Solving the Riccati equation

$$y' = f_0 + f_1 y + f_2 y^2$$

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Contents

1.1 Flow diagram

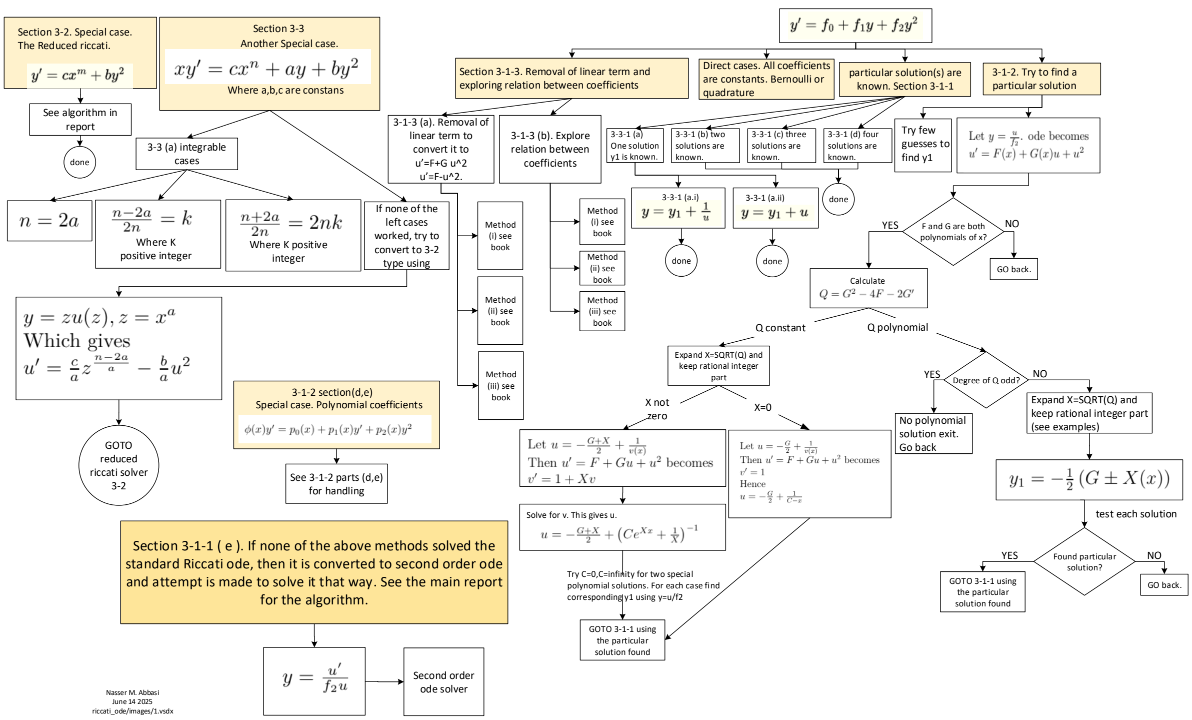

Figure 1: Top level algorithm for Riccati ode

1.2 Algorithm pseudocode

Local contents

1.2.1 Main solver (section 3-1 in Murphy)

```
INPUT y'= f0 + f1 y + f2 y^2 where f2 /= 0

RICCATI_SOLVER := proc(ode)
    IF all f0,f1,f2 are constants OR f0=0 then
        -- this is either quadrature or Bernoulli.
        SOL:= solve as Bernoulli or quadrature
    ELSE
        IF f1=0 THEN -- linear term is missing
            IF f0=c x^m AND f2=b (where b,m,c are constants) THEN
                -- ode now is y' = c x^m + b y^2 , This is the studied by Riccati
                -- called REDUCED riccati ode
                SOL:= reduced_riccati_solver(ode) -- Section 3-2 in Murphy book.
            END IF
        ELIF ode has form  x y' = c x^n + a y - b y^2 THEN
            -- here, c,n,a,b are CONSTANTS
            SOL:= CALL riccati_solver_3_3(ode)  -- section 3-3 in Murphy book
        END IF

        IF still not solved from above THEN
            -- try to transform the ode y'=f0 + f1 y + f2 y^2
            -- to form u'=F(x)+u^2 by removing the linear term
            SOL:= riccati_solver_3_1_3(ode) -- Section 3-1-3 in Murphy book.
        END IF

        IF still not solved from above THEN
            -- try special solutions. Murphy section 3-1-2
            SOL:= riccati_solver_3_1_2(ode)
        END IF


        IF still NOT solved then
           SOL:= riccati_solver_3_1_1(ode) -- section 3-1-1 in Murphy
        END IF
    END IF

    return SOL

END PROC;
```

1.2.2 First special case. The reduced Riccati solver (section 3-2 in Murphy book)

```
-- solves y' = c x^m + b y^2. Section 3-2 in Morphy book.
reduced_riccati_solver:=proc
 -- see my other notes below. two cases. n=-2 and n/= -2
end proc
```

1.2.3 Second special case. (section 3-3 in Murphy book)

```
-- handle ode form  x y' = c x^n + a y - b y^2 -- section 3-3
-- where c,n,a,b are CONSTANTS

riccati_solver_3_3 :=proc(ode)
    IF n = 2 a THEN
        LET y=x^a u
        ode becomes u'=x^(a-1) ( c - b u^2) separable.
        SOL:= riccati_special_i(a,b,c,n,y,x)

    ELIF (n-2a)/(2n) = k, where k positive integer THEN
        -- example  x y' = 2 x^4 - 6 y - 5 y^2
        -- n=4, a=-6, hence (n-2a)=4+12=16 and 16/(8) = k = 2.
        IF odd(k) THEN
            SOL:= riccati_special_i(n/2,c,b,n,y,x)
        ELSE
            SOL:= riccati_special_i(n/2,b,c,n,y,x)
        END IF

        FOR i from k-1 to 1 by -1 DO
            IF odd(i) THEN
               SOL:= (a+i n)/c + x^n/SOL
            ELSE
               SOL:= (a+i n)/b + x^n/SOL
            END IF
        END DO

        SOL:= y = a/b + x^n/SOL
    ELIF  (n+2 a)= 2 n k, where k is positive integer.
          -- example x y' = 3 x^4 + 6 y + 3 y^2
        IF odd(k) THEN
            SOL:= riccati_special_i(n/2,c,b,n,y,x)
        ELSE
            SOL:= riccati_special_i(n/2,b,c,n,y,x)
        END IF

        FOR i from k-1 to 1 by -1 DO
            IF odd(i) THEN
               SOL:= (i n-a)/c + x^n/SOL
            ELSE
               SOL:= (i n-a)/b + x^n/SOL
            END IF
        END DO

        SOL:= y = x^n/SOL
    END IF

    return SOL
END PROC:
```

1.2.4 Transform to $u' = F(x) + u^2$. remove linear term. section 3-1-3 in Murphy

```
riccati_solver_3_1_3 :=proc(ode)
    -- transform y' = f0 + f1 y + f2 y^2 to u'=F(x)+u^2 by removing linear term
    -- to transform to either 3-2 (reduced, y'=c x^m + b y^2)
    -- or 3-3 form (x y' = c x^n + a y - b y^2)

    -- this is (a) in section 3-1-3
    Let y=u exp(phi), phi=INT(f1,x). ode becomes  u'=F(x)+G(x) u^2 -- (i)
        where F=f0 exp(-phi), G=f2 exp(phi)

    IF F(x) is propertional to G(x) THEN
       ode u'=F(x)+G(x) u^2 is separable
       -- example u' = 3 x + 6 x u^2, becomes u' = x( 3 + 6 u^2)
    ELSE
        IF G is constant THEN
            IF F(x) = c x^m THEN
                 -- this is reduced riccati. Solved.
            ELIF F(x) is polynomial of x with more than one term (say x+x^2) THEN
                IF degree of F(x) is odd (say u'=x+x^3 + b u^2) THEN
                    NO SOLUTION exist
                ELSE degree of F(x) is even (say u'=x+x^2 + b u^2) THEN
                    SOLUTION can exist. see Murphy page 17
                END IF
            ELSE
                -- transform to second order and try
            END IF
        ELSE
            -- transform to second order and try
        END IF
    END IF

    IF still not solved THEN
        Let y=u-v, where v=f1/(2*f2). ode becomes  u'=F(x)+G(x) u^2 -- (ii)
            where F=f0+v'-f1^2/(4 f2), G=f2

        Try same as above.
    END IF

    IF still not solved THEN
        Let y=u(z) exp(phi), phi(x)=INT(f1,x), z=-INT(h exp(phi),x) --(iii)
        ode becomes  u'(z)=F(z)-u^2(z), with F(z)=-f_0 exp(-2 phi)

        Try same as above.
    END IF

    -- this is (b) in section 3-1-3
    IF still Not solved THEN -- look at relations between coefficients.
        -- case i
        Try to find a,b with  |a|+|b|>0 s.t. (a^2 f0+ a b f1 + b^2 f2) = 0
        IF a/=0 then y1=b/a is particular solution of Riccati THEN
            SOL:= riccati_solver_3_1_1() using this y1 to find general solution.
        ELIF f0+f1+f2=0 THEN
            a=1,b=1 and y1=1.??
            SOL:= riccati_solver_3_1_1() using this y1 to find general solution.
```

```
            -- But this does not seem correct. Example
            -- y' = x -1/2 y -1/2 y^2. But y1=1 does not satisfy this ode??
        END IF

        -- case ii -- need to make example
        IF f0= A^2 f2 exp(2 INT(f1,x)) THEN
            IF f0 f2>0 THEN
               SOL := SQRT(f0/f2)  tanh( INT( SQRT(f0 f2),x) + constant )
            ELSE
               SOL := SQRT(-f0/f2)  tan( INT( SQRT(-f0 f2),x) + constant )
            END IF
        END IF

        try case iii
            -- to do
    END IF
END PROC
```

1.2.5 Special solutions. section 3-1-2 in Murphy

```
-- tries special cases
-- input y' = f0 + f1 y + f2 y^2

riccati_solver_3_1_2 :=proc(ode)

    -- Let y=f0/f1, this converts the ode to
    -- u' = F + G u + u^2
    -- where F=f0/f2, G=f1+f2'/f2

    IF G=0 THEN -- (a) case in Murphy, page 17
        u' = F + u^2
        IF F(x) = c x^m THEN
            -- this is reduced riccati. Solved.
        ELIF F(x) is polynomial of x with more than one term (say x+x^2) THEN
            IF degree of F(x) is odd (say u'=x+x^3 + b u^2) THEN
                NO SOLUTION exist  -- (a.i)
            ELSE degree of F(x) is even (say u'=x+x^2 + b u^2) THEN
                SOLUTION can exist. see Murphy page 17  -- (a.ii, page 17)
            END IF
        END IF
    ELSE -- case where G /=0 (b case in Murphy, page 17)
        let u= w - G/2 and   u' = F + G u + u^2 becomes
            -- w' = H + w^2
            -- where 4 H + G^2 = 4 F + 2 G'
        Let Q = G^2 -4 F -2 G'

        IF Q polynomial of odd degree THEN -- (b.i)
            NO solution.
        ELIF Q polynomial of even degree THEN
            Try as on page 18, Murphy in the hope to find solution. -- (b.ii)
        ELIF Q is constant THEN
             see page 18, part i  -- (c.i case, page 17)
             see page 18, part ii  -- (c.ii case, page 17)
        END IF
```

```
    END IF

    IF still not solved THEN
       IF ode has form phi(x) y' = f0 + f1 y + f2 y^2 THEN
            IF all coefficients f_i are polynomials in x THEN -- case (d. page 17)
                Let particular solution be  y1=R(x).
                -- let y=u+y1, the ode becomes
                -- phi(x) y' = F u + f2 u^2, where F=f1 + 2 f2 R
                -- see rest case d.
            END IF
        END IF
    END IF

    -- case (e). See page 19
    -- to finish
```

1.2.6 Helper function. Used by Second special case

```
#solves x y' = c x^n + a y - b y^2 THEN
riccati_special_i :=proc (a,b,c,n,y,x)

    -- note sign difference from book, typo in book.
    IF bc>0
        SOL:= y = SQRT(c/b) x^a tanh(C + SQRT(bc) x^2/a)
    ELIF bc<0 THEN
        SOL:= y = SQRT(-c/b) x^a tan(C + SQRT(-bc) x^2/a)
    END IF

    RETURN(SOL)
end proc
```

1.2.7 General solver. Converts to second order. Used if all above fail

```
--solves y' = f0 + f1 y + f2 y^2
--called when all other methods above failed
general_riccati_solver :=proc (ode)

    -- convert to second order ode using transformation and see if can solve that.
    IF can not solve the second order ode then
       STOP.
    ELSE -- reverse transformation to obtain solution in y(x)
        RETURN (sol)
    END IF
END PROC
```

1.3 Examples

Local contents

1.3.1 Examples for section 3-2 (Special case, Reduced Riccati ode) $y' = ax^n + by^2$

Local contents

This is special case of the general Riccati ode $y' = c_0(x) + c_1(x)\,y + c_2(x)\,y^2$ where now $c_0(x) = ax^n$ and $c_2(x) = b$ where a, b, n are constants. The reduced Riccati ode do not have y term in it. Only x and y^2 in the RHS of the ode.

1.3.1.1 Reduced Riccati with $n = -2$

Local contents

1.3.1.1.1 Algorithm For the special case of $n = -2$ the solution can be written directly as given by Eqworld ode0106 as

$$y = \frac{\lambda}{x} - \frac{x^{2b\lambda}}{\frac{bx}{2b\lambda+1}x^{2b\lambda} + c_1} \tag{1}$$

Where in the above λ is a root of $b\lambda^2 + \lambda + a = 0$.

There is another way to solve the above with $n = -2$. This can be solved using the substitution

$$y = \frac{1}{u} \tag{2}$$

Hence $y' = -\frac{u'}{u^2}$ and the ode becomes

$$\begin{aligned}
-\frac{u'}{u^2} &= ax^{-2} + b\frac{1}{u^2}\\
-u' &= a\frac{u^2}{x^2} + b\\
u' &= -a\frac{u^2}{x^2} - b
\end{aligned}$$

Which is first order Homogeneous ode type (see earlier section). But using (1) is much simpler method as solution can be written directly. The following example shows that using (1) and (2) give same solution.

1.3.1.1.2 Example $y' = -x^{-2} + 2y^2$

$$y' = -x^{-2} + 2y^2$$

Comparing this to $y' = ax^n + by^2$ shows that $a = -1, b = 2, n = -2$. We will first solve this using (1). The quadratic equation is

$$\begin{aligned} b\lambda^2 + \lambda + a &= 0 \\ 2\lambda^2 + \lambda - 1 &= 0 \end{aligned}$$

The roots are $\frac{1}{2}, -1$. Let us pick first $\lambda = -1$. Hence the solution using (1) is

$$\begin{aligned} y &= \frac{\lambda}{x} - \frac{x^{2b\lambda}}{\frac{bx}{2b\lambda+1}x^{2b\lambda} + c_1} \\ &= \frac{-1}{x} - \frac{x^{-4}}{\frac{2x}{-4+1}x^{-4} + c_1} \\ &= \frac{-1}{x} - \frac{x^{-4}}{\frac{2}{-3}x^{-3} + c_1} \\ &= \frac{1 + 3c_1x^3}{2x - 3x^4c_1} \\ &= \frac{1 + c_2x^3}{2x - x^4c_2} \end{aligned}$$

Let us now try $\lambda = \frac{1}{2}$. The solution becomes

$$\begin{aligned} y &= \frac{\lambda}{x} - \frac{x^{2b\lambda}}{\frac{bx}{2b\lambda+1}x^{2b\lambda} + c_1} \\ &= \frac{1}{2x} - \frac{x^2}{\frac{2x}{2+1}x^2 + c_1} \\ &= \frac{1}{2x} - \frac{x^2}{\frac{2x^3}{3} + c_1} \\ &= \frac{3c_1 - 4x^3}{4x^4 + 6c_1x} \end{aligned}$$

Both these solution verified OK. Now we will solve the same using the transformation $y = \frac{1}{u}$.This results in the ode $y' = ax^n + by^2$ becoming

$$\begin{aligned} u' &= -a\frac{u^2}{x^2} - b \\ u' &= \frac{u^2}{x^2} - 2 \end{aligned}$$

We see that this transformation made the ode a homogeneous type which can be easily solved now. This only works for $n = -2$. Solving this ode gives

$$u = \frac{-x(2 + c_1x^3)}{-1 + c_1x^3}$$

Hence

$$\begin{aligned} y &= \frac{1}{u} \\ &= \frac{1 - c_1x^3}{2x + c_1x^4} \end{aligned}$$

Which is the same as first solution above.

1.3.1.2 Reduced Riccati with $n \neq -2$

Local contents

1.3.1.2.1 Algorithm For $n \neq -2$ there is direct solution to the reduced Riccati given by Eqworld ode0106 and Dr Dobrushkin web page as

$$
\begin{aligned}
w &= \sqrt{x}\begin{cases} c_1 \operatorname{BesselJ}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{ab}x^k\right) + c_2 \operatorname{BesselY}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{ab}x^k\right) & ab > 0 \\ c_1 \operatorname{BesselI}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{-ab}x^k\right) + c_2 \operatorname{BesselK}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{-ab}x^k\right) & ab < 0 \end{cases} \\
y &= -\frac{1}{b}\frac{w'}{w} \\
k &= 1 + \frac{n}{2}
\end{aligned} \tag{2}
$$

If n satisfies constraint that

$$\frac{n}{2n+4}$$

Is an integer, then the solution $y(x)$ will come out using algebraic, exponential and logarithmic functions (including circular functions, such as sin and cosine). If however, n does not satisfy the above constraint, then (2) can still be used but the solution will come out using Bessel function (also called cylindrical functions).

Hence (2) can be used for any n to solve the special or reduced Riccati ode.

The constraint that $\frac{n}{2n+4}$ is an integer, can also be given by saying that $n = \frac{4k}{1-2k}$ where $k = \pm 1, \pm 2, \cdots$.

When n satisfies this, then as mentioned above Eq (2) gives the solution in algebraic, exponential and logarithmic functions. For all other values, Liouville proved no solution exist in terms of elementary functions.

These n values come out to be $n = \left\{\cdots, -\frac{40}{21}, \cdots, -\frac{8}{5}, \frac{-4}{3}, -4, -\frac{8}{3}, -\frac{12}{5}, \cdots, -\frac{40}{19}\right\}$. We notice that the limit on both ends goes to $n = -2$ which is the first special case above. Below are two examples to illustrate this. First example will use n that meets this constraint, and the second example will use n that does not meet the constraint.

1.3.1.2.2 Example $y' = x^{-4} + y^2$

$$y' = x^{-4} + y^2$$

Comparing this to $y' = ax^n + by^2$ shows that $a = 1, b = 1, n = -4$. We see that n satisfies that $\frac{n}{2n+4} = 1$ which is integer. Hence we expect that applying (2) will give solution in elementary functions. Since $ab > 0$ then applying

$$
\begin{aligned}
w &= \sqrt{x}c_1 \operatorname{BesselJ}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{ab}x^k\right) + c_2 \operatorname{BesselY}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{ab}x^k\right) \\
k &= 1 + \frac{-4}{2} = 1 - 2 = -1
\end{aligned}
$$

Hence

$$w = \sqrt{x}c_1 \operatorname{BesselJ}\left(\frac{-1}{2}, -x^{-1}\right) + c_2 \operatorname{BesselY}\left(-\frac{1}{2}, -x^{-1}\right)$$

Hence

$$y = -\frac{w'}{w}$$

Simplifying the above gives

$$y = \frac{1}{x^2}\left(\tan\left(-\frac{1}{x} + c_1\right) - x\right)$$

1.3.1.2.3 Example $y' = x^3 + y^2$

$$y' = x^3 + y^2$$

Comparing this to $y' = ax^n + by^2$ shows that $a = 1, b = 1, n = 3$. We see that n do not satisfy that $\frac{n}{2n+4} = \frac{3}{6+4} = \frac{3}{10}$ being an integer. Hence we expect that applying (2) will give solution in cylindrical functions and not elementary functions. Since $ab > 0$ then applying

$$\begin{aligned} w &= \sqrt{x}c_1 \operatorname{BesselJ}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{ab}x^k\right) + c_2 \operatorname{BesselY}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{ab}x^k\right) \\ k &= 1 + \frac{3}{2} = \frac{5}{2} \end{aligned}$$

Hence

$$w = \sqrt{x}c_1 \operatorname{BesselJ}\left(\frac{1}{5}, \frac{2}{5}x^{\frac{5}{2}}\right) + c_2 \operatorname{BesselY}\left(\frac{1}{5}, \frac{2}{5}x^{\frac{5}{2}}\right)$$

Hence

$$y = -\frac{w'}{w}$$

Simplifying the above gives

$$y = \frac{x^{\frac{3}{2}}\left(-c_1 \operatorname{BesselJ}\left(\frac{-4}{5}, \frac{2}{5}x^{\frac{5}{2}}\right) - \operatorname{BesselY}\left(\frac{-4}{5}, \frac{2}{5}x^{\frac{5}{2}}\right)\right)}{c_1 \operatorname{BesselJ}\left(\frac{1}{5}, \frac{2}{5}x^{\frac{5}{2}}\right) + \operatorname{BesselY}\left(\frac{1}{5}, \frac{2}{5}x^{\frac{5}{2}}\right)}$$

We see that the solution is in terms of cylindrical functions. Because n did not satisfy that $\frac{n}{2n+4}$ is integer. But the main point is that (2) can still be used to solve the special Riccati ode.

1.3.2 Conversion of Riccati to second order ode (section 3-1-1 e)

Solved using transformation $y = \frac{-u'}{f_2 u}$ which generates second order ode in u. This is solved for u (if possible) then y is found.

1.3.2.1 Example 1

$$y' = -x + \frac{1}{x}y^2 \tag{1}$$

Comparing to $y' = f_0 + f_1 y + f_2 y^2$ form shows that $f_0 = -x, f_1 = 0, f_2 = \frac{1}{x}$. We will use the method of converting to second order ode. Let $y = \frac{-u'}{f_2 u} = x\frac{u'}{u}$. Using this substitution results in

$$\begin{aligned} f_2 u'' - \left(f_2' + f_1 f_2\right) u' + f_2^2 f_0 u &= 0 \\ \frac{1}{x}u'' - \left(-\frac{1}{x^2}\right) u' + \left(\frac{1}{x^2}\right)(-x) u &= 0 \\ \frac{1}{x}u'' + \frac{1}{x^2}u' - \frac{1}{x}u &= 0 \\ xu'' + u' - xu &= 0 \end{aligned}$$

This is Bessel ode the solution is

$$u = c_1 \operatorname{BesselI}(0, x) + c_2 \operatorname{BesselK}(0, x)$$

But $y = x\frac{u'}{u}$, hence

$$y = x\frac{(c_1 \operatorname{BesselI}(1, x) - c_2 \operatorname{BesselK}(1, x))}{c_1 \operatorname{BesselI}(0, x) + c_2 \operatorname{BesselK}(0, x)}$$

1.3.3 Another special case. $xy' = cx^n + ay - by^2$. (section 3-3)

Local contents

This is used when the input is $xy' = cx^n + ay + by^2$ with c, n, a, b are all constants. There are 4 sub cases to solving this. The first three if there is some special relation between a, n and if there is none found, then we convert this to reduced riccati and try again.

1.3.3.1 Case when $n = 2a$ (section 3-3 a(i))

1.3.3.1.1 Example $xy' = cx^4 + 2y - by^2$

Given

$$xy' = cx^n + ay - by^2$$

Where $n = 2a$ as in this example. In this case let $y = x^a u$. Then $y' = ax^{a-1}u + x^a u'$ and the ode becomes

$$\begin{aligned}
x\left(ax^{a-1}u + x^a u'\right) &= cx^{2a} + a(x^a u) - b\left(x^{2a}u^2\right) \\
ax^a u + x^{a+1}u' &= cx^{2a} + a(x^a u) - b\left(x^{2a}u^2\right) \\
x^{a+1}u' &= cx^{2a} - ax^a u + ax^a u - b\left(x^{2a}u^2\right) \\
x^{a+1}u' &= cx^{2a} - bx^{2a}u^2 \\
u' &= cx^{2a-a-1} - bx^{2a-a-1}u^2 \\
u' &= cx^{a-1} - bx^{a-1}u^2 \\
&= x^{a-1}\left(c - bu^2\right) \qquad (1)
\end{aligned}$$

We see that (1) is separable. Solving (1) gives

$$u = \frac{1}{b}\tanh\left(\frac{\sqrt{cb}(C_1 a + x^a)}{a}\right)\sqrt{cb}$$

But $y = x^a u$, hence $u = yx^{-a}$. Therefore

$$\begin{aligned}
yx^{-a} &= \frac{1}{b}\tanh\left(\frac{\sqrt{cb}(C_1 a + x^a)}{a}\right)\sqrt{cb} \\
y &= \sqrt{cb}\frac{x^a}{b}\tanh\left(\frac{\sqrt{cb}(C_1 a + x^a)}{a}\right)
\end{aligned}$$

Is the final solution for $xy = cx^n + ay - by^2$ when $n = 2a$. The above can be simplified more if we know the specific numerical values of b, c. Note that in method, we have to know the values of a, n always in order to decide.

1.3.3.2 Case when $\frac{(n-2a)}{2n} = k$ with k positive integer. (section 3-3 a(ii))

Local contents

1.3.3.2.1 Example $xy' = 2x^4 - 6y - 5y^2$

Comparing to

$$xy' = cx^n + ay - by^2$$

Shows that $n = 4, a = -6$, hence $\frac{(n-2a)}{2n} = \frac{(4-2(-6))}{8} = \frac{16}{8} = 2$. Hence $\frac{(n-2a)}{2n} = k = 2$ a positive integer. Therefore

$$k = 2$$

A positive integer. Notice that the values of b, c do not matter for this solution method, but they have to be constants.

$$xy' = 2x^4 - 6y - 5y^2$$

The solution is finite continued fraction

$$\begin{aligned}
y &= \frac{a}{b} + \frac{x^n}{y_1} \\
y_1 &= \frac{a+n}{c} + \frac{x^n}{y_2} \\
y_2 &= \frac{a+2n}{b} + \frac{x^n}{y_3} \\
y_3 &= \frac{a+3n}{c} + \frac{x^n}{y_4} \\
y_4 &= \frac{a+4n}{b} + \frac{x^n}{y_5} \\
&\vdots \\
y_{k-1} &= \frac{a+(k-1)\,n}{\Delta} + \frac{x^n}{y_k}
\end{aligned}$$

Where $\Delta = c$ when index i on y_i is odd and $\Delta = b$ when index i on y_i is even. In this example, we found that $k = 2$. Therefore we have (we stop at $k - 1 = 1$)

$$\begin{aligned}
y &= \frac{a}{b} + \frac{x^n}{y_1} \\
y_1 &= \frac{a+n}{c} + \frac{x^n}{y_2}
\end{aligned} \tag{1}$$

Now, we just need to find $y_2 = y_k$ to finish. To find y_2 we use either

$$xy_2' = bx^n + (a+nk)\,y_2 - cy_2^2 \tag{A}$$

Or

$$xy_2' = cx^n + (a+nk)\,y_2 - by_2^2 \tag{B}$$

ODE (A) is used when k is odd and ode (B) is used when k is even. Since $k = 2$ is even, then we use (B). Hence the ode to solve for y_2 is

$$xy_2' = cx^n + (a+nk)\,y_2 - by_2^2$$

But $n = 4, a = -6, b = 5, c = 2$, then the above becomes

$$\begin{aligned}
xy_2' &= 2x^4 + (-6+8)\,y_2 - 5y_2^2 \\
&= 2x^4 + 2y_2 - 5y_2^2
\end{aligned}$$

Notice that this has form

$$xy_2' = Cx^N + Ay_2 - By_2^2$$

(used upper case letters now, so not to confuse with lower case letters used for the original ode).

Now we see that $N=4, A=2, C=2, B=5$ which means $N=2A$. But this is case (i). This always happens with case (ii) that we end up having to solve one ode using case (i) to finish. If we do not end up with case(i) ode, then we have made a mistake

But we know how to solve case (i) which we did above. This gives us y_2. We plug this into (1) and now we have the solution for y.

Now we solve (B) as we did in part (i). Let $y_2 = x^A u$. Then ode (B) becomes (as was shown in case (i) example)

$$u' = x^{A-1}(C - Bu^2) \tag{3}$$

We see that (3) is separable. Solving (3) gives

$$u = \frac{1}{B}\tanh\left(\frac{\sqrt{CB}(C_1 A + x^A)}{A}\right)\sqrt{CB}$$

But $y_2 = x^A u$, hence $u = y_2 x^{-A}$. Therefore

$$\begin{aligned} y_2 x^{-A} &= \frac{1}{B}\tanh\left(\frac{\sqrt{CB}(C_1 A + x^A)}{A}\right)\sqrt{CB} \\ y_2 &= \sqrt{CB}\frac{x^A}{B}\tanh\left(\frac{\sqrt{CB}(C_1 A + x^A)}{A}\right) \end{aligned}$$

Plugging in values for $N=4, A=2, B=5, C=2$ gives

$$\begin{aligned} y_2 &= \sqrt{10}\frac{x^A}{5}\tanh\left(\frac{\sqrt{10}(2C_1 + x^2)}{2}\right) \\ &= \frac{\sqrt{2}}{\sqrt{5}}x^2\tanh\left(\frac{\sqrt{10}x^2}{2} + C_2\right) \end{aligned} \tag{4}$$

Now we go back to (1) and find y

$$\begin{aligned} y &= \frac{a}{b} + \frac{x^n}{y_1} \\ y_1 &= \frac{a+n}{c} + \frac{x^n}{y_2} \end{aligned}$$

Hence

$$y = \frac{a}{b} + \frac{x^n}{\frac{a+n}{c} + \frac{x^n}{y_2}}$$

Substituting all parameters into the above and using y_2 from (4) gives, using $n=4, a=-6, b=5, c=2$

$$\begin{aligned} y &= \frac{-6}{5} + \frac{x^4}{\frac{-6+4}{2} + \frac{x^4}{\frac{\sqrt{2}}{\sqrt{5}}x^2\tanh\left(\frac{\sqrt{10}x^2}{2} + C_2\right)}} \\ &= \frac{-6}{5} + \frac{x^4}{\frac{x^2\sqrt{5}}{\sqrt{2}\tanh\left(\frac{\sqrt{10}x^2}{2} + C_2\right)} - 1} \end{aligned}$$

1.3.3.2.2 Example $xy' = 2x^4 - 10y - 5y^2$ Comparing to

$$xy' = cx^n + ay - by^2$$

Shows that $c = 2, n = 4, a = -10, b = 5$. Hence $\frac{(n-2a)}{2n} = \frac{(4-2(-10))}{8} = \frac{24}{8} = 3$. Therefore $k = 3$ which is positive integer.

The solution is finite continued fraction

$$\begin{aligned}
y &= \frac{a}{b} + \frac{x^n}{y_1} \\
y_1 &= \frac{a+n}{c} + \frac{x^n}{y_2} \\
y_2 &= \frac{a+2n}{b} + \frac{x^n}{y_3} \\
y_3 &= \frac{a+3n}{c} + \frac{x^n}{y_4} \\
y_4 &= \frac{a+4n}{b} + \frac{x^n}{y_5} \\
&\vdots \\
y_{k-1} &= \frac{a+(k-1)\,n}{\Delta} + \frac{x^n}{y_k}
\end{aligned}$$

Where $\Delta = c$ when index i on y_i is odd and $\Delta = b$ when index i on y_i is even. In this example, we found that $k = 3$. Therefore we have (we stop at $k - 1 = 2$)

$$\begin{aligned}
y &= \frac{a}{b} + \frac{x^n}{y_1} \\
y_1 &= \frac{a+n}{c} + \frac{x^n}{y_2} \\
y_2 &= \frac{a+2n}{b} + \frac{x^n}{y_3}
\end{aligned} \tag{1}$$

Now, we just need to find $y_3 = y_k$ to finish. To find y_3 we use either

$$xy_3' = bx^n + (a+nk)\,y_3 - cy_3^2 \tag{A}$$

Or

$$xy_3' = cx^n + (a+nk)\,y_3 - by_3^2 \tag{B}$$

ODE (A) is used when k is odd and ode (B) is used when k is even. Since $k = 3$ is even, then we use (A). Hence the ode to solve for y_3 is

$$xy_3' = bx^n + (a+nk)\,y_3 - cy_3^2$$

But in this problem, $c = 2, n = 4, a = -10, b = 5$, then the above becomes

$$\begin{aligned}
xy_3' &= 5x^4 + (-10+12)\,y_3 - 2y_3^2 \\
&= 5x^4 + 2y_3 - 2y_3^2
\end{aligned}$$

Notice that this has form

$$xy_3' = Cx^N + Ay_3 - By_3^2$$

(used upper case letters now, so not to confuse with lower case letters used for the original ode).

Now we see that $N=4, A=2, C=5, B=2$ which means $N=2A$. But this is case (i). This always happens with case (ii) that we end up having to solve one ode using case (i) to finish. If we do not end up with case(i) ode, then we have made a mistake

But we know how to solve case (i) which we did above. This gives us y_3. We plug this into (1) and now we have the solution for y.

Now we solve (B) as we did in part (i). Let $y_3 = x^A u$. Then ode (B) becomes (as was shown in case (i) example)

$$u' = x^{A-1}(C - Bu^2) \tag{3}$$

We see that (3) is separable. Solving (3) gives

$$u = \frac{1}{B}\tanh\left(\frac{\sqrt{CB}(C_1 A + x^A)}{A}\right)\sqrt{CB}$$

But $y_3 = x^A u$, hence $u = y_3 x^{-A}$. Therefore

$$\begin{aligned}
y_3 x^{-A} &= \frac{1}{B}\tanh\left(\frac{\sqrt{CB}(C_1 A + x^A)}{A}\right)\sqrt{CB} \\
y_3 &= \sqrt{CB}\frac{x^A}{B}\tanh\left(\frac{\sqrt{CB}(C_1 A + x^A)}{A}\right)
\end{aligned}$$

Plugging in values for $N=4, A=2, C=5, B=2$ gives

$$\begin{aligned}
y_3 &= \sqrt{10}\frac{x^2}{2}\tanh\left(\frac{\sqrt{10}(2C_1 + x^2)}{2}\right) \\
&= \sqrt{10}\frac{x^2}{2}\tanh\left(\frac{\sqrt{10}x^2}{2} + C_2\right) \qquad (4)
\end{aligned}$$

Now we go back to (1) and find y

$$\begin{aligned}
y &= \frac{a}{b} + \frac{x^n}{y_1} \\
y_1 &= \frac{a+n}{c} + \frac{x^n}{y_2} \\
y_2 &= \frac{a+2n}{b} + \frac{x^n}{y_3} \qquad (1)
\end{aligned}$$

Hence

$$y = \frac{a}{b} + \frac{x^n}{\frac{a+n}{c} + \frac{x^n}{\frac{a+2n}{b} + \frac{x^n}{y_3}}}$$

Substituting all parameters into the above and using y_2 from (4) gives, using $c=2, n=4, a=-10, b=5$

$$\begin{aligned}
y &= \frac{-10}{5} + \frac{x^4}{\frac{-10+4}{2} + \frac{x^4}{\frac{-10+8}{5} + \frac{x^4}{\sqrt{10}\frac{x^2}{2}\tanh\left(\frac{\sqrt{10}x^2}{2} + C_2\right)}}} \\
&= -2 + \frac{x^4}{-3 + \frac{x^4}{\frac{-2}{5} + \frac{2x^2}{\sqrt{10}\tanh\left(\frac{\sqrt{10}x^2}{2} + C_2\right)}}}
\end{aligned}$$

1.3.3.2.3 Example $xy' = 2x^4 - 18y + 5y^2$ Comparing to

$$xy' = cx^n + ay - by^2$$

Shows that $c = 2, n = 4, a = -18, b = -5$. Hence $\frac{(n-2a)}{2n} = \frac{(4-2(-18))}{8} = \frac{40}{8} = 5$. Therefore $k = 5$ which is positive integer.

The solution is finite continued fraction

$$\begin{aligned} y &= \frac{a}{b} + \frac{x^n}{y_1} \\ y_1 &= \frac{a+n}{c} + \frac{x^n}{y_2} \\ y_2 &= \frac{a+2n}{b} + \frac{x^n}{y_3} \\ y_3 &= \frac{a+3n}{c} + \frac{x^n}{y_4} \\ y_4 &= \frac{a+4n}{b} + \frac{x^n}{y_5} \\ &\vdots \\ y_{k-1} &= \frac{a+(k-1)\,n}{\Delta} + \frac{x^n}{y_k} \end{aligned}$$

Where $\Delta = c$ when index i on y_i is odd and $\Delta = b$ when index i on y_i is even. In this example, we found that $k = 5$. Therefore we have (we stop at $k - 1 = 4$)

$$\begin{aligned} y &= \frac{a}{b} + \frac{x^n}{y_1} \\ y_1 &= \frac{a+n}{c} + \frac{x^n}{y_2} \\ y_2 &= \frac{a+2n}{b} + \frac{x^n}{y_3} \\ y_3 &= \frac{a+3n}{c} + \frac{x^n}{y_4} \\ y_4 &= \frac{a+4n}{b} + \frac{x^n}{y_5} \end{aligned} \tag{1}$$

Now, we just need to find $y_5 = y_k$ to finish. To find y_5 we use either

$$xy_5' = bx^n + (a+nk)\,y_5 - cy_5^2 \tag{A}$$

Or

$$xy_5' = cx^n + (a+nk)\,y_5 - by_5^2 \tag{B}$$

ODE (A) is used when k is odd and ode (B) is used when k is even. Since $k = 5$ is odd, then we use (A). Hence the ode to solve for y_5 is

$$xy_5' = bx^n + (a+nk)\,y_5 - cy_5^2$$

But in this problem, $c = 2, n = 4, a = -18, b = -5$, then the above becomes

$$\begin{aligned} xy_5' &= -5x^4 + (-18+20)\,y_5 - 2y_5^2 \\ &= -5x^4 + 2y_5 - 2y_5^2 \end{aligned}$$

Notice that this has form

$$xy_5' = Cx^N + Ay_5 - By_5^2$$

(used upper case letters now, so not to confuse with lower case letters used for the original ode).

Now we see that $N = 4, A = 2, C = -5, B = 2$ which means $N = 2A$. But this is case (i). This always happens with case (ii) that we end up having to solve one ode using case (i) to finish. If we do not end up with case(i) ode, then we have made a mistake

But we know how to solve case (i) which we did above. This gives us y_2. We plug this into (1) and now we have the solution for y.

Now we solve (B) as we did in part (i). Let $y_2 = x^A u$. Then ode (B) becomes (as was shown in case (i) example)

$$u' = x^{A-1}(C - Bu^2) \tag{3}$$

We see that (3) is separable. Solving (3) gives

$$u = \frac{1}{B}\tanh\left(\frac{\sqrt{CB}(C_1 A + x^A)}{A}\right)\sqrt{CB}$$

But $y_5 = x^A u$, hence $u = y_5 x^{-A}$. Therefore

$$\begin{aligned} y_5 x^{-A} &= \frac{1}{B}\tanh\left(\frac{\sqrt{CB}(C_1 A + x^A)}{A}\right)\sqrt{CB} \\ y_5 &= \sqrt{CB}\frac{x^A}{B}\tanh\left(\frac{\sqrt{CB}(C_1 A + x^A)}{A}\right) \end{aligned}$$

Plugging in values for $N = 4, A = 2, C = -5, B = 2$ gives

$$\begin{aligned} y_5 &= \sqrt{-10}\frac{x^2}{2}\tanh\left(\frac{\sqrt{-10}(2C_1 + x^2)}{2}\right) \\ &= \frac{\sqrt{-10}}{2}x^2\tanh\left(\frac{\sqrt{-10}x^2}{2} + C_2\right) \\ &= \frac{i\sqrt{10}}{2}x^2\tanh\left(\frac{i\sqrt{10}x^2}{2} + C_2\right) \end{aligned} \tag{4}$$

But $\tanh(iz) = i\tan(z)$, hence the above becomes

$$\begin{aligned} y_5 &= \frac{i^2\sqrt{10}}{2}x^2\tan\left(\frac{\sqrt{10}x^2}{2} + C_2\right) \\ &= \frac{-\sqrt{10}}{2}x^2\tan\left(\frac{\sqrt{10}x^2}{2} + C_2\right) \end{aligned}$$

Now we go back to (1) and find y

$$\begin{aligned} y &= \frac{a}{b} + \frac{x^n}{y_1} \\ y_1 &= \frac{a+n}{c} + \frac{x^n}{y_2} \\ y_2 &= \frac{a+2n}{b} + \frac{x^n}{y_3} \\ y_3 &= \frac{a+3n}{c} + \frac{x^n}{y_4} \\ y_4 &= \frac{a+4n}{b} + \frac{x^n}{y_5} \end{aligned} \tag{1}$$

Hence

$$y=\frac{a}{b}+\frac{x^n}{\frac{a+n}{c}+\frac{x^n}{\frac{a+2n}{b}+\frac{x^n}{\frac{a+3n}{c}+\frac{x^n}{\frac{a+4n}{b}+\frac{x^n}{y_5}}}}}$$

Substituting all parameters into the above and using y_2 from (4) gives, using $c=2,n=4,a=-18,b=-5$

$$y=\frac{-18}{-5}+\frac{x^4}{\frac{-18+4}{2}+\frac{x^4}{\frac{-18+8}{-5}+\frac{x^4}{\frac{-18+12}{2}+\frac{x^4}{\frac{-18+16}{-5}+\frac{x^4}{\frac{-\sqrt{10}}{2}x^2\tan\left(\frac{\sqrt{10}x^2}{2}+C_2\right)}}}}}$$

Or

$$y=\frac{18}{5}+\frac{x^4}{-7+\frac{x^4}{2+\frac{x^4}{-3+\frac{x^4}{\frac{2}{5}-\frac{2x^2}{\sqrt{10}\tan\left(\frac{\sqrt{10}x^2}{2}+C_2\right)}}}}}$$

1.3.3.3 Case when $\frac{(n+2a)}{2n}=k$ with k positive integer. (section 3-3 a(iii))

1.3.3.3.1 Example $xy'=2x^4+6y-5y^2$

Comparing to

$$xy'=cx^n+ay-by^2$$

Shows that $n=4,c=2,a=6,b=5$. Hence $\frac{(n+2a)}{2n}=\frac{(4+12)}{8}=\frac{16}{8}=2$ a positive integer. Therefore $k=2$. This is similar to case (ii) above, where here also the solution is finite continued fraction, but now $-a$ is used in place of a as in case (ii) and first term y is different that in case (ii). But everything else is the same. Hence the solution now becomes

$$\begin{aligned}
y&=\frac{x^n}{y_1}\\
y_1&=\frac{n-a}{c}+\frac{x^n}{y_2}\\
y_2&=\frac{2n-a}{b}+\frac{x^n}{y_3}\\
y_3&=\frac{3n-a}{c}+\frac{x^n}{y_4}\\
y_4&=\frac{4n-a}{b}+\frac{x^n}{y_5}\\
&\vdots\\
y_{k-1}&=\frac{(k-1)\,n-a}{\Delta}+\frac{x^n}{y_k}
\end{aligned}$$

Where $\Delta=c$ when index i on y_i is odd and $\Delta=b$ when index i on y_i is even. In this example, we found that $k=2$. Therefore we have (we stop at $k-1=1$)

$$\begin{aligned}
y&=\frac{x^n}{y_1}\\
y_1&=\frac{n-a}{c}+\frac{x^n}{y_2}
\end{aligned}\tag{1}$$

Now, we just need to find $y_2=y_k$ to finish. To find y_2 we use either

$$xy_2'=bx^n+(nk-a)\,y_2-cy_2^2\tag{A}$$

Or

$$xy_2' = cx^n + (nk - a)\, y_2 - by_2^2 \tag{B}$$

ODE (A) is used when k is odd and ode (B) is used when k is even. Since $k = 2$ is even, then we use (B). Hence the ode to solve for y_2 is

$$xy_2' = cx^n + (nk - a)\, y_2 - by_2^2$$

But in this problem, $n = 4, c = 2, a = 6, b = 5$, hence the above becomes

$$\begin{aligned} xy_2' &= 2x^4 + (8 - 6)\, y_2 - 5y_2^2 \\ &= 2x^4 + 2y_2 - 5y_2^2 \end{aligned} \tag{B1}$$

Notice that this has form

$$xy_2' = Cx^N + Ay_2 - By_2^2$$

(used upper case letters now, so not to confuse with lower case letters used for the original ode).

Now we see that $N = 4, A = 2, C = 2, B = 5$ which means $N = 2A$. Then we use case (i) to solve (B1). Let $y_2 = x^A u$. Then ode (B1)

$$u' = x^{A-1}\left(C - Bu^2\right) \tag{3}$$

We see that (3) is separable. Solving (3) gives

$$u = \frac{1}{B} \tanh\left(\frac{\sqrt{CB}\,(C_1 A + x^A)}{A}\right) \sqrt{CB}$$

But $y_2 = x^A u$, hence $u = y_2 x^{-A}$. Therefore

$$\begin{aligned} y_2 x^{-A} &= \frac{1}{B} \tanh\left(\frac{\sqrt{CB}\,(C_1 A + x^A)}{A}\right) \sqrt{CB} \\ y_2 &= \sqrt{CB}\frac{x^A}{B} \tanh\left(\frac{\sqrt{CB}\,(C_1 A + x^A)}{A}\right) \end{aligned}$$

Plugging in values for $N = 4, A = 2, C = 2, B = 5$ gives

$$\begin{aligned} y_2 &= \sqrt{10}\frac{x^2}{5} \tanh\left(\frac{\sqrt{10}\,(2C_1 + x^2)}{2}\right) \\ &= \sqrt{10}\frac{x^2}{5} \tanh\left(\frac{\sqrt{10}\,x^2}{2} + C_2\right) \end{aligned} \tag{4}$$

Now we go back to (1) and find y

$$\begin{aligned} y &= \frac{x^n}{y_1} \\ y_1 &= \frac{n - a}{c} + \frac{x^n}{y_2} \end{aligned}$$

Substituting all parameters into the above and using y_2 from (4) gives, using $n = 4, c = 2, a = 6, b = 5$,

$$\begin{aligned} y &= \frac{x^4}{\frac{4-6}{2} + \frac{x^4}{\sqrt{10}\frac{x^2}{5} \tanh\left(\frac{\sqrt{10}x^2}{2} + C_2\right)}} \\ &= \frac{x^4}{-1 + \frac{5x^2}{\sqrt{10} \tanh\left(\frac{\sqrt{10}x^2}{2} + C_2\right)}} \end{aligned}$$

1.3.3.4 Case of conversion to reduced riccati. section 3-3 (b)

If none of the above three cases apply, then we convert it to reduced Riccati.

1.3.3.4.1 Example $xy' = 2x^5 + 6y - 5y^2$ Comparing to

$$xy' = cx^n + ay - by^2 \tag{1}$$

Shows that $c = 2, n = 5, a = 6, b = 5$. Checking case (i), where condition is $n = 2a$. We see this does not apply. Check case (ii) where condition is $\frac{(n-2a)}{2n} = k$ with k positive integer. We see that $\frac{(n-2a)}{2n} = \frac{(5-12)}{10} = \frac{-7}{10}$ which is not positive integer. Finally checking case (iii) where condition is $\frac{(n+2a)}{2n}$ being positive integer. But $\frac{(n+2a)}{2n} = \frac{(5+12)}{10} = \frac{17}{10}$ which is not positive integer. Since all three cases failed, we convert the ode to reduced Riccati using

$$\begin{aligned} y &= zu(z) \\ z &= x^a \end{aligned}$$

Or $x = z^{\frac{1}{a}}$. Hence

$$\begin{aligned} y'(x) &= z'u + z\frac{du}{dz}z' \\ &= ax^{a-1}u + z\frac{du}{dz}ax^{a-1} \\ &= a\left(z^{\frac{1}{a}}\right)^{a-1}u + z\frac{du}{dz}a\left(z^{\frac{1}{a}}\right)^{a-1} \\ &= az^{\frac{a-1}{a}}u + z\frac{du}{dz}az^{\frac{a-1}{a}} \\ &= az^{\frac{a-1}{a}}u + az^{1+\frac{a-1}{a}}\frac{du}{dz} \\ &= az^{\frac{a-1}{a}}u + az^{\frac{a+a-1}{a}}\frac{du}{dz} \\ &= az^{\frac{a-1}{a}}u + az^{\frac{2a-1}{a}}\frac{du}{dz} \end{aligned}$$

The original ode $xy' = cx^n + ay - by^2$ now becomes

$$\begin{aligned} z^{\frac{1}{a}}\left(az^{\frac{a-1}{a}}u + az^{\frac{2a-1}{a}}\frac{du}{dz}\right) &= c\left(z^{\frac{1}{a}}\right)^n + azu - bz^2u^2 \\ azu + az^{\frac{2a-1}{a}+\frac{1}{a}}\frac{du}{dz} &= cz^{\frac{n}{a}} + azu - bz^2u^2 \\ az^{\frac{2a-1+1}{a}}\frac{du}{dz} &= cz^{\frac{n}{a}} - bz^2u^2 \\ az^2\frac{du}{dz} &= cz^{\frac{n}{a}} - bz^2u^2 \\ a\frac{du}{dz} &= cz^{\frac{n}{a}-2} - bu^2 \\ \frac{du}{dz} &= \frac{c}{a}z^{\frac{n}{a}-2} - \frac{b}{a}u^2 \end{aligned} \tag{2}$$

Which is reduced Riccati form

$$y' = Ax^N + By^2$$

Where here $A = \frac{c}{a}, N = \frac{n}{a} - 2, B = -\frac{b}{a}$ which can be solved as shown in the section of reduced Riccati ode above. Using $c = 2, n = 5, a = 6, b = 5$ ode (2) becomes

$$\begin{aligned} \frac{du}{dz} &= \frac{2}{6}z^{\frac{5}{6}-2} - \frac{5}{6}u^2 \\ &= \frac{1}{3}z^{-\frac{7}{6}} - \frac{5}{6}u^2 \end{aligned}$$

Hence $A = \frac{1}{3}, B = -\frac{5}{6}$ and $N = -\frac{7}{6}$. Since N is not the special case -2. we use Reduced Riccati for the case where $N \neq -2$, which gives

$$
\begin{aligned}
w &= \sqrt{z}\begin{cases} c_1 \operatorname{BesselJ}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{AB}z^k\right) + c_2 \operatorname{BesselY}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{AB}z^k\right) & AB > 0 \\ c_1 \operatorname{BesselI}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{-AB}z^k\right) + c_2 \operatorname{BesselK}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{-AB}z^k\right) & AB < 0 \end{cases} \qquad (3) \\
u &= -\frac{1}{B}\frac{w'}{w} \\
k &= 1 + \frac{N}{2}
\end{aligned}
$$

Since $AB = -\frac{5}{18} < 0$ then the solution is

$$
w = \sqrt{z}c_1 \operatorname{BesselI}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{-AB}z^k\right) + c_2 \operatorname{BesselK}\left(\frac{1}{2k}, \frac{1}{k}\sqrt{-AB}z^k\right)
$$

But $k = 1 + \frac{-\frac{7}{6}}{2} = \frac{5}{12}$ and the above becomes

$$
\begin{aligned}
w &= \sqrt{z}c_1 \operatorname{BesselI}\left(\frac{1}{2\frac{5}{12}}, \frac{1}{\frac{5}{12}}\sqrt{\frac{5}{18}}z^{\frac{5}{12}}\right) + c_2 \operatorname{BesselK}\left(\frac{1}{2\frac{5}{12}}, \frac{1}{\frac{5}{12}}\sqrt{\frac{5}{18}}z^{\frac{5}{12}}\right) \\
&= \sqrt{z}c_1 \operatorname{BesselI}\left(\frac{5}{6}, \frac{12}{5}\sqrt{\frac{5}{18}}z^{\frac{5}{12}}\right) + c_2 \operatorname{BesselK}\left(\frac{5}{6}, \frac{12}{5}\sqrt{\frac{5}{18}}z^{\frac{5}{12}}\right)
\end{aligned}
$$

Hence

$$
\begin{aligned}
u(z) &= -\frac{1}{B}\frac{w'(z)}{w(z)} \\
&= -\frac{1}{-\frac{5}{6}}\frac{w'(z)}{w(z)} \\
&= \frac{6}{5}\frac{w'(z)}{w(z)}
\end{aligned}
$$

Carrying out the simplification above gives

$$
u(z) = \frac{1}{z^{\frac{1}{6}}} \frac{2\sqrt{10}\left(\operatorname{BesselI}\left(\frac{1}{5}, \frac{2\sqrt{10}}{5}z^{\frac{5}{12}}\right)c_1 - \operatorname{BesselK}\left(\frac{1}{5}, \frac{2\sqrt{10}}{5}z^{\frac{5}{12}}\right)\right)}{-\operatorname{BesselI}\left(\frac{1}{5}, \frac{2\sqrt{10}}{5}z^{\frac{5}{12}}\right)\sqrt{10}c_1 + \sqrt{10}\operatorname{BesselK}\left(\frac{1}{5}, \frac{2\sqrt{10}}{5}z^{\frac{5}{12}}\right) + 10z^{\frac{5}{12}}\left(\operatorname{BesselK}\left(\frac{4}{5}, \frac{2\sqrt{10}}{5}z^{\frac{5}{12}}\right) + \operatorname{BesselI}\left(-\frac{4}{5}, \frac{2\sqrt{10}}{5}z^{\frac{5}{1}}\right.\right.}
$$

Now that we found u, then y for the original ode is found, since

$$
y = zu(z)
$$

This completes the solution. We need to also change all z in the solution to x using $z = x^a$ after applying the above. This results in

$$
y(x) = -x^5 \frac{2\sqrt{10}\left(\operatorname{BesselI}\left(\frac{1}{5}, \frac{2\sqrt{10}}{5}x^{\frac{5}{2}}\right)c_1 - \operatorname{BesselK}\left(\frac{1}{5}, \frac{2\sqrt{10}}{5}x^{\frac{5}{2}}\right)\right)}{\operatorname{BesselI}\left(\frac{1}{5}, \frac{2\sqrt{10}}{5}x^{\frac{5}{2}}\right)\sqrt{10}c_1 + \sqrt{10}\operatorname{BesselK}\left(\frac{1}{5}, \frac{2\sqrt{10}}{5}x^{\frac{5}{2}}\right) + 10x^{\frac{5}{2}}\left(\operatorname{BesselK}\left(\frac{4}{5}, \frac{2\sqrt{10}}{5}x^{\frac{5}{2}}\right) + \operatorname{BesselI}\left(-\frac{4}{5}, \frac{2\sqrt{10}}{5}x^{\frac{5}{2}}\right)c_1\right.}
$$

1.3.4 Examples for section 3-1-3 (Removal of linear term and exploring relations between coefficients)

Local contents

This section is used If none of the above algorithms worked in solving $y' = f_0 + f_1 y + f_2 y^2$. They broken into part (a) and part(b) with 3 subcases under each part. Part (a) is for removal of linear term (this is $f_1 y$ term) and part (b) is for exploring some relations between f_0, f_1, f_2 that can lead to solutions.

1.3.4.1 Examples for section 3-1-3 (a) Removal of linear term, subcase (i)

1.3.4.1.1 Algorithm Given $y' = f_0 + f_1 y + f_2 y^2$ let $y = u(x)\, e^{\phi}$ where $\phi = \int f_1 dx$ and the ode becomes

$$u' = F(x) + G(x)\, u^2$$

Where $F(x) = f_0 e^{-\phi}, G(x) = f_2 e^{\phi}$. There is one special subcase here to consider. If $F(x)$ turns out to be proportional to $G(x)$ then the ode is separable. So this check should be done after completing the above. See second example below of one such example.

1.3.4.1.2 Example $y' = x + 3y + 7y^2$ Comparing to $y' = f_0 + f_1 y + f_2 y^2$ shows that

$$\begin{aligned} f_0 &= x \\ f_1 &= 3 \\ f_2 &= 7 \end{aligned}$$

Let

$$\begin{aligned} y &= u(x)\, e^{\phi} \\ \phi &= \int f_1 dx \end{aligned}$$

Therefore $\phi = \int 3dx = 3x$ and we have

$$y = e^{3x} u(x)$$

Therefore

$$y' = 3e^{3x} u + e^{3x} u'$$

The input ode becomes

$$\begin{aligned} y' &= x + 3y + 7y^2 \\ 3e^{3x} u + e^{3x} u' &= x + 3\left(e^{3x} u\right) + 7\left(e^{3x} u\right)^2 \\ e^{3x} u' &= x + 7e^{6x} u^2 \\ u' &= xe^{-3x} + 7e^{3x} u^2 \end{aligned} \tag{1}$$

We see that the transformed u ode will always have the general form

$$\begin{aligned} u' &= F(x) + G(x)\, u^2 \\ &= f_0 e^{-\phi} + f_2 e^{\phi} u^2 \end{aligned}$$

Where

$$F(x) = f_0 e^{-\phi}$$
$$G(x) = f_2 e^{\phi}$$

Hence the linear term f_1 is removed. Now we will try to solve (1). It is not one of the special cases we have solved before. We can either try to find a particular solution, or if we can't, then convert it to second order ode and solve it that way.

1.3.4.1.3 Example $y' = 5xe^{3x} + 3y + 10xe^{-3x}y^2$

Comparing to $y' = f_0 + f_1 y + f_2 y^2$ shows that

$$f_0 = 5xe^{3x}$$
$$f_1 = 3$$
$$f_2 = 10xe^{-3x}$$

Let

$$y = u(x)\, e^{\phi}$$
$$\phi = \int f_1 dx$$

Therefore $\phi = \int 3dx = 3x$ and we have

$$u' = F(x) + G(x)\, u^2$$

Where

$$F(x) = f_0 e^{-\phi} = 5xe^{3x}e^{-3x} = 5x$$
$$G(x) = f_2 e^{\phi} = 10xe^{-3x}e^{3x} = 10x$$

Hence

$$u' = 5x + 10xu^2$$

Since $F(x)$ is proportional to $G(x)$, then this is separable ode which is easily solved. Once u is known, then y is found since $y = u(x)\, e^{\phi}$.

1.3.4.2 Examples for section 3-1-3 (a) Removal of linear term, subcase (ii)

1.3.4.2.1 Example $y' = x + 3y + 7y^2$

Comparing to $y' = f_0 + f_1 y + f_2 y^2$ shows that

$$f_0 = x$$
$$f_1 = 3$$
$$f_2 = 7$$

Let

$$y = u(x) - v(x)$$
$$v = \frac{f_1}{2f_2}$$
$$= \frac{3}{14}$$

Then $y = u - \frac{3}{14}$ and $y' = u'$. The ode becomes

$$
\begin{aligned}
y' &= x + 3y + 7y^2 \\
u' &= x + 3\left(u - \frac{3}{14}\right) + 7\left(u - \frac{3}{14}\right)^2 \\
&= x + 3u - \frac{9}{14} + 7\left(u^2 + \frac{9}{14^2} - \frac{6}{14}u\right) \\
&= x + 3u - \frac{9}{14} + 7u^2 + \frac{(7)(9)}{14^2} - \frac{6}{2}u \\
&= x - \frac{9}{14} + 7u^2 + \frac{9}{28} \\
&= \left(x - \frac{9}{28}\right) + 7u^2 \qquad (1)
\end{aligned}
$$

We see that the linear term is removed. In the above $f_0 = x - \frac{9}{28}, f_1 = 0, f_2 = 7$.

Now we will try to solve (1). It is not one of the special cases we have solved before. We can either try to find a particular solution, or if we can't, then convert it to second order ode and solve it that way.

1.3.4.3 Examples for section 3-1-3 (a) Removal of linear term, subcase (iii)

1.3.4.3.1 Example $y' = x + 3y + 7y^2$ Comparing to $y' = f_0 + f_1 y + f_2 y^2$ shows that

$$
\begin{aligned}
f_0 &= x \\
f_1 &= 3 \\
f_2 &= 7
\end{aligned}
$$

Let

$$
\begin{aligned}
y &= u(z)\, e^{\phi} \\
\phi &= \int f_1 dx \\
z &= -\int f_2 e^{\phi} dx
\end{aligned}
$$

Hence $\phi = \int 3dx = 3x$. Applying this transformation results in new ode in $u(z)$ as

$$
u'(z) = F(z) - u^2(z)
$$

Where

$$
\begin{aligned}
F(z) &= -\frac{f_0(z)\, e^{-2\phi}}{f_2\,(z)} \\
&= \frac{-x}{7} e^{-6x}
\end{aligned}
$$

But $z = -\int f_2 e^{\phi} dx = -\int 7e^{3x} dx = -\frac{7}{3}e^{3x}$. Hence $-\frac{3z}{7} = e^{3x}$ or $\ln\left(-\frac{3z}{7}\right) = 3x$ or

$$
x = \frac{1}{3} \ln\left(-\frac{3z}{7}\right)
$$

Therefore

$$
\begin{aligned}
F(z) &= \frac{-1}{7}\frac{1}{3}\ln\left(-\frac{3z}{7}\right)\left(e^{3x}\right)^{-2} \\
&= \frac{-1}{21}\ln\left(-\frac{3z}{7}\right)\left(-\frac{3z}{7}\right)^{-2} \\
&= \frac{-1}{21}\ln\left(-\frac{3z}{7}\right)\left(\frac{7^2}{9z^2}\right) \\
&= \frac{-7}{3}\ln\left(-\frac{3z}{7}\right)\left(\frac{1}{9z^2}\right) \\
&= \frac{-7}{27z^2}\ln\left(-\frac{3z}{7}\right)
\end{aligned}
$$

The new ode in u becomes

$$
u' = \frac{-7}{27z^2}\ln\left(-\frac{3z}{7}\right) - u^2 \tag{1}
$$

We see that the linear term is removed. In the above $f_0 = \frac{-7}{27z^2}\ln\left(-\frac{3z}{7}\right), f_1 = 0, f_2 = -1$.

Now we will try to solve (1). It is not one of the special cases we have solved before. We can either try to find a particular solution, or if we can't, then convert it to second order ode and solve it that way.

1.3.4.4 Examples for section 3-1-3 (b) Relations between the coefficients, subcase (i)

Local contents

1.3.4.4.1 Algorithm Given $y' = f_0 + f_1 y + f_2 y^2$, we look for a, b constants not both a, b zero such that

$$
a^2 f_0 + ab f_1 + b^2 f_2 = 0
$$

If we can find such a, b, then there are three cases to consider.

1. $a \neq 0$. In this case a particular solution is $y_1 = \frac{b}{a}$. Hence now we can solve the Riccati ode since a particular solution is known.
2. $a = 1, b = 1$, then this means $f_0 + f_1 + f_2 = 0$ where now a particular solution is $y_1 = 1$, and now we can solve the Riccati ode since a particular solution is known
3. case $a = 0$ and $b \neq 0$ can not show up, since this implies $f_2 = 0$ and hence the ode is not Riccati to start with.

1.3.4.4.2 Example $y' = x - 2xy + xy^2$

Comparing to $y' = f_0 + f_1 y + f_2 y^2$ shows that

$$\begin{aligned} f_0 &= x \\ f_1 &= -2x \\ f_2 &= x \end{aligned}$$

Hence

$$\begin{aligned} a^2 f_0 + ab f_1 + b^2 f_2 &= 0 \\ a^2 x - 2xab + b^2 x &= 0 \\ a^2 - 2ab + b^2 &= 0 \end{aligned}$$

A solution is $a = 1, b = 1$. This is case 2. Hence a particular solution is

$$y_1 = 1$$

Now that we know a particular solution, finding the general solution to the Riccati ode is easy. See the section below on how this is done.

1.3.4.4.3 Example $y' = x + xy - 2xy^2$

Comparing to $y' = f_0 + f_1 y + f_2 y^2$ shows that

$$\begin{aligned} f_0 &= x \\ f_1 &= x \\ f_2 &= -2x \end{aligned}$$

Hence

$$\begin{aligned} a^2 f_0 + ab f_1 + b^2 f_2 &= 0 \\ a^2 x + xab - 2b^2 x &= 0 \\ a^2 + ab - 2b^2 &= 0 \end{aligned}$$

A solution is $a = 2, b = -1$. Since $a \neq 0$ then a particular solution is

$$\begin{aligned} y_1 &= \frac{b}{a} \\ &= \frac{-1}{2} \end{aligned}$$

Now that we know a particular solution, finding the general solution to the Riccati ode is easy. See the section below on how this is done.

1.3.4.5 Examples for section 3-1-3 (b) Relations between the coefficients, subcase (ii)

1.3.4.5.1 Algorithm

This is covered in section 3-1-3 (a) Removal of linear term, subcase (i), where the linear term is removed giving

$$u' = F(x) + G(x)\,u^2$$

And it turns out that $F(x)$ is proportional to $G(x)$ making it separable. See example above.

1.3.4.6 Examples for section 3-1-3 (b) Relations between the coefficients, subcase (iii)

1.3.4.6.1 Algorithm

Assuming particular solution of $y' = f_0 + f_1 y + f_2 y^2$ is y_1 where

$$2 f_2 y_1 = X(x) - f_1 \tag{1}$$

Where $X(x)$ satisfies

$$f_0 = f_2 y_1^2 - X y_1 + y_1' \tag{2}$$

Murphy book suggests to try

$$\begin{aligned} X &= 0 \\ X &= -\frac{f_2'}{f_2} \\ X &= f_1 - 2\sqrt{f_0 f_2} \end{aligned}$$

For each such case of X, we end up with y_1 from (1), which now we check if it satisfies (2) or not. If it does, then y_1 can be used to solve the Riccati ode

1.3.5 Examples for section 3-1-1. One known particular solution is given

Local contents

1.3.5.1 Algorithm

Given $y' = f_0 + f_1 y + f_2 y^2$, and also given one known particular solution y_1. This is easy case. See section (3-1-2) on how to try to find a particular solution if one is not given. Once one particular solution is known (either given or by using 3-1-2) then finding the solution is done as follows. Let

$$y = y_1 + u(x)$$

The ode becomes Bernoulli as shown below

Assuming we are given a particular solution y_1 to the general Riccati ode $y' = f_0(x) + f_1(x)\,y + f_2(x)\,y^2$. Then we can either let $y = y_1 + u$ or $y = y_1 + \frac{1}{u}$.

Using $y = y_1 + u$ method, then the the Riccati ode $y' = f_0 + f_1 y + f_2 y^2$ becomes a Bernoulli ode.

$$\begin{aligned} (y_1 + u)' &= f_0 + f_1(y_1 + u) + f_2(y_1 + u)^2 \\ y_1' + u' &= f_0 + f_1 y_1 + f_1 u + f_2\left(y_1^2 + u^2 + 2y_1 u\right) \\ y_1' + u' &= f_0 + f_1 y_1 + f_1 u + f_2 y_1^2 + f_2 u^2 + 2 f_2 y_1 u \\ y_1' + u' &= \overbrace{f_0 + f_1 y_1 + f_2 y_1^2} + f_1 u + f_2 u^2 + 2 f_2 y_1 u \\ u' &= f_1 u + f_2 u^2 + 2 f_2 y_1 u \\ &= u(f_1 + 2 f_2 y_1) + f_2 u^2 \end{aligned}$$

Which is Bernoulli ode. Solving this for u then $y = y_1 + u$. Another possibility is to assume that $y = y_1 + \frac{1}{u(x)}$ which results in Linear ode instead of Bernoulli which is a little simpler to solve. A direct formula to obtain the general solution if particular solution y_1 is known is given on page 105 of the handbook of exact solutions of ordinary differential equation as

$$\begin{aligned} y &= y_1 + \Phi \frac{1}{c_1 - \int \Phi f_2 dx} \\ \Phi &= e^{\int 2 f_2 y_1 + f_1 dx} \end{aligned} \tag{1}$$

If the input ode was $g(x)\, y' = f_0 + f_1 y + f_2 y^2$ instead, then (1) is modified to be

$$
\begin{aligned}
y &= y_1 + \Phi \frac{1}{c_1 - \int \Phi \frac{f_2}{g} dx} \\
\Phi &= e^{\int \frac{2 f_2 y_1 + f_1}{g} dx}
\end{aligned} \tag{2}
$$

Examples below show how to use these formulas. The above formula can be derived from using $y = y_1 + \frac{1}{u(x)}$ and using an integrating factor to solve for u.

1.3.5.2 Example $y' = x^5 + \left(\frac{1}{x} - 2x^4\right) y + x^3 y^2$

Comparing to $y' = f_0 + f_1 y + f_2 y^2$ shows that

$$
\begin{aligned}
f_0 &= x^5 \\
f_1 &= \left(\frac{1}{x} - 2x^4\right) \\
f_2 &= x^3
\end{aligned}
$$

We see that $y_1 = x$ is a solution. Hence let $y = y_1 + \frac{1}{u}$. The ode becomes

$$
u' - \frac{u}{x} = -x^3
$$

Which is linear. Solving using integrating factor gives

$$
u = \frac{1}{x}\left(c_1 - \frac{x^5}{5}\right)
$$

Hence

$$
\begin{aligned}
y &= y_1 + \frac{1}{u} \\
&= x + \frac{1}{\frac{1}{x}\left(c_1 - \frac{x^5}{5}\right)} \\
&= x + \frac{5x}{(5c_1 - x^5)} \\
&= x + \frac{5x}{c_2 - x^5} \\
&= \frac{x(x^5 - c_2 - 5)}{x^5 - c_2}
\end{aligned} \tag{A}
$$

Using the direct formula (1) given earlier, we can write

$$
\begin{aligned}
y &= y_1 + \Phi \frac{1}{c_1 - \int \Phi f_2 dx} \\
&= x + \Phi \frac{1}{c_1 - \int \Phi x^3 dx}
\end{aligned} \tag{B}
$$

Where

$$
\begin{aligned}
\Phi &= e^{\int 2 f_2 y_1 + f_1 dx} \\
&= e^{\int 2x^4 + \left(\frac{1}{x} - 2x^4\right) dx} \\
&= e^{\ln x} \\
&= x
\end{aligned}
$$

Hence (B) becomes

$$\begin{aligned}
y &= x + \frac{x}{c_1 - \int x^4 dx} \\
&= x + \frac{x}{c_1 - \frac{x^5}{5}} \\
&= x + \frac{5x}{5c_1 - x^5} \\
&= \frac{x(5c_1 - x^5) + 5x}{5c_1 - x^5} \\
&= \frac{x(5c_1 - x^5 + 5)}{5c_1 - x^5} \\
&= \frac{x(c_2 - x^5 + 5)}{c_2 - x^5} \\
&= \frac{x(x^5 - c_2 - 5)}{x^5 - c_2}
\end{aligned}$$

Which is the same solution in (A). The direct formula might be easier to use.

1.3.5.3 Example $y' = 3a - a^2x^2 + y^2$

Comparing to $y' = f_0 + f_1 y + f_2 y^2$ shows that

$$\begin{aligned}
f_0 &= 3a - a^2x^2 \\
f_1 &= 0 \\
f_2 &= 1
\end{aligned}$$

A particular solution is $y_1 = ax - \frac{1}{x}$. Using the direct formula (1) given earlier

$$\begin{aligned}
y &= y_1 + \Phi \frac{1}{c_1 - \int \Phi f_2 dx} \\
&= ax - \frac{1}{x} + \frac{\Phi}{c_1 - \int \Phi dx} \qquad \text{(B)}
\end{aligned}$$

Where

$$\begin{aligned}
\Phi &= e^{\int 2f_2 y_1 + f_1 dx} \\
&= e^{\int 2\left(ax - \frac{1}{x}\right) dx} \\
&= e^{ax^2 - 2\ln x} \\
&= \frac{e^{ax^2}}{x^2}
\end{aligned}$$

Hence (B) becomes

$$\begin{aligned}
y &= ax - \frac{1}{x} + \frac{\frac{e^{ax^2}}{x^2}}{c_1 - \int \frac{e^{ax^2}}{x^2} dx} \\
&= ax - \frac{1}{x} + \frac{e^{ax^2}}{x^2 \left(c_1 - \int \frac{e^{ax^2}}{x^2} dx\right)}
\end{aligned}$$

1.3.5.4 Example $y' = ax^{n-1} + ax^n y + y^2$

Comparing to $y' = f_0 + f_1 y + f_2 y^2$ shows that

$$f_0 = ax^{n-1}$$
$$f_1 = ax^n$$
$$f_2 = 1$$

A particular solution is $y_1 = -\frac{1}{x}$. Using the direct formula (1) given earlier

$$\begin{aligned} y &= y_1 + \Phi \frac{1}{c_1 - \int \Phi f_2 dx} \\ &= -\frac{1}{x} + \frac{\Phi}{c_1 - \int \Phi dx} \end{aligned} \tag{B}$$

Where

$$\begin{aligned} \Phi &= e^{\int 2f_2 y_1 + f_1 dx} \\ &= e^{\int -\frac{2}{x} + ax^n dx} \\ &= e^{a\frac{x^{n+1}}{n+1} - 2\ln x} \\ &= \frac{1}{x^2} e^{a\frac{x^{n+1}}{n+1}} \end{aligned}$$

Hence (B) becomes

$$\begin{aligned} y &= -\frac{1}{x} + \frac{\frac{1}{x^2} e^{a\frac{x^{n+1}}{n+1}}}{c_1 - \int \frac{e^{a\frac{x^{n+1}}{n+1}}}{x^2} dx} \\ &= -\frac{1}{x} + \frac{e^{a\frac{x^{n+1}}{n+1}}}{x^2 \left(c_1 - \int \frac{e^{a\frac{x^{n+1}}{n+1}}}{x^2} dx\right)} \end{aligned}$$

1.3.5.5 Example $xy' = -ab^2 x^{n+2m} + my + ax^n y^2$

Comparing to $gy' = f_0 + f_1 y + f_2 y^2$ shows that

$$f_0 = -ab^2 x^{n+2m}$$
$$f_1 = m$$
$$f_2 = ax^n$$
$$g = x$$

A particular solution is $y_1 = bx^m$. Using the direct formula (1) given earlier

$$\begin{aligned} y &= y_1 + \Phi \frac{1}{c_1 - \int \Phi \frac{f_2}{g} dx} \\ \Phi &= e^{\int \frac{2f_2 y_1 + f_1}{g} dx} \end{aligned} \tag{2}$$

Then

$$\begin{aligned} \Phi &= e^{\int \frac{2(ax^n)(bx^m) + m}{x} dx} \\ &= x^m e^{\frac{2abx^{n+m}}{n+m}} \end{aligned}$$

Hence the solution is

$$
\begin{aligned}
y &= y_1 + \Phi \frac{1}{c_1 - \int \Phi \frac{f_2}{g} dx} \\
&= bx^m + \frac{x^m e^{\frac{2abx^{n+m}}{n+m}}}{c_1 - \int x^m e^{\frac{2abx^{n+m}}{n+m}} \frac{(ax^n)}{x} dx} \\
&= bx^m + \frac{x^m e^{\frac{2abx^{n+m}}{n+m}}}{c_1 - a \int x^{m+n-1} e^{\frac{2abx^{n+m}}{n+m}} dx} \\
&= bx^m + \frac{x^m e^{\frac{2abx^{n+m}}{n+m}}}{c_1 - a \left(\frac{e^{\frac{2abx^{n+m}}{n+m}}}{2ab} \right)} \\
&= bx^m + \frac{2bx^m e^{\frac{2abx^{n+m}}{n+m}}}{2bc_1 - e^{\frac{2abx^{n+m}}{n+m}}} \\
&= bx^m + \frac{2bx^m \Delta}{2bc_1 - \Delta}
\end{aligned}
$$

Where

$$
\Delta = e^{\frac{2abx^{n+m}}{n+m}}
$$

1.3.6 Examples for section 3-1-2. Trying to find a particular solution

Local contents

1.3.6.1 Algorithm

Given $y' = f_0 + f_1 y + f_2 y^2$, this section shows how to attempt to find a particular solution. This is probably the most important part, since if we can find a particular solution, then the Riccati ode is solved. But there is no algorithm which works all the time to find a particular solution, since if there is, then Riccati can now be solved in all cases. But it is not.

The main algorithm shown by Murphy and Kamke books starts by assuming that $y = \frac{u(x)}{f_2}$ and transforms to new ode in u which is

$$
u' = F(x) + G(x)\, u + u^2
$$

Where

$$
\begin{aligned}
F &= f_0 f_2 \\
G &= f_1 + \frac{f_2'}{f_2}
\end{aligned}
$$

Only if F, G are polynomials in x, then more progress can be made as shown in the flow chart. There are two main subcases. When $G = 0$ and when $G \neq 0$. Examples below go over each subcase. If F, G are not polynomials then no progress can be made. Guessing a particular solution can be tried but guessing is not an algorithm. Note that even if f_0, f_1, f_2 are polynomials, this does not necessarily implies that F, G will be polynomials.

1.3.6.2 Example when G is zero $y' = 5 - x - \frac{2}{x}y - x^2y^2$

Comparing to $y' = f_0 + f_1 y + f_2 y^2$

$$\begin{aligned} f_0 &= 5 - x \\ f_1 &= -\frac{2}{x} \\ f_2 &= -x^2 \end{aligned}$$

Hence

$$\begin{aligned} F &= f_0 f_2 = (5 - x)\left(-x^2\right) \\ &= -5x^2 + x^3 \\ G &= f_1 + \frac{f_2'}{f_2} = -\frac{2}{x} + \frac{-2x}{-x^2} = -\frac{2}{x} + \frac{2}{x} \\ &= 0 \end{aligned}$$

The new ode is (after substituting $y = \frac{u}{f_2}$) becomes

$$\begin{aligned} u' &= F(x) + G(x)\,u + u^2 \\ &= \left(-5x^2 + x^3\right) + u^2 \end{aligned}$$

Now we look at the degree of $F(x)$. Since degree is odd, then no polynomial solution exist for the above Riccati ode in u.

1.3.6.3 Example when G is zero $y' = 3 + x^2 - \frac{2}{x}y - x^2y^2$

Comparing to $y' = f_0 + f_1 y + f_2 y^2$

$$\begin{aligned} f_0 &= 3 + x^2 \\ f_1 &= -\frac{2}{x} \\ f_2 &= -x^2 \end{aligned}$$

Hence

$$\begin{aligned} F &= f_0 f_2 = \left(3 + x^2\right)\left(-x^2\right) \\ &= -3x^2 - x^4 \\ G &= f_1 + \frac{f_2'}{f_2} = -\frac{2}{x} + \frac{-2x}{-x^2} = -\frac{2}{x} + \frac{2}{x} \\ &= 0 \end{aligned}$$

The new ode is (after substituting $y = \frac{u}{f_2}$) becomes

$$\begin{aligned} u' &= F(x) + G(x)\,u + u^2 \\ &= \left(-3x^2 - x^4\right) + u^2 \end{aligned} \tag{1}$$

Now we look at the degree of $F(x)$. Since degree is even, then polynomial solution might exist for the above ode. Now we expand $\sqrt{-F(x)}$ in series and stop at the constant term. The degree of $F(x)$ is 4. Hence $2n = 4$ and $n = 2$.

$$\sqrt{-F(x)} = \sqrt{3x^2 + x^4}$$
$$X(x) = \overbrace{a_2x^2 + a_1x + a_0}+\frac{b_1}{x} + \frac{b_2}{x^2} + \cdots$$

To find $X(x)$, we use method of undetermined coefficients. Let

$$\begin{aligned}
\left(a_2x^2 + a_1x + a_0\right)^2 &= 3x^2 + x^4 \\
\left(a_2x^2 + (a_1x + a_0)\right)^2 &= 3x^2 + x^4 \\
a_2x^4 + (a_1x + a_0)^2 + 2a_2x^2(a_1x + a_0) &= 3x^2 + x^4 \\
a_2x^4 + \left(a_1^2x^2 + a_0^2 + 2a_0a_1x\right) + 2a_1a_2x^3 + 2a_0a_2x^2 &= 3x^2 + x^4 \\
x^4(a_2) + x^3(2a_1a_2) + x^2\left(a_1^2 + 2a_0a_2\right) + x(2a_0a_1) + a_0^2 &= 3x^2 + x^4
\end{aligned}$$

Hence $a_2 = 1, 2a_1a_2 = 0$ or $a_1 = 0$ and $a_1^2 + 2a_0a_2 = 3$ or $2a_0 = 3$ or $a_0 = \frac{3}{2}$. Hence

$$X(x) = x^2 + \frac{3}{2}$$

Now we test if $\pm X(x)$ satisfies (1). It does not. Hence no polynomial solution.

1.3.6.4 Example when G not zero $y' = 1 + x^2 - 2xy + y^2$

Comparing to $y' = f_0 + f_1y + f_2y^2$

$$\begin{aligned}
f_0 &= 1 + x^2 \\
f_1 &= -2x \\
f_2 &= 1
\end{aligned}$$

Hence

$$\begin{aligned}
F &= f_0f_2 \\
&= 1 + x^2 \\
G &= f_1 + \frac{f_2'}{f_2} \\
&= -2x
\end{aligned}$$

Since F, G are polynomials, and $G \neq 0$ then let $y = \frac{u}{f_2}$ which gives

$$\begin{aligned}
u' &= F(x) + G(x)\,u + u^2 \\
&= \left(1 + x^2\right) - 2xu + u^2
\end{aligned} \tag{1}$$

Since $G \neq 0$ then let

$$\begin{aligned}
u &= w - \frac{G}{2} \\
&= w + x
\end{aligned} \tag{2}$$

When $G \neq 0$, book says to calculate Q given by

$$\begin{aligned}
Q &= G^2 - 4F - 2G' \\
&= (-2x)^2 - 4\left(1 + x^2\right) - 2(-2) \\
&= 0
\end{aligned}$$

There are 3 cases, either Q is polynomial of even or odd degree, or Q is constant as in this case.

Substituting (2) into (1) gives

$$w' = w^2$$

We see this is now separable. And we can solve it. This gives

$$w = \frac{1}{c_1 - x}$$

Hence

$$\begin{aligned} u &= w + x \\ &= \frac{1}{c_1 - x} + x \end{aligned}$$

And since $y = \frac{u}{f_2}$ then

$$\begin{aligned} y &= \frac{\frac{1}{c_1 - x} + x}{1} \\ &= \frac{x^2 + c_2 x - 1}{x + c_2} \end{aligned}$$

1.3.6.5 Example when G not zero $y' = -2 - 4x^2 - 4xy - y^2$

Comparing to $y' = f_0 + f_1 y + f_2 y^2$

$$\begin{aligned} f_0 &= -2 - 4x^2 \\ f_1 &= -4x \\ f_2 &= -1 \end{aligned}$$

Hence

$$\begin{aligned} F &= f_0 f_2 \\ &= 2 + 4x^2 \\ G &= f_1 + \frac{f_2'}{f_2} \\ &= -4x \end{aligned}$$

Since F, G are polynomials, and $G \neq 0$ then let $y = \frac{u}{f_2}$ which gives

$$\begin{aligned} u' &= F(x) + G(x)\,u + u^2 \qquad (1) \\ &= 4x^2 + 2 - 4xu + u^2 \end{aligned}$$

Since $G \neq 0$ then let

$$\begin{aligned} u &= w - \frac{G}{2} \\ &= w - \frac{-4x}{2} \\ &= w + 2x \qquad (2) \end{aligned}$$

When $G \neq 0$, book says to calculate Q given by

$$\begin{aligned} Q &= G^2 - 4F - 2G' \\ &= (-4x)^2 - 4(2 + 4x^2) - 2(-4) \\ &= 0 \end{aligned}$$

There are 3 cases, either Q is polynomial of even or odd degree, or Q is constant as in this case.

Substituting (2) into (1) gives

$$w' = w^2$$

We see this is now separable. And we can solve it. This gives

$$w = \frac{1}{c_1 - x}$$

Hence

$$\begin{aligned} u &= w + 2x \\ &= \frac{1}{c_1 - x} + 2x \end{aligned}$$

And since $y = \frac{u}{f_2}$ then

$$\begin{aligned} y &= \frac{\frac{1}{c_1 - x} + 2x}{-1} \\ &= \frac{1 + 2c_1 x - 2x^2}{x - c_1} \end{aligned}$$

1.3.6.6 Example when G not zero $y' = \frac{1}{2}(1+\sqrt{5})\,x + (\sqrt{5}+x)\,y + y^2$

Comparing to $y' = f_0 + f_1 y + f_2 y^2$

$$\begin{aligned} f_0 &= \frac{1}{2}\left(1+\sqrt{5}\right) x \\ f_1 &= \left(\sqrt{5}+x\right) \\ f_2 &= 1 \end{aligned}$$

Since $f_1 = 1$ already, we can calculate Q directly now

$$\begin{aligned} Q &= f_1^2 - 4f_0 - 2f_1' \\ &= \left(\sqrt{5}+x\right)^2 - 4\left(\frac{1}{2}\left(1+\sqrt{5}\right) x\right) - 2 \\ &= x^2 - 2x + 3 \end{aligned}$$

Since the degree is even, then now we expand $\sqrt{Q}$ and keep rational integer part and call that X

$$\begin{aligned} x^2 - 2x + 3 &= (a + bx)^2 \\ &= a^2 + b^2x^2 + 2abx \end{aligned}$$

Comparing terms, then $b = 1$ and $2ab = -2$. Hence $a = -1$. Therefore

$$\begin{aligned} \sqrt{Q} &= X \\ &= x - 1 \end{aligned}$$

Since X is not constant, then there are two possible particular polynomial solutions of $y' = f_0 + f_1 y + f_2 y^2$. These are

$$\begin{aligned} y &= -\frac{1}{2}(f_1 \pm X) \\ &= -\frac{1}{2}\left(\sqrt{5} + x \pm (x - 1)\right) \end{aligned}$$

These are

$$
\begin{aligned}
y_1 &= -\frac{1}{2}\left(\sqrt{5}+2x-1\right) \\
y_2 &= -\frac{1}{2}\left(\sqrt{5}+1\right)
\end{aligned}
$$

Testing each shows that y_1 satsifies the ode but not the second one. Hence we will use y_1 particular solution in order now to find general solution to the given ode.

Using the direct formula given earlier then the solution is

$$
\begin{aligned}
y &= y_1 + \Phi \frac{1}{c_1 - \int \Phi f_2 dx} \\
\Phi &= e^{\int 2 f_2 y_1 + f_1 dx}
\end{aligned} \tag{2}
$$

Then

$$
\begin{aligned}
\Phi &= e^{\int 2\left(-\frac{1}{2}\left(\sqrt{5}+2x-1\right)\right)+\left(\sqrt{5}+x\right)dx} \\
&= e^{x-\frac{1}{2}x^2}
\end{aligned}
$$

Hence the solution is

$$
\begin{aligned}
y &= y_1 + \Phi \frac{1}{c_1 - \int \Phi f_2 dx} \\
&= -\frac{1}{2}\left(\sqrt{5}+2x-1\right) + \frac{e^{x-\frac{1}{2}x^2}}{c_1 - \int e^{x-\frac{1}{2}x^2} dx} \\
&= -\frac{1}{2}\left(\sqrt{5}+2x-1\right) + \frac{e^{x-\frac{1}{2}x^2}}{c_1 - \left(\frac{\sqrt{\pi}e^{\frac{1}{2}}\sqrt{2}\operatorname{erf}\left(\frac{\sqrt{2}}{2}x-\frac{\sqrt{2}}{2}\right)}{2}\right)}
\end{aligned}
$$

Hence

$$
y = -\frac{1}{2}\left(\sqrt{5}+2x-1\right) + \frac{2e^{x-\frac{1}{2}x^2}}{c_2 - \sqrt{\pi}e^{\frac{1}{2}}\sqrt{2}\operatorname{erf}\left(\frac{\sqrt{2}}{2}x-\frac{\sqrt{2}}{2}\right)}
$$

1.3.6.7 Example when G not zero $y' = -x + \frac{33}{16} + \left(x^2 - x - \frac{1}{2}\right)y + y^2$

Comparing to $y' = f_0 + f_1 y + f_2 y^2$

$$
\begin{aligned}
f_0 &= -x + \frac{33}{16} \\
f_1 &= \left(x^2 - x - \frac{1}{2}\right) \\
f_2 &= 1
\end{aligned}
$$

Since $f_1 = 1$ already, we can calculate Q directly now

$$
\begin{aligned}
Q &= f_1^2 - 4f_0 - 2f_1' \\
&= \left(x^2 - x - \frac{1}{2}\right)^2 - 4\left(-x + \frac{33}{16}\right) - 2(2x-1) \\
&= x^4 - 2x^3 + x - 6
\end{aligned}
$$

Since the degree is even, then now we expand $\sqrt{Q}$ and keep rational integer part and call that X

$$
\begin{aligned}
x^4 - 2x^3 + x - 6 &= \left(ax^2 + bx + c\right)^2 \\
&= a^2x^4 + (bx + c)^2 + 2ax^2(bx + c) \\
&= a^2x^4 + \left(b^2x^2 + c^2 + 2bcx\right) + 2abx^3 + 2acx^2 \\
&= a^2x^4 + x^3(2ab) + x^2\left(b^2 + 2ac\right) + x(2bc) + c^2
\end{aligned}
$$

Comparing terms, then $a = 1, 2ab = -2$ or $b = -1$. And $2bc = 1$ or $c = -\frac{1}{2}$. Hence

$$
\begin{aligned}
\sqrt{Q} &= X \\
&= ax^2 + bx + c \\
&= x^2 - x - \frac{1}{2}
\end{aligned}
$$

Since X is not constant, then there are two possible particular polynomial solutions of $y' = f_0 + f_1y + f_2y^2$. These are

$$
\begin{aligned}
y &= -\frac{1}{2}(f_1 \pm X) \\
&= -\frac{1}{2}\left(\left(x^2 - x - \frac{1}{2}\right) \pm \left(x^2 - x - \frac{1}{2}\right)\right)
\end{aligned}
$$

These are

$$
\begin{aligned}
y_1 &= -\frac{1}{2}\left(\left(x^2 - x - \frac{1}{2}\right) + \left(x^2 - x - \frac{1}{2}\right)\right) \\
&= -x^2 + x + \frac{1}{2} \\
y_2 &= -\frac{1}{2}\left(\left(x^2 - x - \frac{1}{2}\right) - \left(x^2 - x - \frac{1}{2}\right)\right) \\
&= 0
\end{aligned}
$$

Testing each shows that neither satisfies the ode. Hence this approach did not work in this example.

1.3.6.8 Example when G,F not polynomials $y' = \frac{7}{x} - \frac{7}{x}y - y^2$

Comparing to $y' = f_0 + f_1y + f_2y^2$

$$
\begin{aligned}
f_0 &= \frac{7}{x} \\
f_1 &= -\frac{7}{x} \\
f_2 &= -1
\end{aligned}
$$

Hence

$$
\begin{aligned}
F &= f_0f_2 = -\frac{7}{x} \\
G &= f_1 + \frac{f_2'}{f_2} = -\frac{7}{x}
\end{aligned}
$$

G, F are not polynomials. Hence can not use this section method. This however can be solved by transforming to second order ode.

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