Singular solutions of first order differential equations

Nasser M. Abbasi

January 3, 2025 Compiled on January 3, 2025 at 1:32am

Contents

1 Introduction

Let the ode be given as $F(x, y, p) = 0$ where $p = y'(x)$, and let the solution be given as $\Psi(x, y, c) = 0$ where c is the constant of integration.

Given first order ode $F(x, y, y') = 0$ the goal is find its singular solutions (if one exists). This method applies to first order ode's which is not linear in y'. By singular solution here we mean the envelope (called E below).

These are solutions that satisfy the ode which can not be obtained from the general solution for any value of the constant of integration c (including $\pm\infty$).

A first order ode which is linear in y' do not have such singular solutions, in the sense of singular solution being an envelope of the general solution.

Singular solution will be called $y_c(x)$. This singular solution will be the envelope of the family of solutions of the general solution. It will have no constant in it, unlike the general solution.

If the ode is an initial value problem, and if the uniqueness theorem says there is a unique solution in an interval around (x_0, y_0) then no singular solution exists in that interval as this will violate the uniqueness theorem.

There are two methods for finding the singular solution. Either from the ode itself (without even knowing the solution) or from the general solution. The first is called the p-discriminant method and the second is called the c-discriminant method. Both of these methods are based on elimination.

In the p-discriminant method, p is eliminated. In the c-discriminant method, the constant c is eliminated. Clearly the p-discriminant method is preferred as it does not require solving the ode first.

In both elimination methods, we set up two equations and solve for y. This can result in more than one solution for y. Only those solutions that satisfy the ode itself are the envelope singular solutions we want. Others are important but they are not singular solution as they do not satisfy the ode and are not an envelope for the general solution. This diagram below shows the types of solutions that can be found by each elimination method. Only the envelope is a valid solution singular solution which is common.

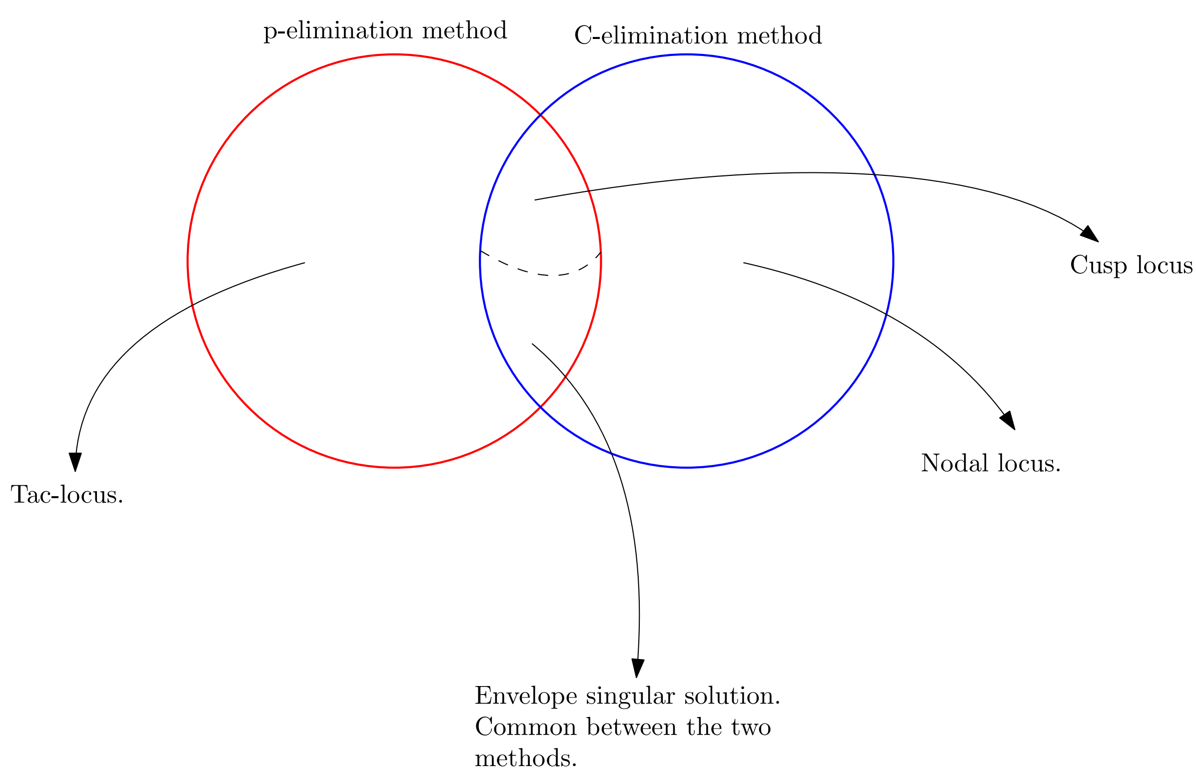

Figure 1: Different type of solutions obtain by elimination methods

In the above we see that there are 4 different type of solutions that show up. Only one type satisfies the ode. This is the envelope singular solution. Let us give some letters to refer to these solutions. Let E be the envelope. Let T be the Tac locus, let N be the nodal locus and let C be the cusp locus.

What happens in this. When doing the elimination process using either method, we can end up with more than one solution y. These we write in factored form.

The general form from the p-elimination will look like $ET^2C = 0$ and the general form from the c-elimination will look like $EN^2C^3 = 0$ (ref: Math24.net).

This is only the most general form. T or C or N could be missing.

We see that both E and C can show up from both methods. The factor that satisfies the ode is the E factor only, which is the envelope singular solution. All others do not satisfy the ode but they have interesting geometrical meanings when plotted against the family of general solution. Will give examples below to show this.

The E solution from the p-elimination is the locus of points satisfied by $F(x, y, p) = 0$ such that at each point the $p's$ have equal roots. And the E solution from the c-elimination is the locus of points satisfied by $\Psi(x, y, c) = 0$ such that at each point the $c's$ have equal roots. Both E solution from both methods is the same. So any one of these methods can be used to find E.

For the special case, when $F(x, y, p) = 0$ is quadratic in p or the solution $\Psi(x, y, c) = 0$ is quadratic in c, we can just use the quadratic discriminant $b^2 - 4ac = 0$ to find solutions for either p or c. This is simpler than using elimination method. But this only works in this special case. See examples below.

In summary, these are the main steps for finding singular solution for first order ode:

1. Find y_s using p-discriminant method by eliminating p from $F(x, y, p) = 0$ and $\frac{\partial F}{\partial p} = 0$ (or by direct use of quadratic formula discriminant in the special case when $F(x, y, p)$ is quadratic in p).

2. Substitue each value of p found from (1) into $F(x, y, p) = 0$ and solve for y. If more than one solution y is found above, pick the one that satisfies the ode. This is the E singular solution.

3. Find general solution to the ode. Written as $\Psi(x, y, c) = 0$

4. Eliminate c from this and the equation $\frac{\partial \Psi}{\partial c} = 0$ and find y_s. (or by direct use of quadratic formula discriminant in the special case when $\Psi(x, y, c)$ is quadratic in c). Substitue each value of c found into $\Psi(x, y, c) = 0$ and solve for y. If more than one solution found above, pick the one that satisfies the ode. This is the E singular solution.

5. Verify we obtain same E from step (2) and (4). We do not have to do both methods, as either method will give same E.

The Examples below show how to use these methods. In all the following examples, the plots will show the singular solution(s) as thick red dashed lines.

Given ode $F(x, y, p) = 0$ the necessary and sufficient conditions that a singular solution exist are the following (see E.L.Ince page 88)

1. $F = 0$

2. $\frac{\partial F}{\partial p} = 0$

3. $\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial y} = 0$

The above should be satisfied simultaneously. However, I am not able to verify these now. Ince says that $\frac{\partial F}{\partial y} \neq 0$ is necessary for singular solution to exist. So will only use this check in all examples below.

2 Examples

2.1 Example 1 $9(y')^2(2-y)^2 = 4(3-y)$

$$
\begin{aligned}
9(y')^2(2-y)^2 &= 4(3-y) \\
9p^2(2-y)^2 - 4(3-y) &= 0
\end{aligned}
$$

Since this is quadratic in p, we do not have to use elimination and can just use the quadratic discriminant

$$
\begin{aligned}
b^2 - 4ac &= 0 \\
0 - 4\left(9(2-y)^2\right)\left(-4(3-y)\right) &= 0 \\
9(2-y)^2(3-y) &= 0 \\
(2-y)^2(3-y) &= 0
\end{aligned}
$$

Comparing this to the form $ET^2C = 0$ we see that $y = 3$ is E (the envelop) and $y = 2$ is T (the Tac locus). This does not satisfy the ode. Let us see what happens if we use elimination method.

$$
\begin{aligned}
F = 9p^2(2-y)^2 - 4(3-y) &= 0 \\
\frac{\partial F}{\partial p} = 18p(2-y)^2 &= 0
\end{aligned}
$$

We first check that $\frac{\partial F}{\partial y} = -18p^2(2-y) + 4 \neq 0$. Eliminating p. Second equation gives $p = 0$ and $y = 2$. Substituting $p = 0$ into the first equation gives

$$
y = 3
$$

We now have to check if this solution satisfies the ode. We see it does. Hence it is the envelope. The second solution is $y = 2$ which does not satisfy the ode. This happens to be the Tac locus. We see that we obtain the same result using elimination as when using the quadratic discriminant directly.

To do the same thing use the solution, we have to first solve the ode. The general solution (also called the primitive) can be found to be

$$
\begin{aligned}
\Psi(x,y,c) &= (x+c)^2 - y^2(3-y) \\
&= x^2 + c^2 + 2xc - y^2(3-y) \\
&= 0
\end{aligned}
$$

Since this is already quadratic in c, we can use the the quadratic discriminant directly. (will use C instead of c so not to confuse it with the constant of integration c in the solution)

$$
\begin{aligned}
b^2 - 4aC &= 0 \\
(2x)^2 - 4(1)\left(-y^2(3-y) + x^2\right) &= 0 \\
4x^2 + 4y^2(3-y) - 4x^2 &= 0 \\
4y^2(3-y) &= 0 \\
y^2(3-y) &= 0
\end{aligned}
$$

Comparing this to the form $EN^2C^3 = 0$ shows that $E = 3$ which is same as found earlier using p-discriminant and $y = 0$ is N (nodal locus).

Hence in summary, we see that $E = 3, T = 2, N = 0$. Only the E (envelope solution $y = 3$) satisfies the ode. The others do not.

The following plot shows the singular solution as the envelope of the family of general solution plotted using different values of c and also shows the N and T curves.

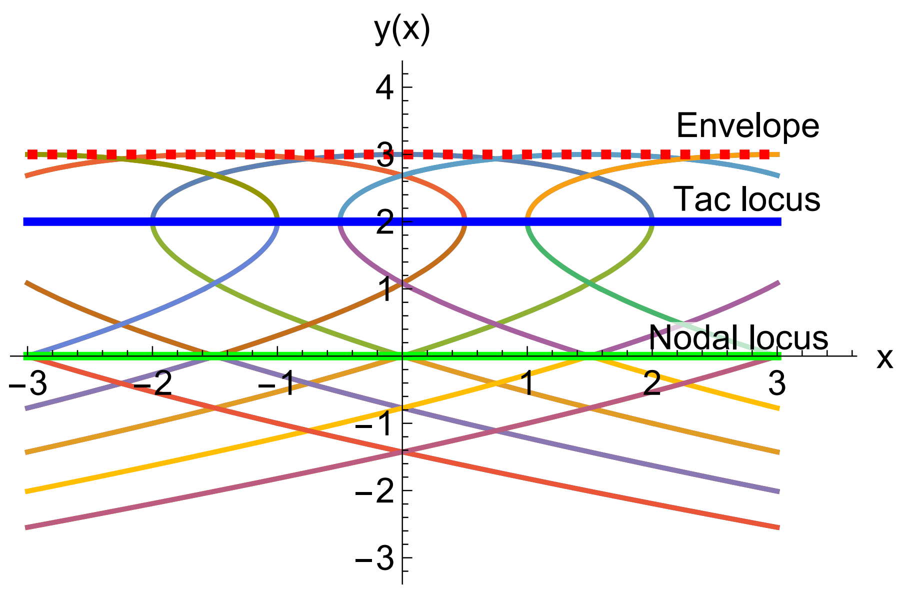

Figure 2: Singular solution envelope with general solution curves

2.2 Example 2 $(y')^2 = xy$

$$
\begin{aligned}
(y')^2 &= xy \\
p^2 - xy &= 0
\end{aligned}
$$

Since this is quadratic in p, we do not have to use elimination and can just use the quadratic discriminant

$$
\begin{aligned}
b^2 - 4ac &= 0 \\
0 - 4(1)(-xy) &= 0 \\
4xy &= 0 \\
y &= 0
\end{aligned}
$$

This has form $ET^2C = 0$. We see that $y = 0$ satisfies the ode, hence it is E (because C do not satisfy the ode). We found no T nor C.

To apply c-discriminant we have to find the general solution. It will be

$$
\begin{aligned}
y_1 &= \frac{1}{36}\left(4x^3 - 12x^{\frac{3}{2}}c + 9c^2\right) \\
y_2 &= \frac{1}{36}\left(4x^3 + 12x^{\frac{3}{2}}c + 9c^2\right)
\end{aligned}
$$

Hence we have two general solutions. These can be written as

$$
\begin{aligned}
\Psi_1(x,y,c) &= y - \frac{1}{36}\left(4x^3 - 12x^{\frac{3}{2}}c + 9c^2\right) = 0 \\
&= y + \frac{1}{3}cx^{\frac{3}{2}} - \frac{1}{4}c^2 - \frac{1}{9}x^3 = 0 \\
\Psi_2(x,y,c) &= y - \frac{1}{36}\left(4x^3 + 12x^{\frac{3}{2}}c + 9c^2\right) = 0 \\
&= y - \frac{1}{3}cx^{\frac{3}{2}} - \frac{1}{4}c^2 - \frac{1}{9}x^3 = 0
\end{aligned}
$$

Since these are quadratic in c, we can use the quadratic discriminant. First equation above gives

$$\begin{aligned} b^2 - 4aC &= 0 \\ \left(\frac{1}{3}x^{\frac{3}{2}}\right)^2 - 4\left(-\frac{1}{4}\right)\left(y - \frac{1}{9}x^3\right) &= 0 \\ \frac{1}{9}x^3 + y - \frac{1}{9}x^3 &= 0 \\ y &= 0 \end{aligned}$$

Comparing to $EN^2C^3 = 0$ shows this is E. Same as before. Second solution will give same result.

Hence $y = 0$ is the singular solution. No T, N, C were found. The following plot shows the singular solution as the envelope of the family of general solution plotted using different values of c.

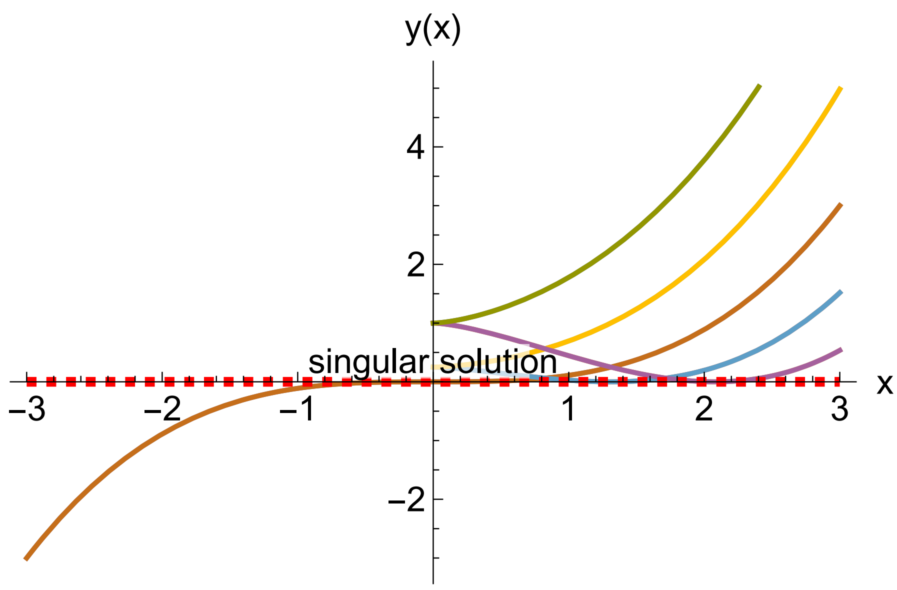

Figure 3: Singular solution envelope with general solution curves

2.3 Example 3 $27y - 8(y')^3 = 0$

$$\begin{aligned} 27y - 8(y')^3 &= 0 \\ 27y - 8p^3 &= 0 \end{aligned}$$

Since this is not quadratic in p, we can not use the discriminant dirctly and have to use elimination.

$$\begin{aligned} F = 27y - 8p^3 &= 0 \\ \frac{\partial F}{\partial p} = -24p^2 &= 0 \end{aligned}$$

We first check that $\frac{\partial F}{\partial y} = 27 \neq 0$. Second equation gives $p = 0$. Substituting this into first equation gives $27y = 0$ or $y = 0$. We see this also satisfies the ode. Hence it is E (the envelope). The general solution can be found as

$$\Psi(x, y, c) = y^2 - (x + c)^3 = 0$$

Applying c-discriminant

$$\begin{aligned} \Psi(x, y, c) &= y^2 - (x + c)^3 = 0 \\ \frac{\partial \Psi(x, y, c)}{\partial c} &= -3(x + c)^2 = 0 \end{aligned}$$

Second equation gives $(x+c)^2 = 0$ or $c = -x$. From first equation this gives $y^2 = 0$ or $y = 0$. This is the same as y_s found from p-discriminant, hence

$$y = 0$$

The following plot shows the singular solution as the envelope of the family of general solution plotted using different values of c.

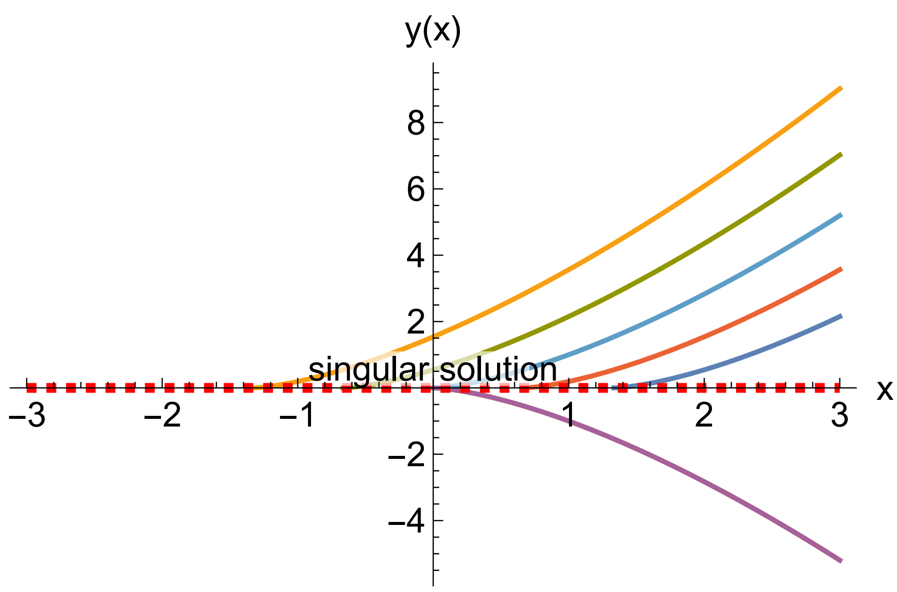

Figure 4: Singular solution envelope with general solution curves

2.4 Example 4 $y - 2xy' - \ln y' = 0$

$$\begin{aligned} y - 2xy' - \ln y' &= 0 \\ y - 2xp - \ln p &= 0 \end{aligned}$$

Applying p-discriminant method via elimination gives

$$\begin{aligned} F = y - 2xp - \ln p = 0 &= 0 \\ \frac{\partial F}{\partial p} = -2x - \frac{1}{p} &= 0 \end{aligned}$$

Second equation gives $p = -\frac{1}{2x}$. Substituting in the first equation gives

$$\begin{aligned} y + 1 - \ln \frac{-1}{2x} &= 0 \\ y &= \ln\left(\frac{-1}{2x}\right) - 1 \end{aligned}$$

This does not satisfy the ode. Hence no singular solution exist.

2.5 Example 5 $y - x(1+y') - (y')^2 = 0$

$$\begin{aligned} y - x(1+y') - (y')^2 &= 0 \\ F = y - x - xp - p^2 \\ &= 0 \end{aligned}$$

We first check that $\frac{\partial F}{\partial y} = 1 \neq 0$. Since this is quadratic in p, we do not have to use elimination and can just use the quadratic discriminant

$$\begin{aligned} b^2 - 4ac &= 0 \\ (-x)^2 - 4(-1)(y-x) &= 0 \\ x^2 + 4y - 4x &= 0 \\ y &= x - \frac{1}{4}x^2 \end{aligned}$$

This does not satisfy the ode. Hence no singular solution exist.

2.6 Example 6 $y - 2xy' - \sin(y') = 0$

$$
\begin{aligned}
y - 2xy' - \sin(y') &= 0 \\
y - 2xp - \sin(p) &= 0
\end{aligned}
$$

Applying p-discriminant method gives

$$
\begin{aligned}
F &= y - 2xp - \sin(p) = 0 \\
\frac{\partial F}{\partial y'} &= -2x - \cos(p) = 0
\end{aligned}
$$

We first check that $\frac{\partial F}{\partial y} = 1 \neq 0$. Now we apply p-discriminant. Second equation gives $-2x - \cos(p) = 0$ or $p = \arccos(-2x)$. Substituting in the first equation gives $y - 2x\arccos(-2x) - \sin(\arccos(-2x)) = 0$. I need to look at this more. This should give $y_s = 0$ but now it does not.

2.7 Example 7 $y - (y')^2 x + \frac{1}{y'} = 0$

$$
\begin{aligned}
y - (y')^2 x + \frac{1}{y'} &= 0 \\
y - p^2 x + \frac{1}{p} &= 0
\end{aligned}
$$

Applying p-discriminant method gives

$$
\begin{aligned}
F &= y - p^2 x + \frac{1}{p} = 0 \\
\frac{\partial F}{\partial y'} &= -2xp - \frac{1}{p^2} = 0
\end{aligned}
$$

We first check that $\frac{\partial F}{\partial y} = 1 \neq 0$. Now we apply p-discriminant. Second equation gives 3 solutions for p.

$$
\begin{aligned}
p_1 &= \frac{\left(-\frac{1}{2}\right)^{\frac{1}{3}}}{x^{\frac{1}{3}}} \\
p_2 &= \frac{1}{2^{\frac{1}{3}} x^{\frac{1}{3}}} \\
p_3 &= -\frac{(-1)^{\frac{2}{3}}}{2^{\frac{1}{3}} x^{\frac{1}{3}}}
\end{aligned}
$$

Using the first solution, then the first equation gives

$$
\begin{aligned}
y - \left(\frac{\left(-\frac{1}{2}\right)^{\frac{1}{3}}}{x^{\frac{1}{3}}}\right)^2 x + \frac{1}{\left(\frac{\left(-\frac{1}{2}\right)^{\frac{1}{3}}}{x^{\frac{1}{3}}}\right)} &= 0 \\
y_s &= \frac{3}{2}(-1)^{\frac{2}{3}} \sqrt[3]{2}\sqrt[3]{x}
\end{aligned}
$$

Now we check if this satisfies the ode $F = 0$. It does not. Trying the second solution $p_2 = \frac{1}{2^{\frac{1}{3}} x^{\frac{1}{3}}}$. Substituting into $F = 0$ gives

$$
\begin{aligned}
y - \left(\frac{1}{2^{\frac{1}{3}} x^{\frac{1}{3}}}\right)^2 x + \frac{1}{\left(\frac{1}{2^{\frac{1}{3}} x^{\frac{1}{3}}}\right)} &= 0 \\
y &= -\frac{1}{2}\sqrt[3]{2}\sqrt[3]{x}
\end{aligned}
$$

Now we check if this satisfies the ode $F=0$. It does not. Trying the third solution $p_3=-\frac{(-1)^{\frac{2}{3}}}{2^{\frac{1}{3}}x^{\frac{1}{3}}}$. Substituting into $F=0$ gives

$$
\begin{aligned}
y-\left(-\frac{(-1)^{\frac{2}{3}}}{2^{\frac{1}{3}}x^{\frac{1}{3}}}\right)^2 x+\frac{1}{\left(-\frac{(-1)^{\frac{2}{3}}}{2^{\frac{1}{3}}x^{\frac{1}{3}}}\right)}&=0\\
y&=-\frac{3}{2}\sqrt[3]{-1}\sqrt[3]{2}\sqrt[3]{x}
\end{aligned}
$$

Now we check if this satisfies the ode $F=0$. It does not. Hence no singular exist.

2.8 Example 8 $xy'+y'-(y')^2-y=0$

$$
\begin{aligned}
xy'+y'-(y')^2-y&=0\\
xp+p-p^2-y&=0\\
p^2-p(1+x)+y&=0
\end{aligned}
$$

This is quadratic in p.

$$
\begin{aligned}
b^2-4ac&=0\\
(-(1+x))^2-4(1)(y)&=0\\
(1+x)^2-4y&=0\\
y&=\frac{(1+x)^2}{4}
\end{aligned}
$$

Now we verify this satisfies the ode. We see it does. Now we have to find the general solution. It will be

$$
\begin{aligned}
y&=c+cx-c^2\\
0&=y-c(1+x)+c^2
\end{aligned}
$$

This is quadratic in c.

$$
\begin{aligned}
b^2-4aC&=0\\
(-(1+x))^2-4(1)(y)&=0\\
(1+x)^2-4y&=0\\
y&=\frac{(1+x)^2}{4}
\end{aligned}
$$

We see this is the same as y from the p-discriminant method. Hence it is a singular solution. The following plot shows the singular solution as the envelope of the family of general solution plotted using different values of c.

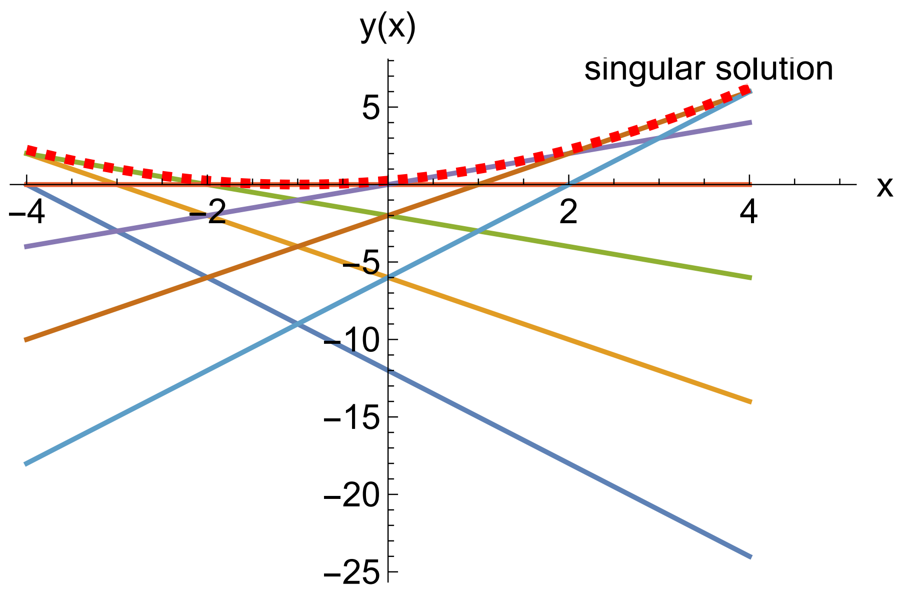

Figure 5: Singular solution envelope with general solution curves

2.9 Example 9 $y = y'x + \sqrt{4 + y'^2}$

$$
\begin{aligned}
y &= y'x + \sqrt{4 + y'^2} \\
y - px - \sqrt{4 + p^2} &= 0
\end{aligned}
$$

Applying p-discriminant method gives

$$
\begin{aligned}
F = y - px - \sqrt{4 + p^2} &= 0 \\
\frac{\partial F}{\partial p} = -x - \frac{p}{\sqrt{4 + p^2}} &= 0
\end{aligned}
$$

We first check that $\frac{\partial F}{\partial y} = 1 \neq 0$. Now we apply p-discriminant. Second equation gives $x\sqrt{4 + p^2} + p = 0$ which gives

$$
\begin{aligned}
x\sqrt{4 + p^2} + p &= 0 \\
-\frac{p}{x} &= \sqrt{4 + p^2} \\
\frac{p^2}{x^2} &= 4 + p^2 \\
p^2\left(1 - \frac{1}{x^2}\right) + 4 &= 0 \\
p^2 &= \frac{-4x^2}{x^2 - 1} \\
p^2 &= \frac{4x^2}{1 - x^2} \\
p &= \pm \frac{2x}{\sqrt{1 - x^2}}
\end{aligned}
$$

Trying the negative root and substituting it in $F=0$ gives

$$
\begin{aligned}
y - px - \sqrt{4+p^2} &= 0\\
y - \left(-\frac{2x}{\sqrt{1-x^2}}\right)x - \sqrt{4+\left(-\frac{2x}{\sqrt{1-x^2}}\right)^2} &= 0\\
y + \frac{2x^2}{\sqrt{1-x^2}} - \sqrt{4+\frac{4x^2}{1-x^2}} &= 0\\
y + \frac{2x^2}{\sqrt{1-x^2}} - 2\frac{\sqrt{1-x^2+x^2}}{\sqrt{1-x^2}} &= 0\\
y + \frac{2x^2}{\sqrt{1-x^2}} - \frac{2}{\sqrt{1-x^2}} &= 0\\
y + \frac{2(x^2-1)}{\sqrt{1-x^2}} &= 0\\
y + \frac{2(x^2-1)\sqrt{1-x^2}}{1-x^2} &= 0\\
y + \frac{-2(1-x^2)\sqrt{1-x^2}}{1-x^2} &= 0\\
y - 2\sqrt{1-x^2} &= 0\\
y &= 2\sqrt{1-x^2}
\end{aligned}
$$

Which satisfies the ode. The general solution can be found to be

$$
\Psi(x,y,c) = y - xc - \sqrt{4+c^2} = 0
$$

Now we have to eliminate c using the c-discriminant method

$$
\begin{aligned}
\Psi(x,y,c) &= y - xc - \sqrt{4+c^2} = 0\\
\frac{\partial \Psi(x,y,c)}{\partial c} &= -x - \frac{2c}{2\sqrt{4+c^2}} = 0
\end{aligned}
$$

Second equation gives

$$
c = \pm \frac{2x}{\sqrt{1-x^2}}
$$

Taking the negative root, and substituting into the first equation gives

$$
\begin{aligned}
y - x\left(\frac{-2x}{\sqrt{1-x^2}}\right) - \sqrt{4+\left(\frac{2x}{\sqrt{1-x^2}}\right)^2} &= 0\\
y + \left(\frac{2x^2}{\sqrt{1-x^2}}\right) - 2\sqrt{1+\frac{x^2}{1-x^2}} &= 0\\
y + \left(\frac{2x^2}{\sqrt{1-x^2}}\right) - \frac{2}{\sqrt{1-x^2}} &= 0\\
y - \frac{2(1-x^2)}{\sqrt{1-x^2}} &= 0\\
y - \frac{2(1-x^2)\sqrt{1-x^2}}{1-x^2} &= 0\\
y - 2\sqrt{1-x^2} &= 0\\
y_s &= 2\sqrt{1-x^2}
\end{aligned}
$$

Which is the same obtained using the p-discriminant. Hence

$$
y = 2\sqrt{1-x^2}
$$

Is singular solution. We have to try the other root also. But graphically, the above seems to be the only valid singular solution. The following plot shows the singular solution as the envelope of the family of general solution plotted using different values of c.

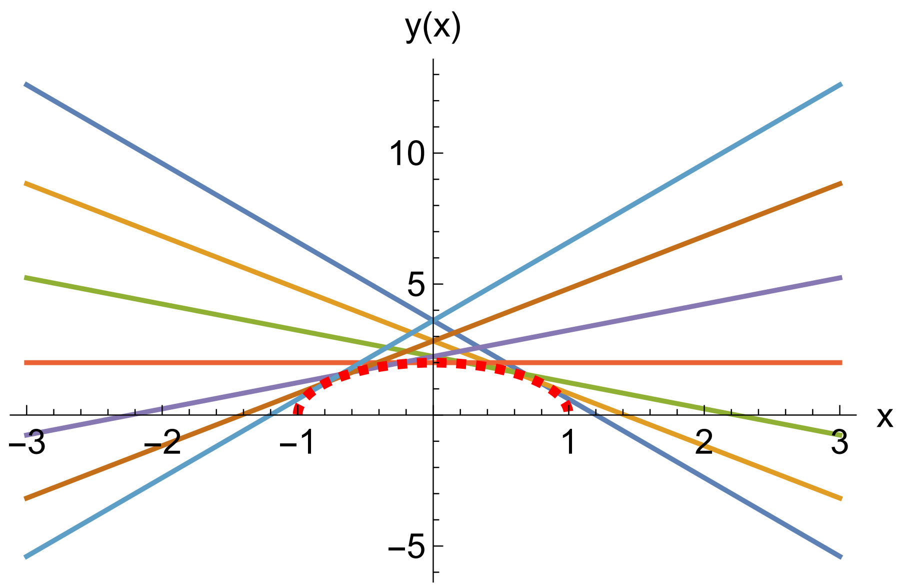

Figure 6: Singular solution envelope with general solution curves

2.10 Example 10 $(y')^2 - xy' + y = 0$

$$
\begin{aligned}
(y')^2 - xy' + y &= 0 \\
p^2 - xp + y &= 0 \\
F &= p^2 - xp + y \\
&= 0
\end{aligned}
$$

We first check that $\frac{\partial F}{\partial y} = 1 \neq 0$. This is quadratic in p.

$$
\begin{aligned}
b^2 - 4ac &= 0 \\
(-x)^2 - 4(1)(y) &= 0 \\
x^2 - 4y &= 0 \\
y &= \frac{x^2}{4}
\end{aligned}
$$

This also satisfies the ode. Hence it is E. Now we check using p-discriminant method. General solution can be found to be

$$
\Psi(x, y, c) = y - xc + c^2 = 0
$$

This is quadratic in c.

$$
\begin{aligned}
b^2 - 4aC &= 0 \\
(-x)^2 - 4(1)(y) &= 0 \\
x^2 - 4y &= 0 \\
y &= \frac{x^2}{4}
\end{aligned}
$$

This is the same as y obtained using p-discriminant method then it is singular solution. The following plot shows the singular solution as the envelope of the family of general solution plotted using different values of c.

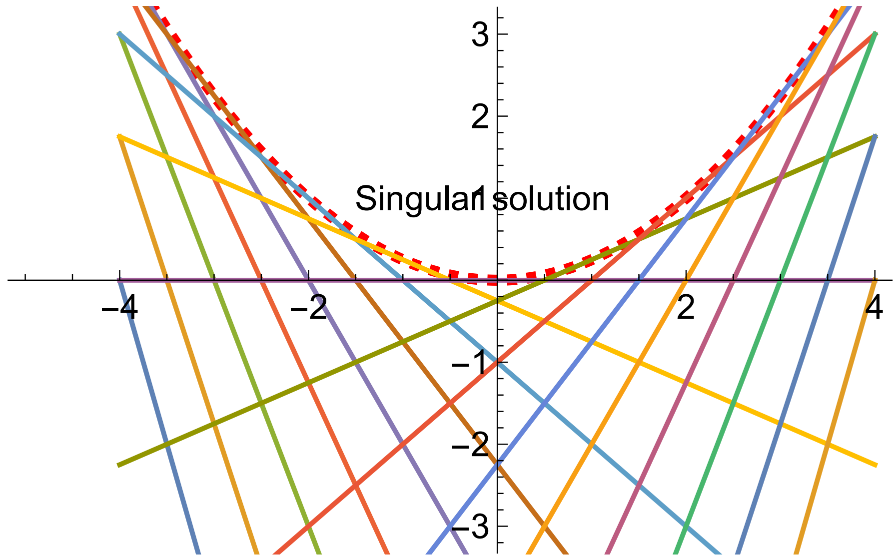

Figure 7: Singular solution envelope with general solution curves

2.11 Example 11 $y' = y(1-y)$

$$\begin{aligned} y' &= y(1-y) \\ F &= p - y(1-y) \\ &= 0 \end{aligned}$$

Since this is linear in p, then there is no singular solution in the sense of envelope. We can not use the p-discriminant method. Mathematica gives singular solutions, but these are not the envelope. It calls them equilibrium solutions. Lets find them. By inspection we see this is separable. Hence the candidate singular solutions are obtain when $y(1-y) = 0$. This is because this is what we have to divide both sides by to integrate. Therefore

$$\begin{aligned} y_s &= 0 \\ y_s &= 1 \end{aligned}$$

The general solution is found from

$$\begin{aligned} \int \frac{dy}{y\,(1-y)} &= \int dx \\ -\ln(y-1) + \ln y &= x + c \\ \ln \frac{y}{y-1} &= x + c \\ \frac{y}{y-1} &= c_1 e^x \\ y &= (y-1)\,c_1 e^x \\ y - y c_1 e^x &= -c_1 e^x \\ y(1 - c_1 e^x) &= -c_1 e^x \\ y &= \frac{c_1 e^x}{c_1 e^x - 1} \\ y &= \frac{c_1}{c_1 - e^{-x}} \\ y &= \frac{1}{1 - c_2 e^{-x}} \end{aligned}$$

Now we ask, can the singular solutions $y_s = 0, y_s = 1$ be obtained from the above general solution for any value of c_2? We see when $c_2 = 0$ then $y = 1$. Also when $c_2 = \infty$ then $y = 0$. So these are not really singular solutions. Mathematica call these equilibrium solutions.

But these should not be called singular solutions. Mathematica generates these when using the option *IncludeSingularSolutions*. But Maple does not give these when using the option *singsol=all*.

The following plot shows these equilibrium solutions with the general solution plotted using different values of c.

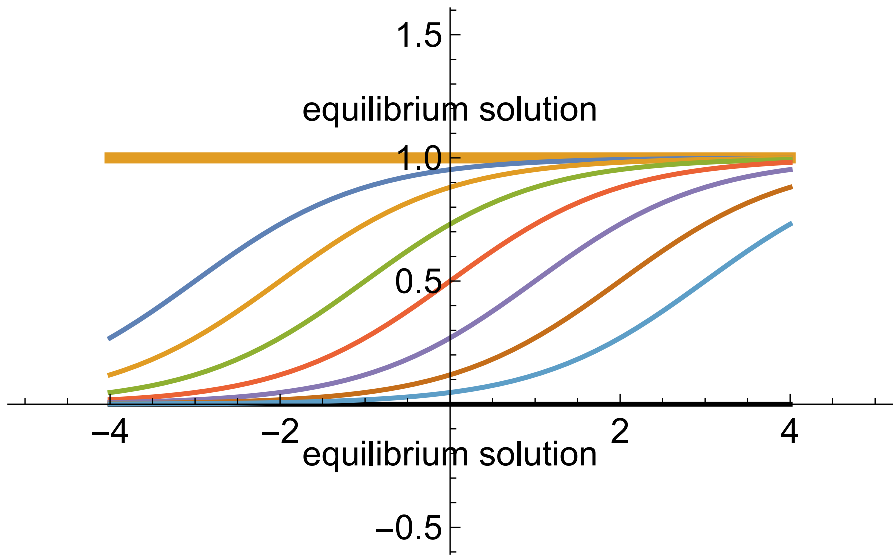

Figure 8: Equilibrium solution envelope with general solution curves

2.12 Example 12 $(y')^2 x + y'y\ln y - y^2(\ln y)^4 = 0$

$$\begin{aligned}(y')^2 x + y'y\ln y - y^2(\ln y)^4 &= 0\\ p^2x + py\ln y - y^2(\ln y)^4 &= 0\end{aligned}$$

Applying p-discriminant method gives

$$\begin{aligned}F = p^2x + py\ln y - y^2(\ln y)^4 &= 0\\ \frac{\partial F}{\partial p} = 2px + y\ln y &= 0\end{aligned}$$

We first check that $\frac{\partial F}{\partial y} \neq 0$. Now we apply p-discriminant. Eliminating p. Second equation gives $p = -\frac{y}{2x}\ln y$. Substituting into the first equation gives

$$\begin{aligned}\left(-\frac{y}{2x}\ln y\right)^2 x + \left(-\frac{y}{2x}\ln y\right) y\ln y - y^2(\ln y)^4 &= 0\\ \frac{y^2}{4x}(\ln y)^2 - \frac{y^2}{2x}(\ln y)^2 - y^2(\ln y)^4 &= 0\\ y^2\ln(y)^2\left(\frac{1}{4x} - \frac{1}{2x} - (\ln y)^2\right) &= 0\end{aligned}$$

Hence we obtain the solutions

$$\begin{aligned}y &= 0\\ y &= 1\\ \frac{1}{4x} - \frac{1}{2x} - (\ln y)^2 &= 0\end{aligned}$$

Or

$$\begin{aligned}y &= 0 \qquad (1)\\ y &= 1 \qquad (2)\\ 1 + 4x(\ln y)^2 &= 0 \qquad (3)\end{aligned}$$

The solution $y=0$ does not satisfy the ode. But $y=1$ does. The solution $1+4x(\ln y)^2=0$ gives $y=\begin{cases} e^{\frac{-i}{2\sqrt{x}}} \\ e^{\frac{i}{2\sqrt{x}}} \end{cases}$ But these do not satisfy the ode.

The primitive can be found to be

$$\Psi(x,y,c)=y-e^{\frac{c}{c^2-x}}=0$$

Now we have to eliminate c using the c-discriminant method

$$\begin{aligned}\Psi(x,y,c)&=y-e^{\frac{c}{c^2-x}}=0\\ \frac{\partial\Psi(x,y,c)}{\partial c}&=\left(\frac{1}{c^2-x}-\frac{2c^2}{(c^2-x)^2}\right)e^{\frac{c}{c^2-x}}=0\end{aligned}$$

Second equation gives $\frac{1}{c^2-x}-\frac{2c^2}{(c^2-x)^2}=0$ or $e^{\frac{c}{c^2-x}}=0$. For the first one, $c=\pm\sqrt{-x}$. Substituting $\sqrt{-x}$ in first equation gives

$$\begin{aligned}y-e^{\frac{\sqrt{-x}}{-x-x}}&=0\\ y&=e^{\frac{\sqrt{-x}}{-2x}}\\ \ln y&=\frac{\sqrt{-x}}{-2x}\\ (\ln y)^2&=\frac{-x}{4x^2}\\ 4x(\ln y)^2+1&=0\\ y_s&=\begin{cases} e^{\frac{-i}{2\sqrt{x}}} \\ e^{\frac{i}{2\sqrt{x}}} \end{cases}\end{aligned} \tag{4}$$

These do not satisfy the ode. Can not obtain $y=1$ solution using c-discriminant.

See paper by C.N. SRINIVASINGAR, example 4. The following plot shows the singular solution above as the envelope of the family of general solution plotted using different values of c. Added also $y_s=1$.

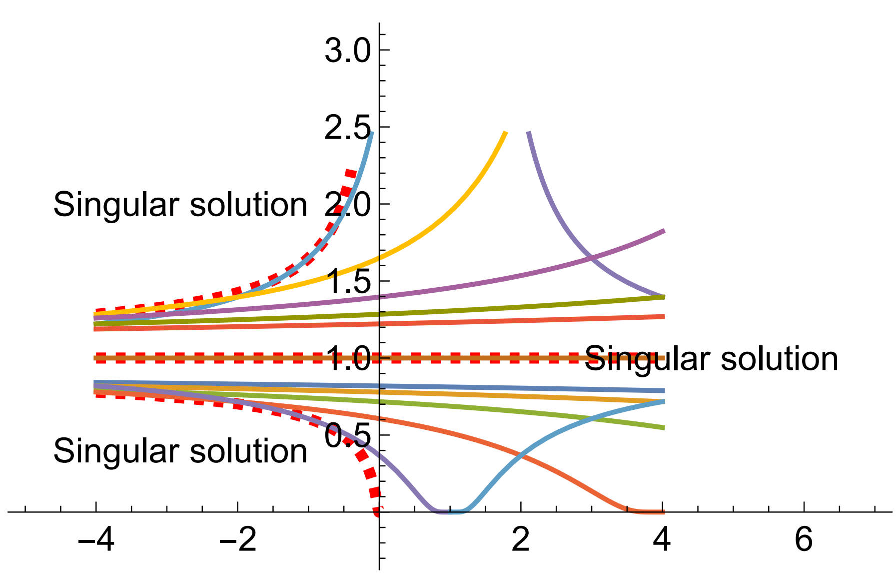

Figure 9: Singular solution envelope with general solution curves

2.13 Example 13 $(y')^2 - 4y = 0$

$$\begin{aligned}(y')^2 - 4y &= 0\\ F &= p^2 - 4y\\ &= 0\end{aligned}$$

This is quadratic in p.

$$\begin{aligned}b^2 - 4ac &= 0\\ (0) - 4(1)(-4y) &= 0\\ y &= 0\end{aligned}$$

We see this also satisfies the ode. Hence it is the envelope. The primitive can be found to be

$$\begin{aligned}\Psi(x,y,c) &= y - (x+c)^2 = 0\\ &= y - \left(x^2 + c^2 + 2xc\right) = 0\\ &= y - x^2 - c^2 - 2xc = 0\end{aligned}$$

This is quadratic in c.

$$\begin{aligned}b^2 - 4aC &= 0\\ (-2x)^2 - 4(-1)\left(y - x^2\right) &= 0\\ 4x^2 + 4y - 4x^2 &= 0\\ y &= 0\end{aligned}$$

This is the same as found by p-discriminant method then this is the singular solution. The following plot shows the singular solution as the envelope of the family of general solution plotted using different values of c.

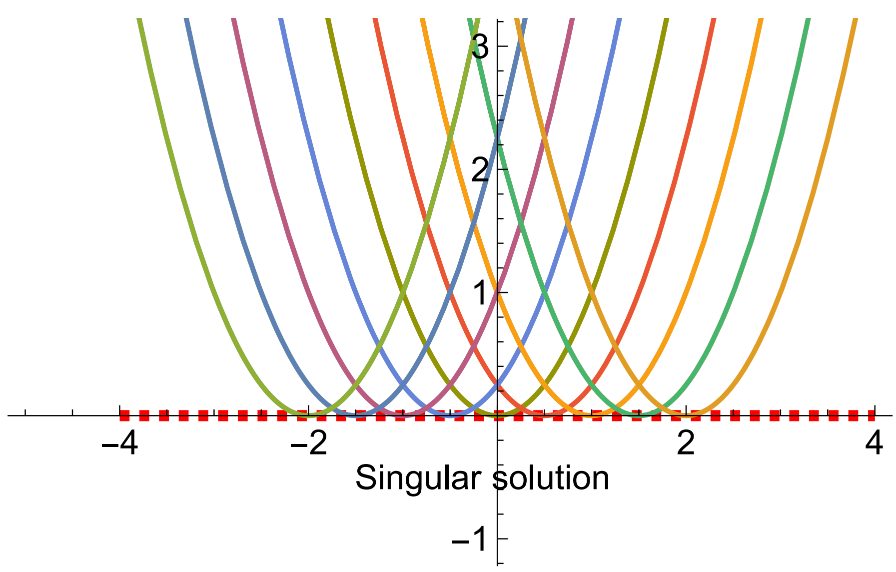

Figure 10: Singular solution envelope with general solution curves

2.14 Example 14 $1 + (y')^2 - \frac{1}{y^2} = 0$

$$
\begin{aligned}
1 + (y')^2 - \frac{1}{y^2} &= 0 \\
F &= 1 + p^2 - \frac{1}{y^2} \\
&= 0
\end{aligned}
$$

We first check that $\frac{\partial F}{\partial y} = 2\frac{1}{y^3} \neq 0$. This is quadratic in p.

$$
\begin{aligned}
b^2 - 4ac &= 0 \\
(0) - 4(1)\left(1 - \frac{1}{y^2}\right) &= 0 \\
1 - \frac{1}{y^2} &= 0 \\
y^2 &= 1 \\
y &= \pm 1
\end{aligned}
$$

We see both solutions also satisfy the ode. The primitive can be found to be

$$
\begin{aligned}
\Psi(x, y, c) &= y^2 + (x + c)^2 - 1 \\
&= y^2 + x^2 + c^2 + 2xc - 1 \\
&= 0
\end{aligned}
$$

This is quadratic in c.

$$
\begin{aligned}
b^2 - 4aC &= 0 \\
(2x)^2 - 4(1)\left(y^2 + x^2 - 1\right) &= 0 \\
4x^2 - 4y^2 - 4x^2 + 4 &= 0 \\
y^2 &= 1 \\
y &= \pm 1
\end{aligned}
$$

Which agrees with the p-discriminant. Hence these are the singular solutions. The following plot shows the singular solution as the envelope of the family of general solution plotted using different values of c.

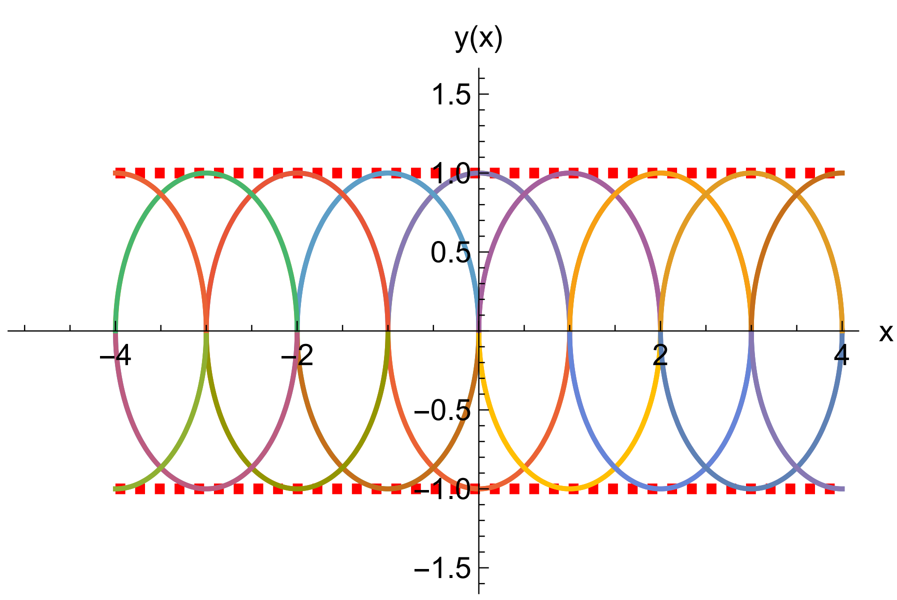

Figure 11: Singular solution envelope with general solution curves

2.15 Example 15 $y-(y')^2+3xy'-3x^2=0$

$$
\begin{aligned}
y-(y')^2+3xy'-3x^2&=0\\
y-p^2+3xp-3x^2&=0\\
F&=y-p^2+3xp-3x^2\\
&=0
\end{aligned}
$$

We first check that $\frac{\partial F}{\partial y}=1\neq 0$. This is quadratic in p.

$$
\begin{aligned}
b^2-4ac&=0\\
(3x)^2-4(-1)\left(y-3x^2\right)&=0\\
9x^2+4y-12x^2&=0\\
y&=\frac{3}{4}x^2
\end{aligned}
$$

Which satisfies the ode. Hence it is the envelope. The primitive can be found to be

$$\Psi(x,y,c)=y-cx-c^2-x^2=0$$

This is quadratic in c.

$$
\begin{aligned}
b^2-4aC&=0\\
(-x)^2-4(-1)\left(y-x^2\right)&=0\\
x^2+4y-4x^2&=0\\
y&=\frac{3}{4}x^2
\end{aligned}
$$

Which agrees with the p-discriminant curve. Hence this is a singular solution. The following plot shows the singular solution as the envelope of the family of general solution plotted using different values of c.

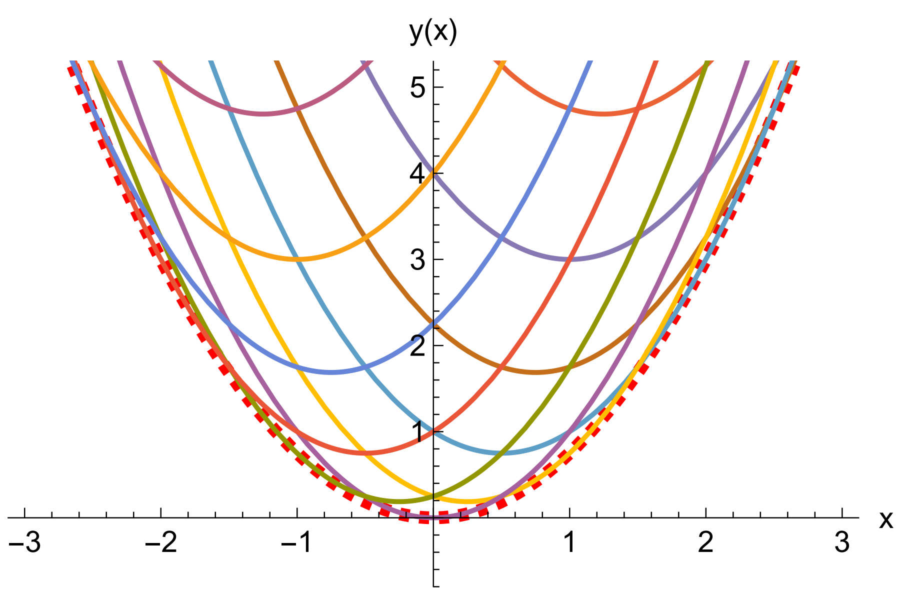

Figure 12: Singular solution envelope with general solution curves

2.16 Example 16 $(y')^2(1-y)^2-2+y=0$

$$
\begin{aligned}
(y')^2(1-y)^2-2+y&=0\\
p^2(1-y)^2-2+y&=0
\end{aligned}
$$

Hence

$$
\begin{aligned}
F&=p^2(1-y)^2-2+y \qquad (1)\\
&=0
\end{aligned}
$$

We first check that $\frac{\partial F}{\partial y}=-2(y')^2(1-y)+1\neq 0$. Since the ode is quadratic in p, we can use the more direct method which is the discriminant of the quadratic equation instead of the elimination method. The discriminant of (1) is

$$
\begin{aligned}
b^2-4aC&=0\\
0-4(1-y)^2(y-2)&=0\\
(1-y)^2(y-2)&=0
\end{aligned}
$$

Comparing this to $ET^2C=0$ shows that $y=2$ is E, i.e. the envelope singular solution which also satisfies the ode, while $y=1$ is T which is Tac locus. This does not satisfy the ode but it plotted below to show geometrically what it means.

The primitive can be found to be

$$
\Psi(x,y,c)=4(2-y)(y+1)^2-9(x+c)^2=0
$$

Since this is also quadratic in c, we can use the more direct method which is the discriminant of the quadratic equation instead of the elimination method. Hence

$$
\begin{aligned}
4(2-y)(y+1)^2-9\left(x^2+c^2+2cx\right)&=0\\
4(2-y)(y+1)^2-9x^2-9c^2-18cx&=0
\end{aligned}
$$

Therefore

$$
\begin{aligned}
b^2-4aC&=0\\
(-18x)^2-4(-9)\left(4(2-y)(y+1)^2-9x^2\right)&=0\\
(-18x)^2+144(2-y)(y+1)^2-324x^2&=0\\
144(2-y)(y+1)^2&=0\\
(2-y)(y+1)^2&=0
\end{aligned}
$$

Comparing to general form $EN^2C^3=0$ shows that $y=2$ is E which agrees with what was found using p-discriminant as expected, and $y=-1$ is N which is nodal locus N since it shows to power of two. Notice that N can only show up using c-discriminant method.

The following plot shows the singular solution as the envelope of the family of general solution plotted using different values of c, and also show the nodal locus N shown as solid green line and T (Tac locus) drawn as solid blue line.

The above result of E,N,T can also be found using elimination method. But since the equation are already quadratic in p,c, it can be easier to just use the quadratic equation discriminant directly. As metioned before, elimination method is more general since it works for any equation and not just quadratic.

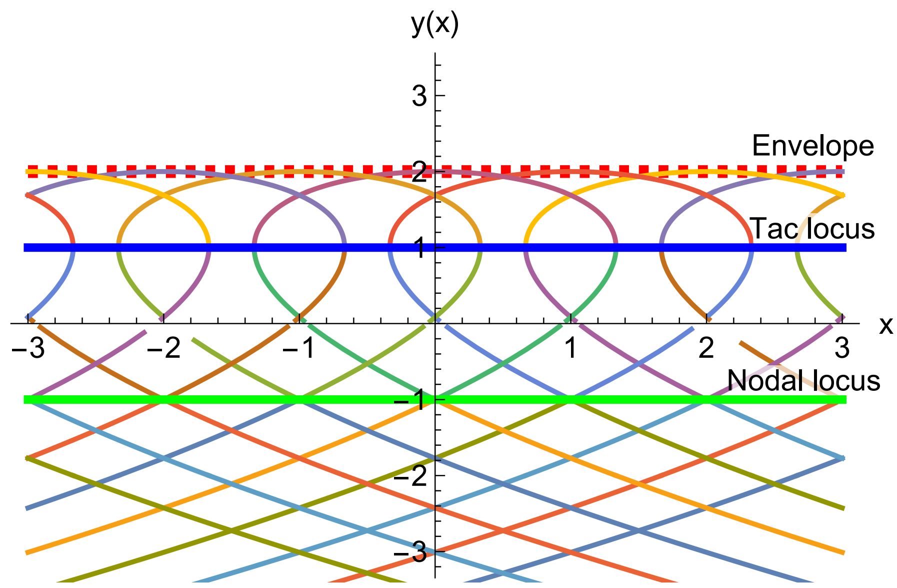

Figure 13: Singular solution envelope with general solution curves

2.17 Example 17 $(y - xy')^2 - (y')^2 = 1$

$$
\begin{aligned}
(y - xy')^2 - (y')^2 &= 1\\
(y - xp)^2 - p^2 &= 1\\
F &= 1 + p^2 - (y - xp)^2\\
&= 1 + p^2 - \left(y^2 + x^2p^2 - 2yxp\right)\\
&= 1 + p^2 - y^2 - x^2p^2 + 2yxp\\
&= p^2\left(1 - x^2\right) + p(2yx) + 1 - y^2\\
&= 0
\end{aligned}
$$

We first check that $\frac{\partial F}{\partial y} = 2(y - xp) \neq 0$. This is quadratic in p. Hence

$$
\begin{aligned}
b^2 - 4ac &= 0\\
(2yx)^2 - 4\left(1 - x^2\right)\left(1 - y^2\right) &= 0\\
4x^2 + 4y^2 - 4 &= 0\\
x^2 + y^2 - 1 &= 0\\
y &= \pm\sqrt{1 - x^2}
\end{aligned}
$$

Both of these solutions verify the ode. Hence they are both singular solutions. The primitive can be found to be

$$
\begin{aligned}
\Psi(x, y, c) &= y - xc \pm \sqrt{1 + c^2}\\
&= 0
\end{aligned}
$$

Eliminating c. First solution gives

$$
\begin{aligned}
\Psi_1(x, y, c) &= y - xc + \sqrt{1 + c^2} = 0\\
\frac{\partial \Psi_1(x, y, c)}{\partial c} &= -x + \frac{1}{2}\frac{2c}{\sqrt{1 + c^2}} = 0
\end{aligned}
$$

Second equation gives $-x + \frac{1}{2}\frac{2c}{\sqrt{1+c^2}} = 0$ or $c = x\sqrt{\frac{1}{1-x^2}}$. Substituting into the first equation above gives

$$
\begin{aligned}
y - x\left(x\sqrt{\frac{1}{1-x^2}}\right) + \sqrt{1 + \left(x\sqrt{\frac{1}{1-x^2}}\right)^2} &= 0 \\
y - x^2\sqrt{\frac{1}{1-x^2}} + \sqrt{1 + \frac{x^2}{1-x^2}} &= 0 \\
y - x^2\sqrt{\frac{1}{1-x^2}} + \sqrt{\frac{1-x^2+x^2}{1-x^2}} &= 0 \\
y - x^2\sqrt{\frac{1}{1-x^2}} + \sqrt{\frac{1}{1-x^2}} &= 0 \\
y + \sqrt{\frac{1}{1-x^2}}\left(1-x^2\right) &= 0
\end{aligned}
$$

$$
\begin{aligned}
y &= \sqrt{\frac{1}{1-x^2}}\left(x^2-1\right) \\
&= \frac{\left(x^2-1\right)}{\sqrt{1-x^2}} \\
&= \frac{\left(x^2-1\right)\sqrt{1-x^2}}{1-x^2} \\
&= -\sqrt{1-x^2}
\end{aligned}
$$

Which is given by p-discriminant above. Hence it is singular solution. If we try $\Psi_2(x,y,c) = y - xc - \sqrt{1+c^2} = 0$ we also can verify the second singular solution. Hence

$$
y = \pm\sqrt{1-x^2}
$$

The following plot shows the singular solution as the envelope of the family of general solution plotted using different values of c.

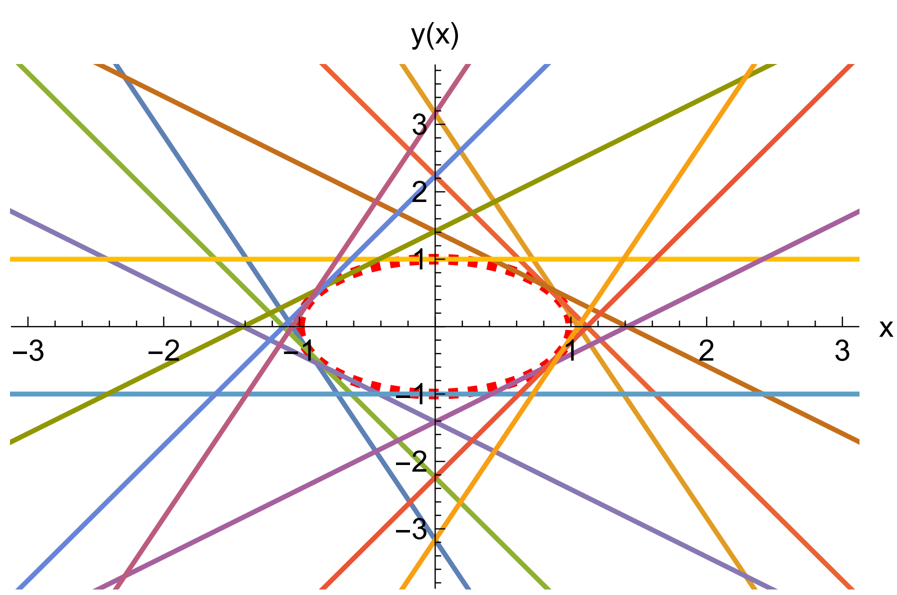

Figure 14: Singular solution envelope with general solution curves

2.18 Example 18 $y = xy' + ay'(1 - y')$

$$\begin{aligned}
y &= xy' + ay'(1 - y') \\
y &= xp + ap(1 - p) \\
F &= xp + ap - ap^2 - y \\
&= -ap^2 + p(a + x) - y \\
&= 0
\end{aligned}$$

We first check that $\frac{\partial F}{\partial y} = -1 \neq 0$. This is quadratic in p.

$$\begin{aligned}
b^2 - 4ac &= 0 \\
(a + x)^2 - 4(-a)(-y) &= 0 \\
(a + x)^2 - 4ay &= 0 \\
y &= \frac{(x + a)^2}{4a} \qquad a \neq 0
\end{aligned}$$

Now we check if this satisfies the ode itself. We see it does. The general solution can be found to be

$$\begin{aligned}
\Psi(x, y, c) &= y - cx - ac(1 - c) \\
&= y - cx - ac + ac^2 \\
&= y - c(x + a) + ac^2 \\
&= 0
\end{aligned}$$

This is quadratic in c.

$$\begin{aligned}
b^2 - 4aC &= 0 \\
(x + a)^2 - 4(a)(y) &= 0 \\
y &= \frac{(x + a)^2}{4a} \qquad a \neq 0
\end{aligned}$$

Which is the same obtained using p-discriminant. Hence this is the singular solution. The following plot shows the singular solution as the envelope of the family of general solution plotted using different values of c. For this, $a = 1$ was used.

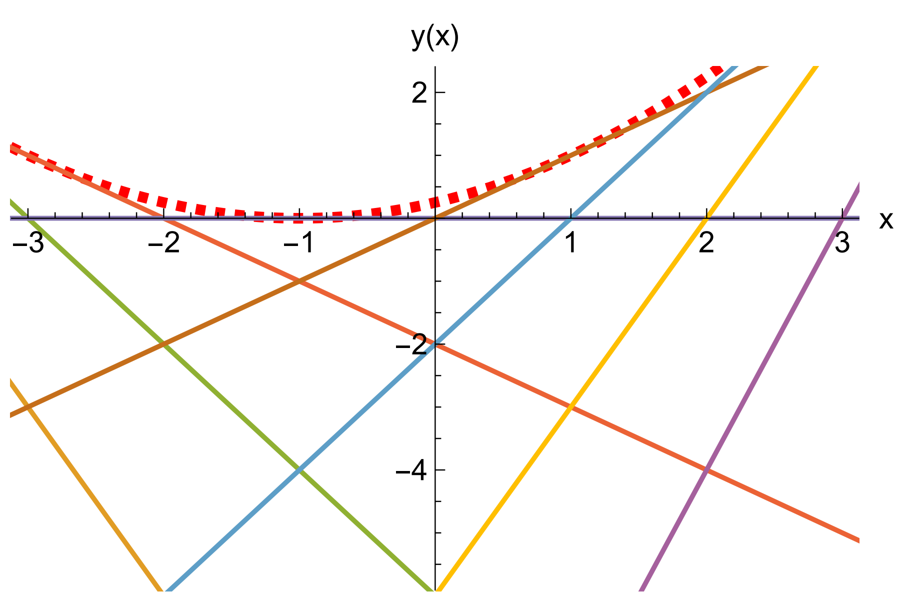

Figure 15: Singular solution envelope with general solution curves

2.19 Example 19 $(y')^3 - 4xyy' + 8y^2 = 0$

$$\begin{aligned}(y')^3 - 4xyy' + 8y^2 &= 0\\ p^3 - 4xyp + 8y^2 &= 0\end{aligned}$$

Applying p-discriminant method gives

$$\begin{aligned}F = p^3 - 4xyp + 8y^2 &= 0\\ \frac{\partial F}{\partial p} = 3p^2 - 4xy &= 0\end{aligned}$$

We first check that $\frac{\partial F}{\partial y} = -4xy' + 16y \neq 0$. Now we apply p-discriminant. Eliminating p. Second equation gives $p = \pm\left(\frac{4xy}{3}\right)^{\frac{1}{2}}$. Substituting first solution in the first equation gives

$$\begin{aligned}\left(\frac{4xy}{3}\right)^{\frac{3}{2}} - 4xy\left(\frac{4xy}{3}\right)^{\frac{1}{2}} + 8y^2 &= 0\\ y_s &= \frac{4}{27}x^3\end{aligned}$$

Which satisfies the ode. The general solution can be found to be

$$\Psi(x,y,c) = y - \frac{1}{4}\frac{x^2}{c} + \frac{1}{8}\frac{x}{c^2} - \frac{1}{64c^3} = 0$$

Hence

$$\begin{aligned}\Psi(x,y,c) = y - \frac{1}{4}\frac{x^2}{c} + \frac{1}{8}\frac{x}{c^2} - \frac{1}{64c^3} &= 0\\ \frac{\partial\Psi(x,y,c)}{\partial c} = \frac{1}{4}\frac{x^2}{c^2} - \frac{1}{4}\frac{x}{c^3} + \frac{3}{64c^4} &= 0\end{aligned}$$

Eliminating c. Second equation gives $c = \frac{1}{4x}$ or $c = \frac{3}{4x}$. Substituting $c = \frac{1}{4x}$ in the first equation above gives

$$\begin{aligned}y - \frac{1}{4}\frac{x^2}{\left(\frac{1}{4x}\right)} + \frac{1}{8}\frac{x}{\left(\frac{1}{4x}\right)^2} - \frac{1}{64\left(\frac{1}{4x}\right)^3} &= 0\\ y_s &= 0\end{aligned}$$

Which satisfies the ode. But $y = 0$ can be obtained from the general solution above when $c = \infty$ so it is not singular solution. Substituting $c = \frac{3}{4x}$ in the first equation above gives

$$\begin{aligned}y - \frac{1}{4}\frac{x^2}{\left(\frac{3}{4x}\right)} + \frac{1}{8}\frac{x}{\left(\frac{3}{4x}\right)^2} - \frac{1}{64\left(\frac{3}{4x}\right)^3} &= 0\\ y &= \frac{4}{27}x^3\end{aligned}$$

Which is the same obtained by p-discriminant. Hence this is the singular solution. The following plot shows the singular solution as the envelope of the family of general solution plotted using different values of c.

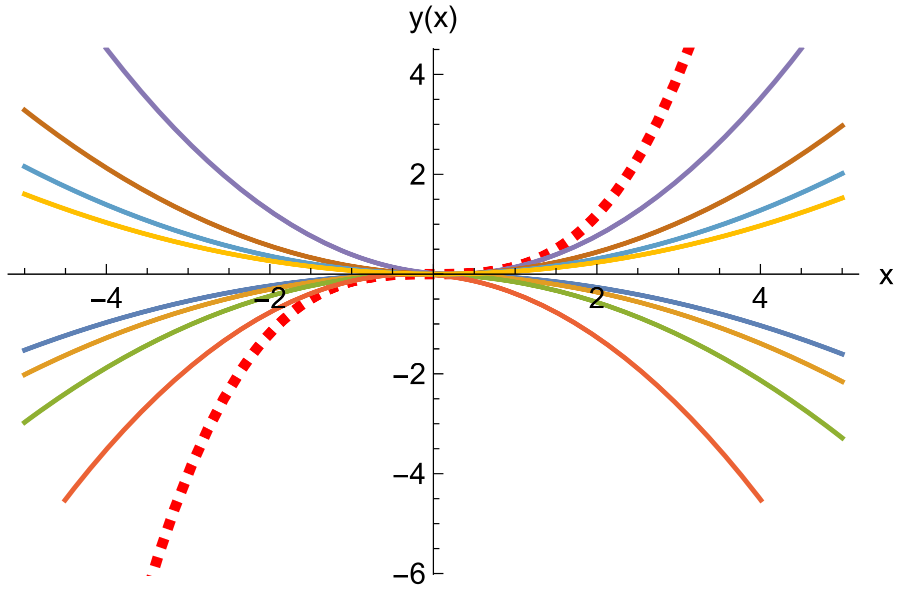

Figure 16: Singular solution envelope with general solution curves

2.20 Example 20 $4x(y')^2 = (2x-1)^2$

$$
\begin{aligned}
4x(y')^2 &= (2x-1)^2 \\
4xp^2 &= (2x-1)^2 \\
F &= -4xp^2 + (2x-1)^2 \\
&= 0
\end{aligned}
$$

We first check that $\frac{\partial F}{\partial y}$ and see it is zero. Hence no singular solution exists.

2.21 Example 21 $y' = 2x(1-y^2)^{\frac{1}{2}}$

$$
\begin{aligned}
y' &= 2x\left(1-y^2\right)^{\frac{1}{2}} \\
p &= 2x\left(1-y^2\right)^{\frac{1}{2}} \\
F &= -p + 2x\left(1-y^2\right)^{\frac{1}{2}} \\
&= 0
\end{aligned}
$$

Checking

$$
\begin{aligned}
\frac{\partial F}{\partial y} &= \frac{x}{2}\frac{-2y}{\sqrt{1-y^2}} \\
&= -\frac{xy}{\sqrt{1-y^2}}
\end{aligned}
$$

Not zero. But this is linear in p. Hence no singular solution will exist using p-discriminant. Lets see. Applying p-discriminant method gives

$$
\begin{aligned}
F = p - 2x\left(1-y^2\right)^{\frac{1}{2}} &= 0 \\
\frac{\partial F}{\partial p} = 1 &= 0
\end{aligned}
$$

The p-discriminant does not yield result since it gives $1 = 0$. Lets try C-discriminant.

$$
\begin{aligned}
\Psi(x,y,c) = y - \sin\left(x^2+2c\right) &= 0 \\
\frac{\partial \Psi(x,y,c)}{\partial c} = -2\cos\left(x^2+2c\right) &= 0
\end{aligned}
$$

Hence $x^2+2c=\frac{\pi}{2}$ (there are infinite solutions). Hence $c=\frac{\pi}{4}-\frac{x^2}{2}$. Substituting in the first equation gives

$$y-\sin\left(x^2+2\left(\frac{\pi}{4}-\frac{x^2}{2}\right)\right)=0$$
$$\begin{aligned} y &= \sin\left(x^2+2\left(\frac{\pi}{4}-\frac{x^2}{2}\right)\right)\\ &= \sin\left(\frac{\pi}{2}\right)\\ &= 1 \end{aligned}$$

And if we took $x^2+2c=-\frac{\pi}{2}$ then we now obtain $y_s=-1$. Now we check that $y=\pm 1$ satisfy the ode itself. We see that they do. This is an example where c-discriminant found singular solution but not p-discriminant. This is strange as books say that same E should result using both methods. Need to look more into this example.

The following plot shows the singular solution as the envelope of the family of general solution plotted using different values of c. p-discriminant does not yield result but C-discriminant does.

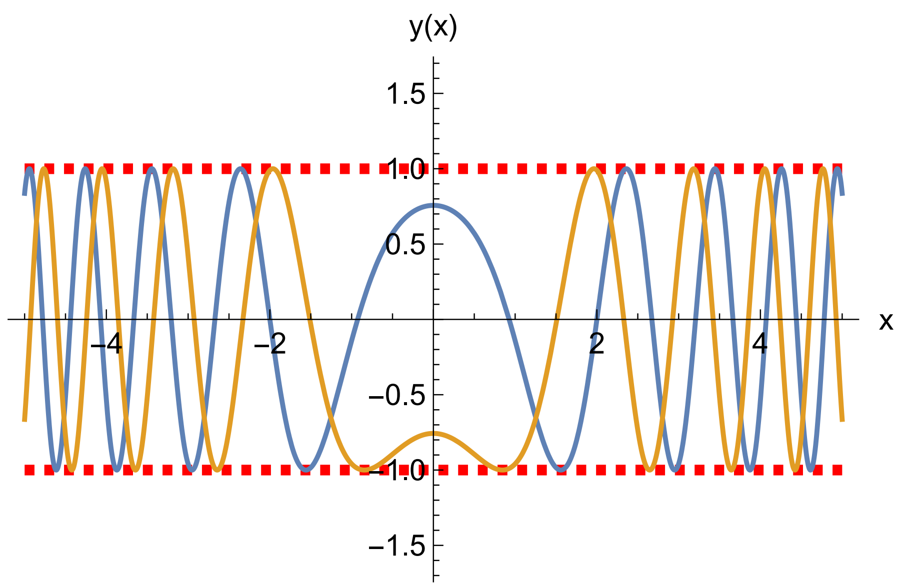

Figure 17: Singular solution envelope with general solution curves

2.22 Example 22 $(y')^2+2xy'-y=0$

$$\begin{aligned} (y')^2+2xy'-y &= 0\\ p^2+2xp-y &= 0\\ F &= p^2+2xp-y\\ &= 0 \end{aligned}$$

Checking $\frac{\partial F}{\partial y}=-1\neq 0$. Since quadratic in p then

$$\begin{aligned} b^2-4ac &= 0\\ (2x)^2-4(1)(-y) &= 0\\ 4x^2+4y &= 0\\ y &= -x^2 \end{aligned}$$

Now we check that this satisfies the ode itself. We see it does not. Now we try the c-discriminant method. The general solution is too complicated to write here. But Mathematica and Maple claim there is no singular solution. So will leave it there for now. The paper I took this example from is wrong. It claimed $y = x^2$ is the envelope. It is not.

2.23 Example 23 $(y')^2 (2-3y)^2 - 4(1-y) = 0$

$$
\begin{aligned}
(y')^2 (2-3y)^2 - 4(1-y) &= 0 \\
p^2(2-3y)^2 - 4(1-y) &= 0
\end{aligned}
$$

Since quadratic in p then

$$
\begin{aligned}
b^2 - 4ac &= 0 \\
0 - 4(2-3y)^2 (-4(1-y)) &= 0 \\
(2-3y)^2 (1-y) &= 0
\end{aligned}
$$

Comparing to $ET^2C = 0$ shows that $y = \frac{2}{3}$ is Tac locus and $y = 1$ is E since it verifies the ode (C will not verify the ode).

The following plot shows the singular solution as the envelope of the family of general solution plotted using different values of c.

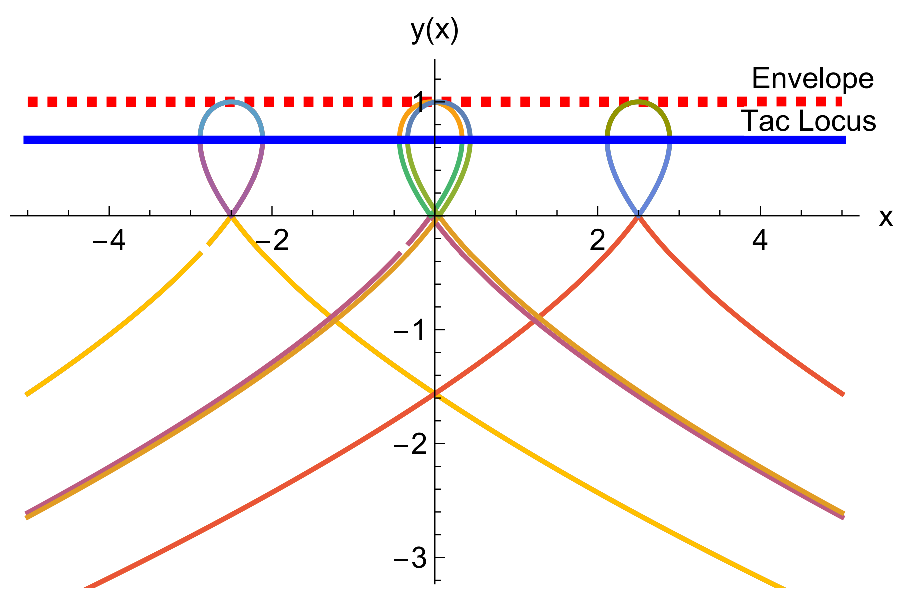

Figure 18: Singular solution envelope with general solution curves

2.24 Example 24 $8(y')^3 x = y\left(12(y')^2 - 9\right)$

$$
\begin{aligned}
8(y')^3 x &= y\left(12(y')^2 - 9\right) \\
8p^3 x &= y\left(12p^2 - 9\right)
\end{aligned}
$$

Hence

$$
F = 8p^3 x - y\left(12p^2 - 9\right) = 0 \tag{1}
$$

$$
\frac{\partial F}{\partial p} = p^2 x - yp = 0 \tag{2}
$$

We first check that $\frac{\partial F}{\partial y} = 12(y')^2 - 9 \neq 0$. Now we apply p-discriminant. Eliminating p. EQ (2) gives $p(px - y) = 0$ or $p = 0, px - y = 0$. Hence

$$
\begin{aligned}
p_1 &= 0 \\
p_2 &= \frac{y}{x}
\end{aligned}
$$

Substituting into the EQ (1) gives these candidate singular solutions

$$y = 0$$

$$8\left(\frac{y}{x}\right)^3 x - y\left(12\left(\frac{y}{x}\right)^2 - 9\right) = 0$$

Or

$$y = 0$$

$$8\frac{y^3}{x^2} - 12\frac{y^3}{x^2} + 9y = 0$$

Or

$$y = 0$$

$$8\frac{y^2}{x^2} - 12\frac{y^2}{x^2} + 9 = 0$$

Or

$$y = 0 \tag{13}$$

$$y = -\frac{3}{2}x \tag{4}$$

$$y = \frac{3}{2}x \tag{5}$$

We now have to check if this solution satisfies the ode. We see it does. Now we have to find the general solution (also called the primitive). This comes out to be

$$\Psi(x, y, c) = y - \frac{(x + 3c_1)^{\frac{3}{2}}}{3\sqrt{c_1}} \tag{6}$$

$$= y + \frac{(x + 3c_1)^{\frac{3}{2}}}{3\sqrt{c_1}} \tag{7}$$

Looking at (7)

$$\Psi(x, y, c) = 0 = y + \frac{(x + 3c_1)^{\frac{3}{2}}}{3\sqrt{c_1}}$$

$$\frac{\partial \Psi}{\partial c} = 0 = \frac{3\sqrt{3c_1 + x}}{2\sqrt{c}} - \frac{(3c_1 + x)^{\frac{3}{2}}}{6c_1^{\frac{3}{2}}}$$

Eliminating c. Second equation gives $c_1 = -\frac{1}{3}x$ or $x = \frac{x}{6}$. Using $x = \frac{x}{6}$ and Substituting into the first equation above gives

$$0 = y + \frac{\left(x + 3\frac{x}{6}\right)^{\frac{3}{2}}}{3\sqrt{\frac{x}{6}}}$$

$$y = -\frac{\left(x + \frac{x}{2}\right)^{\frac{3}{2}}}{3\sqrt{\frac{x}{6}}}$$

Hence the c-discriminant method gives

$$y_s = 0 \tag{2}$$

$$y_s = 3$$

Now we take the common y_s from the p-discriminant and the c-discriminant from (1,2). We see that $y_s = 3$ is common. Hence

$$y_s = 3$$

And $y_s = 0$ is removed. We also see that $y_s = 0$ does not even satisfy the ode. But even if it did, it is removed since it is not common with the p-discriminant .

If there is no common y_s found from applying the two method (p-discriminant and the c-discriminant) then it means there is no singular solution. The following plot shows the singular solution as the envelope of the family of general solution plotted using different values of c.

2.25 Example 25 Clairaut $y = xp + a\sqrt{1+p^2}$

$$y = xp + a\sqrt{1+p^2} \tag{1}$$

In this problem, will use new method to find singular solution. Which is to write the equation in rational form and find the discriminant from $b^2 - 4ac$ from the quadratic equation. This is simpler method than using elimination as was done in all the problems above. I just learned this method from old book by Daniel A. Murray. But this only works if equation for p and c can be written as quadratic equation. In General, the method of elimination works for all cases.

In this quadratic method, we write the ode as quadratic in p and find the discriminant and set it to zero.

$$\begin{aligned}
-y + xp + a\sqrt{1+p^2} &= 0\\
\frac{y}{a} - \frac{xp}{a} &= \sqrt{1+p^2}\\
1 + p^2 &= \frac{y^2}{a^2} + \frac{x^2p^2}{a^2} - 2\frac{xyp}{a^2}\\
p^2\left(1 - \frac{x^2}{a^2}\right) + p\left(2\frac{xy}{a^2}\right) + 1 - \frac{y^2}{a^2} &= 0
\end{aligned}$$

Hence p-discriminant is given by $b^2 - 4ac = 0$ or

$$\begin{aligned}
\left(2\frac{xy}{a^2}\right)^2 - 4\left(1 - \frac{x^2}{a^2}\right)\left(1 - \frac{y^2}{a^2}\right) &= 0\\
\frac{4}{a^2}\left(-a^2 + x^2 + y^2\right) &= 0\\
y^2 &= a^2 - x^2
\end{aligned}$$

The singular solutions are given by the above using p-discriminant method. Now we do the same using c-discriminant. The general solution is

$$\Psi(x,y,c) = y - cx - a\sqrt{1+c^2} = 0$$

Writing the above as

$$\begin{aligned}
\sqrt{1+c^2} &= \frac{y - cx}{a}\\
1 + c^2 &= \frac{(y-cx)^2}{a^2}\\
a^2c^2 + a^2 &= y^2 + c^2x^2 - 2cxy\\
c^2\left(a^2 - x^2\right) + 2cxy + a^2 - y^2 &= 0
\end{aligned}$$

The discriminant zero condition is $b^2 - 4ac = 0$. Or

$$\begin{aligned}
(2xy)^2 - 4\left(a^2 - x^2\right)\left(a^2 - y^2\right) &= 0\\
4x^2y^2 - 4\left(a^4 - a^2y^2 - a^2x^2 + x^2y^2\right) &= 0\\
4x^2y^2 - 4a^4 + 4a^2y^2 + 4a^2x^2 - 4x^2y^2 &= 0\\
-4a^4 + 4a^2y^2 + 4a^2x^2 &= 0\\
-a^2 + x^2 + y^2 &= 0\\
x^2 + y^2 &= a^2 \qquad (3)
\end{aligned}$$

Which satisfies the ode and is the same singular solution as found using the p-discriminant. Now will do the same but using elimination method to see if we get same result as above. We will use c-discriminant and p-discriminant. Both should give same singular solution as above, which is $x^2 + y^2 = a^2$. To use c-discriminant we start with general solution which is

$$\Psi(x,y,c) = y - cx - a\sqrt{1+c^2} = 0 \tag{4}$$

$$\frac{\partial \Psi}{\partial c} = -x - \frac{ac}{\sqrt{1+c^2}} = 0 \tag{5}$$

Eliminating c. From (5) $-x\sqrt{1+c^2}-ac=0$ or $(ac)^2=x^2(1+c^2)$ or $a^2c^2-x^2c^2-x^2=0$ or $c^2(a^2-x^2)=x^2$ or $c=\frac{\pm x}{\sqrt{a^2-x^2}}$. Substituting $\frac{x}{\sqrt{a^2-x^2}}$ into (4) gives

$$
\begin{aligned}
y-\left(\frac{x}{\sqrt{a^2-x^2}}\right)x-a\sqrt{1+\left(\frac{x}{\sqrt{a^2-x^2}}\right)^2} &= 0\\
y-\frac{x^2}{\sqrt{a^2-x^2}}-a\sqrt{1+\frac{x^2}{a^2-x^2}} &= 0\\
y-\frac{x^2}{\sqrt{a^2-x^2}}-\frac{a}{\sqrt{a^2-x^2}}\sqrt{a^2-x^2+x^2} &= 0\\
y-\frac{x^2}{\sqrt{a^2-x^2}}-\frac{a^2}{\sqrt{a^2-x^2}} &= 0\\
y &= \frac{x^2+a^2}{\sqrt{a^2-x^2}}
\end{aligned}
$$

Which does not satisfy the ode. Trying now $c=\frac{-x}{\sqrt{a^2-x^2}}$ then (4) gives

$$
\begin{aligned}
y-\left(\frac{-x}{\sqrt{a^2-x^2}}\right)x-a\sqrt{1+\left(\frac{-x}{\sqrt{a^2-x^2}}\right)^2} &= 0\\
y+\frac{x^2}{\sqrt{a^2-x^2}}-a\sqrt{1+\frac{x^2}{a^2-x^2}} &= 0\\
y+\frac{x^2}{\sqrt{a^2-x^2}}-\frac{a}{\sqrt{a^2-x^2}}\sqrt{a^2-x^2+x^2} &= 0\\
y+\frac{x^2}{\sqrt{a^2-x^2}}-\frac{a^2}{\sqrt{a^2-x^2}} &= 0\\
y &= \frac{x^2-a^2}{\sqrt{a^2-x^2}}\\
&= \frac{(x^2-a^2)\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}\sqrt{a^2-x^2}}\\
&= \frac{(x^2-a^2)\sqrt{a^2-x^2}}{a^2-x^2}\\
y &= -\sqrt{a^2-x^2}\\
y^2 &= a^2-x^2
\end{aligned}
$$

Which satisfies the ode. This is the same as found earlier. Now we will use p-discriminant elimination method.

$$
F=y-xp-a\sqrt{1+p^2}=0 \tag{6}
$$

$$
\frac{\partial F}{\partial p}=x-\frac{ap}{\sqrt{1+p^2}}=0 \tag{7}
$$

Eliminating p from (7).

$$
\begin{aligned}
x\sqrt{1+p^2}-ap &= 0\\
\sqrt{1+p^2} &= \frac{ap}{x}\\
1+p^2 &= \frac{a^2p^2}{x^2}\\
\frac{a^2p^2}{x^2}-p^2-1 &= 0\\
a^2p^2-x^2p^2-x^2 &= 0\\
p^2(a^2-x^2) &= x^2\\
p &= \pm\frac{x}{\sqrt{a^2-x^2}}
\end{aligned}
$$

Using $p = \frac{x}{\sqrt{a^2-x^2}}$ in (6) gives

$$
\begin{aligned}
y - x\left(\frac{x}{\sqrt{a^2-x^2}}\right) - a\sqrt{1+\left(\frac{x}{\sqrt{a^2-x^2}}\right)^2} &= 0\\
y - \frac{x^2}{\sqrt{a^2-x^2}} - a\sqrt{1+\frac{x^2}{a^2-x^2}} &= 0\\
y - \frac{x^2}{\sqrt{a^2-x^2}} - \frac{a}{\sqrt{a^2-x^2}}\sqrt{a^2-x^2+x^2} &= 0\\
y - \frac{x^2}{\sqrt{a^2-x^2}} - \frac{a^2}{\sqrt{a^2-x^2}} &= 0\\
y &= \frac{x^2+a^2}{\sqrt{a^2-x^2}}
\end{aligned}
$$

Which does not satisfy the ode. Using $p = \frac{-x}{\sqrt{a^2-x^2}}$ in (6) gives

$$
\begin{aligned}
y - x\left(\frac{-x}{\sqrt{a^2-x^2}}\right) - a\sqrt{1+\left(\frac{-x}{\sqrt{a^2-x^2}}\right)^2} &= 0\\
y + \frac{x^2}{\sqrt{a^2-x^2}} - a\sqrt{1+\frac{x^2}{a^2-x^2}} &= 0\\
y + \frac{x^2}{\sqrt{a^2-x^2}} - \frac{a}{\sqrt{a^2-x^2}}\sqrt{a^2-x^2+x^2} &= 0\\
y + \frac{x^2}{\sqrt{a^2-x^2}} - \frac{a^2}{\sqrt{a^2-x^2}} &= 0\\
y &= \frac{a^2-x^2}{\sqrt{a^2-x^2}}\\
&= \frac{a^2-x^2}{\sqrt{a^2-x^2}}\frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}}\\
&= \frac{a^2-x^2}{a^2-x^2}\sqrt{a^2-x^2}\\
&= \sqrt{a^2-x^2}
\end{aligned}
$$

Hence $y^2 = a^2 - x^2$. This is the same as found earlier.

The above shows that using Elimination or quadratic equation discriminant gives same singular solution and both p or c discriminant give same result. The elimination method is more general as it works for equations in p, c which are not quadratic. If the equation in p, c come out to be quadratic, then it is better not to use elimination and use direct discriminant as it is simpler.

The following is a plot of general solution and the above singular solution for $a = 1$.

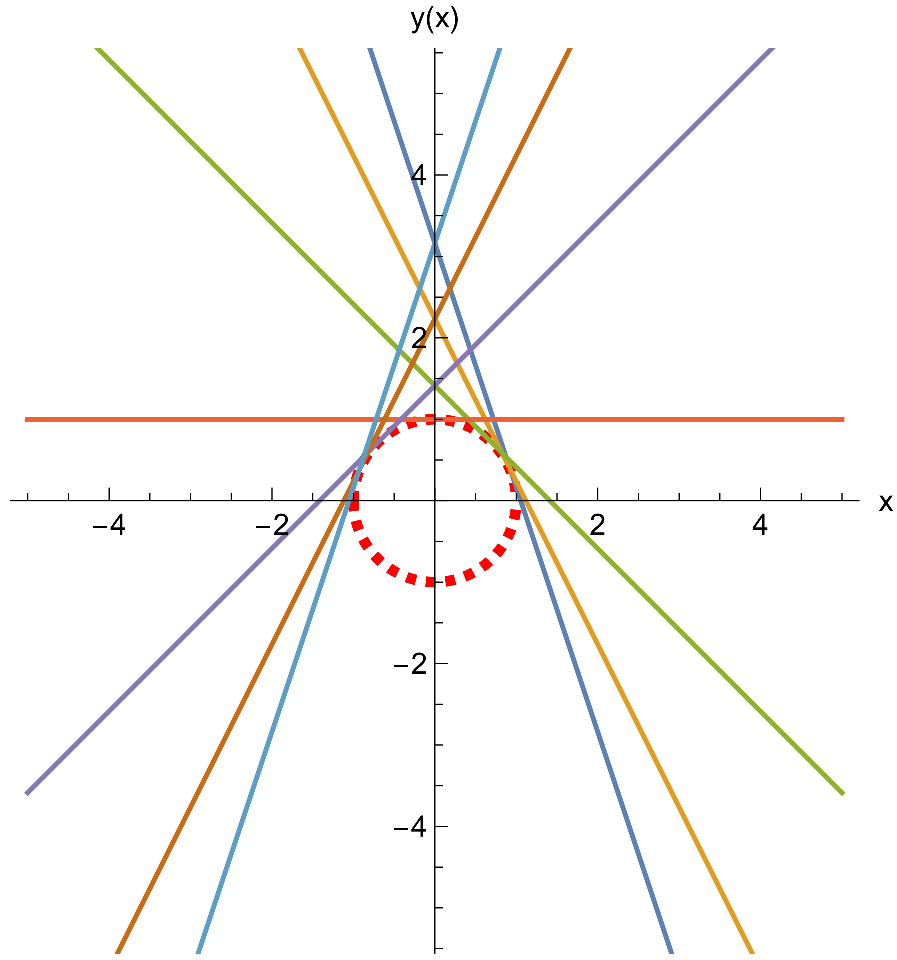

Figure 19: Singular solution envelope with general solution curves

Since this is Clairaut, let us use the standard Clairaut ode method to find the singular solution also and compare. We should obtain the same singular solution as above.

As we know, Clairaut ode has form

$$y = xf(p) + g(p) \tag{1}$$

Comparing the ode we are given $y = xp + a\sqrt{1+p^2}$, we see that

$$\begin{aligned} f(p) &= p \\ g(p) &= a\sqrt{1+p^2} \end{aligned}$$

The singular solution comes from solving

$$x + g'(p) = 0 \tag{2}$$

For p and then substituting this back into (1). But (2) is

$$\begin{aligned}
x + \frac{d}{dp}\left(a\sqrt{1+p^2}\right) &= 0 \\
x + \frac{a}{2\sqrt{1+p^2}}2p &= 0 \\
x\sqrt{1+p^2} + ap &= 0 \\
\sqrt{1+p^2} &= -\frac{ap}{x} \\
1+p^2 &= \frac{a^2p^2}{x^2} \\
p^2\left(1-\frac{a^2}{x^2}\right) &= -1 \\
p^2 &= \frac{-1}{1-\frac{a^2}{x^2}} \\
&= \frac{-x^2}{x^2-a^2} \\
&= \frac{x^2}{a^2-x^2}
\end{aligned}$$

Hence

$$p = \pm\frac{x}{\sqrt{a^2-x^2}}$$

Substituting this into (1) gives singular solutions (using first solution)

$$\begin{aligned}
y &= xp + a\sqrt{1+p^2} \\
&= x\frac{x}{\sqrt{a^2-x^2}} + a\sqrt{1+\left(\frac{x}{\sqrt{a^2-x^2}}\right)^2} \\
&= \frac{x^2}{\sqrt{a^2-x^2}} + a\sqrt{1+\frac{x^2}{a^2-x^2}} \\
&= \frac{x^2}{\sqrt{a^2-x^2}} + \frac{a}{\sqrt{a^2-x^2}}\sqrt{a^2-x^2+x^2} \\
&= \frac{x^2}{\sqrt{a^2-x^2}} + \frac{a^2}{\sqrt{a^2-x^2}} \\
&= \frac{x^2+a^2}{\sqrt{a^2-x^2}}
\end{aligned}$$

Which does not satisfy the ode. Trying the second root $p = \frac{-x}{\sqrt{a^2-x^2}}$ then (1) becomes

$$\begin{aligned}
y &= xp + a\sqrt{1+p^2} \\
&= x\frac{-x}{\sqrt{a^2-x^2}} + a\sqrt{1+\left(\frac{-x}{\sqrt{a^2-x^2}}\right)^2} \\
&= \frac{-x^2}{\sqrt{a^2-x^2}} + a\sqrt{1+\frac{x^2}{a^2-x^2}} \\
&= \frac{-x^2}{\sqrt{a^2-x^2}} + \frac{a}{\sqrt{a^2-x^2}}\sqrt{a^2-x^2+x^2} \\
&= \frac{-x^2}{\sqrt{a^2-x^2}} + \frac{a^2}{\sqrt{a^2-x^2}} \\
&= \frac{a^2-x^2}{\sqrt{a^2-x^2}} \\
&= \frac{a^2-x^2}{\sqrt{a^2-x^2}}\frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}} \\
&= \frac{a^2-x^2}{a^2-x^2}\sqrt{a^2-x^2} \\
&= \sqrt{a^2-x^2}
\end{aligned}$$

Hence $y^2 = a^2 - x^2$ which is the same solution found using p, c elimination.

2.26 Example 26 Clairaut $y = xp - e^p$

$$y = xp - e^p \tag{1}$$

Using elimination method.

$$F = y - xp + e^p = 0 \tag{2}$$
$$\frac{\partial F}{\partial p} = -x + e^p = 0 \tag{3}$$

We first check that $\frac{\partial F}{\partial y} = 1 \neq 0$. Now we apply p-discriminant. Eliminating p. EQ (3) gives $e^p = x$ or

$$p = \ln(x)$$

Substituting into the EQ (2) gives

$$\begin{aligned} y - x\ln(x) + e^{\ln(x)} &= 0 \\ y - x\ln(x) + x &= 0 \\ y &= x\ln(x) - x \end{aligned}$$

We now have to check if this solution satisfies the ode. We see it does. Now we have to find the general solution (also called the primitive). This comes out to be

$$\begin{aligned} \Psi(x, y, c) &= 0 = y - cx + e^c \\ \frac{\partial \Psi}{\partial c} &= 0 = -x + e^c \end{aligned}$$

Eliminating c. Second equation gives $c = \ln x$ and Substituting into the first equation above gives

$$\begin{aligned} 0 &= y - x\ln(x) + e^{\ln x} \\ 0 &= y - x\ln(x) + x \\ y &= x\ln(x) - x \end{aligned}$$

Which is the same as p-discriminant.

Since this is Clairaut, let us use the standard Clairaut ode method to find the singular solution also and compare. As we know, Clairaut ode has form

$$y = xf(p) + g(p) \tag{4}$$

Comparing the ode we are given $y = xp - e^p$, we see that

$$\begin{aligned} f(p) &= p \\ g(p) &= -e^p \end{aligned}$$

The singular solution comes from solving

$$x + g'(p) = 0 \tag{5}$$

For p and then substituting this back into (4). But (5) is

$$\begin{aligned} x + \frac{d}{dp}(-e^p) &= 0 \\ x - e^p &= 0 \end{aligned}$$

Hence

$$p = \ln(x)$$

Substituting this into (4) gives singular solution

$$
\begin{aligned}
y &= xp - e^p \\
&= x \ln x - e^{\ln x} \\
&= x \ln x - x
\end{aligned}
$$

Which is same as found using elimination methods.

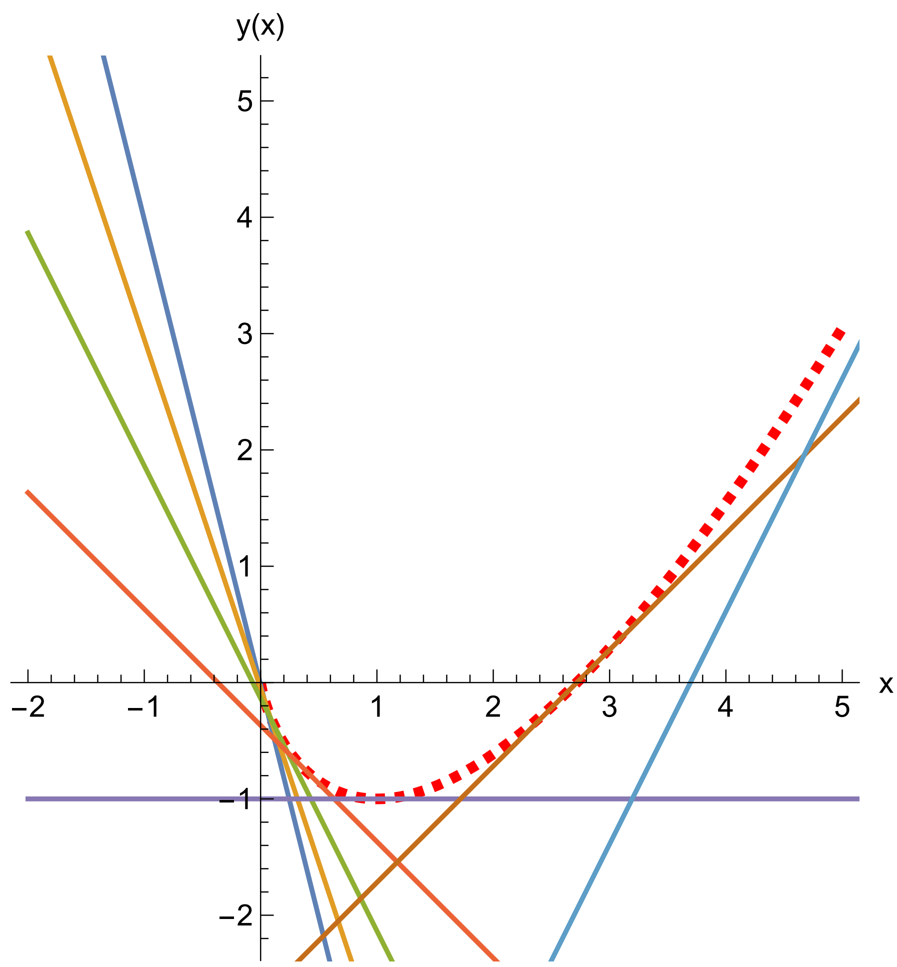

Figure 20: Singular solution envelope with general solution curves

3 References

1. Introductory Course On Differential Equations by Daniel A Murray. Longmans Green and Co. NY. 1924. Chapter IV. Singular solutions

2. `https://math24.net/singular-solutions-differential-equations.html`