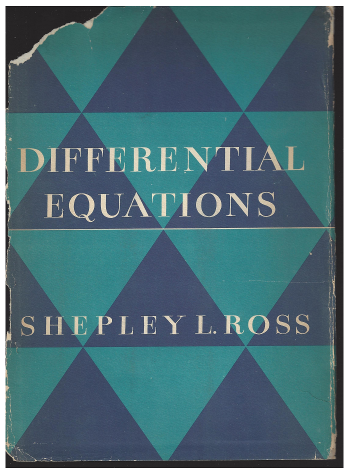
A Solution Manual For

Differential equations, Shepley L. Ross, 1964



Nasser M. Abbasi December 31, 2024 Compiled on December 31, 2024 at 1:57am

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CHAPTER 1

Lookup tables for all problems in current book

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1.1 2.4, page 55

Table 1.1: Lookup table for all problems in current section

ID	problem	ODE
4077	1	$5yx + 4y^2 + 1 + (x^2 + 2yx)\,y' = 0$
4078	2	$2x\tan(y) + (x - x^2\tan(y))\,y' = 0$
4079	3	$y^2(x^2 + 1) + y + (2yx + 1)\,y' = 0$
4080	4	$4xy^2 + 6y + (5x^2y + 8x)\,y' = 0$
4081	5	$5x + 2y + 1 + (2x + y + 1)\,y' = 0$
4082	6	$3x - y + 1 - (6x - 2y - 3)\,y' = 0$
4083	7	$x - 2y - 3 + (2x + y - 1)\,y' = 0$
4084	8	$6x + 4y + 1 + (4x + 2y + 2)\,y' = 0$
4085	9	$3x - y - 6 + (x + y + 2)\,y' = 0$
4086	10	$2x + 3y + 1 + (4x + 6y + 1)\,y' = 0$

CHAPTER 2

Book Solved Problems

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2.1 2.4, page 55

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2.1.1 problem 1

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Internal problem ID [4077]
Book : Differential equations, Shepley L. Ross, 1964
Section : 2.4, page 55
Problem number : 1
Date solved : `Tuesday, December 17, 2024 at 06:21:25 AM`
CAS classification : `[_rational, [_Abel, '2nd type', 'class B']]`

Solve

$$5xy + 4y^2 + 1 + \left(x^2 + 2xy\right) y' = 0$$

Solved as first order Exact ode

Time used: 0.392 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x^2+2xy\right)\mathrm{d}y &= \left(-5xy-4y^2-1\right)\mathrm{d}x\\ \left(5xy+4y^2+1\right)\mathrm{d}x + \left(x^2+2xy\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 5xy+4y^2+1\\ N(x,y) &= x^2+2xy\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(5xy+4y^2+1\right)\\ &= 5x+8y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2+2xy\right)\\ &= 2x+2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x\left(x+2y\right)}\left(\left(5x+8y\right)-\left(2x+2y\right)\right)\\ &= \frac{3}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \frac{3}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{3\ln(x)} \\ &= x^3\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= x^3\left(5xy + 4y^2 + 1\right) \\ &= \left(5xy + 4y^2 + 1\right)x^3\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= x^3\left(x^2 + 2xy\right) \\ &= x^4(x + 2y)\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\left(5xy + 4y^2 + 1\right)x^3\right) + \left(x^4(x + 2y)\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \left(5xy + 4y^2 + 1\right)x^3 \, \mathrm{d}x\end{aligned}$$

$$\phi = x^5y + x^4y^2 + \frac{1}{4}x^4 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned} \frac{\partial \phi}{\partial y} &= x^5 + 2y\, x^4 + f'(y) \qquad (4) \\ &= x^4(x+2y) + f'(y) \end{aligned}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^4(x+2y)$. Therefore equation (4) becomes

$$x^4(x+2y) = x^4(x+2y) + f'(y) \qquad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x^5 y + x^4 y^2 + \frac{1}{4} x^4 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = x^5 y + x^4 y^2 + \frac{1}{4} x^4$$

Solving for y gives

$$y = \frac{-x^3 - \sqrt{x^6 - x^4 + 4c_1}}{2x^2}$$

$$y = \frac{-x^3 + \sqrt{x^6 - x^4 + 4c_1}}{2x^2}$$

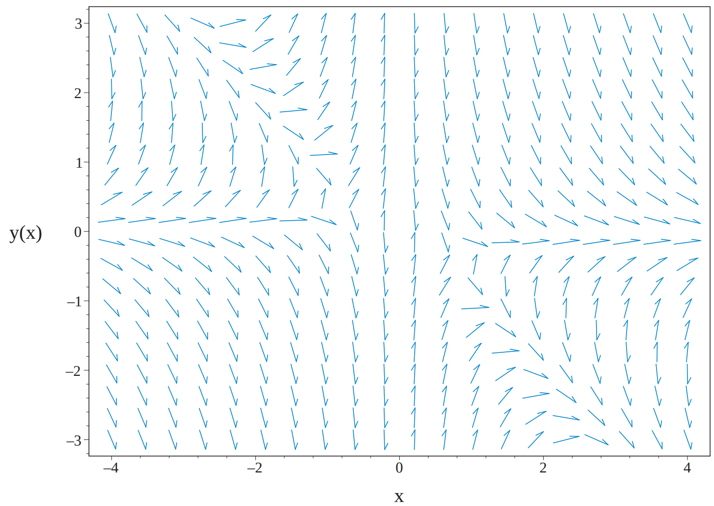

Figure 2.1: Slope field plot
$5xy + 4y^2 + 1 + (x^2 + 2xy) y' = 0$

Summary of solutions found

$$y = \frac{-x^3 - \sqrt{x^6 - x^4 + 4c_1}}{2x^2}$$

$$y = \frac{-x^3 + \sqrt{x^6 - x^4 + 4c_1}}{2x^2}$$

Maple step by step solution

Let's solve

$5xy(x) + 4y(x)^2 + 1 + (x^2 + 2xy(x)) \left(\frac{d}{dx}y(x)\right) = 0$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = \frac{-5xy(x)-4y(x)^2-1}{x^2+2xy(x)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

Maple dsolve solution

Solving time : 0.004 (sec)
Leaf size : 59

```
dsolve(5*x*y(x)+4*y(x)^2+1+(x^2+2*x*y(x))*diff(y(x),x) = 0,
       y(x),singsol=all)
```

$$y(x) = \frac{-x^3 - \sqrt{x^6 - x^4 - 4c_1}}{2x^2}$$
$$y(x) = \frac{-x^3 + \sqrt{x^6 - x^4 - 4c_1}}{2x^2}$$

Mathematica DSolve solution

Solving time : 1.02 (sec)
Leaf size : 84

```
DSolve[{(5*x*y[x]+4*y[x]^2+1)+(x^2+2*x*y[x])*D[y[x],x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{x^5 + \sqrt{x^3}\sqrt{x^7 - x^5 + 4c_1 x}}{2x^4}$$
$$y(x) \to -\frac{x}{2} + \frac{\sqrt{x^3}\sqrt{x^7 - x^5 + 4c_1 x}}{2x^4}$$

2.1.2 problem 2

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Maple trace 20
Maple dsolve solution 20
Mathematica DSolve solution 21

Internal problem ID [4078]
Book : Differential equations, Shepley L. Ross, 1964
Section : 2.4, page 55
Problem number : 2
Date solved : `Tuesday, December 17, 2024 at 06:21:27 AM`
CAS classification : `[[_1st_order, _with_exponential_symmetries]]`

Solve

$$2x\tan(y) + \left(x - x^2\tan(y)\right)y' = 0$$

Solved as first order Exact ode

Time used: 0.494 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-1 + x\tan(y))\,\mathrm{d}y &= (2\tan(y))\,\mathrm{d}x\\(-2\tan(y))\,\mathrm{d}x + (-1 + x\tan(y))\,\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -2\tan(y)\\N(x,y) &= -1 + x\tan(y)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-2\tan(y))\\&= -2\sec(y)^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-1 + x\tan(y))\\&= \tan(y)\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{-1 + x\tan(y)}\left(\left(-2 - 2\tan(y)^2\right) - (\tan(y))\right) \\ &= \frac{-\sin(y) - 2\sec(y)}{x\sin(y) - \cos(y)} \end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= -\frac{\cot(y)}{2}\left((\tan(y)) - \left(-2 - 2\tan(y)^2\right)\right) \\ &= -\cot(y) - \tan(y) - \frac{1}{2} \end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned} \mu &= e^{\int B \, \mathrm{d}y} \\ &= e^{\int -\cot(y) - \tan(y) - \frac{1}{2} \, \mathrm{d}y} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\frac{y}{2} - \ln(\sin(y)) + \ln(\cos(y))} \\ &= \frac{\cos(y)\,\mathrm{e}^{-\frac{y}{2}}}{\sin(y)} \end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{\cos(y)\,\mathrm{e}^{-\frac{y}{2}}}{\sin(y)}(-2\tan(y)) \\ &= -2\,\mathrm{e}^{-\frac{y}{2}} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{\cos(y)\,\mathrm{e}^{-\frac{y}{2}}}{\sin(y)}(-1 + x\tan(y)) \\ &= \mathrm{e}^{-\frac{y}{2}}(x - \cot(y)) \end{aligned}$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\left(-2\,\mathrm{e}^{-\frac{y}{2}}\right) + \left(\mathrm{e}^{-\frac{y}{2}}(x - \cot(y))\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int -2\,\mathrm{e}^{-\frac{y}{2}}\,\mathrm{d}x$$

$$\phi = -2\,\mathrm{e}^{-\frac{y}{2}}x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = \mathrm{e}^{-\frac{y}{2}}x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \mathrm{e}^{-\frac{y}{2}}(x - \cot(y))$. Therefore equation (4) becomes

$$\mathrm{e}^{-\frac{y}{2}}(x - \cot(y)) = \mathrm{e}^{-\frac{y}{2}}x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\mathrm{e}^{-\frac{y}{2}}\cot(y)$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\mathrm{e}^{-\frac{y}{2}}\cot(y)\right)\mathrm{d}y$$

$$f(y) = \int^{y} -\mathrm{e}^{-\frac{\tau}{2}}\cot(\tau)\,d\tau + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -2\,\mathrm{e}^{-\frac{y}{2}}x + \int^{y} -\mathrm{e}^{-\frac{\tau}{2}}\cot(\tau)\,d\tau + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = -2\,\mathrm{e}^{-\frac{y}{2}}x + \int^{y} -\mathrm{e}^{-\frac{\tau}{2}}\cot(\tau)\,d\tau$$

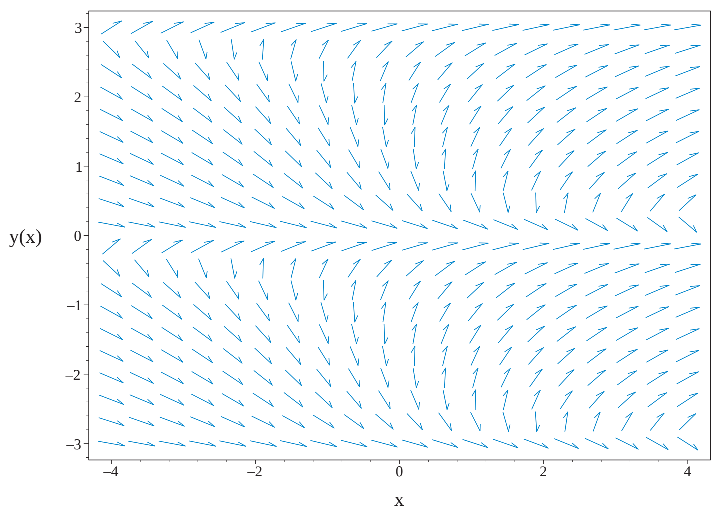

Figure 2.2: Slope field plot
$2x\tan(y) + (x - x^2\tan(y))\,y' = 0$

Summary of solutions found

$$-2\,\mathrm{e}^{-\frac{y}{2}}x + \int^{y} -\mathrm{e}^{-\frac{\tau}{2}}\cot(\tau)\,d\tau = c_1$$

Maple step by step solution

Let's solve

$2x \tan(y(x)) + (x - x^2 \tan(y(x))) \left(\frac{d}{dx}y(x)\right) = 0$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = -\frac{2x \tan(y(x))}{x - x^2 \tan(y(x))}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

Maple dsolve solution

Solving time : 0.071 (sec)
Leaf size : 32

```
dsolve(2*x*tan(y(x))+(x-x^2*tan(y(x)))*diff(y(x),x) = 0,
       y(x),singsol=all)
```

$$\frac{e^{\frac{y(x)}{2}} \left(\int^{y(x)} \cot(\_a)\, e^{-\frac{\_a}{2}} d\_a\right)}{2} - e^{\frac{y(x)}{2}} c_1 + x = 0$$

Mathematica DSolve solution

Solving time : `0.657` (sec)
Leaf size : 78

```
DSolve[{(2*x*Tan[y[x]])+(x-x^2*Tan[y[x]])*D[y[x],x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$\text{Solve}\left[x=\frac{1}{34}\left((8-2i)e^{2iy(x)}\,\text{Hypergeometric2F1}\left(1,1+\frac{i}{4},2+\frac{i}{4},e^{2iy(x)}\right)\right.\right.$$
$$\left.\left.-34i\,\text{Hypergeometric2F1}\left(\frac{i}{4},1,1+\frac{i}{4},e^{2iy(x)}\right)\right)+c_1e^{\frac{y(x)}{2}},y(x)\right]$$

2.1.3 problem 3

Maple step by step solution 22
Maple trace 22
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Mathematica DSolve solution 24

Internal problem ID [4079]
Book : Differential equations, Shepley L. Ross, 1964
Section : 2.4, page 55
Problem number : 3
Date solved : `Tuesday, December 17, 2024 at 06:21:29 AM`
CAS classification : `[_rational, [_Abel, '2nd type', 'class B']]`

Solve

$$y^2(x^2+1)+y+(2xy+1)\,y'=0$$

Unknown ode type.

Maple step by step solution

Let's solve
$y(x)^2\left(x^2+1\right)+y(x)+(1+2xy(x))\left(\frac{d}{dx}y(x)\right)=0$

- Highest derivative means the order of the ODE is 1
$\frac{d}{dx}y(x)$
- Solve for the highest derivative
$\frac{d}{dx}y(x)=-\frac{y(x)^2(x^2+1)+y(x)}{1+2xy(x)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
```

```
trying Chini
differential order: 1; looking for linear symmetries
trying exact
trying Abel
Looking for potential symmetries
Looking for potential symmetries
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 2
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

Maple dsolve solution

Solving time : 0.009 (sec)
Leaf size : maple_leaf_size

```
dsolve(y(x)^2*(x^2+1)+y(x)+(2*x*y(x)+1)*diff(y(x),x) = 0,
       y(x),singsol=all)
```

No solution found

Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{(y[x]^2*(x^2+1)+y[x])+(2*x*y[x]+1)*D[y[x],x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

Not solved

2.1.4 problem 4

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Internal problem ID [4080]
Book : Differential equations, Shepley L. Ross, 1964
Section : 2.4, page 55
Problem number : 4
Date solved : `Tuesday, December 17, 2024 at 06:21:31 AM`
CAS classification :
`[[_homogeneous, ‘class G‘], _rational, [_Abel, ‘2nd type‘, ‘class B‘]]`

Solve

$$4xy^2 + 6y + \left(5x^2y + 8x\right) y' = 0$$

Solved as first order Exact ode

Time used: 0.420 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(5x^2y + 8x\right)\mathrm{d}y &= \left(-4y^2x - 6y\right)\mathrm{d}x \\ \left(4y^2x + 6y\right)\mathrm{d}x + \left(5x^2y + 8x\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 4y^2x + 6y \\ N(x,y) &= 5x^2y + 8x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(4y^2x + 6y\right) \\ &= 8xy + 6\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(5x^2y + 8x\right) \\ &= 10xy + 8\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{5x^2y + 8x}((8xy + 6) - (10xy + 8)) \\ &= \frac{-2xy - 2}{5x^2y + 8x}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{1}{4y^2x+6y}((10xy+8)-(8xy+6)) \\
&= \frac{xy+1}{2y^2x+3y}
\end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned}
R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\
&= \frac{(10xy+8)-(8xy+6)}{x\,(4y^2x+6y) - y\,(5x^2y+8x)} \\
&= \frac{-2xy-2}{xy\,(xy+2)}
\end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = \frac{-2t-2}{t\,(t+2)}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned}
\mu &= e^{\int R \,\mathrm{d}t} \\
&= e^{\int \left(\frac{-2t-2}{t(t+2)}\right) \mathrm{d}t}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-\ln(t(t+2))} \\
&= \frac{1}{t\,(t+2)}
\end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{xy\,(xy+2)}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{xy\,(xy+2)}\left(4y^2x+6y\right) \\
&= \frac{4xy+6}{x\,(xy+2)}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{xy\,(xy+2)}\left(5x^2y+8x\right) \\
&= \frac{5xy+8}{y\,(xy+2)}
\end{aligned}
$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$
\begin{aligned}
\overline{M}+\overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{4xy+6}{x\,(xy+2)}\right)+\left(\frac{5xy+8}{y\,(xy+2)}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{4xy+6}{x\,(xy+2)}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = \ln(xy+2)+3\ln(x)+f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{x}{xy+2}+f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{5xy+8}{y(xy+2)}$. Therefore equation (4) becomes

$$\frac{5xy+8}{y\left(xy+2\right)} = \frac{x}{xy+2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{4}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{4}{y}\right) \mathrm{d}y$$

$$f(y) = 4\ln\left(y\right) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln\left(xy+2\right) + 3\ln\left(x\right) + 4\ln\left(y\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \ln\left(xy+2\right) + 3\ln\left(x\right) + 4\ln\left(y\right)$$

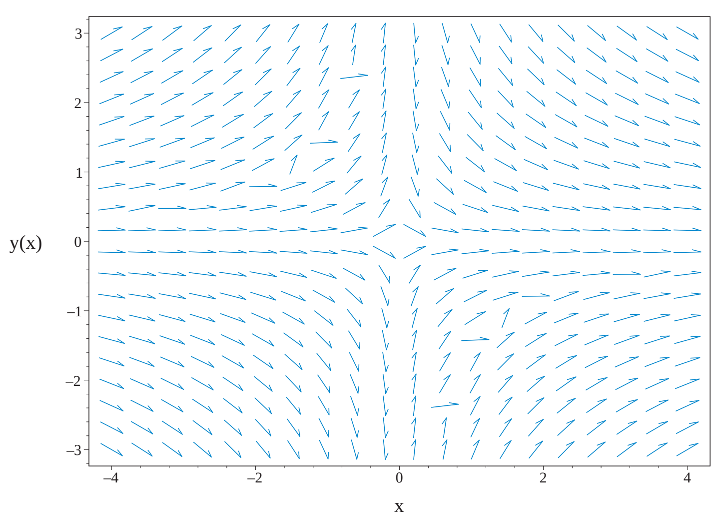


Figure 2.3: Slope field plot
$4xy^2 + 6y + (5x^2y + 8x)\,y' = 0$

Summary of solutions found

$$\ln(xy+2)+3\ln(x)+4\ln(y)=c_1$$

Solved as first order isobaric ode

Time used: 0.260 (sec)

Solving for y' gives

$$y'=-\frac{2y(2xy+3)}{x(5xy+8)} \tag{1}$$

Each of the above ode's is now solved An ode $y'=f(x,y)$ is isobaric if

$$f(tx,t^m y)=t^{m-1}f(x,y) \tag{1}$$

Where here

$$f(x,y)=-\frac{2y(2xy+3)}{x(5xy+8)} \tag{2}$$

m is the order of isobaric. Substituting (2) into (1) and solving for m gives

$$m=-1$$

Since the ode is isobaric of order $m=-1$, then the substitution

$$\begin{aligned} y &= ux^m \\ &= \frac{u}{x} \end{aligned}$$

Converts the ODE to a separable in $u(x)$. Performing this substitution gives

$$-\frac{u(x)}{x^2}+\frac{u'(x)}{x}=-\frac{2u(x)(2u(x)+3)}{x^2(5u(x)+8)}$$

The ode $u'(x)=\frac{u(x)(u(x)+2)}{x(5u(x)+8)}$ is separable as it can be written as

$$\begin{aligned} u'(x) &= \frac{u(x)(u(x)+2)}{x(5u(x)+8)} \\ &= f(x)g(u) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= \frac{1}{x} \\ g(u) &= \frac{u(u+2)}{5u+8} \end{aligned}$$

Integrating gives

$$\int \frac{1}{g(u)}\,du = \int f(x)\,dx$$
$$\int \frac{5u+8}{u\,(u+2)}\,du = \int \frac{1}{x}\,dx$$
$$\ln\left(\left(u(x)+2\right)u(x)^4\right) = \ln(x) + c_1$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $\frac{u(u+2)}{5u+8} = 0$ for $u(x)$ gives

$$u(x) = -2$$
$$u(x) = 0$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\ln\left(\left(u(x)+2\right)u(x)^4\right) = \ln(x) + c_1$$
$$u(x) = -2$$
$$u(x) = 0$$

Converting $\ln\left(\left(u(x)+2\right)u(x)^4\right) = \ln(x) + c_1$ back to y gives

$$\ln\left(\left(xy+2\right)x^4y^4\right) = \ln(x) + c_1$$

Converting $u(x) = -2$ back to y gives

$$xy = -2$$

Converting $u(x) = 0$ back to y gives

$$xy = 0$$

Solving for y gives

$$\ln\left(\left(xy+2\right)x^4y^4\right) = \ln(x) + c_1$$
$$y = 0$$
$$y = -\frac{2}{x}$$

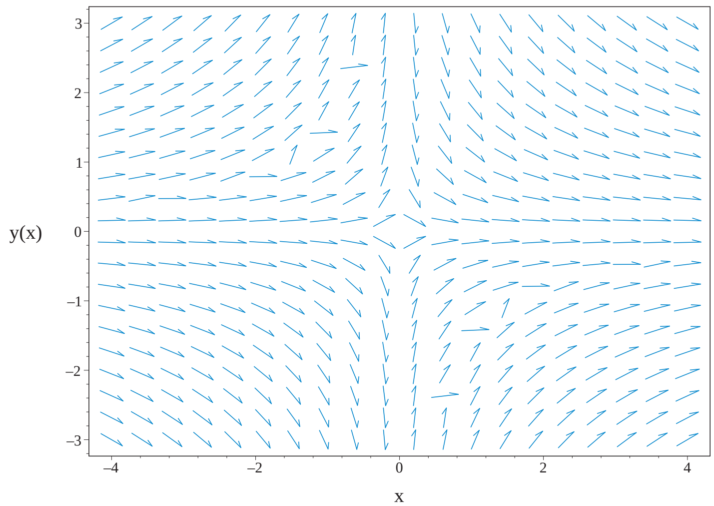


Figure 2.4: Slope field plot
$4xy^2 + 6y + (5x^2y + 8x)\, y' = 0$

Summary of solutions found

$$\ln\left((xy+2)\, x^4 y^4\right) = \ln(x) + c_1$$
$$y = 0$$
$$y = -\frac{2}{x}$$

Solved using Lie symmetry for first order ode

Time used: 0.927 (sec)

Writing the ode as

$$y' = -\frac{2y(2xy+3)}{x\,(5xy+8)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{2y(2xy+3)(b_3-a_2)}{x(5xy+8)} - \frac{4y^2(2xy+3)^2 a_3}{x^2(5xy+8)^2} \\ &- \left(-\frac{4y^2}{x(5xy+8)} + \frac{2y(2xy+3)}{x^2(5xy+8)} + \frac{10y^2(2xy+3)}{x(5xy+8)^2}\right)(xa_2 + ya_3 + a_1) \\ &- \left(-\frac{2(2xy+3)}{x(5xy+8)} - \frac{4y}{5xy+8} + \frac{10y(2xy+3)}{(5xy+8)^2}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{45x^4y^2b_2 - 36x^2y^4a_3 + 20x^3y^2b_1 - 20x^2y^3a_1 + 144x^3yb_2 + 2x^2y^2a_2 + 2x^2y^2b_3 - 108x\,y^3a_3 + 64x^2yb_1 - 6\cdots}{x^2(5xy+8)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &45x^4y^2b_2 - 36x^2y^4a_3 + 20x^3y^2b_1 - 20x^2y^3a_1 + 144x^3yb_2 + 2x^2y^2a_2 + 2x^2y^2b_3 \\ &- 108x\,y^3a_3 + 64x^2yb_1 - 60x\,y^2a_1 + 112b_2x^2 - 84y^2a_3 + 48xb_1 - 48ya_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-36a_3v_1^2v_2^4 + 45b_2v_1^4v_2^2 - 20a_1v_1^2v_2^3 + 20b_1v_1^3v_2^2 + 2a_2v_1^2v_2^2 - 108a_3v_1v_2^3 + 144b_2v_1^3v_2 \\ &+ 2b_3v_1^2v_2^2 - 60a_1v_1v_2^2 + 64b_1v_1^2v_2 - 84a_3v_2^2 + 112b_2v_1^2 - 48a_1v_2 + 48b_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}&45b_2v_1^4v_2^2 + 20b_1v_1^3v_2^2 + 144b_2v_1^3v_2 - 36a_3v_1^2v_2^4 - 20a_1v_1^2v_2^3 + (2a_2 + 2b_3)\, v_1^2v_2^2 \\ &+ 64b_1v_1^2v_2 + 112b_2v_1^2 - 108a_3v_1v_2^3 - 60a_1v_1v_2^2 + 48b_1v_1 - 84a_3v_2^2 - 48a_1v_2 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-60a_1 &= 0\\ -48a_1 &= 0\\ -20a_1 &= 0\\ -108a_3 &= 0\\ -84a_3 &= 0\\ -36a_3 &= 0\\ 20b_1 &= 0\\ 48b_1 &= 0\\ 64b_1 &= 0\\ 45b_2 &= 0\\ 112b_2 &= 0\\ 144b_2 &= 0\\ 2a_2 + 2b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= 0\\ a_2 &= -b_3\\ a_3 &= 0\\ b_1 &= 0\\ b_2 &= 0\\ b_3 &= b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= -x\\ \eta &= y\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{2y(2xy+3)}{x\,(5xy+8)}\right)(-x) \\
&= \frac{y^2x+2y}{5xy+8} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta}dy \\
&= \int \frac{1}{\frac{y^2x+2y}{5xy+8}}dy
\end{aligned}$$

Which results in

$$S = \ln(xy+2) + 4\ln(y)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2y(2xy+3)}{x\,(5xy+8)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{y}{xy+2} \\
S_y &= \frac{x}{xy+2} + \frac{4}{y}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{3}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{3}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

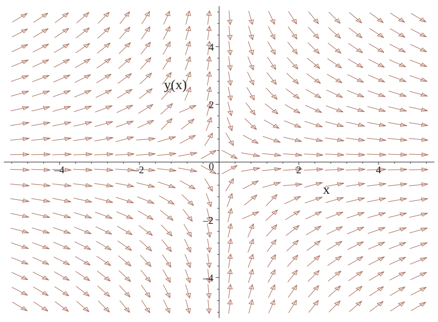
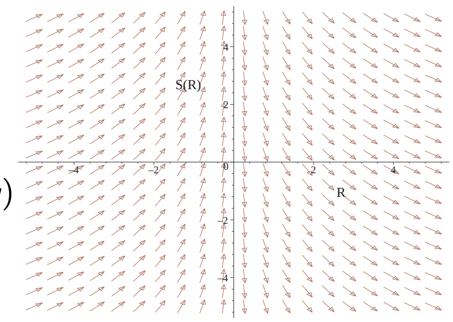
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$
\begin{aligned}
\int dS &= \int -\frac{3}{R}\, dR \\
S(R) &= -3\ln(R) + c_2
\end{aligned}
$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$
\ln(xy+2) + 4\ln(y) = -3\ln(x) + c_2
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2y(2xy+3)}{x(5xy+8)}$ 	$R = x$ $S = \ln(xy + 2) + 4\ln(y)$	$\frac{dS}{dR} = -\frac{3}{R}$ 

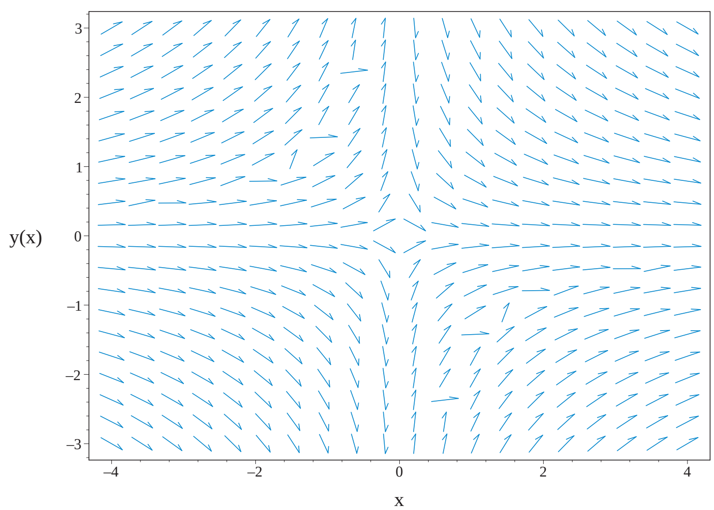


Figure 2.5: Slope field plot
$4xy^2 + 6y + (5x^2y + 8x)\, y' = 0$

Summary of solutions found

$$\ln(xy + 2) + 4\ln(y) = -3\ln(x) + c_2$$

Maple step by step solution

Let's solve

$4xy(x)^2 + 6y(x) + (5x^2y(x) + 8x)\left(\frac{d}{dx}y(x)\right) = 0$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = \frac{-4xy(x)^2 - 6y(x)}{5x^2y(x) + 8x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
<- homogeneous successful`
```

Maple dsolve solution

Solving time : `0.051` (sec)
Leaf size : 23

```
dsolve(4*y(x)^2*x+6*y(x)+(5*x^2*y(x)+8*x)*diff(y(x),x) = 0,
       y(x),singsol=all)
```

$$y(x) = \frac{\operatorname{RootOf}\left(-\ln(x) + c_1 + \ln(\_Z + 2) + 4\ln(\_Z)\right)}{x}$$

Mathematica DSolve solution

Solving time : 2.933 (sec)
Leaf size : 156

```
DSolve[{(4*x*y[x]^2+6*y[x])+(5*x^2*y[x]+8*x)*D[y[x],x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \text{Root}\left[-\#1^5 - \frac{2\#1^4}{x} + \frac{e^{c_1}}{x^4}\&, 1\right]$$
$$y(x) \to \text{Root}\left[-\#1^5 - \frac{2\#1^4}{x} + \frac{e^{c_1}}{x^4}\&, 2\right]$$
$$y(x) \to \text{Root}\left[-\#1^5 - \frac{2\#1^4}{x} + \frac{e^{c_1}}{x^4}\&, 3\right]$$
$$y(x) \to \text{Root}\left[-\#1^5 - \frac{2\#1^4}{x} + \frac{e^{c_1}}{x^4}\&, 4\right]$$
$$y(x) \to \text{Root}\left[-\#1^5 - \frac{2\#1^4}{x} + \frac{e^{c_1}}{x^4}\&, 5\right]$$

2.1.5 problem 5

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Internal problem ID [4081]
Book : Differential equations, Shepley L. Ross, 1964
Section : 2.4, page 55
Problem number : 5
Date solved : `Tuesday, December 17, 2024 at 06:21:33 AM`
CAS classification :
`[[_homogeneous, ‘class C‘], _exact, _rational, [_Abel, ‘2nd type‘, ‘class A‘]]`

Solve

$$5x + 2y + 1 + (2x + y + 1)\, y' = 0$$

Summary of solutions found

$$y = -2x - 1 - \sqrt{-\left(x - 1\right)^{2} + e^{2c_1}}$$

$$y = -2x - 1 + \sqrt{-\left(x - 1\right)^{2} + e^{2c_1}}$$

$$y = -ix - 2x + i - 1$$

$$y = ix - 2x - i - 1$$

Solved as first order homogeneous class Maple C ode

Time used: 0.813 (sec)

Let $Y = y - y_0$ and $X = x - x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{2Y(X) + 2y_0 + 5x_0 + 5X + 1}{2x_0 + 2X + Y(X) + y_0 + 1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$\begin{aligned} x_0 &= 1 \\ y_0 &= -3 \end{aligned}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{2Y(X) + 5X}{2X + Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= -\frac{2Y + 5X}{2X + Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = -2Y - 5X$ and $N = 2X + Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{dY}{dX} = \frac{du}{dX}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{du}{dX}X + u &= \frac{-2u - 5}{u + 2} \\ \frac{du}{dX} &= \frac{\frac{-2u(X)-5}{u(X)+2} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{-2u(X)-5}{u(X)+2} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right) Xu(X) + 2\left(\frac{d}{dX}u(X)\right) X + u(X)^2 + 4u(X) + 5 = 0$$

Or

$$X(u(X)+2)\left(\frac{d}{dX}u(X)\right) + u(X)^2 + 4u(X) + 5 = 0$$

Which is now solved as separable in $u(X)$.

The ode $\frac{d}{dX}u(X) = -\frac{u(X)^2+4u(X)+5}{X(u(X)+2)}$ is separable as it can be written as

$$\begin{aligned}\frac{d}{dX}u(X) &= -\frac{u(X)^2 + 4u(X) + 5}{X\left(u\left(X\right)+2\right)}\\ &= f(X)g(u)\end{aligned}$$

Where

$$\begin{aligned}f(X) &= -\frac{1}{X}\\ g(u) &= \frac{u^2+4u+5}{u+2}\end{aligned}$$

Integrating gives

$$\begin{aligned}\int \frac{1}{g(u)}\,du &= \int f(X)\,dX\\ \int \frac{u+2}{u^2+4u+5}\,du &= \int -\frac{1}{X}\,dX\\ \frac{\ln\left(u(X)^2+4u(X)+5\right)}{2} &= \ln\left(\frac{1}{X}\right) + c_1\end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $\frac{u^2+4u+5}{u+2} = 0$ for $u(X)$ gives

$$\begin{aligned}u(X) &= -2 - i\\ u(X) &= -2 + i\end{aligned}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\frac{\ln\left(u(X)^2+4u(X)+5\right)}{2}=\ln\left(\frac{1}{X}\right)+c_1$$
$$u(X)=-2-i$$
$$u(X)=-2+i$$

Solving for $u(X)$ gives

$$u(X)=-2-i$$
$$u(X)=-2+i$$
$$u(X)=\frac{-2X-\sqrt{-X^2+e^{2c_1}}}{X}$$
$$u(X)=\frac{-2X+\sqrt{-X^2+e^{2c_1}}}{X}$$

Converting $u(X)=-2-i$ back to $Y(X)$ gives

$$Y(X)=(-2-i)\,X$$

Converting $u(X)=-2+i$ back to $Y(X)$ gives

$$Y(X)=(-2+i)\,X$$

Converting $u(X)=\frac{-2X-\sqrt{-X^2+e^{2c_1}}}{X}$ back to $Y(X)$ gives

$$Y(X)=-2X-\sqrt{-X^2+e^{2c_1}}$$

Converting $u(X)=\frac{-2X+\sqrt{-X^2+e^{2c_1}}}{X}$ back to $Y(X)$ gives

$$Y(X)=-2X+\sqrt{-X^2+e^{2c_1}}$$

Using the solution for $Y(X)$

$$Y(X)=(-2-i)\,X \tag{A}$$

And replacing back terms in the above solution using

$$Y=y+y_0$$
$$X=x+x_0$$

Or

$$Y = y - 3$$
$$X = x + 1$$

Then the solution in y becomes using EQ (A)

$$y + 3 = (-2 - i)(x - 1)$$

Using the solution for $Y(X)$

$$Y(X) = (-2 + i) X \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - 3$$
$$X = x + 1$$

Then the solution in y becomes using EQ (A)

$$y + 3 = (-2 + i)(x - 1)$$

Using the solution for $Y(X)$

$$Y(X) = -2X - \sqrt{-X^2 + e^{2c_1}} \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - 3$$
$$X = x + 1$$

Then the solution in y becomes using EQ (A)

$$y + 3 = -2x + 2 - \sqrt{-(x - 1)^2 + e^{2c_1}}$$

Using the solution for $Y(X)$

$$Y(X) = -2X + \sqrt{-X^2 + e^{2c_1}} \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - 3$$
$$X = x + 1$$

Then the solution in y becomes using EQ (A)

$$y + 3 = -2x + 2 + \sqrt{-(x-1)^2 + e^{2c_1}}$$

Solving for y gives

$$y = -2x - 1 - \sqrt{-(x-1)^2 + e^{2c_1}}$$
$$y = -2x - 1 + \sqrt{-(x-1)^2 + e^{2c_1}}$$
$$y = -ix - 2x + i - 1$$
$$y = ix - 2x - i - 1$$

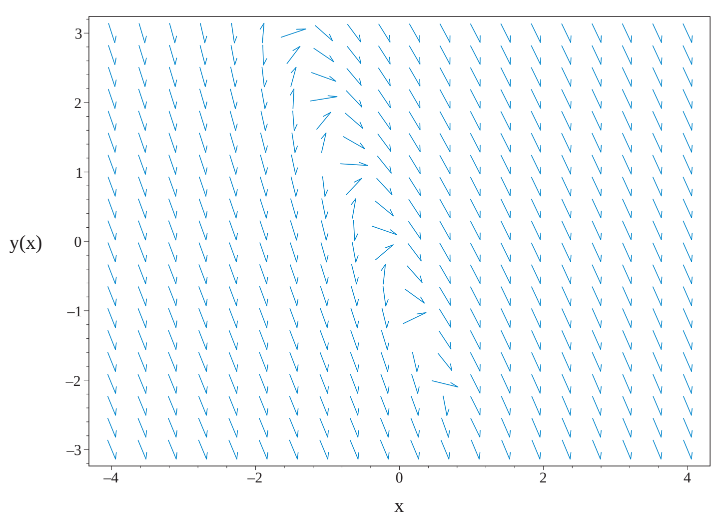


Figure 2.6: Slope field plot
$5x + 2y + 1 + (2x + y + 1)\,y' = 0$

Solved as first order Exact ode

Time used: 0.192 (sec)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(2x+y+1)\,\mathrm{d}y&=(-2y-5x-1)\,\mathrm{d}x\\ (2y+5x+1)\,\mathrm{d}x+(2x+y+1)\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=2y+5x+1\\ N(x,y)&=2x+y+1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2y + 5x + 1)\\ &= 2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2x + y + 1)\\ &= 2\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int 2y + 5x + 1\,\mathrm{d}x\end{aligned}$$

$$\phi = \frac{x(4y + 5x + 2)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2x + y + 1$. Therefore equation (4) becomes

$$2x + y + 1 = 2x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y + 1$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (y+1)\,\mathrm{d}y$$

$$f(y) = \frac{1}{2}y^2 + y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(4y + 5x + 2)}{2} + \frac{y^2}{2} + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

$$c_1 = \frac{x(4y + 5x + 2)}{2} + \frac{y^2}{2} + y$$

Solving for y gives

$$y = -2x - 1 - \sqrt{-x^2 + 2c_1 + 2x + 1}$$

$$y = -2x - 1 + \sqrt{-x^2 + 2c_1 + 2x + 1}$$

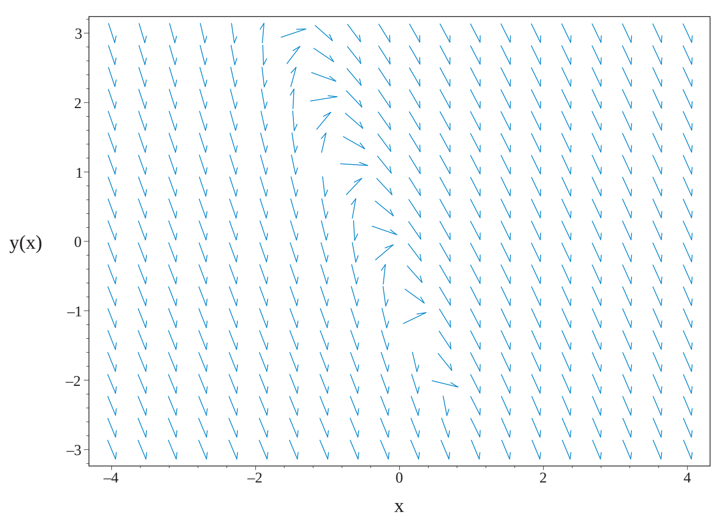


Figure 2.7: Slope field plot
$5x + 2y + 1 + (2x + y + 1)\,y' = 0$

Summary of solutions found

$$y = -2x - 1 - \sqrt{-x^2 + 2c_1 + 2x + 1}$$
$$y = -2x - 1 + \sqrt{-x^2 + 2c_1 + 2x + 1}$$

Solved using Lie symmetry for first order ode

Time used: 0.533 (sec)

Writing the ode as

$$y' = -\frac{2y + 5x + 1}{2x + y + 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 &- \frac{(2y + 5x + 1)(b_3 - a_2)}{2x + y + 1} - \frac{(2y + 5x + 1)^2 a_3}{(2x + y + 1)^2} \\ &- \left(-\frac{5}{2x + y + 1} + \frac{4y + 10x + 2}{(2x + y + 1)^2}\right)(x a_2 + y a_3 + a_1) \\ &- \left(-\frac{2}{2x + y + 1} + \frac{2y + 5x + 1}{(2x + y + 1)^2}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{10x^2a_2 - 25x^2a_3 + 3x^2b_2 - 10x^2b_3 + 10xya_2 - 20xya_3 + 4xyb_2 - 10xyb_3 + 2y^2a_2 - 3y^2a_3 + y^2b_2 - 2y^2b_3}{(2x + y}$$
$$= 0$$

Setting the numerator to zero gives

$$10x^2a_2 - 25x^2a_3 + 3x^2b_2 - 10x^2b_3 + 10xya_2 - 20xya_3 + 4xyb_2 - 10xyb_3 + 2y^2a_2 - 3y^2a_3 + y^2b_2 - 2y^2b_3 + 10xa_2 - 10xa_3 - xb_1 + 5xb_2 - 7xb_3 + ya_1 + 3ya_2 - ya_3 + 2yb_2 - 2yb_3 + 3a_1 + a_2 - a_3 + b_1 + b_2 - b_3 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$10a_2v_1^2 + 10a_2v_1v_2 + 2a_2v_2^2 - 25a_3v_1^2 - 20a_3v_1v_2 - 3a_3v_2^2 + 3b_2v_1^2 + 4b_2v_1v_2 + b_2v_2^2 - 10b_3v_1^2 - 10b_3v_1v_2 - 2b_3v_2^2 + a_1v_2 + 10a_2v_1 + 3a_2v_2 - 10a_3v_1 - a_3v_2 - b_1v_1 + 5b_2v_1 + 2b_2v_2 - 7b_3v_1 - 2b_3v_2 + 3a_1 + a_2 - a_3 + b_1 + b_2 - b_3 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(10a_2 - 25a_3 + 3b_2 - 10b_3)\, v_1^2 + (10a_2 - 20a_3 + 4b_2 - 10b_3)\, v_1v_2 + (10a_2 - 10a_3 - b_1 + 5b_2 - 7b_3)\, v_1 + (2a_2 - 3a_3 + b_2 - 2b_3)\, v_2^2 + (a_1 + 3a_2 - a_3 + 2b_2 - 2b_3)\, v_2 + 3a_1 + a_2 - a_3 + b_1 + b_2 - b_3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
2a_2 - 3a_3 + b_2 - 2b_3 &= 0 \\
10a_2 - 25a_3 + 3b_2 - 10b_3 &= 0 \\
10a_2 - 20a_3 + 4b_2 - 10b_3 &= 0 \\
a_1 + 3a_2 - a_3 + 2b_2 - 2b_3 &= 0 \\
10a_2 - 10a_3 - b_1 + 5b_2 - 7b_3 &= 0 \\
3a_1 + a_2 - a_3 + b_1 + b_2 - b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= -a_3 - b_3 \\
a_2 &= 4a_3 + b_3 \\
a_3 &= a_3 \\
b_1 &= 5a_3 + 3b_3 \\
b_2 &= -5a_3 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x - 1 \\
\eta &= y + 3
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y + 3 - \left(-\frac{2y + 5x + 1}{2x + y + 1}\right)(x - 1) \\
&= \frac{5x^2 + 4xy + y^2 + 2x + 2y + 2}{2x + y + 1} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{5x^2+4xy+y^2+2x+2y+2}{2x+y+1}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln\left(5x^2 + 4xy + y^2 + 2x + 2y + 2\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2y + 5x + 1}{2x + y + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2y + 5x + 1}{5x^2 + (4y + 2)\,x + y^2 + 2y + 2} \\ S_y &= \frac{2x + y + 1}{y^2 + (4x + 2)\,y + 5x^2 + 2x + 2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

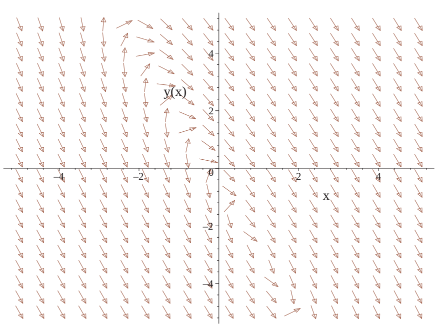
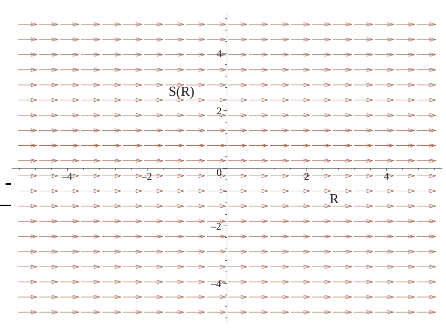
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int 0\, dR + c_2$$
$$S(R) = c_2$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\frac{\ln\left(5x^2 + \left(4y + 2\right)x + y^2 + 2y + 2\right)}{2} = c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2y+5x+1}{2x+y+1}$ 	$R = x$ $S = \frac{\ln\left(5x^2 + \left(4y + 2\right)x -\right.}{2}$	$\frac{dS}{dR} = 0$ 

Solving for y gives

$$y = -2x - 1 - \sqrt{-x^2 + e^{2c_2} + 2x - 1}$$
$$y = -2x - 1 + \sqrt{-x^2 + e^{2c_2} + 2x - 1}$$

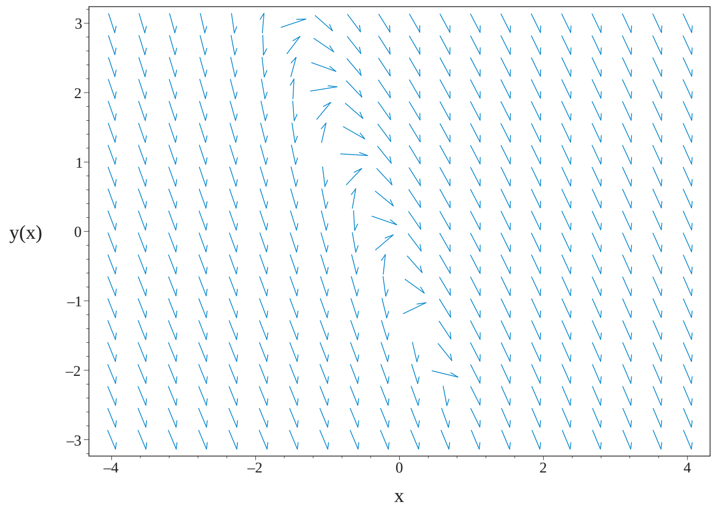


Figure 2.8: Slope field plot
$5x + 2y + 1 + (2x + y + 1)\, y' = 0$

Summary of solutions found

$$y = -2x - 1 - \sqrt{-x^2 + e^{2c_2} + 2x - 1}$$
$$y = -2x - 1 + \sqrt{-x^2 + e^{2c_2} + 2x - 1}$$

Solved as first order ode of type dAlembert

Time used: 0.399 (sec)

Let $p = y'$ the ode becomes

$$5x + 2y + 1 + (2x + y + 1)\, p = 0$$

Solving for y from the above results in

$$y = -\frac{(2p + 5)\, x}{2 + p} - \frac{p + 1}{2 + p} \tag{1}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= \frac{-2p-5}{2+p} \\ g &= \frac{-p-1}{2+p} \end{aligned}$$

Hence (2) becomes

$$p - \frac{-2p-5}{2+p} = \left(-\frac{2x}{2+p} + \frac{2xp}{(2+p)^2} + \frac{5x}{(2+p)^2} - \frac{1}{2+p} + \frac{p}{(2+p)^2} + \frac{1}{(2+p)^2}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{-2p-5}{2+p} = 0$$

Solving the above for p results in

$$\begin{aligned} p_1 &= -2 + i \\ p_2 &= -2 - i \end{aligned}$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$\begin{aligned} y &= (-2+i)\,x - 1 - i \\ y &= (-2-i)\,x - 1 + i \end{aligned}$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{-2p(x)-5}{2+p(x)}}{-\frac{2x}{2+p(x)} + \frac{2xp(x)}{(2+p(x))^2} + \frac{5x}{(2+p(x))^2} - \frac{1}{2+p(x)} + \frac{p(x)}{(2+p(x))^2} + \frac{1}{(2+p(x))^2}} \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed. The ode $p'(x) = \frac{(2+p(x))\left(p(x)^2+4p(x)+5\right)}{x-1}$ is separable as it can be written as

$$\begin{aligned} p'(x) &= \frac{(2+p(x))\left(p(x)^2+4p(x)+5\right)}{x-1} \\ &= f(x)g(p) \end{aligned}$$

Where

$$
\begin{aligned}
f(x) &= \frac{1}{x-1} \\
g(p) &= (2+p)\left(p^2+4p+5\right)
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(p)}\, dp &= \int f(x)\, dx \\
\int \frac{1}{(2+p)\left(p^2+4p+5\right)}\, dp &= \int \frac{1}{x-1}\, dx \\
\ln\left(\frac{2+p(x)}{\sqrt{p\left(x\right)^2+4p\left(x\right)+5}}\right) &= \ln\left(x-1\right)+c_1
\end{aligned}
$$

We now need to find the singular solutions, these are found by finding for what values $g(p)$ is zero, since we had to divide by this above. Solving $g(p)=0$ or $(2+p)\left(p^2+4p+5\right)=0$ for $p(x)$ gives

$$
\begin{aligned}
p(x) &= -2 \\
p(x) &= -2-i \\
p(x) &= -2+i
\end{aligned}
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
\ln\left(\frac{2+p(x)}{\sqrt{p\left(x\right)^2+4p\left(x\right)+5}}\right) &= \ln\left(x-1\right)+c_1 \\
p(x) &= -2 \\
p(x) &= -2-i \\
p(x) &= -2+i
\end{aligned}
$$

Substituing the above solution for p in (2A) gives

$$
y = \frac{x\left(-2x\,\mathrm{e}^{c_1}\sqrt{-\frac{1}{x^2\mathrm{e}^{2c_1}-2x\,\mathrm{e}^{2c_1}+\mathrm{e}^{2c_1}-1}}+2\,\mathrm{e}^{c_1}\sqrt{-\frac{1}{x^2\mathrm{e}^{2c_1}-2x\,\mathrm{e}^{2c_1}+\mathrm{e}^{2c_1}-1}}-1\right)}{x\,\mathrm{e}^{c_1}\sqrt{-\frac{1}{x^2\mathrm{e}^{2c_1}-2x\,\mathrm{e}^{2c_1}+\mathrm{e}^{2c_1}-1}}-\mathrm{e}^{c_1}\sqrt{-\frac{1}{x^2\mathrm{e}^{2c_1}-2x\,\mathrm{e}^{2c_1}+\mathrm{e}^{2c_1}-1}}} + \frac{-x\,\mathrm{e}^{c_1}\sqrt{-\frac{1}{x^2\mathrm{e}^{2c_1}-2x\,\mathrm{e}^{2c_1}+\mathrm{e}^{2c_1}-1}}}{x\,\mathrm{e}^{c_1}\sqrt{-\frac{1}{x^2\mathrm{e}^{2c_1}-2x\,\mathrm{e}^{2c_1}+\mathrm{e}^{2c_1}-}}}
$$

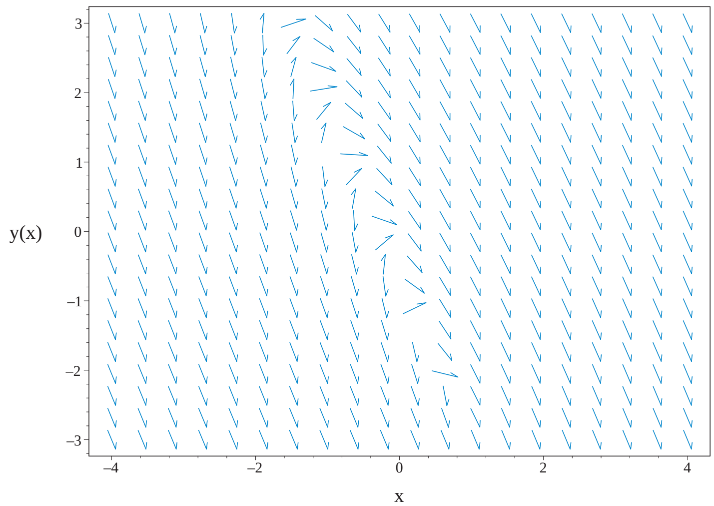


Figure 2.9: Slope field plot
$5x + 2y + 1 + (2x + y + 1)\, y' = 0$

Summary of solutions found

$$y = (-2 - i)\, x - 1 + i$$

$$y = (-2 + i)\, x - 1 - i$$

$$y = \frac{x \left(-2x\, e^{c_1} \sqrt{-\frac{1}{x^2 e^{2c_1} - 2x\, e^{2c_1} + e^{2c_1} - 1}} + 2\, e^{c_1} \sqrt{-\frac{1}{x^2 e^{2c_1} - 2x\, e^{2c_1} + e^{2c_1} - 1}} - 1\right)}{x\, e^{c_1} \sqrt{-\frac{1}{x^2 e^{2c_1} - 2x\, e^{2c_1} + e^{2c_1} - 1}} - e^{c_1} \sqrt{-\frac{1}{x^2 e^{2c_1} - 2x\, e^{2c_1} + e^{2c_1} - 1}}} + \frac{-x\, e^{c_1} \sqrt{-\frac{1}{x^2 e^{2c_1} - 2x\, e^{2c_1} + e^{2c_1} - 1}} + e^{c_1} \sqrt{-\frac{1}{x^2 e^{2c_1} - 2x\, e^{2c_1} + e^{2c_1} - 1}} + 1}{x\, e^{c_1} \sqrt{-\frac{1}{x^2 e^{2c_1} - 2x\, e^{2c_1} + e^{2c_1} - 1}} - e^{c_1} \sqrt{-\frac{1}{x^2 e^{2c_1} - 2x\, e^{2c_1} + e^{2c_1} - 1}}}$$

Maple step by step solution

Let's solve
$5x + 2y(x) + 1 + (2x + y(x) + 1) \left(\frac{d}{dx} y(x)\right) = 0$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx} y(x)$
- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function
    $\frac{d}{dx} F(x, y(x)) = 0$
  - Compute derivative of lhs

$\frac{\partial}{\partial x}F(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)\left(\frac{d}{dx}y(x)\right)=0$

- Evaluate derivatives

  $2=2$
- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=C1, M(x,y)=\frac{\partial}{\partial x}F(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$
- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int(5x+2y+1)\,dx+\_F1(y)$
- Evaluate integral

  $F(x,y)=\frac{5x^2}{2}+2xy+x+\_F1(y)$
- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$
- Compute derivative

  $2x+y+1=2x+\frac{d}{dy}\_F1(y)$
- Isolate for $\frac{d}{dy}\_F1(y)$

  $\frac{d}{dy}\_F1(y)=y+1$
- Solve for $\_F1(y)$

  $\_F1(y)=\frac{1}{2}y^2+y$
- Substitute $\_F1(y)$ into equation for $F(x,y)$

  $F(x,y)=\frac{5}{2}x^2+2xy+x+\frac{1}{2}y^2+y$
- Substitute $F(x,y)$ into the solution of the ODE

  $\frac{5}{2}x^2+2xy+x+\frac{1}{2}y^2+y=C1$
- Solve for $y(x)$

  $\left\{y(x)=-1-2x-\sqrt{-x^2+2C1+2x+1}, y(x)=-1-2x+\sqrt{-x^2+2C1+2x+1}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
```

```
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

Maple dsolve solution

Solving time : 0.395 (sec)
Leaf size : 32

```
dsolve(5*x+2*y(x)+1+(2*x+y(x)+1)*diff(y(x),x) = 0,
       y(x),singsol=all)
```

$$y(x) = \frac{-\sqrt{-\left(x-1\right)^{2} c_1^{2}+1}+\left(-2x-1\right) c_1}{c_1}$$

Mathematica DSolve solution

Solving time : 0.219 (sec)
Leaf size : 53

```
DSolve[{(5*x+2*y[x]+1)+(2*x+y[x]+1)*D[y[x],x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\sqrt{-x^2+2x+1+c_1}-2x-1$$
$$y(x) \to \sqrt{-x^2+2x+1+c_1}-2x-1$$

2.1.6 problem 6

Solved using Lie symmetry for first order ode 60
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Maple step by step solution 68
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Internal problem ID [4082]
Book : Differential equations, Shepley L. Ross, 1964
Section : 2.4, page 55
Problem number : 6
Date solved : `Tuesday, December 17, 2024 at 06:21:36 AM`
CAS classification :
`[[_homogeneous, ‘class C‘], _rational, [_Abel, ‘2nd type‘, ‘class A‘]]`

Solve

$$3x - y + 1 - (6x - 2y - 3)\,y' = 0$$

Solved using Lie symmetry for first order ode

Time used: 0.659 (sec)

Writing the ode as

$$y' = \frac{y - 3x - 1}{2y - 6x + 3}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
b_2 &+ \frac{(y-3x-1)\,(b_3-a_2)}{2y-6x+3} - \frac{(y-3x-1)^2\,a_3}{(2y-6x+3)^2} \\
&- \left(-\frac{3}{2y-6x+3} + \frac{6y-18x-6}{(2y-6x+3)^2}\right)(xa_2+ya_3+a_1) \\
&- \left(\frac{1}{2y-6x+3} - \frac{2(y-3x-1)}{(2y-6x+3)^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{18x^2a_2+9x^2a_3-36x^2b_2-18x^2b_3-12xya_2-6xya_3+24xyb_2+12xyb_3+2y^2a_2+y^2a_3-4y^2b_2-2y^2}{(-2y+6}
$$

$= 0$

Setting the numerator to zero gives

$$
\begin{aligned}
&-18x^2a_2-9x^2a_3+36x^2b_2+18x^2b_3+12xya_2+6xya_3-24xyb_2-12xyb_3 \\
&-2y^2a_2-y^2a_3+4y^2b_2+2y^2b_3+18xa_2-6xa_3-41xb_2-3xb_3-ya_2 \\
&+17ya_3+12yb_2-4yb_3+15a_1+3a_2-a_3-5b_1+9b_2-3b_3 = 0
\end{aligned}
\tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-18a_2v_1^2+12a_2v_1v_2-2a_2v_2^2-9a_3v_1^2+6a_3v_1v_2-a_3v_2^2+36b_2v_1^2-24b_2v_1v_2 \\
&+4b_2v_2^2+18b_3v_1^2-12b_3v_1v_2+2b_3v_2^2+18a_2v_1-a_2v_2-6a_3v_1+17a_3v_2 \\
&-41b_2v_1+12b_2v_2-3b_3v_1-4b_3v_2+15a_1+3a_2-a_3-5b_1+9b_2-3b_3 = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&(-18a_2 - 9a_3 + 36b_2 + 18b_3)\, v_1^2 + (12a_2 + 6a_3 - 24b_2 - 12b_3)\, v_1 v_2 \\
&+ (18a_2 - 6a_3 - 41b_2 - 3b_3)\, v_1 + (-2a_2 - a_3 + 4b_2 + 2b_3)\, v_2^2 \\
&+ (-a_2 + 17a_3 + 12b_2 - 4b_3)\, v_2 + 15a_1 + 3a_2 - a_3 - 5b_1 + 9b_2 - 3b_3 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-18a_2 - 9a_3 + 36b_2 + 18b_3 &= 0 \\
-2a_2 - a_3 + 4b_2 + 2b_3 &= 0 \\
-a_2 + 17a_3 + 12b_2 - 4b_3 &= 0 \\
12a_2 + 6a_3 - 24b_2 - 12b_3 &= 0 \\
18a_2 - 6a_3 - 41b_2 - 3b_3 &= 0 \\
15a_1 + 3a_2 - a_3 - 5b_1 + 9b_2 - 3b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= a_1 \\
a_2 &= 2b_2 \\
a_3 &= -\frac{2b_2}{3} \\
b_1 &= 3a_1 + \frac{10b_2}{3} \\
b_2 &= b_2 \\
b_3 &= -\frac{b_2}{3}
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 1 \\
\eta &= 3
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= 3 - \left(\frac{y - 3x - 1}{2y - 6x + 3}\right)(1) \\
&= \frac{-5y + 15x - 10}{-2y + 6x - 3} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R,S. The canonical coordinates map $(x,y)\to(R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi}=\frac{dy}{\eta}=dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}\right)S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R=x$$

S is found from

$$\begin{aligned} S&=\int\frac{1}{\eta}dy\\ &=\int\frac{1}{\frac{-5y+15x-10}{-2y+6x-3}}dy \end{aligned}$$

Which results in

$$S=\frac{2y}{5}-\frac{\ln\left(-3x+y+2\right)}{5}$$

Now that R,S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR}=\frac{S_x+\omega(x,y)S_y}{R_x+\omega(x,y)R_y} \tag{2}$$

Where in the above R_x,R_y,S_x,S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y)=\frac{y-3x-1}{2y-6x+3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x&=1\\ R_y&=0\\ S_x&=-\frac{3}{-5y+15x-10}\\ S_y&=\frac{2}{5}+\frac{1}{-5y+15x-10} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{5} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{5}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int \frac{1}{5}\, dR$$
$$S(R) = \frac{R}{5} + c_2$$

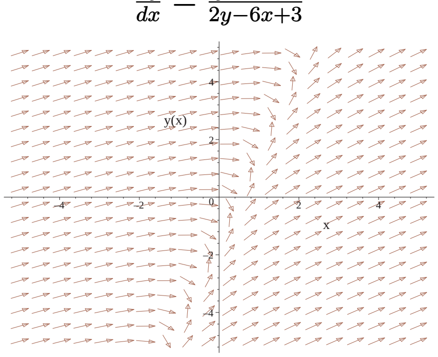
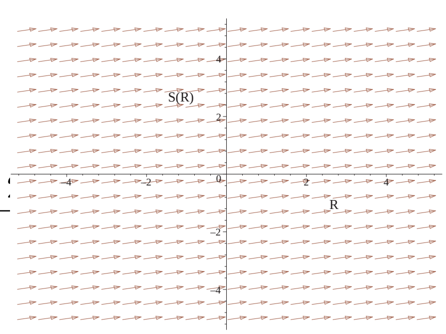
To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\frac{2y}{5} - \frac{\ln(-3x + y + 2)}{5} = \frac{x}{5} + c_2$$

Which gives

$$y = -\frac{\operatorname{LambertW}\left(-2\, e^{5x-4-5c_2}\right)}{2} + 3x - 2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y-3x-1}{2y-6x+3}$ 	$R = x$ $S = \frac{2y}{5} - \frac{\ln(-3x + y + }{5}$	$\frac{dS}{dR} = \frac{1}{5}$ 

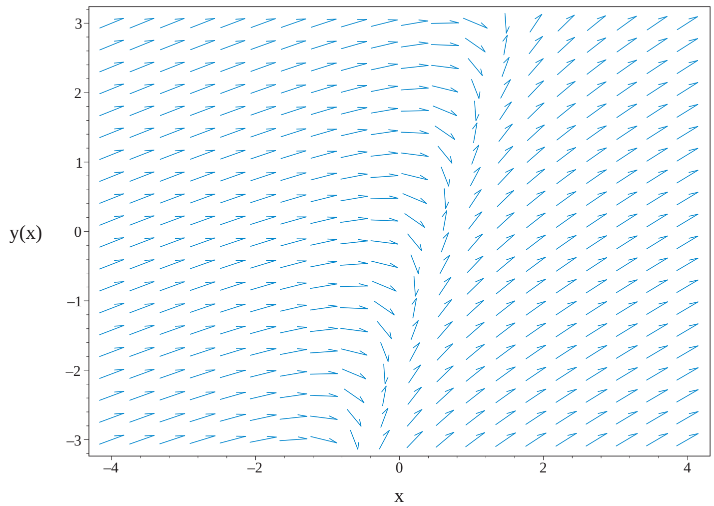


Figure 2.10: Slope field plot
$3x - y + 1 - (6x - 2y - 3)\, y' = 0$

Summary of solutions found

$$y = -\frac{\operatorname{LambertW}\left(-2\, e^{5x-4-5c_2}\right)}{2} + 3x - 2$$

Solved as first order ode of type dAlembert

Time used: 0.335 (sec)

Let $p = y'$ the ode becomes

$$3x - y + 1 - (-2y + 6x - 3)\, p = 0$$

Solving for y from the above results in

$$y = \frac{(6p - 3)\, x}{-1 + 2p} + \frac{-3p - 1}{-1 + 2p} \tag{1}$$

This has the form

$$y = x f(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. x gives

$$
\begin{aligned}
p &= f+(xf'+g')\frac{dp}{dx} \\
p-f &= (xf'+g')\frac{dp}{dx} \qquad (2)
\end{aligned}
$$

Comparing the form $y=xf+g$ to (1A) shows that

$$
\begin{aligned}
f &= 3 \\
g &= \frac{-3p-1}{-1+2p}
\end{aligned}
$$

Hence (2) becomes

$$
p-3=\left(-\frac{3}{-1+2p}+\frac{6p}{(-1+2p)^2}+\frac{2}{(-1+2p)^2}\right)p'(x) \qquad (2A)
$$

The singular solution is found by setting $\frac{dp}{dx}=0$ in the above which gives

$$
p-3=0
$$

Solving the above for p results in

$$
p_1=3
$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$
y=3x-2
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x}\neq 0$. From eq. (2A). This results in

$$
p'(x)=\frac{p(x)-3}{-\frac{3}{-1+2p(x)}+\frac{6p(x)}{(-1+2p(x))^2}+\frac{2}{(-1+2p(x))^2}} \qquad (3)
$$

This ODE is now solved for $p(x)$. No inversion is needed. Integrating gives

$$
\begin{aligned}
\int \frac{5}{(p-3)(-1+2p)^2}dp &= dx \\
\frac{\ln(p-3)}{5}+\frac{1}{-1+2p}-\frac{\ln(-1+2p)}{5} &= x+c_1
\end{aligned}
$$

Singular solutions are found by solving

$$
\frac{(p-3)(-1+2p)^2}{5}=0
$$

for $p(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$p(x) = 3$$
$$p(x) = \frac{1}{2}$$

Substituing the above solution for p in (2A) gives

$$y = 3x + \frac{-3\,\mathrm{e}^{\mathrm{RootOf}\left(2\ln\left(2\,\mathrm{e}^{-Z}+5\right)\mathrm{e}^{-Z}+10c_1\,\mathrm{e}^{-Z}-2\_Z\mathrm{e}^{-Z}+10x\,\mathrm{e}^{-Z}+5\ln\left(2\,\mathrm{e}^{-Z}+5\right)+25c_1-5\_Z+25x-5\right)}-10}{2\,\mathrm{e}^{\mathrm{RootOf}\left(2\ln\left(2\,\mathrm{e}^{-Z}+5\right)\mathrm{e}^{-Z}+10c_1\,\mathrm{e}^{-Z}-2\_Z\mathrm{e}^{-Z}+10x\,\mathrm{e}^{-Z}+5\ln\left(2\,\mathrm{e}^{-Z}+5\right)+25c_1-5\_Z+25x-5\right)}+5}$$
$$y = 3x - 2$$

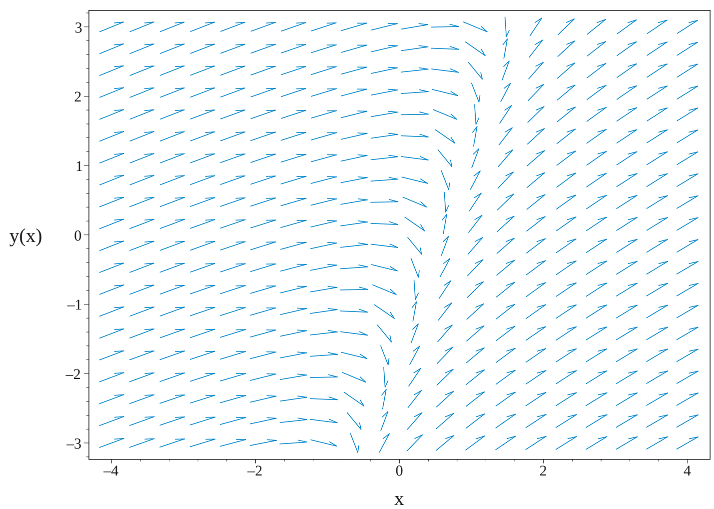


Figure 2.11: Slope field plot
$3x - y + 1 - (6x - 2y - 3)\,y' = 0$

Summary of solutions found

$$y = 3x - 2$$
$$y = 3x + \frac{-3\,\mathrm{e}^{\mathrm{RootOf}\left(2\ln\left(2\,\mathrm{e}^{-Z}+5\right)\mathrm{e}^{-Z}+10c_1\,\mathrm{e}^{-Z}-2\_Z\mathrm{e}^{-Z}+10x\,\mathrm{e}^{-Z}+5\ln\left(2\,\mathrm{e}^{-Z}+5\right)+25c_1-5\_Z+25x-5\right)}-10}{2\,\mathrm{e}^{\mathrm{RootOf}\left(2\ln\left(2\,\mathrm{e}^{-Z}+5\right)\mathrm{e}^{-Z}+10c_1\,\mathrm{e}^{-Z}-2\_Z\mathrm{e}^{-Z}+10x\,\mathrm{e}^{-Z}+5\ln\left(2\,\mathrm{e}^{-Z}+5\right)+25c_1-5\_Z+25x-5\right)}+5}$$

Maple step by step solution

Let's solve

$3x - y(x) + 1 - (6x - 2y(x) - 3)\left(\frac{d}{dx}y(x)\right) = 0$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = -\frac{-3x+y(x)-1}{6x-2y(x)-3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = 3, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

Maple dsolve solution

Solving time : 0.024 (sec)
Leaf size : 23

```
dsolve(3*x-y(x)+1-(6*x-2*y(x)-3)*diff(y(x),x) = 0,
       y(x),singsol=all)
```

$$y(x) = -\frac{\text{LambertW}\left(-2\,\mathrm{e}^{5x-4-5c_1}\right)}{2} + 3x - 2$$

Mathematica DSolve solution

Solving time : 4.378 (sec)
Leaf size : 35

```
DSolve[{(3*x-y[x]+1)-(6*x-2*y[x]-3)*D[y[x],x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to -\frac{1}{2}W\left(-e^{5x-1+c_1}\right) + 3x - 2$$
$$y(x) \to 3x - 2$$

2.1.7 problem 7

Solved as first order homogeneous class Maple C ode 70
Solved using Lie symmetry for first order ode 75
Maple step by step solution 80
Maple trace 80
Maple dsolve solution 80
Mathematica DSolve solution 81

Internal problem ID [4083]
Book : Differential equations, Shepley L. Ross, 1964
Section : 2.4, page 55
Problem number : 7
Date solved : `Tuesday, December 17, 2024 at 06:21:38 AM`
CAS classification :
`[[_homogeneous, ‘class C‘], _rational, [_Abel, ‘2nd type‘, ‘class A‘]]`

Solve

$$x - 2y - 3 + (2x + y - 1)\, y' = 0$$

Summary of solutions found

$$\frac{\ln\left(\frac{(y+1)^2+(x-1)^2}{(x-1)^2}\right)}{2} + 2\arctan\left(\frac{y+1}{x-1}\right) = \ln\left(\frac{1}{x-1}\right) + c_1$$

$$y = -ix + i - 1$$

$$y = ix - i - 1$$

Solved as first order homogeneous class Maple C ode

Time used: 0.601 (sec)

Let $Y = y - y_0$ and $X = x - x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{2Y(X) + 2y_0 - x_0 - X + 3}{2x_0 + 2X + Y(X) + y_0 - 1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 1$$

$$y_0 = -1$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{2Y(X) - X}{2X + Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= \frac{2Y - X}{2X + Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = 2Y - X$ and $N = 2X + Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{2u - 1}{u + 2} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{2u(X)-1}{u(X)+2} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{2u(X)-1}{u(X)+2} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)Xu(X) + 2\left(\frac{d}{dX}u(X)\right)X + u(X)^2 + 1 = 0$$

Or

$$X(u(X) + 2)\left(\frac{d}{dX}u(X)\right) + u(X)^2 + 1 = 0$$

Which is now solved as separable in $u(X)$.

The ode $\frac{d}{dX}u(X) = -\frac{u(X)^2+1}{X(u(X)+2)}$ is separable as it can be written as

$$
\begin{aligned}
\frac{d}{dX}u(X) &= -\frac{u(X)^2+1}{X\left(u\left(X\right)+2\right)} \\
&= f(X)g(u)
\end{aligned}
$$

Where

$$
\begin{aligned}
f(X) &= -\frac{1}{X} \\
g(u) &= \frac{u^2+1}{u+2}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(u)}\,du &= \int f(X)\,dX \\
\int \frac{u+2}{u^2+1}\,du &= \int -\frac{1}{X}\,dX \\
\frac{\ln\left(u(X)^2+1\right)}{2} + 2\arctan\left(u(X)\right) &= \ln\left(\frac{1}{X}\right) + c_1
\end{aligned}
$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $\frac{u^2+1}{u+2} = 0$ for $u(X)$ gives

$$
\begin{aligned}
u(X) &= -i \\
u(X) &= i
\end{aligned}
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
\frac{\ln\left(u(X)^2+1\right)}{2} + 2\arctan\left(u(X)\right) = \ln\left(\frac{1}{X}\right) + c_1 \\
u(X) = -i \\
u(X) = i
\end{aligned}
$$

Converting $\frac{\ln\left(u(X)^2+1\right)}{2} + 2\arctan\left(u(X)\right) = \ln\left(\frac{1}{X}\right) + c_1$ back to $Y(X)$ gives

$$
\frac{\ln\left(\frac{Y(X)^2+X^2}{X^2}\right)}{2} + 2\arctan\left(\frac{Y(X)}{X}\right) = \ln\left(\frac{1}{X}\right) + c_1
$$

Converting $u(X) = -i$ back to $Y(X)$ gives

$$Y(X) = -iX$$

Converting $u(X) = i$ back to $Y(X)$ gives

$$Y(X) = iX$$

Using the solution for $Y(X)$

$$\frac{\ln\left(\frac{Y(X)^2+X^2}{X^2}\right)}{2} + 2\arctan\left(\frac{Y(X)}{X}\right) = \ln\left(\frac{1}{X}\right) + c_1 \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - 1$$
$$X = x + 1$$

Then the solution in y becomes using EQ (A)

$$\frac{\ln\left(\frac{(y+1)^2+(x-1)^2}{(x-1)^2}\right)}{2} + 2\arctan\left(\frac{y+1}{x-1}\right) = \ln\left(\frac{1}{x-1}\right) + c_1$$

Using the solution for $Y(X)$

$$Y(X) = -iX \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - 1$$
$$X = x + 1$$

Then the solution in y becomes using EQ (A)

$$y + 1 = -i(x - 1)$$

Using the solution for $Y(X)$

$$Y(X) = iX \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - 1$$
$$X = x + 1$$

Then the solution in y becomes using EQ (A)

$$y + 1 = i(x - 1)$$

Solving for y gives

$$\frac{\ln\left(\frac{(y+1)^2+(x-1)^2}{(x-1)^2}\right)}{2} + 2\arctan\left(\frac{y+1}{x-1}\right) = \ln\left(\frac{1}{x-1}\right) + c_1$$
$$y = -ix + i - 1$$
$$y = ix - i - 1$$

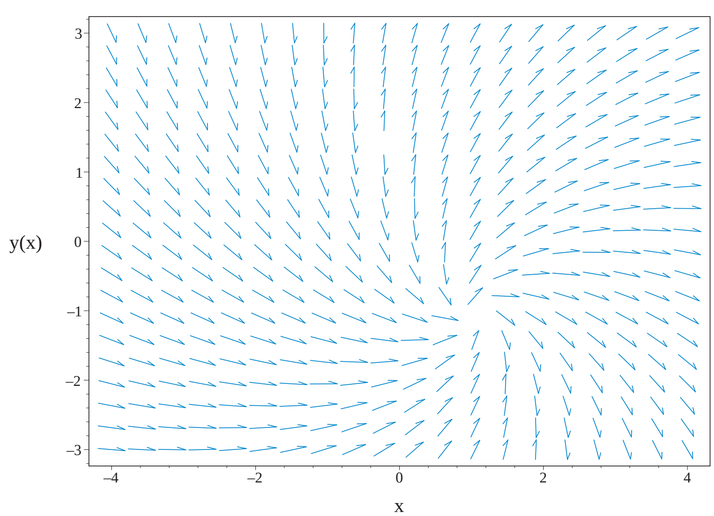


Figure 2.12: Slope field plot
$x - 2y - 3 + (2x + y - 1)\, y' = 0$

Solved using Lie symmetry for first order ode

Time used: 0.669 (sec)

Writing the ode as

$$y' = \frac{2y - x + 3}{2x + y - 1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{(2y - x + 3)(b_3 - a_2)}{2x + y - 1} - \frac{(2y - x + 3)^2 a_3}{(2x + y - 1)^2} - \left(-\frac{1}{2x + y - 1} - \frac{2(2y - x + 3)}{(2x + y - 1)^2}\right)(xa_2 + ya_3 + a_1) - \left(\frac{2}{2x + y - 1} - \frac{2y - x + 3}{(2x + y - 1)^2}\right)(xb_2 + yb_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{2x^2a_2 - x^2a_3 - x^2b_2 - 2x^2b_3 + 2xya_2 + 4xya_3 + 4xyb_2 - 2xyb_3 - 2y^2a_2 + y^2a_3 + y^2b_2 + 2y^2b_3 - 2xa_2 +}{(2x + y -} = 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&2x^2a_2 - x^2a_3 - x^2b_2 - 2x^2b_3 + 2xya_2 + 4xya_3 + 4xyb_2 - 2xyb_3 - 2y^2a_2 \\
&+ y^2a_3 + y^2b_2 + 2y^2b_3 - 2xa_2 + 6xa_3 - 5xb_1 + xb_2 + 7xb_3 + 5ya_1 \\
&- ya_2 - 7ya_3 - 2yb_2 + 6yb_3 + 5a_1 + 3a_2 - 9a_3 + 5b_1 + b_2 - 3b_3 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&2a_2v_1^2 + 2a_2v_1v_2 - 2a_2v_2^2 - a_3v_1^2 + 4a_3v_1v_2 + a_3v_2^2 - b_2v_1^2 + 4b_2v_1v_2 + b_2v_2^2 \\
&- 2b_3v_1^2 - 2b_3v_1v_2 + 2b_3v_2^2 + 5a_1v_2 - 2a_2v_1 - a_2v_2 + 6a_3v_1 - 7a_3v_2 - 5b_1v_1 \\
&+ b_2v_1 - 2b_2v_2 + 7b_3v_1 + 6b_3v_2 + 5a_1 + 3a_2 - 9a_3 + 5b_1 + b_2 - 3b_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(2a_2 - a_3 - b_2 - 2b_3)\, v_1^2 + (2a_2 + 4a_3 + 4b_2 - 2b_3)\, v_1v_2 \\
&+ (-2a_2 + 6a_3 - 5b_1 + b_2 + 7b_3)\, v_1 + (-2a_2 + a_3 + b_2 + 2b_3)\, v_2^2 \\
&+ (5a_1 - a_2 - 7a_3 - 2b_2 + 6b_3)\, v_2 + 5a_1 + 3a_2 - 9a_3 + 5b_1 + b_2 - 3b_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-2a_2 + a_3 + b_2 + 2b_3 &= 0 \\
2a_2 - a_3 - b_2 - 2b_3 &= 0 \\
2a_2 + 4a_3 + 4b_2 - 2b_3 &= 0 \\
5a_1 - a_2 - 7a_3 - 2b_2 + 6b_3 &= 0 \\
-2a_2 + 6a_3 - 5b_1 + b_2 + 7b_3 &= 0 \\
5a_1 + 3a_2 - 9a_3 + 5b_1 + b_2 - 3b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= -b_2 - b_3 \\
a_2 &= b_3 \\
a_3 &= -b_2 \\
b_1 &= -b_2 + b_3 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x - 1 \\
\eta &= y + 1
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y + 1 - \left(\frac{2y - x + 3}{2x + y - 1}\right)(x - 1) \\
&= \frac{x^2 + y^2 - 2x + 2y + 2}{2x + y - 1} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x^2+y^2-2x+2y+2}{2x+y-1}} dy
\end{aligned}
$$

Which results in

$$S = \frac{\ln \left(x^2 + y^2 - 2x + 2y + 2\right)}{2} + 2\arctan\left(\frac{2y+2}{2x-2}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{2y - x + 3}{2x + y - 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{-2y + x - 3}{x^2 + y^2 - 2x + 2y + 2} \\ S_y &= \frac{2x + y - 1}{x^2 + y^2 - 2x + 2y + 2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

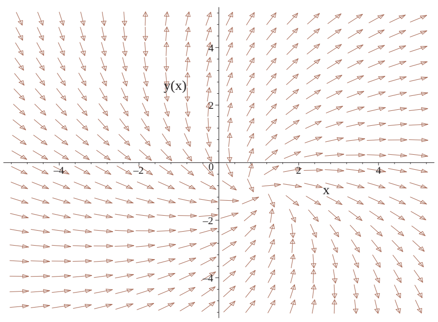
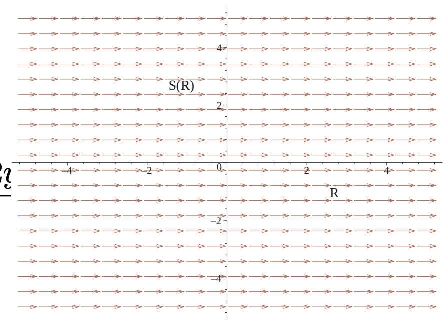
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned} \int dS &= \int 0\, dR + c_2 \\ S(R) &= c_2 \end{aligned}$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\frac{\ln(y^2+x^2+2y-2x+2)}{2}+2\arctan\left(\frac{y+1}{x-1}\right)=c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx}=\frac{2y-x+3}{2x+y-1}$
</td><td>$R = x$
$S = \frac{\ln(x^2+y^2-2x+2y}{2}$</td><td>$\frac{dS}{dR}=0$
</td></tr>
</table>

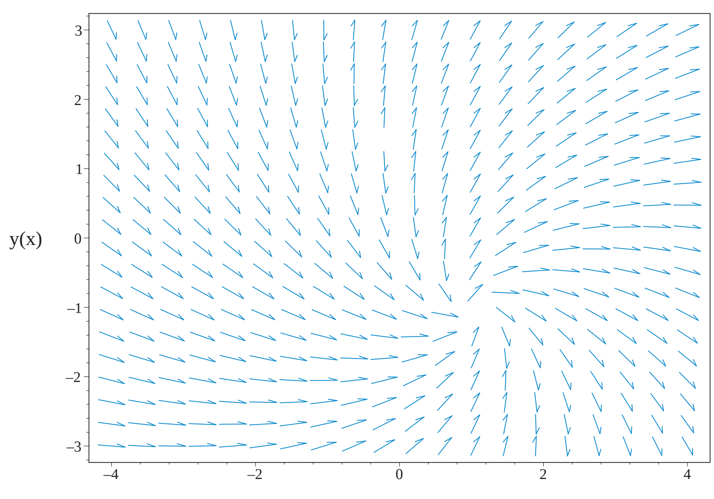


Figure 2.13: Slope field plot
$x-2y-3+(2x+y-1)\,y'=0$

Summary of solutions found

$$\frac{\ln(y^2+x^2+2y-2x+2)}{2}+2\arctan\left(\frac{y+1}{x-1}\right)=c_2$$

Maple step by step solution

Let's solve

$x - 2y(x) - 3 + (2x + y(x) - 1)\left(\frac{d}{dx}y(x)\right) = 0$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = \frac{-x+2y(x)+3}{2x+y(x)-1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

Maple dsolve solution

Solving time : `0.034` (sec)
Leaf size : 31

```
dsolve(x-2*y(x)-3+(2*x+y(x)-1)*diff(y(x),x) = 0,
       y(x),singsol=all)
```

$$y(x) = -1 - \tan\left(\operatorname{RootOf}\left(-4\_Z + \ln\left(\sec\left(\_Z\right)^2\right) + 2\ln\left(x-1\right) + 2c_1\right)\right)(x-1)$$

Mathematica DSolve solution

Solving time : 0.091 (sec)
Leaf size : 66

```
DSolve[{(x-2*y[x]-3)+(2*x+y[x]-1)*D[y[x],x]==0,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$\text{Solve}\left[32\arctan\left(\frac{2y(x)-x+3}{y(x)+2x-1}\right)+8\log\left(\frac{x^2+y(x)^2+2y(x)-2x+2}{5(x-1)^2}\right)+16\log(x-1)+5c_1=0,y(x)\right]$$

2.1.8 problem 8

Existence and uniqueness analysis 82
Solved as first order homogeneous class Maple C ode 83
Solved as first order Exact ode 89
Solved using Lie symmetry for first order ode 92
Solved as first order ode of type dAlembert 98
Maple step by step solution 101
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Mathematica DSolve solution 103

Internal problem ID [4084]
Book : Differential equations, Shepley L. Ross, 1964
Section : 2.4, page 55
Problem number : 8
Date solved : `Tuesday, December 17, 2024 at 06:21:40 AM`
CAS classification :
`[[_homogeneous, 'class C'], _exact, _rational, [_Abel, '2nd type', 'class A']]`

Solve

$$6x + 4y + 1 + (4x + 2y + 2)\, y' = 0$$

With initial conditions

$$y\left(\frac{1}{2}\right) = 3$$

Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{4y + 6x + 1}{2\,(2x + y + 1)} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 3$ is

$$\{x < -2 \lor -2 < x\}$$

And the point $x_0 = \frac{1}{2}$ is inside this domain. The y domain of $f(x, y)$ when $x = \frac{1}{2}$ is

$$\{y < -2 \lor -2 < y\}$$

And the point $y_0 = 3$ is inside this domain. Now we will look at the continuity of

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{4y+6x+1}{2\left(2x+y+1\right)}\right)$$
$$= -\frac{2}{2x+y+1} + \frac{4y+6x+1}{2\left(2x+y+1\right)^2}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 3$ is

$$\{x < -2 \lor -2 < x\}$$

And the point $x_0 = \frac{1}{2}$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = \frac{1}{2}$ is

$$\{y < -2 \lor -2 < y\}$$

And the point $y_0 = 3$ is inside this domain. Therefore solution exists and is unique.

Summary of solutions found

$$y = -2x - 1 + \sqrt{\left(x+\frac{3}{2}\right)^2 + \mathrm{e}^{\ln(3)+\ln(7)}}$$

Solved as first order homogeneous class Maple C ode

Time used: 1.074 (sec)

Let $Y = y - y_0$ and $X = x - x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{4Y(X)+4y_0+6x_0+6X+1}{2\left(2x_0+2X+Y\left(X\right)+y_0+1\right)}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = -\frac{3}{2}$$
$$y_0 = 2$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{4Y(X)+6X}{2\left(2X+Y\left(X\right)\right)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= -\frac{2Y+3X}{2X+Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = -2Y - 3X$ and $N = 2X + Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{-2u-3}{u+2} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{-2u(X)-3}{u(X)+2} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{-2u(X)-3}{u(X)+2} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right) Xu(X) + 2\left(\frac{d}{dX}u(X)\right) X + u(X)^2 + 4u(X) + 3 = 0$$

Or

$$X(u(X)+2)\left(\frac{d}{dX}u(X)\right) + u(X)^2 + 4u(X) + 3 = 0$$

Which is now solved as separable in $u(X)$.

The ode $\frac{d}{dX}u(X) = -\frac{u(X)^2+4u(X)+3}{X(u(X)+2)}$ is separable as it can be written as

$$\begin{aligned} \frac{d}{dX}u(X) &= -\frac{u(X)^2 + 4u(X) + 3}{X\,(u\,(X)+2)} \\ &= f(X)g(u) \end{aligned}$$

Where

$$
\begin{aligned}
f(X) &= -\frac{1}{X} \\
g(u) &= \frac{u^2+4u+3}{u+2}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(u)}\, du &= \int f(X)\, dX \\
\int \frac{u+2}{u^2+4u+3}\, du &= \int -\frac{1}{X}\, dX \\
\frac{\ln\left(u(X)^2+4u(X)+3\right)}{2} &= \ln\left(\frac{1}{X}\right) + c_1
\end{aligned}
$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $\frac{u^2+4u+3}{u+2} = 0$ for $u(X)$ gives

$$
\begin{aligned}
u(X) &= -3 \\
u(X) &= -1
\end{aligned}
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
\frac{\ln\left(u(X)^2+4u(X)+3\right)}{2} &= \ln\left(\frac{1}{X}\right) + c_1 \\
u(X) &= -3 \\
u(X) &= -1
\end{aligned}
$$

Solving for $u(X)$ gives

$$
\begin{aligned}
u(X) &= -3 \\
u(X) &= -1 \\
u(X) &= \frac{-2X - \sqrt{X^2+e^{2c_1}}}{X} \\
u(X) &= \frac{-2X + \sqrt{X^2+e^{2c_1}}}{X}
\end{aligned}
$$

Converting $u(X) = -3$ back to $Y(X)$ gives

$$Y(X) = -3X$$

Converting $u(X) = -1$ back to $Y(X)$ gives

$$Y(X) = -X$$

Converting $u(X) = \frac{-2X-\sqrt{X^2+e^{2c_1}}}{X}$ back to $Y(X)$ gives

$$Y(X) = -2X - \sqrt{X^2 + e^{2c_1}}$$

Converting $u(X) = \frac{-2X+\sqrt{X^2+e^{2c_1}}}{X}$ back to $Y(X)$ gives

$$Y(X) = -2X + \sqrt{X^2 + e^{2c_1}}$$

Using the solution for $Y(X)$

$$Y(X) = -3X \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y + 2$$
$$X = x - \frac{3}{2}$$

Then the solution in y becomes using EQ (A)

$$y - 2 = -3x - \frac{9}{2}$$

Using the solution for $Y(X)$

$$Y(X) = -X \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y + 2$$
$$X = x - \frac{3}{2}$$

Then the solution in y becomes using EQ (A)

$$y - 2 = -x - \frac{3}{2}$$

Using the solution for $Y(X)$

$$Y(X) = -2X - \sqrt{X^2 + e^{2c_1}} \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y + 2$$
$$X = x - \frac{3}{2}$$

Then the solution in y becomes using EQ (A)

$$y - 2 = -2x - 3 - \sqrt{\left(x + \frac{3}{2}\right)^2 + e^{2c_1}}$$

Using the solution for $Y(X)$

$$Y(X) = -2X + \sqrt{X^2 + e^{2c_1}} \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y + 2$$
$$X = x - \frac{3}{2}$$

Then the solution in y becomes using EQ (A)

$$y-2=-2x-3+\sqrt{\left(x+\frac{3}{2}\right)^{2}+e^{2c_1}}$$

Solving for the constant of integration from initial conditions, the solution becomes

$$y-2=-2x-3-\sqrt{\left(x+\frac{3}{2}\right)^{2}+21}$$

Solving for the constant of integration from initial conditions, the solution becomes

$$y-2=-2x-3+\sqrt{\left(x+\frac{3}{2}\right)^{2}+e^{\ln(3)+\ln(7)}}$$

Solving for y gives

$$\begin{aligned}
y-2&=-3x-\frac{9}{2}\\
y-2&=-x-\frac{3}{2}\\
y-2&=-2x-3-\sqrt{\left(x+\frac{3}{2}\right)^{2}+21}\\
y&=-2x-1+\sqrt{\left(x+\frac{3}{2}\right)^{2}+e^{\ln(3)+\ln(7)}}
\end{aligned}$$

The solution

$$y-2=-3x-\frac{9}{2}$$

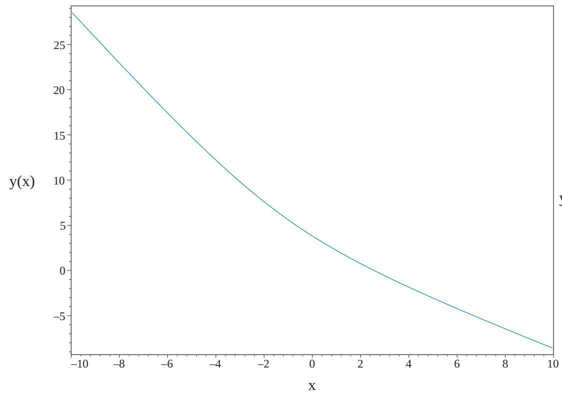
was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$y-2=-x-\frac{3}{2}$$

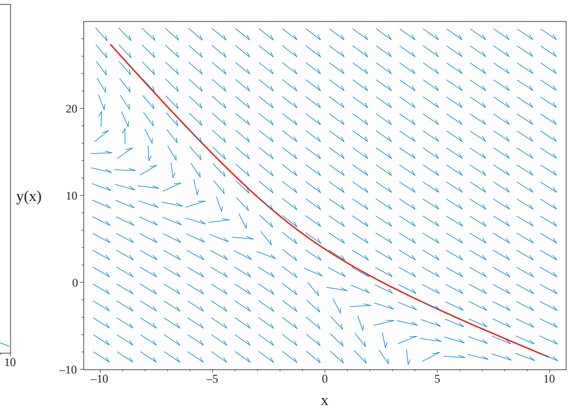
was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$y-2=-2x-3-\sqrt{\left(x+\frac{3}{2}\right)^{2}+21}$$

was found not to satisfy the ode or the IC. Hence it is removed.



(a) Solution plot
$y = -2x - 1 + \sqrt{\left(x + \frac{3}{2}\right)^2 + e^{\ln(3)+\ln(7)}}$



(b) Slope field plot
$6x + 4y + 1 + (4x + 2y + 2)\, y' = 0$

Solved as first order Exact ode

Time used: 0.110 (sec)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(4x + 2y + 2)\,\mathrm{d}y &= (-4y - 6x - 1)\,\mathrm{d}x\\ (4y + 6x + 1)\,\mathrm{d}x + (4x + 2y + 2)\,\mathrm{d}y &= 0 \qquad (2\mathrm{A})\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 4y + 6x + 1\\ N(x, y) &= 4x + 2y + 2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(4y + 6x + 1)\\ &= 4\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(4x + 2y + 2)\\ &= 4\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int M\, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int 4y + 6x + 1\, \mathrm{d}x$$

$$\phi = x(3x + 4y + 1) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 4x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 4x + 2y + 2$. Therefore equation (4) becomes

$$4x + 2y + 2 = 4x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2y + 2$$

Integrating the above w.r.t y gives

$$\int f'(y)\, \mathrm{d}y = \int (2y + 2)\, \mathrm{d}y$$

$$f(y) = y^2 + 2y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x(3x + 4y + 1) + y^2 + 2y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into the constant c_1 gives the solution as

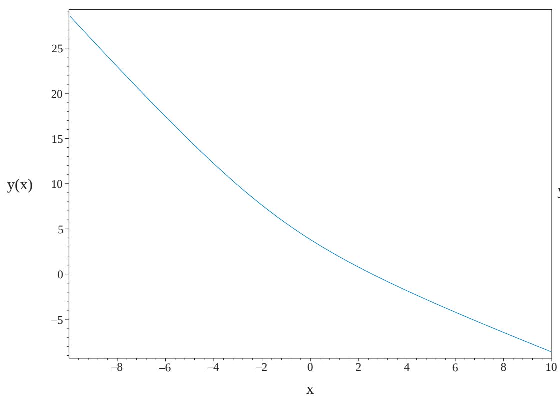
$$c_1 = x(3x + 4y + 1) + y^2 + 2y$$

Solving for the constant of integration from initial conditions, the solution becomes

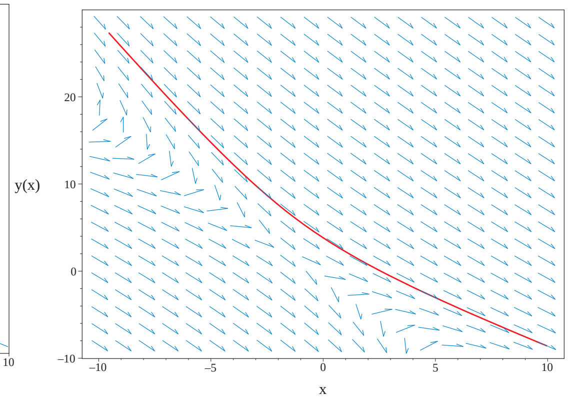
$$x(3x+4y+1)+y^2+2y=\frac{89}{4}$$

Solving for y gives

$$y=-2x-1+\frac{\sqrt{4x^2+12x+93}}{2}$$



(a) Solution plot
$y=-2x-1+\frac{\sqrt{4x^2+12x+93}}{2}$



(b) Slope field plot
$6x+4y+1+(4x+2y+2)\,y'=0$

Summary of solutions found

$$y=-2x-1+\frac{\sqrt{4x^2+12x+93}}{2}$$

Solved using Lie symmetry for first order ode

Time used: 0.966 (sec)

Writing the ode as

$$y'=-\frac{4y+6x+1}{2\,(2x+y+1)}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 - \frac{(4y+6x+1)(b_3-a_2)}{2(2x+y+1)} - \frac{(4y+6x+1)^2 a_3}{4(2x+y+1)^2} \\ - \left(-\frac{3}{2x+y+1} + \frac{4y+6x+1}{(2x+y+1)^2}\right)(xa_2+ya_3+a_1) \\ - \left(-\frac{2}{2x+y+1} + \frac{4y+6x+1}{2(2x+y+1)^2}\right)(xb_2+yb_3+b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{24x^2a_2 - 36x^2a_3 + 20x^2b_2 - 24x^2b_3 + 24xya_2 - 48xya_3 + 16xyb_2 - 24xyb_3 + 8y^2a_2 - 20y^2a_3 + 4y^2b_2 - 8}{4(2}$$

$$= 0$$

Setting the numerator to zero gives

$$\begin{aligned} 24x^2a_2 - 36x^2a_3 + 20x^2b_2 - 24x^2b_3 + 24xya_2 - 48xya_3 + 16xyb_2 - 24xyb_3 \\ + 8y^2a_2 - 20y^2a_3 + 4y^2b_2 - 8y^2b_3 + 24xa_2 - 12xa_3 + 4xb_1 + 22xb_2 - 16xb_3 \\ - 4ya_1 + 10ya_2 + 8yb_2 - 4yb_3 + 8a_1 + 2a_2 - a_3 + 6b_1 + 4b_2 - 2b_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}24a_2v_1^2+24a_2v_1v_2+8a_2v_2^2-36a_3v_1^2-48a_3v_1v_2-20a_3v_2^2+20b_2v_1^2+16b_2v_1v_2\\+4b_2v_2^2-24b_3v_1^2-24b_3v_1v_2-8b_3v_2^2-4a_1v_2+24a_2v_1+10a_2v_2-12a_3v_1+4b_1v_1\\+22b_2v_1+8b_2v_2-16b_3v_1-4b_3v_2+8a_1+2a_2-a_3+6b_1+4b_2-2b_3=0\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}(24a_2-36a_3+20b_2-24b_3)\,v_1^2+(24a_2-48a_3+16b_2-24b_3)\,v_1v_2\\+(24a_2-12a_3+4b_1+22b_2-16b_3)\,v_1+(8a_2-20a_3+4b_2-8b_3)\,v_2^2\\+(-4a_1+10a_2+8b_2-4b_3)\,v_2+8a_1+2a_2-a_3+6b_1+4b_2-2b_3=0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}-4a_1+10a_2+8b_2-4b_3&=0\\8a_2-20a_3+4b_2-8b_3&=0\\24a_2-48a_3+16b_2-24b_3&=0\\24a_2-36a_3+20b_2-24b_3&=0\\24a_2-12a_3+4b_1+22b_2-16b_3&=0\\8a_1+2a_2-a_3+6b_1+4b_2-2b_3&=0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1&=4a_3+\frac{3b_3}{2}\\a_2&=4a_3+b_3\\a_3&=a_3\\b_1&=-\frac{9a_3}{2}-2b_3\\b_2&=-3a_3\\b_3&=b_3\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi&=x+\frac{3}{2}\\\eta&=y-2\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= y - 2 - \left(-\frac{4y+6x+1}{2\,(2x+y+1)}\right)\left(x+\frac{3}{2}\right) \\ &= \frac{12x^2+16xy+4y^2+4x+8y-5}{8x+4y+4} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta}dy \\ &= \int \frac{1}{\frac{12x^2+16xy+4y^2+4x+8y-5}{8x+4y+4}}dy\end{aligned}$$

Which results in

$$S = \frac{\ln\left(12x^2+16xy+4y^2+4x+8y-5\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{4y+6x+1}{2\,(2x+y+1)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{1}{2x+2y-1} + \frac{3}{6x+2y+5} \\
S_y &= \frac{1}{2x+2y-1} + \frac{1}{6x+2y+5}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

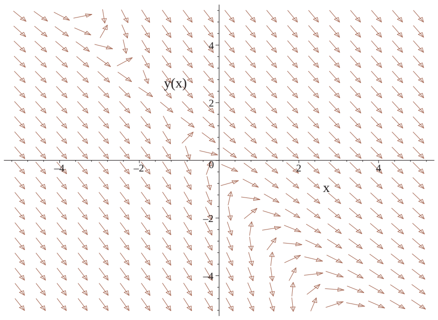
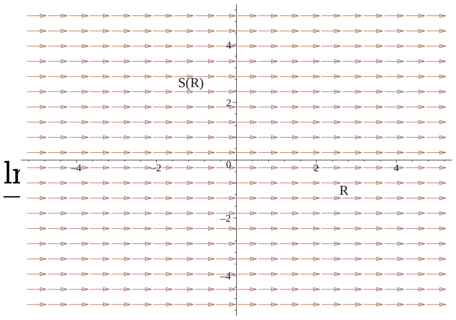
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$
\begin{aligned}
\int dS &= \int 0\, dR + c_2 \\
S(R) &= c_2
\end{aligned}
$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\frac{\ln(2x+2y-1)}{2} + \frac{\ln(6x+2y+5)}{2} = c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

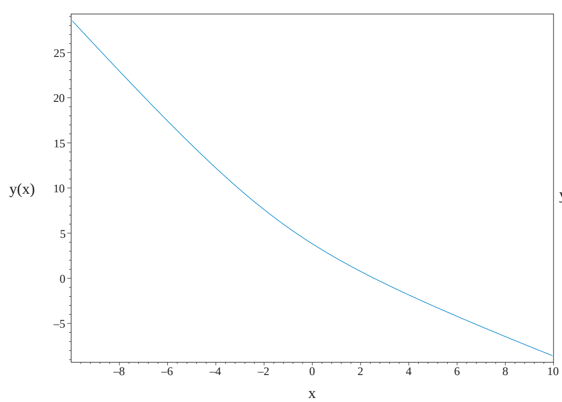
Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{4y+6x+1}{2(2x+y+1)}$ 	$R = x$ $S = \frac{\ln(2x+2y-1)}{2} + \ln$	$\frac{dS}{dR} = 0$ 

Solving for the constant of integration from initial conditions, the solution becomes

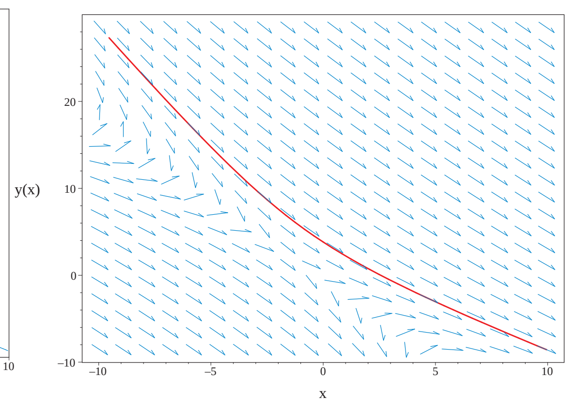
$$\frac{\ln(2x+2y-1)}{2} + \frac{\ln(6x+2y+5)}{2} = \ln(2) + \frac{\ln(3)}{2} + \frac{\ln(7)}{2}$$

Solving for y gives

$$y = -2x - 1 + \frac{\sqrt{4x^2+12x+93}}{2}$$



(a) Solution plot
$y = -2x - 1 + \frac{\sqrt{4x^2+12x+93}}{2}$



(b) Slope field plot
$6x + 4y + 1 + (4x + 2y + 2)\, y' = 0$

Summary of solutions found

$$y = -2x - 1 + \frac{\sqrt{4x^2+12x+93}}{2}$$

Solved as first order ode of type dAlembert

Time used: 0.613 (sec)

Let $p = y'$ the ode becomes

$$6x + 4y + 1 + (4x + 2y + 2)\,p = 0$$

Solving for y from the above results in

$$y = -\frac{(4p + 6)\,x}{2\,(2 + p)} - \frac{2p + 1}{2\,(2 + p)} \tag{1}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$\begin{aligned} f &= \frac{-2p - 3}{2 + p} \\ g &= \frac{-2p - 1}{4 + 2p} \end{aligned}$$

Hence (2) becomes

$$p - \frac{-2p - 3}{2 + p} = \left(-\frac{2x}{2 + p} + \frac{2xp}{(2 + p)^2} + \frac{3x}{(2 + p)^2} - \frac{2}{4 + 2p} + \frac{4p}{(4 + 2p)^2} + \frac{2}{(4 + 2p)^2}\right) p'(x) \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$p - \frac{-2p - 3}{2 + p} = 0$$

No valid singular solutions found.

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$p'(x) = \frac{p(x) - \frac{-2p(x)-3}{2+p(x)}}{-\frac{2x}{2+p(x)} + \frac{2xp(x)}{(2+p(x))^2} + \frac{3x}{(2+p(x))^2} - \frac{2}{4+2p(x)} + \frac{4p(x)}{(4+2p(x))^2} + \frac{2}{(4+2p(x))^2}} \tag{3}$$

This ODE is now solved for $p(x)$. No inversion is needed. The ode $p'(x) = -\frac{2(2+p(x))(p(x)+3)(p(x)+1)}{2x+3}$ is separable as it can be written as

$$\begin{aligned} p'(x) &= -\frac{2(2+p(x))\,(p(x)+3)\,(p(x)+1)}{2x+3} \\ &= f(x)g(p) \end{aligned}$$

Where

$$\begin{aligned} f(x) &= -\frac{2}{2x+3} \\ g(p) &= (2+p)\,(p+3)\,(p+1) \end{aligned}$$

Integrating gives

$$\begin{aligned} \int \frac{1}{g(p)}\,dp &= \int f(x)\,dx \\ \int \frac{1}{(2+p)\,(p+3)\,(p+1)}\,dp &= \int -\frac{2}{2x+3}\,dx \\ \ln\left(\frac{\sqrt{p(x)+3}\,\sqrt{p(x)+1}}{2+p(x)}\right) &= \ln\left(\frac{1}{2x+3}\right) + c_1 \end{aligned}$$

We now need to find the singular solutions, these are found by finding for what values $g(p)$ is zero, since we had to divide by this above. Solving $g(p) = 0$ or $(2+p)\,(p+3)\,(p+1) = 0$ for $p(x)$ gives

$$\begin{aligned} p(x) &= -3 \\ p(x) &= -2 \\ p(x) &= -1 \end{aligned}$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$\begin{aligned} \ln\left(\frac{\sqrt{p(x)+3}\,\sqrt{p(x)+1}}{2+p(x)}\right) &= \ln\left(\frac{1}{2x+3}\right) + c_1 \\ p(x) &= -3 \\ p(x) &= -2 \\ p(x) &= -1 \end{aligned}$$

Substituing the above solution for p in (2A) gives

$$y = \frac{x\left(\frac{-8x^2-24x+2\,e^{2c_1}-18+2\sqrt{16x^4-4\,e^{2c_1}x^2+96x^3-12\,e^{2c_1}x+216x^2-9\,e^{2c_1}+216x+81}}{-4x^2+e^{2c_1}-12x-9}-1\right)}{-\frac{-4x^2+\sqrt{16x^4-4\,e^{2c_1}x^2+96x^3-12\,e^{2c_1}x+216x^2-9\,e^{2c_1}+216x+81}+e^{2c_1}-12x-9}{-4x^2+e^{2c_1}-12x-9}+1} + \frac{-8x^2-24x+2\,e^{2c_1}-18+2\sqrt{16x^4-4 \cdots}}{2-\frac{2\left(-4x^2+\sqrt{16x^4-4\,e^{2c_1}x^2+9\cdots}\right.}{\cdots}}$$

Solving for the constant of integration from initial conditions, the solution becomes

$$y = \frac{x\left(\frac{-8x^2+2\sqrt{16x^4+4\,e^{2\ln(2)+\ln(3)+\ln(7)}x^2+96x^3+12\,e^{2\ln(2)+\ln(3)+\ln(7)}x+216x^2+9\,e^{2\ln(2)+\ln(3)+\ln(7)}+216x+81}-2\,e^{2\ln(2)+\ln(3)+\ln(7)}-\cdots}{-4x^2-e^{2\ln(2)+\ln(3)+\ln(7)}-12x-9}\right.}{-\frac{-4x^2+\sqrt{16x^4+4\,e^{2\ln(2)+\ln(3)+\ln(7)}x^2+96x^3+12\,e^{2\ln(2)+\ln(3)+\ln(7)}x+216x^2+9\,e^{2\ln(2)+\ln(3)+\ln(7)}+216x+81}-e^{2\ln(2)+\ln(3)+\ln(7)}-\cdots}{-4x^2-e^{2\ln(2)+\ln(3)+\ln(7)}-12x-9}}$$

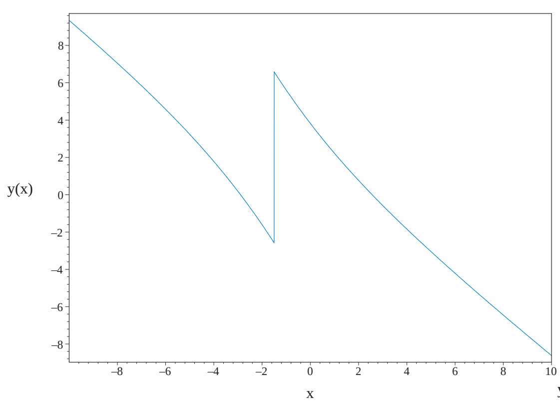
The solution

$$y = -3x - \frac{5}{2}$$

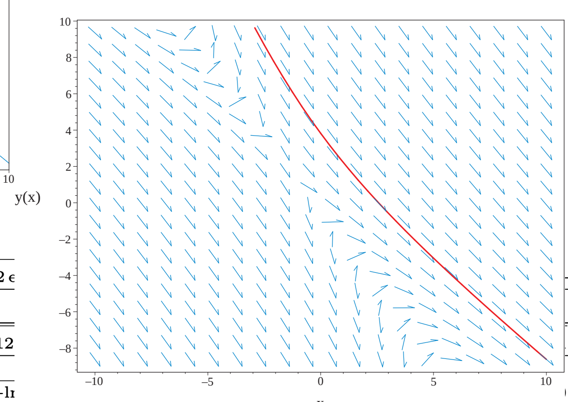
was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$y = -x + \frac{1}{2}$$

was found not to satisfy the ode or the IC. Hence it is removed.



(a) Solution plot



(b) Slope field plot
$6x + 4y + 1 + (4x + 2y + 2)\,y' = 0$

$$y = \frac{x\left(\frac{-8x^2+2\sqrt{16x^4+4\,e^{2\ln(2)+\ln(3)+\ln(7)}x^2+96x^3+12\,e^{\cdots}}\cdots-2\,e^{2\ln(2)+\ln(3)+\ln(7)}-24x-\cdots}{\cdots}\right.}{-\frac{-4x^2+\sqrt{16x^4+4\,e^{2\ln(2)+\ln(3)+\ln(7)}x^2+96x^3+12\cdots}\cdots-e^{2\ln(2)+\ln(3)+\ln(7)}-12x-9}{\cdots}}$$

$$\frac{\frac{-8x^2+2\sqrt{16x^4+4\,e^{2\ln(2)+\ln(3)+\ln(7)}x^2+96x^3+12\,e^{2\ln(2)+\ln\cdots}}\cdots+\ln(3)+\ln(7)-24x-18}{-4x^2-e^{2\ln(2)+\ln(3)+\ln(7)}-12x-9}+1}{2-\frac{2\left(-4x^2+\sqrt{16x^4+4\,e^{2\ln(2)+\ln(3)+\ln(7)}x^2+96x^3+12\,e^{2\ln(2)+\ln(3)+\ln(7)}x+216x^2+9\,e^{2\ln(2)+\ln(3)+\ln(7)}+216x+81}-e^{2\ln(2)+\ln(3)+\ln(7)}-12x-9\right)}{-4x^2-e^{2\ln(3)+\ln(3)+\ln(7)}-12x-9}}$$

Summary of solutions found

$$y = \frac{x\left(\frac{-8x^2+2\sqrt{16x^4+4\,\mathrm{e}^{2\ln(2)+\ln(3)+\ln(7)}x^2+96x^3+12\,\mathrm{e}^{2\ln(2)+\ln(3)+\ln(7)}x+216x^2+9\,\mathrm{e}^{2\ln(2)+\ln(3)+\ln(7)}+216x+81}-2\,\mathrm{e}^{2\ln(2)+\ln(3)+\ln(7)}-24\cdots}{-4x^2-\mathrm{e}^{2\ln(2)+\ln(3)+\ln(7)}-12x-9}\cdots\right.}{-\frac{-4x^2+\sqrt{16x^4+4\,\mathrm{e}^{2\ln(2)+\ln(3)+\ln(7)}x^2+96x^3+12\,\mathrm{e}^{2\ln(2)+\ln(3)+\ln(7)}x+216x^2+9\,\mathrm{e}^{2\ln(2)+\ln(3)+\ln(7)}+216x+81}-\mathrm{e}^{2\ln(2)+\ln(3)+\ln(7)}-12\cdots}{-4x^2-\mathrm{e}^{2\ln(2)+\ln(3)+\ln(7)}-12x-9}\cdots} + \frac{\frac{-8x^2+2\sqrt{16x^4+4\,\mathrm{e}^{2\ln(2)+\ln(3)+\ln(7)}x^2+96x^3+12\,\mathrm{e}^{2\ln(2)+\ln(3)+\ln(7)}x+216x^2+9\,\mathrm{e}^{2\ln(2)+\ln(3)+\ln(7)}+216x+81}-2\,\mathrm{e}^{2\ln(2)+\ln(3)+\ln(7)}-24\cdots}{-4x^2-\mathrm{e}^{2\ln(2)+\ln(3)+\ln(7)}-12x-9}\cdots}{2-\frac{2\left(-4x^2+\sqrt{16x^4+4\,\mathrm{e}^{2\ln(2)+\ln(3)+\ln(7)}x^2+96x^3+12\,\mathrm{e}^{2\ln(2)+\ln(3)+\ln(7)}x+216x^2+9\,\mathrm{e}^{2\ln(2)+\ln(3)+\ln(7)}+216x+81}-\mathrm{e}^{2\ln(2)+\ln(3)+\ln(7)}\cdots\right.}{-4x^2-\mathrm{e}^{2\ln(2)+\ln(3)+\ln(7)}-12x-9}\cdots}$$

Maple step by step solution

Let's solve

$\left[6x + 4y(x) + 1 + (4x + 2y(x) + 2)\left(\frac{d}{dx}y(x)\right) = 0, y\left(\frac{1}{2}\right) = 3\right]$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function
    $\frac{d}{dx}F(x, y(x)) = 0$
  - Compute derivative of lhs
    $\frac{\partial}{\partial x}F(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right)\left(\frac{d}{dx}y(x)\right) = 0$
  - Evaluate derivatives
    $4 = 4$
  - Condition met, ODE is exact
- Exact ODE implies solution will be of this form
  $\left[F(x, y) = C1, M(x, y) = \frac{\partial}{\partial x}F(x, y), N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$
- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x
  $F(x, y) = \int (6x + 4y + 1)\, dx + \_F1(y)$
- Evaluate integral
  $F(x, y) = 3x^2 + 4xy + x + \_F1(y)$
- Take derivative of $F(x, y)$ with respect to y
  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$
- Compute derivative
  $4x + 2y + 2 = 4x + \frac{d}{dy}\_F1(y)$
- Isolate for $\frac{d}{dy}\_F1(y)$

$\frac{d}{dy}$_$F1(y) = 2y + 2$

- Solve for _$F1(y)$

  _$F1(y) = y^2 + 2y$

- Substitute _$F1(y)$ into equation for $F(x, y)$

  $F(x, y) = 3x^2 + 4xy + y^2 + x + 2y$

- Substitute $F(x, y)$ into the solution of the ODE

  $3x^2 + 4xy + y^2 + x + 2y = C1$

- Solve for $y(x)$

  $\left\{y(x) = -1 - 2x - \sqrt{x^2 + C1 + 3x + 1}, y(x) = -1 - 2x + \sqrt{x^2 + C1 + 3x + 1}\right\}$

- Use initial condition $y\left(\frac{1}{2}\right) = 3$

  $3 = -2 - \sqrt{C1 + \frac{11}{4}}$

- Solve for _$C1$

  $C1 = ()$

- Solution does not satisfy initial condition
- Use initial condition $y\left(\frac{1}{2}\right) = 3$

  $3 = -2 + \sqrt{C1 + \frac{11}{4}}$

- Solve for _$C1$

  $C1 = \frac{89}{4}$

- Substitute _$C1 = \frac{89}{4}$ into general solution and simplify

  $y(x) = -1 - 2x + \frac{\sqrt{4x^2+12x+93}}{2}$

- Solution to the IVP

  $y(x) = -1 - 2x + \frac{\sqrt{4x^2+12x+93}}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
```

```
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

Maple dsolve solution

Solving time : 0.211 (sec)
Leaf size : 23

```
dsolve([6*x+4*y(x)+1+(4*x+2*y(x)+2)*diff(y(x),x) = 0,
       op([y(1/2) = 3])],y(x),singsol=all)
```

$$y(x) = -2x - 1 + \frac{\sqrt{4x^2 + 12x + 93}}{2}$$

Mathematica DSolve solution

Solving time : 0.189 (sec)
Leaf size : 28

```
DSolve[{(6*x+4*y[x]+1)+(4*x+2*y[x]+2)*D[y[x],x]==0,y[1/2]==3},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}\left(\sqrt{4x^2 + 12x + 93} - 4x - 2\right)$$

2.1.9 problem 9

Existence and uniqueness analysis 104
Solved as first order homogeneous class Maple C ode 105
Solved using Lie symmetry for first order ode 110
Maple step by step solution 115
Maple trace 116
Maple dsolve solution 116
Mathematica DSolve solution 117

Internal problem ID [4085]
Book : Differential equations, Shepley L. Ross, 1964
Section : 2.4, page 55
Problem number : 9
Date solved : `Tuesday, December 17, 2024 at 06:21:44 AM`
CAS classification :
`[[_homogeneous, ‘class C‘], _rational, [_Abel, ‘2nd type‘, ‘class A‘]]`

Solve

$$3x - y - 6 + (x + y + 2)\, y' = 0$$

With initial conditions

$$y(2) = -2$$

Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{y - 3x + 6}{x + y + 2} \end{aligned}$$

The x domain of $f(x,y)$ when $y = -2$ is

$$\{x < 0 \vee 0 < x\}$$

And the point $x_0 = 2$ is inside this domain. The y domain of $f(x,y)$ when $x = 2$ is

$$\{y < -4 \vee -4 < y\}$$

And the point $y_0 = -2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{y - 3x + 6}{x + y + 2}\right)\\ &= \frac{1}{x + y + 2} - \frac{y - 3x + 6}{(x + y + 2)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = -2$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 2$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 2$ is

$$\{y < -4 \lor -4 < y\}$$

And the point $y_0 = -2$ is inside this domain. Therefore solution exists and is unique.

Summary of solutions found

$$\frac{\sqrt{3}\left(\sqrt{3}\,\ln\left(\frac{(y+3)^2+3(x-1)^2}{(x-1)^2}\right) + 2\arctan\left(\frac{(y+3)\sqrt{3}}{3x-3}\right)\right)}{6} = \ln\left(\frac{1}{x-1}\right) + \frac{\sqrt{3}\,(6\sqrt{3}\,\ln(2) + \pi)}{18}$$

Solved as first order homogeneous class Maple C ode

Time used: 1.641 (sec)

Let $Y = y - y_0$ and $X = x - x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{Y(X) + y_0 - 3x_0 - 3X + 6}{x_0 + X + Y(X) + y_0 + 2}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$\begin{aligned}x_0 &= 1\\ y_0 &= -3\end{aligned}$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{Y(X) - 3X}{X + Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= \frac{Y-3X}{X+Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = Y - 3X$ and $N = X + Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{u-3}{u+1} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{u(X)-3}{u(X)+1} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{u(X)-3}{u(X)+1} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right) Xu(X) + \left(\frac{d}{dX}u(X)\right) X + u(X)^2 + 3 = 0$$

Or

$$X(u(X)+1)\left(\frac{d}{dX}u(X)\right) + u(X)^2 + 3 = 0$$

Which is now solved as separable in $u(X)$.

The ode $\frac{d}{dX}u(X) = -\frac{u(X)^2+3}{X(u(X)+1)}$ is separable as it can be written as

$$\begin{aligned} \frac{d}{dX}u(X) &= -\frac{u(X)^2+3}{X\left(u\left(X\right)+1\right)} \\ &= f(X)g(u) \end{aligned}$$

Where

$$
\begin{aligned}
f(X) &= -\frac{1}{X} \\
g(u) &= \frac{u^2+3}{u+1}
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(u)}\, du &= \int f(X)\, dX \\
\int \frac{u+1}{u^2+3}\, du &= \int -\frac{1}{X}\, dX \\
\frac{\ln\left(u(X)^2+3\right)}{2} + \frac{\sqrt{3}\, \arctan\left(\frac{u(X)\sqrt{3}}{3}\right)}{3} &= \ln\left(\frac{1}{X}\right) + c_1
\end{aligned}
$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $\frac{u^2+3}{u+1} = 0$ for $u(X)$ gives

$$
\begin{aligned}
u(X) &= -i\sqrt{3} \\
u(X) &= i\sqrt{3}
\end{aligned}
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
\frac{\ln\left(u(X)^2+3\right)}{2} + \frac{\sqrt{3}\, \arctan\left(\frac{u(X)\sqrt{3}}{3}\right)}{3} &= \ln\left(\frac{1}{X}\right) + c_1 \\
u(X) &= -i\sqrt{3} \\
u(X) &= i\sqrt{3}
\end{aligned}
$$

Converting $\frac{\ln\left(u(X)^2+3\right)}{2} + \frac{\sqrt{3}\, \arctan\left(\frac{u(X)\sqrt{3}}{3}\right)}{3} = \ln\left(\frac{1}{X}\right) + c_1$ back to $Y(X)$ gives

$$
\frac{\sqrt{3}\left(\sqrt{3}\, \ln\left(\frac{Y(X)^2+3X^2}{X^2}\right) + 2\arctan\left(\frac{Y(X)\sqrt{3}}{3X}\right)\right)}{6} = \ln\left(\frac{1}{X}\right) + c_1
$$

Converting $u(X) = -i\sqrt{3}$ back to $Y(X)$ gives

$$
Y(X) = -iX\sqrt{3}
$$

Converting $u(X) = i\sqrt{3}$ back to $Y(X)$ gives

$$Y(X) = iX\sqrt{3}$$

Using the solution for $Y(X)$

$$\frac{\sqrt{3}\left(\sqrt{3}\,\ln\left(\frac{Y(X)^2+3X^2}{X^2}\right)+2\arctan\left(\frac{Y(X)\sqrt{3}}{3X}\right)\right)}{6} = \ln\left(\frac{1}{X}\right)+c_1 \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - 3$$
$$X = x + 1$$

Then the solution in y becomes using EQ (A)

$$\frac{\sqrt{3}\left(\sqrt{3}\,\ln\left(\frac{(y+3)^2+3(x-1)^2}{(x-1)^2}\right)+2\arctan\left(\frac{(y+3)\sqrt{3}}{3x-3}\right)\right)}{6} = \ln\left(\frac{1}{x-1}\right)+c_1$$

Using the solution for $Y(X)$

$$Y(X) = -iX\sqrt{3} \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - 3$$
$$X = x + 1$$

Then the solution in y becomes using EQ (A)

$$y + 3 = -i(x-1)\sqrt{3}$$

Using the solution for $Y(X)$

$$Y(X) = iX\sqrt{3} \tag{A}$$

And replacing back terms in the above solution using

$$Y = y + y_0$$
$$X = x + x_0$$

Or

$$Y = y - 3$$
$$X = x + 1$$

Then the solution in y becomes using EQ (A)

$$y + 3 = i(x - 1)\sqrt{3}$$

Solving for the constant of integration from initial conditions, the solution becomes

$$\frac{\sqrt{3}\left(\sqrt{3}\ln\left(\frac{(y+3)^2+3(x-1)^2}{(x-1)^2}\right)+2\arctan\left(\frac{(y+3)\sqrt{3}}{3x-3}\right)\right)}{6} = \ln\left(\frac{1}{x-1}\right) + \frac{\sqrt{3}\left(6\sqrt{3}\ln(2)+\pi\right)}{18}$$

The solution

$$y + 3 = -i(x - 1)\sqrt{3}$$

was found not to satisfy the ode or the IC. Hence it is removed. The solution

$$y + 3 = i(x - 1)\sqrt{3}$$

was found not to satisfy the ode or the IC. Hence it is removed.

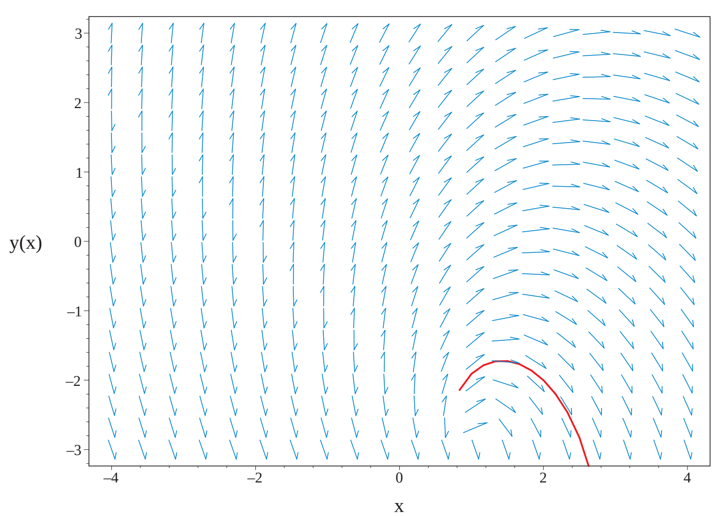


Figure 2.18: Slope field plot
$3x - y - 6 + (x + y + 2)\,y' = 0$

Solved using Lie symmetry for first order ode

Time used: 6.339 (sec)

Writing the ode as

$$y' = \frac{y - 3x + 6}{x + y + 2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{(y - 3x + 6)(b_3 - a_2)}{x + y + 2} - \frac{(y - 3x + 6)^2 a_3}{(x + y + 2)^2} - \left(-\frac{3}{x + y + 2} - \frac{y - 3x + 6}{(x + y + 2)^2}\right)(x a_2 + y a_3 + a_1) - \left(\frac{1}{x + y + 2} - \frac{y - 3x + 6}{(x + y + 2)^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{3x^2 a_2 - 9x^2 a_3 - 3x^2 b_2 - 3x^2 b_3 + 6xy a_2 + 6xy a_3 + 2xy b_2 - 6xy b_3 - y^2 a_2 + 3y^2 a_3 + y^2 b_2 + y^2 b_3 + 12x a_2}{(x + y + 2)} = 0$$

Setting the numerator to zero gives

$$\begin{aligned}
&3x^2a_2 - 9x^2a_3 - 3x^2b_2 - 3x^2b_3 + 6xya_2 + 6xya_3 + 2xyb_2 - 6xyb_3 \\
&- y^2a_2 + 3y^2a_3 + y^2b_2 + y^2b_3 + 12xa_2 + 36xa_3 - 4xb_1 + 8xb_2 + 4ya_1 \\
&- 8ya_2 + 4yb_2 + 12yb_3 + 12a_1 - 12a_2 - 36a_3 + 4b_1 + 4b_2 + 12b_3 = 0
\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}
&3a_2v_1^2 + 6a_2v_1v_2 - a_2v_2^2 - 9a_3v_1^2 + 6a_3v_1v_2 + 3a_3v_2^2 - 3b_2v_1^2 + 2b_2v_1v_2 + b_2v_2^2 \\
&- 3b_3v_1^2 - 6b_3v_1v_2 + b_3v_2^2 + 4a_1v_2 + 12a_2v_1 - 8a_2v_2 + 36a_3v_1 - 4b_1v_1 \\
&+ 8b_2v_1 + 4b_2v_2 + 12b_3v_2 + 12a_1 - 12a_2 - 36a_3 + 4b_1 + 4b_2 + 12b_3 = 0
\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&(3a_2 - 9a_3 - 3b_2 - 3b_3)\, v_1^2 + (6a_2 + 6a_3 + 2b_2 - 6b_3)\, v_1v_2 \\
&+ (12a_2 + 36a_3 - 4b_1 + 8b_2)\, v_1 + (-a_2 + 3a_3 + b_2 + b_3)\, v_2^2 \\
&+ (4a_1 - 8a_2 + 4b_2 + 12b_3)\, v_2 + 12a_1 - 12a_2 - 36a_3 + 4b_1 + 4b_2 + 12b_3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
4a_1 - 8a_2 + 4b_2 + 12b_3 &= 0 \\
-a_2 + 3a_3 + b_2 + b_3 &= 0 \\
3a_2 - 9a_3 - 3b_2 - 3b_3 &= 0 \\
6a_2 + 6a_3 + 2b_2 - 6b_3 &= 0 \\
12a_2 + 36a_3 - 4b_1 + 8b_2 &= 0 \\
12a_1 - 12a_2 - 36a_3 + 4b_1 + 4b_2 + 12b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= -b_3 + 3a_3 \\
a_2 &= b_3 \\
a_3 &= a_3 \\
b_1 &= 3a_3 + 3b_3 \\
b_2 &= -3a_3 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x - 1 \\
\eta &= y + 3
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y + 3 - \left(\frac{y - 3x + 6}{x + y + 2}\right)(x - 1) \\
&= \frac{3x^2 + y^2 - 6x + 6y + 12}{x + y + 2} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{3x^2 + y^2 - 6x + 6y + 12}{x + y + 2}} dy
\end{aligned}
$$

Which results in

$$S = \frac{\ln\left(3x^2+y^2-6x+6y+12\right)}{2} + \frac{\sqrt{3}\,\arctan\left(\frac{(2y+6)\sqrt{3}}{6x-6}\right)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y-3x+6}{x+y+2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{-y+3x-6}{3x^2+y^2-6x+6y+12} \\ S_y &= \frac{x+y+2}{3x^2+y^2-6x+6y+12} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

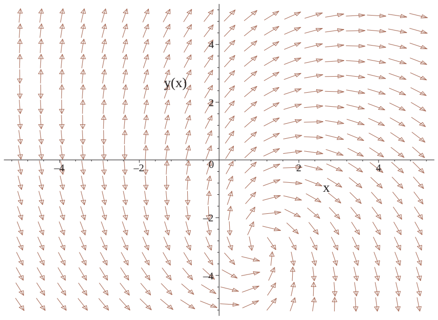
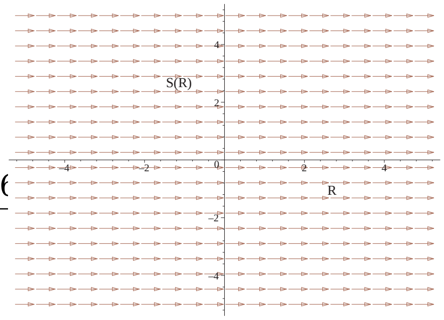
Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\begin{aligned} \int dS &= \int 0\,dR + c_2 \\ S(R) &= c_2 \end{aligned}$$

To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$\frac{\ln(y^2+3x^2+6y-6x+12)}{2}+\frac{\sqrt{3}\arctan\left(\frac{(y+3)\sqrt{3}}{3x-3}\right)}{3}=c_2$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx}=\frac{y-3x+6}{x+y+2}$ 	$R = x$ $S = \frac{\ln(3x^2+y^2-6x+6}{2}$	$\frac{dS}{dR}=0$ 

Solving for the constant of integration from initial conditions, the solution becomes

$$\frac{\ln(y^2+3x^2+6y-6x+12)}{2}+\frac{\sqrt{3}\arctan\left(\frac{(y+3)\sqrt{3}}{3x-3}\right)}{3}=\ln(2)+\frac{\sqrt{3}\,\pi}{18}$$

Solving for y gives

$$y$$
$$=\Big(\tan\Big(\mathrm{RootOf}\Big(6\sqrt{3}\ln(2)-3\sqrt{3}\ln\left(3x^2\tan(\_Z)^2-6x\tan(\_Z)^2+3x^2+3\tan(\_Z)^2-6x+3\right)+\pi-6\_Z$$
$$-\tan\Big(\mathrm{RootOf}\Big(6\sqrt{3}\ln(2)-3\sqrt{3}\ln\left(3x^2\tan(\_Z)^2-6x\tan(\_Z)^2+3x^2+3\tan(\_Z)^2-6x+3\right)+\pi-6\_$$
$$-\sqrt{3}\Big)\sqrt{3}$$

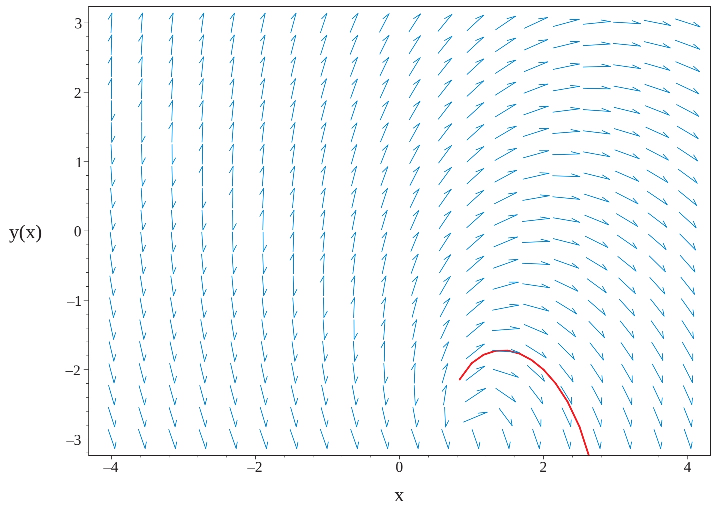


Figure 2.19: Slope field plot
$3x - y - 6 + (x + y + 2)\, y' = 0$

Summary of solutions found

$$y = \Big(\tan\Big(\mathrm{RootOf}\Big(6\sqrt{3}\,\ln(2) - 3\sqrt{3}\,\ln\left(3x^2\tan(\_Z)^2 - 6x\tan(\_Z)^2 + 3x^2 + 3\tan(\_Z)^2 - 6x + 3\right) + \pi - 6\_Z$$
$$-\tan\Big(\mathrm{RootOf}\Big(6\sqrt{3}\,\ln(2) - 3\sqrt{3}\,\ln\left(3x^2\tan(\_Z)^2 - 6x\tan(\_Z)^2 + 3x^2 + 3\tan(\_Z)^2 - 6x + 3\right) + \pi - 6\_$$
$$- \sqrt{3}\Big)\sqrt{3}$$

Maple step by step solution

Let's solve
$\left[3x - y(x) - 6 + (x + y(x) + 2)\left(\frac{d}{dx}y(x)\right) = 0, y(2) = -2\right]$

- Highest derivative means the order of the ODE is 1
  $\frac{d}{dx}y(x)$
- Solve for the highest derivative
  $\frac{d}{dx}y(x) = \frac{-3x + y(x) + 6}{x + y(x) + 2}$
- Use initial condition $y(2) = -2$
  0
- Solve for 0
  $0 = 0$
- Substitute $0 = 0$ into general solution and simplify
  0

- Solution to the IVP
  0

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

Maple dsolve solution

Solving time : `12.461` (sec)
Leaf size : 51

```
dsolve([3*x-y(x)-6+(x+y(x)+2)*diff(y(x),x) = 0,
       op([y(2) = -2])],y(x),singsol=all)
```

$$y(x) = -3 - \tan\left(\operatorname{RootOf}\left(-3\sqrt{3}\,\ln\left(\sec\left(\_Z\right)^{2}\left(x-1\right)^{2}\right) - 3\sqrt{3}\,\ln\left(3\right) + 6\sqrt{3}\,\ln\left(2\right) + \pi + 6\_Z\right)\right)\sqrt{3}\left(x-1\right)$$

Mathematica DSolve solution

Solving time : 0.217 (sec)
Leaf size : 90

```
DSolve[{(3*x-y[x]-6)+(x+y[x]+2)*D[y[x],x]==0,y[2]==-2},
       y[x],x,IncludeSingularSolutions->True]
```

$$\text{Solve}\left[\frac{\arctan\left(\frac{-y(x)+3x-6}{\sqrt{3}(y(x)+x+2)}\right)}{\sqrt{3}}+\log(2)=\frac{1}{2}\log\left(\frac{3x^2+y(x)^2+6y(x)-6x+12}{(x-1)^2}\right)\right.$$
$$\left.+\log(x-1)+\frac{1}{18}\left(\sqrt{3}\pi+18\log(2)-9\log(4)\right),y(x)\right]$$

2.1.10 problem 10

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Solved using Lie symmetry for first order ode . . . 119
Solved as first order ode of type dAlembert . . . 124
Maple step by step solution . . . 127
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Maple dsolve solution . . . 128
Mathematica DSolve solution . . . 128

Internal problem ID [4086]
Book : Differential equations, Shepley L. Ross, 1964
Section : 2.4, page 55
Problem number : 10
Date solved : `Tuesday, December 17, 2024 at 06:21:58 AM`
CAS classification :
`[[_homogeneous, ‘class C‘], _rational, [_Abel, ‘2nd type‘, ‘class A‘]]`

Solve

$$2x + 3y + 1 + (4x + 6y + 1)\, y' = 0$$

With initial conditions

$$y(-2) = 2$$

Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= -\frac{3y + 2x + 1}{4x + 6y + 1} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 2$ is

$$\left\{x < -\frac{13}{4} \vee -\frac{13}{4} < x\right\}$$

And the point $x_0 = -2$ is inside this domain. The y domain of $f(x, y)$ when $x = -2$ is

$$\left\{y < \frac{7}{6} \vee \frac{7}{6} < y\right\}$$

And the point $y_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{3y+2x+1}{4x+6y+1}\right)\\ &= -\frac{3}{4x+6y+1}+\frac{18y+12x+6}{(4x+6y+1)^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 2$ is

$$\left\{x < -\frac{13}{4} \lor -\frac{13}{4} < x\right\}$$

And the point $x_0 = -2$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = -2$ is

$$\left\{y < \frac{7}{6} \lor \frac{7}{6} < y\right\}$$

And the point $y_0 = 2$ is inside this domain. Therefore solution exists and is unique.

Solved using Lie symmetry for first order ode

Time used: 0.796 (sec)

Writing the ode as

$$\begin{aligned}y' &= -\frac{3y+2x+1}{4x+6y+1}\\ y' &= \omega(x,y)\end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{(3y+2x+1)(b_3-a_2)}{4x+6y+1} - \frac{(3y+2x+1)^2 a_3}{(4x+6y+1)^2} \\
&- \left(-\frac{2}{4x+6y+1} + \frac{12y+8x+4}{(4x+6y+1)^2}\right)(xa_2+ya_3+a_1) \\
&- \left(-\frac{3}{4x+6y+1} + \frac{18y+12x+6}{(4x+6y+1)^2}\right)(xb_2+yb_3+b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{8x^2a_2 - 4x^2a_3 + 16x^2b_2 - 8x^2b_3 + 24xya_2 - 12xya_3 + 48xyb_2 - 24xyb_3 + 18y^2a_2 - 9y^2a_3 + 36y^2b_2 - 18y}{(4x+6y+} = 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&8x^2a_2 - 4x^2a_3 + 16x^2b_2 - 8x^2b_3 + 24xya_2 - 12xya_3 + 48xyb_2 - 24xyb_3 \\
&+ 18y^2a_2 - 9y^2a_3 + 36y^2b_2 - 18y^2b_3 + 4xa_2 - 4xa_3 + 5xb_2 - 6xb_3 \\
&+ 9ya_2 - 8ya_3 + 12yb_2 - 12yb_3 - 2a_1 + a_2 - a_3 - 3b_1 + b_2 - b_3 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x = v_1, y = v_2\}
$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&8a_2v_1^2 + 24a_2v_1v_2 + 18a_2v_2^2 - 4a_3v_1^2 - 12a_3v_1v_2 - 9a_3v_2^2 + 16b_2v_1^2 + 48b_2v_1v_2 \\
&+ 36b_2v_2^2 - 8b_3v_1^2 - 24b_3v_1v_2 - 18b_3v_2^2 + 4a_2v_1 + 9a_2v_2 - 4a_3v_1 - 8a_3v_2 \\
&+ 5b_2v_1 + 12b_2v_2 - 6b_3v_1 - 12b_3v_2 - 2a_1 + a_2 - a_3 - 3b_1 + b_2 - b_3 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2\}
$$

Equation (7E) now becomes

$$\begin{aligned}(8a_2 - 4a_3 + 16b_2 - 8b_3)\, v_1^2 + (24a_2 - 12a_3 + 48b_2 - 24b_3)\, v_1 v_2 \\ + (4a_2 - 4a_3 + 5b_2 - 6b_3)\, v_1 + (18a_2 - 9a_3 + 36b_2 - 18b_3)\, v_2^2 \\ + (9a_2 - 8a_3 + 12b_2 - 12b_3)\, v_2 - 2a_1 + a_2 - a_3 - 3b_1 + b_2 - b_3 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}4a_2 - 4a_3 + 5b_2 - 6b_3 &= 0\\ 8a_2 - 4a_3 + 16b_2 - 8b_3 &= 0\\ 9a_2 - 8a_3 + 12b_2 - 12b_3 &= 0\\ 18a_2 - 9a_3 + 36b_2 - 18b_3 &= 0\\ 24a_2 - 12a_3 + 48b_2 - 24b_3 &= 0\\ -2a_1 + a_2 - a_3 - 3b_1 + b_2 - b_3 &= 0\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}a_1 &= a_1\\ a_2 &= -8a_1 - 12b_1\\ a_3 &= -12a_1 - 18b_1\\ b_1 &= b_1\\ b_2 &= 4a_1 + 6b_1\\ b_3 &= 6a_1 + 9b_1\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}\xi &= -12x - 18y\\ \eta &= 6x + 9y + 1\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi\\ &= 6x + 9y + 1 - \left(-\frac{3y + 2x + 1}{4x + 6y + 1}\right)(-12x - 18y)\\ &= \frac{-2x - 3y + 1}{4x + 6y + 1}\\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-2x-3y+1}{4x+6y+1}} dy \end{aligned}$$

Which results in

$$S = -2y - \ln(2x + 3y - 1)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{3y + 2x + 1}{4x + 6y + 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2}{2x + 3y - 1} \\ S_y &= -2 - \frac{3}{2x + 3y - 1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$\int dS = \int 1\, dR$$
$$S(R) = R + c_2$$

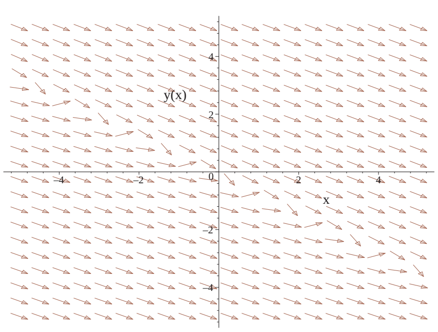
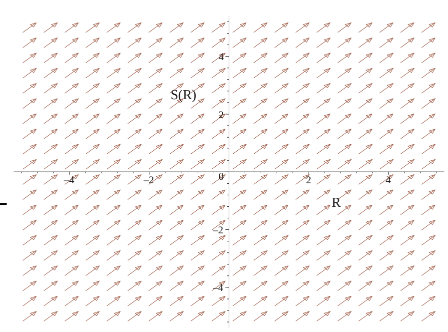
To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$-2y - \ln(2x + 3y - 1) = x + c_2$$

Which gives

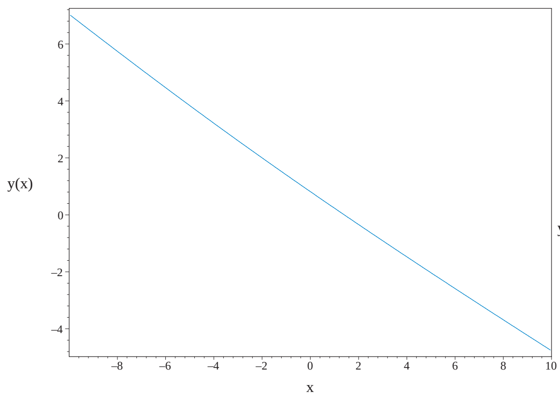
$$y = -\frac{2x}{3} + \frac{\operatorname{LambertW}\left(\frac{2\,e^{\frac{x}{3}-\frac{2}{3}-c_2}}{3}\right)}{2} + \frac{1}{3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

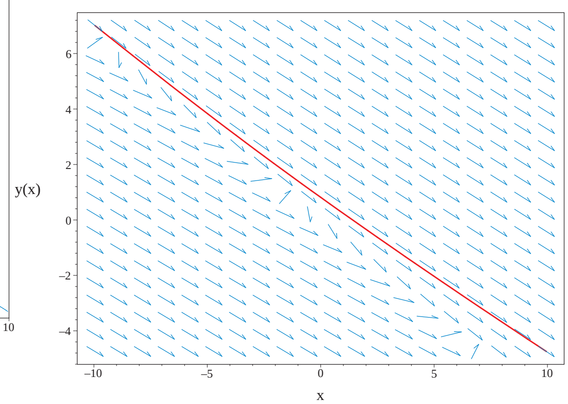
Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3y+2x+1}{4x+6y+1}$ 	$R = x$ $S = -2y - \ln(2x + 3y -$	$\frac{dS}{dR} = 1$ 

Solving for the constant of integration from initial conditions, the solution becomes

$$y = -\frac{2x}{3} + \frac{\text{LambertW}\left(\frac{2\,e^{\frac{x}{3}+\frac{4}{3}}}{3}\right)}{2} + \frac{1}{3}$$



(a) Solution plot
$y = -\frac{2x}{3} + \frac{\text{LambertW}\left(\frac{2\,e^{\frac{x}{3}+\frac{4}{3}}}{3}\right)}{2} + \frac{1}{3}$



(b) Slope field plot
$2x + 3y + 1 + (4x + 6y + 1)\, y' = 0$

Summary of solutions found

$$y = -\frac{2x}{3} + \frac{\text{LambertW}\left(\frac{2\,e^{\frac{x}{3}+\frac{4}{3}}}{3}\right)}{2} + \frac{1}{3}$$

Solved as first order ode of type dAlembert

Time used: 0.420 (sec)

Let $p = y'$ the ode becomes

$$2x + 3y + 1 + (4x + 6y + 1)\, p = 0$$

Solving for y from the above results in

$$y = -\frac{(4p+2)\, x}{3\,(1+2p)} - \frac{p+1}{3\,(1+2p)} \tag{1}$$

This has the form

$$y = x f(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. x gives

$$
\begin{aligned}
p &= f + (xf' + g')\frac{dp}{dx} \\
p - f &= (xf' + g')\frac{dp}{dx} \qquad (2)
\end{aligned}
$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= -\frac{2}{3} \\
g &= \frac{-p-1}{3+6p}
\end{aligned}
$$

Hence (2) becomes

$$
p + \frac{2}{3} = \left(-\frac{1}{3+6p} + \frac{6p}{(3+6p)^2} + \frac{6}{(3+6p)^2}\right) p'(x) \qquad (2A)
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p + \frac{2}{3} = 0
$$

No valid singular solutions found.

The general solution is found when $\frac{dp}{dx} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) + \frac{2}{3}}{-\frac{1}{3+6p(x)} + \frac{6p(x)}{(3+6p(x))^2} + \frac{6}{(3+6p(x))^2}} \qquad (3)
$$

This ODE is now solved for $p(x)$. No inversion is needed. Integrating gives

$$
\begin{aligned}
\int \frac{1}{(3p+2)(1+2p)^2} dp &= dx \\
3\ln(3p+2) - \frac{1}{1+2p} - 3\ln(1+2p) &= x + c_1
\end{aligned}
$$

Singular solutions are found by solving

$$
(3p+2)(1+2p)^2 = 0
$$

for $p(x)$. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$p(x) = -\frac{2}{3}$$
$$p(x) = -\frac{1}{2}$$

Substituing the above solution for p in (2A) gives

$$y = -\frac{2x}{3} + \frac{-\frac{e^{\operatorname{RootOf}\left(6\ln\left(\frac{2 e^{\_Z}}{3}-\frac{1}{3}\right)e^{\_Z}+2c_1 e^{\_Z}-6\_Z e^{\_Z}+2x e^{\_Z}-3\ln\left(\frac{2 e^{\_Z}}{3}-\frac{1}{3}\right)-c_1+3\_Z-x+3\right)}}{3}-\frac{1}{3}}{2 e^{\operatorname{RootOf}\left(6\ln\left(\frac{2 e^{\_Z}}{3}-\frac{1}{3}\right)e^{\_Z}+2c_1 e^{\_Z}-6\_Z e^{\_Z}+2x e^{\_Z}-3\ln\left(\frac{2 e^{\_Z}}{3}-\frac{1}{3}\right)-c_1+3\_Z-x+3\right)}-1}$$
$$y = -\frac{2x}{3} + \frac{1}{3}$$

Solving for the constant of integration from initial conditions, the solution becomes

$$y = -\frac{2x}{3} + \frac{-\frac{e^{\operatorname{RootOf}\left(6i\pi e^{\_Z}-3i\pi-6\ln\left(\frac{2 e^{\_Z}}{3}-\frac{1}{3}\right)e^{\_Z}+6\_Z e^{\_Z}-2x e^{\_Z}-14 e^{\_Z}+3\ln\left(\frac{2 e^{\_Z}}{3}-\frac{1}{3}\right)-3\_Z+x+4\right)}}{3}-\frac{1}{3}}{2 e^{\operatorname{RootOf}\left(6i\pi e^{\_Z}-3i\pi-6\ln\left(\frac{2 e^{\_Z}}{3}-\frac{1}{3}\right)e^{\_Z}+6\_Z e^{\_Z}-2x e^{\_Z}-14 e^{\_Z}+3\ln\left(\frac{2 e^{\_Z}}{3}-\frac{1}{3}\right)-3\_Z+x+4\right)}-1}$$

The solution

$$y = -\frac{2x}{3} + \frac{1}{3}$$

was found not to satisfy the ode or the IC. Hence it is removed.

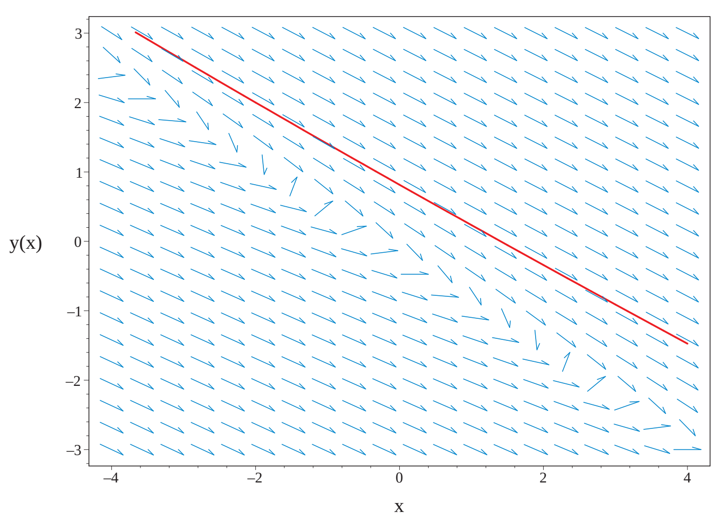


Figure 2.21: Slope field plot
$2x + 3y + 1 + (4x + 6y + 1)\, y' = 0$

Summary of solutions found

$$y = -\frac{2x}{3} + \frac{-\frac{e^{\operatorname{RootOf}\left(6i\pi\,e^{\_Z}-3i\pi-6\ln\left(\frac{2\,e^{\_Z}}{3}-\frac{1}{3}\right)e^{\_Z}+6\_Z\,e^{\_Z}-2x\,e^{\_Z}-14\,e^{\_Z}+3\ln\left(\frac{2\,e^{\_Z}}{3}-\frac{1}{3}\right)-3\_Z+x+4\right)}}{3}-\frac{1}{3}}{2\,e^{\operatorname{RootOf}\left(6i\pi\,e^{\_Z}-3i\pi-6\ln\left(\frac{2\,e^{\_Z}}{3}-\frac{1}{3}\right)e^{\_Z}+6\_Z\,e^{\_Z}-2x\,e^{\_Z}-14\,e^{\_Z}+3\ln\left(\frac{2\,e^{\_Z}}{3}-\frac{1}{3}\right)-3\_Z+x+4\right)}-1}$$

Maple step by step solution

Let's solve

$\left[2x+3y(x)+1+(4x+6y(x)+1)\left(\frac{d}{dx}y(x)\right)=0, y(-2)=2\right]$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x)=\frac{-2x-3y(x)-1}{4x+6y(x)+1}$

- Use initial condition $y(-2)=2$

  0

- Solve for 0

  $0=0$

- Substitute $0=0$ into general solution and simplify

  0

- Solution to the IVP

  0

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -2/3, y(x)`      *** Sublevel 2 *
```

```
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

Maple dsolve solution

Solving time : 0.144 (sec)
Leaf size : 20

```
dsolve([2*x+3*y(x)+1+(4*x+6*y(x)+1)*diff(y(x),x) = 0,
       op([y(-2) = 2])],y(x),singsol=all)
```

$$y(x) = \frac{1}{3} - \frac{2x}{3} + \frac{\operatorname{LambertW}\left(\frac{2\,\mathrm{e}^{\frac{x}{3}+\frac{4}{3}}}{3}\right)}{2}$$

Mathematica DSolve solution

Solving time : 5.576 (sec)
Leaf size : 30

```
DSolve[{(2*x+3*y[x]+1)+(4*x+6*y[x]+1)*D[y[x],x]==0,y[-2]==2},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{6}\left(3W\left(\frac{2}{3}e^{\frac{x+4}{3}}\right) - 4x + 2\right)$$