Highly Constrained Image Reconstruction (HYPR)

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Contents

1 Notations and definitions

1. MLEM Maximum-Likelihood Expectation-Maximization
2. PET Positron Emission Tomography
3. SPECT Single-Photon Emission Computed Tomography
4. I A 2-D image. This represent the original user image at which the HYPR algorithm is applied to.
5. I_t When the original image content changes during the process, we add a subscript to indicate the image I at time instance t.
6. R radon transform.
7. R_ϕ radon transform used at a projection angle ϕ.
8. ϕ_t When the projection angle ϕ is not constant but changes with time during the MRI acquisition process, we add a subscript to indicate the angle at time instance t.
9. R_{ϕ_t} radon transform used at an angle ϕ_t.
10. $s = R_\phi[I]$. radon transform applied to an image I at angle ϕ. This results in a projection vector s.
11. H Forward projection matrix. The Matrix equivalent to the radon transform R.
12. θ Estimate of an image I.
13. $H\theta$ Multiply the forward projection matrix H with an image estimate θ.
14. $g = H\theta$ Multiply the forward projection matrix H with an image estimate θ to obtain a projection vector g.
15. $R_\phi^u[s]$ The inverse radon transform applied in unfiltered mode to a projection s which was taken at angle ϕ. This results in a 2D image.
16. $R_\phi^f[s]$ The inverse radon transform applied in filtered mode to a projection s which was taken at angle ϕ. This results in a 2D image.

17. $H^T g$ The transpose of the forward projection matrix H multiplied by the projection vector g. This is the matrix equivalent of applying the inverse radon transform in an unfiltered mode to a projection s (see item 12 above).

18. $H^+ g$ The pseudo inverse of the forward projection matrix H being multiplied by the projection vector g. This is the matrix equivalent of applying the inverse radon transform in filtered mode to a projection s (see item 13 above).

19. C Composite image generated by summing all the filtered back projections from projections s_t of the original images I_t. Hence $C = \sum_{i=1}^{N} R^f_{\phi_{t_i}}[s_{t_i}]$

20. P_t The unfiltered backprojection 2D image as a result of applying $R^u_{\phi_t}[s_t]$ where s_t is projection from user image I_t taken at angle ϕ_t.

21. P_{c_t} The unfiltered backprojection 2D image as a result of applying $R^u_{\phi_t}[s_t]$ where s_t is projection from the composite image C taken at angle ϕ_t.

22. N_p Number of projections used to generate one HYPR frame image. This is the same as the number of projections per one time frame.

23. N The total number of projections used. This is the number of time frames multiplied by N_p

24. J_k The k^{th} HYPR frame image. A 2-D image generate at the end of the HYPR algorithm. There will be as many HYPR frame images J_k as there are time frames.

25. Image fidelity: " (inferred by the ability to discriminate between two images)" reference: The relationship between image fidelity and image quality by Silverstein, D.A.; Farrell, J.E

    Sci-Tech Encyclopedia: Fidelity

    "The degree to which the output of a system accurately reproduces the essential characteristics of its input signal. Thus, high fidelity in a sound system means that the reproduced sound is virtually indistinguishable from that picked up by the microphones in the recording or broadcasting studio. Similarly, a television system has a high fidelity when the picture seen on the screen of a receiver corresponds in essential respects to that picked up by the television camera. Fidelity is achieved by designing each part of a system to have minimum distortion, so that the waveform of the signal is unchanged as it travels through the system. "

26. "image quality (inferred by the preference for one image over another)". Same reference as above

27. TE (Echo Time) "represents the time in milliseconds between the application of the 90° pulse and the peak of the echo signal in Spin Echo and Inversion Recovery pulse sequences." reference: http://www.fonar.com/glossary.htm

28. TR (Repetition Time) "the amount of time that exists between successive pulse sequences applied to the same slice." reference: http://www.fonar.com/glossary.htm

2 HYPR mathematical formulation

2.1 Original HYPR

This mathematics of this algorithm will be presented by using the radon transform R notation and not the matrix projection matrix H notation.

The projection s_t is obtained by applying radon transform R on the image I_t at some angle ϕ_t

$$s_t = R_{\phi_t}[I_t]$$

When the original object image does not change with time then we can drop the subscript t from I_t and just write $s_t = R_{\phi_t}[I]$

The composite image C is found from the filtered back projection applied to all the s_t

$$C = \sum_{i=1}^{N} R^f_{\phi_{t_i}}[s_{t_i}]$$

Notice that the sum above is taken over N and not over N. Next a projection s_c is taken from C at angle ϕ as follows

$$s_{c_t} = R_{\phi_t}[C]$$

The the unfiltered back projection 2-D image P_t is generated

$$P_t = R^u_{\phi_t}[s_t]$$

And the unfiltered back projection 2-D image P_{c_t} is found

$$P_{c_t} = R^u_{\phi_t}[s_{c_t}]$$

Then the ratio of $\frac{P_t}{P_{c_t}}$ is summed and averaged over the time frame and multiplied by C to generate a HYPR frame J for the time frame.Hence for the k^{th} time frame we obtain

$$\begin{aligned} J_k &= C\left(\frac{1}{N_p}\sum_{i=1}^{N_p}\frac{P_{t_i}}{P_{c_{t_i}}}\right) \\ &= \frac{1}{N_p}\left(\sum_{i=1}^{N} R^f_{\phi_{t_i}}[s_{t_i}]\right)\sum_{j=1}^{N_p}\frac{R^u_{\phi_{t_j}}[s_{t_j}]}{R^u_{\phi_{t_j}}\left[s_{c_{t_j}}\right]} \end{aligned}$$

2.2 Wright HYPR

This mathematics of this algorithm will be presented by using the radon transform R notation and not the matrix projection matrix H notation. The conversion between the notation can be easily made by referring to the notation page at the end of this report.

The projection s_t is obtained by applying radon transform R on the image I_t at some angle ϕ_t

$$s_t = R_{\phi_t}[I_t]$$

When the original object image does not change with time then we can drop the subscript t from I_t and just write $s_t = R_{\phi_t}[I]$

The composite image C is found from the filtered back projection applied to all the s_t

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$$s_{c_t} = R_{\phi_t}[C]$$

The the unfiltered back projection 2-D image P_t is generated

$$P_t = R^u_{\phi_t}[s_t]$$

And the unfiltered back projection 2-D image P_{c_t} is found

$$P_{c_t} = R^u_{\phi_t}[s_{c_t}]$$

Now the set of P_t and P_{c_t} over one time frame are summed the their ratio multiplied by C to obtain the k^{th} HYPR frame

$$\begin{aligned} J_k &= C \frac{\sum_{i=1}^{N_p} P_{t_i}}{\sum_{i=1}^{N_p} P_{c_{t_i}}} \\ &= C \frac{\sum_{i=1}^{N_{pr}} R^u_{\phi_t}[s_t]}{\sum_{i=1}^{N_{pr}} R^u_{\phi_t}[s_{c_t}]} \end{aligned}$$

3 Derivation of Wright HYPR from normal equation

We start with the same starting equation used to derive the HYPR formulation as in the above section.

$$s_t = H_{\phi_t}[I_t] + \mathbf{n}$$

Where $\mathbf{n}$ is noise vector from Gaussian distribution with zero mean. H_{ϕ_t} is forward projection operator at an angle ϕ at time t, and I_t is the original image at time t, and s_t is the one dimensional projection vector that results from the above operation.

Now apply the H^T operator to the above equation, we obtain

$$H^T[s_t] = H^T[H_{\phi_t}[I_t] + \mathbf{n}]$$

Since H^T is linear, the above becomes

$$H^T[s_t] = H^T[H_{\phi_t}[I_t]] + H^T[\mathbf{n}]$$

Pre multiply the above with I_t

$$I_t\ H^T[s_t] = I_t\ H^T[H_{\phi_t}[I_t]] + I_t\ H^T[\mathbf{n}]$$

Divide both side by $H^T[H_{\phi_t}[I_t]]$

$$\frac{I_t H^T[s_t]}{H^T\left[H_{\phi_t}\left[I_t\right]\right]} = \frac{I_t H^T[H_{\phi_t}[I_t]]}{H^T\left[H_{\phi_t}\left[I_t\right]\right]} + \frac{I_t H^T[\mathbf{n}]}{H^T\left[H_{\phi_t}\left[I_t\right]\right]}$$

Under the condition that noise vector can be ignored the above becomes (after canceling out the $H^T[H_{\phi_t}[I_t]]$ terms)

$$\frac{I_t H^T[s_t]}{H^T\left[H_{\phi_t}\left[I_t\right]\right]} = I_t$$

Or

$$I_t = I_t \left(\frac{H^T[s_t]}{H^T\left[H_{\phi_t}\left[I_t\right]\right]} \right)$$

If we select the composite C as representing the initial estimate of the true image I_t, the above becomes, after replacing I_t in the R.H.S. of the above equation by C

$$I_t = C \left(\frac{H^T[s_t]}{H^T\left[H_{\phi_t}\left[C\right]\right]} \right) \tag{1}$$

But $H^T[s_t]$ is the unfiltered backprojection of the projection s_t, hence this term represents the term P_t shown in the last section, which is the unfiltered backprojection 2D image, and $H^T[H_{\phi_t}[C]]$ is the unfiltered backprojection of the projection $H_{\phi_t}[C]$, which is the term P_{c_t} in the last section. Hence we see that (1) is the same equation as

$$I_t = C\frac{P_t}{P_{c_t}} \tag{2}$$

Once I_t is computed from (1), we can repeat (1) again, using this computed I_t as the new estimate of the true image in the RHS of (1), and repeat the process again.

4 References

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4. Time-Resolved MR Angiography With Limited Projections by Yuexi Huang1,and Graham A. Wright
5. GE medical PPT dated 6/6/2008
6. Book principles of computerized Tomographic imaging by Kak and Staney