}}}\, dy = \left(\sqrt{\frac{1}{x}}\right)\, dx$$
$$-\frac{1}{\sqrt{\frac{1}{y}}}\, dy = \left(\sqrt{\frac{1}{x}}\right)\, dx$$

Integrating now gives the following solutions

$$\int \frac{1}{\sqrt{\frac{1}{y}}}dy = \int \sqrt{\frac{1}{x}}dx + c_1$$

$$\frac{2y^2\sqrt{\frac{1}{y}}}{3} = 2x\sqrt{\frac{1}{x}}$$

$$\int -\frac{1}{\sqrt{\frac{1}{y}}}dy = \int \sqrt{\frac{1}{x}}dx + c_1$$

$$-\frac{2y^2\sqrt{\frac{1}{y}}}{3} = 2x\sqrt{\frac{1}{x}}$$

Therefore

$$\frac{2y^2\sqrt{\frac{1}{y}}}{3} = 2x\sqrt{\frac{1}{x}} + c_1$$

$$-\frac{2y^2\sqrt{\frac{1}{y}}}{3} = 2x\sqrt{\frac{1}{x}} + c_1$$

Solving for y gives

$$y = \frac{1}{\left(-\frac{\left(-18\left(2x\sqrt{\frac{1}{x}}+c_1\right)^2\right)^{1/3}}{6\left(2x\sqrt{\frac{1}{x}}+c_1\right)} - \frac{i\sqrt{3}\left(-18\left(2x\sqrt{\frac{1}{x}}+c_1\right)^2\right)^{1/3}}{6\left(2x\sqrt{\frac{1}{x}}+c_1\right)}\right)^2}$$

$$y = \frac{1}{\left(-\frac{\left(-18\left(2x\sqrt{\frac{1}{x}}+c_1\right)^2\right)^{1/3}}{6\left(2x\sqrt{\frac{1}{x}}+c_1\right)} + \frac{i\sqrt{3}\left(-18\left(2x\sqrt{\frac{1}{x}}+c_1\right)^2\right)^{1/3}}{12x\sqrt{\frac{1}{x}}+6c_1}\right)^2}$$

$$y = \frac{1}{\left(-\frac{18^{1/3}\left(\left(2x\sqrt{\frac{1}{x}}+c_1\right)^2\right)^{1/3}}{6\left(2x\sqrt{\frac{1}{x}}+c_1\right)} - \frac{i\sqrt{3}\,18^{1/3}\left(\left(2x\sqrt{\frac{1}{x}}+c_1\right)^2\right)^{1/3}}{6\left(2x\sqrt{\frac{1}{x}}+c_1\right)}\right)^2}$$

$$y = \frac{1}{\left(-\frac{18^{1/3}\left(\left(2x\sqrt{\frac{1}{x}}+c_1\right)^2\right)^{1/3}}{6\left(2x\sqrt{\frac{1}{x}}+c_1\right)} + \frac{i\sqrt{3}\,18^{1/3}\left(\left(2x\sqrt{\frac{1}{x}}+c_1\right)^2\right)^{1/3}}{12x\sqrt{\frac{1}{x}}+6c_1}\right)^2}$$

$$y = \frac{9\left(2x\sqrt{\frac{1}{x}}+c_1\right)^2}{\left(-18\left(2x\sqrt{\frac{1}{x}}+c_1\right)^2\right)^{2/3}}$$

$$y = \frac{\left(2x\sqrt{\frac{1}{x}}+c_1\right)^2 18^{1/3}}{2\left(\left(2x\sqrt{\frac{1}{x}}+c_1\right)^2\right)^{2/3}}$$

Summary of solutions found

$$y = \frac{1}{\left(-\frac{\left(-18\left(2x\sqrt{\frac{1}{x}}+c_1\right)^2\right)^{1/3}}{6\left(2x\sqrt{\frac{1}{x}}+c_1\right)} - \frac{i\sqrt{3}\left(-18\left(2x\sqrt{\frac{1}{x}}+c_1\right)^2\right)^{1/3}}{6\left(2x\sqrt{\frac{1}{x}}+c_1\right)}\right)^2}$$

$$y = \frac{1}{\left(-\frac{\left(-18\left(2x\sqrt{\frac{1}{x}}+c_1\right)^2\right)^{1/3}}{6\left(2x\sqrt{\frac{1}{x}}+c_1\right)} + \frac{i\sqrt{3}\left(-18\left(2x\sqrt{\frac{1}{x}}+c_1\right)^2\right)^{1/3}}{12x\sqrt{\frac{1}{x}}+6c_1}\right)^2}$$

$$y = \frac{1}{\left(-\frac{18^{1/3}\left(\left(2x\sqrt{\frac{1}{x}}+c_1\right)^2\right)^{1/3}}{6\left(2x\sqrt{\frac{1}{x}}+c_1\right)} - \frac{i\sqrt{3}\,18^{1/3}\left(\left(2x\sqrt{\frac{1}{x}}+c_1\right)^2\right)^{1/3}}{6\left(2x\sqrt{\frac{1}{x}}+c_1\right)}\right)^2}$$

$$y = \frac{1}{\left(-\frac{18^{1/3}\left(\left(2x\sqrt{\frac{1}{x}}+c_1\right)^2\right)^{1/3}}{6\left(2x\sqrt{\frac{1}{x}}+c_1\right)} + \frac{i\sqrt{3}\,18^{1/3}\left(\left(2x\sqrt{\frac{1}{x}}+c_1\right)^2\right)^{1/3}}{12x\sqrt{\frac{1}{x}}+6c_1}\right)^2}$$

$$y = \frac{9\left(2x\sqrt{\frac{1}{x}}+c_1\right)^2}{\left(-18\left(2x\sqrt{\frac{1}{x}}+c_1\right)^2\right)^{2/3}}$$

$$y = \frac{\left(2x\sqrt{\frac{1}{x}}+c_1\right)^2 18^{1/3}}{2\left(\left(2x\sqrt{\frac{1}{x}}+c_1\right)^2\right)^{2/3}}$$

Maple step by step solution

Let's solve
$\left(\frac{d}{dx}y(x)\right)^2 = \frac{1}{xy(x)}$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\left[\frac{d}{dx}y(x) = \frac{1}{\sqrt{xy(x)}}, \frac{d}{dx}y(x) = -\frac{1}{\sqrt{xy(x)}}\right]$
- Solve the equation $\frac{d}{dx}y(x) = \frac{1}{\sqrt{xy(x)}}$
- Solve the equation $\frac{d}{dx}y(x) = -\frac{1}{\sqrt{xy(x)}}$
- Set of solutions
  $\{workingODE, workingODE\}$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying simple symmetries for implicit equations
Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting OD
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying homogeneous G
```

```
   1st order, trying the canonical coordinates of the invariance group
   <- 1st order, canonical coordinates successful
   <- homogeneous successful
-------------------
* Tackling next ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying homogeneous G
   1st order, trying the canonical coordinates of the invariance group
   <- 1st order, canonical coordinates successful
   <- homogeneous successful`
```

Maple dsolve solution

Solving time : 0.085 (sec)
Leaf size : 51

```
dsolve(diff(y(x),x)^2 = 1/x/y(x),
       y(x),singsol=all)
```

$$\frac{y\sqrt{xy} - c_1\sqrt{x} - 3x}{\sqrt{x}} = 0$$
$$\frac{y\sqrt{xy} - c_1\sqrt{x} + 3x}{\sqrt{x}} = 0$$

Mathematica DSolve solution

Solving time : 3.342 (sec)
Leaf size : 53

```
DSolve[{(D[y[x],x])^2==1/(y[x]*x),{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \left(\frac{3}{2}\right)^{2/3} \left(-2\sqrt{x} + c_1\right)^{2/3}$$
$$y(x) \to \left(\frac{3}{2}\right)^{2/3} \left(2\sqrt{x} + c_1\right)^{2/3}$$

2.1.56 problem 56

Internal problem ID [8716]
Book : First order enumerated odes
Section : section 1
Problem number : 56
Date solved : `Tuesday, December 17, 2024 at 12:58:22 PM`
CAS classification : `[[_homogeneous, ‘class G‘]]`

Solve

$$y'^2 = \frac{1}{x y^3}$$

Solved as first order ode of type nonlinear p but separable

Time used: 0.294 (sec)

The ode has the form

$$(y')^{\frac{n}{m}} = f(x) g(y) \tag{1}$$

Where $n = 2, m = 1, f = \frac{1}{x}, g = \frac{1}{y^3}$. Hence the ode is

$$(y')^2 = \frac{1}{x\, y^3}$$

Solving for y' from (1) gives

$$y' = \sqrt{fg}$$
$$y' = -\sqrt{fg}$$

To be able to solve as separable ode, we have to now assume that $f > 0, g > 0$.

$$\frac{1}{x} > 0$$
$$\frac{1}{y^3} > 0$$

Under the above assumption the differential equations become separable and can be written as

$$y' = \sqrt{f}\,\sqrt{g}$$
$$y' = -\sqrt{f}\,\sqrt{g}$$

Therefore

$$\frac{1}{\sqrt{g}}\, dy = \left(\sqrt{f}\right)\, dx$$
$$-\frac{1}{\sqrt{g}}\, dy = \left(\sqrt{f}\right)\, dx$$

Replacing $f(x), g(y)$ by their values gives

$$\frac{1}{\sqrt{\frac{1}{y^3}}}\, dy = \left(\sqrt{\frac{1}{x}}\right)\, dx$$
$$-\frac{1}{\sqrt{\frac{1}{y^3}}}\, dy = \left(\sqrt{\frac{1}{x}}\right)\, dx$$

Integrating now gives the following solutions

$$\int \frac{1}{\sqrt{\frac{1}{y^3}}} dy = \int \sqrt{\frac{1}{x}} dx + c_1$$

$$\frac{2y^4\sqrt{\frac{1}{y^3}}}{5} = 2x\sqrt{\frac{1}{x}}$$

$$\int -\frac{1}{\sqrt{\frac{1}{y^3}}} dy = \int \sqrt{\frac{1}{x}} dx + c_1$$

$$-\frac{2y^4\sqrt{\frac{1}{y^3}}}{5} = 2x\sqrt{\frac{1}{x}}$$

Therefore

$$\frac{2y^4\sqrt{\frac{1}{y^3}}}{5} = 2x\sqrt{\frac{1}{x}} + c_1$$

$$-\frac{2y^4\sqrt{\frac{1}{y^3}}}{5} = 2x\sqrt{\frac{1}{x}} + c_1$$

Summary of solutions found

$$-\frac{2y^4\sqrt{\frac{1}{y^3}}}{5} = 2x\sqrt{\frac{1}{x}} + c_1$$

$$\frac{2y^4\sqrt{\frac{1}{y^3}}}{5} = 2x\sqrt{\frac{1}{x}} + c_1$$

Maple step by step solution

Let's solve

$\left(\frac{d}{dx}y(x)\right)^2 = \frac{1}{xy(x)^3}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$
- Solve for the highest derivative

  $\left[\frac{d}{dx}y(x) = \frac{1}{\sqrt{xy(x)}\,y(x)}, \frac{d}{dx}y(x) = -\frac{1}{\sqrt{xy(x)}\,y(x)}\right]$
- Solve the equation $\frac{d}{dx}y(x) = \frac{1}{\sqrt{xy(x)}\,y(x)}$
- Solve the equation $\frac{d}{dx}y(x) = -\frac{1}{\sqrt{xy(x)}\,y(x)}$
- Set of solutions

  $\{workingODE, workingODE\}$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying simple symmetries for implicit equations
Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting OD
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying homogeneous G
   1st order, trying the canonical coordinates of the invariance group
   <- 1st order, canonical coordinates successful
   <- homogeneous successful
-------------------
* Tackling next ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying homogeneous G
   1st order, trying the canonical coordinates of the invariance group
   <- 1st order, canonical coordinates successful
   <- homogeneous successful`
```

Maple dsolve solution

Solving time : 0.109 (sec)
Leaf size : 55

```
dsolve(diff(y(x),x)^2 = 1/x/y(x)^3,
       y(x),singsol=all)
```

$$\frac{\sqrt{xy}\, y^2 - c_1\sqrt{x} - 5x}{\sqrt{x}} = 0$$
$$\frac{\sqrt{xy}\, y^2 - c_1\sqrt{x} + 5x}{\sqrt{x}} = 0$$

Mathematica DSolve solution

Solving time : `0.109` (sec)
Leaf size : 53

```
DSolve[{(D[y[x],x])^2==1/(x*y[x]^3),{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \left(\frac{5}{2}\right)^{2/5} \left(-2\sqrt{x} + c_1\right)^{2/5}$$
$$y(x) \to \left(\frac{5}{2}\right)^{2/5} \left(2\sqrt{x} + c_1\right)^{2/5}$$

2.1.57 problem 57

Internal problem ID [8717]
Book : First order enumerated odes
Section : section 1
Problem number : 57
Date solved : `Tuesday, December 17, 2024 at 12:58:23 PM`
CAS classification : `[_separable]`

Solve

$$y'^2 = \frac{1}{x^2 y^3}$$

Solved as first order ode of type nonlinear p but separable

Time used: 0.325 (sec)

The ode has the form

$$(y')^{\frac{n}{m}} = f(x)g(y) \tag{1}$$

Where $n = 2, m = 1, f = \frac{1}{x^2}, g = \frac{1}{y^3}$. Hence the ode is

$$(y')^2 = \frac{1}{x^2 y^3}$$

Solving for y' from (1) gives

$$y' = \sqrt{fg}$$
$$y' = -\sqrt{fg}$$

To be able to solve as separable ode, we have to now assume that $f > 0, g > 0$.

$$\frac{1}{x^2} > 0$$
$$\frac{1}{y^3} > 0$$

Under the above assumption the differential equations become separable and can be written as

$$y' = \sqrt{f}\sqrt{g}$$
$$y' = -\sqrt{f}\sqrt{g}$$

Therefore

$$\frac{1}{\sqrt{g}}\, dy = \left(\sqrt{f}\right)\, dx$$
$$-\frac{1}{\sqrt{g}}\, dy = \left(\sqrt{f}\right)\, dx$$

Replacing $f(x), g(y)$ by their values gives

$$\frac{1}{\sqrt{\frac{1}{y^3}}}\, dy = \left(\sqrt{\frac{1}{x^2}}\right)\, dx$$
$$-\frac{1}{\sqrt{\frac{1}{y^3}}}\, dy = \left(\sqrt{\frac{1}{x^2}}\right)\, dx$$

Integrating now gives the following solutions

$$\int \frac{1}{\sqrt{\frac{1}{y^3}}}dy = \int \sqrt{\frac{1}{x^2}}dx + c_1$$
$$\frac{2y^4\sqrt{\frac{1}{y^3}}}{5} = \sqrt{\frac{1}{x^2}}\, x\ln(x)$$
$$\int -\frac{1}{\sqrt{\frac{1}{y^3}}}dy = \int \sqrt{\frac{1}{x^2}}dx + c_1$$
$$-\frac{2y^4\sqrt{\frac{1}{y^3}}}{5} = \sqrt{\frac{1}{x^2}}\, x\ln(x)$$

Therefore

$$\frac{2y^4\sqrt{\frac{1}{y^3}}}{5} = \sqrt{\frac{1}{x^2}}\, x\ln(x) + c_1$$
$$-\frac{2y^4\sqrt{\frac{1}{y^3}}}{5} = \sqrt{\frac{1}{x^2}}\, x\ln(x) + c_1$$

Summary of solutions found

$$-\frac{2y^4\sqrt{\frac{1}{y^3}}}{5} = \sqrt{\frac{1}{x^2}}\, x\ln(x) + c_1$$
$$\frac{2y^4\sqrt{\frac{1}{y^3}}}{5} = \sqrt{\frac{1}{x^2}}\, x\ln(x) + c_1$$

Maple step by step solution

Let's solve
$\left(\frac{d}{dx}y(x)\right)^2 = \frac{1}{x^2 y(x)^3}$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\left[\frac{d}{dx}y(x) = \frac{1}{y(x)^{3/2}x}, \frac{d}{dx}y(x) = -\frac{1}{y(x)^{3/2}x}\right]$

□ Solve the equation $\frac{d}{dx}y(x) = \frac{1}{y(x)^{3/2}x}$
  - Separate variables
    $\left(\frac{d}{dx}y(x)\right)y(x)^{3/2} = \frac{1}{x}$
  - Integrate both sides with respect to x
    $\int \left(\frac{d}{dx}y(x)\right)y(x)^{3/2}dx = \int \frac{1}{x}dx + \_C1$
  - Evaluate integral
    $\frac{2y(x)^{5/2}}{5} = \ln(x) + \_C1$
  - Solve for $y(x)$
    $y(x) = \frac{\left(80\ln(x)+80\_C1\right)^{2/5}}{4}$

□ Solve the equation $\frac{d}{dx}y(x) = -\frac{1}{y(x)^{3/2}x}$
  - Separate variables
    $\left(\frac{d}{dx}y(x)\right)y(x)^{3/2} = -\frac{1}{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}y(x)\right) y(x)^{3/2} dx = \int -\frac{1}{x}dx + \_C1$
- Evaluate integral

  $\frac{2y(x)^{5/2}}{5} = -\ln(x) + \_C1$
- Solve for $y(x)$

  $y(x) = \frac{\left(-80\ln(x)+80\_C1\right)^{2/5}}{4}$

- Set of solutions

  $\left\{y(x) = \frac{(-80\ln(x)+80C1)^{2/5}}{4}, y(x) = \frac{(80\ln(x)+80C1)^{2/5}}{4}\right\}$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying simple symmetries for implicit equations
<- symmetries for implicit equations successful`
```

Maple dsolve solution

Solving time : 0.158 (sec)
Leaf size : 29

```
dsolve(diff(y(x),x)^2 = 1/x^2/y(x)^3,
       y(x),singsol=all)
```

$$\ln(x) - \frac{2y^{5/2}}{5} - c_1 = 0$$
$$\ln(x) + \frac{2y^{5/2}}{5} - c_1 = 0$$

Mathematica DSolve solution

Solving time : 0.132 (sec)
Leaf size : 45

```
DSolve[{(D[y[x],x])^2==1/(x^2*y[x]^3),{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \left(\frac{5}{2}\right)^{2/5} (-\log(x) + c_1)^{2/5}$$
$$y(x) \to \left(\frac{5}{2}\right)^{2/5} (\log(x) + c_1)^{2/5}$$

2.1.58 problem 58

Internal problem ID [8718]
Book : First order enumerated odes
Section : section 1
Problem number : 58
Date solved : `Tuesday, December 17, 2024 at 12:58:24 PM`
CAS classification : `[[_homogeneous, 'class G'], _rational]`

Solve

$$y'^4 = \frac{1}{x y^3}$$

Solved as first order ode of type nonlinear p but separable

Time used: 0.720 (sec)

The ode has the form

$$(y')^{\frac{n}{m}} = f(x)g(y) \tag{1}$$

Where $n = 4, m = 1, f = \frac{1}{x}, g = \frac{1}{y^3}$. Hence the ode is

$$(y')^4 = \frac{1}{x\, y^3}$$

Solving for y' from (1) gives

$$\begin{aligned} y' &= (fg)^{1/4} \\ y' &= i(fg)^{1/4} \\ y' &= -(fg)^{1/4} \\ y' &= -i(fg)^{1/4} \end{aligned}$$

To be able to solve as separable ode, we have to now assume that $f > 0, g > 0$.

$$\frac{1}{x} > 0$$
$$\frac{1}{y^3} > 0$$

Under the above assumption the differential equations become separable and can be written as

$$\begin{aligned} y' &= f^{1/4} g^{1/4} \\ y' &= i f^{1/4} g^{1/4} \\ y' &= -f^{1/4} g^{1/4} \\ y' &= -i f^{1/4} g^{1/4} \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{g^{1/4}}\, dy &= \left(f^{1/4}\right) dx \\ -\frac{i}{g^{1/4}}\, dy &= \left(f^{1/4}\right) dx \\ -\frac{1}{g^{1/4}}\, dy &= \left(f^{1/4}\right) dx \\ \frac{i}{g^{1/4}}\, dy &= \left(f^{1/4}\right) dx \end{aligned}$$

Replacing $f(x), g(y)$ by their values gives

$$
\begin{aligned}
\frac{1}{\left(\frac{1}{y^3}\right)^{1/4}} dy &= \left(\left(\frac{1}{x}\right)^{1/4}\right) dx \\
-\frac{i}{\left(\frac{1}{y^3}\right)^{1/4}} dy &= \left(\left(\frac{1}{x}\right)^{1/4}\right) dx \\
-\frac{1}{\left(\frac{1}{y^3}\right)^{1/4}} dy &= \left(\left(\frac{1}{x}\right)^{1/4}\right) dx \\
\frac{i}{\left(\frac{1}{y^3}\right)^{1/4}} dy &= \left(\left(\frac{1}{x}\right)^{1/4}\right) dx
\end{aligned}
$$

Integrating now gives the following solutions

$$
\begin{aligned}
\int \frac{1}{\left(\frac{1}{y^3}\right)^{1/4}} dy &= \int \left(\frac{1}{x}\right)^{1/4} dx + c_1 \\
\frac{4y^4\left(\frac{1}{y^3}\right)^{3/4}}{7} &= \frac{4x\left(\frac{1}{x}\right)^{1/4}}{3} \\
\int -\frac{i}{\left(\frac{1}{y^3}\right)^{1/4}} dy &= \int \left(\frac{1}{x}\right)^{1/4} dx + c_1 \\
-\frac{4iy^4\left(\frac{1}{y^3}\right)^{3/4}}{7} &= \frac{4x\left(\frac{1}{x}\right)^{1/4}}{3} \\
\int -\frac{1}{\left(\frac{1}{y^3}\right)^{1/4}} dy &= \int \left(\frac{1}{x}\right)^{1/4} dx + c_1 \\
-\frac{4y^4\left(\frac{1}{y^3}\right)^{3/4}}{7} &= \frac{4x\left(\frac{1}{x}\right)^{1/4}}{3} \\
\int \frac{i}{\left(\frac{1}{y^3}\right)^{1/4}} dy &= \int \left(\frac{1}{x}\right)^{1/4} dx + c_1 \\
\frac{4iy^4\left(\frac{1}{y^3}\right)^{3/4}}{7} &= \frac{4x\left(\frac{1}{x}\right)^{1/4}}{3}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\frac{4y^4\left(\frac{1}{y^3}\right)^{3/4}}{7} &= \frac{4x\left(\frac{1}{x}\right)^{1/4}}{3} + c_1 \\
-\frac{4iy^4\left(\frac{1}{y^3}\right)^{3/4}}{7} &= \frac{4x\left(\frac{1}{x}\right)^{1/4}}{3} + c_1 \\
-\frac{4y^4\left(\frac{1}{y^3}\right)^{3/4}}{7} &= \frac{4x\left(\frac{1}{x}\right)^{1/4}}{3} + c_1 \\
\frac{4iy^4\left(\frac{1}{y^3}\right)^{3/4}}{7} &= \frac{4x\left(\frac{1}{x}\right)^{1/4}}{3} + c_1
\end{aligned}
$$

Summary of solutions found

$$-\frac{4 i y^{4} \left(\frac{1}{y^{3}}\right)^{3/4}}{7} = \frac{4 x \left(\frac{1}{x}\right)^{1/4}}{3}+c_{1}$$

$$\frac{4 i y^{4} \left(\frac{1}{y^{3}}\right)^{3/4}}{7} = \frac{4 x \left(\frac{1}{x}\right)^{1/4}}{3}+c_{1}$$

$$-\frac{4 y^{4} \left(\frac{1}{y^{3}}\right)^{3/4}}{7} = \frac{4 x \left(\frac{1}{x}\right)^{1/4}}{3}+c_{1}$$

$$\frac{4 y^{4} \left(\frac{1}{y^{3}}\right)^{3/4}}{7} = \frac{4 x \left(\frac{1}{x}\right)^{1/4}}{3}+c_{1}$$

Maple step by step solution

Let's solve
$\left(\frac{d}{dx}y(x)\right)^{4} = \frac{1}{x y(x)^{3}}$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\left[\frac{d}{dx}y(x) = \frac{\left(x^{3} y(x)\right)^{1/4}}{x y(x)}, \frac{d}{dx}y(x) = -\frac{\left(x^{3} y(x)\right)^{1/4}}{x y(x)}, \frac{d}{dx}y(x) = \frac{-\mathrm{I}\left(x^{3} y(x)\right)^{1/4}}{x y(x)}, \frac{d}{dx}y(x) = \frac{\mathrm{I}\left(x^{3} y(x)\right)^{1/4}}{x y(x)}\right]$
- Solve the equation $\frac{d}{dx}y(x) = \frac{\left(x^{3} y(x)\right)^{1/4}}{x y(x)}$
- Solve the equation $\frac{d}{dx}y(x) = -\frac{\left(x^{3} y(x)\right)^{1/4}}{x y(x)}$
- Solve the equation $\frac{d}{dx}y(x) = \frac{-\mathrm{I}\left(x^{3} y(x)\right)^{1/4}}{x y(x)}$
- Solve the equation $\frac{d}{dx}y(x) = \frac{\mathrm{I}\left(x^{3} y(x)\right)^{1/4}}{x y(x)}$
- Set of solutions
  $\{workingODE, workingODE, workingODE, workingODE\}$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying simple symmetries for implicit equations
Successful isolation of dy/dx: 4 solutions were found. Trying to solve each resulting
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying homogeneous G
   1st order, trying the canonical coordinates of the invariance group
   <- 1st order, canonical coordinates successful
   <- homogeneous successful
-------------------
```

```
* Tackling next ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying homogeneous G
   1st order, trying the canonical coordinates of the invariance group
   <- 1st order, canonical coordinates successful
   <- homogeneous successful
-------------------
* Tackling next ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying homogeneous G
   1st order, trying the canonical coordinates of the invariance group
   <- 1st order, canonical coordinates successful
   <- homogeneous successful
-------------------
* Tackling next ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying homogeneous types:
   trying homogeneous G
   1st order, trying the canonical coordinates of the invariance group
   <- 1st order, canonical coordinates successful
   <- homogeneous successful`
```

Maple dsolve solution

Solving time : `0.223` (sec)
Leaf size : 121

```
dsolve(diff(y(x),x)^4 = 1/x/y(x)^3,
       y(x),singsol=all)
```

$$-\frac{7x^3 - 3(yx^3)^{3/4} y + c_1 x^{9/4}}{x^{9/4}} = 0$$
$$\frac{-7x^3 + 3i(yx^3)^{3/4} y - c_1 x^{9/4}}{x^{9/4}} = 0$$
$$\frac{7x^3 + 3i(yx^3)^{3/4} y - c_1 x^{9/4}}{x^{9/4}} = 0$$
$$\frac{7x^3 + 3(yx^3)^{3/4} y - c_1 x^{9/4}}{x^{9/4}} = 0$$

Mathematica DSolve solution

Solving time : 6.693 (sec)
Leaf size : 129

```
DSolve[{(D[y[x],x])^4==1/(x*y[x]^3),{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{\left(-\frac{28x^{3/4}}{3} + 7c_1\right)^{4/7}}{2\sqrt[7]{2}}$$
$$y(x) \to \frac{\left(7c_1 - \frac{28}{3}ix^{3/4}\right)^{4/7}}{2\sqrt[7]{2}}$$
$$y(x) \to \frac{\left(\frac{28}{3}ix^{3/4} + 7c_1\right)^{4/7}}{2\sqrt[7]{2}}$$
$$y(x) \to \frac{\left(\frac{28x^{3/4}}{3} + 7c_1\right)^{4/7}}{2\sqrt[7]{2}}$$

2.1.59 problem 59

Internal problem ID [8719]
Book : First order enumerated odes
Section : section 1
Problem number : 59
Date solved : `Tuesday, December 17, 2024 at 12:58:25 PM`
CAS classification : `[_separable]`

Solve

$$y'^2 = \frac{1}{x^3 y^4}$$

Solved as first order ode of type nonlinear p but separable

Time used: 0.469 (sec)

The ode has the form

$$(y')^{\frac{n}{m}} = f(x)g(y) \tag{1}$$

Where $n = 2, m = 1, f = \frac{1}{x^3}, g = \frac{1}{y^4}$. Hence the ode is

$$(y')^2 = \frac{1}{x^3 y^4}$$

Solving for y' from (1) gives

$$y' = \sqrt{fg}$$
$$y' = -\sqrt{fg}$$

To be able to solve as separable ode, we have to now assume that $f > 0, g > 0$.

$$\frac{1}{x^3} > 0$$
$$\frac{1}{y^4} > 0$$

Under the above assumption the differential equations become separable and can be written as

$$y' = \sqrt{f}\,\sqrt{g}$$
$$y' = -\sqrt{f}\,\sqrt{g}$$

Therefore

$$\frac{1}{\sqrt{g}}\,dy = \left(\sqrt{f}\right)\,dx$$
$$-\frac{1}{\sqrt{g}}\,dy = \left(\sqrt{f}\right)\,dx$$

Replacing $f(x), g(y)$ by their values gives

$$\frac{1}{\sqrt{\frac{1}{y^4}}}\,dy = \left(\sqrt{\frac{1}{x^3}}\right)\,dx$$
$$-\frac{1}{\sqrt{\frac{1}{y^4}}}\,dy = \left(\sqrt{\frac{1}{x^3}}\right)\,dx$$

Integrating now gives the following solutions

$$\int \frac{1}{\sqrt{\frac{1}{y^4}}}dy = \int \sqrt{\frac{1}{x^3}}dx + c_1$$
$$\frac{y^5\sqrt{\frac{1}{y^4}}}{3} = -2x\sqrt{\frac{1}{x^3}}$$
$$\int -\frac{1}{\sqrt{\frac{1}{y^4}}}dy = \int \sqrt{\frac{1}{x^3}}dx + c_1$$
$$-\frac{y^5\sqrt{\frac{1}{y^4}}}{3} = -2x\sqrt{\frac{1}{x^3}}$$

Therefore

$$\frac{y^5\sqrt{\frac{1}{y^4}}}{3} = -2x\sqrt{\frac{1}{x^3}} + c_1$$
$$-\frac{y^5\sqrt{\frac{1}{y^4}}}{3} = -2x\sqrt{\frac{1}{x^3}} + c_1$$

Summary of solutions found

$$-\frac{y^5\sqrt{\frac{1}{y^4}}}{3} = -2x\sqrt{\frac{1}{x^3}} + c_1$$
$$\frac{y^5\sqrt{\frac{1}{y^4}}}{3} = -2x\sqrt{\frac{1}{x^3}} + c_1$$

Maple step by step solution

Let's solve
$\left(\frac{d}{dx}y(x)\right)^2 = \frac{1}{y(x)^4 x^3}$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\left[\frac{d}{dx}y(x) = \frac{1}{x^{3/2}y(x)^2}, \frac{d}{dx}y(x) = -\frac{1}{x^{3/2}y(x)^2}\right]$

☐ Solve the equation $\frac{d}{dx}y(x) = \frac{1}{x^{3/2}y(x)^2}$
  - Separate variables
    $y(x)^2\left(\frac{d}{dx}y(x)\right) = \frac{1}{x^{3/2}}$
  - Integrate both sides with respect to x
    $\int y(x)^2\left(\frac{d}{dx}y(x)\right)dx = \int \frac{1}{x^{3/2}}dx + \_C1$
  - Evaluate integral
    $\frac{y(x)^3}{3} = -\frac{2}{\sqrt{x}} + \_C1$
  - Solve for $y(x)$
    $y(x) = \left(\frac{3\sqrt{x}\_C1 - 6}{\sqrt{x}}\right)^{1/3}$

☐ Solve the equation $\frac{d}{dx}y(x) = -\frac{1}{x^{3/2}y(x)^2}$
  - Separate variables
    $y(x)^2\left(\frac{d}{dx}y(x)\right) = -\frac{1}{x^{3/2}}$

- Integrate both sides with respect to x

  $\int y(x)^2 \left(\frac{d}{dx}y(x)\right) dx = \int -\frac{1}{x^{3/2}}dx + \_C1$
- Evaluate integral

  $\frac{y(x)^3}{3} = \frac{2}{\sqrt{x}} + \_C1$
- Solve for $y(x)$

  $y(x) = \left(\frac{3\sqrt{x}\_C1+6}{\sqrt{x}}\right)^{1/3}$

- Set of solutions

  $\left\{y(x) = \left(\frac{3\sqrt{x}\,C1-6}{\sqrt{x}}\right)^{1/3}, y(x) = \left(\frac{3\sqrt{x}\,C1+6}{\sqrt{x}}\right)^{1/3}\right\}$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
trying simple symmetries for implicit equations
Successful isolation of dy/dx: 2 solutions were found. Trying to solve each resulting OD
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   <- Bernoulli successful
-------------------
* Tackling next ODE.
   *** Sublevel 2 ***
   Methods for first order ODEs:
   --- Trying classification methods ---
   trying a quadrature
   trying 1st order linear
   trying Bernoulli
   <- Bernoulli successful`
```

Maple dsolve solution

Solving time : 0.070 (sec)
Leaf size : 133

```
dsolve(diff(y(x),x)^2 = 1/x^3/y(x)^4,
       y(x),singsol=all)
```

$$y = \left(\frac{c_1\sqrt{x}-6}{\sqrt{x}}\right)^{1/3}$$
$$y = -\frac{\left(\frac{c_1\sqrt{x}-6}{\sqrt{x}}\right)^{1/3}\left(1+i\sqrt{3}\right)}{2}$$
$$y = \frac{\left(\frac{c_1\sqrt{x}-6}{\sqrt{x}}\right)^{1/3}\left(i\sqrt{3}-1\right)}{2}$$
$$y = \left(\frac{c_1\sqrt{x}+6}{\sqrt{x}}\right)^{1/3}$$
$$y = -\frac{\left(\frac{c_1\sqrt{x}+6}{\sqrt{x}}\right)^{1/3}\left(1+i\sqrt{3}\right)}{2}$$
$$y = \frac{\left(\frac{c_1\sqrt{x}+6}{\sqrt{x}}\right)^{1/3}\left(i\sqrt{3}-1\right)}{2}$$

Mathematica DSolve solution

Solving time : 3.383 (sec)
Leaf size : 157

```
DSolve[{(D[y[x],x])^2==1/(x^3*y[x]^4),{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\sqrt[3]{-3}\sqrt[3]{-\frac{2}{\sqrt{x}}+c_1}$$
$$y(x) \to \sqrt[3]{3}\sqrt[3]{-\frac{2}{\sqrt{x}}+c_1}$$
$$y(x) \to (-1)^{2/3}\sqrt[3]{3}\sqrt[3]{-\frac{2}{\sqrt{x}}+c_1}$$
$$y(x) \to -\sqrt[3]{-3}\sqrt[3]{\frac{2}{\sqrt{x}}+c_1}$$
$$y(x) \to \sqrt[3]{3}\sqrt[3]{\frac{2}{\sqrt{x}}+c_1}$$
$$y(x) \to (-1)^{2/3}\sqrt[3]{3}\sqrt[3]{\frac{2}{\sqrt{x}}+c_1}$$

2.1.60 problem 60

Internal problem ID [8720]
Book : First order enumerated odes
Section : section 1
Problem number : 60
Date solved : `Tuesday, December 17, 2024 at 12:58:26 PM`
CAS classification : `[[_homogeneous, 'class C'], _dAlembert]`

Solve

$$y' = \sqrt{1 + 6x + y}$$

Solved as first order homogeneous class C ode

Time used: 0.536 (sec)

Let

$$z = 1 + 6x + y \tag{1}$$

Then

$$z'(x) = 6 + y'$$

Therefore

$$y' = z'(x) - 6$$

Hence the given ode can now be written as

$$z'(x) - 6 = \sqrt{z}$$

This is separable first order ode. Integrating

$$\int dx = \int \frac{1}{\sqrt{z} + 6} dz$$
$$x + c_1 = 2\sqrt{z} - 6\ln\left(\sqrt{z} + 6\right) + 6\ln\left(-6 + \sqrt{z}\right) - 6\ln\left(-36 + z\right)$$

Replacing z back by its value from (1) then the above gives the solution as Solving for y gives

$$y = \mathrm{e}^{-2\,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{-1-\frac{x}{12}-\frac{c_1}{12}}}{6}\right)-2-\frac{x}{6}-\frac{c_1}{6}} - 12\,\mathrm{e}^{-\mathrm{LambertW}\left(-\frac{\mathrm{e}^{-1-\frac{x}{12}-\frac{c_1}{12}}}{6}\right)-1-\frac{x}{12}-\frac{c_1}{12}} - 6x + 35$$

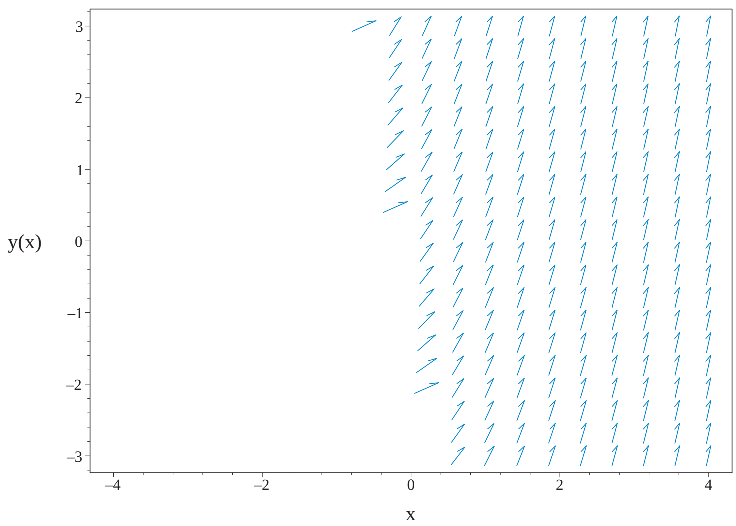


Figure 2.77: Slope field plot
$y' = \sqrt{1 + 6x + y}$

Summary of solutions found

$$y = e^{-2\,\mathrm{LambertW}\left(-\frac{e^{-1-\frac{x}{12}-\frac{c_1}{12}}}{6}\right)-2-\frac{x}{6}-\frac{c_1}{6}} - 12\,e^{-\mathrm{LambertW}\left(-\frac{e^{-1-\frac{x}{12}-\frac{c_1}{12}}}{6}\right)-1-\frac{x}{12}-\frac{c_1}{12}} - 6x + 35$$

Solved using Lie symmetry for first order ode

Time used: 1.181 (sec)

Writing the ode as

$$y' = \sqrt{1 + 6x + y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \sqrt{1 + 6x + y}\,(b_3 - a_2) - (1 + 6x + y)\,a_3 - \frac{3(xa_2 + ya_3 + a_1)}{\sqrt{1 + 6x + y}} - \frac{xb_2 + yb_3 + b_1}{2\sqrt{1 + 6x + y}} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{12a_3\sqrt{1 + 6x + y}\,x + 2a_3\sqrt{1 + 6x + y}\,y + 2a_3\sqrt{1 + 6x + y} - 2b_2\sqrt{1 + 6x + y} + 18xa_2 + xb_2 - 12b_3x +}{2\sqrt{1 + 6x + y}}$$
$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-12a_3\sqrt{1 + 6x + y}\,x - 2a_3\sqrt{1 + 6x + y}\,y - 2a_3\sqrt{1 + 6x + y} + 2b_2\sqrt{1 + 6x + y} \\ &- 18xa_2 - xb_2 + 12b_3x - 2a_2y - 6ya_3 + yb_3 - 6a_1 - 2a_2 - b_1 + 2b_3 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} &-2(1+6x+y)\,a_2+2(1+6x+y)\,b_3-12a_3\sqrt{1+6x+y}\,x-2a_3\sqrt{1+6x+y}\,y \\ &-2a_3\sqrt{1+6x+y}+2b_2\sqrt{1+6x+y}-6xa_2-xb_2-6ya_3-yb_3-6a_1-b_1=0 \end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned} &-12a_3\sqrt{1+6x+y}\,x-2a_3\sqrt{1+6x+y}\,y-2a_3\sqrt{1+6x+y}+2b_2\sqrt{1+6x+y} \\ &-18xa_2-xb_2+12b_3x-2a_2y-6ya_3+yb_3-6a_1-2a_2-b_1+2b_3=0 \end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$\left\{x,y,\sqrt{1+6x+y}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$\left\{x=v_1,y=v_2,\sqrt{1+6x+y}=v_3\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-12a_3v_3v_1-2a_3v_3v_2-18v_1a_2-2a_2v_2-6v_2a_3-2a_3v_3 \\ &-v_1b_2+2b_2v_3+12b_3v_1+v_2b_3-6a_1-2a_2-b_1+2b_3=0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1,v_2,v_3\}$$

Equation (7E) now becomes

$$\begin{aligned} &-12a_3v_3v_1+(-18a_2-b_2+12b_3)\,v_1-2a_3v_3v_2+(-2a_2-6a_3+b_3)\,v_2 \\ &+(-2a_3+2b_2)\,v_3-6a_1-2a_2-b_1+2b_3=0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -12a_3&=0 \\ -2a_3&=0 \\ -2a_3+2b_2&=0 \\ -18a_2-b_2+12b_3&=0 \\ -2a_2-6a_3+b_3&=0 \\ -6a_1-2a_2-b_1+2b_3&=0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1&=a_1 \\ a_2&=0 \\ a_3&=0 \\ b_1&=-6a_1 \\ b_2&=0 \\ b_3&=0 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = 1$$
$$\eta = -6$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= -6 - \left(\sqrt{1+6x+y}\right)(1) \\ &= -\sqrt{1+6x+y} - 6 \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-\sqrt{1+6x+y} - 6} dy\end{aligned}$$

Which results in

$$S = -2\sqrt{1+6x+y} + 6\ln\left(\sqrt{1+6x+y}+6\right) - 6\ln\left(-6+\sqrt{1+6x+y}\right) + 6\ln\left(-35+6x+y\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \sqrt{1+6x+y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{6}{\sqrt{1+6x+y}+6} \\ S_y &= \frac{1}{-\sqrt{1+6x+y}-6}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int -1\, dR$$
$$S(R) = -R + c_2$$

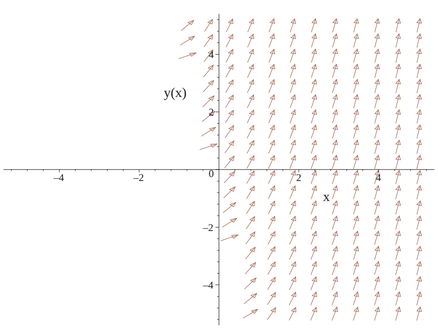
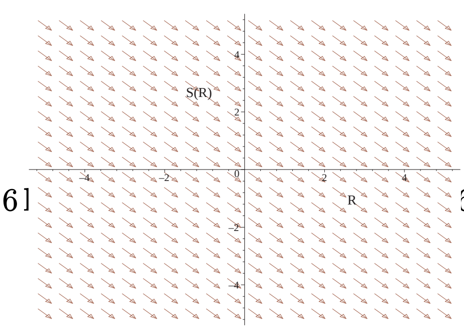
To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$-2\sqrt{1+6x+y} + 6\ln\left(\sqrt{1+6x+y}+6\right) - 6\ln\left(-6+\sqrt{1+6x+y}\right) + 6\ln\left(-35+6x+y\right) = -x + c_2$$

Which gives

$$y = e^{-2\,\mathrm{LambertW}\left(-\frac{e^{-1-\frac{x}{12}+\frac{c_2}{12}}}{6}\right)-2-\frac{x}{6}+\frac{c_2}{6}} - 12\,e^{-\mathrm{LambertW}\left(-\frac{e^{-1-\frac{x}{12}+\frac{c_2}{12}}}{6}\right)-1-\frac{x}{12}+\frac{c_2}{12}} - 6x + 35$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \sqrt{1+6x+y}$ 	$R = x$ $S = -2\sqrt{1+6x+y} + 6\ln\left(\sqrt{1+6x+y}+6\right) - 6\ln\left(-6+\sqrt{1+\ldots}\right)$	$\frac{dS}{dR} = -1$ 

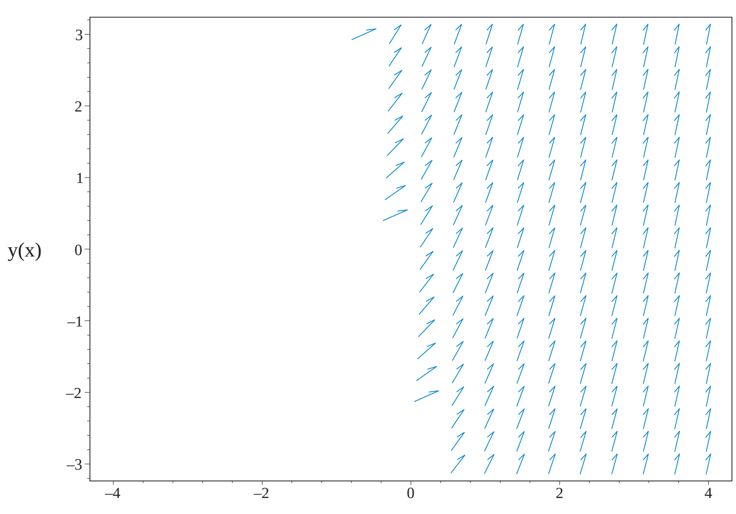


Figure 2.78: Slope field plot
$y' = \sqrt{1+6x+y}$

Summary of solutions found

$$y = \mathrm{e}^{-2\,\mathrm{LambertW}\left(-\frac{\mathrm{e}^{-1-\frac{x}{12}+\frac{c_2}{12}}}{6}\right)-2-\frac{x}{6}+\frac{c_2}{6}} - 12\,\mathrm{e}^{-\mathrm{LambertW}\left(-\frac{\mathrm{e}^{-1-\frac{x}{12}+\frac{c_2}{12}}}{6}\right)-1-\frac{x}{12}+\frac{c_2}{12}} - 6x + 35$$

Solved as first order ode of type dAlembert

Time used: 0.268 (sec)

Let $p = y'$ the ode becomes

$$p = \sqrt{1 + 6x + y}$$

Solving for y from the above results in

$$y = p^2 - 6x - 1 \tag{1}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= -6 \\ g &= p^2 - 1 \end{aligned}$$

Hence (2) becomes

$$p + 6 = 2pp'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p + 6 = 0$$

No valid singular solutions found.

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) + 6}{2p(x)} \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed. Integrating gives

$$\begin{aligned} \int \frac{2p}{p+6} dp &= dx \\ 2p - 12\ln(p+6) &= x + c_1 \end{aligned}$$

Singular solutions are found by solving

$$\frac{p+6}{2p} = 0$$

for $p(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$p(x) = -6$$

Solving for $p(x)$ gives

$$p(x) = -6$$
$$p(x) = -6 \operatorname{LambertW}\left(-\frac{e^{-1-\frac{x}{12}-\frac{c_1}{12}}}{6}\right) - 6$$

Substituing the above solution for p in (2A) gives

$$y = -6x + 35$$
$$y = \left(-6 \operatorname{LambertW}\left(-\frac{e^{-1-\frac{x}{12}-\frac{c_1}{12}}}{6}\right) - 6\right)^2 - 6x - 1$$

The solution

$$y = -6x + 35$$

was found not to satisfy the ode or the IC. Hence it is removed.

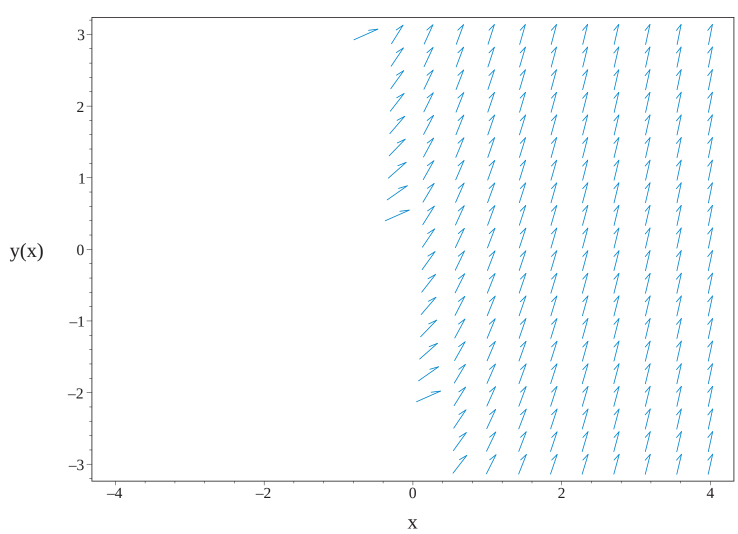


Figure 2.79: Slope field plot
$y' = \sqrt{1 + 6x + y}$

Summary of solutions found

$$y = \left(-6 \operatorname{LambertW}\left(-\frac{e^{-1-\frac{x}{12}-\frac{c_1}{12}}}{6}\right) - 6\right)^2 - 6x - 1$$

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) = \sqrt{1+6x+y(x)}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = \sqrt{1+6x+y(x)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -6, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

Maple dsolve solution

Solving time : 0.029 (sec)
Leaf size : 57

```
dsolve(diff(y(x),x) = (1+6*x+y(x))^(1/2),
       y(x),singsol=all)
```

$$\begin{aligned}&x-2\sqrt{1+6x+y}+6\ln\left(6+\sqrt{1+6x+y}\right)\\&-6\ln\left(-6+\sqrt{1+6x+y}\right)+6\ln\left(-35+y+6x\right)-c_1=0\end{aligned}$$

Mathematica DSolve solution

Solving time : 10.898 (sec)
Leaf size : 112

```
DSolve[{D[y[x],x]==(1+6*x+y[x])^(1/2),{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x)\to 36W\left(-\frac{1}{6}e^{\frac{1}{72}(-6x-73+6c_1)}\right)^2+72W\left(-\frac{1}{6}e^{\frac{1}{72}(-6x-73+6c_1)}\right)-6x+35$$
$$y(x)\to 35-6x$$
$$y(x)\to 36W\left(-\frac{1}{6}e^{\frac{1}{72}(-6x-73)}\right)^2+72W\left(-\frac{1}{6}e^{\frac{1}{72}(-6x-73)}\right)-6x+35$$

2.1.61 problem 61

Internal problem ID [8721]
Book : First order enumerated odes
Section : section 1
Problem number : 61
Date solved : `Tuesday, December 17, 2024 at 12:58:29 PM`
CAS classification : `[[_homogeneous, 'class C'], _dAlembert]`

Solve

$$y' = (1+6x+y)^{1/3}$$

Solved as first order homogeneous class C ode

Time used: 0.240 (sec)

Let

$$z = 1+6x+y \tag{1}$$

Then

$$z'(x) = 6+y'$$

Therefore

$$y' = z'(x)-6$$

Hence the given ode can now be written as

$$z'(x)-6 = z^{1/3}$$

This is separable first order ode. Integrating

$$\int dx = \int \frac{1}{z^{1/3}+6}dz$$

$$x+c_1 = \frac{3z^{2/3}}{2} - 36\ln\left(z^{2/3}-6z^{1/3}+36\right) + 72\ln\left(z^{1/3}+6\right) + 36\ln\left(216+z\right) - 18z^{1/3}$$

Replacing z back by its value from (1) then the above gives the solution as

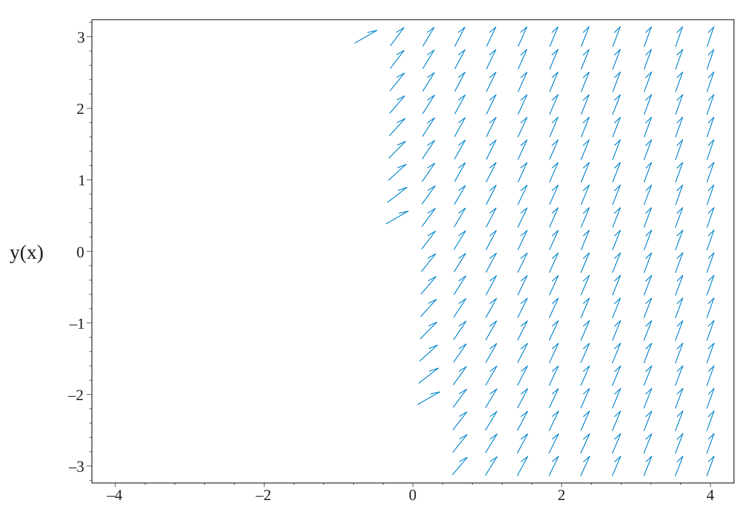


Figure 2.80: Slope field plot
$y' = (1+6x+y)^{1/3}$

Summary of solutions found

$$\frac{3(1+6x+y)^{2/3}}{2} - 36\ln\left((1+6x+y)^{2/3} - 6(1+6x+y)^{1/3}+36\right)+72\ln\left((1+6x+y)^{1/3}+6\right)+36\ln(217+6x+y)-18(1+6x+y)^{1/3}=x+c_1$$

Solved using Lie symmetry for first order ode

Time used: 0.928 (sec)

Writing the ode as

$$\begin{aligned}y' &= (1+6x+y)^{1/3}\\ y' &= \omega(x,y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}b_2 + (1+6x+y)^{1/3}(b_3-a_2) - (1+6x+y)^{2/3}a_3\\ - \frac{2(xa_2+ya_3+a_1)}{(1+6x+y)^{2/3}} - \frac{xb_2+yb_3+b_1}{3(1+6x+y)^{2/3}} = 0\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{3(1+6x+y)^{4/3}a_3 - 3b_2(1+6x+y)^{2/3} + 24xa_2 + xb_2 - 18b_3x + 3a_2y + 6ya_3 - 2yb_3 + 6a_1 + 3a_2 + b_1}{3(1+6x+y)^{2/3}}$$

Setting the numerator to zero gives

$$\begin{aligned}-3(1+6x+y)^{4/3}a_3 + 3b_2(1+6x+y)^{2/3} - 24xa_2 - xb_2\\ + 18b_3x - 3a_2y - 6ya_3 + 2yb_3 - 6a_1 - 3a_2 - b_1 + 3b_3 = 0\end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned}-3(1+6x+y)^{4/3}a_3 - 3(1+6x+y)a_2 + 3(1+6x+y)b_3\\ + 3b_2(1+6x+y)^{2/3} - 6xa_2 - xb_2 - 6ya_3 - yb_3 - 6a_1 - b_1 = 0\end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned}-18(1+6x+y)^{1/3}a_3x + 3b_2(1+6x+y)^{2/3} - 3(1+6x+y)^{1/3}a_3y - 24xa_2 - xb_2\\ + 18b_3x - 3(1+6x+y)^{1/3}a_3 - 3a_2y - 6ya_3 + 2yb_3 - 6a_1 - 3a_2 - b_1 + 3b_3 = 0\end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, (1+6x+y)^{1/3}, (1+6x+y)^{2/3}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, (1+6x+y)^{1/3} = v_3, (1+6x+y)^{2/3} = v_4\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-18v_3a_3v_1 - 3v_3a_3v_2 - 24v_1a_2 - 3a_2v_2 - 6v_2a_3 - 3v_3a_3 \\ &- v_1b_2 + 3b_2v_4 + 18b_3v_1 + 2v_2b_3 - 6a_1 - 3a_2 - b_1 + 3b_3 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$\begin{aligned} &-18v_3a_3v_1 + (-24a_2 - b_2 + 18b_3)\, v_1 - 3v_3a_3v_2 \\ &+ (-3a_2 - 6a_3 + 2b_3)\, v_2 - 3v_3a_3 + 3b_2v_4 - 6a_1 - 3a_2 - b_1 + 3b_3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -18a_3 &= 0 \\ -3a_3 &= 0 \\ 3b_2 &= 0 \\ -24a_2 - b_2 + 18b_3 &= 0 \\ -3a_2 - 6a_3 + 2b_3 &= 0 \\ -6a_1 - 3a_2 - b_1 + 3b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= a_1 \\ a_2 &= 0 \\ a_3 &= 0 \\ b_1 &= -6a_1 \\ b_2 &= 0 \\ b_3 &= 0 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= 1 \\ \eta &= -6 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= -6 - \left((1+6x+y)^{1/3}\right)(1) \\ &= -(1+6x+y)^{1/3} - 6 \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-(1+6x+y)^{1/3} - 6} dy \end{aligned}$$

Which results in

$$S = -\frac{3(1+6x+y)^{2/3}}{2} + 36\ln\left((1+6x+y)^{2/3} - 6(1+6x+y)^{1/3} + 36\right) - 72\ln\left((1+6x+y)^{1/3} + 6\right) -$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = (1+6x+y)^{1/3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{6}{(1+6x+y)^{1/3} + 6} \\ S_y &= \frac{1}{-(1+6x+y)^{1/3} - 6} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

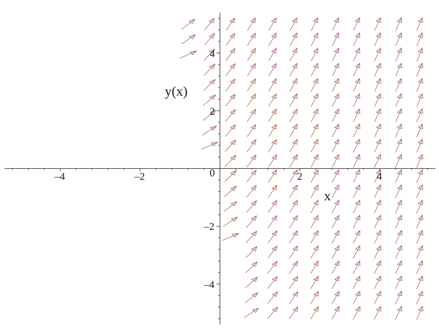
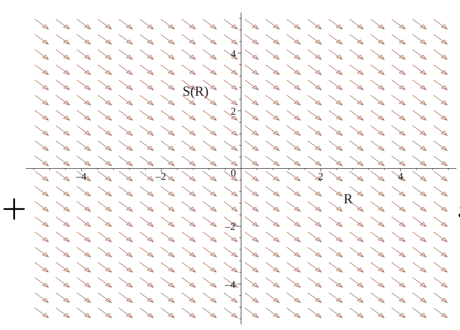
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int -1\, dR$$
$$S(R) = -R + c_2$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$-\frac{3(1+6x+y)^{2/3}}{2} + 36\ln\left((1+6x+y)^{2/3} - 6(1+6x+y)^{1/3} + 36\right) - 72\ln\left((1+6x+y)^{1/3} + 6\right) - 36\ln($$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = (1+6x+y)^{1/3}$ 	$R = x$ $S = -\frac{3(1+6x+y)^{2/3}}{2} + \quad x+y)^{1/3}$	$\frac{dS}{dR} = -1$ 

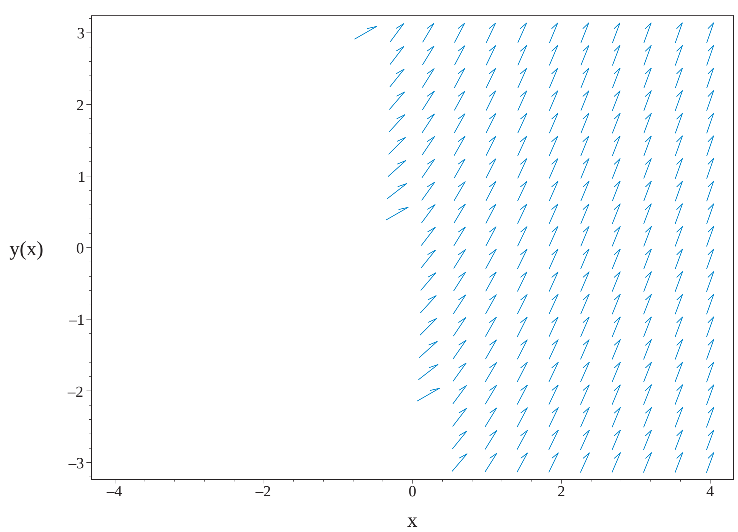


Figure 2.81: Slope field plot
$y' = (1+6x+y)^{1/3}$

Summary of solutions found

$$-\frac{3(1+6x+y)^{2/3}}{2} + 36\ln\left((1+6x+y)^{2/3}\right.$$
$$\left.-6(1+6x+y)^{1/3}+36\right) - 72\ln\left((1+6x+y)^{1/3}+6\right) - 36\ln\left(217+6x+y\right) + 18(1+6x+y)^{1/3} = -x + c_2$$

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) = (1 + 6x + y(x))^{1/3}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = (1 + 6x + y(x))^{1/3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

Maple dsolve solution

Solving time : 0.029 (sec)
Leaf size : 79

```
dsolve(diff(y(x),x) = (1+6*x+y(x))^(1/3),
       y(x),singsol=all)
```

$$x - \frac{3(1 + 6x + y)^{2/3}}{2} - 72 \ln \left(6 + (1 + 6x + y)^{1/3}\right) + 36 \ln \left((1+6x+y)^{2/3} - 6(1+6x+y)^{1/3} + 36\right) - 36 \ln (217+y+6x) + 18(1+6x+y)^{1/3} - c_1 = 0$$

Mathematica DSolve solution

Solving time : 0.229 (sec)
Leaf size : 66

```
DSolve[{D[y[x],x]==(1+6*x+y[x])^(1/3),{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$\text{Solve}\left[\frac{1}{6}\left(y(x) - 9(y(x) + 6x + 1)^{2/3} + 108 \sqrt[3]{y(x) + 6x + 1} - 648 \log \left(\sqrt[3]{y(x) + 6x + 1} + 6\right) + 6x + 1\right) - \frac{y(x)}{6} = c_1, y(x)\right]$$

2.1.62 problem 62

Internal problem ID [8722]
Book : First order enumerated odes
Section : section 1
Problem number : 62
Date solved : `Tuesday, December 17, 2024 at 12:58:31 PM`
CAS classification : `[[_homogeneous, 'class C'], _dAlembert]`

Solve

$$y' = (1 + 6x + y)^{1/4}$$

Solved as first order homogeneous class C ode

Time used: 0.237 (sec)

Let

$$z = 1 + 6x + y \tag{1}$$

Then

$$z'(x) = 6 + y'$$

Therefore

$$y' = z'(x) - 6$$

Hence the given ode can now be written as

$$z'(x) - 6 = z^{1/4}$$

This is separable first order ode. Integrating

$$\begin{aligned}\int dx &= \int \frac{1}{z^{1/4} + 6} dz \\ x + c_1 &= -216 \ln(-z + 1296) - 12\sqrt{z} + 216 \ln\left(\sqrt{z} + 36\right) - 216 \ln\left(\sqrt{z} - 36\right) \\ &\quad + 144 z^{1/4} - 432 \ln\left(z^{1/4} + 6\right) + 432 \ln\left(z^{1/4} - 6\right) + \frac{4 z^{3/4}}{3}\end{aligned}$$

Replacing z back by its value from (1) then the above gives the solution as

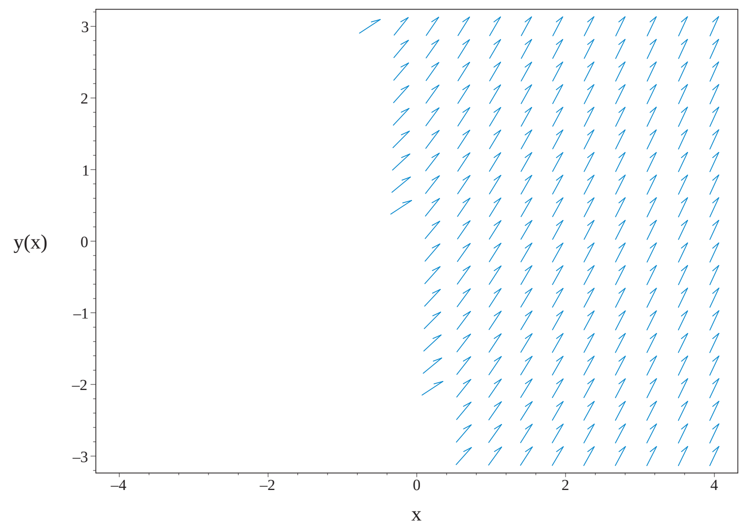


Figure 2.82: Slope field plot
$y' = (1 + 6x + y)^{1/4}$

Summary of solutions found

$$-216 \ln (1295 - 6x - y) - 12\sqrt{1 + 6x + y} + 216 \ln \left(\sqrt{1 + 6x + y} + 36\right) - 216 \ln \left(\sqrt{1 + 6x + y} - 36\right) + 144(1 + 6x + y)^{1/4} - 432 \ln \left((1+6x+y)^{1/4} + 6\right) + 432 \ln \left((1+6x+y)^{1/4} - 6\right) + \frac{4(1 + 6x + y)^{3/4}}{3} = x + c_1$$

Solved using Lie symmetry for first order ode

Time used: 0.756 (sec)

Writing the ode as

$$y' = (1 + 6x + y)^{1/4}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + (1+6x+y)^{1/4} (b_3 - a_2) - \sqrt{1 + 6x + y}\, a_3 - \frac{3(x a_2 + y a_3 + a_1)}{2 (1 + 6x + y)^{3/4}} - \frac{x b_2 + y b_3 + b_1}{4 (1 + 6x + y)^{3/4}} = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{4(1 + 6x + y)^{5/4} a_3 - 4 b_2 (1 + 6x + y)^{3/4} + 30 x a_2 + x b_2 - 24 b_3 x + 4 a_2 y + 6 y a_3 - 3 y b_3 + 6 a_1 + 4 a_2 + b_1}{4 (1 + 6x + y)^{3/4}}$$

Setting the numerator to zero gives

$$-4(1+6x+y)^{5/4}\,a_3+4b_2(1+6x+y)^{3/4}-30xa_2-xb_2 \\ +24b_3x-4a_2y-6ya_3+3yb_3-6a_1-4a_2-b_1+4b_3=0 \tag{6E}$$

Simplifying the above gives

$$-4(1+6x+y)\,a_2+4(1+6x+y)\,b_3-4(1+6x+y)^{5/4}\,a_3 \\ +4b_2(1+6x+y)^{3/4}-6xa_2-xb_2-6ya_3-yb_3-6a_1-b_1=0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$4b_2(1+6x+y)^{3/4}-24(1+6x+y)^{1/4}\,a_3x-4(1+6x+y)^{1/4}\,a_3y-30xa_2-xb_2 \\ +24b_3x-4(1+6x+y)^{1/4}\,a_3-4a_2y-6ya_3+3yb_3-6a_1-4a_2-b_1+4b_3=0$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$\left\{x,y,(1+6x+y)^{1/4},(1+6x+y)^{3/4}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$\left\{x=v_1,y=v_2,(1+6x+y)^{1/4}=v_3,(1+6x+y)^{3/4}=v_4\right\}$$

The above PDE (6E) now becomes

$$-24v_3a_3v_1-4v_3a_3v_2-30v_1a_2-4a_2v_2-6v_2a_3-4v_3a_3 \\ -v_1b_2+4b_2v_4+24b_3v_1+3v_2b_3-6a_1-4a_2-b_1+4b_3=0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1,v_2,v_3,v_4\}$$

Equation (7E) now becomes

$$-24v_3a_3v_1+(-30a_2-b_2+24b_3)\,v_1-4v_3a_3v_2 \\ +(-4a_2-6a_3+3b_3)\,v_2-4v_3a_3+4b_2v_4-6a_1-4a_2-b_1+4b_3=0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-24a_3&=0\\-4a_3&=0\\4b_2&=0\\-30a_2-b_2+24b_3&=0\\-4a_2-6a_3+3b_3&=0\\-6a_1-4a_2-b_1+4b_3&=0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1&=a_1\\a_2&=0\\a_3&=0\\b_1&=-6a_1\\b_2&=0\\b_3&=0\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = 1$$
$$\eta = -6$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi} \\ &= \frac{-6}{1} \\ &= -6\end{aligned}$$

This is easily solved to give

$$y = -6x + c_1$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = 6x + y$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi} \\ &= \frac{dx}{1}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T} \\ &= x\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = (1 + 6x + y)^{1/4}$$

Evaluating all the partial derivatives gives

$$R_x = 6$$
$$R_y = 1$$
$$S_x = 1$$
$$S_y = 0$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{(1+6x+y)^{1/4}+6} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{(1+R)^{1/4}+6}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

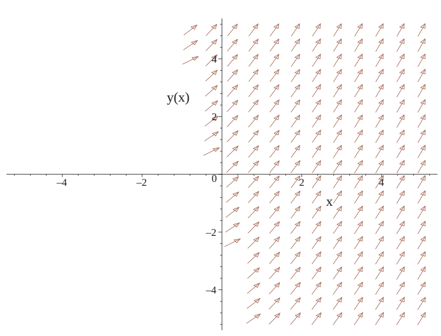
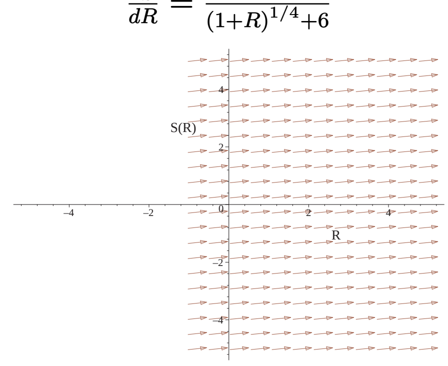
$$\int dS = \int \frac{1}{(1+R)^{1/4}+6}\, dR$$

$$S(R) = \frac{4(1+R)^{3/4}}{3} - 12\sqrt{1+R} + 144(1+R)^{1/4} - 864\ln\left((1+R)^{1/4}+6\right) + c_2$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$x = \frac{4(1+6x+y)^{3/4}}{3} - 12\sqrt{1+6x+y} + 144(1+6x+y)^{1/4} - 864\ln\left((1+6x+y)^{1/4}+6\right) + c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = (1+6x+y)^{1/4}$ 	$R = 6x + y$ $S = x$	$\frac{dS}{dR} = \frac{1}{(1+R)^{1/4}+6}$ 

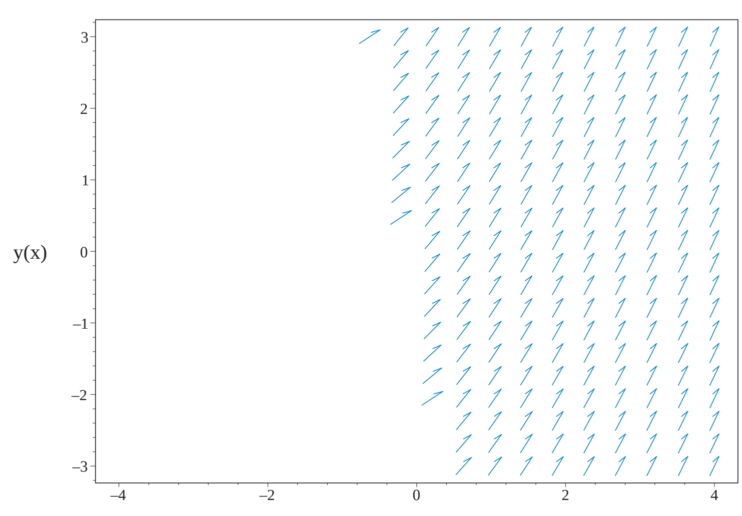


Figure 2.83: Slope field plot
$y' = (1+6x+y)^{1/4}$

Summary of solutions found

$$x = \frac{4(1+6x+y)^{3/4}}{3} - 12\sqrt{1+6x+y} + 144(1+6x+y)^{1/4} - 864\ln\left((1+6x+y)^{1/4}+6\right) + c_2$$

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) = (1+6x+y(x))^{1/4}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = (1+6x+y(x))^{1/4}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

Maple dsolve solution

Solving time : 0.030 (sec)
Leaf size : 109

```
dsolve(diff(y(x),x) = (1+6*x+y(x))^(1/4),
       y(x),singsol=all)
```

$$x + 216\ln(-y-6x+1295) + 12\sqrt{1+6x+y} - 216\ln\left(\sqrt{1+6x+y}+36\right) + 216\ln\left(\sqrt{1+6x+y}-36\right) - 144(1+6x+y)^{1/4} + 432\ln\left(6+(1+6x+y)^{1/4}\right) - 432\ln\left((1+6x+y)^{1/4}-6\right) - \frac{4(1+6x+y)^{3/4}}{3} - c_1 = 0$$

Mathematica DSolve solution

Solving time : 0.325 (sec)
Leaf size : 79

```
DSolve[{D[y[x],x]==(1+6*x+y[x])^(1/4),{}},
      y[x],x,IncludeSingularSolutions->True]
```

$$\text{Solve}\left[\frac{1}{6}\left(y(x) - 8(y(x)+6x+1)^{3/4} + 72\sqrt{y(x)+6x+1} - 864\sqrt[4]{y(x)+6x+1} + 5184\log\left(\sqrt[4]{y(x)+6x+1}+6\right) + 6x + 1\right) - \frac{y(x)}{6} = c_1, y(x)\right]$$

2.1.63 problem 63

Internal problem ID [8723]
Book : First order enumerated odes
Section : section 1
Problem number : 63
Date solved : `Tuesday, December 17, 2024 at 12:58:33 PM`
CAS classification : `[[_homogeneous, 'class C'], _dAlembert]`

Solve

$$y' = (a + bx + y)^4$$

Solved as first order homogeneous class C ode

Time used: 0.510 (sec)

Let

$$z = a + bx + y \tag{1}$$

Then

$$z'(x) = b + y'$$

Therefore

$$y' = z'(x) - b$$

Hence the given ode can now be written as

$$z'(x) - b = z^4$$

This is separable first order ode. Integrating

$$\int dx = \int \frac{1}{z^4 + b} dz$$

$$x + c_1 = \frac{\sqrt{2}\left(\ln\left(\frac{z^2+b^{1/4}z\sqrt{2}+\sqrt{b}}{z^2-b^{1/4}z\sqrt{2}+\sqrt{b}}\right) + 2\arctan\left(\frac{\sqrt{2}\,z}{b^{1/4}} + 1\right) + 2\arctan\left(\frac{\sqrt{2}\,z}{b^{1/4}} - 1\right)\right)}{8b^{3/4}}$$

Replacing z back by its value from (1) then the above gives the solution as

Summary of solutions found

$$\frac{\sqrt{2}\left(\ln\left(\frac{(a+bx+y)^2+b^{1/4}(a+bx+y)\sqrt{2}+\sqrt{b}}{(a+bx+y)^2-b^{1/4}(a+bx+y)\sqrt{2}+\sqrt{b}}\right) + 2\arctan\left(\frac{\sqrt{2}\,(a+bx+y)}{b^{1/4}} + 1\right) + 2\arctan\left(\frac{\sqrt{2}\,(a+bx+y)}{b^{1/4}} - 1\right)\right)}{8b^{3/4}} = x + c_1$$

Solved using Lie symmetry for first order ode

Time used: 0.961 (sec)

Writing the ode as

$$y' = (bx + a + y)^4$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + (bx + a + y)^4 (b_3 - a_2) - (bx + a + y)^8 a_3 \\ - 4(bx + a + y)^3 b(xa_2 + ya_3 + a_1) - 4(bx + a + y)^3 (xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

Expression too large to display (7E)

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&-56a\,b^6a_3v_1^6v_2-168a^2b^5a_3v_1^5v_2-168a\,b^5a_3v_1^5v_2^2\\
&-280a^3b^4a_3v_1^4v_2-420a^2b^4a_3v_1^4v_2^2-280a\,b^4a_3v_1^4v_2^3\\
&-560a^3b^3a_3v_1^3v_2^2-560a^2b^3a_3v_1^3v_2^3-280a\,b^3a_3v_1^3v_2^4\\
&-560a^3b^2a_3v_1^2v_2^3-420a^2b^2a_3v_1^2v_2^4-168a\,b^2a_3v_1^2v_2^5\\
&-280a^3ba_3v_1v_2^4-168a^2ba_3v_1v_2^5-56aba_3v_1v_2^6\\
&+\left(-280a^4b^3a_3-4b^4a_3-16b^3a_2-12b^2b_2\right)v_1^3v_2\\
&+\left(-420a^4b^2a_3-12b^3a_3-18b^2a_2-6b^2b_3-12bb_2\right)v_1^2v_2^2\\
&+\left(-168a^5b^2a_3-12a\,b^3a_3-36a\,b^2a_2\right.\\
&\left.-12b^3a_1-24abb_2-12b^2b_1\right)v_1^2v_2\\
&+\left(-280a^4ba_3-12b^2a_3-8ba_2-8bb_3-4b_2\right)v_1v_2^3\\
&+\left(-168a^5ba_3-24a\,b^2a_3-24aba_2-12abb_3-12b^2a_1-12ab_2\right.\\
&\left.-12bb_1\right)v_1v_2^2+\left(-56a^6ba_3-12a^2b^2a_3-24a^2ba_2-24a\,b^2a_1\right.\\
&\left.-12a^2b_2-24abb_1\right)v_1v_2-28b^2a_3v_1^2v_2^6-8ba_3v_1v_2^7-8a\,b^7a_3v_1^7\\
&-8b^7a_3v_1^7v_2-28a^2b^6a_3v_1^6-28b^6a_3v_1^6v_2^2-56a^3b^5a_3v_1^5\\
&-56b^5a_3v_1^5v_2^3-70b^4a_3v_1^4v_2^4-56b^3a_3v_1^3v_2^5+b_2-4a^3ba_1\\
&+\left(-70a^4b^4a_3-5b^4a_2+b^4b_3-4b^3b_2\right)v_1^4+\left(-56a^5b^3a_3\right.\\
&\left.-16a\,b^3a_2+4a\,b^3b_3-4b^4a_1-12a\,b^2b_2-4b^3b_1\right)v_1^3\\
&+\left(-28a^6b^2a_3-18a^2b^2a_2+6a^2b^2b_3-12a\,b^3a_1-12a^2bb_2\right.\\
&\left.-12a\,b^2b_1\right)v_1^2+\left(-8a^7ba_3-8a^3ba_2+4a^3bb_3-12a^2b^2a_1\right.\\
&\left.-4a^3b_2-12a^2bb_1\right)v_1+\left(-70a^4a_3-4ba_3-a_2-3b_3\right)v_2^4\\
&+\left(-56a^5a_3-12aba_3-4aa_2-8ab_3-4ba_1-4b_1\right)v_2^3\\
&+\left(-28a^6a_3-12a^2ba_3-6a^2a_2-6a^2b_3-12aba_1-12ab_1\right)v_2^2\\
&+\left(-8a^7a_3-4a^3ba_3-4a^3a_2-12a^2ba_1-12a^2b_1\right)v_2\\
&-b^8a_3v_1^8-56a^3a_3v_2^5-28a^2a_3v_2^6-8aa_3v_2^7\\
&-a^4a_2+a^4b_3-a^8a_3-4a^3b_1-a_3v_2^8=0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-a_3 &= 0\\
-8aa_3 &= 0\\
-28a^2a_3 &= 0\\
-56a^3a_3 &= 0\\
-8ba_3 &= 0\\
-28b^2a_3 &= 0\\
-56b^3a_3 &= 0\\
-70b^4a_3 &= 0\\
-56b^5a_3 &= 0\\
-28b^6a_3 &= 0\\
-8b^7a_3 &= 0\\
-b^8a_3 &= 0\\
-56aba_3 &= 0\\
-168a\,b^2a_3 &= 0\\
-280a\,b^3a_3 &= 0\\
-280a\,b^4a_3 &= 0\\
-168a\,b^5a_3 &= 0\\
-56a\,b^6a_3 &= 0\\
-8a\,b^7a_3 &= 0\\
-168a^2ba_3 &= 0\\
-420a^2b^2a_3 &= 0\\
-560a^2b^3a_3 &= 0\\
-420a^2b^4a_3 &= 0\\
-168a^2b^5a_3 &= 0\\
-28a^2b^6a_3 &= 0\\
-280a^3ba_3 &= 0\\
-560a^3b^2a_3 &= 0\\
-560a^3b^3a_3 &= 0\\
-280a^3b^4a_3 &= 0\\
-56a^3b^5a_3 &= 0\\
-280a^4b^3a_3 - 4b^4a_3 - 16b^3a_2 - 12b^2b_2 &= 0\\
-70a^4a_3 - 4ba_3 - a_2 - 3b_3 &= 0\\
-70a^4b^4a_3 - 5b^4a_2 + b^4b_3 - 4b^3b_2 &= 0\\
-8a^7a_3 - 4a^3ba_3 - 4a^3a_2 - 12a^2ba_1 - 12a^2b_1 &= 0\\
-280a^4ba_3 - 12b^2a_3 - 8ba_2 - 8bb_3 - 4b_2 &= 0\\
-420a^4b^2a_3 - 12b^3a_3 - 18b^2a_2 - 6b^2b_3 - 12bb_2 &= 0\\
-168a^5b^2a_3 - 12a\,b^3a_3 - 36a\,b^2a_2 - 12b^3a_1 - 24abb_2 - 12b^2b_1 &= 0\\
-56a^6ba_3 - 12a^2b^2a_3 - 24a^2ba_2 - 24a\,b^2a_1 - 12a^2b_2 - 24abb_1 &= 0\\
-56a^5a_3 - 12aba_3 - 4aa_2 - 8ab_3 - 4ba_1 - 4b_1 &= 0\\
-28a^6a_3 - 12a^2ba_3 - 6a^2a_2 - 6a^2b_3 - 12aba_1 - 12ab_1 &= 0\\
-56a^5b^3a_3 - 16a\,b^3a_2 + 4a\,b^3b_3 - 4b^4a_1 - 12a\,b^2b_2 - 4b^3b_1 &= 0\\
-28a^6b^2a_3 - 18a^2b^2a_2 + 6a^2b^2b_3 - 12a\,b^3a_1 - 12a^2bb_2 - 12a\,b^2b_1 &= 0\\
-8a^7ba_3 - 8a^3ba_2 + 4a^3bb_3 - 12a^2b^2a_1 - 4a^3b_2 - 12a^2bb_1 &= 0\\
-a^8a_3 - a^4a_2 + a^4b_3 - 4a^3ba_1 - 4a^3b_1 + b_2 &= 0\\
-168a^5ba_3 - 24a\,b^2a_3 - 24aba_2 - 12abb_3 - 12b^2a_1 - 12ab_2 - 12bb_1 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= a_1 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= -ba_1 \\
b_2 &= 0 \\
b_3 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 1 \\
\eta &= -b
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{-b}{1} \\
&= -b
\end{aligned}
$$

This is easily solved to give

$$
y = -bx + c_1
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = bx + y
$$

And S is found from

$$
\begin{aligned}
dS &= \frac{dx}{\xi} \\
&= \frac{dx}{1}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
S &= \int \frac{dx}{T} \\
&= x
\end{aligned}
$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = (bx + a + y)^4$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= b \\ R_y &= 1 \\ S_x &= 1 \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{b + (bx + a + y)^4} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{b + (R + a)^4}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int \frac{1}{R^4 + 4R^3a + 6R^2a^2 + 4R\,a^3 + a^4 + b}\,dR$$

$$S(R) = \frac{\left(\sum_{\_R=\text{RootOf}(\_Z^4+4\_Z^3a+6\_Z^2a^2+4a^3\_Z+a^4+b)} \frac{\ln(R-\_R)}{\_R^3+3\_R^2a+3\_R\,a^2+a^3}\right)}{4} + c_2$$

$$S(R) = \int \frac{1}{R^4 + 4R^3a + 6R^2a^2 + 4R\,a^3 + a^4 + b}dR + c_2$$

This results in

$$x = \int^{y} \frac{1}{(bx + \_a)^4 + 4(bx + \_a)^3 a + 6(bx + \_a)^2 a^2 + 4(bx + \_a)a^3 + a^4 + b} d\_a + c_2$$

Summary of solutions found

$$x = \int^{y} \frac{1}{(bx + \_a)^4 + 4(bx + \_a)^3 a + 6(bx + \_a)^2 a^2 + 4(bx + \_a)a^3 + a^4 + b} d\_a + c_2$$

Maple step by step solution

Let's solve
$\frac{d}{dx}y(x) = (a + bx + y(x))^4$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = (a + bx + y(x))^4$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -b, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

Maple dsolve solution

Solving time : 0.040 (sec)
Leaf size : 49

```
dsolve(diff(y(x),x) = (a+b*x+y(x))^4,
       y(x),singsol=all)
```

$$y = -bx + \operatorname{RootOf}\left(-x + \int^{\_Z} \frac{1}{\_a^4 + 4\_a^3 a + 6\_a^2 a^2 + 4\_a\, a^3 + a^4 + b} d\_a + c_1\right)$$

Mathematica DSolve solution

Solving time : 0.413 (sec)
Leaf size : 163

```
DSolve[{D[y[x],x]==(a+b*x+y[x])^(4),{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$\text{Solve}\left[\frac{2\sqrt{2}\arctan\left(1-\frac{\sqrt{2}(a+bx+y(x))}{\sqrt[4]{b}}\right)-2\sqrt{2}\arctan\left(\frac{\sqrt{2}(a+bx+y(x))}{\sqrt[4]{b}}+1\right)+\sqrt{2}\log\left((a+bx+y(x))^2-\sqrt{2}\right.}{8b^{3/4}}\right.$$

2.1.64 problem 64

Internal problem ID [8724]
Book : First order enumerated odes
Section : section 1
Problem number : 64
Date solved : `Tuesday, December 17, 2024 at 01:01:14 PM`
CAS classification : `[[_homogeneous, 'class C'], _dAlembert]`

Solve

$$y' = (\pi + x + 7y)^{7/2}$$

Solved as first order homogeneous class C ode

Time used: 0.861 (sec)

Let

$$z = \pi + x + 7y \tag{1}$$

Then

$$z'(x) = 1 + 7y'$$

Therefore

$$y' = \frac{z'(x)}{7} - \frac{1}{7}$$

Hence the given ode can now be written as

$$\frac{z'(x)}{7} - \frac{1}{7} = z^{7/2}$$

This is separable first order ode. Integrating

$$\int dx = \int \frac{1}{7z^{7/2}+1} dz$$

$$x + c_1 = -\frac{\left(\sum_{\_R=\mathrm{RootOf}(49\_Z^7-1)} \frac{\ln(z-\_\_R)}{\_R^6}\right)}{343} + \frac{\left(\sum_{\_R=\mathrm{RootOf}(7\_Z^7+1)} \frac{\ln(\sqrt{z}-\_\_R)}{\_R^5}\right)}{49} + \frac{\left(\sum_{\_R=\mathrm{RootOf}(7\_Z^7-1)} \frac{\ln(\sqrt{z}-\_\_R)}{\_R^5}\right)}{49}$$

Replacing z back by its value from (1) then the above gives the solution as

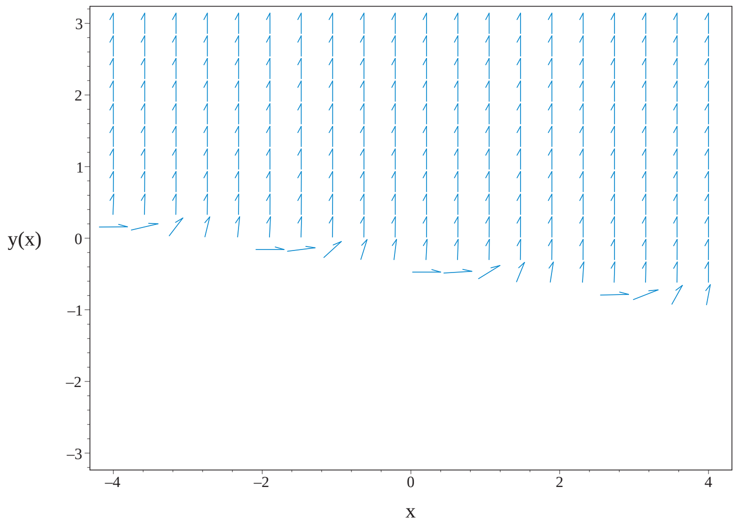


Figure 2.84: Slope field plot
$y' = (\pi + x + 7y)^{7/2}$

Summary of solutions found

$$-\frac{\left(\sum_{\_R=\mathrm{RootOf}(49\_Z^7-1)} \frac{\ln(\pi+x+7y-\_R)}{\_R^6}\right)}{343} + \frac{\left(\sum_{\_R=\mathrm{RootOf}(7\_Z^7+1)} \frac{\ln(\sqrt{\pi+x+7y}-\_R)}{\_R^5}\right)}{49} + \frac{\left(\sum_{\_R=\mathrm{RootOf}(7\_Z^7-1)} \frac{\ln(\sqrt{\pi+x+7y}-\_R)}{\_R^5}\right)}{49} = x + c_1$$

Solved using Lie symmetry for first order ode

Time used: 2.738 (sec)

Writing the ode as

$$y' = (\pi + x + 7y)^{7/2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + (\pi + x + 7y)^{7/2} (b_3 - a_2) - (\pi + x + 7y)^7 a_3 - \frac{7(\pi + x + 7y)^{5/2} (x a_2 + y a_3 + a_1)}{2} - \frac{49(\pi + x + 7y)^{5/2} (x b_2 + y b_3 + b_1)}{2} = 0 \tag{5E}$$

Putting the above in normal form gives

$$\begin{aligned}&-x^7a_3-823543y^7a_3+(\pi+x+7y)^{7/2}\,b_3-(\pi+x+7y)^{7/2}\,a_2-\pi^7a_3\\&-\frac{7(\pi+x+7y)^{5/2}\,a_1}{2}-\frac{49(\pi+x+7y)^{5/2}\,b_1}{2}+b_2-\frac{7(\pi+x+7y)^{5/2}\,xa_2}{2}\\&-\frac{7(\pi+x+7y)^{5/2}\,ya_3}{2}-\frac{49(\pi+x+7y)^{5/2}\,xb_2}{2}-\frac{49(\pi+x+7y)^{5/2}\,yb_3}{2}\\&-7\pi^6xa_3-49\pi^6ya_3-21\pi^5x^2a_3-1029\pi^5y^2a_3-35\pi^4x^3a_3-12005\pi^4y^3a_3\\&-35\pi^3x^4a_3-84035\pi^3y^4a_3-21\pi^2x^5a_3-352947\pi^2y^5a_3-7\pi\,x^6a_3\\&-823543\pi\,y^6a_3-49x^6ya_3-1029x^5y^2a_3-12005x^4y^3a_3-84035x^3y^4a_3\\&-352947x^2y^5a_3-823543x\,y^6a_3-735\pi^4x^2ya_3-5145\pi^4x\,y^2a_3-980\pi^3x^3ya_3\\&-10290\pi^3x^2y^2a_3-48020\pi^3x\,y^3a_3-735\pi^2x^4ya_3-10290\pi^2x^3y^2a_3\\&-72030\pi^2x^2y^3a_3-252105\pi^2x\,y^4a_3-294\pi\,x^5ya_3-5145\pi\,x^4y^2a_3\\&-48020\pi\,x^3y^3a_3-252105\pi\,x^2y^4a_3-705894\pi x\,y^5a_3-294\pi^5xya_3=0\end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned}&-2x^7a_3-1647086y^7a_3+2(\pi+x+7y)^{7/2}\,b_3-2(\pi+x+7y)^{7/2}\,a_2-2\pi^7a_3-7(\pi+x+7y)^{5/2}\,a_1\\&-49(\pi+x+7y)^{5/2}\,b_1+2b_2-7(\pi+x+7y)^{5/2}\,xa_2-7(\pi+x+7y)^{5/2}\,ya_3-49(\pi+x+7y)^{5/2}\,xb_2-49(\pi+x+7y)^{5}\end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&-2x^7a_3 - 1647086y^7a_3 - 2\pi^7a_3 + 2b_2 \\
&- 14\pi x\sqrt{\pi + x + 7y}\, yb_3 - 2\pi^3\sqrt{\pi + x + 7y}\, a_2 \\
&+ 2\pi^3\sqrt{\pi + x + 7y}\, b_3 - 9x^3\sqrt{\pi + x + 7y}\, a_2 \\
&- 49x^3\sqrt{\pi + x + 7y}\, b_2 + 2x^3\sqrt{\pi + x + 7y}\, b_3 \\
&- 686\sqrt{\pi + x + 7y}\, y^3a_2 - 343\sqrt{\pi + x + 7y}\, y^3a_3 \\
&- 1715\sqrt{\pi + x + 7y}\, y^3b_3 - 7\pi^2\sqrt{\pi + x + 7y}\, a_1 \\
&- 49\pi^2\sqrt{\pi + x + 7y}\, b_1 - 7x^2\sqrt{\pi + x + 7y}\, a_1 \\
&- 49x^2\sqrt{\pi + x + 7y}\, b_1 - 343\sqrt{\pi + x + 7y}\, y^2a_1 \\
&- 2401\sqrt{\pi + x + 7y}\, y^2b_1 - 182\pi x\sqrt{\pi + x + 7y}\, ya_2 \\
&- 14\pi x\sqrt{\pi + x + 7y}\, ya_3 - 686\pi x\sqrt{\pi + x + 7y}\, yb_2 \\
&- 14\pi^6xa_3 - 98\pi^6ya_3 - 42\pi^5x^2a_3 - 2058\pi^5y^2a_3 \\
&- 70\pi^4x^3a_3 - 24010\pi^4y^3a_3 - 70\pi^3x^4a_3 - 168070\pi^3y^4a_3 \\
&- 42\pi^2x^5a_3 - 705894\pi^2y^5a_3 - 14\pi x^6a_3 - 1647086\pi y^6a_3 \\
&- 98x^6ya_3 - 2058x^5y^2a_3 - 24010x^4y^3a_3 \\
&- 168070x^3y^4a_3 - 705894x^2y^5a_3 - 1647086x y^6a_3 \\
&- 13\pi^2x\sqrt{\pi + x + 7y}\, a_2 - 49\pi^2x\sqrt{\pi + x + 7y}\, b_2 \\
&+ 6\pi^2x\sqrt{\pi + x + 7y}\, b_3 - 42\pi^2\sqrt{\pi + x + 7y}\, ya_2 \\
&- 7\pi^2\sqrt{\pi + x + 7y}\, ya_3 - 7\pi^2\sqrt{\pi + x + 7y}\, yb_3 \\
&- 20\pi x^2\sqrt{\pi + x + 7y}\, a_2 - 98\pi x^2\sqrt{\pi + x + 7y}\, b_2 \\
&+ 6\pi x^2\sqrt{\pi + x + 7y}\, b_3 - 294\pi\sqrt{\pi + x + 7y}\, y^2a_2 \\
&- 98\pi\sqrt{\pi + x + 7y}\, y^2a_3 - 392\pi\sqrt{\pi + x + 7y}\, y^2b_3 \\
&- 140x^2\sqrt{\pi + x + 7y}\, ya_2 - 7x^2\sqrt{\pi + x + 7y}\, ya_3 \\
&- 686x^2\sqrt{\pi + x + 7y}\, yb_2 - 7x^2\sqrt{\pi + x + 7y}\, yb_3 \\
&- 637x\sqrt{\pi + x + 7y}\, y^2a_2 - 98x\sqrt{\pi + x + 7y}\, y^2a_3 \\
&- 2401x\sqrt{\pi + x + 7y}\, y^2b_2 - 392x\sqrt{\pi + x + 7y}\, y^2b_3 \\
&- 14\pi x\sqrt{\pi + x + 7y}\, a_1 - 98\pi x\sqrt{\pi + x + 7y}\, b_1 \\
&- 98\pi\sqrt{\pi + x + 7y}\, ya_1 - 686\pi\sqrt{\pi + x + 7y}\, yb_1 \\
&- 98x\sqrt{\pi + x + 7y}\, ya_1 - 686x\sqrt{\pi + x + 7y}\, yb_1 \\
&- 1470\pi^4x^2ya_3 - 10290\pi^4x y^2a_3 - 1960\pi^3x^3ya_3 \\
&- 20580\pi^3x^2y^2a_3 - 96040\pi^3x y^3a_3 - 1470\pi^2x^4ya_3 \\
&- 20580\pi^2x^3y^2a_3 - 144060\pi^2x^2y^3a_3 - 504210\pi^2x y^4a_3 \\
&- 588\pi x^5ya_3 - 10290\pi x^4y^2a_3 - 96040\pi x^3y^3a_3 \\
&- 504210\pi x^2y^4a_3 - 1411788\pi x y^5a_3 - 588\pi^5xya_3 = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{\pi + x + 7y}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\left\{x = v_1, y = v_2, \sqrt{\pi + x + 7y} = v_3\right\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-2\pi^7 a_3 - 14\pi^6 v_1 a_3 - 98\pi^6 v_2 a_3 - 42\pi^5 v_1^2 a_3 - 588\pi^5 v_1 v_2 a_3 \\
&- 2058\pi^5 v_2^2 a_3 - 70\pi^4 v_1^3 a_3 - 1470\pi^4 v_1^2 v_2 a_3 - 10290\pi^4 v_1 v_2^2 a_3 \\
&- 24010\pi^4 v_2^3 a_3 - 70\pi^3 v_1^4 a_3 - 1960\pi^3 v_1^3 v_2 a_3 - 20580\pi^3 v_1^2 v_2^2 a_3 \\
&- 96040\pi^3 v_1 v_2^3 a_3 - 168070\pi^3 v_2^4 a_3 - 42\pi^2 v_1^5 a_3 - 1470\pi^2 v_1^4 v_2 a_3 \\
&- 20580\pi^2 v_1^3 v_2^2 a_3 - 144060\pi^2 v_1^2 v_2^3 a_3 - 504210\pi^2 v_1 v_2^4 a_3 \\
&- 705894\pi^2 v_2^5 a_3 - 14\pi v_1^6 a_3 - 588\pi v_1^5 v_2 a_3 - 10290\pi v_1^4 v_2^2 a_3 \\
&- 96040\pi v_1^3 v_2^3 a_3 - 504210\pi v_1^2 v_2^4 a_3 - 1411788\pi v_1 v_2^5 a_3 \\
&- 1647086\pi v_2^6 a_3 - 2v_1^7 a_3 - 98v_1^6 v_2 a_3 - 2058 v_1^5 v_2^2 a_3 \\
&- 24010 v_1^4 v_2^3 a_3 - 168070 v_1^3 v_2^4 a_3 - 705894 v_1^2 v_2^5 a_3 \\
&- 1647086 v_1 v_2^6 a_3 - 1647086 v_2^7 a_3 - 2\pi^3 v_3 a_2 + 2\pi^3 v_3 b_3 \\
&- 13\pi^2 v_1 v_3 a_2 - 42\pi^2 v_3 v_2 a_2 - 7\pi^2 v_3 v_2 a_3 - 49\pi^2 v_1 v_3 b_2 \\
&+ 6\pi^2 v_1 v_3 b_3 - 7\pi^2 v_3 v_2 b_3 - 20\pi v_1^2 v_3 a_2 - 182\pi v_1 v_3 v_2 a_2 \\
&- 294\pi v_3 v_2^2 a_2 - 14\pi v_1 v_3 v_2 a_3 - 98\pi v_3 v_2^2 a_3 - 98\pi v_1^2 v_3 b_2 \\
&- 686\pi v_1 v_3 v_2 b_2 + 6\pi v_1^2 v_3 b_3 - 14\pi v_1 v_3 v_2 b_3 - 392\pi v_3 v_2^2 b_3 \\
&- 9v_1^3 v_3 a_2 - 140 v_1^2 v_3 v_2 a_2 - 637 v_1 v_3 v_2^2 a_2 - 686 v_3 v_2^3 a_2 \\
&- 7v_1^2 v_3 v_2 a_3 - 98 v_1 v_3 v_2^2 a_3 - 343 v_3 v_2^3 a_3 - 49 v_1^3 v_3 b_2 \\
&- 686 v_1^2 v_3 v_2 b_2 - 2401 v_1 v_3 v_2^2 b_2 + 2v_1^3 v_3 b_3 - 7v_1^2 v_3 v_2 b_3 \\
&- 392 v_1 v_3 v_2^2 b_3 - 1715 v_3 v_2^3 b_3 - 7\pi^2 v_3 a_1 - 49\pi^2 v_3 b_1 - 14\pi v_1 v_3 a_1 \\
&- 98\pi v_3 v_2 a_1 - 98\pi v_1 v_3 b_1 - 686\pi v_3 v_2 b_1 - 7v_1^2 v_3 a_1 - 98 v_1 v_3 v_2 a_1 \\
&- 343 v_3 v_2^2 a_1 - 49 v_1^2 v_3 b_1 - 686 v_1 v_3 v_2 b_1 - 2401 v_3 v_2^2 b_1 + 2b_2 = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&-2\pi^7 a_3 - 1470\pi^4 v_1^2 v_2 a_3 - 10290\pi^4 v_1 v_2^2 a_3 - 1960\pi^3 v_1^3 v_2 a_3 \\
&- 20580\pi^3 v_1^2 v_2^2 a_3 - 96040\pi^3 v_1 v_2^3 a_3 - 1470\pi^2 v_1^4 v_2 a_3 \\
&- 20580\pi^2 v_1^3 v_2^2 a_3 - 144060\pi^2 v_1^2 v_2^3 a_3 - 504210\pi^2 v_1 v_2^4 a_3 \\
&- 588\pi v_1^5 v_2 a_3 - 10290\pi v_1^4 v_2^2 a_3 - 96040\pi v_1^3 v_2^3 a_3 - 504210\pi v_1^2 v_2^4 a_3 \\
&- 1411788\pi v_1 v_2^5 a_3 - 588\pi^5 v_1 v_2 a_3 + (-9a_2 - 49b_2 + 2b_3) v_1^3 v_3 \\
&+ (-20\pi a_2 - 98\pi b_2 + 6\pi b_3 - 7a_1 - 49b_1) v_1^2 v_3 \\
&+ \left(-13\pi^2 a_2 - 49\pi^2 b_2 + 6\pi^2 b_3 - 14\pi a_1 - 98\pi b_1\right) v_1 v_3 \\
&+ (-686a_2 - 343a_3 - 1715b_3) v_2^3 v_3 \\
&+ (-294\pi a_2 - 98\pi a_3 - 392\pi b_3 - 343a_1 - 2401b_1) v_2^2 v_3 \\
&+ \left(-42\pi^2 a_2 - 7\pi^2 a_3 - 7\pi^2 b_3 - 98\pi a_1 - 686\pi b_1\right) v_2 v_3 + 2b_2 \\
&+ \left(-2\pi^3 a_2 + 2\pi^3 b_3 - 7\pi^2 a_1 - 49\pi^2 b_1\right) v_3 - 2v_1^7 a_3 - 1647086 v_2^7 a_3 \\
&- 14\pi^6 v_1 a_3 - 98\pi^6 v_2 a_3 - 42\pi^5 v_1^2 a_3 - 2058\pi^5 v_2^2 a_3 - 70\pi^4 v_1^3 a_3 \\
&- 24010\pi^4 v_2^3 a_3 - 70\pi^3 v_1^4 a_3 - 168070\pi^3 v_2^4 a_3 - 42\pi^2 v_1^5 a_3 \\
&- 705894\pi^2 v_2^5 a_3 - 14\pi v_1^6 a_3 - 1647086\pi v_2^6 a_3 - 98 v_1^6 v_2 a_3 \\
&- 2058 v_1^5 v_2^2 a_3 - 24010 v_1^4 v_2^3 a_3 - 168070 v_1^3 v_2^4 a_3 - 705894 v_1^2 v_2^5 a_3 \\
&- 1647086 v_1 v_2^6 a_3 + (-140a_2 - 7a_3 - 686b_2 - 7b_3) v_1^2 v_2 v_3 \\
&+ (-637a_2 - 98a_3 - 2401b_2 - 392b_3) v_1 v_2^2 v_3 \\
&+ (-182\pi a_2 - 14\pi a_3 - 686\pi b_2 - 14\pi b_3 - 98a_1 - 686b_1) v_1 v_2 v_3 = 0
\end{aligned}
\tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-1647086a_3 &= 0 \\
-705894a_3 &= 0 \\
-168070a_3 &= 0 \\
-24010a_3 &= 0 \\
-2058a_3 &= 0 \\
-98a_3 &= 0 \\
-2a_3 &= 0 \\
-1647086\pi a_3 &= 0 \\
-1411788\pi a_3 &= 0 \\
-504210\pi a_3 &= 0 \\
-96040\pi a_3 &= 0 \\
-10290\pi a_3 &= 0 \\
-588\pi a_3 &= 0 \\
-14\pi a_3 &= 0 \\
-705894\pi^2 a_3 &= 0 \\
-504210\pi^2 a_3 &= 0 \\
-144060\pi^2 a_3 &= 0 \\
-20580\pi^2 a_3 &= 0 \\
-1470\pi^2 a_3 &= 0 \\
-42\pi^2 a_3 &= 0 \\
-168070\pi^3 a_3 &= 0 \\
-96040\pi^3 a_3 &= 0 \\
-20580\pi^3 a_3 &= 0 \\
-1960\pi^3 a_3 &= 0 \\
-70\pi^3 a_3 &= 0 \\
-24010\pi^4 a_3 &= 0 \\
-10290\pi^4 a_3 &= 0 \\
-1470\pi^4 a_3 &= 0 \\
-70\pi^4 a_3 &= 0 \\
-2058\pi^5 a_3 &= 0 \\
-588\pi^5 a_3 &= 0 \\
-42\pi^5 a_3 &= 0 \\
-98\pi^6 a_3 &= 0 \\
-14\pi^6 a_3 &= 0 \\
-686a_2 - 343a_3 - 1715b_3 &= 0 \\
-9a_2 - 49b_2 + 2b_3 &= 0 \\
-637a_2 - 98a_3 - 2401b_2 - 392b_3 &= 0 \\
-140a_2 - 7a_3 - 686b_2 - 7b_3 &= 0 \\
-2\pi^7 a_3 + 2b_2 &= 0 \\
-2\pi^3 a_2 + 2\pi^3 b_3 - 7\pi^2 a_1 - 49\pi^2 b_1 &= 0 \\
-294\pi a_2 - 98\pi a_3 - 392\pi b_3 - 343a_1 - 2401b_1 &= 0 \\
-42\pi^2 a_2 - 7\pi^2 a_3 - 7\pi^2 b_3 - 98\pi a_1 - 686\pi b_1 &= 0 \\
-20\pi a_2 - 98\pi b_2 + 6\pi b_3 - 7a_1 - 49b_1 &= 0 \\
-13\pi^2 a_2 - 49\pi^2 b_2 + 6\pi^2 b_3 - 14\pi a_1 - 98\pi b_1 &= 0 \\
-182\pi a_2 - 14\pi a_3 - 686\pi b_2 - 14\pi b_3 - 98a_1 - 686b_1 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= -7b_1 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= b_1 \\
b_2 &= 0 \\
b_3 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -7 \\
\eta &= 1
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{1}{-7} \\
&= -\frac{1}{7}
\end{aligned}
$$

This is easily solved to give

$$
y = -\frac{x}{7} + c_1
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = \frac{x}{7} + y
$$

And S is found from

$$
\begin{aligned}
dS &= \frac{dx}{\xi} \\
&= \frac{dx}{-7}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
S &= \int \frac{dx}{T} \\
&= -\frac{x}{7}
\end{aligned}
$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = (\pi + x + 7y)^{7/2}$$

Evaluating all the partial derivatives gives

$$R_x = \frac{1}{7}$$
$$R_y = 1$$
$$S_x = -\frac{1}{7}$$
$$S_y = 0$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{1 + 7(\pi + x + 7y)^{7/2}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{1 + 7(\pi + 7R)^{7/2}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int -\frac{1}{1 + 7(\pi + 7R)^{7/2}}\, dR$$

$$S(R) = -\frac{2\left(\sum_{\_R=\text{RootOf}(7\_Z^7+1)} \frac{\ln(\sqrt{\pi+7R}-\_R)}{\_R^5}\right)}{343} + c_2$$

$$S(R) = \int -\frac{1}{1 + 7(\pi + 7R)^{7/2}} dR + c_2$$

This results in

$$-\frac{x}{7} = \int^y -\frac{1}{1 + 7(\pi + x + 7\_a)^{7/2}} d\_a + c_2$$

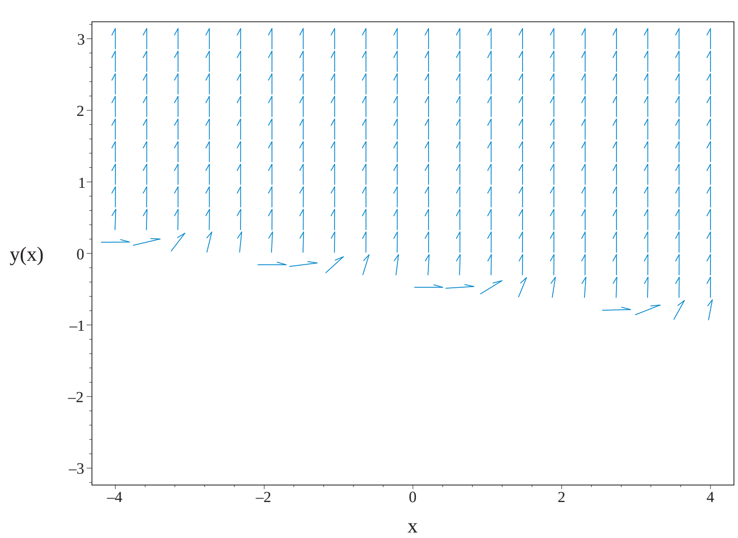


Figure 2.85: Slope field plot
$y' = (\pi + x + 7y)^{7/2}$

Summary of solutions found

$$-\frac{x}{7} = \int^y -\frac{1}{1 + 7(\pi + x + 7\_a)^{7/2}} d\_a + c_2$$

Solved as first order ode of type dAlembert

Time used: 12.446 (sec)

Let $p = y'$ the ode becomes

$$p = (\pi + x + 7y)^{7/2}$$

Solving for y from the above results in

$$y = \frac{p^{2/7}}{7} - \frac{\pi}{7} - \frac{x}{7} \tag{1}$$

$$y = \frac{\left(\cos\left(\frac{2\pi}{7}\right) + i\cos\left(\frac{3\pi}{14}\right)\right)^2 p^{2/7}}{7} - \frac{\pi}{7} - \frac{x}{7} \tag{2}$$

$$y = \frac{\left(-\cos\left(\frac{3\pi}{7}\right) + i\cos\left(\frac{\pi}{14}\right)\right)^2 p^{2/7}}{7} - \frac{\pi}{7} - \frac{x}{7} \tag{3}$$

$$y = \frac{\left(-\cos\left(\frac{\pi}{7}\right) + i\cos\left(\frac{5\pi}{14}\right)\right)^2 p^{2/7}}{7} - \frac{\pi}{7} - \frac{x}{7} \tag{4}$$

$$y = \frac{\left(-\cos\left(\frac{\pi}{7}\right) - i\cos\left(\frac{5\pi}{14}\right)\right)^2 p^{2/7}}{7} - \frac{\pi}{7} - \frac{x}{7} \tag{5}$$

$$y = \frac{\left(-\cos\left(\frac{3\pi}{7}\right) - i\cos\left(\frac{\pi}{14}\right)\right)^2 p^{2/7}}{7} - \frac{\pi}{7} - \frac{x}{7} \tag{6}$$

$$y = \frac{\left(\cos\left(\frac{2\pi}{7}\right) - i\cos\left(\frac{3\pi}{14}\right)\right)^2 p^{2/7}}{7} - \frac{\pi}{7} - \frac{x}{7} \tag{7}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. Each of the above ode's is dAlembert ode which is now solved.

<u>Solving ode 1A</u>

Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= -\frac{1}{7} \\ g &= \frac{p^{2/7}}{7} - \frac{\pi}{7} \end{aligned}$$

Hence (2) becomes

$$p + \frac{1}{7} = \frac{2p'(x)}{49p^{5/7}} \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p + \frac{1}{7} = 0$$

Solving the above for p results in

$$p_1 = -\frac{1}{7}$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$y = \frac{(-1)^{2/7} 7^{5/7}}{49} - \frac{\pi}{7} - \frac{x}{7}$$

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{49\left(p(x) + \frac{1}{7}\right) p(x)^{5/7}}{2} \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed. Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{p(x)} \frac{2}{7(7\tau + 1)\tau^{5/7}} d\tau = x + c_1$$

Singular solutions are found by solving

$$\frac{7(7p+1)p^{5/7}}{2} = 0$$

for $p(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$p(x) = 0$$
$$p(x) = -\frac{1}{7}$$

Substituing the above solution for p in (2A) gives

$$y = \frac{\operatorname{RootOf}\left(-\int^{-Z} \frac{2}{7(7\tau+1)\tau^{5/7}} d\tau + x + c_1\right)^{2/7}}{7} - \frac{\pi}{7} - \frac{x}{7}$$
$$y = -\frac{\pi}{7} - \frac{x}{7}$$
$$y = \frac{(-1)^{2/7} 7^{5/7}}{49} - \frac{\pi}{7} - \frac{x}{7}$$

Solving ode 2A

Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = -\frac{1}{7}$$
$$g = \frac{p^{2/7}\left(-\cos\left(\frac{3\pi}{7}\right) + i\sin\left(\frac{3\pi}{7}\right)\right)}{7} - \frac{\pi}{7}$$

Hence (2) becomes

$$p + \frac{1}{7} = \left(-\frac{2\cos\left(\frac{3\pi}{7}\right)}{49p^{5/7}} + \frac{2i\sin\left(\frac{3\pi}{7}\right)}{49p^{5/7}}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p + \frac{1}{7} = 0$$

Solving the above for p results in

$$p_1 = -\frac{1}{7}$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$y = \frac{7^{5/7}(-1)^{6/7}}{49} - \frac{\pi}{7} - \frac{x}{7}$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) + \frac{1}{7}}{-\frac{2\cos\left(\frac{3\pi}{7}\right)}{49p(x)^{5/7}} + \frac{2i\sin\left(\frac{3\pi}{7}\right)}{49p(x)^{5/7}}} \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed. Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{p(x)} \frac{2(-1)^{4/7}}{7\left(7\tau + 1\right)\tau^{5/7}} d\tau = x + c_2$$

Singular solutions are found by solving

$$-\frac{7(-1)^{3/7}\left(7p + 1\right)p^{5/7}}{2} = 0$$

for $p(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$p(x) = 0$$
$$p(x) = -\frac{1}{7}$$

Substituing the above solution for p in (2A) gives

$$y = -\frac{x}{7} + \frac{\operatorname{RootOf}\left(-\int^{\_Z} \frac{2(-1)^{4/7}}{7(7\tau+1)\tau^{5/7}} d\tau + x + c_2\right)^{2/7}\left(-\cos\left(\frac{3\pi}{7}\right) + i\sin\left(\frac{3\pi}{7}\right)\right)}{7} - \frac{\pi}{7}$$
$$y = -\frac{\pi}{7} - \frac{x}{7}$$
$$y = -\frac{x}{7} + \frac{(-1)^{2/7}\,7^{5/7}\left(-\cos\left(\frac{3\pi}{7}\right) + i\sin\left(\frac{3\pi}{7}\right)\right)}{49} - \frac{\pi}{7}$$

<u>Solving ode 3A</u>

Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = -\frac{1}{7}$$
$$g = \frac{p^{2/7}\left(-i\sin\left(\frac{\pi}{7}\right) - \cos\left(\frac{\pi}{7}\right)\right)}{7} - \frac{\pi}{7}$$

Hence (2) becomes

$$p+\frac{1}{7}=\left(-\frac{2i\sin\left(\frac{\pi}{7}\right)}{49p^{5/7}}-\frac{2\cos\left(\frac{\pi}{7}\right)}{49p^{5/7}}\right)p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx}=0$ in the above which gives

$$p+\frac{1}{7}=0$$

No valid singular solutions found.

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x}\neq 0$. From eq. (2A). This results in

$$p'(x)=\frac{p(x)+\frac{1}{7}}{-\frac{2i\sin\left(\frac{\pi}{7}\right)}{49p(x)^{5/7}}-\frac{2\cos\left(\frac{\pi}{7}\right)}{49p(x)^{5/7}}} \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed. Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{p(x)}-\frac{2(-1)^{1/7}}{7\left(7\tau+1\right)\tau^{5/7}}d\tau=x+c_3$$

Singular solutions are found by solving

$$\frac{7(-1)^{6/7}\left(7p+1\right)p^{5/7}}{2}=0$$

for $p(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$p(x)=0$$
$$p(x)=-\frac{1}{7}$$

Substituing the above solution for p in (2A) gives

$$y=-\frac{x}{7}+\frac{\mathrm{RootOf}\left(-\int^{-Z}-\frac{2(-1)^{1/7}}{7(7\tau+1)\tau^{5/7}}d\tau+x+c_3\right)^{2/7}\left(-i\sin\left(\frac{\pi}{7}\right)-\cos\left(\frac{\pi}{7}\right)\right)}{7}-\frac{\pi}{7}$$
$$y=-\frac{\pi}{7}-\frac{x}{7}$$
$$y=-\frac{x}{7}+\frac{(-1)^{2/7}\,7^{5/7}\left(-i\sin\left(\frac{\pi}{7}\right)-\cos\left(\frac{\pi}{7}\right)\right)}{49}-\frac{\pi}{7}$$

Solving ode 4A

Taking derivative of (*) w.r.t. x gives

$$p=f+(xf'+g')\frac{dp}{dx}$$
$$p-f=(xf'+g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y=xf+g$ to (1A) shows that

$$f=-\frac{1}{7}$$
$$g=-\frac{\pi}{7}+\frac{p^{2/7}\left(-i\sin\left(\frac{2\pi}{7}\right)+\cos\left(\frac{2\pi}{7}\right)\right)}{7}$$

Hence (2) becomes

$$p+\frac{1}{7}=\left(-\frac{2i\sin\left(\frac{2\pi}{7}\right)}{49p^{5/7}}+\frac{2\cos\left(\frac{2\pi}{7}\right)}{49p^{5/7}}\right)p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p + \frac{1}{7} = 0$$

No valid singular solutions found.

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) + \frac{1}{7}}{-\frac{2i \sin\left(\frac{2\pi}{7}\right)}{49 p(x)^{5/7}} + \frac{2\cos\left(\frac{2\pi}{7}\right)}{49 p(x)^{5/7}}} \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed. Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{p(x)} -\frac{2(-1)^{5/7}}{7\left(7\tau + 1\right) \tau^{5/7}} d\tau = x + c_4$$

Singular solutions are found by solving

$$\frac{7(-1)^{2/7} \left(7p + 1\right) p^{5/7}}{2} = 0$$

for $p(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$p(x) = 0$$
$$p(x) = -\frac{1}{7}$$

Substituing the above solution for p in (2A) gives

$$y = -\frac{x}{7} - \frac{\pi}{7} + \frac{\operatorname{RootOf}\left(-\int^{\_Z} -\frac{2(-1)^{5/7}}{7(7\tau+1)\tau^{5/7}} d\tau + x + c_4\right)^{2/7} \left(-i \sin\left(\frac{2\pi}{7}\right) + \cos\left(\frac{2\pi}{7}\right)\right)}{7}$$
$$y = -\frac{\pi}{7} - \frac{x}{7}$$
$$y = -\frac{x}{7} - \frac{\pi}{7} + \frac{(-1)^{2/7} 7^{5/7} \left(-i \sin\left(\frac{2\pi}{7}\right) + \cos\left(\frac{2\pi}{7}\right)\right)}{49}$$

Solving ode 5A

Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = -\frac{1}{7}$$
$$g = -\frac{\pi}{7} + \frac{p^{2/7}\left(i \sin\left(\frac{2\pi}{7}\right) + \cos\left(\frac{2\pi}{7}\right)\right)}{7}$$

Hence (2) becomes

$$p + \frac{1}{7} = \left(\frac{2i \sin\left(\frac{2\pi}{7}\right)}{49 p^{5/7}} + \frac{2\cos\left(\frac{2\pi}{7}\right)}{49 p^{5/7}}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p + \frac{1}{7} = 0$$

No valid singular solutions found.

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) + \frac{1}{7}}{\frac{2i\sin\left(\frac{2\pi}{7}\right)}{49p(x)^{5/7}} + \frac{2\cos\left(\frac{2\pi}{7}\right)}{49p(x)^{5/7}}} \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed. Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{p(x)} \frac{2(-1)^{2/7}}{7\left(7\tau+1\right)\tau^{5/7}} d\tau = x + c_5$$

Singular solutions are found by solving

$$-\frac{7(-1)^{5/7}\left(7p+1\right)p^{5/7}}{2} = 0$$

for $p(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$p(x) = 0$$
$$p(x) = -\frac{1}{7}$$

Substituing the above solution for p in (2A) gives

$$y = -\frac{x}{7} - \frac{\pi}{7} + \frac{\operatorname{RootOf}\left(-\int^{\_Z} \frac{2(-1)^{2/7}}{7(7\tau+1)\tau^{5/7}} d\tau + x + c_5\right)^{2/7} \left(i\sin\left(\frac{2\pi}{7}\right) + \cos\left(\frac{2\pi}{7}\right)\right)}{7}$$
$$y = -\frac{\pi}{7} - \frac{x}{7}$$
$$y = -\frac{x}{7} - \frac{\pi}{7} + \frac{(-1)^{2/7}\, 7^{5/7}\left(i\sin\left(\frac{2\pi}{7}\right) + \cos\left(\frac{2\pi}{7}\right)\right)}{49}$$

Solving ode 6A

Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = -\frac{1}{7}$$
$$g = \frac{p^{2/7}\left(i\sin\left(\frac{\pi}{7}\right) - \cos\left(\frac{\pi}{7}\right)\right)}{7} - \frac{\pi}{7}$$

Hence (2) becomes

$$p + \frac{1}{7} = \left(\frac{2i\sin\left(\frac{\pi}{7}\right)}{49p^{5/7}} - \frac{2\cos\left(\frac{\pi}{7}\right)}{49p^{5/7}}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p + \frac{1}{7} = 0$$

Solving the above for p results in

$$p_1 = -\frac{1}{7}$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$y = -\frac{7^{5/7}(-1)^{1/7}}{49} - \frac{\pi}{7} - \frac{x}{7}$$

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) + \frac{1}{7}}{\frac{2i\sin\left(\frac{\pi}{7}\right)}{49p(x)^{5/7}} - \frac{2\cos\left(\frac{\pi}{7}\right)}{49p(x)^{5/7}}} \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed. Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{p(x)} \frac{2(-1)^{6/7}}{7\,(7\tau+1)\,\tau^{5/7}} d\tau = x + c_6$$

Singular solutions are found by solving

$$-\frac{7(-1)^{1/7}\,(7p+1)\,p^{5/7}}{2} = 0$$

for $p(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$p(x) = 0$$
$$p(x) = -\frac{1}{7}$$

Substituing the above solution for p in (2A) gives

$$y = -\frac{x}{7} + \frac{\operatorname{RootOf}\left(-\int^{-Z} \frac{2(-1)^{6/7}}{7(7\tau+1)\tau^{5/7}} d\tau + x + c_6\right)^{2/7} \left(i\sin\left(\frac{\pi}{7}\right) - \cos\left(\frac{\pi}{7}\right)\right)}{7} - \frac{\pi}{7}$$
$$y = -\frac{\pi}{7} - \frac{x}{7}$$
$$y = -\frac{x}{7} + \frac{(-1)^{2/7}\,7^{5/7}\left(i\sin\left(\frac{\pi}{7}\right) - \cos\left(\frac{\pi}{7}\right)\right)}{49} - \frac{\pi}{7}$$

Solving ode 7A

Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = -\frac{1}{7}$$
$$g = \frac{p^{2/7}\left(-\cos\left(\frac{3\pi}{7}\right) - i\sin\left(\frac{3\pi}{7}\right)\right)}{7} - \frac{\pi}{7}$$

Hence (2) becomes

$$p + \frac{1}{7} = \left(-\frac{2\cos\left(\frac{3\pi}{7}\right)}{49p^{5/7}} - \frac{2i\sin\left(\frac{3\pi}{7}\right)}{49p^{5/7}}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p + \frac{1}{7} = 0$$

Solving the above for p results in

$$p_1 = -\frac{1}{7}$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$y = -\frac{(-7)^{5/7}}{49} - \frac{\pi}{7} - \frac{x}{7}$$

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) + \frac{1}{7}}{-\frac{2\cos\left(\frac{3\pi}{7}\right)}{49p(x)^{5/7}} - \frac{2i\sin\left(\frac{3\pi}{7}\right)}{49p(x)^{5/7}}} \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed. Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{p(x)} -\frac{2(-1)^{3/7}}{7\,(7\tau+1)\,\tau^{5/7}} d\tau = x + c_7$$

Singular solutions are found by solving

$$\frac{7(-1)^{4/7}\,(7p+1)\,p^{5/7}}{2} = 0$$

for $p(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$p(x) = 0$$
$$p(x) = -\frac{1}{7}$$

Substituing the above solution for p in (2A) gives

$$y = -\frac{x}{7} + \frac{\operatorname{RootOf}\left(-\int^{\_Z} -\frac{2(-1)^{3/7}}{7(7\tau+1)\tau^{5/7}} d\tau + x + c_7\right)^{2/7} \left(-\cos\left(\frac{3\pi}{7}\right) - i\sin\left(\frac{3\pi}{7}\right)\right)}{7} - \frac{\pi}{7}$$
$$y = -\frac{\pi}{7} - \frac{x}{7}$$
$$y = -\frac{x}{7} + \frac{(-1)^{2/7}\,7^{5/7}\left(-\cos\left(\frac{3\pi}{7}\right) - i\sin\left(\frac{3\pi}{7}\right)\right)}{49} - \frac{\pi}{7}$$

The solution

$$y = -\frac{\pi}{7} - \frac{x}{7}$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$y = -\frac{x}{7} - \frac{\pi}{7} + \frac{\operatorname{RootOf}\left(-\int^{\_Z} \frac{2(-1)^{2/7}}{7(7\tau+1)\tau^{5/7}} d\tau + x + c_5\right)^{2/7} \left(i\sin\left(\frac{2\pi}{7}\right) + \cos\left(\frac{2\pi}{7}\right)\right)}{7}$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$y = -\frac{x}{7} - \frac{\pi}{7} + \frac{\operatorname{RootOf}\left(-\int^{\_Z} -\frac{2(-1)^{5/7}}{7(7\tau+1)\tau^{5/7}} d\tau + x + c_4\right)^{2/7} \left(-i\sin\left(\frac{2\pi}{7}\right) + \cos\left(\frac{2\pi}{7}\right)\right)}{7}$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$y = -\frac{x}{7} - \frac{\pi}{7} + \frac{(-1)^{2/7}\,7^{5/7}\left(-i\sin\left(\frac{2\pi}{7}\right) + \cos\left(\frac{2\pi}{7}\right)\right)}{49}$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$y = -\frac{x}{7} - \frac{\pi}{7} + \frac{(-1)^{2/7} 7^{5/7} \left(i \sin\left(\frac{2\pi}{7}\right) + \cos\left(\frac{2\pi}{7}\right)\right)}{49}$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$y = -\frac{x}{7} + \frac{\operatorname{RootOf}\left(-\int^{\_Z} -\frac{2(-1)^{1/7}}{7(7\tau+1)\tau^{5/7}} d\tau + x + c_3\right)^{2/7} \left(-i \sin\left(\frac{\pi}{7}\right) - \cos\left(\frac{\pi}{7}\right)\right)}{7} - \frac{\pi}{7}$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$y = -\frac{x}{7} + \frac{\operatorname{RootOf}\left(-\int^{\_Z} -\frac{2(-1)^{3/7}}{7(7\tau+1)\tau^{5/7}} d\tau + x + c_7\right)^{2/7} \left(-\cos\left(\frac{3\pi}{7}\right) - i \sin\left(\frac{3\pi}{7}\right)\right)}{7} - \frac{\pi}{7}$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$y = -\frac{x}{7} + \frac{\operatorname{RootOf}\left(-\int^{\_Z} \frac{2(-1)^{4/7}}{7(7\tau+1)\tau^{5/7}} d\tau + x + c_2\right)^{2/7} \left(-\cos\left(\frac{3\pi}{7}\right) + i \sin\left(\frac{3\pi}{7}\right)\right)}{7} - \frac{\pi}{7}$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$y = -\frac{x}{7} + \frac{\operatorname{RootOf}\left(-\int^{\_Z} \frac{2(-1)^{6/7}}{7(7\tau+1)\tau^{5/7}} d\tau + x + c_6\right)^{2/7} \left(i \sin\left(\frac{\pi}{7}\right) - \cos\left(\frac{\pi}{7}\right)\right)}{7} - \frac{\pi}{7}$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$y = -\frac{x}{7} + \frac{(-1)^{2/7} 7^{5/7} \left(-i \sin\left(\frac{\pi}{7}\right) - \cos\left(\frac{\pi}{7}\right)\right)}{49} - \frac{\pi}{7}$$

was found not to satisfy the ode or the IC. Hence it is removed.

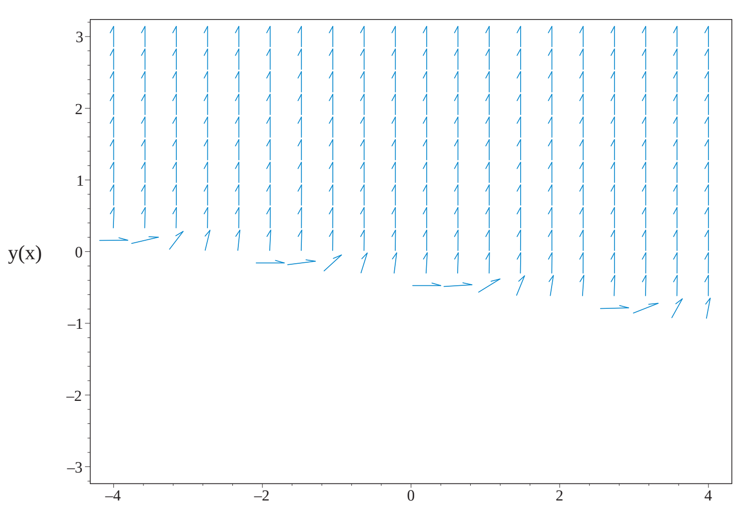


Figure 2.86: Slope field plot
$y' = (\pi + x + 7y)^{7/2}$

Summary of solutions found

$$y = -\frac{x}{7} + \frac{(-1)^{2/7} 7^{5/7} \left(i \sin\left(\frac{\pi}{7}\right) - \cos\left(\frac{\pi}{7}\right)\right)}{49} - \frac{\pi}{7}$$

$$y = -\frac{x}{7} + \frac{(-1)^{2/7} 7^{5/7} \left(-\cos\left(\frac{3\pi}{7}\right) - i \sin\left(\frac{3\pi}{7}\right)\right)}{49} - \frac{\pi}{7}$$

$$y = -\frac{x}{7} + \frac{(-1)^{2/7} 7^{5/7} \left(-\cos\left(\frac{3\pi}{7}\right) + i \sin\left(\frac{3\pi}{7}\right)\right)}{49} - \frac{\pi}{7}$$

$$y = -\frac{(-7)^{5/7}}{49} - \frac{\pi}{7} - \frac{x}{7}$$

$$y = \frac{\operatorname{RootOf}\left(-\int^{\_Z} \frac{2}{7(7\tau+1)\tau^{5/7}} d\tau + x + c_1\right)^{2/7}}{7} - \frac{\pi}{7} - \frac{x}{7}$$

$$y = \frac{(-1)^{2/7} 7^{5/7}}{49} - \frac{\pi}{7} - \frac{x}{7}$$

$$y = -\frac{7^{5/7}(-1)^{1/7}}{49} - \frac{\pi}{7} - \frac{x}{7}$$

$$y = \frac{7^{5/7}(-1)^{6/7}}{49} - \frac{\pi}{7} - \frac{x}{7}$$

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) = (\pi + x + 7y(x))^{7/2}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = (\pi + x + 7y(x))^{7/2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -1/7, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

Maple dsolve solution

Solving time : 0.061 (sec)
Leaf size : 33

```
dsolve(diff(y(x),x) = (Pi+x+7*y(x))^(7/2),
       y(x),singsol=all)
```

$$y = -\frac{x}{7} + \text{RootOf}\left(-x + 7\left(\int^{\_Z} \frac{1}{1 + 7\left(\pi + 7\_a\right)^{7/2}} d\_a\right) + c_1\right)$$

Mathematica DSolve solution

Solving time : 30.453 (sec)
Leaf size : 43

```
DSolve[{D[y[x],x]==(Pi+x+7*y[x])^(7/2),{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$\text{Solve}\left[-(7y(x) + x + \pi)\left(\text{Hypergeometric2F1}\left(\frac{2}{7}, 1, \frac{9}{7}, -7(x + 7y(x) + \pi)^{7/2}\right) - 1\right) - 7y(x) = c_1, y(x)\right]$$

2.1.65 problem 65

Internal problem ID [8725]
Book : First order enumerated odes
Section : section 1
Problem number : 65
Date solved : `Tuesday, December 17, 2024 at 01:01:31 PM`
CAS classification : `[[_homogeneous, ‘class C‘], _dAlembert]`

Solve

$$y' = (a+bx+cy)^6$$

Solved as first order homogeneous class C ode

Time used: 1.395 (sec)

Let

$$z = a+bx+cy \tag{1}$$

Then

$$z'(x) = b+cy'$$

Therefore

$$y' = \frac{z'(x)-b}{c}$$

Hence the given ode can now be written as

$$\frac{z'(x)-b}{c} = z^6$$

This is separable first order ode. Integrating

$$\int dx = \int \frac{1}{c\,z^6+b}dz$$

$$x+c_1 = \frac{\sqrt{3}\left(\frac{b}{c}\right)^{1/6}\ln\left(z^2+\sqrt{3}\left(\frac{b}{c}\right)^{1/6}z+\left(\frac{b}{c}\right)^{1/3}\right)}{12b} + \frac{\left(\frac{b}{c}\right)^{1/6}\arctan\left(\frac{2z}{\left(\frac{b}{c}\right)^{1/6}}+\sqrt{3}\right)}{6b} - \frac{\sqrt{3}\left(\frac{b}{c}\right)^{1/6}\ln\left(z^2-\sqrt{3}\left(\frac{b}{c}\right)^{1/6}z+\left(\frac{b}{c}\right)^{1/3}\right)}{12b} + \frac{\left(\frac{b}{c}\right)^{1/6}\arctan\left(\frac{2z}{\left(\frac{b}{c}\right)^{1/6}}-\sqrt{3}\right)}{6b} + \frac{\left(\frac{b}{c}\right)^{1/6}\arctan\left(\frac{z}{\left(\frac{b}{c}\right)^{1/6}}\right)}{3b}$$

Replacing z back by its value from (1) then the above gives the solution as

Summary of solutions found

$$\frac{\sqrt{3}\left(\frac{b}{c}\right)^{1/6}\ln\left(\left(a+bx+cy\right)^{2}+\sqrt{3}\left(\frac{b}{c}\right)^{1/6}\left(a+bx+cy\right)+\left(\frac{b}{c}\right)^{1/3}\right)}{12b}+\frac{\left(\frac{b}{c}\right)^{1/6}\arctan\left(\frac{2cy+2bx+2a}{\left(\frac{b}{c}\right)^{1/6}}+\sqrt{3}\right)}{6b}-\frac{\sqrt{3}\left(\frac{b}{c}\right)^{1/6}\ln\left(\left(a+bx+cy\right)^{2}-\sqrt{3}\left(\frac{b}{c}\right)^{1/6}\left(a+bx+cy\right)+\left(\frac{b}{c}\right)^{1/3}\right)}{12b}+\frac{\left(\frac{b}{c}\right)^{1/6}\arctan\left(\frac{2cy+2bx+2a}{\left(\frac{b}{c}\right)^{1/6}}-\sqrt{3}\right)}{6b}+\frac{\left(\frac{b}{c}\right)^{1/6}\arctan\left(\frac{a+bx+cy}{\left(\frac{b}{c}\right)^{1/6}}\right)}{3b}=x+c_1$$

Solved using Lie symmetry for first order ode

Time used: 1.539 (sec)

Writing the ode as

$$y'=(bx+cy+a)^6$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1,a_2,a_3,b_1,b_2,b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2+(bx+cy+a)^6(b_3-a_2)-(bx+cy+a)^{12}a_3-6(bx+cy+a)^5b(xa_2+ya_3+a_1)-6(bx+cy+a)^5c(xb_2+yb_3+b_1)=0 \tag{5E}$$

Putting the above in normal form gives

Expression too large to display

Setting the numerator to zero gives

Expression too large to display (6E)

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$\{x,y\}$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$\{x=v_1,y=v_2\}$$

The above PDE (6E) now becomes

$$\text{Expression too large to display} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\text{Expression too large to display} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-b^{12}a_3 &= 0\\
-c^{12}a_3 &= 0\\
-12a\,b^{11}a_3 &= 0\\
-12a\,c^{11}a_3 &= 0\\
-66a^2b^{10}a_3 &= 0\\
-66a^2c^{10}a_3 &= 0\\
-220a^3b^9a_3 &= 0\\
-220a^3c^9a_3 &= 0\\
-495a^4b^8a_3 &= 0\\
-495a^4c^8a_3 &= 0\\
-792a^5b^7a_3 &= 0\\
-792a^5c^7a_3 &= 0\\
-12b\,c^{11}a_3 &= 0\\
-66b^2c^{10}a_3 &= 0\\
-220b^3c^9a_3 &= 0\\
-495b^4c^8a_3 &= 0\\
-792b^5c^7a_3 &= 0\\
-924b^6c^6a_3 &= 0\\
-792b^7c^5a_3 &= 0\\
-495b^8c^4a_3 &= 0\\
-220b^9c^3a_3 &= 0\\
-66b^{10}c^2a_3 &= 0\\
-12b^{11}ca_3 &= 0\\
-132ab\,c^{10}a_3 &= 0\\
-660a\,b^2c^9a_3 &= 0\\
-1980a\,b^3c^8a_3 &= 0\\
-3960a\,b^4c^7a_3 &= 0\\
-5544a\,b^5c^6a_3 &= 0\\
-5544a\,b^6c^5a_3 &= 0\\
-3960a\,b^7c^4a_3 &= 0\\
-1980a\,b^8c^3a_3 &= 0\\
-660a\,b^9c^2a_3 &= 0\\
-132a\,b^{10}ca_3 &= 0\\
-660a^2b\,c^9a_3 &= 0\\
-2970a^2b^2c^8a_3 &= 0\\
-7920a^2b^3c^7a_3 &= 0\\
-13860a^2b^4c^6a_3 &= 0\\
-16632a^2b^5c^5a_3 &= 0\\
-13860a^2b^6c^4a_3 &= 0\\
-7920a^2b^7c^3a_3 &= 0\\
-2970a^2b^8c^2a_3 &= 0\\
-660a^2b^9ca_3 &= 0\\
-1980a^3b\,c^8a_3 &= 0\\
-7920a^3b^2c^7a_3 &= 0\\
-18480a^3b^3c^6a_3 &= 0\\
-27720a^3b^4c^5a_3 &= 0\\
-27720a^3b^5c^4a_3 &= 0\\
-18480a^3b^6c^3a_3 &= 0\\
-7920a^3b^7c^2a_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= -\frac{cb_1}{b} \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= b_1 \\
b_2 &= 0 \\
b_3 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -\frac{c}{b} \\
\eta &= 1
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{1}{-\frac{c}{b}} \\
&= -\frac{b}{c}
\end{aligned}
$$

This is easily solved to give

$$
y = -\frac{bx}{c} + c_1
$$

Where now the coordinate R is taken as the constant of integration. Hence

$$
R = \frac{bx + cy}{c}
$$

And S is found from

$$
\begin{aligned}
dS &= \frac{dx}{\xi} \\
&= \frac{dx}{-\frac{c}{b}}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
S &= \int \frac{dx}{T} \\
&= -\frac{bx}{c}
\end{aligned}
$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = (bx + cy + a)^6$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= \frac{b}{c} \\ R_y &= 1 \\ S_x &= -\frac{b}{c} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{b}{c\left(\frac{b}{c} + (bx + cy + a)^6\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{b}{R^6c^7 + 6R^5a\,c^6 + 15R^4a^2c^5 + 20R^3a^3c^4 + 15R^2a^4c^3 + 6R\,a^5c^2 + a^6c + b}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int -\frac{b}{R^6c^7 + 6R^5a\,c^6 + 15R^4a^2c^5 + 20R^3a^3c^4 + 15R^2a^4c^3 + 6R\,a^5c^2 + a^6c + b}\,dR$$

$$S(R) = -\frac{b\left(\sum_{\_R=\mathrm{RootOf}\left(c^7\_Z^6+6\_Z^5a\,c^6+15\_Z^4a^2c^5+20\_Z^3a^3c^4+15\_Z^2a^4c^3+6a^5c^2\_Z+a^6c+b\right)} \frac{\ln\left(R-\_\right.}{\_R^5c^5+5\_R^4a\,c^4+10\_R^3a^2c\cdots}\right.}{6c^2}$$

$$S(R) = \int -\frac{b}{R^6c^7 + 6R^5a\,c^6 + 15R^4a^2c^5 + 20R^3a^3c^4 + 15R^2a^4c^3 + 6R\,a^5c^2 + a^6c + b}dR + c_2$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$-\frac{bx}{c} = \int^{\frac{cy+bx}{c}} -\frac{b}{\_a^6c^7 + 6\_a^5a\,c^6 + 15\_a^4a^2c^5 + 20\_a^3a^3c^4 + 15\_a^2a^4c^3 + 6\_a\,a^5c^2 + a^6c + b}d\_a + c_2$$

Summary of solutions found

$$-\frac{bx}{c} = \int^{\frac{cy+bx}{c}} -\frac{b}{\_a^6c^7 + 6\_a^5a\,c^6 + 15\_a^4a^2c^5 + 20\_a^3a^3c^4 + 15\_a^2a^4c^3 + 6\_a\,a^5c^2 + a^6c + b}d\_a + c_2$$

Solved as first order ode of type dAlembert

Time used: 2.337 (sec)

Let $p = y'$ the ode becomes

$$p = (bx + cy + a)^6$$

Solving for y from the above results in

$$y = -\frac{bx}{c} + \frac{p^{1/6} - a}{c} \tag{1}$$

$$y = -\frac{bx}{c} + \frac{\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) p^{1/6} - a}{c} \tag{2}$$

$$y = -\frac{bx}{c} + \frac{\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) p^{1/6} - a}{c} \tag{3}$$

$$y = -\frac{bx}{c} + \frac{-p^{1/6} - a}{c} \tag{4}$$

$$y = -\frac{bx}{c} + \frac{\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) p^{1/6} - a}{c} \tag{5}$$

$$y = -\frac{bx}{c} + \frac{\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) p^{1/6} - a}{c} \tag{6}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. Each of the above ode's is dAlembert ode which is now solved.

Solving ode 1A

Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= -\frac{b}{c} \\ g &= \frac{p^{1/6} - a}{c} \end{aligned}$$

Hence (2) becomes

$$p + \frac{b}{c} = \frac{p'(x)}{6p^{5/6}c} \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p + \frac{b}{c} = 0$$

Solving the above for p results in

$$p_1 = -\frac{b}{c}$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$y = \frac{-bx + \left(-\frac{b}{c}\right)^{1/6} - a}{c}$$

The general solution is found when $\frac{\mathrm{dp}}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = 6\left(p(x) + \frac{b}{c}\right) p(x)^{5/6} c \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed. Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{p(x)} \frac{1}{6\left(c\tau + b\right)\tau^{5/6}} d\tau = x + c_1$$

Singular solutions are found by solving

$$6(pc + b)\, p^{5/6} = 0$$

for $p(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$p(x) = 0$$
$$p(x) = -\frac{b}{c}$$

Substituing the above solution for p in (2A) gives

$$y = -\frac{bx}{c} + \frac{\operatorname{RootOf}\left(-\int^{-Z} \frac{1}{6(c\tau+b)\tau^{5/6}} d\tau + x + c_1\right)^{1/6} - a}{c}$$
$$y = -\frac{bx}{c} - \frac{a}{c}$$
$$y = -\frac{bx}{c} + \frac{\left(-\frac{b}{c}\right)^{1/6} - a}{c}$$

Solving ode 2A

Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = -\frac{b}{c}$$
$$g = \frac{ip^{1/6}\sqrt{3} + p^{1/6} - 2a}{2c}$$

Hence (2) becomes

$$p + \frac{b}{c} = \left(\frac{i\sqrt{3}}{12c\,p^{5/6}} + \frac{1}{12p^{5/6}c}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p + \frac{b}{c} = 0$$

Solving the above for p results in

$$p_1 = -\frac{b}{c}$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$y = \frac{i\left(-\frac{b}{c}\right)^{1/6}\sqrt{3} - 2bx + \left(-\frac{b}{c}\right)^{1/6} - 2a}{2c}$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) + \frac{b}{c}}{\frac{i\sqrt{3}}{12cp(x)^{5/6}} + \frac{1}{12p(x)^{5/6}c}} \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed. Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{p(x)} \frac{1 + i\sqrt{3}}{12\left(c\tau + b\right)\tau^{5/6}} d\tau = x + c_2$$

Singular solutions are found by solving

$$\frac{12(pc + b)\, p^{5/6}}{1 + i\sqrt{3}} = 0$$

for $p(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$p(x) = 0$$
$$p(x) = -\frac{b}{c}$$

Substituing the above solution for p in (2A) gives

$$y = -\frac{bx}{c} + \frac{i\,\mathrm{RootOf}\left(-\int^{\_Z} \frac{1+i\sqrt{3}}{12(c\tau+b)\tau^{5/6}} d\tau + x + c_2\right)^{1/6}\sqrt{3} + \mathrm{RootOf}\left(-\int^{\_Z} \frac{1+i\sqrt{3}}{12(c\tau+b)\tau^{5/6}} d\tau + x + c_2\right)^{1/6} - 2}{2c}$$

$$y = -\frac{bx}{c} -$$

$$y = -\frac{bx}{c} + \frac{i\left(-\frac{b}{c}\right)^{1/6}\sqrt{3} + \left(-\frac{b}{c}\right)^{1/6} - 2}{2c}$$

Solving ode 3A

Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = -\frac{b}{c}$$
$$g = \frac{ip^{1/6}\sqrt{3} - p^{1/6} - 2a}{2c}$$

Hence (2) becomes

$$p + \frac{b}{c} = \left(\frac{i\sqrt{3}}{12c\,p^{5/6}} - \frac{1}{12p^{5/6}c}\right)p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p + \frac{b}{c} = 0$$

Solving the above for p results in

$$p_1 = -\frac{b}{c}$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$y = \frac{i\left(-\frac{b}{c}\right)^{1/6}\sqrt{3} - 2bx - \left(-\frac{b}{c}\right)^{1/6} - 2a}{2c}$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) + \frac{b}{c}}{\frac{i\sqrt{3}}{12cp(x)^{5/6}} - \frac{1}{12p(x)^{5/6}c}} \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed. Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{p(x)} \frac{i\sqrt{3}-1}{12\,(c\tau+b)\,\tau^{5/6}} d\tau = x + c_3$$

Singular solutions are found by solving

$$\frac{12(pc+b)\,p^{5/6}}{i\sqrt{3}-1} = 0$$

for $p(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$p(x) = 0$$
$$p(x) = -\frac{b}{c}$$

Substituing the above solution for p in (2A) gives

$$y = -\frac{bx}{c} + \frac{i\,\mathrm{RootOf}\left(-\int^{\_Z} \frac{i\sqrt{3}-1}{12(c\tau+b)\tau^{5/6}} d\tau + x + c_3\right)^{1/6}\sqrt{3} - \mathrm{RootOf}\left(-\int^{\_Z} \frac{i\sqrt{3}-1}{12(c\tau+b)\tau^{5/6}} d\tau + x + c_3\right)^{1/6}}{2c}$$

$$y = -\frac{bx}{c}$$

$$y = -\frac{bx}{c} + \frac{i\left(-\frac{b}{c}\right)^{1/6}\sqrt{3} - \left(-\frac{b}{c}\right)^{1/6}}{2c}$$

Solving ode 4A

Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = -\frac{b}{c}$$
$$g = \frac{-p^{1/6} - a}{c}$$

Hence (2) becomes

$$p+\frac{b}{c}=-\frac{p'(x)}{6p^{5/6}c} \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx}=0$ in the above which gives

$$p+\frac{b}{c}=0$$

Solving the above for p results in

$$p_1=-\frac{b}{c}$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$y=\frac{-bx-\left(-\frac{b}{c}\right)^{1/6}-a}{c}$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x}\neq 0$. From eq. (2A). This results in

$$p'(x)=-6\left(p(x)+\frac{b}{c}\right)p(x)^{5/6}\,c \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed. Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{p(x)}-\frac{1}{6\left(c\tau+b\right)\tau^{5/6}}d\tau=x+c_4$$

Singular solutions are found by solving

$$-6(pc+b)\,p^{5/6}=0$$

for $p(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$p(x)=0$$
$$p(x)=-\frac{b}{c}$$

Substituing the above solution for p in (2A) gives

$$y=-\frac{bx}{c}+\frac{-\operatorname{RootOf}\left(-\int^{-Z}-\frac{1}{6(c\tau+b)\tau^{5/6}}d\tau+x+c_4\right)^{1/6}-a}{c}$$
$$y=-\frac{bx}{c}-\frac{a}{c}$$
$$y=-\frac{bx}{c}+\frac{-\left(-\frac{b}{c}\right)^{1/6}-a}{c}$$

Solving ode 5A

Taking derivative of (*) w.r.t. x gives

$$p=f+(xf'+g')\frac{dp}{dx}$$
$$p-f=(xf'+g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y=xf+g$ to (1A) shows that

$$f=-\frac{b}{c}$$
$$g=\frac{-ip^{1/6}\sqrt{3}-p^{1/6}-2a}{2c}$$

Hence (2) becomes

$$p+\frac{b}{c}=\left(-\frac{i\sqrt{3}}{12c\,p^{5/6}}-\frac{1}{12p^{5/6}c}\right)p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx}=0$ in the above which gives

$$p+\frac{b}{c}=0$$

Solving the above for p results in

$$p_1=-\frac{b}{c}$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$y=\frac{-i\left(-\frac{b}{c}\right)^{1/6}\sqrt{3}-2bx-\left(-\frac{b}{c}\right)^{1/6}-2a}{2c}$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x}\neq 0$. From eq. (2A). This results in

$$p'(x)=\frac{p(x)+\frac{b}{c}}{-\frac{i\sqrt{3}}{12cp(x)^{5/6}}-\frac{1}{12p(x)^{5/6}c}} \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed. Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$\int^{p(x)}-\frac{1+i\sqrt{3}}{12\,(c\tau+b)\,\tau^{5/6}}d\tau=x+c_5$$

Singular solutions are found by solving

$$-\frac{12(pc+b)\,p^{5/6}}{1+i\sqrt{3}}=0$$

for $p(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$p(x)=0$$
$$p(x)=-\frac{b}{c}$$

Substituing the above solution for p in (2A) gives

$$y=-\frac{bx}{c}+\frac{-i\,\mathrm{RootOf}\left(-\int^{\_Z}-\frac{1+i\sqrt{3}}{12(c\tau+b)\tau^{5/6}}d\tau+x+c_5\right)^{1/6}\sqrt{3}-\mathrm{RootOf}\left(-\int^{\_Z}-\frac{1+i\sqrt{3}}{12(c\tau+b)\tau^{5/6}}d\tau+x+c\right.}{2c}$$

$$y=$$

$$y=-\frac{bx}{c}+\frac{-i\left(-\frac{b}{c}\right)^{1/6}\sqrt{3}-(-}{2c}$$

<u>Solving ode 6A</u>

Taking derivative of (*) w.r.t. x gives

$$p=f+(xf'+g')\frac{dp}{dx}$$
$$p-f=(xf'+g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= -\frac{b}{c} \\
g &= \frac{-ip^{1/6}\sqrt{3} + p^{1/6} - 2a}{2c}
\end{aligned}
$$

Hence (2) becomes

$$
p + \frac{b}{c} = \left(-\frac{i\sqrt{3}}{12c\,p^{5/6}} + \frac{1}{12p^{5/6}c}\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p + \frac{b}{c} = 0
$$

Solving the above for p results in

$$
p_1 = -\frac{b}{c}
$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$
y = \frac{-i\left(-\frac{b}{c}\right)^{1/6}\sqrt{3} - 2bx + \left(-\frac{b}{c}\right)^{1/6} - 2a}{2c}
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) + \frac{b}{c}}{-\frac{i\sqrt{3}}{12cp(x)^{5/6}} + \frac{1}{12p(x)^{5/6}c}} \tag{3}
$$

This ODE is now solved for $p(x)$. No inversion is needed. Unable to integrate (or intergal too complicated), and since no initial conditions are given, then the result can be written as

$$
\int^{p(x)} -\frac{i\sqrt{3} - 1}{12\,(c\tau + b)\,\tau^{5/6}} d\tau = x + c_6
$$

Singular solutions are found by solving

$$
-\frac{12(pc + b)\,p^{5/6}}{i\sqrt{3} - 1} = 0
$$

for $p(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$
\begin{aligned}
p(x) &= 0 \\
p(x) &= -\frac{b}{c}
\end{aligned}
$$

Substituing the above solution for p in (2A) gives

$$
y = -\frac{bx}{c} + \frac{-i\,\mathrm{RootOf}\left(-\int^{\_Z} -\frac{i\sqrt{3}-1}{12(c\tau+b)\tau^{5/6}} d\tau + x + c_6\right)^{1/6}\sqrt{3} + \mathrm{RootOf}\left(-\int^{\_Z} -\frac{i\sqrt{3}-1}{12(c\tau+b)\tau^{5/6}} d\tau + x + c_6\right)^{1}}{2c}
$$

$$
y = -
$$

$$
y = -\frac{bx}{c} + \frac{-i\left(-\frac{b}{c}\right)^{1/6}\sqrt{3} + \left(-\frac{b}{c}\right)^{1}}{2c}
$$

The solution

$$
y = -\frac{bx}{c} - \frac{a}{c}
$$

was found not to satisfy the ode or the IC. Hence it is removed.

Summary of solutions found

$$y = \frac{-bx - \left(-\frac{b}{c}\right)^{1/6} - a}{c}$$

$$y = \frac{-bx + \left(-\frac{b}{c}\right)^{1/6} - a}{c}$$

$$y = \frac{-i\left(-\frac{b}{c}\right)^{1/6}\sqrt{3} - 2bx - \left(-\frac{b}{c}\right)^{1/6} - 2a}{2c}$$

$$y = \frac{-i\left(-\frac{b}{c}\right)^{1/6}\sqrt{3} - 2bx + \left(-\frac{b}{c}\right)^{1/6} - 2a}{2c}$$

$$y = \frac{i\left(-\frac{b}{c}\right)^{1/6}\sqrt{3} - 2bx - \left(-\frac{b}{c}\right)^{1/6} - 2a}{2c}$$

$$y = \frac{i\left(-\frac{b}{c}\right)^{1/6}\sqrt{3} - 2bx + \left(-\frac{b}{c}\right)^{1/6} - 2a}{2c}$$

$$y = -\frac{bx}{c} + \frac{-\left(-\frac{b}{c}\right)^{1/6} - a}{c}$$

$$y = -\frac{bx}{c} + \frac{\left(-\frac{b}{c}\right)^{1/6} - a}{c}$$

$$y = -\frac{bx}{c} + \frac{-\operatorname{RootOf}\left(-\int^{-Z} -\frac{1}{6(c\tau+b)\tau^{5/6}}d\tau + x + c_4\right)^{1/6} - a}{c}$$

$$y = -\frac{bx}{c} + \frac{\operatorname{RootOf}\left(-\int^{-Z} \frac{1}{6(c\tau+b)\tau^{5/6}}d\tau + x + c_1\right)^{1/6} - a}{c}$$

$$y = -\frac{bx}{c} + \frac{-i\left(-\frac{b}{c}\right)^{1/6}\sqrt{3} - \left(-\frac{b}{c}\right)^{1/6} - 2a}{2c}$$

$$y = -\frac{bx}{c} + \frac{-i\left(-\frac{b}{c}\right)^{1/6}\sqrt{3} + \left(-\frac{b}{c}\right)^{1/6} - 2a}{2c}$$

$$y = -\frac{bx}{c} + \frac{-i\operatorname{RootOf}\left(-\int^{-Z} -\frac{i\sqrt{3}-1}{12(c\tau+b)\tau^{5/6}}d\tau + x + c_6\right)^{1/6}\sqrt{3} + \operatorname{RootOf}\left(-\int^{-Z} -\frac{i\sqrt{3}-1}{12(c\tau+b)\tau^{5/6}}d\tau + x + c_6\right)^{1/6} -}{2c}$$

$$y = -\frac{bx}{c} + \frac{-i\operatorname{RootOf}\left(-\int^{-Z} -\frac{1+i\sqrt{3}}{12(c\tau+b)\tau^{5/6}}d\tau + x + c_5\right)^{1/6}\sqrt{3} - \operatorname{RootOf}\left(-\int^{-Z} -\frac{1+i\sqrt{3}}{12(c\tau+b)\tau^{5/6}}d\tau + x + c_5\right)^{1/6} -}{2c}$$

$$y = -\frac{bx}{c} + \frac{i\left(-\frac{b}{c}\right)^{1/6}\sqrt{3} - \left(-\frac{b}{c}\right)^{1/6} - 2a}{2c}$$

$$y = -\frac{bx}{c} + \frac{i\left(-\frac{b}{c}\right)^{1/6}\sqrt{3} + \left(-\frac{b}{c}\right)^{1/6} - 2a}{2c}$$

$$y = -\frac{bx}{c} + \frac{i\operatorname{RootOf}\left(-\int^{-Z} \frac{1+i\sqrt{3}}{12(c\tau+b)\tau^{5/6}}d\tau + x + c_2\right)^{1/6}\sqrt{3} + \operatorname{RootOf}\left(-\int^{-Z} \frac{1+i\sqrt{3}}{12(c\tau+b)\tau^{5/6}}d\tau + x + c_2\right)^{1/6} - 2a}{2c}$$

$$y = -\frac{bx}{c} + \frac{i \operatorname{RootOf}\left(-\int^{\_Z} \frac{i\sqrt{3}-1}{12(c\tau+b)\tau^{5/6}}d\tau + x + c_3\right)^{1/6}\sqrt{3} - \operatorname{RootOf}\left(-\int^{\_Z} \frac{i\sqrt{3}-1}{12(c\tau+b)\tau^{5/6}}d\tau + x + c_3\right)^{1/6} - 2a}{2c}$$

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) = (a + bx + cy(x))^6$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = (a + bx + cy(x))^6$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -b/c, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

Maple dsolve solution

Solving time : `0.045` (sec)
Leaf size : 94

```
dsolve(diff(y(x),x) = (a+b*x+c*y(x))^6,
       y(x),singsol=all)
```

$$y = \frac{\operatorname{RootOf}\left(\left(\int^{\_Z} \frac{1}{c^7\_a^6+6\_a^5 a c^6+15\_a^4 a^2 c^5+20\_a^3 a^3 c^4+15\_a^2 a^4 c^3+6\_a a^5 c^2+a^6 c+b}d\_a\right)c - x + c_1\right)c - bx}{c}$$

Mathematica DSolve solution

Solving time : `1.783` (sec)
Leaf size : 274

```
DSolve[{D[y[x],x]==(a+b*x+c*y[x])^6,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$\text{Solve}\left[\frac{-4\sqrt[6]{b}\arctan\left(\frac{\sqrt[6]{c}(a+bx+cy(x))}{\sqrt[6]{b}}\right)+2\sqrt[6]{b}\arctan\left(\sqrt{3}-\frac{2\sqrt[6]{c}(a+bx+cy(x))}{\sqrt[6]{b}}\right)-2\sqrt[6]{b}\arctan\left(\frac{2\sqrt[6]{c}(a+bx+cy}{\sqrt[6]{b}}\right.}{}\right.$$

$$\left.-\frac{cy(x)}{b}=c_1,y(x)\right]$$

2.1.66 problem 66

Internal problem ID [8726]
Book : First order enumerated odes
Section : section 1
Problem number : 66
Date solved : `Tuesday, December 17, 2024 at 01:01:37 PM`
CAS classification : `[_separable]`

Solve

$$y' = \mathrm{e}^{x+y}$$

Solved as first order form A1 ode

Time used: 0.232 (sec)

The given ode has the general form

$$y' = B + Cf(ax + by + c) \tag{1}$$

Comparing (1) to the ode given shows the parameters in the ODE have these values

$$\begin{aligned} B &= 0 \\ C &= 1 \\ a &= 1 \\ b &= 1 \\ c &= 0 \end{aligned}$$

This form of ode can be solved by change of variables $u = ax + by + c$ which makes the ode separable.

$$u'(x) = a + by'$$

Or

$$y' = \frac{u'(x) - a}{b}$$

The ode becomes

$$\begin{aligned} \frac{u' - a}{b} &= B + Cf(u) \\ u' &= bB + bCf(u) + a \\ \frac{du}{bB + bCf(u) + a} &= dx \end{aligned}$$

Integrating gives

$$\int \frac{du}{bB + bCf(u) + a} = x + c_1$$
$$\int^{u} \frac{d\tau}{bB + bCf(\tau) + a} = x + c_1$$

Replacing back $u = ax + by + c$ the above becomes

$$\int^{ax+by+c} \frac{d\tau}{bB + bCf(\tau) + a} = x + c_1 \tag{2}$$

If initial conditions are given as $y(x_0) = y_0$, the above becomes

$$\int_0^{ax_0+by_0+c} \frac{d\tau}{bB + bCf(\tau) + a} = x_0 + c_1$$
$$c_1 = \int_0^{ax+by_0+c} \frac{d\tau}{bB + bCf(\tau) + a} - x_0$$

Substituting this into (2) gives

$$\int^{ax+by+c} \frac{d\tau}{bB + bCf(\tau) + a} = x + \int_0^{ax+by_0+c} \frac{d\tau}{bB + bCf(\tau) + a} - x_0 \tag{3}$$

Since no initial conditions are given, then using (2) and replacing the values of the parameters into (2) gives the solution as

$$\int^{x+y} \frac{1}{1 + e^{\tau}} d\tau = x + c_1$$

Which simplifies to

$$-\ln\left(1 + e^{x+y}\right) + \ln\left(e^{x+y}\right) = x + c_1$$

Solving for y gives

$$y = \ln\left(-\frac{1}{-1 + e^{x+c_1}}\right) + c_1$$

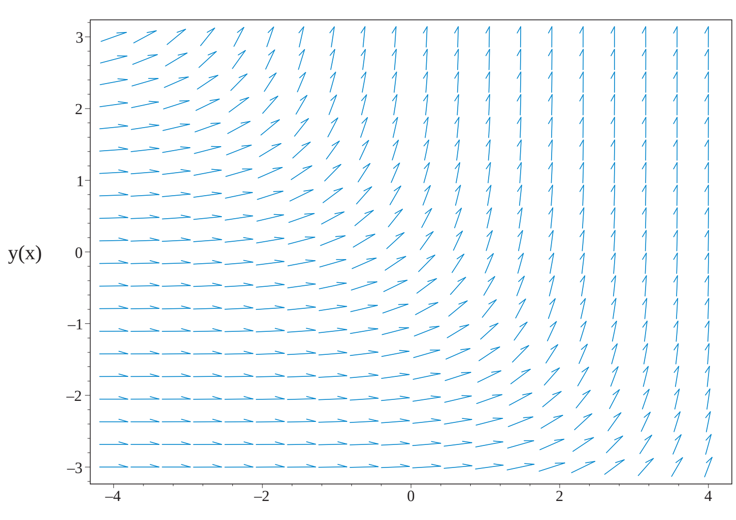


Figure 2.87: Slope field plot
$y' = e^{x+y}$

Summary of solutions found

$$y = \ln\left(-\frac{1}{-1 + e^{x+c_1}}\right) + c_1$$

Solved as first order separable ode

Time used: 0.044 (sec)

The ode $y' = e^{x+y}$ is separable as it can be written as

$$\begin{aligned} y' &= e^{x+y} \\ &= f(x)g(y) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= e^{x} \\ g(y) &= e^{y} \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(y)}\, dy &= \int f(x)\, dx \\ \int e^{-y}\, dy &= \int e^{x}\, dx \\ -e^{-y} &= e^{x} + c_1 \end{aligned}$$

Solving for y gives

$$y = -\ln\left(-e^{x} - c_1\right)$$

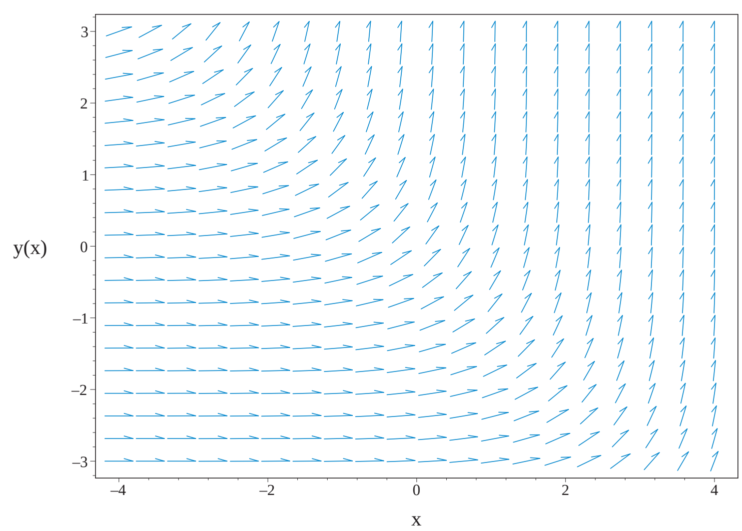


Figure 2.88: Slope field plot
$y' = e^{x+y}$

Summary of solutions found

$$y = -\ln\left(-e^{x} - c_1\right)$$

Solved as first order Exact ode

Time used: 0.164 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(\mathrm{e}^{x+y}\right) \mathrm{d}x$$
$$\left(-\mathrm{e}^{x+y}\right) \mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\mathrm{e}^{x+y}$$
$$N(x,y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\mathrm{e}^{x+y}\right) \\ &= -\mathrm{e}^{x+y}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1\left(\left(-\mathrm{e}^{x+y}\right) - (0)\right) \\ &= -\mathrm{e}^{x+y}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= -\mathrm{e}^{-x-y}\left((0) - \left(-\mathrm{e}^{x+y}\right)\right) \\ &= -1\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B\,\mathrm{d}y} \\ &= e^{\int -1\,\mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-y} \\ &= \mathrm{e}^{-y}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \mathrm{e}^{-y}\left(-\mathrm{e}^{x+y}\right) \\ &= -\mathrm{e}^{x}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \mathrm{e}^{-y}(1) \\ &= \mathrm{e}^{-y}\end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-\mathrm{e}^{x}\right) + \left(\mathrm{e}^{-y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\ \int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int -\mathrm{e}^{x}\,\mathrm{d}x\end{aligned}$$

$$\phi = -\mathrm{e}^{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \mathrm{e}^{-y}$. Therefore equation (4) becomes

$$\mathrm{e}^{-y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \mathrm{e}^{-y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (\mathrm{e}^{-y})\,\mathrm{d}y$$
$$f(y) = -\mathrm{e}^{-y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\mathrm{e}^{x} - \mathrm{e}^{-y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -\mathrm{e}^{x} - \mathrm{e}^{-y}$$

Solving for y gives

$$y = -\ln(-\mathrm{e}^{x} - c_1)$$

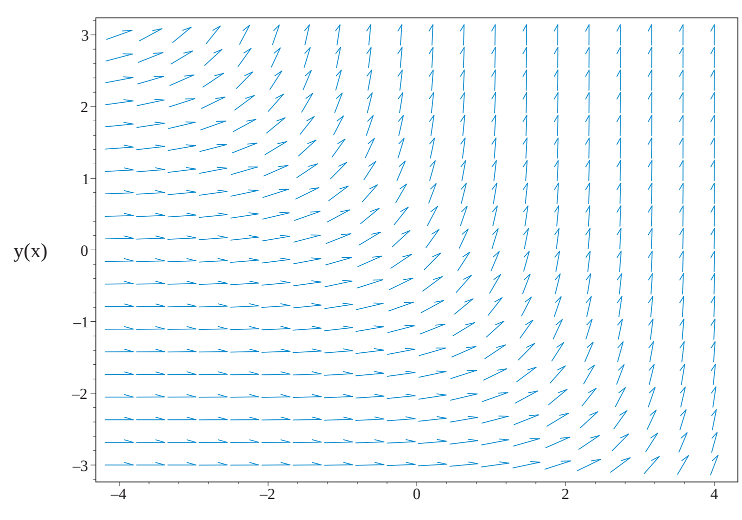


Figure 2.89: Slope field plot
$y' = \mathrm{e}^{x+y}$

<u>Summary of solutions found</u>

$$y = -\ln(-\mathrm{e}^{x} - c_1)$$

Solved using Lie symmetry for first order ode

Time used: 0.805 (sec)

Writing the ode as

$$y' = \mathrm{e}^{x+y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + e^{x+y}(b_3 - a_2) - e^{2x+2y}a_3 - e^{x+y}(xa_2 + ya_3 + a_1) - e^{x+y}(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-e^{2x+2y}a_3 - e^{x+y}xa_2 - e^{x+y}xb_2 - e^{x+y}ya_3 - e^{x+y}yb_3 - e^{x+y}a_1 - e^{x+y}a_2 - e^{x+y}b_1 + e^{x+y}b_3 + b_2 = 0$$

Setting the numerator to zero gives

$$-e^{2x+2y}a_3 - e^{x+y}xa_2 - e^{x+y}xb_2 - e^{x+y}ya_3 - e^{x+y}yb_3 - e^{x+y}a_1 - e^{x+y}a_2 - e^{x+y}b_1 + e^{x+y}b_3 + b_2 = 0 \tag{6E}$$

Simplifying the above gives

$$-e^{2x+2y}a_3 - e^{x+y}xa_2 - e^{x+y}xb_2 - e^{x+y}ya_3 - e^{x+y}yb_3 - e^{x+y}a_1 - e^{x+y}a_2 - e^{x+y}b_1 + e^{x+y}b_3 + b_2 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, e^{x+y}, e^{2x+2y}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, e^{x+y} = v_3, e^{2x+2y} = v_4\}$$

The above PDE (6E) now becomes

$$-v_3v_1a_2 - v_3v_2a_3 - v_3v_1b_2 - v_3v_2b_3 - v_3a_1 - v_3a_2 - v_4a_3 - v_3b_1 + v_3b_3 + b_2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$(-a_2 - b_2)v_1v_3 + (-a_3 - b_3)v_2v_3 + (-a_1 - a_2 - b_1 + b_3)v_3 - v_4a_3 + b_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_2 &= 0 \\
-a_3 &= 0 \\
-a_2 - b_2 &= 0 \\
-a_3 - b_3 &= 0 \\
-a_1 - a_2 - b_1 + b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= -b_1 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= b_1 \\
b_2 &= 0 \\
b_3 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= -1 \\
\eta &= 1
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= 1 - \left(\mathrm{e}^{x+y}\right)(-1) \\
&= 1 + \mathrm{e}^{x}\mathrm{e}^{y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{1 + \mathrm{e}^{x}\mathrm{e}^{y}} dy
\end{aligned}
$$

Which results in

$$
S = \ln\left(\mathrm{e}^{y}\right) - \ln\left(1 + \mathrm{e}^{x}\mathrm{e}^{y}\right)
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = \mathrm{e}^{x+y}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{e^{x+y}}{1+e^{x+y}} \\
S_y &= \frac{1}{1+e^{x+y}}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$
\begin{aligned}
\int dS &= \int 0\, dR + c_2 \\
S(R) &= c_2
\end{aligned}
$$

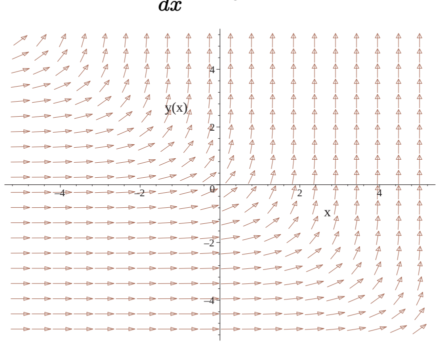
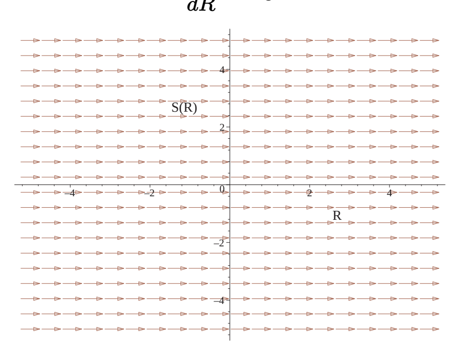
To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$
y - \ln\left(1 + e^{x+y}\right) = c_2
$$

Which gives

$$
y = \ln\left(-\frac{1}{-1 + e^{x+c_2}}\right) + c_2
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = e^{x+y}$ 	$R = x$ $S = y - \ln\left(1 + e^{x+y}\right)$	$\frac{dS}{dR} = 0$ 

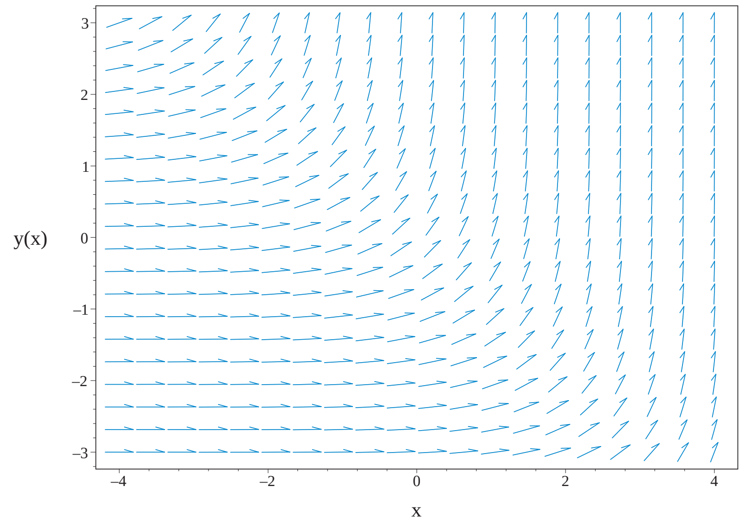


Figure 2.90: Slope field plot
$y' = e^{x+y}$

Summary of solutions found

$$y = \ln\left(-\frac{1}{-1 + e^{x+c_2}}\right) + c_2$$

Solved as first order ode of type ID 1

Time used: 0.106 (sec)

Writing the ode as

$$y' = e^{x+y} \tag{1}$$

And using the substitution $u = e^{-y}$ then

$$u' = -y' e^{-y}$$

The above shows that

$$\begin{aligned} y' &= -u'(x)\, e^{y} \\ &= -\frac{u'(x)}{u} \end{aligned}$$

Substituting this in (1) gives

$$-\frac{u'(x)}{u} = \frac{e^{x}}{u}$$

The above simplifies to

$$u'(x) = -e^{x} \tag{2}$$

Now ode (2) is solved for $u(x)$.

Since the ode has the form $u'(x) = f(x)$, then we only need to integrate $f(x)$.

$$\int du = \int -e^{x}\, dx$$
$$u(x) = -e^{x} + c_1$$

Substituting the solution found for $u(x)$ in $u = e^{-y}$ gives

$$\begin{aligned} y &= -\ln(u(x)) \\ &= -\ln(-\ln(-e^{x} + c_1)) \\ &= -\ln(-e^{x} + c_1) \end{aligned}$$

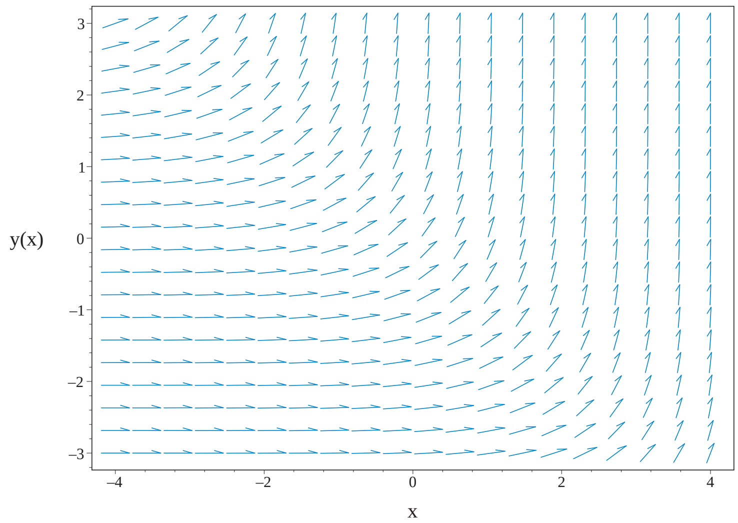


Figure 2.91: Slope field plot
$y' = e^{x+y}$

Summary of solutions found

$$y = -\ln(-e^x + c_1)$$

Solved as first order ode of type dAlembert

Time used: 0.109 (sec)

Let $p = y'$ the ode becomes

$$p = e^{x+y}$$

Solving for y from the above results in

$$y = -x + \ln(p) \tag{1}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$
$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$f = -1$$
$$g = \ln(p)$$

Hence (2) becomes

$$p + 1 = \frac{p'(x)}{p} \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p + 1 = 0$$

Solving the above for p results in

$$p_1 = -1$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$y = i\pi - x$$

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$p'(x) = (p(x) + 1)\, p(x) \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed. Integrating gives

$$\int \frac{1}{(p+1)\, p} dp = dx$$
$$-\ln(p+1) + \ln(p) = x + c_1$$

Singular solutions are found by solving

$$(p+1)\, p = 0$$

for $p(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$p(x) = -1$$
$$p(x) = 0$$

Solving for $p(x)$ gives

$$p(x) = -1$$
$$p(x) = 0$$
$$p(x) = -\frac{e^{x+c_1}}{-1 + e^{x+c_1}}$$

Substituing the above solution for p in (2A) gives

$$y = i\pi - x$$
$$y = -x + \ln\left(-\frac{e^{x+c_1}}{-1 + e^{x+c_1}}\right)$$

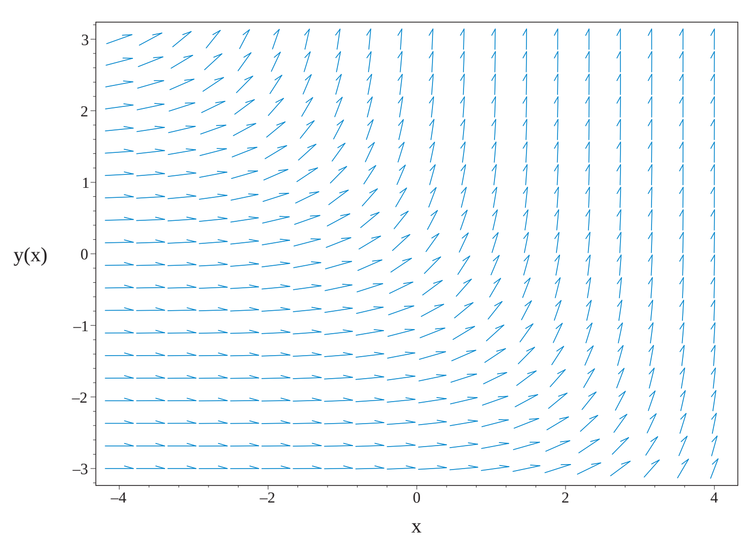


Figure 2.92: Slope field plot
$y' = e^{x+y}$

Summary of solutions found

$$y = i\pi - x$$
$$y = -x + \ln\left(-\frac{e^{x+c_1}}{-1 + e^{x+c_1}}\right)$$

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) = \mathrm{e}^{x+y(x)}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = \mathrm{e}^{x+y(x)}$

- Separate variables

  $\frac{\frac{d}{dx}y(x)}{\mathrm{e}^{y(x)}} = \mathrm{e}^{x}$

- Integrate both sides with respect to x

  $\int \frac{\frac{d}{dx}y(x)}{\mathrm{e}^{y(x)}}dx = \int \mathrm{e}^{x}dx + C1$

- Evaluate integral

  $-\frac{1}{\mathrm{e}^{y(x)}} = \mathrm{e}^{x} + C1$

- Solve for $y(x)$

  $y(x) = \ln\left(-\frac{1}{\mathrm{e}^{x}+C1}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

Maple dsolve solution

Solving time : 0.006 (sec)
Leaf size : 13

```
dsolve(diff(y(x),x) = exp(x+y(x)),
       y(x),singsol=all)
```

$$y = \ln\left(-\frac{1}{c_1 + \mathrm{e}^{x}}\right)$$

Mathematica DSolve solution

Solving time : 0.822 (sec)
Leaf size : 18

```
DSolve[{D[y[x],x]==Exp[x+y[x]],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\log\left(-e^{x} - c_1\right)$$

2.1.67 problem 67

Internal problem ID [8727]
Book : First order enumerated odes
Section : section 1
Problem number : 67
Date solved : `Tuesday, December 17, 2024 at 01:01:39 PM`
CAS classification : `[[_homogeneous, 'class C'], _dAlembert]`

Solve

$$y' = 10 + e^{x+y}$$

Solved as first order form A1 ode

Time used: 0.241 (sec)

The given ode has the general form

$$y' = B + Cf(ax + by + c) \tag{1}$$

Comparing (1) to the ode given shows the parameters in the ODE have these values

$$\begin{aligned} B &= 10 \\ C &= 1 \\ a &= 1 \\ b &= 1 \\ c &= 0 \end{aligned}$$

This form of ode can be solved by change of variables $u = ax + by + c$ which makes the ode separable.

$$u'(x) = a + by'$$

Or

$$y' = \frac{u'(x) - a}{b}$$

The ode becomes

$$\begin{aligned} \frac{u' - a}{b} &= B + Cf(u) \\ u' &= bB + bCf(u) + a \\ \frac{du}{bB + bCf(u) + a} &= dx \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{du}{bB + bCf(u) + a} &= x + c_1 \\ \int^u \frac{d\tau}{bB + bCf(\tau) + a} &= x + c_1 \end{aligned}$$

Replacing back $u = ax + by + c$ the above becomes

$$\int^{ax+by+c} \frac{d\tau}{bB + bCf(\tau) + a} = x + c_1 \tag{2}$$

If initial conditions are given as $y(x_0) = y_0$, the above becomes

$$\int_0^{ax_0+by_0+c} \frac{d\tau}{bB + bCf(\tau) + a} = x_0 + c_1$$

$$c_1 = \int_0^{ax+by_0+c} \frac{d\tau}{bB + bCf(\tau) + a} - x_0$$

Substituting this into (2) gives

$$\int^{ax+by+c} \frac{d\tau}{bB + bCf(\tau) + a} = x + \int_0^{ax+by_0+c} \frac{d\tau}{bB + bCf(\tau) + a} - x_0 \tag{3}$$

Since no initial conditions are given, then using (2) and replacing the values of the parameters into (2) gives the solution as

$$\int^{x+y} \frac{1}{11 + e^{\tau}} d\tau = x + c_1$$

Which simplifies to

$$-\frac{\ln(11 + e^{x+y})}{11} + \frac{\ln(e^{x+y})}{11} = x + c_1$$

Solving for y gives

$$y = 10x + \ln\left(-\frac{11}{-1 + e^{11x+11c_1}}\right) + 11c_1$$

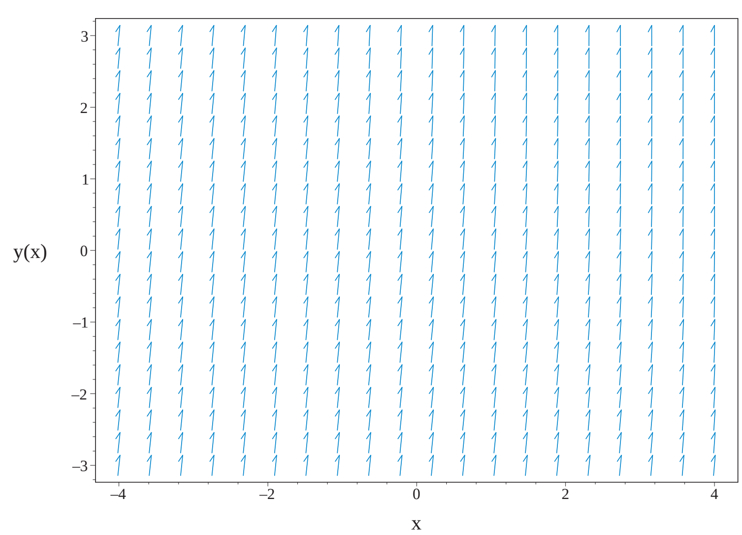


Figure 2.93: Slope field plot
$y' = 10 + e^{x+y}$

Summary of solutions found

$$y = 10x + \ln\left(-\frac{11}{-1 + e^{11x+11c_1}}\right) + 11c_1$$

Solved using Lie symmetry for first order ode

Time used: 0.963 (sec)

Writing the ode as

$$
\begin{aligned}
y' &= 10 + \mathrm{e}^{x+y} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2 + \left(10 + \mathrm{e}^{x+y}\right)(b_3 - a_2) - \left(10 + \mathrm{e}^{x+y}\right)^2 a_3 \\
& - \mathrm{e}^{x+y}(x a_2 + y a_3 + a_1) - \mathrm{e}^{x+y}(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\begin{aligned}
& -\mathrm{e}^{2x+2y} a_3 - \mathrm{e}^{x+y} x a_2 - \mathrm{e}^{x+y} x b_2 - \mathrm{e}^{x+y} y a_3 - \mathrm{e}^{x+y} y b_3 - \mathrm{e}^{x+y} a_1 - \mathrm{e}^{x+y} a_2 \\
& - 20\,\mathrm{e}^{x+y} a_3 - \mathrm{e}^{x+y} b_1 + \mathrm{e}^{x+y} b_3 - 10 a_2 - 100 a_3 + b_2 + 10 b_3 = 0
\end{aligned}
$$

Setting the numerator to zero gives

$$
\begin{aligned}
& -\mathrm{e}^{2x+2y} a_3 - \mathrm{e}^{x+y} x a_2 - \mathrm{e}^{x+y} x b_2 - \mathrm{e}^{x+y} y a_3 - \mathrm{e}^{x+y} y b_3 - \mathrm{e}^{x+y} a_1 - \mathrm{e}^{x+y} a_2 \\
& - 20\,\mathrm{e}^{x+y} a_3 - \mathrm{e}^{x+y} b_1 + \mathrm{e}^{x+y} b_3 - 10 a_2 - 100 a_3 + b_2 + 10 b_3 = 0
\end{aligned} \tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
& -\mathrm{e}^{2x+2y} a_3 - \mathrm{e}^{x+y} x a_2 - \mathrm{e}^{x+y} x b_2 - \mathrm{e}^{x+y} y a_3 - \mathrm{e}^{x+y} y b_3 - \mathrm{e}^{x+y} a_1 - \mathrm{e}^{x+y} a_2 \\
& - 20\,\mathrm{e}^{x+y} a_3 - \mathrm{e}^{x+y} b_1 + \mathrm{e}^{x+y} b_3 - 10 a_2 - 100 a_3 + b_2 + 10 b_3 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \mathrm{e}^{x+y}, \mathrm{e}^{2x+2y}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \mathrm{e}^{x+y} = v_3, \mathrm{e}^{2x+2y} = v_4\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
& -v_3 v_1 a_2 - v_3 v_2 a_3 - v_3 v_1 b_2 - v_3 v_2 b_3 - v_3 a_1 - v_3 a_2 - 20 v_3 a_3 \\
& - v_4 a_3 - v_3 b_1 + v_3 b_3 - 10 a_2 - 100 a_3 + b_2 + 10 b_3 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3, v_4\}$$

Equation (7E) now becomes

$$\begin{aligned}(-a_2 - b_2) v_1 v_3 + (-a_3 - b_3) v_2 v_3 + (-a_1 - a_2 - 20a_3 - b_1 + b_3) v_3 \\ - v_4 a_3 - 10a_2 - 100a_3 + b_2 + 10b_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-a_3 &= 0\\ -a_2 - b_2 &= 0\\ -a_3 - b_3 &= 0\\ -10a_2 - 100a_3 + b_2 + 10b_3 &= 0\\ -a_1 - a_2 - 20a_3 - b_1 + b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= -b_1\\ a_2 &= 0\\ a_3 &= 0\\ b_1 &= b_1\\ b_2 &= 0\\ b_3 &= 0\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= -1\\ \eta &= 1\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi\\ &= 1 - \left(10 + \mathrm{e}^{x+y}\right)(-1)\\ &= 11 + \mathrm{e}^x \mathrm{e}^y\\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{11 + e^x e^y} dy \end{aligned}$$

Which results in

$$S = -\frac{\ln (11 + e^x e^y)}{11} + \frac{\ln (e^y)}{11}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = 10 + e^{x+y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{e^{x+y}}{121 + 11\, e^{x+y}} \\ S_y &= \frac{1}{11 + e^{x+y}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{10}{11} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{10}{11}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR} S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned} \int dS &= \int \frac{10}{11}\, dR \\ S(R) &= \frac{10R}{11} + c_2 \end{aligned}$$

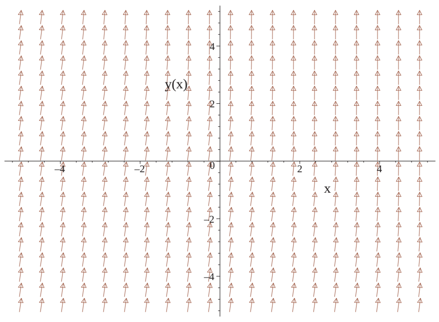
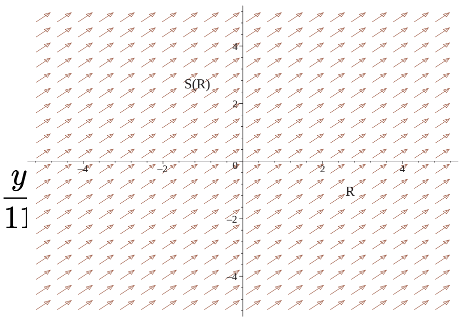
To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$-\frac{\ln (11 + e^{x+y})}{11} + \frac{y}{11} = \frac{10x}{11} + c_2$$

Which gives

$$y = 10x + \ln \left(-\frac{11}{-1 + e^{11x + 11c_2}} \right) + 11c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 10 + e^{x+y}$ 	$R = x$ $S = -\frac{\ln(11 + e^{x+y})}{11} + \frac{y}{11}$	$\frac{dS}{dR} = \frac{10}{11}$ 

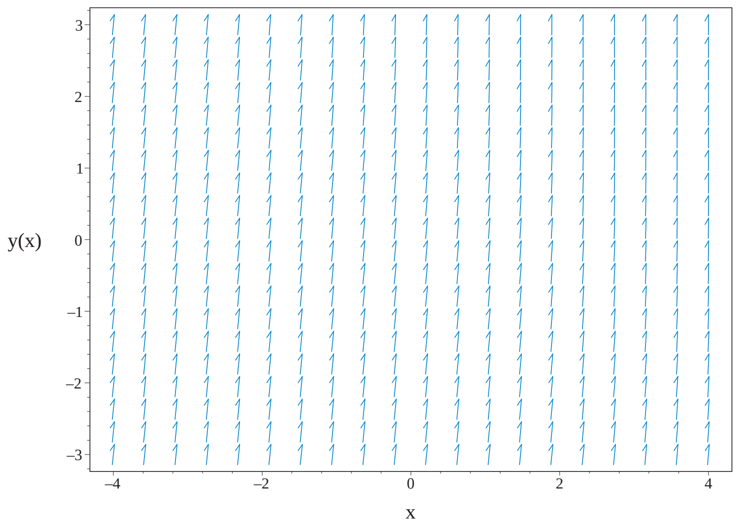


Figure 2.94: Slope field plot
$y' = 10 + e^{x+y}$

Summary of solutions found

$$y = 10x + \ln\left(-\frac{11}{-1 + e^{11x + 11c_2}}\right) + 11c_2$$

Solved as first order ode of type ID 1

Time used: 0.121 (sec)

Writing the ode as

$$y' = 10 + e^{x+y} \tag{1}$$

And using the substitution $u = e^{-y}$ then

$$u' = -y' e^{-y}$$

The above shows that

$$\begin{aligned} y' &= -u'(x)\, e^{y} \\ &= -\frac{u'(x)}{u} \end{aligned}$$

Substituting this in (1) gives

$$-\frac{u'(x)}{u} = \frac{e^{x}}{u} + 10$$

The above simplifies to

$$
\begin{aligned}
-u'(x) &= e^x + 10u(x) \\
u'(x) + 10u(x) &= -e^x \qquad (2)
\end{aligned}
$$

Now ode (2) is solved for $u(x)$.

In canonical form a linear first order is

$$u'(x) + q(x)u(x) = p(x)$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(x) &= 10 \\
p(x) &= -e^x
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\,dx} \\
&= e^{\int 10dx} \\
&= e^{10x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu u) &= \mu p \\
\frac{d}{dx}(\mu u) &= (\mu)\left(-e^x\right) \\
\frac{d}{dx}\left(u\,e^{10x}\right) &= \left(e^{10x}\right)\left(-e^x\right) \\
d\left(u\,e^{10x}\right) &= \left(-e^x e^{10x}\right)\,dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
u\,e^{10x} &= \int -e^x e^{10x}\,dx \\
&= -\frac{e^{11x}}{11} + c_1
\end{aligned}
$$

Dividing throughout by the integrating factor e^{10x} gives the final solution

$$u(x) = -\frac{\left(e^{11x} - 11c_1\right)e^{-10x}}{11}$$

Substituting the solution found for $u(x)$ in $u = e^{-y}$ gives

$$
\begin{aligned}
y &= -\ln\left(u(x)\right) \\
&= -\ln\left(\ln(11) - \ln\left(\left(-e^{11x} + 11c_1\right)e^{-10x}\right)\right) \\
&= \ln(11) - \ln\left(\left(-e^{11x} + 11c_1\right)e^{-10x}\right)
\end{aligned}
$$

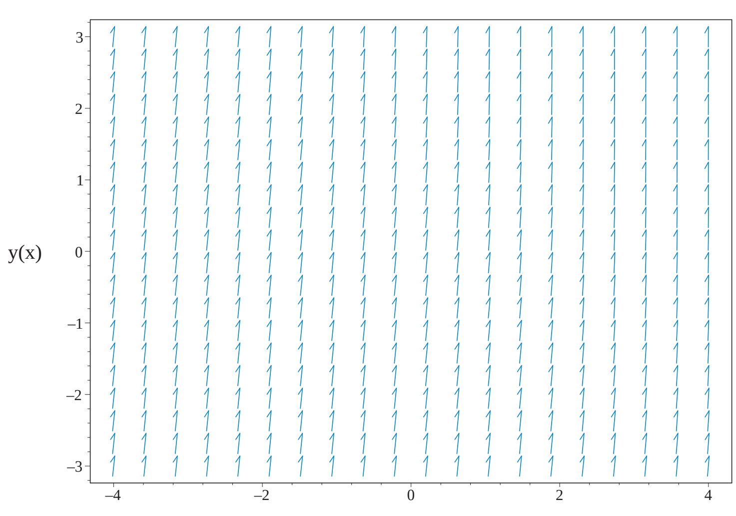


Figure 2.95: Slope field plot
$y' = 10 + e^{x+y}$

Summary of solutions found

$$y = \ln(11) - \ln\left(\left(-e^{11x} + 11c_1\right) e^{-10x}\right)$$

Solved as first order ode of type dAlembert

Time used: 0.155 (sec)

Let $p = y'$ the ode becomes

$$p = 10 + e^{x+y}$$

Solving for y from the above results in

$$y = -x + \ln(p - 10) \tag{1}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= -1 \\ g &= \ln(p - 10) \end{aligned}$$

Hence (2) becomes

$$p + 1 = \frac{p'(x)}{p - 10} \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p + 1 = 0$$

Solving the above for p results in

$$p_1 = -1$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$y = -x + \ln(11) + i\pi$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = (p(x) + 1)(p(x) - 10) \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed. Integrating gives

$$\begin{aligned} \int \frac{1}{(p + 1)(p - 10)} dp &= dx \\ \frac{\ln(p - 10)}{11} - \frac{\ln(p + 1)}{11} &= x + c_1 \end{aligned}$$

Singular solutions are found by solving

$$(p+1)(p-10) = 0$$

for $p(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$p(x) = -1$$
$$p(x) = 10$$

Solving for $p(x)$ gives

$$p(x) = -1$$
$$p(x) = 10$$
$$p(x) = -\frac{10 + e^{11x+11c_1}}{-1 + e^{11x+11c_1}}$$

Substituing the above solution for p in (2A) gives

$$y = -x + \ln(11) + i\pi$$
$$y = -x + \ln\left(-\frac{10 + e^{11x+11c_1}}{-1 + e^{11x+11c_1}} - 10\right)$$

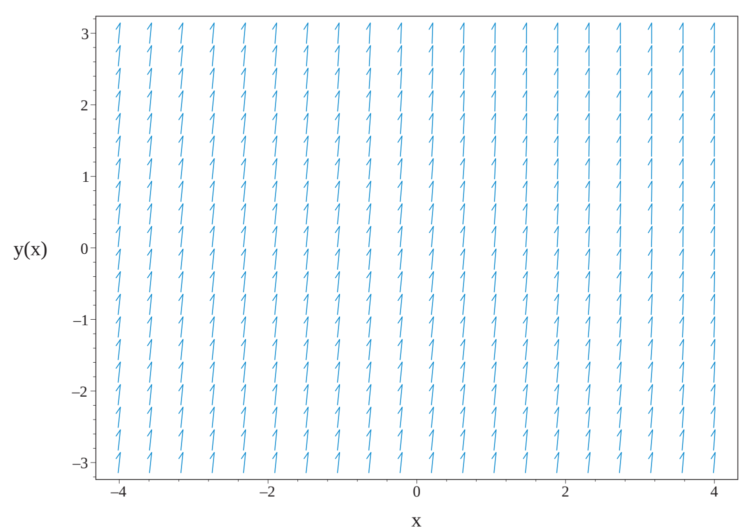


Figure 2.96: Slope field plot
$y' = 10 + e^{x+y}$

Summary of solutions found

$$y = -x + \ln\left(-\frac{10 + e^{11x+11c_1}}{-1 + e^{11x+11c_1}} - 10\right)$$
$$y = -x + \ln(11) + i\pi$$

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) = 10 + e^{x+y(x)}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = 10 + e^{x+y(x)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

Maple dsolve solution

Solving time : `0.072` (sec)
Leaf size : 26

```
dsolve(diff(y(x),x) = 10+exp(x+y(x)),
       y(x),singsol=all)
```

$$y = -x + \ln(11) + \ln\left(\frac{e^{11x}}{-e^{11x} + c_1}\right)$$

Mathematica DSolve solution

Solving time : `3.188` (sec)
Leaf size : 42

```
DSolve[{D[y[x],x]==10+Exp[x+y[x]],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \log\left(-\frac{11e^{10x+11c_1}}{-1 + e^{11(x+c_1)}}\right)$$
$$y(x) \to \log\left(-11e^{-x}\right)$$

2.1.68 problem 68

Internal problem ID [8728]
Book : First order enumerated odes
Section : section 1
Problem number : 68
Date solved : `Tuesday, December 17, 2024 at 01:01:42 PM`
CAS classification : `[[_1st_order, '_with_symmetry_[F(x),G(x)]']]`

Solve

$$y' = 10\,e^{x+y} + x^2$$

Solved as first order ode of type ID 1

Time used: 0.491 (sec)

Writing the ode as

$$y' = 10\,e^{x+y} + x^2 \tag{1}$$

And using the substitution $u = e^{-y}$ then

$$u' = -y' e^{-y}$$

The above shows that

$$\begin{aligned} y' &= -u'(x)\,e^{y} \\ &= -\frac{u'(x)}{u} \end{aligned}$$

Substituting this in (1) gives

$$-\frac{u'(x)}{u} = \frac{10\,e^{x}}{u} + x^2$$

The above simplifies to

$$\begin{aligned} -u'(x) &= 10\,e^{x} + x^2 u(x) \\ u'(x) + x^2 u(x) &= -10\,e^{x} \end{aligned} \tag{2}$$

Now ode (2) is solved for $u(x)$.

In canonical form a linear first order is

$$u'(x) + q(x)u(x) = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= x^2 \\ p(x) &= -10\,e^{x} \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= e^{\int x^2 dx} \\ &= e^{\frac{x^3}{3}} \end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu u) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mu u) &= (\mu)\left(-10\,\mathrm{e}^{x}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(u\,\mathrm{e}^{\frac{x^3}{3}}\right) &= \left(\mathrm{e}^{\frac{x^3}{3}}\right)\left(-10\,\mathrm{e}^{x}\right) \\
\mathrm{d}\left(u\,\mathrm{e}^{\frac{x^3}{3}}\right) &= \left(-10\,\mathrm{e}^{x}\mathrm{e}^{\frac{x^3}{3}}\right)\,\mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
u\,\mathrm{e}^{\frac{x^3}{3}} &= \int -10\,\mathrm{e}^{x}\mathrm{e}^{\frac{x^3}{3}}\,dx \\
&= \int -10\,\mathrm{e}^{x}\mathrm{e}^{\frac{x^3}{3}}\,dx + c_1
\end{aligned}$$

Dividing throughout by the integrating factor $\mathrm{e}^{\frac{x^3}{3}}$ gives the final solution

$$u(x) = \mathrm{e}^{-\frac{x^3}{3}}\left(\int -10\,\mathrm{e}^{x}\mathrm{e}^{\frac{x^3}{3}}\,dx + c_1\right)$$

Substituting the solution found for $u(x)$ in $u = \mathrm{e}^{-y}$ gives

$$\begin{aligned}
y &= -\ln\left(u(x)\right) \\
&= -\ln\left(-\ln\left(\left(-10\left(\int \mathrm{e}^{\frac{x(x^2+3)}{3}}\,dx\right) + c_1\right)\mathrm{e}^{-\frac{x^3}{3}}\right)\right) \\
&= -\ln\left(\left(-10\left(\int \mathrm{e}^{\frac{x(x^2+3)}{3}}\,dx\right) + c_1\right)\mathrm{e}^{-\frac{x^3}{3}}\right)
\end{aligned}$$

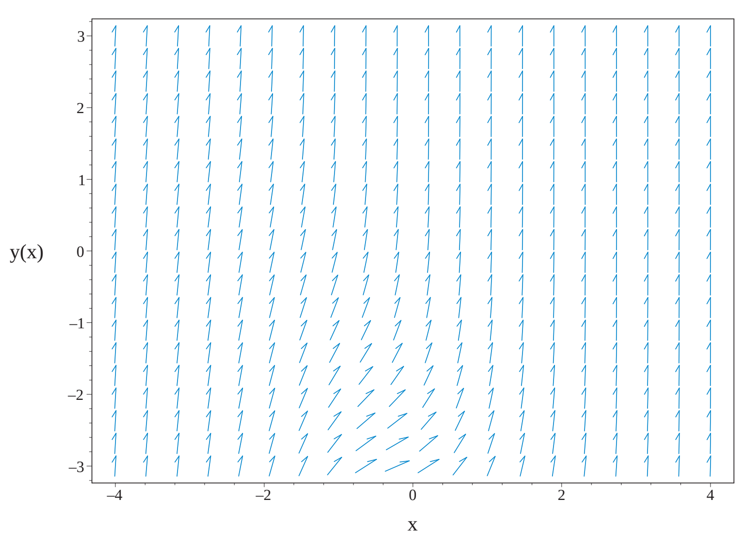


Figure 2.97: Slope field plot
$y' = 10\,\mathrm{e}^{x+y} + x^2$

Summary of solutions found

$$y = -\ln\left(\left(-10\left(\int \mathrm{e}^{\frac{x(x^2+3)}{3}}\,dx\right) + c_1\right)\mathrm{e}^{-\frac{x^3}{3}}\right)$$

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) = 10\,\mathrm{e}^{x+y(x)} + x^2$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = 10\,\mathrm{e}^{x+y(x)} + x^2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
<- symmetry pattern of the form [0, F(x)*G(y)] successful`
```

Maple dsolve solution

Solving time : `0.019` (sec)
Leaf size : 30

```
dsolve(diff(y(x),x) = 10*exp(x+y(x))+x^2,
       y(x),singsol=all)
```

$$y = \frac{x^3}{3} - \ln\left(-c_1 - 10\left(\int \mathrm{e}^{\frac{x\left(x^2+3\right)}{3}}dx\right)\right)$$

Mathematica DSolve solution

Solving time : `0.413` (sec)
Leaf size : 115

```
DSolve[{D[y[x],x]==10*Exp[x+y[x]]+x^2,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$\text{Solve}\left[\int_1^{y(x)} -\frac{1}{10}e^{-K[2]}\left(10e^{K[2]}\int_1^x -\frac{1}{10}e^{\frac{K[1]^3}{3}-K[2]}K[1]^2dK[1]+e^{\frac{x^3}{3}}\right)dK[2] + \int_1^x \left(\frac{1}{10}e^{\frac{K[1]^3}{3}-y(x)}K[1]^2+e^{\frac{K[1]^3}{3}+K[1]}\right)dK[1]=c_1,y(x)\right]$$

2.1.69 problem 69

Internal problem ID [8729]
Book : First order enumerated odes
Section : section 1
Problem number : 69
Date solved : `Tuesday, December 17, 2024 at 01:01:44 PM`
CAS classification : `[[_1st_order, '_with_symmetry_[F(x),G(x)]']]`

Solve

$$y' = x\,\mathrm{e}^{x+y} + \sin(x)$$

Solved as first order ode of type ID 1

Time used: 0.654 (sec)

Writing the ode as

$$y' = x\,\mathrm{e}^{x+y} + \sin(x) \tag{1}$$

And using the substitution $u = \mathrm{e}^{-y}$ then

$$u' = -y'\mathrm{e}^{-y}$$

The above shows that

$$\begin{aligned} y' &= -u'(x)\,\mathrm{e}^{y} \\ &= -\frac{u'(x)}{u} \end{aligned}$$

Substituting this in (1) gives

$$-\frac{u'(x)}{u} = \frac{x\,\mathrm{e}^{x}}{u} + \sin(x)$$

The above simplifies to

$$-u'(x) = x\,\mathrm{e}^{x} + \sin(x)\,u(x)$$
$$u'(x) + \sin(x)\,u(x) = -x\,\mathrm{e}^{x} \tag{2}$$

Now ode (2) is solved for $u(x)$.

In canonical form a linear first order is

$$u'(x) + q(x)u(x) = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= \sin(x) \\ p(x) &= -x\,\mathrm{e}^{x} \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= \mathrm{e}^{\int \sin(x)dx} \\ &= \mathrm{e}^{-\cos(x)} \end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu u) = \mu p$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu u) = (\mu)\left(-x\,\mathrm{e}^{x}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(u\,\mathrm{e}^{-\cos(x)}\right) = \left(\mathrm{e}^{-\cos(x)}\right)\left(-x\,\mathrm{e}^{x}\right)$$

$$\mathrm{d}\left(u\,\mathrm{e}^{-\cos(x)}\right) = \left(-x\,\mathrm{e}^{x}\mathrm{e}^{-\cos(x)}\right)\,\mathrm{d}x$$

Integrating gives

$$\begin{aligned} u\,\mathrm{e}^{-\cos(x)} &= \int -x\,\mathrm{e}^{x}\mathrm{e}^{-\cos(x)}\,dx \\ &= \int -x\,\mathrm{e}^{x}\mathrm{e}^{-\cos(x)}dx + c_1 \end{aligned}$$

Dividing throughout by the integrating factor $\mathrm{e}^{-\cos(x)}$ gives the final solution

$$u(x) = \mathrm{e}^{\cos(x)}\left(\int -x\,\mathrm{e}^{x}\mathrm{e}^{-\cos(x)}dx + c_1\right)$$

Substituting the solution found for $u(x)$ in $u = \mathrm{e}^{-y}$ gives

$$\begin{aligned} y &= -\ln\left(u(x)\right) \\ &= -\ln\left(-\ln\left(\left(-\int x\,\mathrm{e}^{x-\cos(x)}dx + c_1\right)\mathrm{e}^{\cos(x)}\right)\right) \\ &= -\ln\left(\left(-\int x\,\mathrm{e}^{x-\cos(x)}dx + c_1\right)\mathrm{e}^{\cos(x)}\right) \end{aligned}$$

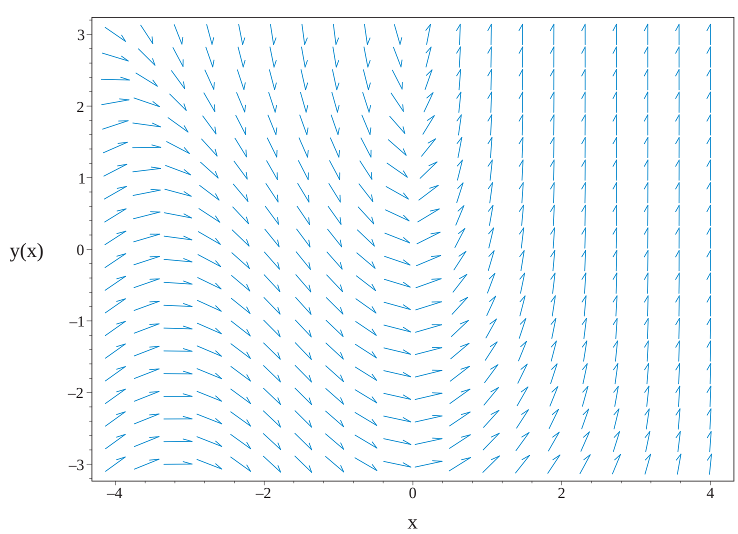


Figure 2.98: Slope field plot
$y' = x\,\mathrm{e}^{x+y} + \sin(x)$

Summary of solutions found

$$y = -\ln\left(\left(-\int x\,\mathrm{e}^{x-\cos(x)}dx + c_1\right)\mathrm{e}^{\cos(x)}\right)$$

Maple step by step solution

Let's solve

$\frac{d}{dx}y(x) = x\,e^{x+y(x)} + \sin(x)$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = x\,e^{x+y(x)} + \sin(x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
<- symmetry pattern of the form [0, F(x)*G(y)] successful`
```

Maple dsolve solution

Solving time : 0.020 (sec)
Leaf size : 29

```
dsolve(diff(y(x),x) = x*exp(x+y(x))+sin(x),
       y(x),singsol=all)
```

$$y = -\cos(x) - \ln\left(-c_1 - \left(\int x\,e^{x-\cos(x)}dx\right)\right)$$

Mathematica DSolve solution

Solving time : 3.151 (sec)
Leaf size : 100

```
DSolve[{D[y[x],x]==x*Exp[x+y[x]]+Sin[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$\text{Solve}\left[\int_1^x \left(-e^{K[1]-\cos(K[1])}K[1]-e^{-\cos(K[1])-y(x)}\sin(K[1])\right)dK[1]+\int_1^{y(x)} -e^{-\cos(x)-K[2]}\left(e^{\cos(x)+K[2]}\int_1^x e^{-\cos(K[1])-K[2]}\sin(K[1])dK[1]-1\right)dK[2]=c_1,y(x)\right]$$

2.1.70 problem 70

Internal problem ID [8730]
Book : First order enumerated odes
Section : section 1
Problem number : 70
Date solved : `Tuesday, December 17, 2024 at 01:01:46 PM`
CAS classification : `[[_1st_order, '_with_symmetry_[F(x),G(x)]']]`

Solve

$$y' = 5\,e^{x^2+20y} + \sin(x)$$

Solved as first order ode of type ID 1

Time used: 0.685 (sec)

Writing the ode as

$$y' = 5\,e^{x^2+20y} + \sin(x) \tag{1}$$

And using the substitution $u = e^{-20y}$ then

$$u' = -20y'e^{-20y}$$

The above shows that

$$\begin{aligned} y' &= -\frac{u'(x)\,e^{20y}}{20} \\ &= -\frac{u'(x)}{20u} \end{aligned}$$

Substituting this in (1) gives

$$-\frac{u'(x)}{20u} = \frac{5\,e^{x^2}}{u} + \sin(x)$$

The above simplifies to

$$-\frac{u'(x)}{20} = 5\,e^{x^2} + \sin(x)\,u(x)$$

$$u'(x) + 20\sin(x)\,u(x) = -100\,e^{x^2} \tag{2}$$

Now ode (2) is solved for $u(x)$.

In canonical form a linear first order is

$$u'(x) + q(x)u(x) = p(x)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(x) &= 20\sin(x) \\ p(x) &= -100\,e^{x^2} \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\,dx} \\ &= e^{\int 20\sin(x)dx} \\ &= e^{-20\cos(x)} \end{aligned}$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu u) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}x}(\mu u) &= (\mu)\left(-100\,\mathrm{e}^{x^2}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(u\,\mathrm{e}^{-20\cos(x)}\right) &= \left(\mathrm{e}^{-20\cos(x)}\right)\left(-100\,\mathrm{e}^{x^2}\right) \\
\mathrm{d}\left(u\,\mathrm{e}^{-20\cos(x)}\right) &= \left(-100\,\mathrm{e}^{x^2}\mathrm{e}^{-20\cos(x)}\right)\,\mathrm{d}x
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
u\,\mathrm{e}^{-20\cos(x)} &= \int -100\,\mathrm{e}^{x^2}\mathrm{e}^{-20\cos(x)}\,dx \\
&= \int -100\,\mathrm{e}^{x^2}\mathrm{e}^{-20\cos(x)}dx + c_1
\end{aligned}
$$

Dividing throughout by the integrating factor $\mathrm{e}^{-20\cos(x)}$ gives the final solution

$$
u(x) = \mathrm{e}^{20\cos(x)}\left(\int -100\,\mathrm{e}^{x^2}\mathrm{e}^{-20\cos(x)}dx + c_1\right)
$$

Substituting the solution found for $u(x)$ in $u = \mathrm{e}^{-20y}$ gives

$$
\begin{aligned}
y &= -\frac{\ln(u(x))}{20} \\
&= -\frac{\ln\left(-\frac{\ln\left(\left(-100\left(\int \mathrm{e}^{x^2-20\cos(x)}dx\right)+c_1\right)\mathrm{e}^{20\cos(x)}\right)}{20}\right)}{20} \\
&= -\frac{\ln\left(\left(-100\left(\int \mathrm{e}^{x^2-20\cos(x)}dx\right)+c_1\right)\mathrm{e}^{20\cos(x)}\right)}{20}
\end{aligned}
$$

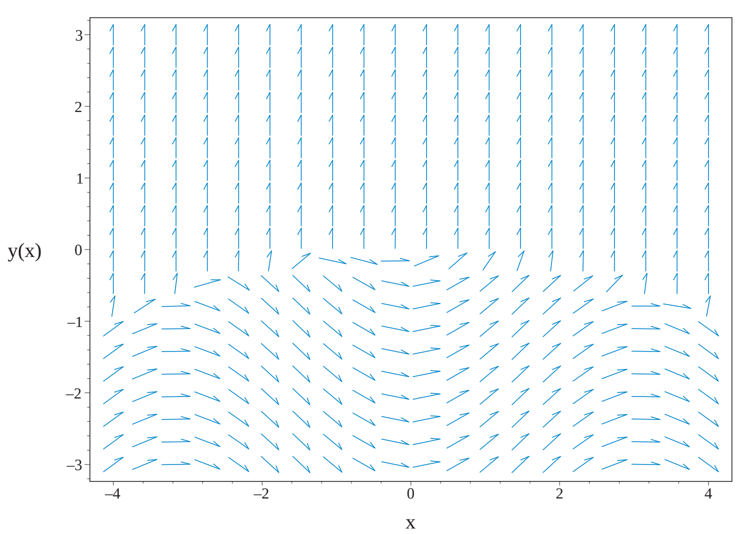


Figure 2.99: Slope field plot
$y' = 5\,\mathrm{e}^{x^2+20y} + \sin(x)$

Summary of solutions found

$$
y = -\frac{\ln\left(\left(-100\left(\int \mathrm{e}^{x^2-20\cos(x)}dx\right)+c_1\right)\mathrm{e}^{20\cos(x)}\right)}{20}
$$

Maple step by step solution

Let's solve

$$\frac{d}{dx}y(x) = 5\,\mathrm{e}^{x^2+20y(x)} + \sin(x)$$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = 5\,\mathrm{e}^{x^2+20y(x)} + \sin(x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
<- symmetry pattern of the form [0, F(x)*G(y)] successful`
```

Maple dsolve solution

Solving time : 0.023 (sec)
Leaf size : 33

```
dsolve(diff(y(x),x) = 5*exp(x^2+20*y(x))+sin(x),
       y(x),singsol=all)
```

$$y = -\cos(x) - \frac{\ln(20)}{20} - \frac{\ln\left(-c_1 - 5\left(\int \mathrm{e}^{x^2-20\cos(x)}dx\right)\right)}{20}$$

Mathematica DSolve solution

Solving time : `7.542` (sec)
Leaf size : 140

```
DSolve[{D[y[x],x]==5*Exp[x^2+20*y[x]]+Sin[x],{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$\text{Solve}\left[\int_1^x -\frac{1}{100}e^{-20\cos(K[1])-20y(x)}\left(\sin(K[1])+5e^{K[1]^2+20y(x)}\right)dK[1]+\int_1^{y(x)}\right.$$
$$-\frac{1}{100}e^{-20\cos(x)-20K[2]}\left(100e^{20\cos(x)+20K[2]}\int_1^x\left(\frac{1}{5}e^{-20\cos(K[1])-20K[2]}\left(\sin(K[1])+5e^{K[1]^2+20K[2]}\right)-e^{K[1]^2-20\cos(K\right.\right.$$
$$\left.\left.-1\right)dK[2]=c_1,y(x)\right]$$

2.2 section 2 (system of first order odes)

2.2.1 problem 1

Internal problem ID [8731]
Book : First order enumerated odes
Section : section 2 (system of first order odes)
Problem number : 1
Date solved : `Thursday, December 12, 2024 at 09:42:36 AM`
CAS classification : `system_of_ODEs`

$$
\begin{aligned}
x' + y' - x &= y + t \\
x' + y' &= 2x + 3y + \mathrm{e}^{t}
\end{aligned}
$$

In canonical form a linear first order is

$$
x' + q(t)x = p(t)
$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(t) &= -1 \\
p(t) &= 3t - 1
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\,dt} \\
&= \mathrm{e}^{\int (-1)dt} \\
&= \mathrm{e}^{-t}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\mu x) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}t}(\mu x) &= (\mu)\,(3t-1) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(x\,\mathrm{e}^{-t}\right) &= \left(\mathrm{e}^{-t}\right)(3t-1) \\
\mathrm{d}\left(x\,\mathrm{e}^{-t}\right) &= \left((3t-1)\,\mathrm{e}^{-t}\right)\,\mathrm{d}t
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
x\,\mathrm{e}^{-t} &= \int (3t-1)\,\mathrm{e}^{-t}\,dt \\
&= -(3t+2)\,\mathrm{e}^{-t} + \_C
\end{aligned}
$$

Dividing throughout by the integrating factor e^{-t} gives the final solution

$$
x = \_C\mathrm{e}^{t} - 3t - 2
$$

The system is

$$
\begin{aligned}
x' + y' &= x + y + t \qquad (1) \\
x' + y' &= 2x + 3y + \mathrm{e}^{t} \qquad (2)
\end{aligned}
$$

Since the left side is the same, this implies

$$x + y + t = 2x + 3y + e^t$$
$$y = -\frac{x}{2} - \frac{e^t}{2} + \frac{t}{2} \tag{3}$$

Taking derivative of the above w.r.t. t gives

$$y' = -\frac{x'}{2} - \frac{e^t}{2} + \frac{1}{2} \tag{4}$$

Substituting (3,4) in (1) to eliminate y, y' gives

$$\frac{x'}{2} - \frac{e^t}{2} + \frac{1}{2} = \frac{x}{2} - \frac{e^t}{2} + \frac{3t}{2}$$
$$x' = x + 3t - 1 \tag{5}$$

Which is now solved for x. Given now that we have the solution

$$x = \_Ce^t - 3t - 2 \tag{6}$$

Then substituting (6) into (3) gives

$$y = -\frac{\_Ce^t}{2} + 2t + 1 - \frac{e^t}{2} \tag{7}$$

Maple step by step solution

Maple dsolve solution

Solving time : 0.044 (sec)
Leaf size : 30

```
dsolve([diff(x(t),t)+diff(y(t),t)-x(t) = y(t)+t, diff(x(t),t)+diff(y(t),t) = 2*x(t)+3*
       ,{op([x(t), y(t)])})
```

$$x(t) = -3t - 2 + c_1 e^t$$
$$y(t) = 2t + 1 - \frac{c_1 e^t}{2} - \frac{e^t}{2}$$

Mathematica DSolve solution

Solving time : 0.026 (sec)
Leaf size : 37

```
DSolve[{{D[x[t],t]+D[y[t],t]-x[t]==y[t]+t,D[x[t],t]+D[y[t],t]==2*x[t]+3*y[t]+Exp[t]},{}},
       {x[t],y[t]},t,IncludeSingularSolutions->True]
```

$$x(t) \to -3t + (1 + 2c_1)e^t - 2$$
$$y(t) \to 2t - (1 + c_1)e^t + 1$$

2.2.2 problem 2

Internal problem ID [8732]
Book : First order enumerated odes
Section : section 2 (system of first order odes)
Problem number : 2
Date solved : `Thursday, December 12, 2024 at 09:42:36 AM`
CAS classification : `system_of_ODEs`

$$
\begin{aligned}
2x' + y' - x &= y + t \\
x' + y' &= 2x + 3y + \mathrm{e}^t
\end{aligned}
$$

Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t - \mathrm{e}^t \\ -t + 2\,\mathrm{e}^t \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{\left(1+\sqrt{3}\right)\mathrm{e}^{-\left(-2+\sqrt{3}\right)t}}{2} - \frac{\mathrm{e}^{\left(2+\sqrt{3}\right)t}\left(\sqrt{3}-1\right)}{2} & \frac{\left(-\mathrm{e}^{\left(2+\sqrt{3}\right)t}+\mathrm{e}^{-\left(-2+\sqrt{3}\right)t}\right)\sqrt{3}}{3} \\ -\frac{\left(-\mathrm{e}^{\left(2+\sqrt{3}\right)t}+\mathrm{e}^{-\left(-2+\sqrt{3}\right)t}\right)\sqrt{3}}{2} & \frac{\left(-\sqrt{3}+1\right)\mathrm{e}^{-\left(-2+\sqrt{3}\right)t}}{2} + \frac{\mathrm{e}^{\left(2+\sqrt{3}\right)t}\left(1+\sqrt{3}\right)}{2} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c}\\
&= \begin{bmatrix} \frac{(1+\sqrt{3})e^{-(-2+\sqrt{3})t}}{2} - \frac{e^{(2+\sqrt{3})t}(\sqrt{3}-1)}{2} & \frac{(-e^{(2+\sqrt{3})t}+e^{-(-2+\sqrt{3})t})\sqrt{3}}{3} \\ -\frac{(-e^{(2+\sqrt{3})t}+e^{-(-2+\sqrt{3})t})\sqrt{3}}{2} & \frac{(-\sqrt{3}+1)e^{-(-2+\sqrt{3})t}}{2} + \frac{e^{(2+\sqrt{3})t}(1+\sqrt{3})}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\\
&= \begin{bmatrix} \left(\frac{(1+\sqrt{3})e^{-(-2+\sqrt{3})t}}{2} - \frac{e^{(2+\sqrt{3})t}(\sqrt{3}-1)}{2}\right)c_1 + \frac{(-e^{(2+\sqrt{3})t}+e^{-(-2+\sqrt{3})t})\sqrt{3}\,c_2}{3} \\ -\frac{(-e^{(2+\sqrt{3})t}+e^{-(-2+\sqrt{3})t})\sqrt{3}\,c_1}{2} + \left(\frac{(-\sqrt{3}+1)e^{-(-2+\sqrt{3})t}}{2} + \frac{e^{(2+\sqrt{3})t}(1+\sqrt{3})}{2}\right)c_2 \end{bmatrix}\\
&= \begin{bmatrix} \frac{((3c_1+2c_2)\sqrt{3}+3c_1)e^{-(-2+\sqrt{3})t}}{6} - \frac{e^{(2+\sqrt{3})t}\left(\left(c_1+\frac{2c_2}{3}\right)\sqrt{3}-c_1\right)}{2} \\ \frac{((-c_1-c_2)\sqrt{3}+c_2)e^{-(-2+\sqrt{3})t}}{2} + \frac{((c_1+c_2)\sqrt{3}+c_2)e^{(2+\sqrt{3})t}}{2} \end{bmatrix}
\end{aligned}
$$

The particular solution given by

$$
\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt
$$

But

$$
\begin{aligned}
e^{-At} &= (e^{At})^{-1}\\
&= \begin{bmatrix} \frac{e^{-4t}\left((-\sqrt{3}+1)e^{-(-2+\sqrt{3})t}+e^{(2+\sqrt{3})t}(1+\sqrt{3})\right)}{2} & -\frac{\sqrt{3}\,e^{-4t}\left(-e^{(2+\sqrt{3})t}+e^{-(-2+\sqrt{3})t}\right)}{3} \\ \frac{\sqrt{3}\,e^{-4t}\left(-e^{(2+\sqrt{3})t}+e^{-(-2+\sqrt{3})t}\right)}{2} & \frac{e^{-4t}\left(\sqrt{3}\,e^{-(-2+\sqrt{3})t}-\sqrt{3}\,e^{(2+\sqrt{3})t}+e^{-(-2+\sqrt{3})t}+e^{(2+\sqrt{3})t}\right)}{2} \end{bmatrix}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} \frac{(1+\sqrt{3})e^{-(-2+\sqrt{3})t}}{2} - \frac{e^{(2+\sqrt{3})t}(\sqrt{3}-1)}{2} & \frac{(-e^{(2+\sqrt{3})t}+e^{-(-2+\sqrt{3})t})\sqrt{3}}{3} \\ -\frac{(-e^{(2+\sqrt{3})t}+e^{-(-2+\sqrt{3})t})\sqrt{3}}{2} & \frac{(-\sqrt{3}+1)e^{-(-2+\sqrt{3})t}}{2} + \frac{e^{(2+\sqrt{3})t}(1+\sqrt{3})}{2} \end{bmatrix} \int \begin{bmatrix} e^{-4t}\left((-\sqrt{3}+1)e^{-(-2\cdots} \\ \sqrt{3}\,e^{-4t}\left(-e^{(2+\cdots}\right. \end{bmatrix}\\
&= \begin{bmatrix} \frac{(1+\sqrt{3})e^{-(-2+\sqrt{3})t}}{2} - \frac{e^{(2+\sqrt{3})t}(\sqrt{3}-1)}{2} & \frac{(-e^{(2+\sqrt{3})t}+e^{-(-2+\sqrt{3})t})\sqrt{3}}{3} \\ -\frac{(-e^{(2+\sqrt{3})t}+e^{-(-2+\sqrt{3})t})\sqrt{3}}{2} & \frac{(-\sqrt{3}+1)e^{-(-2+\sqrt{3})t}}{2} + \frac{e^{(2+\sqrt{3})t}(1+\sqrt{3})}{2} \end{bmatrix} \begin{bmatrix} \frac{((5t+19)\sqrt{3}-9t-33)e^{-\cdots}}{6} \\ \frac{(7+(-4-t)\sqrt{3}+2t)e^{-(2+\cdots}}{2} \end{bmatrix}\\
&= \begin{bmatrix} -3t-11 \\ 2t+7-\frac{e^t}{2} \end{bmatrix}
\end{aligned}
$$

Hence the complete solution is

$$
\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t)\\
&= \begin{bmatrix} \frac{((3c_1+2c_2)\sqrt{3}+3c_1)e^{-(-2+\sqrt{3})t}}{6} + \frac{((-3c_1-2c_2)\sqrt{3}+3c_1)e^{(2+\sqrt{3})t}}{6} - 3t - 11 \\ \frac{((-c_1-c_2)\sqrt{3}+c_2)e^{-(-2+\sqrt{3})t}}{2} + \frac{((c_1+c_2)\sqrt{3}+c_2)e^{(2+\sqrt{3})t}}{2} + 2t + 7 - \frac{e^t}{2} \end{bmatrix}
\end{aligned}
$$

Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t - \mathrm{e}^{t} \\ -t + 2\,\mathrm{e}^{t} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} -1 & -2 \\ 3 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -1-\lambda & -2 \\ 3 & 5-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^{2} - 4\lambda + 1 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2 + \sqrt{3}$$
$$\lambda_2 = 2 - \sqrt{3}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$2 - \sqrt{3}$	1	real eigenvalue
$2 + \sqrt{3}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2 - \sqrt{3}$

We need to solve $A\vec{v} = \lambda \vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -2 \\ 3 & 5 \end{bmatrix} - \left(2 - \sqrt{3}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -3+\sqrt{3} & -2 \\ 3 & 3+\sqrt{3} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3+\sqrt{3} & -2 & 0 \\ 3 & 3+\sqrt{3} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{-3+\sqrt{3}} \Longrightarrow \left[\begin{array}{cc|c} -3+\sqrt{3} & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3+\sqrt{3} & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{-3+\sqrt{3}}\right\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{-3+\sqrt{3}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{-3+\sqrt{3}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{-3+\sqrt{3}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{-3+\sqrt{3}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2 + \sqrt{3}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} -1 & -2 \\ 3 & 5 \end{bmatrix} - \left(2+\sqrt{3}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3-\sqrt{3} & -2 \\ 3 & 3-\sqrt{3} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3-\sqrt{3} & -2 & 0 \\ 3 & 3-\sqrt{3} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{3R_1}{-3-\sqrt{3}} \Longrightarrow \left[\begin{array}{cc|c} -3-\sqrt{3} & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3-\sqrt{3} & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{3+\sqrt{3}}\right\}$

Hence the solution is

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{3+\sqrt{3}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2}{3+\sqrt{3}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{3+\sqrt{3}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} v_1 \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{3+\sqrt{3}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$2+\sqrt{3}$	1	1	No	$\begin{bmatrix} -\frac{2}{3+\sqrt{3}} \\ 1 \end{bmatrix}$
$2-\sqrt{3}$	1	1	No	$\begin{bmatrix} -\frac{2}{3-\sqrt{3}} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $2 + \sqrt{3}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^{\left(2+\sqrt{3}\right)t} \\ &= \begin{bmatrix} -\frac{2}{3+\sqrt{3}} \\ 1 \end{bmatrix} e^{\left(2+\sqrt{3}\right)t} \end{aligned}$$

Since eigenvalue $2 - \sqrt{3}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_2(t) &= \vec{v}_2 e^{\left(2-\sqrt{3}\right)t} \\ &= \begin{bmatrix} -\frac{2}{3-\sqrt{3}} \\ 1 \end{bmatrix} e^{\left(2-\sqrt{3}\right)t} \end{aligned}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} -\frac{2\,\mathrm{e}^{\left(2+\sqrt{3}\right)t}}{3+\sqrt{3}} \\ \mathrm{e}^{\left(2+\sqrt{3}\right)t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{2\,\mathrm{e}^{\left(2-\sqrt{3}\right)t}}{3-\sqrt{3}} \\ \mathrm{e}^{\left(2-\sqrt{3}\right)t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -\frac{2\,\mathrm{e}^{\left(2+\sqrt{3}\right)t}}{3+\sqrt{3}} & -\frac{2\,\mathrm{e}^{\left(2-\sqrt{3}\right)t}}{3-\sqrt{3}} \\ \mathrm{e}^{\left(2+\sqrt{3}\right)t} & \mathrm{e}^{\left(2-\sqrt{3}\right)t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\,dt$$

But

$$\Phi^{-1} = \begin{bmatrix} \frac{\sqrt{3}\,\mathrm{e}^{-\left(2+\sqrt{3}\right)t}}{2} & \frac{\sqrt{3}\left(3+\sqrt{3}\right)\mathrm{e}^{-\left(2+\sqrt{3}\right)t}}{6} \\ -\frac{\sqrt{3}\,\mathrm{e}^{\left(-2+\sqrt{3}\right)t}}{2} & \frac{\mathrm{e}^{\left(-2+\sqrt{3}\right)t}\sqrt{3}\left(-3+\sqrt{3}\right)}{6} \end{bmatrix}$$

Hence

$$\begin{aligned}
\vec{x}_p(t) &= \begin{bmatrix} -\frac{2\,\mathrm{e}^{\left(2+\sqrt{3}\right)t}}{3+\sqrt{3}} & -\frac{2\,\mathrm{e}^{\left(2-\sqrt{3}\right)t}}{3-\sqrt{3}} \\ \mathrm{e}^{\left(2+\sqrt{3}\right)t} & \mathrm{e}^{\left(2-\sqrt{3}\right)t} \end{bmatrix} \int \begin{bmatrix} \frac{\sqrt{3}\,\mathrm{e}^{-\left(2+\sqrt{3}\right)t}}{2} & \frac{\sqrt{3}\left(3+\sqrt{3}\right)\mathrm{e}^{-\left(2+\sqrt{3}\right)t}}{6} \\ -\frac{\sqrt{3}\,\mathrm{e}^{\left(-2+\sqrt{3}\right)t}}{2} & \frac{\mathrm{e}^{\left(-2+\sqrt{3}\right)t}\sqrt{3}\left(-3+\sqrt{3}\right)}{6} \end{bmatrix} \begin{bmatrix} t-\mathrm{e}^{t} \\ -t+2\,\mathrm{e}^{t} \end{bmatrix} dt \\
&= \begin{bmatrix} -\frac{2\,\mathrm{e}^{\left(2+\sqrt{3}\right)t}}{3+\sqrt{3}} & -\frac{2\,\mathrm{e}^{\left(2-\sqrt{3}\right)t}}{3-\sqrt{3}} \\ \mathrm{e}^{\left(2+\sqrt{3}\right)t} & \mathrm{e}^{\left(2-\sqrt{3}\right)t} \end{bmatrix} \int \begin{bmatrix} \frac{\sqrt{3}\,\mathrm{e}^{-t\left(1+\sqrt{3}\right)}}{2} + \mathrm{e}^{-t\left(1+\sqrt{3}\right)} - \frac{\mathrm{e}^{-\left(2+\sqrt{3}\right)t}t}{2} \\ -\frac{\sqrt{3}\,\mathrm{e}^{t\left(\sqrt{3}-1\right)}}{2} + \mathrm{e}^{t\left(\sqrt{3}-1\right)} - \frac{\mathrm{e}^{\left(-2+\sqrt{3}\right)t}t}{2} \end{bmatrix} dt \\
&= \begin{bmatrix} -\frac{2\,\mathrm{e}^{\left(2+\sqrt{3}\right)t}}{3+\sqrt{3}} & -\frac{2\,\mathrm{e}^{\left(2-\sqrt{3}\right)t}}{3-\sqrt{3}} \\ \mathrm{e}^{\left(2+\sqrt{3}\right)t} & \mathrm{e}^{\left(2-\sqrt{3}\right)t} \end{bmatrix} \begin{bmatrix} \frac{5\left(\left(\left(t+\frac{1}{5}\right)\sqrt{3}+\frac{9t}{5}+\frac{3}{5}\right)\mathrm{e}^{-\left(2+\sqrt{3}\right)t}+\mathrm{e}^{-t\left(1+\sqrt{3}\right)}\left(-\frac{26\sqrt{3}}{5}-9\right)\right)\sqrt{3}}{6\left(1+\sqrt{3}\right)\left(2+\sqrt{3}\right)^{2}} \\ -\frac{5\left(\left(\left(t+\frac{1}{5}\right)\sqrt{3}-\frac{9t}{5}-\frac{3}{5}\right)\mathrm{e}^{\left(-2+\sqrt{3}\right)t}+\mathrm{e}^{t\left(\sqrt{3}-1\right)}\left(-\frac{26\sqrt{3}}{5}+9\right)\right)\sqrt{3}}{6\left(\sqrt{3}-1\right)\left(-2+\sqrt{3}\right)^{2}} \end{bmatrix} \\
&= \begin{bmatrix} -3t-11 \\ 2t+7-\frac{\mathrm{e}^{t}}{2} \end{bmatrix}
\end{aligned}$$

Now that we found particular solution, the final solution is

$$\begin{aligned}
\vec{x}(t) &= \vec{x}_h(t) + \vec{x}_p(t) \\
\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} -\frac{2c_1\,\mathrm{e}^{\left(2+\sqrt{3}\right)t}}{3+\sqrt{3}} \\ c_1\,\mathrm{e}^{\left(2+\sqrt{3}\right)t} \end{bmatrix} + \begin{bmatrix} -\frac{2c_2\,\mathrm{e}^{\left(2-\sqrt{3}\right)t}}{3-\sqrt{3}} \\ c_2\,\mathrm{e}^{\left(2-\sqrt{3}\right)t} \end{bmatrix} + \begin{bmatrix} -3t-11 \\ 2t+7-\frac{\mathrm{e}^{t}}{2} \end{bmatrix}
\end{aligned}$$

Which becomes

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{c_2\left(3+\sqrt{3}\right)\mathrm{e}^{-\left(-2+\sqrt{3}\right)t}}{3} + \frac{c_1\left(-3+\sqrt{3}\right)\mathrm{e}^{\left(2+\sqrt{3}\right)t}}{3} - 3t - 11 \\ c_1\,\mathrm{e}^{\left(2+\sqrt{3}\right)t} + c_2\,\mathrm{e}^{-\left(-2+\sqrt{3}\right)t} + 2t + 7 - \frac{\mathrm{e}^{t}}{2} \end{bmatrix}$$

Maple step by step solution

Maple dsolve solution

Solving time : 0.070 (sec)
Leaf size : 94

```
dsolve([2*diff(x(t),t)+diff(y(t),t)-x(t) = y(t)+t, diff(x(t),t)+diff(y(t),t) = 2*x(t)+3*
       ,{op([x(t), y(t)])})
```

$$x(t) = e^{\left(2+\sqrt{3}\right)t} c_2 + e^{-\left(-2+\sqrt{3}\right)t} c_1 - 3t - 11$$
$$y(t) = -\frac{e^{\left(2+\sqrt{3}\right)t} c_2 \sqrt{3}}{2} + \frac{e^{-\left(-2+\sqrt{3}\right)t} c_1 \sqrt{3}}{2} - \frac{3\, e^{\left(2+\sqrt{3}\right)t} c_2}{2} - \frac{3\, e^{-\left(-2+\sqrt{3}\right)t} c_1}{2} - \frac{e^t}{2} + 2t + 7$$

Mathematica DSolve solution

Solving time : 6.59 (sec)
Leaf size : 174

```
DSolve[{{2*D[x[t],t]+D[y[t],t]-x[t]==y[t]+t,D[x[t],t]+D[y[t],t]==2*x[t]+3*y[t]+Exp[t]},{}},
       {x[t],y[t]},t,IncludeSingularSolutions->True]
```

$$x(t) \to \frac{1}{6} e^{-\left(\left(\sqrt{3}-2\right)t\right)} \left(-6e^{\left(\sqrt{3}-2\right)t}(3t+11) + \left(-3\left(\sqrt{3}-1\right)c_1 - 2\sqrt{3}c_2\right) e^{2\sqrt{3}t} + 3\left(1+\sqrt{3}\right)c_1 + 2\sqrt{3}c_2\right)$$
$$y(t) \to \frac{1}{2}\left(4t - e^t + \left(-\sqrt{3}c_1 - \sqrt{3}c_2 + c_2\right) e^{-\left(\left(\sqrt{3}-2\right)t\right)} + \left(\sqrt{3}c_1 + \left(1+\sqrt{3}\right)c_2\right) e^{\left(2+\sqrt{3}\right)t} + 14\right)$$

2.2.3 problem 3

Internal problem ID [8733]
Book : First order enumerated odes
Section : section 2 (system of first order odes)
Problem number : 3
Date solved : `Thursday, December 12, 2024 at 09:42:37 AM`
CAS classification : `system_of_ODEs`

$$
\begin{aligned}
x' + y' - x &= y + t + \sin(t) + \cos(t) \\
x' + y' &= 2x + 3y + e^{t}
\end{aligned}
$$

In canonical form a linear first order is

$$x' + q(t)x = p(t)$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(t) &= -1 \\
p(t) &= 3t + 4\sin(t) + 2\cos(t) - 1
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\,dt} \\
&= e^{\int (-1)dt} \\
&= e^{-t}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dt}(\mu x) &= \mu p \\
\frac{d}{dt}(\mu x) &= (\mu)\,(3t + 4\sin(t) + 2\cos(t) - 1) \\
\frac{d}{dt}\left(x\,e^{-t}\right) &= \left(e^{-t}\right)(3t + 4\sin(t) + 2\cos(t) - 1) \\
d\left(x\,e^{-t}\right) &= \left((3t + 4\sin(t) + 2\cos(t) - 1)\,e^{-t}\right)\,dt
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
x\,e^{-t} &= \int (3t + 4\sin(t) + 2\cos(t) - 1)\,e^{-t}\,dt \\
&= -3\,e^{-t}t - 2\,e^{-t} - 3\,e^{-t}\cos(t) - e^{-t}\sin(t) + \_C
\end{aligned}
$$

Dividing throughout by the integrating factor e^{-t} gives the final solution

$$x = \_C e^{t} - \sin(t) - 3\cos(t) - 3t - 2$$

The system is

$$x' + y' = x + y + t + \sin(t) + \cos(t) \tag{1}$$

$$x' + y' = 2x + 3y + e^{t} \tag{2}$$

Since the left side is the same, this implies

$$x+y+t+\sin(t)+\cos(t)=2x+3y+e^{t}$$
$$y=-\frac{x}{2}-\frac{e^{t}}{2}+\frac{t}{2}+\frac{\sin(t)}{2}+\frac{\cos(t)}{2} \tag{3}$$

Taking derivative of the above w.r.t. t gives

$$y'=-\frac{x'}{2}-\frac{e^{t}}{2}+\frac{1}{2}+\frac{\cos(t)}{2}-\frac{\sin(t)}{2} \tag{4}$$

Substituting (3,4) in (1) to eliminate y, y' gives

$$\frac{x'}{2}-\frac{e^{t}}{2}+\frac{1}{2}+\frac{\cos(t)}{2}-\frac{\sin(t)}{2}=\frac{x}{2}-\frac{e^{t}}{2}+\frac{3t}{2}+\frac{3\sin(t)}{2}+\frac{3\cos(t)}{2}$$
$$x'=x+3t+4\sin(t)+2\cos(t)-1 \tag{5}$$

Which is now solved for x. Given now that we have the solution

$$x=\_Ce^{t}-\sin(t)-3\cos(t)-3t-2 \tag{6}$$

Then substituting (6) into (3) gives

$$y=-\frac{\_Ce^{t}}{2}+\sin(t)+2\cos(t)+2t+1-\frac{e^{t}}{2} \tag{7}$$

Maple step by step solution

Maple dsolve solution

Solving time : `0.198` (sec)
Leaf size : 44

```
dsolve([diff(x(t),t)+diff(y(t),t)-x(t) = y(t)+t+sin(t)+cos(t), diff(x(t),t)+diff(y(t),t)
       ,{op([x(t), y(t)])})
```

$$x(t)=-\sin(t)-3\cos(t)+c_1e^{t}-3t-2$$
$$y(t)=\sin(t)+2\cos(t)-\frac{c_1e^{t}}{2}+2t+1-\frac{e^{t}}{2}$$

Mathematica DSolve solution

Solving time : `0.038` (sec)
Leaf size : 54

```
DSolve[{{D[x[t],t]+D[y[t],t]-x[t]==y[t]+t+Sin[t]+Cos[t],D[x[t],t]+D[y[t],t]==2*x[t]+3*y[t]+Exp
       {x[t],y[t]},t,IncludeSingularSolutions->True]
```

$$x(t)\to -3t+e^{t}-\sin(t)-3\cos(t)+2c_1e^{t}-2$$
$$y(t)\to 2t-e^{t}+\sin(t)+2\cos(t)-c_1e^{t}+1$$

2.3 section 3. First order odes solved using Laplace method

2.3.1 problem 1

Internal problem ID [8734]
Book : First order enumerated odes
Section : section 3. First order odes solved using Laplace method
Problem number : 1
Date solved : `Tuesday, December 17, 2024 at 01:01:49 PM`
CAS classification : `[_linear]`

Solve

$$ty' + y = t$$

With initial conditions

$$y(0) = 5$$

We will now apply Laplace transform to each term in the ode. Since this is time varying, the following Laplace transform property will be used

$$t^n f(t) \xrightarrow{\mathscr{L}} (-1)^n \frac{d^n}{ds^n} F(s)$$

Where in the above $F(s)$ is the laplace transform of $f(t)$. Applying the above property to each term of the ode gives

$$\begin{aligned} y &\xrightarrow{\mathscr{L}} Y(s) \\ ty' &\xrightarrow{\mathscr{L}} -Y(s) - s\left(\frac{d}{ds}Y(s)\right) \\ t &\xrightarrow{\mathscr{L}} \frac{1}{s^2} \end{aligned}$$

Collecting all the terms above, the ode in Laplace domain becomes

$$-sY' = \frac{1}{s^2}$$

The above ode in Y(s) is now solved.

Since the ode has the form $Y' = f(s)$, then we only need to integrate $f(s)$.

$$\begin{aligned} \int dY &= \int -\frac{1}{s^3}\, ds \\ Y &= \frac{1}{2s^2} + c_1 \end{aligned}$$

Applying inverse Laplace transform on the above gives.

$$y = \frac{t}{2} + c_1 \delta(t) \tag{1}$$

Substituting initial conditions $y(0) = 5$ and $y'(0) = 5$ into the above solution Gives

$$5 = c_1 \delta(0)$$

Solving for the constant c_1 from the above equation gives

$$c_1 = \frac{5}{\delta(0)}$$

Substituting the above back into the solution (1) gives

$$y = \frac{t}{2} + \frac{5\delta(t)}{\delta(0)}$$

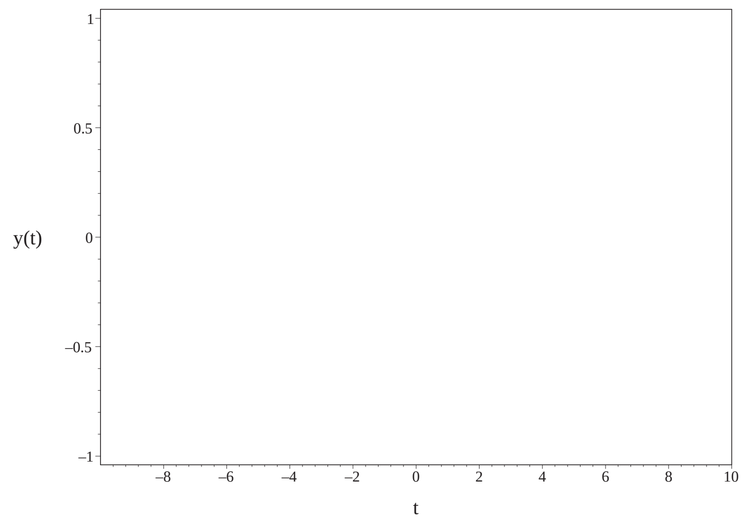


Figure 2.100: Solution plot
$y = \frac{t}{2} + \frac{5\delta(t)}{\delta(0)}$

Maple step by step solution

Let's solve
$[ty' + y = t, y(0) = 5]$

- Highest derivative means the order of the ODE is 1
  y'
- Isolate the derivative
  $y' = 1 - \frac{y}{t}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE
  $y' + \frac{y}{t} = 1$
- The ODE is linear; multiply by an integrating factor $\mu(t)$
  $\mu(t)\left(y' + \frac{y}{t}\right) = \mu(t)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(y\mu(t))$
  $\mu(t)\left(y' + \frac{y}{t}\right) = y'\mu(t) + y\mu'(t)$
- Isolate $\mu'(t)$
  $\mu'(t) = \frac{\mu(t)}{t}$
- Solve to find the integrating factor
  $\mu(t) = t$
- Integrate both sides with respect to t
  $\int \left(\frac{d}{dt}(y\mu(t))\right) dt = \int \mu(t)\, dt + C1$
- Evaluate the integral on the lhs
  $y\mu(t) = \int \mu(t)\, dt + C1$
- Solve for y
  $y = \frac{\int \mu(t)dt + C1}{\mu(t)}$
- Substitute $\mu(t) = t$
  $y = \frac{\int t dt + C1}{t}$
- Evaluate the integrals on the rhs
  $y = \frac{\frac{t^2}{2} + C1}{t}$

- Simplify

  $y = \frac{t^2+2C1}{2t}$
- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : `0.033` (sec)
Leaf size : 16

```
dsolve([t*diff(y(t),t)+y(t) = t,
        op([y(0) = 5])],
        y(t),method=laplace)
```

$$y = \frac{t}{2} + \frac{5\delta(t)}{\delta(0)}$$

Mathematica DSolve solution

Solving time : `0.0` (sec)
Leaf size : 0

```
DSolve[{t*D[y[t],t]+y[t]==t,{y[0]==5}},
       y[t],t,IncludeSingularSolutions->True]
```

Not solved

2.3.2 problem 2

Internal problem ID [8735]
Book : First order enumerated odes
Section : section 3. First order odes solved using Laplace method
Problem number : 2
Date solved : `Tuesday, December 17, 2024 at 01:01:49 PM`
CAS classification : `[_separable]`

Solve

$$y' - ty = 0$$

With initial conditions

$$y(0) = 0$$

We will now apply Laplace transform to each term in the ode. Since this is time varying, the following Laplace transform property will be used

$$t^n f(t) \xrightarrow{\mathscr{L}} (-1)^n \frac{d^n}{ds^n} F(s)$$

Where in the above $F(s)$ is the laplace transform of $f(t)$. Applying the above property to each term of the ode gives

$$-ty \xrightarrow{\mathscr{L}} \frac{d}{ds} Y(s)$$
$$y' \xrightarrow{\mathscr{L}} sY(s) - y(0)$$

Collecting all the terms above, the ode in Laplace domain becomes

$$sY - y(0) + Y' = 0$$

Replacing $y(0) = 0$ in the above results in

$$sY + Y' = 0$$

The above ode in Y(s) is now solved.

In canonical form a linear first order is

$$Y' + q(s)Y = p(s)$$

Comparing the above to the given ode shows that

$$q(s) = s$$
$$p(s) = 0$$

The integrating factor μ is

$$\mu = e^{\int s ds}$$

Therefore the solution is

$$Y = c_1 e^{-\int s ds}$$

Expanding and simplifying $Y(s)$ found above gives

$$Y = c_1 \,\mathrm{e}^{-\frac{s^2}{2}}$$

Applying inverse Laplace transform on the above gives.

$$y = c_1 \mathcal{L}^{-1}\left(\mathrm{e}^{-\frac{s^2}{2}}, s, t\right) \tag{1}$$

Substituting initial conditions $y(0) = 0$ and $y'(0) = 0$ into the above solution Gives

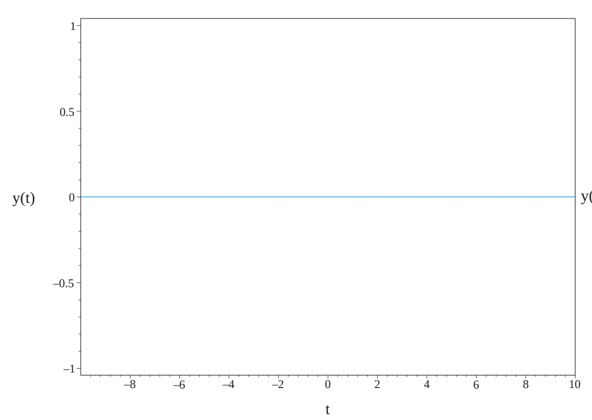
$$0 = c_1 \mathcal{L}^{-1}\left(\mathrm{e}^{-\frac{s^2}{2}}, s, t\right)$$

Solving for the constant c_1 from the above equation gives

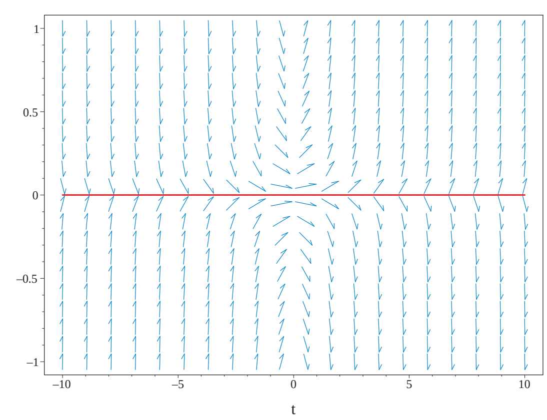
$$c_1 = 0$$

Substituting the above back into the solution (1) gives

$$y = 0$$



(a) Solution plot
$y = 0$



(b) Slope field plot
$y' - ty = 0$

Maple step by step solution

Let's solve
$[y' - yt = 0, y(0) = 0]$

- Highest derivative means the order of the ODE is 1
  y'
- Solve for the highest derivative
  $y' = yt$
- Separate variables
  $\frac{y'}{y} = t$
- Integrate both sides with respect to t
  $\int \frac{y'}{y} dt = \int t dt + C1$
- Evaluate integral
  $\ln(y) = \frac{t^2}{2} + C1$
- Solve for y
  $y = \mathrm{e}^{\frac{t^2}{2} + C1}$
- Use initial condition $y(0) = 0$
  $0 = \mathrm{e}^{C1}$

- Solve for $\_C1$
  $C1 = ()$
- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.032 (sec)
Leaf size : 5

```
dsolve([diff(y(t),t)-y(t)*t = 0,
        op([y(0) = 0])],
        y(t),method=laplace)
```

$$y = 0$$

Mathematica DSolve solution

Solving time : 0.001 (sec)
Leaf size : 6

```
DSolve[{D[y[t],t]-t*y[t]==0,y[0]==0},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to 0$$

2.3.3 problem 3

Internal problem ID [8736]
Book : First order enumerated odes
Section : section 3. First order odes solved using Laplace method
Problem number : 3
Date solved : `Tuesday, December 17, 2024 at 01:01:50 PM`
CAS classification : `[_separable]`

Solve

$$ty' + y = 0$$

With initial conditions

$$y(0) = 0$$

We will now apply Laplace transform to each term in the ode. Since this is time varying, the following Laplace transform property will be used

$$t^n f(t) \xrightarrow{\mathscr{L}} (-1)^n \frac{d^n}{ds^n} F(s)$$

Where in the above $F(s)$ is the laplace transform of $f(t)$. Applying the above property to each term of the ode gives

$$\begin{aligned} y &\xrightarrow{\mathscr{L}} Y(s) \\ ty' &\xrightarrow{\mathscr{L}} -Y(s) - s\left(\frac{d}{ds}Y(s)\right) \end{aligned}$$

Collecting all the terms above, the ode in Laplace domain becomes

$$-sY' = 0$$

The above ode in Y(s) is now solved.

Since the ode has the form $Y' = f(s)$, then we only need to integrate $f(s)$.

$$\int dY = \int 0\, ds + c_1$$

$$Y = c_1$$

Applying inverse Laplace transform on the above gives.

$$y = c_1 \delta(t) \tag{1}$$

Substituting initial conditions $y(0) = 0$ and $y'(0) = 0$ into the above solution Gives

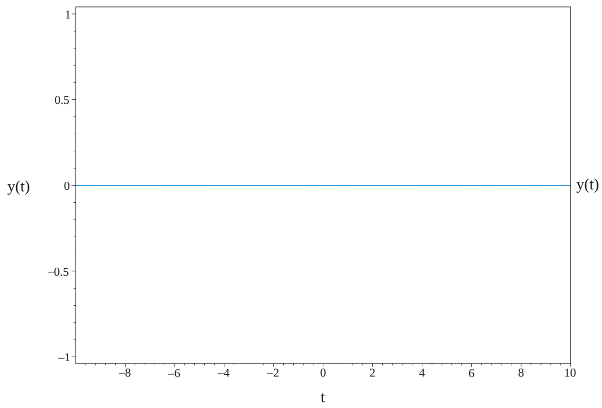
$$0 = c_1 \delta(0)$$

Solving for the constant c_1 from the above equation gives

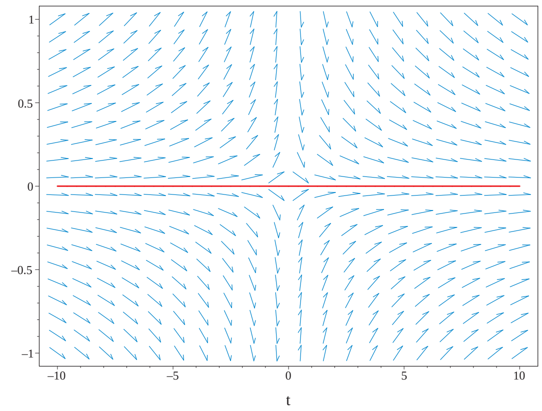
$$c_1 = 0$$

Substituting the above back into the solution (1) gives

$$y = 0$$



(a) Solution plot
$y = 0$



(b) Slope field plot
$ty' + y = 0$

Maple step by step solution

Let's solve
$[ty' + y = 0, y(0) = 0]$

- Highest derivative means the order of the ODE is 1
  y'
- Separate variables
  $\frac{y'}{y} = -\frac{1}{t}$
- Integrate both sides with respect to t
  $\int \frac{y'}{y} dt = \int -\frac{1}{t} dt + C1$
- Evaluate integral
  $\ln(y) = -\ln(t) + C1$
- Solve for y
  $y = \frac{e^{C1}}{t}$
- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : `0.020` (sec)
Leaf size : 5

```
dsolve([t*diff(y(t),t)+y(t) = 0,
        op([y(0) = 0])],
        y(t),method=laplace)
```

$$y = 0$$

Mathematica DSolve solution

Solving time : 0.001 (sec)
Leaf size : 6

```
DSolve[{t*D[y[t],t]+y[t]==0,y[0]==0},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to 0$$

2.3.4 problem 4

Internal problem ID [8737]
Book : First order enumerated odes
Section : section 3. First order odes solved using Laplace method
Problem number : 4
Date solved : `Tuesday, December 17, 2024 at 01:01:51 PM`
CAS classification : `[_separable]`

Solve

$$ty' + y = 0$$

With initial conditions

$$y(0) = y_0$$

We will now apply Laplace transform to each term in the ode. Since this is time varying, the following Laplace transform property will be used

$$t^n f(t) \xrightarrow{\mathscr{L}} (-1)^n \frac{d^n}{ds^n} F(s)$$

Where in the above $F(s)$ is the laplace transform of $f(t)$. Applying the above property to each term of the ode gives

$$\begin{aligned} y &\xrightarrow{\mathscr{L}} Y(s) \\ ty' &\xrightarrow{\mathscr{L}} -Y(s) - s\left(\frac{d}{ds}Y(s)\right) \end{aligned}$$

Collecting all the terms above, the ode in Laplace domain becomes

$$-sY' = 0$$

The above ode in Y(s) is now solved.

Since the ode has the form $Y' = f(s)$, then we only need to integrate $f(s)$.

$$\begin{aligned} \int dY &= \int 0\, ds + c_1 \\ Y &= c_1 \end{aligned}$$

Applying inverse Laplace transform on the above gives.

$$y = c_1 \delta(t) \tag{1}$$

Substituting initial conditions $y(0) = y_0$ and $y'(0) = y_0$ into the above solution Gives

$$y_0 = c_1 \delta(0)$$

Solving for the constant c_1 from the above equation gives

$$c_1 = \frac{y_0}{\delta(0)}$$

Substituting the above back into the solution (1) gives

$$y = \frac{y_0 \delta(t)}{\delta(0)}$$

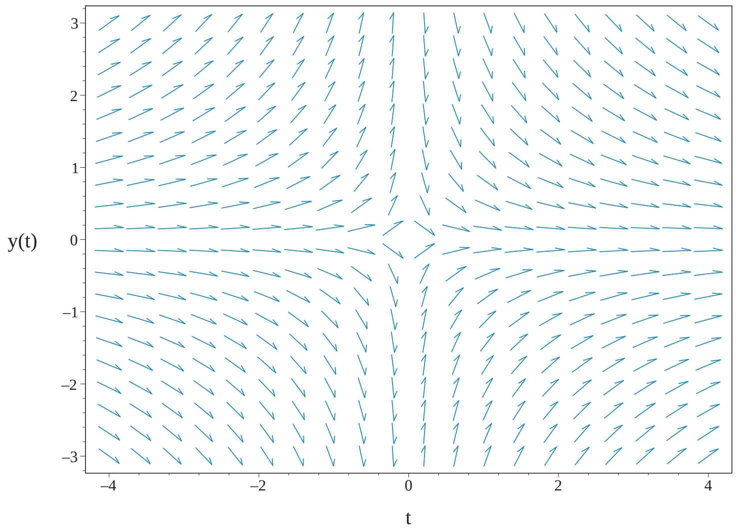


Figure 2.103: Slope field plot
$ty' + y = 0$

Maple step by step solution

Let's solve
$[ty' + y = 0, y(0) = y_0]$

- Highest derivative means the order of the ODE is 1
  y'
- Separate variables
  $\frac{y'}{y} = -\frac{1}{t}$
- Integrate both sides with respect to t
  $\int \frac{y'}{y} dt = \int -\frac{1}{t} dt + C1$
- Evaluate integral
  $\ln(y) = -\ln(t) + C1$
- Solve for y
  $y = \frac{e^{C1}}{t}$
- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.020 (sec)
Leaf size : 12

```
dsolve([t*diff(y(t),t)+y(t) = 0,
        op([y(0) = y__0])],
        y(t),method=laplace)
```

$$y = \frac{y_0 \delta(t)}{\delta(0)}$$

Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{t*D[y[t],t]+y[t]==0,y[0]==y0},
       y[t],t,IncludeSingularSolutions->True]
```

Not solved

2.3.5 problem 5

Internal problem ID [8738]
Book : First order enumerated odes
Section : section 3. First order odes solved using Laplace method
Problem number : 5
Date solved : `Tuesday, December 17, 2024 at 01:01:52 PM`
CAS classification : `[_separable]`

Solve

$$ty' + y = 0$$

With initial conditions

$$y(x_0) = y_0$$

Since initial condition is not at zero, then change of variable is used to transform the ode so that initial condition is at zero.

$$\tau = t - x_0$$

Solve

$$(\tau + x_0)\, y' + y = 0$$

With initial conditions

$$y(0) = y_0$$

We will now apply Laplace transform to each term in the ode. Since this is time varying, the following Laplace transform property will be used

$$\tau^n f(\tau) \xrightarrow{\mathscr{L}} (-1)^n \frac{d^n}{ds^n} F(s)$$

Where in the above $F(s)$ is the laplace transform of $f(\tau)$. Applying the above property to each term of the ode gives

$$\begin{aligned} y(\tau) &\xrightarrow{\mathscr{L}} Y(s) \\ (\tau + x_0)\left(\frac{d}{d\tau} y(\tau)\right) &\xrightarrow{\mathscr{L}} -Y(s) - s\left(\frac{d}{ds} Y(s)\right) + x_0(sY(s) - y(0)) \end{aligned}$$

Collecting all the terms above, the ode in Laplace domain becomes

$$-sY' + x_0(sY - y(0)) = 0$$

Replacing $y(0) = y_0$ in the above results in

$$-sY' + x_0(sY - y_0) = 0$$

The above ode in Y(s) is now solved.

In canonical form a linear first order is

$$Y' + q(s)Y = p(s)$$

Comparing the above to the given ode shows that

$$
\begin{aligned}
q(s) &= -x_0 \\
p(s) &= -\frac{x_0 y_0}{s}
\end{aligned}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int q\, ds} \\
&= \mathrm{e}^{\int -x_0 ds} \\
&= \mathrm{e}^{-x_0 s}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}(\mu Y) &= \mu p \\
\frac{\mathrm{d}}{\mathrm{d}s}(\mu Y) &= (\mu)\left(-\frac{x_0 y_0}{s}\right) \\
\frac{\mathrm{d}}{\mathrm{d}s}\left(Y\,\mathrm{e}^{-x_0 s}\right) &= \left(\mathrm{e}^{-x_0 s}\right)\left(-\frac{x_0 y_0}{s}\right) \\
\mathrm{d}\left(Y\,\mathrm{e}^{-x_0 s}\right) &= \left(-\frac{x_0 y_0 \mathrm{e}^{-x_0 s}}{s}\right)\,\mathrm{d}s
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
Y\,\mathrm{e}^{-x_0 s} &= \int -\frac{x_0 y_0 \mathrm{e}^{-x_0 s}}{s}\, ds \\
&= x_0 y_0 \operatorname{Ei}_1(x_0 s) + c_1
\end{aligned}
$$

Dividing throughout by the integrating factor $\mathrm{e}^{-x_0 s}$ gives the final solution

$$
Y = \mathrm{e}^{x_0 s}(x_0 y_0 \operatorname{Ei}_1(x_0 s) + c_1)
$$

Applying inverse Laplace transform on the above gives.

$$
y = \frac{x_0 y_0}{\tau + x_0} + c_1 \mathcal{L}^{-1}(\mathrm{e}^{x_0 s}, s, \tau) \tag{1}
$$

Substituting initial conditions $y(0) = y_0$ and $y'(0) = y_0$ into the above solution Gives

$$
y_0 = c_1 \mathcal{L}^{-1}(\mathrm{e}^{x_0 s}, s, \tau) + y_0
$$

Solving for the constant c_1 from the above equation gives

$$
c_1 = 0
$$

Substituting the above back into the solution (1) gives

$$
y = \frac{x_0 y_0}{\tau + x_0}
$$

Changing back the solution from τ to t using

$$
\tau = t - x_0
$$

the solution becomes

$$
y(t) = \frac{x_0 y_0}{t}
$$

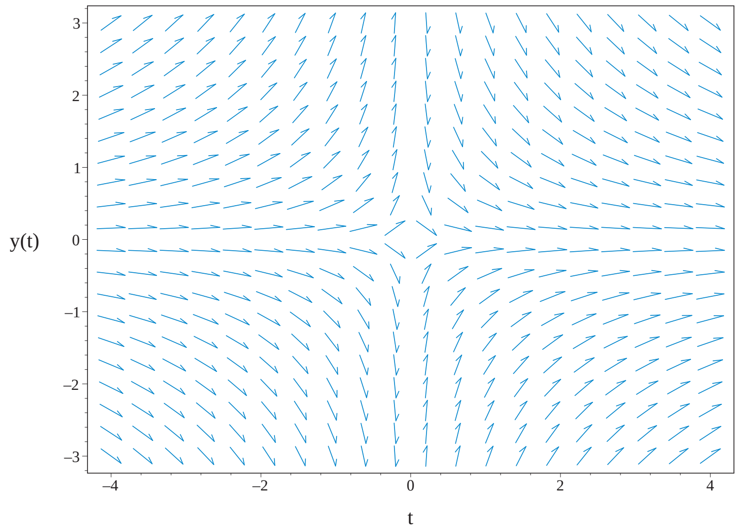


Figure 2.104: Slope field plot
$t\left(\frac{d}{dt}y(t)\right)+y(t)=0$

Maple step by step solution

Let's solve
$[ty'+y=0, y(x_0)=y_0]$

- Highest derivative means the order of the ODE is 1
  y'
- Separate variables
  $\frac{y'}{y}=-\frac{1}{t}$
- Integrate both sides with respect to t
  $\int \frac{y'}{y}dt=\int -\frac{1}{t}dt+C1$
- Evaluate integral
  $\ln(y)=-\ln(t)+C1$
- Solve for y
  $y=\frac{e^{C1}}{t}$
- Use initial condition $y(x_0)=y_0$
  $y_0=\frac{e^{C1}}{x_0}$
- Solve for $\_C1$
  $C1=\ln(x_0y_0)$
- Substitute $\_C1=\ln(x_0y_0)$ into general solution and simplify
  $y=\frac{x_0y_0}{t}$
- Solution to the IVP
  $y=\frac{x_0y_0}{t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.028 (sec)
Leaf size : 10

```
dsolve([t*diff(y(t),t)+y(t) = 0,
        op([y(x__0) = y__0])],
        y(t),method=laplace)
```

$$y = \frac{x_0 y_0}{t}$$

Mathematica DSolve solution

Solving time : 1.702 (sec)
Leaf size : 11

```
DSolve[{t*D[y[t],t]+y[t]==0,y[x0]==y0},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{\text{x0y0}}{t}$$

2.3.6 problem 6

Internal problem ID [8739]
Book : First order enumerated odes
Section : section 3. First order odes solved using Laplace method
Problem number : 6
Date solved : `Tuesday, December 17, 2024 at 01:01:52 PM`
CAS classification : `[_separable]`

Solve

$$ty' + y = 0$$

Since no initial condition is explicitly given, then let

$$y(0) = c_1$$

We will now apply Laplace transform to each term in the ode. Since this is time varying, the following Laplace transform property will be used

$$t^n f(t) \xrightarrow{\mathscr{L}} (-1)^n \frac{d^n}{ds^n} F(s)$$

Where in the above $F(s)$ is the laplace transform of $f(t)$. Applying the above property to each term of the ode gives

$$\begin{aligned} y &\xrightarrow{\mathscr{L}} Y(s) \\ ty' &\xrightarrow{\mathscr{L}} -Y(s) - s\left(\frac{d}{ds}Y(s)\right) \end{aligned}$$

Collecting all the terms above, the ode in Laplace domain becomes

$$-sY' = 0$$

The above ode in Y(s) is now solved.

Since the ode has the form $Y' = f(s)$, then we only need to integrate $f(s)$.

$$\begin{aligned} \int dY &= \int 0\,ds + c_2 \\ Y &= c_2 \end{aligned}$$

Applying inverse Laplace transform on the above gives.

$$y = c_2\delta(t) \tag{1}$$

Substituting initial conditions $y(0) = c_1$ and $y'(0) = c_1$ into the above solution Gives

$$c_1 = c_2\delta(0)$$

Solving for the constant c_2 from the above equation gives

$$c_2 = \frac{c_1}{\delta(0)}$$

Substituting the above back into the solution (1) gives

$$y = \frac{c_1 \delta(t)}{\delta(0)}$$

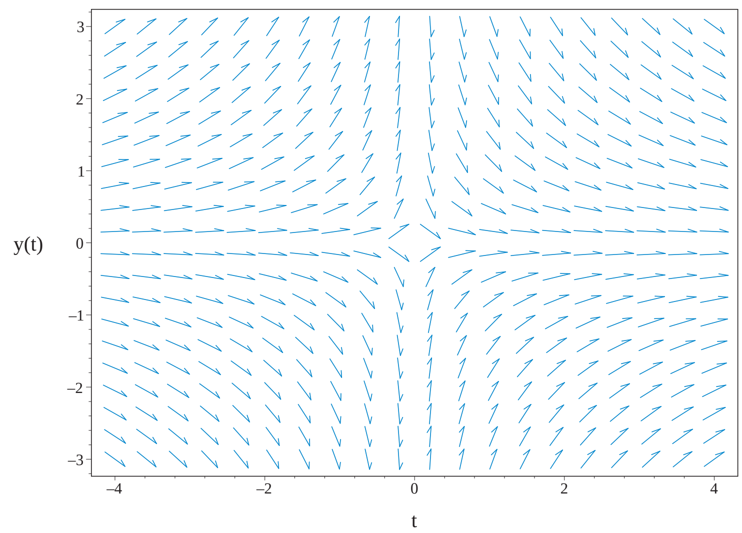


Figure 2.105: Slope field plot
$ty' + y = 0$

Maple step by step solution

Let's solve
$ty' + y = 0$

- Highest derivative means the order of the ODE is 1
  y'
- Separate variables
  $\frac{y'}{y} = -\frac{1}{t}$
- Integrate both sides with respect to t
  $\int \frac{y'}{y} dt = \int -\frac{1}{t} dt + C1$
- Evaluate integral
  $\ln(y) = -\ln(t) + C1$
- Solve for y
  $y = \frac{e^{C1}}{t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.027 (sec)
Leaf size : 8

```
dsolve(t*diff(y(t),t)+y(t) = 0,
       y(t),method=laplace)
```

$$y = c_1 \delta(t)$$

Mathematica DSolve solution

Solving time : 0.02 (sec)
Leaf size : 16

```
DSolve[{t*D[y[t],t]+y[t]==0,{}},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{c_1}{t}$$
$$y(t) \to 0$$

2.3.7 problem 7

Internal problem ID [8740]
Book : First order enumerated odes
Section : section 3. First order odes solved using Laplace method
Problem number : 7
Date solved : `Tuesday, December 17, 2024 at 01:01:53 PM`
CAS classification : `[_separable]`

Solve

$$ty' + y = 0$$

With initial conditions

$$y(1) = 5$$

Since initial condition is not at zero, then change of variable is used to transform the ode so that initial condition is at zero.

$$\tau = t - 1$$

Solve

$$(\tau + 1)\, y' + y = 0$$

With initial conditions

$$y(0) = 5$$

We will now apply Laplace transform to each term in the ode. Since this is time varying, the following Laplace transform property will be used

$$\tau^n f(\tau) \xrightarrow{\mathscr{L}} (-1)^n \frac{d^n}{ds^n} F(s)$$

Where in the above $F(s)$ is the laplace transform of $f(\tau)$. Applying the above property to each term of the ode gives

$$\begin{aligned} y(\tau) &\xrightarrow{\mathscr{L}} Y(s) \\ (\tau + 1)\left(\frac{d}{d\tau} y(\tau)\right) &\xrightarrow{\mathscr{L}} -Y(s) - s\left(\frac{d}{ds} Y(s)\right) + sY(s) - y(0) \end{aligned}$$

Collecting all the terms above, the ode in Laplace domain becomes

$$-sY' + sY - y(0) = 0$$

Replacing $y(0) = 5$ in the above results in

$$-sY' + sY - 5 = 0$$

The above ode in Y(s) is now solved.

In canonical form a linear first order is

$$Y' + q(s)Y = p(s)$$

Comparing the above to the given ode shows that

$$q(s) = -1$$
$$p(s) = -\frac{5}{s}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,ds} \\ &= e^{\int(-1)ds} \\ &= e^{-s}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{ds}(\mu Y) &= \mu p \\ \frac{d}{ds}(\mu Y) &= (\mu)\left(-\frac{5}{s}\right) \\ \frac{d}{ds}\left(Y\,e^{-s}\right) &= \left(e^{-s}\right)\left(-\frac{5}{s}\right) \\ d\left(Y\,e^{-s}\right) &= \left(-\frac{5\,e^{-s}}{s}\right)\,ds\end{aligned}$$

Integrating gives

$$\begin{aligned}Y\,e^{-s} &= \int -\frac{5\,e^{-s}}{s}\,ds \\ &= 5\ \mathrm{Ei}_1(s) + c_1\end{aligned}$$

Dividing throughout by the integrating factor e^{-s} gives the final solution

$$Y = e^{s}(5\ \mathrm{Ei}_1(s) + c_1)$$

Applying inverse Laplace transform on the above gives.

$$y = \frac{5}{\tau+1} + c_1\mathcal{L}^{-1}(e^{s}, s, \tau) \tag{1}$$

Substituting initial conditions $y(0) = 5$ and $y'(0) = 5$ into the above solution Gives

$$5 = c_1\mathcal{L}^{-1}(e^{s}, s, \tau) + 5$$

Solving for the constant c_1 from the above equation gives

$$c_1 = 0$$

Substituting the above back into the solution (1) gives

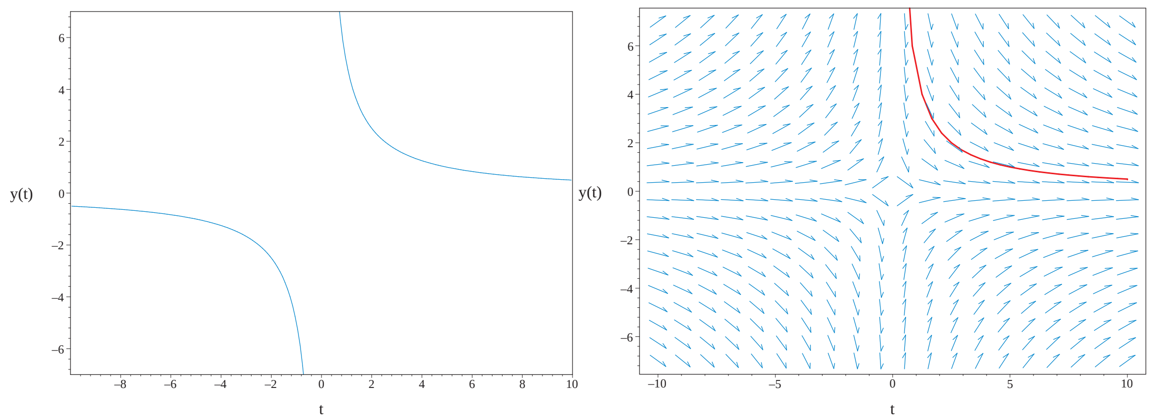
$$y = \frac{5}{\tau+1}$$

Changing back the solution from τ to t using

$$\tau = t - 1$$

the solution becomes

$$y(t) = \frac{5}{t}$$



(a) Solution plot
$y(t) = \frac{5}{t}$

(b) Slope field plot
$t\left(\frac{d}{dt}y(t)\right) + y(t) = 0$

Maple step by step solution

Let's solve
$[ty' + y = 0, y(1) = 5]$

- Highest derivative means the order of the ODE is 1
  y'
- Separate variables
  $\frac{y'}{y} = -\frac{1}{t}$
- Integrate both sides with respect to t
  $\int \frac{y'}{y} dt = \int -\frac{1}{t} dt + C1$
- Evaluate integral
  $\ln(y) = -\ln(t) + C1$
- Solve for y
  $y = \frac{e^{C1}}{t}$
- Use initial condition $y(1) = 5$
  $5 = e^{C1}$
- Solve for $\_C1$
  $C1 = \ln(5)$
- Substitute $\_C1 = \ln(5)$ into general solution and simplify
  $y = \frac{5}{t}$
- Solution to the IVP
  $y = \frac{5}{t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.046 (sec)
Leaf size : 9

```
dsolve([t*diff(y(t),t)+y(t) = 0,
        op([y(1) = 5])],
        y(t),method=laplace)
```

$$y = \frac{5}{t}$$

Mathematica DSolve solution

Solving time : 0.019 (sec)
Leaf size : 10

```
DSolve[{t*D[y[t],t]+y[t]==0,y[1]==5},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{5}{t}$$

2.3.8 problem 8

Internal problem ID [8741]
Book : First order enumerated odes
Section : section 3. First order odes solved using Laplace method
Problem number : 8
Date solved : `Tuesday, December 17, 2024 at 01:01:54 PM`
CAS classification : `[_linear]`

Solve

$$ty' + y = \sin(t)$$

With initial conditions

$$y(1) = 0$$

Since initial condition is not at zero, then change of variable is used to transform the ode so that initial condition is at zero.

$$\tau = t - 1$$

Solve

$$(\tau + 1)\, y' + y = \sin(\tau + 1)$$

With initial conditions

$$y(0) = 0$$

We will now apply Laplace transform to each term in the ode. Since this is time varying, the following Laplace transform property will be used

$$\tau^n f(\tau) \xrightarrow{\mathscr{L}} (-1)^n \frac{d^n}{ds^n} F(s)$$

Where in the above $F(s)$ is the laplace transform of $f(\tau)$. Applying the above property to each term of the ode gives

$$\begin{aligned} y(\tau) &\xrightarrow{\mathscr{L}} Y(s) \\ (\tau + 1)\left(\frac{d}{d\tau} y(\tau)\right) &\xrightarrow{\mathscr{L}} -Y(s) - s\left(\frac{d}{ds} Y(s)\right) + sY(s) - y(0) \\ \sin(\tau + 1) &\xrightarrow{\mathscr{L}} \frac{\sin(1)\, s + \cos(1)}{s^2 + 1} \end{aligned}$$

Collecting all the terms above, the ode in Laplace domain becomes

$$-sY' + sY - y(0) = \frac{\sin(1)\, s + \cos(1)}{s^2 + 1}$$

Replacing $y(0) = 0$ in the above results in

$$-sY' + sY = \frac{\sin(1)\, s + \cos(1)}{s^2 + 1}$$

The above ode in Y(s) is now solved.

In canonical form a linear first order is

$$Y' + q(s)Y = p(s)$$

Comparing the above to the given ode shows that

$$q(s) = -1$$
$$p(s) = \frac{-\sin(1)\, s - \cos(1)}{(s^2+1)\, s}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\, ds}\\ &= \mathrm{e}^{\int(-1)ds}\\ &= \mathrm{e}^{-s}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}(\mu Y) &= \mu p\\ \frac{\mathrm{d}}{\mathrm{d}s}(\mu Y) &= (\mu)\left(\frac{-\sin(1)\, s - \cos(1)}{(s^2+1)\, s}\right)\\ \frac{\mathrm{d}}{\mathrm{d}s}\left(Y\,\mathrm{e}^{-s}\right) &= \left(\mathrm{e}^{-s}\right)\left(\frac{-\sin(1)\, s - \cos(1)}{(s^2+1)\, s}\right)\\ \mathrm{d}\left(Y\,\mathrm{e}^{-s}\right) &= \left(\frac{(-\sin(1)\, s - \cos(1))\,\mathrm{e}^{-s}}{(s^2+1)\, s}\right)\mathrm{d}s\end{aligned}$$

Integrating gives

$$\begin{aligned}Y\,\mathrm{e}^{-s} &= \int \frac{(-\sin(1)\, s - \cos(1))\,\mathrm{e}^{-s}}{(s^2+1)\, s}\, ds\\ &= -\cos(1)\left(\frac{\mathrm{e}^{i}\,\mathrm{Ei}_1(s+i)}{2} + \frac{\mathrm{e}^{-i}\,\mathrm{Ei}_1(s-i)}{2} - \mathrm{Ei}_1(s)\right) + \sin(1)\left(\frac{i\mathrm{e}^{i}\,\mathrm{Ei}_1(s+i)}{2} - \frac{i\mathrm{e}^{-i}\,\mathrm{Ei}_1(s-i)}{2}\right) +\end{aligned}$$

Dividing throughout by the integrating factor e^{-s} gives the final solution

$$Y = -\frac{(-2\cos(1)\,\mathrm{Ei}_1(s) + \mathrm{Ei}_1(s+i) + \mathrm{Ei}_1(s-i) - 2c_1)\,\mathrm{e}^{s}}{2}$$

Applying inverse Laplace transform on the above gives.

$$y = \frac{\cos(1)}{\tau+1} + c_1\mathcal{L}^{-1}(\mathrm{e}^{s}, s, \tau) - \frac{\cos(\tau+1)}{\tau+2} \tag{1}$$

Substituting initial conditions $y(0) = 0$ and $y'(0) = 0$ into the above solution Gives

$$0 = c_1\mathcal{L}^{-1}(\mathrm{e}^{s}, s, \tau) + \frac{\cos(1)}{2}$$

Solving for the constant c_1 from the above equation gives

$$c_1 = -\frac{\cos(1)}{2\mathcal{L}^{-1}(\mathrm{e}^{s}, s, \tau)}$$

Substituting the above back into the solution (1) gives

$$y = \frac{\cos(1)}{\tau+1} - \frac{\cos(1)}{2} - \frac{\cos(\tau+1)}{\tau+2}$$

Changing back the solution from τ to t using

$$\tau = t - 1$$

the solution becomes

$$y(t) = \frac{\cos(1)}{t} - \frac{\cos(1)}{2} - \frac{\cos(t)}{t+1}$$

The solution was found not to satisfy the ode or the IC. Hence it is removed.

Maple step by step solution

Let's solve
$[ty' + y = \sin(t), y(1) = 0]$

- Highest derivative means the order of the ODE is 1
  y'
- Isolate the derivative
  $y' = -\frac{y}{t} + \frac{\sin(t)}{t}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE
  $y' + \frac{y}{t} = \frac{\sin(t)}{t}$
- The ODE is linear; multiply by an integrating factor $\mu(t)$
  $\mu(t)\left(y' + \frac{y}{t}\right) = \frac{\mu(t)\sin(t)}{t}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(y\mu(t))$
  $\mu(t)\left(y' + \frac{y}{t}\right) = y'\mu(t) + y\mu'(t)$
- Isolate $\mu'(t)$
  $\mu'(t) = \frac{\mu(t)}{t}$
- Solve to find the integrating factor
  $\mu(t) = t$
- Integrate both sides with respect to t
  $\int \left(\frac{d}{dt}(y\mu(t))\right) dt = \int \frac{\mu(t)\sin(t)}{t} dt + C1$
- Evaluate the integral on the lhs
  $y\mu(t) = \int \frac{\mu(t)\sin(t)}{t} dt + C1$
- Solve for y
  $y = \frac{\int \frac{\mu(t)\sin(t)}{t} dt + C1}{\mu(t)}$
- Substitute $\mu(t) = t$
  $y = \frac{\int \sin(t) dt + C1}{t}$
- Evaluate the integrals on the rhs
  $y = \frac{-\cos(t) + C1}{t}$
- Use initial condition $y(1) = 0$
  $0 = -\cos(1) + C1$
- Solve for $\_C1$
  $C1 = \cos(1)$
- Substitute $\_C1 = \cos(1)$ into general solution and simplify
  $y = \frac{-\cos(t) + \cos(1)}{t}$
- Solution to the IVP
  $y = \frac{-\cos(t) + \cos(1)}{t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.237 (sec)
Leaf size : maple_leaf_size

```
dsolve([t*diff(y(t),t)+y(t) = sin(t),
        op([y(1) = 0])],
        y(t),method=laplace)
```

No solution found

Mathematica DSolve solution

Solving time : 0.09 (sec)
Leaf size : 16

```
DSolve[{t*D[y[t],t]+y[t]==Sin[t],y[1]==0},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{\cos(1) - \cos(t)}{t}$$

2.3.9 problem 9

Internal problem ID [8742]
Book : First order enumerated odes
Section : section 3. First order odes solved using Laplace method
Problem number : 9
Date solved : `Tuesday, December 17, 2024 at 01:01:54 PM`
CAS classification : `[_linear]`

Solve

$$ty' + y = t$$

With initial conditions

$$y(1) = 0$$

Since initial condition is not at zero, then change of variable is used to transform the ode so that initial condition is at zero.

$$\tau = t - 1$$

Solve

$$(\tau + 1)\, y' + y = \tau + 1$$

With initial conditions

$$y(0) = 0$$

We will now apply Laplace transform to each term in the ode. Since this is time varying, the following Laplace transform property will be used

$$\tau^n f(\tau) \xrightarrow{\mathscr{L}} (-1)^n \frac{d^n}{ds^n} F(s)$$

Where in the above $F(s)$ is the laplace transform of $f(\tau)$. Applying the above property to each term of the ode gives

$$\begin{aligned} y(\tau) &\xrightarrow{\mathscr{L}} Y(s) \\ (\tau + 1)\left(\frac{d}{d\tau} y(\tau)\right) &\xrightarrow{\mathscr{L}} -Y(s) - s\left(\frac{d}{ds} Y(s)\right) + sY(s) - y(0) \\ \tau + 1 &\xrightarrow{\mathscr{L}} \frac{1+s}{s^2} \end{aligned}$$

Collecting all the terms above, the ode in Laplace domain becomes

$$-sY' + sY - y(0) = \frac{1+s}{s^2}$$

Replacing $y(0) = 0$ in the above results in

$$-sY' + sY = \frac{1+s}{s^2}$$

The above ode in Y(s) is now solved.

In canonical form a linear first order is

$$Y' + q(s)Y = p(s)$$

Comparing the above to the given ode shows that

$$q(s) = -1$$
$$p(s) = \frac{-s-1}{s^3}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,ds} \\ &= \mathrm{e}^{\int (-1)ds} \\ &= \mathrm{e}^{-s}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}(\mu Y) &= \mu p \\ \frac{\mathrm{d}}{\mathrm{d}s}(\mu Y) &= (\mu)\left(\frac{-s-1}{s^3}\right) \\ \frac{\mathrm{d}}{\mathrm{d}s}\left(Y\,\mathrm{e}^{-s}\right) &= \left(\mathrm{e}^{-s}\right)\left(\frac{-s-1}{s^3}\right) \\ \mathrm{d}\left(Y\,\mathrm{e}^{-s}\right) &= \left(\frac{(-s-1)\,\mathrm{e}^{-s}}{s^3}\right)\mathrm{d}s\end{aligned}$$

Integrating gives

$$\begin{aligned}Y\,\mathrm{e}^{-s} &= \int \frac{(-s-1)\,\mathrm{e}^{-s}}{s^3}\,ds \\ &= \frac{\mathrm{e}^{-s}}{2s^2} + \frac{\mathrm{e}^{-s}}{2s} - \frac{\mathrm{Ei}_1(s)}{2} + c_1\end{aligned}$$

Dividing throughout by the integrating factor e^{-s} gives the final solution

$$Y = \frac{2c_1\,\mathrm{e}^{s}s^2 - \mathrm{Ei}_1(s)\,\mathrm{e}^{s}s^2 + s + 1}{2s^2}$$

Applying inverse Laplace transform on the above gives.

$$y = c_1\mathcal{L}^{-1}(\mathrm{e}^{s}, s, \tau) - \frac{1}{2(\tau+1)} + \frac{1}{2} + \frac{\tau}{2} \tag{1}$$

Substituting initial conditions $y(0) = 0$ and $y'(0) = 0$ into the above solution Gives

$$0 = c_1\mathcal{L}^{-1}(\mathrm{e}^{s}, s, \tau)$$

Solving for the constant c_1 from the above equation gives

$$c_1 = 0$$

Substituting the above back into the solution (1) gives

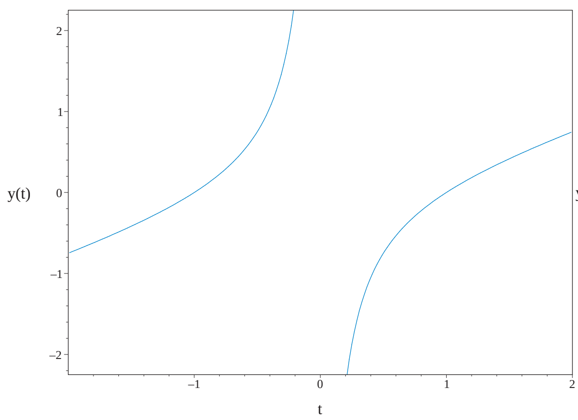
$$y = \frac{1}{2} - \frac{1}{2(\tau+1)} + \frac{\tau}{2}$$

Changing back the solution from τ to t using

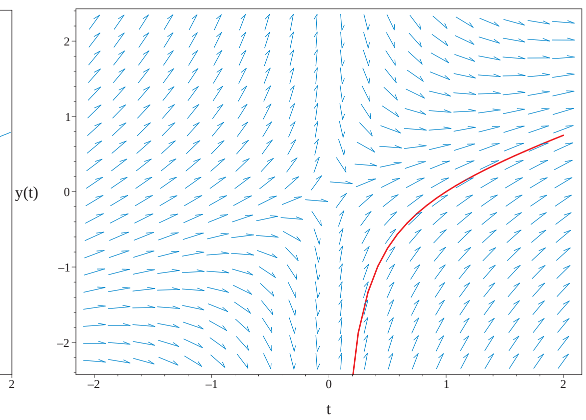
$$\tau = t - 1$$

the solution becomes

$$y(t) = -\frac{1}{2t} + \frac{t}{2}$$



(a) Solution plot
$y(t) = -\frac{1}{2t} + \frac{t}{2}$



(b) Slope field plot
$t\left(\frac{d}{dt}y(t)\right) + y(t) = t$

Maple step by step solution

Let's solve
$[ty' + y = t, y(1) = 0]$

- Highest derivative means the order of the ODE is 1
  y'
- Isolate the derivative
  $y' = 1 - \frac{y}{t}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE
  $y' + \frac{y}{t} = 1$
- The ODE is linear; multiply by an integrating factor $\mu(t)$
  $\mu(t)\left(y' + \frac{y}{t}\right) = \mu(t)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(y\mu(t))$
  $\mu(t)\left(y' + \frac{y}{t}\right) = y'\mu(t) + y\mu'(t)$
- Isolate $\mu'(t)$
  $\mu'(t) = \frac{\mu(t)}{t}$
- Solve to find the integrating factor
  $\mu(t) = t$
- Integrate both sides with respect to t
  $\int \left(\frac{d}{dt}(y\mu(t))\right) dt = \int \mu(t)\, dt + C1$
- Evaluate the integral on the lhs
  $y\mu(t) = \int \mu(t)\, dt + C1$
- Solve for y
  $y = \frac{\int \mu(t)dt + C1}{\mu(t)}$
- Substitute $\mu(t) = t$
  $y = \frac{\int t dt + C1}{t}$
- Evaluate the integrals on the rhs
  $y = \frac{\frac{t^2}{2} + C1}{t}$
- Simplify
  $y = \frac{t^2 + 2C1}{2t}$
- Use initial condition $y(1) = 0$

$0 = C1 + \frac{1}{2}$

- Solve for $\_C1$

  $C1 = -\frac{1}{2}$

- Substitute $\_C1 = -\frac{1}{2}$ into general solution and simplify

  $y = \frac{t^2-1}{2t}$

- Solution to the IVP

  $y = \frac{t^2-1}{2t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : `0.058` (sec)
Leaf size : 13

```
dsolve([t*diff(y(t),t)+y(t) = t,
        op([y(1) = 0])],
        y(t),method=laplace)
```

$$y = -\frac{1}{2t} + \frac{t}{2}$$

Mathematica DSolve solution

Solving time : `0.022` (sec)
Leaf size : 17

```
DSolve[{t*D[y[t],t]+y[t]==t,y[1]==0},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{t^2 - 1}{2t}$$

2.3.10 problem 10

Internal problem ID [8743]
Book : First order enumerated odes
Section : section 3. First order odes solved using Laplace method
Problem number : 10
Date solved : `Tuesday, December 17, 2024 at 01:01:55 PM`
CAS classification : `[_linear]`

Solve

$$ty' + y = t$$

With initial conditions

$$y(1) = 1$$

Since initial condition is not at zero, then change of variable is used to transform the ode so that initial condition is at zero.

$$\tau = t - 1$$

Solve

$$(\tau + 1)\, y' + y = \tau + 1$$

With initial conditions

$$y(0) = 1$$

We will now apply Laplace transform to each term in the ode. Since this is time varying, the following Laplace transform property will be used

$$\tau^n f(\tau) \xrightarrow{\mathscr{L}} (-1)^n \frac{d^n}{ds^n} F(s)$$

Where in the above $F(s)$ is the laplace transform of $f(\tau)$. Applying the above property to each term of the ode gives

$$\begin{aligned} y(\tau) &\xrightarrow{\mathscr{L}} Y(s) \\ (\tau + 1)\left(\frac{d}{d\tau} y(\tau)\right) &\xrightarrow{\mathscr{L}} -Y(s) - s\left(\frac{d}{ds} Y(s)\right) + sY(s) - y(0) \\ \tau + 1 &\xrightarrow{\mathscr{L}} \frac{1+s}{s^2} \end{aligned}$$

Collecting all the terms above, the ode in Laplace domain becomes

$$-sY' + sY - y(0) = \frac{1+s}{s^2}$$

Replacing $y(0) = 1$ in the above results in

$$-sY' + sY - 1 = \frac{1+s}{s^2}$$

The above ode in Y(s) is now solved.

In canonical form a linear first order is

$$Y' + q(s)Y = p(s)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(s) &= -1 \\ p(s) &= \frac{-s^2 - s - 1}{s^3} \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q \, ds} \\ &= e^{\int (-1) ds} \\ &= e^{-s} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{ds}(\mu Y) &= \mu p \\ \frac{d}{ds}(\mu Y) &= (\mu)\left(\frac{-s^2 - s - 1}{s^3}\right) \\ \frac{d}{ds}(Y e^{-s}) &= (e^{-s})\left(\frac{-s^2 - s - 1}{s^3}\right) \\ d(Y e^{-s}) &= \left(\frac{(-s^2 - s - 1) e^{-s}}{s^3}\right) ds \end{aligned}$$

Integrating gives

$$\begin{aligned} Y e^{-s} &= \int \frac{(-s^2 - s - 1) e^{-s}}{s^3} ds \\ &= \frac{e^{-s}}{2s^2} + \frac{e^{-s}}{2s} + \frac{\mathrm{Ei}_1(s)}{2} + c_1 \end{aligned}$$

Dividing throughout by the integrating factor e^{-s} gives the final solution

$$Y = \frac{2c_1 e^s s^2 + \mathrm{Ei}_1(s) e^s s^2 + s + 1}{2s^2}$$

Applying inverse Laplace transform on the above gives.

$$y = c_1 \mathcal{L}^{-1}(e^s, s, \tau) + \frac{1}{2\tau + 2} + \frac{1}{2} + \frac{\tau}{2} \tag{1}$$

Substituting initial conditions $y(0) = 1$ and $y'(0) = 1$ into the above solution Gives

$$1 = c_1 \mathcal{L}^{-1}(e^s, s, \tau) + 1$$

Solving for the constant c_1 from the above equation gives

$$c_1 = 0$$

Substituting the above back into the solution (1) gives

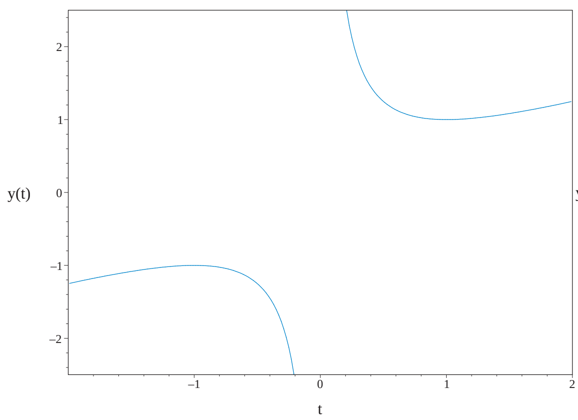
$$y = \frac{1}{2} + \frac{1}{2\tau + 2} + \frac{\tau}{2}$$

Changing back the solution from τ to t using

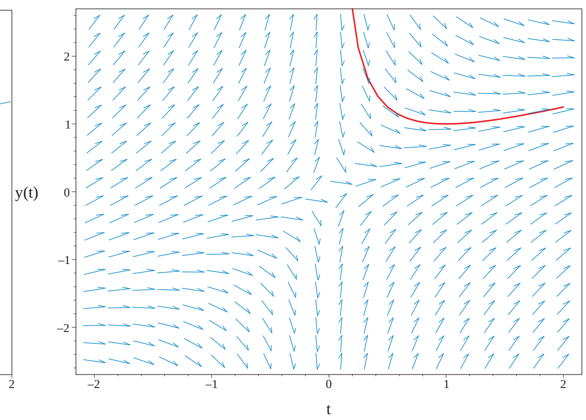
$$\tau = t - 1$$

the solution becomes

$$y(t) = \frac{1}{2t} + \frac{t}{2}$$



(a) Solution plot
$y(t) = \frac{1}{2t} + \frac{t}{2}$



(b) Slope field plot
$t\left(\frac{d}{dt}y(t)\right) + y(t) = t$

Maple step by step solution

Let's solve
$[ty' + y = t, y(1) = 1]$

- Highest derivative means the order of the ODE is 1
  y'
- Isolate the derivative
  $y' = 1 - \frac{y}{t}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE
  $y' + \frac{y}{t} = 1$
- The ODE is linear; multiply by an integrating factor $\mu(t)$
  $\mu(t)\left(y' + \frac{y}{t}\right) = \mu(t)$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(y\mu(t))$
  $\mu(t)\left(y' + \frac{y}{t}\right) = y'\mu(t) + y\mu'(t)$
- Isolate $\mu'(t)$
  $\mu'(t) = \frac{\mu(t)}{t}$
- Solve to find the integrating factor
  $\mu(t) = t$
- Integrate both sides with respect to t
  $\int \left(\frac{d}{dt}(y\mu(t))\right) dt = \int \mu(t)\, dt + C1$
- Evaluate the integral on the lhs
  $y\mu(t) = \int \mu(t)\, dt + C1$
- Solve for y
  $y = \frac{\int \mu(t) dt + C1}{\mu(t)}$
- Substitute $\mu(t) = t$
  $y = \frac{\int t dt + C1}{t}$
- Evaluate the integrals on the rhs
  $y = \frac{\frac{t^2}{2} + C1}{t}$
- Simplify
  $y = \frac{t^2 + 2C1}{2t}$
- Use initial condition $y(1) = 1$

$1 = C1 + \frac{1}{2}$

- Solve for $\_C1$

  $C1 = \frac{1}{2}$

- Substitute $\_C1 = \frac{1}{2}$ into general solution and simplify

  $y = \frac{t^2+1}{2t}$

- Solution to the IVP

  $y = \frac{t^2+1}{2t}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.053 (sec)
Leaf size : 13

```
dsolve([t*diff(y(t),t)+y(t) = t,
        op([y(1) = 1])],
        y(t),method=laplace)
```

$$y = \frac{1}{2t} + \frac{t}{2}$$

Mathematica DSolve solution

Solving time : 0.022 (sec)
Leaf size : 17

```
DSolve[{t*D[y[t],t]+y[t]==t,y[1]==1},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{t^2 + 1}{2t}$$

2.3.11 problem 11

Internal problem ID [8744]
Book : First order enumerated odes
Section : section 3. First order odes solved using Laplace method
Problem number : 11
Date solved : `Tuesday, December 17, 2024 at 01:01:56 PM`
CAS classification : `[_separable]`

Solve

$$y' + t^2 y = 0$$

With initial conditions

$$y(0) = 0$$

We will now apply Laplace transform to each term in the ode. Since this is time varying, the following Laplace transform property will be used

$$t^n f(t) \xrightarrow{\mathscr{L}} (-1)^n \frac{d^n}{ds^n} F(s)$$

Where in the above $F(s)$ is the laplace transform of $f(t)$. Applying the above property to each term of the ode gives

$$t^2 y \xrightarrow{\mathscr{L}} \frac{d^2}{ds^2} Y(s)$$
$$y' \xrightarrow{\mathscr{L}} sY(s) - y(0)$$

Collecting all the terms above, the ode in Laplace domain becomes

$$sY - y(0) + Y'' = 0$$

Replacing $y(0) = 0$ in the above results in

$$sY + Y'' = 0$$

The above ode in Y(s) is now solved.

This is Airy ODE. It has the general form

$$aY'' + bY' + csY = F(s)$$

Where in this case

$$a = 1$$
$$b = 0$$
$$c = 1$$
$$F = 0$$

Therefore the solution to the homogeneous Airy ODE becomes

$$Y = c_1 \operatorname{AiryAi}(-s) + c_2 \operatorname{AiryBi}(-s)$$

Will add steps showing solving for IC soon.

Applying inverse Laplace transform on the above gives.

$$y = c_1\mathcal{L}^{-1}(\text{AiryAi}(-s), s, t) + c_2\mathcal{L}^{-1}(\text{AiryBi}(-s), s, t) \tag{1}$$

Substituting initial conditions $y(0) = 0$ and $y'(0) = 0$ into the above solution Gives

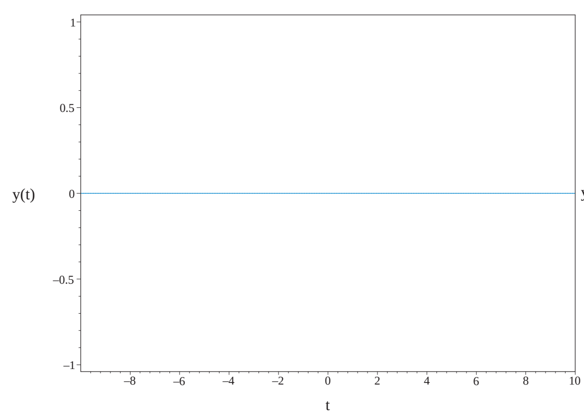
$$0 = c_1\mathcal{L}^{-1}(\text{AiryAi}(-s), s, t) + c_2\mathcal{L}^{-1}(\text{AiryBi}(-s), s, t)$$

Solving for the constant c_1 from the above equation gives

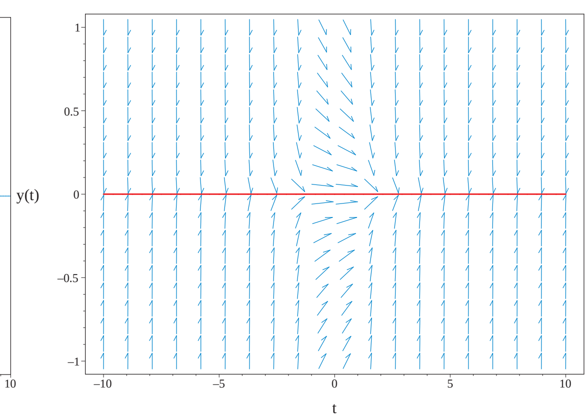
$$c_1 = -\frac{c_2\mathcal{L}^{-1}(\text{AiryBi}(-s), s, t)}{\mathcal{L}^{-1}(\text{AiryAi}(-s), s, t)}$$

Substituting the above back into the solution (1) gives

$$y = 0$$



(a) Solution plot
$y = 0$



(b) Slope field plot
$y' + t^2y = 0$

Maple step by step solution

Let's solve
$[y' + yt^2 = 0, y(0) = 0]$

- Highest derivative means the order of the ODE is 1
  y'
- Solve for the highest derivative
  $y' = -yt^2$
- Separate variables
  $\frac{y'}{y} = -t^2$
- Integrate both sides with respect to t
  $\int \frac{y'}{y}dt = \int -t^2dt + C1$
- Evaluate integral
  $\ln(y) = -\frac{t^3}{3} + C1$
- Solve for y
  $y = e^{-\frac{t^3}{3}+C1}$
- Use initial condition $y(0) = 0$
  $0 = e^{C1}$
- Solve for $\_C1$
  $C1 = ()$
- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : `0.071` (sec)
Leaf size : 40

```
dsolve([diff(y(t),t)+y(t)*t^2 = 0,
        op([y(0) = 0])],
        y(t),method=laplace)
```

$$y = -\frac{c_2\mathcal{L}^{-1}(\mathrm{AiryBi}(-\_s1), \_s1, 0)\, \mathcal{L}^{-1}(\mathrm{AiryAi}(-\_s1), \_s1, t)}{\mathcal{L}^{-1}(\mathrm{AiryAi}(-\_s1), \_s1, 0)} + c_2\mathcal{L}^{-1}(\mathrm{AiryBi}(-\_s1), \_s1, t)$$

Mathematica DSolve solution

Solving time : `0.001` (sec)
Leaf size : 6

```
DSolve[{D[y[t],t]+t^2*y[t]==0,y[0]==0},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to 0$$

2.3.12 problem 12

Internal problem ID [8745]
Book : First order enumerated odes
Section : section 3. First order odes solved using Laplace method
Problem number : 12
Date solved : `Tuesday, December 17, 2024 at 01:01:57 PM`
CAS classification : `[_linear]`

Solve

$$(at+1)\,y'+y=t$$

With initial conditions

$$y(1)=0$$

Since initial condition is not at zero, then change of variable is used to transform the ode so that initial condition is at zero.

$$\tau=t-1$$

Solve

$$(a(\tau+1)+1)\,y'+y=\tau+1$$

With initial conditions

$$y(0)=0$$

We will now apply Laplace transform to each term in the ode. Since this is time varying, the following Laplace transform property will be used

$$\tau^n f(\tau)\xrightarrow{\mathscr{L}}(-1)^n\frac{d^n}{ds^n}F(s)$$

Where in the above $F(s)$ is the laplace transform of $f(\tau)$. Applying the above property to each term of the ode gives

$$\begin{aligned} y(\tau)&\xrightarrow{\mathscr{L}}Y(s)\\ (a\tau+a+1)\left(\frac{d}{d\tau}y(\tau)\right)&\xrightarrow{\mathscr{L}}-a\left(Y(s)+s\left(\frac{d}{ds}Y(s)\right)\right)+a(sY(s)-y(0))+sY(s)-y(0)\\ \tau+1&\xrightarrow{\mathscr{L}}\frac{1+s}{s^2}\end{aligned}$$

Collecting all the terms above, the ode in Laplace domain becomes

$$-a(Y+sY')+a(sY-y(0))+sY-y(0)+Y=\frac{1+s}{s^2}$$

Replacing $y(0)=0$ in the above results in

$$-a(Y+sY')+asY+sY+Y=\frac{1+s}{s^2}$$

The above ode in Y(s) is now solved.

In canonical form a linear first order is

$$Y' + q(s)Y = p(s)$$

Comparing the above to the given ode shows that

$$q(s) = -\frac{(s-1)\,a+1+s}{as}$$
$$p(s) = \frac{-s-1}{s^3 a}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\,ds}\\ &= \mathrm{e}^{\int -\frac{(s-1)a+1+s}{as}ds}\\ &= s^{\frac{a-1}{a}}\,\mathrm{e}^{-\frac{s(a+1)}{a}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}(\mu Y) &= \mu p\\ \frac{\mathrm{d}}{\mathrm{d}s}(\mu Y) &= (\mu)\left(\frac{-s-1}{s^3 a}\right)\\ \frac{\mathrm{d}}{\mathrm{d}s}\left(Y\, s^{\frac{a-1}{a}}\,\mathrm{e}^{-\frac{s(a+1)}{a}}\right) &= \left(s^{\frac{a-1}{a}}\,\mathrm{e}^{-\frac{s(a+1)}{a}}\right)\left(\frac{-s-1}{s^3 a}\right)\\ \mathrm{d}\left(Y\, s^{\frac{a-1}{a}}\,\mathrm{e}^{-\frac{s(a+1)}{a}}\right) &= \left(\frac{(-s-1)\, s^{\frac{a-1}{a}}\,\mathrm{e}^{-\frac{s(a+1)}{a}}}{s^3 a}\right)\mathrm{d}s\end{aligned}$$

Integrating gives

$$\begin{aligned}Y\, s^{\frac{a-1}{a}}\,\mathrm{e}^{-\frac{s(a+1)}{a}} &= \int \frac{(-s-1)\, s^{\frac{a-1}{a}}\,\mathrm{e}^{-\frac{s(a+1)}{a}}}{s^3 a}\,ds\\ &= \frac{s^{-2+\frac{a-1}{a}}\,\mathrm{e}^{-\frac{s(a+1)}{a}}}{a+1} + c_1\end{aligned}$$

Dividing throughout by the integrating factor $s^{\frac{a-1}{a}}\,\mathrm{e}^{-\frac{s(a+1)}{a}}$ gives the final solution

$$Y = \frac{1 + c_1\, s^{\frac{a+1}{a}}\,(a+1)\,\mathrm{e}^{\frac{s(a+1)}{a}}}{s^2\,(a+1)}$$

Applying inverse Laplace transform on the above gives.

$$y = \frac{\tau}{a+1} + c_1\mathcal{L}^{-1}\left(\mathrm{e}^{s+\frac{s}{a}}s^{-1+\frac{1}{a}}, s, \tau\right) \tag{1}$$

Substituting initial conditions $y(0) = 0$ and $y'(0) = 0$ into the above solution Gives

$$0 = c_1\mathcal{L}^{-1}\left(\mathrm{e}^{s+\frac{s}{a}}s^{-1+\frac{1}{a}}, s, \tau\right)$$

Solving for the constant c_1 from the above equation gives

$$c_1 = 0$$

Substituting the above back into the solution (1) gives

$$y = \frac{\tau}{a+1}$$

Changing back the solution from τ to t using

$$\tau = t - 1$$

the solution becomes

$$y(t) = \frac{t-1}{a+1}$$

Maple step by step solution

Let's solve
$[(at+1)\,y'+y=t, y(1)=0]$

- Highest derivative means the order of the ODE is 1
y'
- Solve for the highest derivative
$y'=\frac{-y+t}{at+1}$
- Collect w.r.t. y and simplify
$y'=-\frac{y}{at+1}+\frac{t}{at+1}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE
$y'+\frac{y}{at+1}=\frac{t}{at+1}$
- The ODE is linear; multiply by an integrating factor $\mu(t)$
$\mu(t)\left(y'+\frac{y}{at+1}\right)=\frac{\mu(t)t}{at+1}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dt}(y\mu(t))$
$\mu(t)\left(y'+\frac{y}{at+1}\right)=y'\mu(t)+y\mu'(t)$
- Isolate $\mu'(t)$
$\mu'(t)=\frac{\mu(t)}{at+1}$
- Solve to find the integrating factor
$\mu(t)=(at+1)^{\frac{1}{a}}$
- Integrate both sides with respect to t
$\int\left(\frac{d}{dt}(y\mu(t))\right)dt=\int\frac{\mu(t)t}{at+1}dt+C1$
- Evaluate the integral on the lhs
$y\mu(t)=\int\frac{\mu(t)t}{at+1}dt+C1$
- Solve for y
$y=\frac{\int\frac{\mu(t)t}{at+1}dt+C1}{\mu(t)}$
- Substitute $\mu(t)=(at+1)^{\frac{1}{a}}$
$y=\frac{\int\frac{t(at+1)^{\frac{1}{a}}}{at+1}dt+C1}{(at+1)^{\frac{1}{a}}}$
- Evaluate the integrals on the rhs
$y=\frac{\frac{(t-1)(at+1)^{\frac{1}{a}}}{a+1}+C1}{(at+1)^{\frac{1}{a}}}$
- Simplify
$y=\frac{t-1+(at+1)^{-\frac{1}{a}}C1(a+1)}{a+1}$
- Use initial condition $y(1)=0$
$0=(a+1)^{-\frac{1}{a}}C1$
- Solve for $\_C1$
$C1=0$
- Substitute $\_C1=0$ into general solution and simplify
$y=\frac{t-1}{a+1}$
- Solution to the IVP
$y=\frac{t-1}{a+1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.079 (sec)
Leaf size : 13

```
dsolve([(a*t+1)*diff(y(t),t)+y(t) = t,
        op([y(1) = 0])],
        y(t),method=laplace)
```

$$y = \frac{t-1}{a+1}$$

Mathematica DSolve solution

Solving time : 0.897 (sec)
Leaf size : 14

```
DSolve[{(1+a*t)*D[y[t],t]+y[t]==t,y[1]==0},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to \frac{t-1}{a+1}$$

2.3.13 problem 13

Internal problem ID [8746]
Book : First order enumerated odes
Section : section 3. First order odes solved using Laplace method
Problem number : 13
Date solved : `Tuesday, December 17, 2024 at 01:01:57 PM`
CAS classification : `[_separable]`

Solve

$$y' + (at + bt)\, y = 0$$

With initial conditions

$$y(0) = 0$$

We will now apply Laplace transform to each term in the ode. Since this is time varying, the following Laplace transform property will be used

$$t^n f(t) \xrightarrow{\mathscr{L}} (-1)^n \frac{d^n}{ds^n} F(s)$$

Where in the above $F(s)$ is the laplace transform of $f(t)$. Applying the above property to each term of the ode gives

$$\begin{aligned}(at + bt)\, y &\xrightarrow{\mathscr{L}} -a\left(\frac{d}{ds}Y(s)\right) - b\left(\frac{d}{ds}Y(s)\right) \\ y' &\xrightarrow{\mathscr{L}} Y(s)\, s - y(0)\end{aligned}$$

Collecting all the terms above, the ode in Laplace domain becomes

$$Ys - y(0) - aY' - bY' = 0$$

Replacing $y(0) = 0$ in the above results in

$$Ys - aY' - bY' = 0$$

The above ode in Y(s) is now solved.

In canonical form a linear first order is

$$Y' + q(s)Y = p(s)$$

Comparing the above to the given ode shows that

$$\begin{aligned}q(s) &= -\frac{s}{a+b} \\ p(s) &= 0\end{aligned}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int q\, ds} \\ &= \mathrm{e}^{\int -\frac{s}{a+b} ds} \\ &= \mathrm{e}^{-\frac{s^2}{2a+2b}}\end{aligned}$$

The ode becomes

$$\frac{d}{ds}\mu Y = 0$$
$$\frac{d}{ds}\left(Y\,e^{-\frac{s^2}{2a+2b}}\right) = 0$$

Integrating gives

$$Y\,e^{-\frac{s^2}{2a+2b}} = \int 0\,ds + c_1$$
$$= c_1$$

Dividing throughout by the integrating factor $e^{-\frac{s^2}{2a+2b}}$ gives the final solution

$$Y = c_1\,e^{\frac{s^2}{2a+2b}}$$

Applying inverse Laplace transform on the above gives.

$$y = c_1 \mathcal{L}^{-1}\left(e^{\frac{s^2}{2a+2b}}, s, t\right) \tag{1}$$

Substituting initial conditions $y(0) = 0$ and $y'(0) = 0$ into the above solution Gives

$$0 = c_1 \mathcal{L}^{-1}\left(e^{\frac{s^2}{2a+2b}}, s, t\right)$$

Solving for the constant c_1 from the above equation gives

$$c_1 = 0$$

Substituting the above back into the solution (1) gives

$$y = 0$$

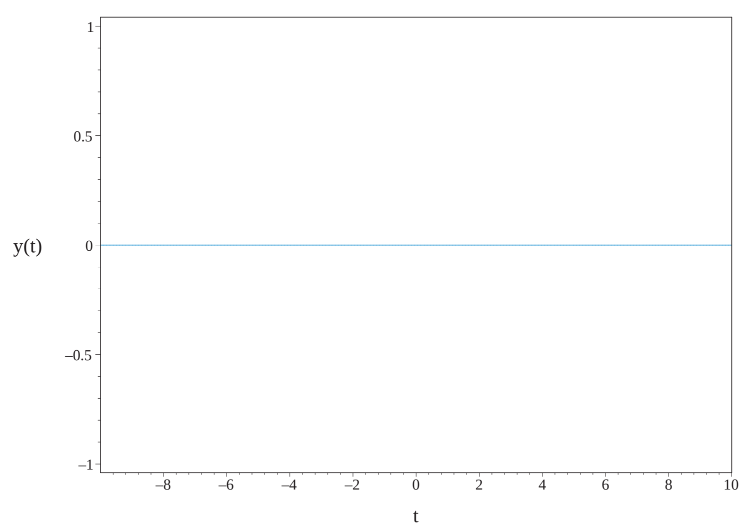


Figure 2.110: Solution plot
$y = 0$

Maple step by step solution

Let's solve
$[y' + (at + bt)\,y = 0, y(0) = 0]$

- Highest derivative means the order of the ODE is 1
y'
- Solve for the highest derivative

$y' = -(at + bt)\, y$

- Separate variables

  $\frac{y'}{y} = -at - bt$

- Integrate both sides with respect to t

  $\int \frac{y'}{y} dt = \int (-at - bt)\, dt + C1$

- Evaluate integral

  $\ln(y) = -\frac{t^2(a+b)}{2} + C1$

- Solve for y

  $y = \mathrm{e}^{-\frac{1}{2}t^2 a - \frac{1}{2}t^2 b + C1}$

- Use initial condition $y(0) = 0$

  $0 = \mathrm{e}^{C1}$

- Solve for $\_C1$

  $C1 = ()$

- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.032 (sec)
Leaf size : 5

```
dsolve([diff(y(t),t)+(a*t+b*t)*y(t) = 0,
        op([y(0) = 0])],
        y(t),method=laplace)
```

$$y = 0$$

Mathematica DSolve solution

Solving time : 0.001 (sec)
Leaf size : 6

```
DSolve[{D[y[t],t]+(a*t+b*t)*y[t]==0,y[0]==0},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to 0$$

2.3.14 problem 14

Internal problem ID [8747]
Book : First order enumerated odes
Section : section 3. First order odes solved using Laplace method
Problem number : 14
Date solved : `Tuesday, December 17, 2024 at 01:01:58 PM`
CAS classification : `[_separable]`

Solve

$$y' + (at + bt)\, y = 0$$

With initial conditions

$$y(-3) = 0$$

Since initial condition is not at zero, then change of variable is used to transform the ode so that initial condition is at zero.

$$\tau = t + 3$$

Solve

$$y' + (a(\tau - 3) + b(\tau - 3))\, y = 0$$

With initial conditions

$$y(0) = 0$$

We will now apply Laplace transform to each term in the ode. Since this is time varying, the following Laplace transform property will be used

$$\tau^n f(\tau) \xrightarrow{\mathscr{L}} (-1)^n \frac{d^n}{ds^n} F(s)$$

Where in the above $F(s)$ is the laplace transform of $f(\tau)$. Applying the above property to each term of the ode gives

$$(a\tau + b\tau - 3a - 3b)\, y(\tau) \xrightarrow{\mathscr{L}} -a\left(\frac{d}{ds} Y(s)\right) - b\left(\frac{d}{ds} Y(s)\right) - 3aY(s) - 3bY(s)$$
$$\frac{d}{d\tau} y(\tau) \xrightarrow{\mathscr{L}} Y(s)\, s - y(0)$$

Collecting all the terms above, the ode in Laplace domain becomes

$$Ys - y(0) - aY' - bY' - 3aY - 3bY = 0$$

Replacing $y(0) = 0$ in the above results in

$$Ys - aY' - bY' - 3aY - 3bY = 0$$

The above ode in Y(s) is now solved.

In canonical form a linear first order is

$$Y' + q(s)Y = p(s)$$

Comparing the above to the given ode shows that

$$\begin{aligned} q(s) &= -\frac{-3a - 3b + s}{a + b} \\ p(s) &= 0 \end{aligned}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int q\, ds} \\ &= \mathrm{e}^{\int -\frac{-3a-3b+s}{a+b} ds} \\ &= \mathrm{e}^{\frac{s(6a+6b-s)}{2a+2b}} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}\mu Y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}s}\left(Y\,\mathrm{e}^{\frac{s(6a+6b-s)}{2a+2b}}\right) &= 0 \end{aligned}$$

Integrating gives

$$\begin{aligned} Y\,\mathrm{e}^{\frac{s(6a+6b-s)}{2a+2b}} &= \int 0\, ds + c_1 \\ &= c_1 \end{aligned}$$

Dividing throughout by the integrating factor $\mathrm{e}^{\frac{s(6a+6b-s)}{2a+2b}}$ gives the final solution

$$Y = c_1\,\mathrm{e}^{-\frac{s(6a+6b-s)}{2a+2b}}$$

Applying inverse Laplace transform on the above gives.

$$y = c_1 \mathcal{L}^{-1}\left(\mathrm{e}^{-\frac{s(6a+6b-s)}{2a+2b}}, s, \tau\right) \tag{1}$$

Substituting initial conditions $y(0) = 0$ and $y'(0) = 0$ into the above solution Gives

$$0 = c_1 \mathcal{L}^{-1}\left(\mathrm{e}^{-\frac{s(6a+6b-s)}{2a+2b}}, s, \tau\right)$$

Solving for the constant c_1 from the above equation gives

$$c_1 = 0$$

Substituting the above back into the solution (1) gives

$$y = 0$$

Changing back the solution from τ to t using

$$\tau = t + 3$$

the solution becomes

$$y(t) = 0$$

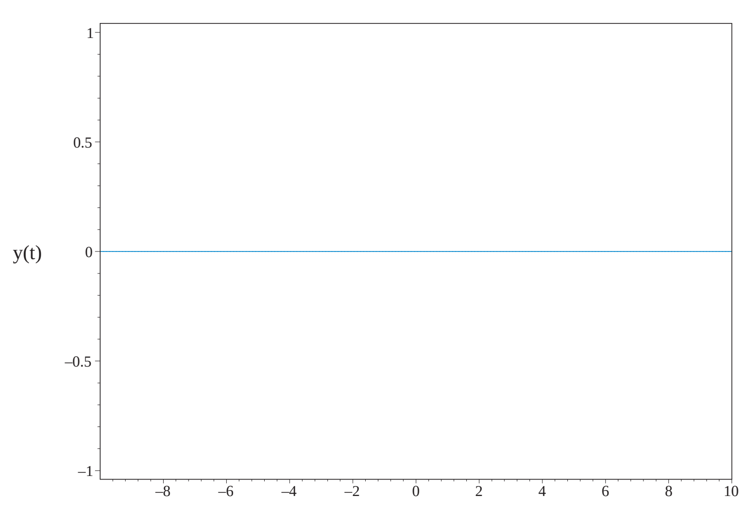


Figure 2.111: Solution plot
$y(t) = 0$

Maple step by step solution

Let's solve
$[y' + (at + bt)\, y = 0, y(-3) = 0]$

- Highest derivative means the order of the ODE is 1
  y'
- Solve for the highest derivative
  $y' = -(at + bt)\, y$
- Separate variables
  $\frac{y'}{y} = -at - bt$
- Integrate both sides with respect to t
  $\int \frac{y'}{y} dt = \int (-at - bt)\, dt + C1$
- Evaluate integral
  $\ln(y) = -\frac{t^2(a+b)}{2} + C1$
- Solve for y
  $y = \mathrm{e}^{-\frac{1}{2}t^2a - \frac{1}{2}t^2b + C1}$
- Use initial condition $y(-3) = 0$
  $0 = \mathrm{e}^{-\frac{9a}{2} - \frac{9b}{2} + C1}$
- Solve for _$C1$
  $C1 = ()$
- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

Maple dsolve solution

Solving time : 0.032 (sec)
Leaf size : 5

```
dsolve([diff(y(t),t)+(a*t+b*t)*y(t) = 0,
        op([y(-3) = 0])],
        y(t),method=laplace)
```

$$y = 0$$

Mathematica DSolve solution

Solving time : 0.001 (sec)
Leaf size : 6

```
DSolve[{D[y[t],t]+(a*t+b*t)*y[t]==0,y[-3]==0},
       y[t],t,IncludeSingularSolutions->True]
```

$$y(t) \to 0$$