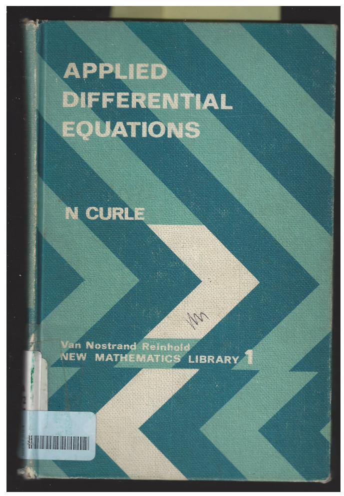
A Solution Manual For

Applied Differential equations, Newby Curle. Van Nostrand Reinhold. 1972



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Contents

CHAPTER 1

Lookup tables for all problems in current book

1.1 Examples, page 35

Table 1.1: Lookup table for all problems in current section

ID	problem	ODE
4087	1	$y = y' + \frac{y'^2}{2}$
4088	2	$(y - xy')^2 = 1 + y'^2$
4089	3	$y - x = y'^2 \left(1 - \frac{2y'}{3}\right)$
4090	4	$x^2 y' = x(y - 1) + (y - 1)^2$

CHAPTER 2

Book Solved Problems

2.1 Examples, page 35

2.1.1 problem 1

Internal problem ID [4087]
Book : Applied Differential equations, Newby Curle. Van Nostrand Reinhold. 1972
Section : Examples, page 35
Problem number : 1
Date solved : `Tuesday, December 17, 2024 at 06:22:00 AM`
CAS classification : `[_quadrature]`

Solve

$$y = y' + \frac{y'^2}{2}$$

Solving for the derivative gives these ODE's to solve

$$y' = -1 + \sqrt{1+2y} \tag{1}$$
$$y' = -1 - \sqrt{1+2y} \tag{2}$$

Now each of the above is solved separately.

Solving Eq. (1)

Integrating gives

$$\int \frac{1}{-1+\sqrt{1+2y}} dy = dx$$
$$\sqrt{1+2y} + \ln\left(-1+\sqrt{1+2y}\right) = x + c_1$$

Singular solutions are found by solving

$$-1+\sqrt{1+2y} = 0$$

for y. This is because we had to divide by this in the above step. This gives the following singular solution(s), which also have to satisfy the given ODE.

$$y = 0$$

Solving for y gives

$$
\begin{aligned}
y &= 0 \\
y &= \frac{\mathrm{LambertW}\left(\mathrm{e}^{-1+x+c_1}\right)^2}{2} + \mathrm{LambertW}\left(\mathrm{e}^{-1+x+c_1}\right)
\end{aligned}
$$

We now need to find the singular solutions, these are found by finding for what values $(-1+\sqrt{1+2y})$ is zero. These give

$$
y = 0
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

The solution $y = 0$ satisfies the ode and initial conditions.

Solving Eq. (2)

Integrating gives

$$
\begin{aligned}
\int \frac{1}{-1-\sqrt{1+2y}} dy &= dx \\
-\sqrt{1+2y} + \ln\left(\sqrt{1+2y}+1\right) &= x + c_2
\end{aligned}
$$

Solving for y gives

$$
y = \frac{\mathrm{LambertW}\left(-\mathrm{e}^{-1+x+c_2}\right)^2}{2} + \mathrm{LambertW}\left(-\mathrm{e}^{-1+x+c_2}\right)
$$

Maple step by step solution

Let's solve

$y(x) = \frac{d}{dx}y(x) + \frac{\left(\frac{d}{dx}y(x)\right)^2}{2}$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\left[\frac{d}{dx}y(x) = -1 - \sqrt{1+2y(x)}, \frac{d}{dx}y(x) = -1 + \sqrt{1+2y(x)}\right]$

□ Solve the equation $\frac{d}{dx}y(x) = -1 - \sqrt{1+2y(x)}$

  ◦ Separate variables

$$\frac{\frac{d}{dx}y(x)}{-1-\sqrt{1+2y(x)}} = 1$$

- Integrate both sides with respect to x

$$\int \frac{\frac{d}{dx}y(x)}{-1-\sqrt{1+2y(x)}}dx = \int 1dx + \_C1$$

- Evaluate integral

$$\frac{\ln(y(x))}{2} - \sqrt{1+2y(x)} - \frac{\ln\left(-1+\sqrt{1+2y(x)}\right)}{2} + \frac{\ln\left(1+\sqrt{1+2y(x)}\right)}{2} = x + \_C1$$

□ Solve the equation $\frac{d}{dx}y(x) = -1 + \sqrt{1+2y(x)}$

- Separate variables

$$\frac{\frac{d}{dx}y(x)}{-1+\sqrt{1+2y(x)}} = 1$$

- Integrate both sides with respect to x

$$\int \frac{\frac{d}{dx}y(x)}{-1+\sqrt{1+2y(x)}}dx = \int 1dx + \_C1$$

- Evaluate integral

$$\frac{\ln(y(x))}{2} + \sqrt{1+2y(x)} + \frac{\ln\left(-1+\sqrt{1+2y(x)}\right)}{2} - \frac{\ln\left(1+\sqrt{1+2y(x)}\right)}{2} = x + \_C1$$

- Set of solutions

$$\left\{\frac{\ln(y(x))}{2} - \sqrt{1+2y(x)} - \frac{\ln\left(-1+\sqrt{1+2y(x)}\right)}{2} + \frac{\ln\left(1+\sqrt{1+2y(x)}\right)}{2} = x + C1, \frac{\ln(y(x))}{2} + \sqrt{1+2y(x)} + \frac{\ln}{}\right.$$

Maple trace

```
`Methods for first order ODEs:
-> Solving 1st order ODE of high degree, 1st attempt
trying 1st order WeierstrassP solution for high degree ODE
trying 1st order WeierstrassPPrime solution for high degree ODE
trying 1st order JacobiSN solution for high degree ODE
trying 1st order ODE linearizable_by_differentiation
trying differential order: 1; missing variables
<- differential order: 1; missing  x  successful`
```

Maple dsolve solution

Solving time : 0.024 (sec)
Leaf size : 106

```
dsolve(y(x) = diff(y(x),x)+1/2*diff(y(x),x)^2,
       y(x),singsol=all)
```

$$y(x) = \frac{\operatorname{LambertW}\left(-\sqrt{2}\,\mathrm{e}^{x-1-c_1}\right)\left(\operatorname{LambertW}\left(-\sqrt{2}\,\mathrm{e}^{x-1-c_1}\right)+2\right)}{2}$$
$$y(x) = \frac{\mathrm{e}^{2\operatorname{RootOf}\left(-\_Z-2x+2\,\mathrm{e}^{-\_Z}-2+2c_1-\ln(2)+\ln\left(\mathrm{e}^{-\_Z}\left(\mathrm{e}^{-\_Z}-2\right)^2\right)\right)}}{2} - \mathrm{e}^{\operatorname{RootOf}\left(-\_Z-2x+2\,\mathrm{e}^{-\_Z}-2+2c_1-\ln(2)+\ln\left(\mathrm{e}^{-\_Z}\left(\mathrm{e}^{-\_Z}-2\right)^2\right)\right)}$$

Mathematica DSolve solution

Solving time : 22.693 (sec)
Leaf size : 66

```
DSolve[{y[x]==D[y[x],x]+1/2*(D[y[x],x])^2,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to \frac{1}{2}W\left(-e^{x-1-c_1}\right)\left(2+W\left(-e^{x-1-c_1}\right)\right)$$
$$y(x) \to \frac{1}{2}W\left(e^{x-1+c_1}\right)\left(2+W\left(e^{x-1+c_1}\right)\right)$$
$$y(x) \to 0$$

2.1.2 problem 2

Internal problem ID [4088]
Book : Applied Differential equations, Newby Curle. Van Nostrand Reinhold. 1972
Section : Examples, page 35
Problem number : 2
Date solved : `Tuesday, December 17, 2024 at 06:22:01 AM`
CAS classification :
`[[_1st_order, _with_linear_symmetries], _rational, _Clairaut]`

Solve

$$(y - xy')^2 = 1 + {y'}^2$$

Solved as first order Clairaut ode

Time used: 0.196 (sec)

This is Clairaut ODE. It has the form

$$y = xy' + g(y')$$

Where g is function of $y'(x)$. Let $p = y'$ the ode becomes

$$(-xp + y)^2 = p^2 + 1$$

Solving for y from the above results in

$$y = xp + \sqrt{p^2 + 1} \tag{1A}$$
$$y = xp - \sqrt{p^2 + 1} \tag{2A}$$

Each of the above ode's is a Clairaut ode which is now solved.

Solving ode 1A We start by replacing y' by p which gives

$$\begin{aligned} y &= xp + \sqrt{p^2 + 1} \\ &= xp + \sqrt{p^2 + 1} \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = \sqrt{p^2 + 1}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xp + g) \\
p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= p + (x + g')\frac{dp}{dx} \\
0 &= (x + g')\frac{dp}{dx}
\end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p.

The general solution is given by

$$\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_1 x + \sqrt{c_1^2 + 1}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = \sqrt{p^2 + 1}$, then the above equation becomes

$$\begin{aligned}
x + g'(p) &= x + \frac{p}{\sqrt{p^2 + 1}} \\
&= 0
\end{aligned}$$

Solving the above for p results in

$$p_1 = y = (-x^2 + 1)\sqrt{-\frac{1}{x^2 - 1}}$$

Substituting the above back in (1) results in

$$y = (-x^2 + 1)\sqrt{-\frac{1}{x^2 - 1}}$$

Solving ode 2A We start by replacing y' by p which gives

$$\begin{aligned} y &= xp - \sqrt{p^2 + 1} \\ &= xp - \sqrt{p^2 + 1} \end{aligned}$$

Writing the ode as

$$y = xp + g(p)$$

We now write $g \equiv g(p)$ to make notation simpler but we should always remember that g is function of p which in turn is function of x. Hence the above becomes

$$y = xp + g \tag{1}$$

Then we see that

$$g = -\sqrt{p^2 + 1}$$

Taking derivative of (1) w.r.t. x gives

$$\begin{aligned} p &= \frac{d}{dx}(xp + g) \\ p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\ p &= p + (x + g')\frac{dp}{dx} \\ 0 &= (x + g')\frac{dp}{dx} \end{aligned}$$

Where g' is derivative of $g(p)$ w.r.t. p.

The general solution is given by

$$\begin{aligned} \frac{dp}{dx} &= 0 \\ p &= c_1 \end{aligned}$$

Substituting this in (1) gives the general solution as

$$y = c_2 x - \sqrt{c_2^2 + 1}$$

The singular solution is found from solving for p from

$$x + g'(p) = 0$$

And substituting the result back in (1). Since we found above that $g = -\sqrt{p^2+1}$, then the above equation becomes

$$\begin{aligned} x + g'(p) &= x - \frac{p}{\sqrt{p^2+1}} \\ &= 0 \end{aligned}$$

Solving the above for p results in

$$p_1 = y = \sqrt{-\frac{1}{x^2-1}}\,(x^2-1)$$

Substituting the above back in (1) results in

$$y = \sqrt{-\frac{1}{x^2-1}}\,(x^2-1)$$

Summary of solutions found

$$y = \sqrt{-\frac{1}{x^2-1}}\,(x^2-1)$$

$$y = (-x^2+1)\sqrt{-\frac{1}{x^2-1}}$$

$$y = c_1 x + \sqrt{c_1^2+1}$$

$$y = c_2 x - \sqrt{c_2^2+1}$$

Maple step by step solution

Let's solve

$\left(-x\left(\frac{d}{dx}y(x)\right)+y(x)\right)^2=1+\left(\frac{d}{dx}y(x)\right)^2$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\left[\frac{d}{dx}y(x)=\frac{xy(x)-\sqrt{y(x)^2+x^2-1}}{x^2-1}, \frac{d}{dx}y(x)=\frac{xy(x)+\sqrt{y(x)^2+x^2-1}}{x^2-1}\right]$

- Solve the equation $\frac{d}{dx}y(x)=\frac{xy(x)-\sqrt{y(x)^2+x^2-1}}{x^2-1}$

- Solve the equation $\frac{d}{dx}y(x)=\frac{xy(x)+\sqrt{y(x)^2+x^2-1}}{x^2-1}$

- Set of solutions

  $\{workingODE, workingODE\}$

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful
   <- dAlembert successful`
```

Maple dsolve solution

Solving time : 0.077 (sec)
Leaf size : 57

```
dsolve((-diff(y(x),x)*x+y(x))^2 = 1+diff(y(x),x)^2,
       y(x),singsol=all)
```

$$
\begin{aligned}
y(x) &= \sqrt{-x^2+1} \\
y(x) &= -\sqrt{-x^2+1} \\
y(x) &= c_1 x - \sqrt{c_1^2+1} \\
y(x) &= c_1 x + \sqrt{c_1^2+1}
\end{aligned}
$$

Mathematica DSolve solution

Solving time : 0.174 (sec)
Leaf size : 73

```
DSolve[{(y[x]-x*D[y[x],x])^2==1+(D[y[x],x])^2,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$
\begin{aligned}
y(x) &\to c_1 x - \sqrt{1+c_1{}^2} \\
y(x) &\to c_1 x + \sqrt{1+c_1{}^2} \\
y(x) &\to -\sqrt{1-x^2} \\
y(x) &\to \sqrt{1-x^2}
\end{aligned}
$$

2.1.3 problem 3

Internal problem ID [4089]
Book : Applied Differential equations, Newby Curle. Van Nostrand Reinhold. 1972
Section : Examples, page 35
Problem number : 3
Date solved : `Tuesday, December 17, 2024 at 06:22:02 AM`
CAS classification : `[[_homogeneous, ‘class C‘], _dAlembert]`

Solve

$$y - x = y'^2 \left(1 - \frac{2y'}{3}\right)$$

Solved as first order ode of type dAlembert

Time used: 1.317 (sec)

Let $p = y'$ the ode becomes

$$y - x = p^2 \left(1 - \frac{2p}{3}\right)$$

Solving for y from the above results in

$$y = p^2 - \frac{2}{3}p^3 + x \tag{1}$$

This has the form

$$y = xf(p) + g(p) \tag{*}$$

Where f, g are functions of $p = y'(x)$. The above ode is dAlembert ode which is now solved.

Taking derivative of (*) w.r.t. x gives

$$p = f + (xf' + g')\frac{dp}{dx}$$

$$p - f = (xf' + g')\frac{dp}{dx} \tag{2}$$

Comparing the form $y = xf + g$ to (1A) shows that

$$
\begin{aligned}
f &= 1 \\
g &= p^2 - \frac{2}{3}p^3
\end{aligned}
$$

Hence (2) becomes

$$
p - 1 = \left(-2p^2 + 2p\right) p'(x) \tag{2A}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in the above which gives

$$
p - 1 = 0
$$

Solving the above for p results in

$$
p_1 = 1
$$

Substituting these in (1A) and keeping singular solution that verifies the ode gives

$$
y = \frac{1}{3} + x
$$

The general solution is found when $\frac{\mathrm{d}p}{\mathrm{d}x} \neq 0$. From eq. (2A). This results in

$$
p'(x) = \frac{p(x) - 1}{-2p(x)^2 + 2p(x)} \tag{3}
$$

This ODE is now solved for $p(x)$. No inversion is needed. Integrating gives

$$
\begin{aligned}
\int -2pdp &= dx \\
-p^2 &= x + c_1
\end{aligned}
$$

Solving for $p(x)$ gives

$$
\begin{aligned}
p(x) &= \sqrt{-c_1 - x} \\
p(x) &= -\sqrt{-c_1 - x}
\end{aligned}
$$

Substituing the above solution for p in (2A) gives

$$
\begin{aligned}
y &= -c_1 - \frac{2(-c_1 - x)^{3/2}}{3} \\
y &= -c_1 + \frac{2(-c_1 - x)^{3/2}}{3}
\end{aligned}
$$

Summary of solutions found

$$y = \frac{1}{3} + x$$

$$y = -c_1 - \frac{2(-c_1 - x)^{3/2}}{3}$$

$$y = -c_1 + \frac{2(-c_1 - x)^{3/2}}{3}$$

Maple step by step solution

Let's solve

$y(x) - x = \left(\frac{d}{dx}y(x)\right)^2 \left(1 - \frac{2\frac{d}{dx}y(x)}{3}\right)$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\left[\frac{d}{dx}y(x) = \frac{\left(1-6y(x)+6x+2\sqrt{-3y(x)+3x+9y(x)^2-18xy(x)+9x^2}\right)^{1/3}}{2} + \frac{1}{2\left(1-6y(x)+6x+2\sqrt{-3y(x)+3x+9y(x)^2-18xy(x)+9}\right)}\right.$

- Solve the equation $\frac{d}{dx}y(x) = \frac{\left(1-6y(x)+6x+2\sqrt{-3y(x)+3x+9y(x)^2-18xy(x)+9x^2}\right)^{1/3}}{2} + \frac{}{2\left(1-6y(x)+6x+2\sqrt{-3y(x)+}\right)}$

- Solve the equation $\frac{d}{dx}y(x) = -\frac{\left(1-6y(x)+6x+2\sqrt{-3y(x)+3x+9y(x)^2-18xy(x)+9x^2}\right)^{1/3}}{4} - \frac{}{4\left(1-6y(x)+6x+2\sqrt{-3y(x}\right)}$

- Solve the equation $\frac{d}{dx}y(x) = -\frac{\left(1-6y(x)+6x+2\sqrt{-3y(x)+3x+9y(x)^2-18xy(x)+9x^2}\right)^{1/3}}{4} - \frac{}{4\left(1-6y(x)+6x+2\sqrt{-3y(x}\right)}$

- Set of solutions

  $\{workingODE, workingODE, workingODE\}$

Maple trace

```
`Methods for first order ODEs:
   *** Sublevel 2 ***
   Methods for first order ODEs:
   -> Solving 1st order ODE of high degree, 1st attempt
   trying 1st order WeierstrassP solution for high degree ODE
   trying 1st order WeierstrassPPrime solution for high degree ODE
   trying 1st order JacobiSN solution for high degree ODE
   trying 1st order ODE linearizable_by_differentiation
   trying differential order: 1; missing variables
   trying dAlembert
   <- dAlembert successful`
```

Maple dsolve solution

Solving time : 0.033 (sec)
Leaf size : 49

```
dsolve(-x+y(x) = diff(y(x),x)^2*(1-2/3*diff(y(x),x)),
       y(x),singsol=all)
```

$$
\begin{aligned}
y(x) &= x+\frac{1}{3} \\
y(x) &= \frac{(2x-2c_1)\sqrt{-x+c_1}}{3}+c_1 \\
y(x) &= \frac{(-2x+2c_1)\sqrt{-x+c_1}}{3}+c_1
\end{aligned}
$$

Mathematica DSolve solution

Solving time : 0.0 (sec)
Leaf size : 0

```
DSolve[{y[x]-x==D[y[x],x]^2*(1-2/3* D[y[x],x]),{}},
       y[x],x,IncludeSingularSolutions->True]
```

Timed out

2.1.4 problem 4

Internal problem ID [4090]
Book : Applied Differential equations, Newby Curle. Van Nostrand Reinhold. 1972
Section : Examples, page 35
Problem number : 4
Date solved : `Tuesday, December 17, 2024 at 06:22:04 AM`
CAS classification : `[[_homogeneous, ‘class C‘], _rational, _Riccati]`

Solve

$$x^2y' = x(y-1) + (y-1)^2$$

Summary of solutions found

$$y = 1$$
$$y = \frac{-x + \ln(x) + c_1}{\ln(x) + c_1}$$

Solved as first order homogeneous class Maple C ode

Time used: 0.356 (sec)

Let $Y = y - y_0$ and $X = x - x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{(Y(X) + y_0 - 1)(x_0 + X + Y(X) + y_0 - 1)}{(x_0 + X)^2}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 0$$
$$y_0 = 1$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{Y(X)\,X + Y(X)^2}{X^2}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= \frac{Y(Y+X)}{X^2} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = Y(Y+X)$ and $N = X^2$ are both homogeneous and of the same order $n = 2$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= u^2 + u \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{u(X)^2}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{u(X)^2}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)X - u(X)^2 = 0$$

Which is now solved as separable in $u(X)$.

The ode $\frac{d}{dX}u(X) = \frac{u(X)^2}{X}$ is separable as it can be written as

$$\begin{aligned} \frac{d}{dX}u(X) &= \frac{u(X)^2}{X} \\ &= f(X)g(u) \end{aligned}$$

Where

$$
\begin{aligned}
f(X) &= \frac{1}{X} \\
g(u) &= u^2
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\int \frac{1}{g(u)}\, du &= \int f(X)\, dX \\
\int \frac{1}{u^2}\, du &= \int \frac{1}{X}\, dX \\
-\frac{1}{u(X)} &= \ln(X) + c_1
\end{aligned}
$$

We now need to find the singular solutions, these are found by finding for what values $g(u)$ is zero, since we had to divide by this above. Solving $g(u) = 0$ or $u^2 = 0$ for $u(X)$ gives

$$
u(X) = 0
$$

Now we go over each such singular solution and check if it verifies the ode itself and any initial conditions given. If it does not then the singular solution will not be used.

Therefore the solutions found are

$$
\begin{aligned}
-\frac{1}{u(X)} &= \ln(X) + c_1 \\
u(X) &= 0
\end{aligned}
$$

Solving for $u(X)$ gives

$$
\begin{aligned}
u(X) &= 0 \\
u(X) &= -\frac{1}{\ln(X) + c_1}
\end{aligned}
$$

Converting $u(X) = 0$ back to $Y(X)$ gives

$$
Y(X) = 0
$$

Converting $u(X) = -\frac{1}{\ln(X)+c_1}$ back to $Y(X)$ gives

$$
Y(X) = -\frac{X}{\ln(X) + c_1}
$$

Using the solution for $Y(X)$

$$Y(X) = 0 \tag{A}$$

And replacing back terms in the above solution using

$$\begin{aligned} Y &= y + y_0 \\ X &= x + x_0 \end{aligned}$$

Or

$$\begin{aligned} Y &= y + 1 \\ X &= x \end{aligned}$$

Then the solution in y becomes using EQ (A)

$$y - 1 = 0$$

Using the solution for $Y(X)$

$$Y(X) = -\frac{X}{\ln(X) + c_1} \tag{A}$$

And replacing back terms in the above solution using

$$\begin{aligned} Y &= y + y_0 \\ X &= x + x_0 \end{aligned}$$

Or

$$\begin{aligned} Y &= y + 1 \\ X &= x \end{aligned}$$

Then the solution in y becomes using EQ (A)

$$y - 1 = -\frac{x}{\ln(x) + c_1}$$

Solving for y gives

$$\begin{aligned} y &= 1 \\ y &= \frac{-x + \ln(x) + c_1}{\ln(x) + c_1} \end{aligned}$$

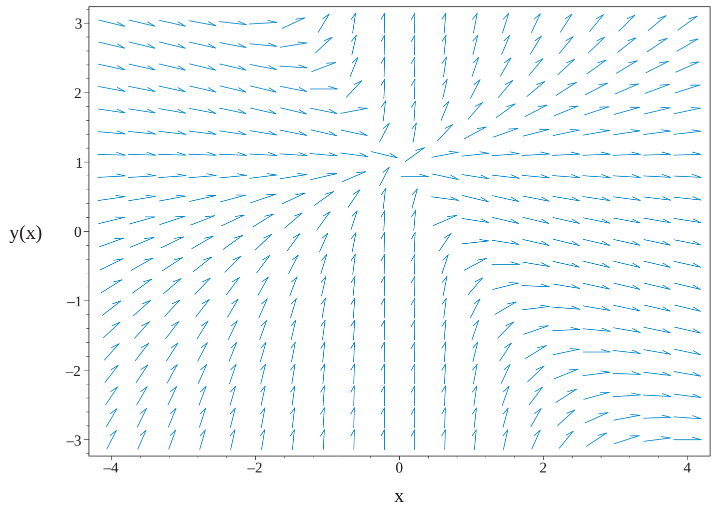


Figure 2.1: Slope field plot
$x^2y' = x(y-1) + (y-1)^2$

Solved using Lie symmetry for first order ode

Time used: 0.506 (sec)

Writing the ode as

$$
\begin{aligned}
y' &= \frac{xy + y^2 - x - 2y + 1}{x^2} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2+\frac{\left(x y+y^2-x-2 y+1\right)\left(b_3-a_2\right)}{x^2}-\frac{\left(x y+y^2-x-2 y+1\right)^2 a_3}{x^4} \\
& -\left(\frac{y-1}{x^2}-\frac{2\left(x y+y^2-x-2 y+1\right)}{x^3}\right)\left(x a_2+y a_3+a_1\right) \\
& -\frac{(2 y+x-2)\left(x b_2+y b_3+b_1\right)}{x^2}=0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{2x^3yb_2 - x^2y^2a_2 + x^2y^2b_3 + y^4a_3 + x^3b_1 - 2x^3b_2 + x^3b_3 - x^2ya_1 + 2x^2ya_2 - x^2ya_3 + 2x^2yb_1 - 2x\,y^2a_1 -}{}
$$

$$= 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
& -2 x^3 y b_2+x^2 y^2 a_2-x^2 y^2 b_3-y^4 a_3-x^3 b_1+2 x^3 b_2-x^3 b_3+x^2 y a_1-2 x^2 y a_2 \\
& +x^2 y a_3-2 x^2 y b_1+2 x\, y^2 a_1+2 x\, y^2 a_3+4 y^3 a_3-x^2 a_1+x^2 a_2-x^2 a_3 \\
& +2 x^2 b_1+x^2 b_3-4 x y a_1-4 x y a_3-6 y^2 a_3+2 x a_1+2 x a_3+4 y a_3-a_3=0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x=v_1, y=v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
& a_2 v_1^2 v_2^2-a_3 v_2^4-2 b_2 v_1^3 v_2-b_3 v_1^2 v_2^2+a_1 v_1^2 v_2+2 a_1 v_1 v_2^2-2 a_2 v_1^2 v_2+a_3 v_1^2 v_2 \\
& +2 a_3 v_1 v_2^2+4 a_3 v_2^3-b_1 v_1^3-2 b_1 v_1^2 v_2+2 b_2 v_1^3-b_3 v_1^3-a_1 v_1^2-4 a_1 v_1 v_2+a_2 v_1^2 \\
& -a_3 v_1^2-4 a_3 v_1 v_2-6 a_3 v_2^2+2 b_1 v_1^2+b_3 v_1^2+2 a_1 v_1+2 a_3 v_1+4 a_3 v_2-a_3=0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-2b_2v_1^3v_2 + (-b_1 + 2b_2 - b_3)\, v_1^3 + (-b_3 + a_2)\, v_1^2v_2^2 + (a_1 - 2a_2 + a_3 - 2b_1)\, v_1^2v_2 \\ + (-a_1 + a_2 - a_3 + 2b_1 + b_3)\, v_1^2 + (2a_1 + 2a_3)\, v_1v_2^2 + (-4a_1 - 4a_3)\, v_1v_2 \\ + (2a_1 + 2a_3)\, v_1 - a_3v_2^4 + 4a_3v_2^3 - 6a_3v_2^2 + 4a_3v_2 - a_3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -6a_3 &= 0 \\ -a_3 &= 0 \\ 4a_3 &= 0 \\ -2b_2 &= 0 \\ -4a_1 - 4a_3 &= 0 \\ 2a_1 + 2a_3 &= 0 \\ -b_3 + a_2 &= 0 \\ -b_1 + 2b_2 - b_3 &= 0 \\ a_1 - 2a_2 + a_3 - 2b_1 &= 0 \\ -a_1 + a_2 - a_3 + 2b_1 + b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= -b_3 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y - 1 \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - 1 - \left(\frac{xy + y^2 - x - 2y + 1}{x^2}\right)(x) \\
&= \frac{-y^2 + 2y - 1}{x} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-y^2+2y-1}{x}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{x}{y-1}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{xy + y^2 - x - 2y + 1}{x^2}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{1}{y-1} \\
S_y &= -\frac{x}{(y-1)^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S.

Since the ode has the form $\frac{d}{dR}S(R) = f(R)$, then we only need to integrate $f(R)$.

$$
\begin{aligned}
\int dS &= \int -\frac{1}{R}\, dR \\
S(R) &= -\ln(R) + c_2
\end{aligned}
$$

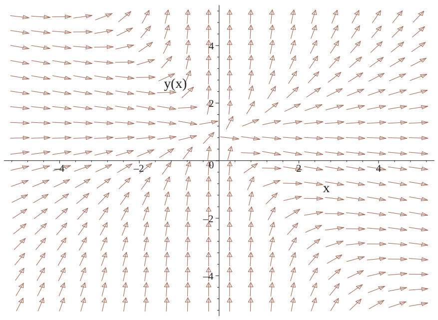
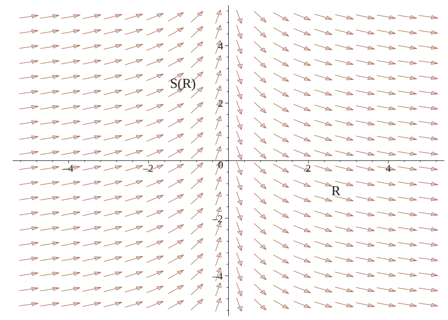
To complete the solution, we just need to transform the above back to x, y coordinates. This results in

$$
\frac{x}{y-1} = -\ln(x) + c_2
$$

Which gives

$$
y = \frac{\ln(x) - c_2 - x}{\ln(x) - c_2}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{xy+y^2-x-2y+1}{x^2}$ 	$R = x$ $S = \frac{x}{y-1}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

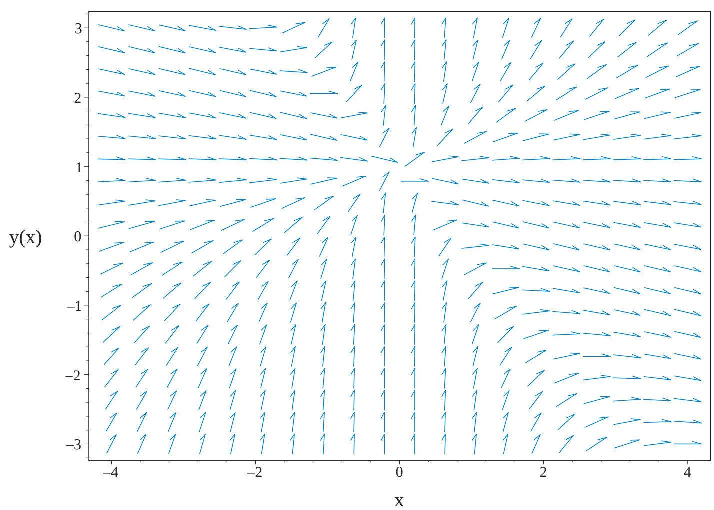


Figure 2.2: Slope field plot
$x^2y' = x(y-1) + (y-1)^2$

<u>Summary of solutions found</u>

$$y = \frac{\ln(x) - c_2 - x}{\ln(x) - c_2}$$

Solved as first order ode of type Riccati

Time used: 0.309 (sec)

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= \frac{xy+y^2-x-2y+1}{x^2}\end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y}{x}+\frac{y^2}{x^2}-\frac{1}{x}-\frac{2y}{x^2}+\frac{1}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x)+f_1(x)y+f_2(x)y^2$$

Shows that $f_0(x)=\frac{1-x}{x^2}$, $f_1(x)=\frac{x-2}{x^2}$ and $f_2(x)=\frac{1}{x^2}$. Let

$$\begin{aligned}y &= \frac{-u'}{f_2 u}\\ &= \frac{-u'}{\frac{u}{x^2}}\end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x)-(f_2'+f_1f_2)\,u'(x)+f_2^2f_0u(x)=0 \tag{2}$$

But

$$\begin{aligned}f_2' &= -\frac{2}{x^3}\\ f_1f_2 &= \frac{x-2}{x^4}\\ f_2^2f_0 &= \frac{1-x}{x^6}\end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x^2}-\left(-\frac{2}{x^3}+\frac{x-2}{x^4}\right)u'(x)+\frac{(1-x)\,u(x)}{x^6}=0$$

Writing the ode as

$$\frac{\frac{d^2u}{dx^2}}{x^2}+\frac{(x+2)\left(\frac{du}{dx}\right)}{x^4}+\frac{(1-x)\,u}{x^6}=0 \tag{1}$$

$$A\frac{d^2u}{dx^2}+B\frac{du}{dx}+Cu=0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= \frac{1}{x^2} \\ B &= \frac{x+2}{x^4} \\ C &= \frac{1-x}{x^6} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ue^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then u is found using the inverse transformation

$$u = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 2.4: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1}\right) + (-)[\sqrt{r}]_\infty \\
&= \frac{1}{2x} + (-)\,(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{1}{2x} dx} \\
&= \sqrt{x}
\end{aligned}
$$

The first solution to the original ode in u is found from

$$
\begin{aligned}
u_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{\frac{x+2}{x^4}}{\frac{1}{x^2}} \, dx} \\
&= z_1 e^{-\frac{\ln(x)}{2} + \frac{1}{x}} \\
&= z_1 \left(\frac{\mathrm{e}^{\frac{1}{x}}}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$u_1 = \mathrm{e}^{\frac{1}{x}}$$

The second solution u_2 to the original ode is found using reduction of order

$$u_2 = u_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{u_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
u_2 &= u_1 \int \frac{e^{\int -\frac{\frac{x+2}{x^4}}{\frac{1}{x^2}}\, dx}}{(u_1)^2}\, dx \\
&= u_1 \int \frac{e^{-\ln(x)+\frac{2}{x}}}{(u_1)^2}\, dx \\
&= u_1 \left(-\mathrm{e}^{-\ln\left(\frac{1}{x}\right)-\ln(x)} \ln\left(\frac{1}{x}\right) \right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
u &= c_1 u_1 + c_2 u_2 \\
&= c_1 \left(\mathrm{e}^{\frac{1}{x}}\right) + c_2 \left(\mathrm{e}^{\frac{1}{x}} \left(-\mathrm{e}^{-\ln\left(\frac{1}{x}\right)-\ln(x)} \ln\left(\frac{1}{x}\right)\right)\right)
\end{aligned}$$

Will add steps showing solving for IC soon.

Taking derivative gives

$$u'(x) = -\frac{c_1 \mathrm{e}^{\frac{1}{x}}}{x^2} + \frac{c_2 \mathrm{e}^{\frac{1}{x}} \ln\left(\frac{1}{x}\right)}{x^2} + \frac{c_2 \mathrm{e}^{\frac{1}{x}}}{x}$$

Doing change of constants, the solution becomes

$$y = -\frac{\left(-\frac{c_3 \mathrm{e}^{\frac{1}{x}}}{x^2} + \frac{\mathrm{e}^{\frac{1}{x}} \ln\left(\frac{1}{x}\right)}{x^2} + \frac{\mathrm{e}^{\frac{1}{x}}}{x}\right) x^2}{c_3 \mathrm{e}^{\frac{1}{x}} - \mathrm{e}^{\frac{1}{x}} \ln\left(\frac{1}{x}\right)}$$

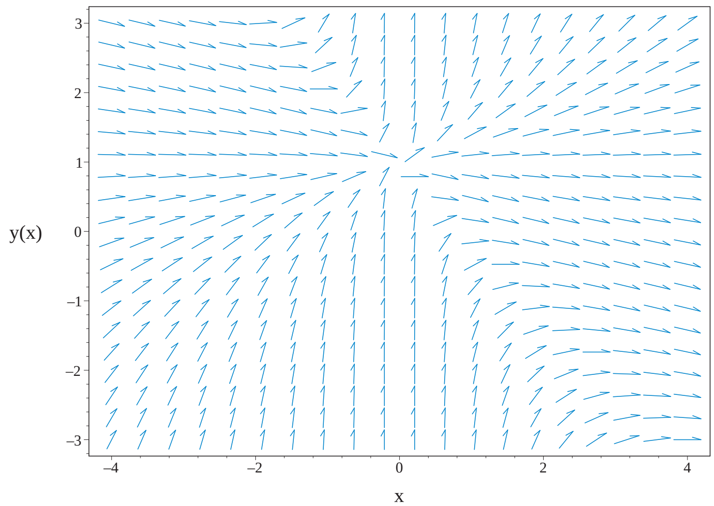


Figure 2.3: Slope field plot
$x^2y' = x(y-1) + (y-1)^2$

Summary of solutions found

$$y = -\frac{\left(-\frac{c_3\,\mathrm{e}^{\frac{1}{x}}}{x^2} + \frac{\mathrm{e}^{\frac{1}{x}}\ln\left(\frac{1}{x}\right)}{x^2} + \frac{\mathrm{e}^{\frac{1}{x}}}{x}\right)x^2}{c_3\,\mathrm{e}^{\frac{1}{x}} - \mathrm{e}^{\frac{1}{x}}\ln\left(\frac{1}{x}\right)}$$

Maple step by step solution

Let's solve

$x^2\left(\frac{d}{dx}y(x)\right) = x(y(x)-1) + (y(x)-1)^2$

- Highest derivative means the order of the ODE is 1

  $\frac{d}{dx}y(x)$

- Solve for the highest derivative

  $\frac{d}{dx}y(x) = \frac{x(y(x)-1)+(y(x)-1)^2}{x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
```

```
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
trying homogeneous types:
trying homogeneous D
<- homogeneous successful
<- homogeneous successful`
```

Maple dsolve solution

Solving time : 0.017 (sec)
Leaf size : 15

```
dsolve(diff(y(x),x)*x^2 = x*(y(x)-1)+(y(x)-1)^2,
       y(x),singsol=all)
```

$$y(x) = 1 - \frac{x}{\ln(x) + c_1}$$

Mathematica DSolve solution

Solving time : 0.312 (sec)
Leaf size : 23

```
DSolve[{x^2*D[y[x],x]==x*(y[x]-1)+(y[x]-1)^2,{}},
       y[x],x,IncludeSingularSolutions->True]
```

$$y(x) \to 1 + \frac{x}{-\log(x) + c_1}$$
$$y(x) \to 1$$