Solving first and second order linear differential equations using power series method

Regular singular point using Frobenius method

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1 Frobenius series. Indicial equation with repeated root

1.1 Example 1. homogeneous ode $x^2y''(x) + xy'(x) + x^2y(x) = 0$

Solve

$$x^2y''(x) + xy'(x) + x^2y(x) = 0 \tag{1}$$

Using power series method by expanding around $x = 0$. Writing the ode as

$$y''(x) + \frac{1}{x}y'(x) + y(x) = 0$$

Shows that $x = 0$ is a singular point. But $\lim_{x\to 0} x\frac{1}{x} = 1$. Hence the singularity is removable. This means $x = 0$ is a regular singular point. In this case the Frobenius power series will be used instead of the standard power series. Let

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Where r is to be determined. It is the root of the indicial equation.

$$y'(x) = \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1}$$

$$y''(x) = \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2}$$

Substituting the above in (1) gives

$$x^2 \sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r-2} + x \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r-1} + x^2 \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n + r)\,(n + r - 1)\, a_n x^{n+r} + \sum_{n=0}^{\infty} (n + r)\, a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} = 0 \tag{2}$$

Here, we need to make all powers on x the same, without making the sums start below zero. This can be done by adjusting the last term above as follows

$$\sum_{n=0}^{\infty} a_n x^{n+r+2} = \sum_{n=2}^{\infty} a_{n-2} x^{n+r}$$

Eq (2) becomes

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0 \tag{3}$$

Eq (3) is the base equation which is used to find roots of indicial equation. For $n = 0$ the above gives

$$\begin{aligned}
(n+r)(n+r-1) a_0 x^r + (n+r) a_0 x^r &= 0\\
(r)(r-1) a_0 x^r + (r) a_0 x^r &= 0\\
(r(r-1) a_0 + r a_0) x^r &= 0\\
(r(r-1) + r) a_0 x^r &= 0
\end{aligned} \tag{4}$$

Since $a_0 \neq 0$ then (4) gives

$$\begin{aligned}
r(r-1) + r &= 0\\
r^2 - r + r &= 0\\
r^2 &= 0
\end{aligned}$$

Hence the roots of the indicial equation are $r = 0$ which is a double root. Hence $r_1 = r_2 = 0$. This is the case when roots of indicial equation are repeated. In this case the solution $y_h(x)$ is given by

$$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$$

Where $y_1(x)$ is the first solution, which is assumed to be

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{5}$$

This is the standard Frobenius power series, just like we did to find the indicial equation, the only difference is that now we use $r = r_1$, and hence it is a known value. Once we find $y_1(x)$, then the second solution is

$$y_2(x) = y_1(x) \ln(x) + \sum_{n=1}^{\infty} b_n x^{n+r} \tag{6}$$

Something important to notice. In the sum above, it starts from 1 and not from 0. The main issue is how to find b_n. Since that is the only thing we need to be able to complete the solution as $y_1(x)$ is easily found. It turns out that there is a relation between the b_n and the a_n. The b_n can be found by taking just derivative of a_n as function of r for each n and then evaluating the result at $r = r_1$. How this is done will be shown below.

First we need to find $y_1(x)$. Substituting (5) in the ODE gives (3) again (but now with r having specific value r_1).

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0 \tag{7}$$

Now we are ready to find a_n for $n > 0$. We skip $n = 0$ since that was used to obtain the indicial equation.

For $n = 1$. Eq (7) gives

$$\begin{aligned}
(n+r)(n+r-1) a_1 + (n+r) a_1 &= 0\\
(1+r)(1+r-1) a_1 + (1+r) a_1 &= 0\\
((1+r)(1+r-1) + (1+r)) a_1 &= 0\\
(r+1)^2 a_1 &= 0
\end{aligned}$$

But $r = r_1 = 0$. The above becomes $\underline{a_1 = 0}$. It is a good idea to use a table to keep record of the a_n values as function of r, since this will be used later to find b_n.

n	a_n	$a_n(r = r_1)$
0	a_0	a_0
1	0	0

For $n \geq 2$ now we obtain the recursive equation. Notice that the recursive equation starts from n which is the largest lower summation index. In this case it is $n = 2$. For all lower index, we have to find a_n without the use of recursive equation as we did above for a_1. Using (7), the recursive equation is

$$(n+r)(n+r-1)a_n + (n+r)a_n + a_{n-2} = 0$$

$$a_n = -\frac{a_{n-2}}{(n+r)(n+r-1)+(n+r)} \tag{8}$$

The above EQ. (8) is very important, since we will use it to find all a_n. It is valid only for $n \geq 2$. Now we find few more a_n terms. From above and for $n = 2$

$$\begin{aligned} a_2 &= -\frac{a_0}{(2+r)(2+r-1)+(2+r)} \\ &= -\frac{a_0}{(r+2)^2} \end{aligned}$$

and $r = r_1 = 0$ then the above becomes

$$a_2 = -\frac{a_0}{(2)^2} = -\frac{a_0}{4}$$

The table now becomes

n	$a_n(r)$	$a_n(r = r_1)$
0	a_0	1
1	0	0
2	$-\frac{a_0}{(r+2)^2}$	$-\frac{a_0}{4}$

And for $n = 3$

$$a_3 = -\frac{a_1}{(3+r)(3+r-1)+(3+r)}$$

But $a_1 = 0$. Then $a_3 = 0$. The table becomes

n	$a_n(r)$	$a_n(r = r_1)$
0	a_0	a_0
1	0	0
2	$-\frac{a_0}{(r+2)^2}$	$-\frac{a_0}{4}$
3	0	0

For $n = 4$ Eq (8) gives

$$a_4 = -\frac{a_2}{(4+r)(4+r-1)+(4+r)}$$

But a_2 from the table is $-\frac{1}{(r+2)^2}$. Hence

$$a_4(r) = -\frac{-\frac{a_0}{(r+2)^2}}{(4+r)(4+r-1)+(4+r)} = \frac{a_0}{(r^2+6r+8)^2}$$

The above becomes at $r = r_1 = 0$

$$a_4 = \frac{a_0}{(8)^2} = \frac{a_0}{64}$$

The Table now becomes

n	$a_n(r)$	$a_n(r = r_1)$
0	a_0	a_0
1	0	0
2	$-\frac{a_0}{(r+2)^2}$	$-\frac{a_0}{4}$
3	0	0
4	$\frac{a_0}{(r^2+6r+8)^2}$	$\frac{a_0}{64}$

For $n = 5$ Eq (8) gives

$$a_5 = -\frac{a_3}{(n+r)(n+r-1)+(n+r)}$$

But $a_3 = 0$, hence $a_5 = 0$. The table becomes

n	$a_n(r)$	$a_n(r = r_1)$
0	a_0	a_0
1	0	0
2	$-\frac{a_0}{(r+2)^2}$	$-\frac{a_0}{4}$
3	0	0
4	$\frac{a_0}{(r^2+6r+8)^2}$	$\frac{a_0}{64}$
5	0	0

For $n = 6$ Eq (8) gives

$$a_6(r) = -\frac{a_4(r)}{(6+r)(6+r-1)+(6+r)}$$

But from the table $a_4 = \frac{a_0}{(r^2+6r+8)^2}$, so the above becomes

$$a_6(r) = -\frac{\frac{a_0}{(r^2+6r+8)^2}}{(6+r)(6+r-1)+(6+r)} = -\frac{a_0}{(r+6)^2(r^2+6r+8)^2}$$

At $r = r_1 = 0$ the above becomes

$$a_6 = -\frac{a_0}{(6)^2(8)^2} = -\frac{a_0}{2304}$$

The table becomes

n	a_n	$a_n(r = r_1)$
0	a_0	a_0
1	0	0
2	$-\frac{a_0}{(r+2)^2}$	$-\frac{a_0}{4}$
3	0	0
4	$\frac{a_0}{(r^2+6r+8)^2}$	$\frac{a_0}{64}$
5	0	0
6	$-\frac{a_0}{(r+6)^2(r^2+6r+8)^2}$	$-\frac{a_0}{2304}$

And so on. Hence $y_1(x)$ is

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

But $r = r_1 = 0$. Therefore

$$\begin{aligned} y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots \\ &= a_0\left(1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{1}{2304}x^6 + \cdots\right) \end{aligned} \tag{9}$$

We are done finding $y_1(x)$. This was not bad at all. Now comes the hard part. Which is finding $y_2(x)$. From (6) it is given by

$$y_2(x) = y_1(x)\ln(x) + \sum_{n=1}^{\infty} b_n x^{n+r}$$

To find b_n, we will use the following

$$b_n = \frac{d}{dr}(a_n) \tag{10}$$

Notice that n starts from 1. Hence

$$b_1(r) = \frac{d}{dr}(a_1(r))\bigg|_{r=r_1}$$

What the above says, is that we first take derivative of a_n w.r.t. r and evaluate the result at the root of the indicial equation. Using the table above, we obtain (recalling that $r_1 = 0$ in this example)

n	a_n	$a_n(r = r_1)$	$b_n = \frac{d}{dr}(a_n)$	$b_n(r = r_1)$
0	a_0	a_0	N/A since b starts from $n = 1$	N/A
1	0	0	0	0
2	$-\frac{a_0}{(r+2)^2}$	$-\frac{a_0}{4}$	$\frac{d}{dr}\left(-\frac{a_0}{(r+2)^2}\right) = \frac{2a_0}{(r+2)^3}$	$\frac{2a_0}{(2)^3} = \frac{a_0}{4}$
3	0	0	0	0
4	$\frac{a_0}{(r^2+6r+8)^2}$	$\frac{a_0}{64}$	$\frac{d}{dr}\left(\frac{a_0}{(r^2+6r+8)^2}\right) = -2a_0\frac{2r+6}{(r^2+6r+8)^3}$	$-2a_0\frac{6}{(8)^3} = -\frac{3a_0}{128}$
5	0	0	0	0
6	$-\frac{a_0}{(r+6)^2(r^2+6r+8)^2}$	$-\frac{a_0}{2304}$	$\frac{d}{dr}\left(-\frac{a_0}{(r+6)^2(r^2+6r+8)^2}\right) = 2a_0\frac{3r^2+24r+44}{(r^3+12r^2+44r+48)^3}$	$2a_0\frac{44}{(48)^3} = \frac{11a_0}{13824}$

We have found all b_n terms. Hence using (6) and since $r = r_1 = 0$ then

$$y_2(x) = y_1(x)\ln(x) + \left(b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + \cdots\right)$$

But from the above table, we see that $b_1 = 0, b_2 = \frac{a_0}{4}, b_3 = 0, b_4 = -\frac{3a_0}{128}, b_5 = 0, b_6 = \frac{11a_0}{13824}$. The above becomes

$$y_2(x) = y_1(x)\ln(x) + a_0\left(\frac{1}{4}x^2 - \frac{3}{128}x^4 + \frac{11}{13824}x^6 + O(x^8)\right) \tag{11}$$

And we know what $y_1(x)$ is from Eq (9). Hence the above becomes

$$y_2(x) = a_0\left(1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{1}{2304}x^6 + O(x^8)\right)\ln(x) + a_0\left(\frac{1}{4}x^2 - \frac{3}{128}x^4 + \frac{11}{13824}x^6 + O(x^8)\right)$$

Therefore the general solution is

$$\begin{aligned}
y(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= a_0 c_1 \left(1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{1}{2304}x^6 + O(x^8)\right) \\
&+ a_0 c_2 \left(\left(1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{1}{2304}x^6 + O(x^8)\right)\ln(x) + \left(\frac{1}{4}x^2 - \frac{3}{128}x^4 + \frac{11}{13824}x^6 + O(x^8)\right)\right)
\end{aligned}$$

We can now absorb a_0 into the constants c_1, c_2 and the above becomes

$$\begin{aligned}
y(x) &= c_1 \left(1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{1}{2304}x^6 + O(x^8)\right) \\
&+ c_2 \left(\left(1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 - \frac{1}{2304}x^6 + O(x^8)\right)\ln(x) + \left(\frac{1}{4}x^2 - \frac{3}{128}x^4 + \frac{11}{13824}x^6 + O(x^8)\right)\right)
\end{aligned}$$

It it easier in implementation to just make $a_0 = 1$ at the start of this process so we do not have to carry it around, and that is what we will do from now on.

This completes the solution. The only difficulty in this method, is to make sure when finding the b_n is to have access to the a_n with r being unevaluated form in order to take derivatives correctly. This was done above by keeping a table of these quantities updated.

1.2 Example 2. inhomogeneous ode example $x^2y''(x) + xy'(x) + x^2y(x) = \sin(x)$

Solve

$$x^2y''(x) + xy'(x) + x^2y(x) = \sin(x) \tag{1}$$

This is the same example as example 1, but with non-zero on RHS. Expanding $\sin(x)$ gives

$$x^2y''(x) + xy'(x) + x^2y(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \cdots \tag{1A}$$

The solution is

$$y = y_h + y_p$$

Where we found y_h above. We just need to find y_p now. This is done by finding y_p that corresponds to each separate term on the RHS at a time. i.e. we need to find y_p for each of the following problems

$$x^2y''(x) + xy'(x) + x^2y(x) = x \tag{2A}$$

$$x^2y''(x) + xy'(x) + x^2y(x) = -\frac{1}{6}x^3 \tag{2B}$$

$$x^2y''(x) + xy'(x) + x^2y(x) = \frac{1}{120}x^5 \tag{2C}$$

$$\vdots$$

Then add all the y_p found from each solution.

Starting with (2A). Let $y_{p_1} = \sum_{n=0}^{\infty} c_n x^{n+r}$. Substituting in (2A) and simplifying, we get as was done in EQ (3) in the first problem, which is the following (with now a_n changed to c_n and with the RHS not zero anymore)

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} + \sum_{n=2}^{\infty} c_{n-2}x^{n+r} = x \tag{*}$$

For $n = 0$

$$\begin{aligned}
(r)(r-1)c_0x^r + rc_0x^r &= x \\
(r(r-1) + r)c_0x^r &= x
\end{aligned}$$

Balance gives $r = 1$. And $(r(r-1)+r)c_0 = 1$ or $c_0 = 1$. Since we found $r = 1$ then (*) becomes

$$\sum_{n=0}^{\infty} n(n+1)c_n x^{n+1} + \sum_{n=0}^{\infty}(n+1)c_n x^{n+1} + \sum_{n=2}^{\infty} c_{n-2}x^{n+1} = x \tag{3}$$

Now that we used $n = 0$ to find r by matching against the RHS which is x, from now on, for all $n > 0$ we will use (3) to solve for all other c_n. From now on, we just need to solve with RHS zero, since there can be no more matches for any x on the right. For $n = 1$ then (3) gives

$$\begin{aligned}
2c_1x^2 + 2c_1x^2 &= 0 \\
4c_1x^2 &= 0
\end{aligned}$$

Hence $c_1 = 0$. For $n = 2$ EQ (3) gives

$$\begin{aligned}
2(3)c_3x^3 + 3c_3x^3 + c_0x^3 &= 0 \\
x^3(9c_3 + c_0) &= 0
\end{aligned}$$

Hence $9c_3 + c_0 = 0$ or $c_3 = -\frac{1}{9}$ since we found $c_1 = 1$. We continue this way and find that

$c_3 = 0, c_4 = \frac{1}{225}, c_5 = 0, c_6 = -\frac{1}{11\,025}$ and so on. Hence

$$\begin{aligned} y_{p_1} &= \sum_{n=0}^{\infty} c_n x^{n+r} \\ &= \sum_{n=0}^{\infty} c_n x^{n+1} \\ &= c_0 x + c_1 x^2 + c_2 x^3 + c_3 x^4 + c_4 x^5 + c_5 x^6 + c_6 x^7 + \cdots \\ &= x - \frac{1}{9} x^3 + \frac{1}{225} x^5 - \frac{1}{11\,025} x^7 + \cdots \end{aligned}$$

Now we repeat the above for the second problem (2B)

$$x^2 y''(x) + x y'(x) + x^2 y(x) = -\frac{1}{6} x^3$$

Let $y_{p_2} = \sum_{n=0}^{\infty} c_n x^{n+r}$. Substituting in above and simplifying, we get as was done in Eq (3) in the first problem, the following (with now a_n changed to c_n and with the RHS not zero anymore)

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) c_n x^{n+r} + \sum_{n=2}^{\infty} c_{n-2} x^{n+r} = -\frac{1}{6} x^3 \tag{*}$$

For $n = 0$

$$\begin{aligned} (r)(r-1) c_0 x^r + r c_0 x^r &= -\frac{1}{6} x^3 \\ (r(r-1) + r) c_0 x^r &= -\frac{1}{6} x^3 \end{aligned}$$

Balance gives $r = 3$ and $(r(r-1)+r) c_0 = -\frac{1}{6}$ or $(6+3) c_0 = -\frac{1}{6}$ or $c_0 = -\frac{1}{54}$. Since we found $r = 3$ then (*) becomes

$$\sum_{n=0}^{\infty} (n+3)(n+2) c_n x^{n+3} + \sum_{n=0}^{\infty} (n+3) c_n x^{n+3} + \sum_{n=2}^{\infty} c_{n-2} x^{n+3} = -\frac{1}{6} x^3 \tag{4}$$

Now that we used $n = 0$ to find r by matching against the RHS which is x, from now on, for all $n > 0$ we will use (4) to solve for all other c_n. From now on, we just need to solve with RHS zero, since there can be no more matches for any x^3 on the right. For $n = 1$ the above gives

$$\begin{aligned} (4)(3) c_1 x^4 + 4 c_1 x^4 &= 0 \\ 16 c_1 x^4 &= 0 \end{aligned}$$

Hence $c_1 = 0$. For $n = 2$ EQ (4) gives

$$\begin{aligned} (5)(4) c_2 x^5 + 5 c_2 x^5 + c_0 x^5 &= 0 \\ x^5 (25 c_2 + c_0) &= 0 \end{aligned}$$

Hence $25 c_2 + c_0 = 0$ or $c_2 = -\frac{c_0}{25} = -\frac{-\frac{1}{54}}{25} = \frac{1}{1350}$. For $n = 3$ then (4) gives

$$(6)(5) c_3 x^6 + 6 c_3 x^6 + c_1 x^6 = 0$$

Which gives $c_3 = 0$ since $c_1 = 0$. For $n = 4$ then (4) gives

$$\begin{aligned} (7)(6) c_4 x^7 + 7 c_4 x^7 + c_2 x^7 &= 0 \\ x^7 (49 c_4 + c_2) &= 0 \end{aligned}$$

Hence $49 c_4 + c_2 = 0$ or $c_4 = -\frac{c_2}{49} = -\frac{\frac{1}{1350}}{49} = -\frac{1}{66\,150}$. We continue this way. Hence

$$\begin{aligned} y_{p_2} &= \sum_{n=0}^{\infty} c_n x^{n+r} \\ &= \sum_{n=0}^{\infty} c_n x^{n+3} \\ &= c_0 x^3 + c_1 x^4 + c_2 x^5 + c_3 x^6 + c_4 x^7 + \cdots \\ &= -\frac{1}{54} x^3 + \frac{1}{1350} x^5 - \frac{1}{66\,150} x^7 + \cdots \end{aligned}$$

Now we repeat the above for the next problem (2C)

$$x^2 y''(x) + x y'(x) + x^2 y(x) = \frac{1}{120} x^5$$

And if we carry the same steps as above we will find that

$$
\begin{aligned}
y_{p3} &= \sum_{n=0}^{\infty} c_n x^{n+r} \\
&= \sum_{n=0}^{\infty} c_n x^{n+5} \\
&= c_0 x^5 + c_1 x^6 + c_2 x^7 + c_3 x^8 + c_4 x^9 + \cdots \\
&= \frac{1}{3000} x^5 - \frac{1}{147\,000} x^7 + \frac{1}{11\,907\,000} x^9 + \cdots
\end{aligned}
$$

We keep doing this for as many terms as we have on the right side. At the end, all y_{p_i} are added. This gives the final y_p

$$
\begin{aligned}
y_p &= y_{p_1} + y_{p2} + y_{p3} + \cdots \\
&= x - \frac{1}{9} x^3 + \frac{1}{225} x^5 - \frac{1}{11\,025} x^7 + \cdots \\
&- \frac{1}{54} x^3 + \frac{1}{1350} x^5 - \frac{1}{66\,150} x^7 + \cdots \\
&+ \frac{1}{3000} x^5 - \frac{1}{147\,000} x^7 + \frac{1}{11\,907\,000} x^9 + \cdots \\
&- \frac{1}{246960} x^7 + \frac{1}{20003760} x^9 + \cdots
\end{aligned}
$$

Which results in

$$
\begin{aligned}
y_p &= x + x^3 \left(-\frac{1}{9} - \frac{1}{54} \right) + x^5 \left(\frac{1}{225} + \frac{1}{1350} + \frac{1}{3000} \right) + x^7 \left(-\frac{1}{11\,025} - \frac{1}{66\,150} - \frac{1}{147\,000} - \frac{1}{246960} \right) + \cdots \\
&= x + x^3 \left(-\frac{7}{54} \right) + x^5 \left(\frac{149}{27\,000} \right) + x^7 \left(-\frac{2161}{18\,522\,000} \right) + \cdots
\end{aligned}
$$

Hence the solution is

$$
y = y_h + y_p
$$

Using y_h from the above problem gives the total solution as

$$
\begin{aligned}
y &= c_1 \left(1 - \frac{1}{4} x^2 + \frac{1}{64} x^4 - \frac{1}{2304} x^6 + O(x^8) \right) \\
&+ c_2 \left(\left(1 - \frac{1}{4} x^2 + \frac{1}{64} x^4 - \frac{1}{2304} x^6 + O(x^8) \right) \ln(x) + \left(\frac{1}{4} x^2 - \frac{3}{128} x^4 + \frac{11}{13824} x^6 + O(x^8) \right) \right) \\
&+ \left(x - \frac{7}{54} x^3 + \frac{149}{27000} x^5 - \frac{2161}{18522000} x^7 + \cdots \right)
\end{aligned}
$$

1.3 Example 3. homogeneous ode example $(e^x - 1) y''(x) + e^x y'(x) + y(x) = 0$

Solve

$$
(e^x - 1) y''(x) + e^x y'(x) + y(x) = 0 \tag{1}
$$

Using power series method by expanding around $x = 0$. Writing the ode as

$$
y''(x) + \frac{e^x}{(e^x - 1)} y'(x) + \frac{1}{(e^x - 1)} y(x) = 0
$$

Shows that $x = 0$ is a singular point. But $\lim_{x \to 0} x \frac{e^x}{(e^x - 1)} = 1$ and $\lim_{x \to 0} x^2 \frac{1}{(e^x - 1)} = 0$ Hence the singularity is removable. This means $x = 0$ is a regular singular point. In this case the Frobenius power series will be used instead of the standard power series. Let

$$
y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}
$$

Where r is to be determined. It is the root of the indicial equation. Therefore

$$
\begin{aligned}
y'(x) &= \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1} \\
y''(x) &= \sum_{n=0}^{\infty} (n + r)(n + r - 1) a_n x^{n+r-2}
\end{aligned}
$$

Substituting the above in (1) gives

$$
(e^x - 1) \sum_{n=0}^{\infty} (n + r)(n + r - 1) a_n x^{n+r-2} + e^x \sum_{n=0}^{\infty} (n + r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0
$$

Expanding e^x in Taylor series around x gives $e^x = 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\cdots$. The above becomes

$$\left(x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\cdots\right)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} \\ +\left(1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\cdots\right)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+\sum_{n=0}^{\infty}a_n x^{n+r}=0 \quad (1)$$

Expanding gives (and keeping only terns up to x^4 gives

$$x\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+\frac{1}{2}x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+\frac{1}{6}x^3\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} \\ +\frac{1}{24}x^4\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}+\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+\frac{1}{2}x^2\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} \\ +\frac{1}{6}x^3\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+\frac{1}{24}x^4\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

Moving the x inside the sum, the above becomes

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1}+\sum_{n=0}^{\infty}\frac{1}{2}(n+r)(n+r-1)a_n x^{n+r}+\sum_{n=0}^{\infty}\frac{1}{6}(n+r)(n+r-1)a_n x^{n+r+1} \\ +\sum_{n=0}^{\infty}\frac{1}{24}(n+r)(n+r-1)a_n x^{n+r+2}+\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+\sum_{n=0}^{\infty}(n+r)a_n x^{n+r}+\sum_{n=0}^{\infty}\frac{1}{2}(n+r)a_n x^{n+r+1} \\ +\sum_{n=0}^{\infty}\frac{1}{6}(n+r)a_n x^{n+r+2}+\sum_{n=0}^{\infty}\frac{1}{24}(n+r)a_n x^{n+r+3}+\sum_{n=0}^{\infty}a_n x^{n+r}=0$$

Here, we need to make all powers on x the same, without making the sums start below zero. This can be done by adjusting the last term above as follows

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1}+\sum_{n=1}^{\infty}\frac{1}{2}(n+r-1)(n+r-2)a_{n-1}x^{n+r-1}+\sum_{n=2}^{\infty}\frac{1}{6}(n+r-2)(n+r-3)a_{n-2}x^{n+} \\ +\sum_{n=3}^{\infty}\frac{1}{24}(n+r-3)(n+r-4)a_{n-3}x^{n+r-1}+\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}+\sum_{n=1}^{\infty}(n+r-1)a_{n-1}x^{n+r-1}+\sum_{n=2}^{\infty}\frac{1}{2}(n+r-2)a_n \\ +\sum_{n=3}^{\infty}\frac{1}{6}(n+r-3)a_{n-3}x^{n+r-1}+\sum_{n=4}^{\infty}\frac{1}{24}(n+r-4)a_{n-4}x^{n+r-1}+\sum_{n=1}^{\infty}a_{n-1}x^{n+r-1}=0 \quad (2)$$

The case $n=0$ gives the indicial equation

$$\begin{aligned}(n+r)(n+r-1)+(n+r)&=0\\ (r)(r-1)+(r)&=0\\ r^2&=0\end{aligned}$$

Hence the roots of the indicial equation are $r=0$ which is a double root. Hence $r_1=r_2=0$. When this happens, the solution is given by

$$y(x)=c_1y_1(x)+c_2y_2(x)$$

Where $y_1(x)$ is the first solution, which is assumed to be

$$y_1(x)=\sum_{n=0}^{\infty}a_n x^{n+r} \quad (3)$$

Where we take $a_0=1$ as it is arbitrary and where $r=r_1$. This is the standard Frobenius power series, just like we did to find the indicial equation, the only difference is that now we use $r=r_1$, and hence it is a known value. Once we find $y_1(x)$, then the second solution is

$$y_2(x)=y_1(x)\ln(x)+\sum_{n=1}^{\infty}b_n x^{n+r} \quad (3)$$

Something important to notice. In the sum above, it starts from 1 and not from 0. The main issue is how to find b_n. Since that is the only thing we need to be able to complete the solution as $y_1(x)$ is easily found. It turns out that there is a relation between the b_n and the a_n. The b_n can be found by taking just derivative of a_n as function of r for each n and then evaluate the result at $r=r_1$. How this is done will be shown below. First we need to find $y_1(x)$. We take Eq(3) and substitute it in the original ODE. This will result in Eq (2) which we found above so no need to repeat that. We just need to remember that now we now what r is. It has a numerical value unlike the above phase where we still did not know its value.

Now we are ready to find a_n. We skip $n=0$ since that was used to obtain the indicial equation, and we know that $a_0=1$ is an arbitrary value to choose. We start from $n=1$.

For $n = 1$ only, using Eq (2) gives

$$(n+r)(n+r-1)a_1 + \frac{1}{2}(n+r-1)(n+r-2)a_0 + (n+r)a_1 + (n+r-1)a_0 + a_0 = 0$$
$$(1+r)(1+r-1)a_1 + \frac{1}{2}(1+r-1)(1+r-2)a_0 + (1+r)a_1 + (1+r-1)a_0 + a_0 = 0$$
$$((1+r)(1+r-1) + (1+r))a_1 + \left(\frac{1}{2}(1+r-1)(1+r-2) + (1+r-1) + 1\right)a_0 = 0$$

But $a_0 = 1$. The above becomes

$$((1+r)(1+r-1) + (1+r))a_1 = -\left(\frac{1}{2}(1+r-1)(1+r-2) + (1+r-1) + 1\right)$$
$$a_1 = \frac{-\left(\frac{1}{2}(1+r-1)(1+r-2) + (1+r-1) + 1\right)}{((1+r)(1+r-1) + (1+r))} = -\frac{(r^2+r+2)}{2r^2+4r+2}$$

Which at $r = 0$ gives

$$a_1 = -1$$

It is a good idea to use a table to keep record of the a_n values as function of r, since this will be used later to find b_n.

n	$a_n(r)$	$a_n(r = r_1)$
0	1	1
1	$-\frac{(r^2+r+2)}{2r^2+4r+2}$	-1

For $n = 2$ only, using Eq (2) gives

$$(n+r)(n+r-1)a_2 + \frac{1}{2}(n+r-1)(n+r-2)a_1 + \frac{1}{6}(n+r-2)(n+r-3)a_0 + (n+r)a_2 + (n+r-1)a_1 + \frac{1}{2}(n$$
$$(2+r)(2+r-1)a_2 + \frac{1}{2}(2+r-1)(2+r-2)a_1 + \frac{1}{6}(2+r-2)(2+r-3)a_0 + (2+r)a_2 + (2+r-1)a_1 + \frac{1}{2}(2$$
$$((2+r)(2+r-1) + (2+r))a_2 + \left(\frac{1}{2}(2+r-1)(2+r-2) + (2+r-1) + 1\right)a_1 + \left(\frac{1}{6}(2+r-2)(2+r-3) + \right.$$
$$(r+2)^2 a_2 + \left(\frac{1}{2}r^2 + \frac{3}{2}r + 2\right) a$$

But $a_0 = 1$ and $a_1 = -\frac{(r^2+r+2)}{2r^2+4r+2}$. The above becomes

$$(r+2)^2 a_2 + \left(\frac{1}{2}r^2 + \frac{3}{2}r + 2\right)\left(-\frac{(r^2+r+2)}{2r^2+4r+2}\right) + \frac{1}{6}r(r+2) = 0$$
$$(r+2)^2 a_2 = \frac{r^4 + 4r^3 + 17r^2 + 26r + 24}{2(r+1)^2}$$
$$a_2 = \frac{r^4 + 4r^3 + 17r^2 + 26r + 24}{12(r+1)^2(r+2)^2}$$

At $r = 0$ the above becomes

$$a_2 = \frac{24}{12(2)^2} = \frac{1}{2}$$

The table becomes

n	$a_n(r)$	$a_n(r = r_1)$
0	1	1
1	$-\frac{(r^2+r+2)}{2r^2+4r+2}$	-1
2	$\frac{r^4+4r^3+17r^2+26r+24}{12(r+1)^2(r+2)^2}$	$\frac{1}{2}$

For $n = 3$ only, using Eq (2) gives

$$(n+r)(n+r-1)a_3 + \frac{1}{2}(n+r-1)(n+r-2)a_2 + \frac{1}{6}(n+r-2)(n+r-3)a_1$$
$$+\frac{1}{24}(n+r-3)(n+r-4)a_0 + (n+r)a_3 + (n+r-1)a_2 + \frac{1}{2}(n+r-2)a_1 + \frac{1}{6}(n+r-3)a_0 + a_2 = 0$$

Or

$$(3+r)(2+r)a_3 + \frac{1}{2}(2+r)(1+r)a_2 + (3+r)a_3 + (2+r)a_2 + \frac{1}{2}(1+r)a_1 + a_2 = 0$$

Or

$$((3+r)(2+r) + (3+r))a_3 + \left(\frac{1}{2}(2+r)(1+r) + (2+r) + 1\right)a_2 + \frac{1}{2}(1+r)a_1 = 0$$
$$(r+3)^2 a_3 + \left(\frac{1}{2}r^2 + \frac{5}{2}r + 4\right)a_2 + \frac{1}{2}(1+r)a_1 = 0$$

But $a_1 = -\frac{(r^2+r+2)}{2r^2+4r+2}, a_2 = \frac{r^4+4r^3+17r^2+26r+24}{12(r+1)^2(r+2)^2}$. The above becomes

$$
\begin{aligned}
(r+3)^2 a_3 &= -\left(\frac{1}{2}r^2 + \frac{5}{2}r + 4\right) a_2 - \frac{1}{2}(1+r)\, a_1 \\
&= -\left(\frac{1}{2}r^2 + \frac{5}{2}r + 4\right)\left(\frac{r^4 + 4r^3 + 17r^2 + 26r + 24}{12\,(r+1)^2\,(r+2)^2}\right) - \frac{1}{2}(1+r)\left(-\frac{(r^2+r+2)}{2r^2+4r+2}\right) \\
&= -\frac{(r^6 + 3r^5 + 9r^4 + 53r^3 + 158r^2 + 208r + 144)}{24\,(r^2+3r+2)^2} \\
a_3 &= -\frac{(r^6 + 3r^5 + 9r^4 + 53r^3 + 158r^2 + 208r + 144)}{24\,(r^2+3r+2)^2\,(r+3)^2}
\end{aligned}
$$

For $r = 0$ the above reduces to

$$
a_3 = -\frac{144}{24\,(2)^2\,(3)^2} = -\frac{1}{6}
$$

The table becomes

n	$a_n(r)$	$a_n(r = r_1)$
0	1	1
1	$-\frac{(r^2+r+2)}{2r^2+4r+2}$	-1
2	$\frac{r^4+4r^3+17r^2+26r+24}{12(r+1)^2(r+2)^2}$	$\frac{1}{2}$
3	$-\frac{(r^6+3r^5+9r^4+53r^3+158r^2+208r+144)}{24(r^2+3r+2)^2(r+3)^2}$	$-\frac{1}{6}$

And so on. Recursion starts at $n \geq 5$ but we have enough terms, so we stop here. $y_1(x)$ is

$$
y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r}
$$

But $r = r_1 = 0$. Therefore

$$
\begin{aligned}
y_1(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \cdots \qquad (6A) \\
&= 1 - x + \frac{1}{2}x^3 - \frac{1}{6}x^3 + \cdots
\end{aligned}
$$

We are done finding $y_1(x)$. This was not bad at all. Now comes the hard part. Which is finding $y_2(x)$. From (3) it is given by

$$
y_2(x) = y_1(x)\ln(x) + \sum_{n=1}^{\infty} b_n x^{n+r}
$$

To find b_n, we will use the following

$$
b_n(r) = \frac{d}{dr}(a_n(r)) \qquad (7)
$$

Notice that n starts from 1. Hence

$$
b_1(r) = \frac{d}{dr}(a_1(r))\bigg|_{r=r_1}
$$

What the above says, is that we first take derivative of $a_n(r)$ w.r.t. r and evaluate the result at the root of the indicial equation. Using the table above, we obtain (recalling that $r_1 = 0$ in this example)

n	$a_n(r)$	$a_n(r = r_1)$	$b_n(r) = \frac{d}{dr}(a_n(r))$
0	1	1	N/A since b starts from $n = 1$
1	$-\frac{(r^2+r+2)}{2r^2+4r+2}$	-1	$\frac{d}{dr}\left(-\frac{(r^2+r+2)}{2r^2+4r+2}\right) = -\frac{(r-3)}{2(r+1)^3}$
2	$\frac{r^4+4r^3+17r^2+26r+24}{12(r+1)^2(r+2)^2}$	$\frac{1}{2}$	$\frac{d}{dr}\left(\frac{r^4+4r^3+17r^2+26r+24}{12(r+1)^2(r+2)^2}\right) = -\frac{(-r^4+7r^3+27r^2+53r+46)}{6(r^2+3r+2)^3}$
3	$-\frac{(r^6+3r^5+9r^4+53r^3+158r^2+208r+144)}{24(r^2+3r+2)^2(r+3)^2}$	$-\frac{1}{6}$	$\frac{d}{dr}\left(-\frac{(r^6+3r^5+9r^4+53r^3+158r^2+208r+144)}{24(r^2+3r+2)^2(r+3)^2}\right) = \frac{(-9r^7-44r^6+24r^5+6}{24}$

We have found all b_n terms. Hence

$$
y_2(x) = y_1(x)\ln(x) + \sum_{n=1}^{\infty} b_n x^{n+r}
$$

And since $r = r_1 = 0$ then

$$
y_2(x) = y_1(x)\ln(x) + \left(b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + \cdots\right)
$$

But from the above table, we see that $b_1 = \frac{3}{2}, b_2 = -\frac{23}{24}, b_3 = \frac{3}{8}$, The above becomes

$$
y_2(x) = y_1(x)\ln(x) + \left(\frac{3}{2}x - \frac{23}{24}x^2 + \frac{10}{27}x^3 + O(x^4)\right)
$$

And we know what $y_1(x)$ is from Eq (6A). Hence

$$
y_2(x) = \left(1 - x + \frac{1}{2}x^3 - \frac{1}{6}x^3 + O(x^4)\right)\ln(x) + \left(\frac{3}{2}x - \frac{23}{24}x^2 + \frac{10}{27}x^3 + O(x^4)\right)
$$

Therefore the general solution is

$$
\begin{aligned}
y(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= c_1\left(1 - x + \frac{1}{2}x^3 - \frac{1}{6}x^3 + O(x^4)\right) + c_2\left(\left(1 - x + \frac{1}{2}x^3 - \frac{1}{6}x^3 + O(x^4)\right)\ln(x) + \left(\frac{3}{2}x - \frac{23}{24}x^2 + \frac{10}{27}x^3 + O(x^4
\end{aligned}
$$

This completes the solution.

2 Frobenius series where indicial equation roots differ by an integer

2.1 Example 1. homogeneous ode where log term is needed $x^2y''(x) - xy(x) = 0$

Solve

$$
x^2y''(x) - xy(x) = 0 \tag{1}
$$

Using power series method by expanding around $x = 0$. Writing the ode as

$$
y''(x) - \frac{1}{x}y(x) = 0
$$

Shows that $x = 0$ is a singular point. But $\lim_{x\to 0} x^2\frac{1}{x} = 0$. Hence the singularity is removable. This means $x = 0$ is a regular singular point. In this case the Frobenius power series will be used instead of the standard power series. Let

$$
y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}
$$

Where r is to be determined. It is the root of the indicial equation. Therefore

$$
\begin{aligned}
y'(x) &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \\
y''(x) &= \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-2}
\end{aligned}
$$

Substituting the above in (1) gives

$$
\begin{aligned}
x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-2} - x\sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+1} &= 0 \qquad \text{(1A)}
\end{aligned}
$$

Here, we need to make all powers on x the same, without making the sums start below zero. This can be done by adjusting the last term above as follows

$$
\sum_{n=0}^{\infty} a_n x^{n+r+1} = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}
$$

And now Eq (1A) becomes

$$
\sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r} = 0 \tag{1B}
$$

$n = 0$ gives the indicial equation

$$
\begin{aligned}
(n+r)(n+r-1)\, a_n x^r &= 0 \\
(r)(r-1)\, a_0 x^r &= 0
\end{aligned}
$$

Since $a_0 \neq 0$ then the above becomes

$$
(r)(r-1)\, x^r = 0
$$

Since this is true for all x, then

$$
(r)(r-1) = 0
$$

Hence the roots of the indicial equation are $r_1 = 1, r_2 = 0$. Or $r_1 = r_2 + N$ where $N = 1$. We always take r_1 to be the larger of the roots.

When this happens, the solution is given by

$$
y(x) = c_1 y_1(x) + c_2 y_2(x)
$$

Where $y_1(x)$ is the first solution, which is assumed to be

$$
y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r} \tag{2}
$$

Where we take $a_0 = 1$ as it is arbitrary and where $r = r_1 = 1$. This is the standard Frobenius power series, just like we did to find the indicial equation, the only difference is that now we use $r = r_1$, and hence it is a known value. Once we find $y_1(x)$, then the second solution is

$$y_2(x) = Cy_1(x)\ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r} \tag{3}$$

We will show below how to find C and b_n. First, let us find $y_1(x)$. From Eq(2)

$$y_1'(x) = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y_1''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-2}$$

We need to remember that in the above r is not a symbol any more. It will have the indicial root value, which is $r = r_1 = 1$ in this case. But we keep r as symbol for now, in order to obtain $a_n(r)$ as function of r first and use this to find $b_n(r)$. At the very end we then evaluate everything at $r = r_1 = 1$. Substituting the above in (1) gives Eq (1B) above (We are following pretty much the same process we did to find the indicial equation here)

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r} - \sum_{n=1}^{\infty} a_{n-1} x^{n+r} = 0 \tag{1B}$$

Now we are ready to find a_n. Now we skip $n = 0$ since that was used to obtain the indicial equation, and we know that $\underline{a_0 = 1}$ is an arbitrary value to choose. We start from $n = 1$. For $n \geq 1$ we obtain the recursion equation

$$(n+r)(n+r-1)\, a_n - a_{n-1} = 0$$
$$a_n = \frac{a_{n-1}}{(n+r)(n+r-1)}$$

To more clearly indicate that a_n is function of r, we write the above as

$$a_n(r) = \frac{a_{n-1}(r)}{(n+r)(n+r-1)} \tag{4}$$

The above is very important, since we will use it to find $b_n(r)$ later on. For now, we are just finding the a_n. Now we find few more a_n terms. From (4) for $n = 1$

$$a_1(r) = \frac{a_0(r)}{(1+r)(r)} = \frac{1}{(1+r)(r)} \qquad a_0 = 1$$

and $r = r_1 = 1$ then the above becomes

$$a_1 = \frac{1}{2}$$

It is a good idea to use a table to keep record of the a_n values as function of r, since this will be used later to find b_n.

n	$a_n(r)$	$a_n(r = r_1)$
0	1	1
1	$\frac{1}{(1+r)(r)}$	$\frac{1}{2}$

And for $n = 2$ from Eq(4)

$$a_2(r) = \frac{a_1(r)}{(2+r)(1+r)}$$

But $a_1(r) = \frac{1}{(1+r)(r)}$. Then

$$a_2(r) = \frac{\frac{1}{(1+r)(r)}}{(2+r)(1+r)} = \frac{1}{r(r+1)^2(r+2)}$$

When $r = r_1 = 1$ the above becomes

$$a_2 = \frac{1}{(2)^2(3)} = \frac{1}{12}$$

The table becomes

n	$a_n(r)$	$a_n(r = r_1)$
0	1	1
1	$\frac{1}{(1+r)(r)}$	$\frac{1}{2}$
2	$\frac{1}{r(r+1)^2(r+2)}$	$\frac{1}{12}$

For $n = 3$ Eq (4) gives

$$a_3(r) = \frac{a_2(r)}{(3+r)(2+r)}$$

Using the value of $a_2(r)$ from the the above becomes

$$a_3(r) = \frac{\frac{1}{r(r+1)^2(r+2)}}{(3+r)(2+r)} = \frac{1}{r(r+1)^2(r+2)^2(r+3)}$$

When $r = r_1 = 1$ the above becomes

$$a_3 = \frac{1}{(2)^2(3)^2(4)} = \frac{1}{144}$$

The Table now becomes

n	$a_n(r)$	$a_n(r=r_1)$
0	1	1
1	$\frac{1}{(1+r)(r)}$	$\frac{1}{2}$
2	$\frac{1}{r(r+1)^2(r+2)}$	$\frac{1}{12}$
3	$\frac{1}{r(r+1)^2(r+2)^2(r+3)}$	$\frac{1}{144}$

And so on. Hence $y_1(x)$ is

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

But $r = r_1 = 1$. Therefore

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= x\sum_{n=0}^{\infty} a_n x^n \\ &= x(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) \\ &= x\left(1 + \frac{1}{2}x + \frac{1}{12}x^2 + \frac{1}{144}x^3 + \cdots\right) \end{aligned} \tag{5}$$

We are done finding $y_1(x)$. This was not bad at all. Now comes the hard part. Which is finding $y_2(x)$. From (3) it is given by

$$y_2(x) = Cy_1(x)\ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r} \tag{3}$$

The first thing to do is to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_N(r)$. If this limit exist, then $C = 0$, else we need to keep the log term. From the above above we see that $a_N(r) = a_1(r) = \frac{1}{(1+r)(r)}$. Recall that $N = 1$ since this was the difference between the two roots and $r_2 = 0$ (the smaller root). Therefore

$$\lim_{r\to r_2} \frac{1}{(1+r)(r)} = \lim_{r\to 0} \frac{1}{(1+r)(r)}$$

Which does not exist. Therefore we need to keep the log term. In this case, we replace Eq. (3) back in the original ODE.

$$\begin{aligned} y_2'(x) &= Cy_1'\ln(x) + Cy_1\frac{1}{x} + \sum_{n=0}^{\infty}(n+r)b_n x^{n+r-1} \\ y_2''(x) &= Cy_1''\ln(x) + Cy_1'\frac{1}{x} + Cy_1'\frac{1}{x} - Cy_1\frac{1}{x^2} + \sum_{n=0}^{\infty}(n+r)(n+r-1)b_n x^{n+r-2} \\ &= Cy_1''\ln(x) + 2Cy_1'\frac{1}{x} - Cy_1\frac{1}{x^2} + \sum_{n=0}^{\infty}(n+r)(n+r-1)b_n x^{n+r-2} \end{aligned}$$

Substituting the above in $x^2y''(x) - xy(x) = 0$ gives

$$\begin{aligned} x^2\left(Cy_1''\ln(x) + 2Cy_1'\frac{1}{x} - Cy_1\frac{1}{x^2} + \sum_{n=0}^{\infty}(n+r)(n+r-1)b_n x^{n+r-2}\right) - x\left(Cy_1\ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r}\right) &= 0 \\ Cx^2y_1''\ln(x) + 2x^2Cy_1'\frac{1}{x} - Cy_1 + x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)b_n x^{n+r-2} - Cxy_1\ln(x) - x\sum_{n=0}^{\infty} b_n x^{n+r} &= 0 \\ Cx^2y_1''\ln(x) + 2xCy_1' - Cy_1 + \sum_{n=0}^{\infty}(n+r)(n+r-1)b_n x^{n+r} - Cxy_1\ln(x) - \sum_{n=0}^{\infty} b_n x^{n+r+1} &= 0 \\ C\ln(x)\left(x^2y_1'' - xy_1\right) + 2xCy_1' - Cy_1 + \sum_{n=0}^{\infty}(n+r)(n+r-1)b_n x^{n+r} - \sum_{n=0}^{\infty} b_n x^{n+r+1} &= 0 \end{aligned}$$

But $x^2y_1'' - xy_1 = 0$ since y_1 is solution to the ode. The above simplifies to

$$C(2xy_1' - y_1) + \sum_{n=0}^{\infty}(n+r)(n+r-1)b_n x^{n+r} - \sum_{n=0}^{\infty} b_n x^{n+r+1} = 0 \tag{6}$$

The above is what we will use to determine C and all the b_n. Remembering that $r = r_2 = 0$ in the above, since this is for the second solution associated with the second root which we found above to be zero. But we found $y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$ then

$$y_1' = \sum_{n=0}^{\infty} (n+1)\, a_n x^n$$

Eq (6) now becomes

$$C\left(2x\sum_{n=0}^{\infty}(n+1)\, a_n x^n\right) - C\left(\sum_{n=0}^{\infty} a_n x^{n+1}\right) + \sum_{n=0}^{\infty}(n+r)\,(n+r-1)\, b_n x^{n+r} - \sum_{n=0}^{\infty} b_n x^{n+r+1} = 0$$
$$2C\sum_{n=0}^{\infty}(n+1)\, a_n x^{n+1} - C\sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty}(n+r)\,(n+r-1)\, b_n x^{n+r} - \sum_{n=0}^{\infty} b_n x^{n+r+1} = 0$$

But $r = r_2 = 0$. The above becomes

$$2C\sum_{n=0}^{\infty}(n+1)\, a_n x^{n+1} - C\sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} n(n-1)\, b_n x^n - \sum_{n=0}^{\infty} b_n x^{n+1} = 0$$

Adjusting the index of terms above, so so all x powers are the same gives

$$2C\sum_{n=1}^{\infty} n a_{n-1} x^n - C\sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=0}^{\infty} n(n-1)\, b_n x^n - \sum_{n=1}^{\infty} b_{n-1} x^n = 0 \tag{7}$$

$n = 0$ is skipped, since b_0 is arbitrary and can be taken as say

$$b_0 = 1$$

At $n = 1$, Eq(7) gives

$$2Ca_0 - Ca_0 - b_0 = 0$$

But $a_0 = 1, b_0 = 1$ hence the above becomes

$$C = 1$$

For $b_N = b_1$ we are free to select any value since it is arbitrary. The standard way is to choose

$$b_1 = 0$$

Now we find the rest of the b_n terms. From Eq(7), for $n = 2$, it gives

$$2C(2a_1) - Ca_1 + 2b_2 - b_1 = 0$$

But $C = 1, b_1 = 0$ and $a_1 = \frac{1}{2}$ from table. Hence the above becomes

$$2\left(2\frac{1}{2}\right) - \frac{1}{2} + 2b_2 = 0$$
$$2 - \frac{1}{2} + 2b_2 = 0$$
$$b_2 = -\frac{3}{4}$$

And for $n = 3$ from Eq. (7) it gives

$$2C(3a_2) - Ca_2 + (3)\,(2)\, b_3 - b_2 = 0$$

But $C = 1, b_2 = -\frac{3}{4}, a_2 = \frac{1}{12}$. The above becomes

$$2\left(3\left(\frac{1}{12}\right)\right) - \frac{1}{12} + (3)\,(2)\, b_3 + \frac{3}{4} = 0$$
$$b_3 = -\frac{7}{36}$$

And so on. Hence the second solution is, for $r = 0, C = 1$

$$\begin{aligned}
y_2(x) &= Cy_1(x)\ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r} \\
&= y_1(x)\ln(x) + \sum_{n=0}^{\infty} b_n x^n \\
&= y_1(x)\ln(x) + \left(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots\right) \\
&= y_1(x)\ln(x) + \left(1 + (0)\, x - \frac{3}{4}x^2 - \frac{7}{36}x^3 + \cdots\right) \\
&= \left(x + \frac{1}{2}x^2 + \frac{1}{12}x^3 + \frac{1}{144}x^4 + \cdots\right)\ln x + \left(1 + \frac{3}{4}x^2 - \frac{7}{36}x^3 + \cdots\right) \\
&= x\left(1 + \frac{1}{2}x + \frac{1}{12}x^2 + \frac{1}{144}x^3 + O(x^4)\right)\ln x + \left(1 + \frac{3}{4}x^2 - \frac{7}{36}x^3 + O(x^4)\right)
\end{aligned}$$

Some observations: b_N is always taken as zero. Where N is the difference between the roots. In this case it is $b_1 = 0$. Now that we found y_1, y_2 then the general solution is

$$
\begin{aligned}
y &= C_1 y_1 + C_2 y_2 \\
&= C_1 x \left(1+\frac{1}{2}x+\frac{1}{12}x^2+\frac{1}{144}x^3+O(x^4)\right) + C_2 \left(x\left(1+\frac{1}{2}x+\frac{1}{12}x^2+\frac{1}{144}x^3+O(x^4)\right)\ln x + \left(1+\frac{3}{4}x^2-\frac{7}{36}x^3+O(x^4)\right)\right)
\end{aligned}
$$

This completes the solution.

2.2 Example 2. homogeneous ode where log term is needed $x^{\frac{3}{2}}y'' + y = 0$

Solve

$$
x^{\frac{3}{2}}y'' + y = 0
$$

Since $x = 0$ is regular singular point, then Frobenius power series must be used. Let the solution be represented as Frobenius power series of the form

$$
y = \sum_{n=0}^{\infty} a_n x^{\frac{n}{2}+r}
$$

Then

$$
\begin{aligned}
y' &= \sum_{n=0}^{\infty}\left(\frac{n}{2}+r\right) a_n x^{\frac{n}{2}+r-1} \\
y'' &= \sum_{n=0}^{\infty}\left(\frac{n}{2}+r\right)\left(\frac{n}{2}+r-1\right) a_n x^{\frac{n}{2}+r-2}
\end{aligned}
$$

Substituting the above back into the ode gives

$$
\begin{aligned}
x^{\frac{3}{2}}\left(\sum_{n=0}^{\infty}\left(\frac{n}{2}+r\right)\left(\frac{n}{2}+r-1\right) a_n x^{\frac{n}{2}+r-2}\right) + \left(\sum_{n=0}^{\infty} a_n x^{\frac{n}{2}+r}\right) &= 0 \\
\sum_{n=0}^{\infty}\left(\frac{n}{2}+r\right)\left(\frac{n}{2}+r-1\right) a_n x^{\frac{n}{2}+r-\frac{1}{2}} + \sum_{n=0}^{\infty} a_n x^{\frac{n}{2}+r} &= 0 \qquad \text{(1A)}
\end{aligned}
$$

$n = 0$ gives the indicial equation

$$
\begin{aligned}
(r)(r-1)\,a_0 x^{r-\frac{1}{2}} + a_0 x^r &= 0 \\
\left((r)(r-1)\,x^{r-\frac{1}{2}} + x^r\right) a_0 &= 0 \\
(r)(r-1)\,x^{r-\frac{1}{2}} + x^r &= 0 \\
\left((r)(r-1)\,x^{-\frac{1}{2}} + 1\right) x^r &= 0 \\
(r)(r-1)\frac{1}{\sqrt{x}} + 1 &= 0
\end{aligned}
$$

Not possible to obtain indicial equation in r only. How to handle this? Maple can't solve this using series solution either.

2.3 Example 3. homogeneous ode where log term is not needed $x^2y'' + 3xy' + 4x^4y = 0$

Solve

$$
x^2y'' + 3xy' + 4x^4y = 0 \qquad (1)
$$

Using power series method by expanding around $x = 0$. Writing the ode as

$$
y''(x) + \frac{3}{x}y' + 4x^2y = 0
$$

Shows that $x = 0$ is a singular point. But $\lim_{x\to 0} x\frac{3}{x} = 3$. Hence the singularity is removable. This means $x = 0$ is a regular singular point. In this case the Frobenius power series will be used instead of the standard power series. Let

$$
y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}
$$

Where r is to be determined. It is the root of the indicial equation. Therefore

$$
\begin{aligned}
y'(x) &= \sum_{n=0}^{\infty}(n+r)\,a_n x^{n+r-1} \\
y''(x) &= \sum_{n=0}^{\infty}(n+r)(n+r-1)\,a_n x^{n+r-2}
\end{aligned}
$$

Substituting the above in (1) gives

$$x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}+3x\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}+4x^4\sum_{n=0}^{\infty}a_nx^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r}+\sum_{n=0}^{\infty}3(n+r)a_nx^{n+r}+\sum_{n=0}^{\infty}4a_nx^{n+r+4}=0 \tag{1A}$$

Here, we need to make all powers on x the same, without making the sums start below zero. This can be done by adjusting the last term above as follows

$$\sum_{n=0}^{\infty}4a_nx^{n+r+4}=\sum_{n=4}^{\infty}4a_{n-4}x^{n+r}$$

And now Eq (1A) becomes

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r}+\sum_{n=0}^{\infty}3(n+r)a_nx^{n+r}+\sum_{n=4}^{\infty}4a_{n-4}x^{n+r}=0 \tag{1B}$$

$n=0$ gives the indicial equation

$$\begin{aligned}(n+r)(n+r-1)a_n+3(n+r)a_n&=0\\(r)(r-1)a_0+3ra_0&=0\\((r)(r-1)+3r)a_0&=0\end{aligned}$$

Since $a_0\neq 0$ then the above becomes

$$(r)(r-1)+3r=0$$

Hence the roots of the indicial equation are $r_1=0, r_2=-2$. Or $r_1=r_2+N$ where $N=2$. We always take r_1 to be the larger of the roots.

When this happens, the solution is given by

$$y(x)=c_1y_1(x)+c_2y_2(x)$$

Where $y_1(x)$ is the first solution, which is assumed to be

$$y_1(x)=\sum_{n=0}^{\infty}a_nx^{n+r} \tag{2}$$

Where we take $a_0=1$ as it is arbitrary and where $r=r_1=0$. This is the standard Frobenius power series, just like we did to find the indicial equation, the only difference is that now we use $r=r_1$, and hence it is a known value. Once we find $y_1(x)$, then the second solution is

$$y_2(x)=Cy_1(x)\ln(x)+\sum_{n=0}^{\infty}b_nx^{n+r} \tag{3}$$

We will show below how to find C and b_n. First, let us find $y_1(x)$. From Eq(2)

$$\begin{aligned}y_1'(x)&=\sum_{n=0}^{\infty}(n+r)a_nx^{n+r-1}\\y_1''(x)&=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}\end{aligned}$$

We need to remember that in the above r is not a symbol any more. It will have the indicial root value, which is $r=r_1=0$ in this case. But we keep r as symbol for now, in order to obtain $a_n(r)$ as function of r first and use this to find $b_n(r)$. At the very end we then evaluate everything at $r=r_1=0$. Substituting the above in (1) gives Eq (1B) above (We are following pretty much the same process we did to find the indicial equation here)

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r}+\sum_{n=0}^{\infty}3(n+r)a_nx^{n+r}+\sum_{n=4}^{\infty}4a_{n-4}x^{n+r}=0 \tag{1B}$$

Now we are ready to find a_n. Now we skip $n=0$ since that was used to obtain the indicial equation, and we know that $\underline{a_0=1}$ is an arbitrary value to choose.

For $n=1$, Eq (1B) gives

$$\begin{aligned}(1+r)(1+r-1)a_1+3(1+r)a_1&=0\\((1+r)(1+r-1)+3(1+r))a_1&=0\\(r^2+4r+3)a_1&=0\end{aligned}$$

But $r=r_1=0$. The above becomes

$$3a_1=0$$

Hence $a_1=0$.

It is a good idea to use a table to keep record of the a_n values as function of r, since this will be used later to find b_n.

n	$a_n(r)$	$a_n(r=r_1)$
0	1	1
1	0	0

For $n=2$, Eq (1B) gives

$$
\begin{aligned}
(2+r)(2+r-1)a_2+3(2+r)a_2&=0\\
((2+r)(2+r-1)+3(2+r))a_2&=0
\end{aligned}
$$

But $r=r_1=0$. The above becomes

$$
\begin{aligned}
((2)(1)+3(2))a_2&=0\\
8a_2&=0
\end{aligned}
$$

Hence $a_2=0$. The table becomes

n	$a_n(r)$	$a_n(r=0)$
0	1	1
1	0	0
2	0	0

For $n=3$, Eq (1B) gives

$$
(3+r)(3+r-1)a_3+3(3+r)a_3=0
$$

But $r=0$. The above becomes

$$
\begin{aligned}
(3)(2)a_3+3(3)a_3&=0\\
15a_3&=0
\end{aligned}
$$

Hence $a_3=0$ and the table becomes

n	$a_n(r)$	$a_n(r=0)$
0	1	1
1	0	0
2	0	0
3	0	0

For $n\geq 4$ we obtain the recursion equation

$$
\begin{aligned}
(n+r)(n+r-1)a_n+3(n+r)a_n+4a_{n-4}&=0\\
((n+r)(n+r-1)+3(n+r))a_n+4a_{n-4}&=0
\end{aligned}
$$

$$
a_n(r)=-\frac{4a_{n-4}(r)}{(n+r)(n+r-1)+3(n+r)} \tag{4}
$$

The above is very important, since we will use it to find $b_n(r)$ later on. For now, we are just finding the a_n. Now we find few more a_n terms. From (4) for $n=4$

$$
a_4(r)=-\frac{4a_0(r)}{(4+r)(4+r-1)+3(4+r)}
$$

and $r=r_1=0$ and $a_0=1$, then the above becomes

$$
a_4=-\frac{4}{(4)(3)+3(4)}=-\frac{1}{6}
$$

The table becomes

n	$a_n(r)$	$a_n(r=0)$
0	1	1
1	0	0
2	0	0
3	0	0
4	$-\frac{4}{(4+r)(4+r-1)+3(4+r)}$	$-\frac{1}{6}$

And for $n=5$ from Eq(4)

$$
\begin{aligned}
a_5(r)&=-\frac{4a_1(r)}{(n+r)(n+r-1)+3(n+r)}\\
&=0
\end{aligned}
$$

Since $a_1=0$. Similarly $a_6=0, a_7=0$. For $n=8$

$$
a_8(r)=-\frac{4a_4(r)}{(8+r)(8+r-1)+3(8+r)}
$$

But $a_4(r) = -\frac{4}{(4+r)(4+r-1)+3(4+r)}$. The above becomes

$$a_8(r) = \frac{4\frac{4}{(4+r)(4+r-1)+3(4+r)}}{(8+r)(8+r-1)+3(8+r)} = \frac{16}{r^4+28r^3+284r^2+1232r+1920}$$

When $r = r_1 = 0$ the above becomes

$$a_8(r) = \frac{1}{120}$$

And so on. The table becomes

n	$a_n(r)$	$a_n(r=0)$
0	1	1
1	0	0
2	0	0
3	0	0
4	$-\frac{4}{(4+r)(4+r-1)+3(4+r)}$	$-\frac{1}{6}$
5	0	0
6	0	0
7	0	0
8	$\frac{16}{r^4+28r^3+284r^2+1232r+1920}$	$\frac{1}{120}$

Hence $y_1(x)$ is

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

But $r = r_1 = 0$. Therefore

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 + \cdots \end{aligned} \tag{5}$$

Using values found for a_n in the above table, then (5) becomes

$$\begin{aligned} y_1(x) &= 1 + a_4x^4 + a_8x^8 + \cdots \\ &= 1 - \frac{1}{6}x^4 + \frac{1}{120}x^8 + O(x^9) \end{aligned}$$

We are done finding $y_1(x)$. This was not bad at all. Now comes the hard part. Which is finding $y_2(x)$. From (3) it is given by

$$y_2(x) = Cy_1(x)\ln(x) + \sum_{n=0}^{\infty} b_n x^{n+r} \tag{3}$$

The first thing to do is to determine if C is zero or not. This is done by finding $\lim_{r\to r_2} a_N(r)$. If this limit exist, then $C = 0$, else we need to keep the log term. From the above above we see that $a_N(r) = a_2(r) = 0$. Recall that $N = 2$ since this was the difference between the two roots and $r_2 = -2$ (the smaller root). Therefore

$$\lim_{r\to r_2} 0 = \lim_{r\to 0} 0 = 0$$

Hence the limit exist. Therefore we do not need the log term. This means we can let $C = 0$. This is the easy case. Hence (3) becomes

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= x^{-2}\sum_{n=0}^{\infty} b_n x^n \end{aligned} \tag{3A}$$

Since $r = r_2 = -2$. Let $b_0 = 1$. We have to remember now that $b_N = b_2 = 0$. This is the same we did when the log term was needed in the above example, since b_N is arbitrary, and used to generate $y_1(x)$. Common practice is to use $b_N = 0$. The rest of the b_n are found in similar way, from recursive relation as was done above. Substituting (3A) into $x^2y'' + 3xy' + 4x^4y = 0$ gives Eq. (1B) again, but with a_n replaced by b_n

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)b_n x^{n+r} + \sum_{n=0}^{\infty} 3(n+r)b_n x^{n+r} + \sum_{n=4}^{\infty} 4b_{n-4}x^{n+r} = 0 \tag{1B}$$

For $n = 0$, we skip and let $b_0 = 1$. For $n = 1$ the above gives $b_1 = 0$. And $b_2 = 0$ since it is the special term b_N. And for $n = 3$, we get $b_3 = 0$. The table for b_n is now

n	$b_n(r)$	$b_n(r=-2)$
0	1	1
1	0	0
2	0	0
3	0	0

For $n \geq 4$, the recursion relation is

$$(n+r)(n+r-1)b_n + 3(n+r)b_n + 4b_{n-4} = 0$$

$$b_n(r) = -\frac{4b_{n-4}(r)}{(n+r)(n+r-1)+3(n+r)}$$

For $n = 4$

$$\begin{aligned} b_4(r) &= -\frac{4b_0(r)}{(4+r)(4+r-1)+3(4+r)} \\ &= -\frac{4}{(4+r)(4+r-1)+3(4+r)} \qquad b_0 = 1 \end{aligned}$$

but $r = -2$. The above becomes

$$b_4 = -\frac{4}{(4-2)(4-2-1)+3(4-2)} = -\frac{1}{2}$$

The table becomes

n	$b_n(r)$	$b_n(r=-2)$
0	1	1
1	0	0
2*	0	0
3	0	0
4	$-\frac{4}{(4+r)(4+r-1)+3(4+r)}$	$-\frac{1}{2}$

We will find that $b_5 = b_6 = b_7 = 0$. And for $n = 8$

$$b_8(r) = -\frac{4b_4(r)}{(8+r)(7+r)+3(8+r)}$$

But $b_4(r) = -\frac{4}{(4+r)(4+r-1)+3(4+r)}$. Hence

$$b_8(r) = \frac{4\frac{4}{(4+r)(4+r-1)+3(4+r)}}{(8+r)(7+r)+3(8+r)} = \frac{16}{r^4+28r^3+284r^2+1232r+1920}$$

But $r = -2$.

$$b_8(r) = \frac{16}{(-2)^4+28(-2)^3+284(-2)^2+1232(-2)+1920} = \frac{1}{24}$$

The table becomes

n	$b_n(r)$	$b_n(r=-2)$
0	1	1
1	0	0
2*	0	0
3	0	0
4	$-\frac{4}{(4+r)(4+r-1)+3(4+r)}$	$-\frac{1}{2}$
5	0	0
6	0	0
7	0	0
8	$\frac{16}{r^4+28r^3+284r^2+1232r+1920}$	$\frac{1}{24}$

And so on. Hence the second solution is

$$\begin{aligned} y_2(x) &= \sum_{n=0}^{\infty} b_n x^{n+r} \\ &= \sum_{n=0}^{\infty} b_n x^{n-2} \\ &= x^{-2}\sum_{n=0}^{\infty} b_n x^n \\ &= x^{-2}\left(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5 + b_6x^6 + b_7x^7 + b_8x^8 + \cdots\right) \\ &= x^{-2}\left(1 - \frac{1}{2}x^4 + \frac{1}{24}b_8x^8 + O(x^9)\right) \end{aligned}$$

Therefore the general solution is

$$\begin{aligned} y &= C_1y_1 + C_2y_2 \\ &= C_1\left(1 - \frac{1}{6}x^4 + \frac{1}{120}x^8 + O(x^9)\right) + C_2\left(x^{-2}\left(1 - \frac{1}{2}x^4 + \frac{1}{24}b_8x^8 + O(x^9)\right)\right) \end{aligned}$$

The following are important items to remember. Always let $b_N = 0$ where N is the difference between the roots. When the log term is not needed (as in this problem), y_2 is found in very similar way to y_1 where $b_0 = 1$ and the recursion formula is used to find all b_n. But when the log term is needed (as in the above problem), it is a little more complicated and need to find C and b_1 values by comparing coefficients as was done).

This completes the solution.

2.4 Example 4 inhomogeneous ode example where log term is needed $xy'' + y = x$

$$xy'' + y = x \tag{1}$$

Let solution be $y = y_h + y_p$. We always start by finding y_h then find y_p using balance method. $x = 0$ is regular singular point. Hence Frobenius series gives

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Where r is to be determined. It is the root of the indicial equation. Therefore

$$y'(x) = \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1}$$
$$y''(x) = \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-2}$$

Substituting the above in (1) gives

$$x \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-2} + \sum_{n=0}^{\infty} a_n x^{n+r} = x$$
$$\sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = x \tag{1A}$$

Adjusting indices to all powers of x are the same gives

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = x \tag{1B}$$

The indicial equation is found from only the terms with the expansion of the dependent variable y. This means by making the LHS of (3) vanish. We only consider

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+r-1} = 0 \tag{1C}$$

For $n = 0$

$$(r)(r-1)\, a_n x^{r-1} = 0$$

EQ (1D) is used to find r. Since $a_0 \neq 0$ then (1D) gives

$$(r)(r-1) = 0$$

Hence roots are $r_1 = 1, r_2 = 0$. Hence the two basis solution for y_h are

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1}$$
$$y_2 = C_1 y_1 \ln x + x^{r_2} \sum_{n=0}^{\infty} b_n x^n = C_1 y_1 \ln x + \sum_{n=0}^{\infty} b_n x^n$$

To find y_1, we can find the recursive equation to be for $n > 0$

$$a_n = -\frac{a_{n-1}}{n(n+1)}$$

Which results in

$$y_1 = x - \frac{x^2}{2} + \frac{x^3}{12} - \frac{x^4}{144} + \frac{x^5}{2880} + \cdots$$

Finding y_2 is a little more involved because we need to determine C. This can be found to be $C = -1$. Using this we can find

$$y_2 = \left(-x + \frac{x^2}{2} - \frac{x^3}{12} + \frac{x^4}{144} - \frac{x^5}{2880} + \cdots\right) \ln x + \left(1 - \frac{3}{4}x^2 + \frac{7x^3}{36} - \cdots\right)$$

Hence

$$\begin{aligned} y_h &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(x - \frac{x^2}{2} + \frac{x^3}{12} - \frac{x^4}{144} + \frac{x^5}{2880} + \cdots\right) + c_2\left(\left(-x + \frac{x^2}{2} - \frac{x^3}{12} + \frac{x^4}{144} - \frac{x^5}{2880} + \cdots\right)\ln x + \left(1 - \frac{3}{4}x^2 + \frac{7x^3}{36} - \cdots\right)\right) \end{aligned} \tag{2}$$

What is left is to find y_p. Let

$$y_p = \sum_{n=0}^{\infty} c_n x^{n+r}$$

Substituting this into the ode $xy'' + y = x$ and simplifying as we did above results in

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-1} + \sum_{n=1}^{\infty} c_{n-1} x^{n+r-1} = x \tag{3A}$$

For $n = 0$

$$(r)(r-1) c_0 x^{r-1} = x$$

Hence for balance we need $r - 1 = 1$ or $r = 2$. Therefore $(r)(r-1) c_0 = 1$ and solving for c_0 gives $c_0 = \frac{1}{2}$. Therefore (3A) becomes

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_n x^{n+1} + \sum_{n=1}^{\infty} c_{n-1} x^{n+1} = x \tag{3B}$$

Now that we used $n = 0$ to find r by matching against the RHS which is x, from now on, for all $n > 0$ we will use (3B) to solve for all other c_n. From now on, we just need to solve with RHS zero, since there can be no more matches for any x on the right. For $n = 1$ then (3B) gives

$$6c_1 x^2 + c_0 x^2 = 0$$

Hence $6c_1 + c_0 = 0$ or $c_1 = -\frac{c_0}{6} = \frac{-1}{12}$. For $n = 2$, EQ(3B) gives

$$12c_2 x^3 + c_1 x^3 = 0$$

Hence $12c_2 + c_1 = 0$ or $c_2 = -\frac{c_1}{12} = -\frac{\frac{-1}{12}}{12} = \frac{1}{144}$. For $n = 3$, EQ(3B) gives

$$20c_3 x^4 + c_2 x^4 = 0$$

Hence $20c_3 + c_2 = 0$ or $c_3 = -\frac{c_2}{20} = -\frac{\frac{1}{144}}{20} = -\frac{1}{2880}$ and so one. Therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+r} \\ &= \sum_{n=0}^{\infty} c_n x^{n+2} \\ &= c_0 x^2 + c_1 x^3 + c_2 x^4 + c_3 x^5 + \cdots \\ &= \frac{1}{2} x^2 - \frac{1}{12} x^3 + \frac{1}{144} x^4 - \frac{1}{2880} x^5 + \cdots \end{aligned} \tag{4}$$

Hence the final solution is

$$y = y_h + y_p$$

Where y_h is given by (2) and y_p is given by (4).

3 Frobenius series. Indicial equation with root that differ by non integer

3.1 Example 1. homogeneous ode $2x^2 y''(x) - xy'(x) + (1 - x^2)\, y(x) = 0$

With expansion around $x = 0$. This is a regular singular ODE. Let

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Therefore

$$\begin{aligned} y'(x) &= \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \\ y''(x) &= \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \end{aligned}$$

Substituting the above into the ode and simplifying gives

$$\begin{aligned} 2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} - x \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + (1 - x^2) \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\ \sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} - x^2 \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\ \sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r+2} &= 0 \end{aligned}$$

The next step is to make all powers of x to be $n + r$. This results in

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} - \sum_{n=2}^{\infty} a_{n-2} x^{n+r} = 0 \tag{1}$$

The indicial equation is obtained from $n = 0$

$$
\begin{aligned}
2(n+r)(n+r-1)a_n - (n+r)a_n + a_n &= 0 \\
a_n(2(n+r)(n+r-1) - (n+r) + 1) &= 0 \\
2(n+r)(n+r-1) - (n+r) + 1 &= 0 \\
2(0+r)(0+r-1) - (0+r) + 1 &= 0 \\
2r^2 - 3r + 1 &= 0
\end{aligned}
$$

The roots are

$$
\begin{aligned}
r_1 &= 1 \\
r_2 &= \frac{1}{2}
\end{aligned}
$$

Since the roots differ by non integer, then the solutions are given by

$$
\begin{aligned}
y_1 &= \sum_{n=0}^{\infty} a_n x^{n+r_1} = \sum_{n=0}^{\infty} a_n x^{n+1} \\
y_2 &= \sum_{n=0}^{\infty} a_n x^{n+r_2} = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}}
\end{aligned}
$$

We start by finding y_1. EQ. (1) gives for $r = 1$

$$
\sum_{n=0}^{\infty} 2(n+1)(n)a_n x^{n+1} - \sum_{n=0}^{\infty} (n+1)a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1} - \sum_{n=2}^{\infty} a_{n-2} x^{n+1} = 0
$$

For $n = 1$ (we skip $n = 0$ as that was used to find r) and we always let $a_0 = 1$ as it is arbitrary. Hence

$$
\begin{aligned}
2(n+1)(n)a_n x^{n+1} - (n+1)a_n x^{n+1} + a_n x^{n+1} &= 0 \\
2(n+1)(n)a_n - (n+1)a_n + a_n &= 0 \\
4a_1 - 2a_1 + a_1 &= 0 \\
3a_1 &= 0 \\
a_1 &= 0
\end{aligned}
$$

For $n \geq 2$, we have recursion relation. From it we can find that $a_2 = \frac{1}{10}, a_3 = 0, a_4 = \frac{1}{360}$ and so on. Hence

$$
\begin{aligned}
y_1 &= \sum_{n=0}^{\infty} a_n x^{n+1} \\
&= a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + a_4 x^5 + \cdots \\
&= x + \frac{1}{10}x^3 + \frac{1}{360}x^5 + \cdots
\end{aligned}
$$

Now we do the same for y_2. EQ. (1) now becomes for $r_2 = \frac{1}{2}$

$$
\sum_{n=0}^{\infty} 2\left(n+\frac{1}{2}\right)\left(n+\frac{1}{2}-1\right)a_n x^{n+\frac{1}{2}} - \sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)a_n x^{n+\frac{1}{2}} + \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} - \sum_{n=2}^{\infty} a_{n-2} x^{n+\frac{1}{2}} = 0
$$

For $n = 1$ (we skip $n = 0$ as that was used to find r) and we always let $a_0 = 1$ as it is arbitrary. Hence

$$
\begin{aligned}
2\left(1+\frac{1}{2}\right)\left(1+\frac{1}{2}-1\right)a_1 - \left(1+\frac{1}{2}\right)a_1 + a_1 &= 0 \\
a_1 &= 0
\end{aligned}
$$

For $n \geq 2$, we have recursion relation. From it we can find that $a_2 = \frac{1}{6}, a_3 = 0, a_4 = \frac{1}{168}$ and so on. Hence

$$
\begin{aligned}
y_2 &= \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} \\
&= x^{\frac{1}{2}} \sum_{n=0}^{\infty} a_n x^n \\
&= x^{\frac{1}{2}}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots\right) \\
&= x^{\frac{1}{2}}\left(1 + \frac{1}{6}x^2 + \frac{1}{168}x^4 + \cdots\right)
\end{aligned}
$$

Hence the final solution is

$$
\begin{aligned}
y_h &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x + \frac{1}{10}x^3 + \frac{1}{360}x^5 + \cdots\right) + c_2\left(x^{\frac{1}{2}}\left(1 + \frac{1}{6}x^2 + \frac{1}{168}x^4 + \cdots\right)\right)
\end{aligned}
$$

3.2 Example 2. homogeneous ode $2x^2y''(x) - xy'(x) + (1-x^2)\,y(x) = 1 + x$

With expansion around $x = 0$. This is a regular singular ODE. This is same ode solved in example 1, but now with non-zero on the right side. Hence solution is given by

$$y = y_h + y_p$$

Where we found y_h to be

$$y_h = c_1\left(x + \frac{1}{10}x^3 + \frac{1}{360}x^5 + \cdots\right) + c_2\left(x^{\frac{1}{2}}\left(1 + \frac{1}{6}x^2 + \frac{1}{168}x^4 + \cdots\right)\right)$$

To find y_p, we see there are two terms on the right side. we find y_{p_1} that corresponds to 1 on the right side and then y_{p_2} that corresponds to x on the right side. We will find that y_{p_2} does not exist. Hence no solution exist using series. Let us now find y_{p_1}. The ode now is

$$2x^2y''(x) - xy'(x) + (1-x^2)\,y(x) = 1$$

Let

$$y_{p_1} = \sum_{n=0}^{\infty} c_n x^{n+r}$$

Using same expansion as in example 1, but now using c_n instead of a_n gives EQ. (1) from example 1 as

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)\,c_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)\,c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} - \sum_{n=2}^{\infty} c_{n-2}x^{n+r} = 1 \tag{1A}$$

For $n = 0$

$$\begin{aligned}
2r(r-1)\,c_0x^r - rc_0x^r + c_0x^r &= x^0\\
(2r(r-1)\,c_0 - rc_0 + c_0)\,x^r &= x^0\\
(2r(r-1) - r + 1)\,c_0x^r &= x^0\\
(2r^2 - 3r + 1)\,c_0x^r &= x^0
\end{aligned}$$

We see that for balance, then $r = 0$. Which implies $(2r^2 - 3r + 1)\,c_0 = 1$ or $c_0 = 1$. Hence

$$y_{p_1} = \sum_{n=0}^{\infty} c_n x^n$$

Now that we used $n = 0$ to find r by matching against the RHS which is 1, from now on, for all $n > 0$ we will use (1A) to solve for all other c_n. From now on, we just need to solve with RHS zero, since there can be no more matches for any x^n on the right.

Then (1A) becomes (using $r = 0$)

$$\sum_{n=0}^{\infty} 2n(n-1)\,c_n x^n - \sum_{n=0}^{\infty} nc_n x^n + \sum_{n=0}^{\infty} c_n x^n - \sum_{n=2}^{\infty} c_{n-2}x^n = 0 \tag{2}$$

For $n = 1$ the above gives

$$-c_1x + c_1x = 0$$

c_1 is arbitrary Let $c_1 = 0$. For $n = 2$ EQ. (2) gives

$$\begin{aligned}
4c_2x^2 - 2c_2x^2 + c_2x^2 - c_0x^2 &= 0\\
(4c_2 - 2c_2 + c_2 - c_0)\,x^2 &= 0\\
(3c_2 - 1)\,x^2 &= 0
\end{aligned}$$

Hence $3c_2 - 1 = 0$ since there is no x^2 term on the right side. Hence $c_2 = \frac{1}{3}$. For $n = 3$ EQ. (2) gives

$$\begin{aligned}
(12c_3 - 3c_3 + c_3 - c_1)\,x^3 &= 0\\
(10c_3)\,x^3 &= 0
\end{aligned}$$

Hence $c_3 = 0$. For $n = 4$ EQ. (2) gives

$$\begin{aligned}
24c_4x^4 - 4c_4x^4 + c_4x^4 - c_2x^4 &= 0\\
\left(21c_4 - \frac{1}{3}\right)x^4 &= 0
\end{aligned}$$

Hence $21c_4 - \frac{1}{3} = 0$ or $c_4 = \frac{1}{63}$ and so on. We find that

$$\begin{aligned}
y_{p_1} &= \sum_{n=0}^{\infty} c_n x^n\\
&= c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \cdots\\
&= 1 + \frac{1}{3}x^2 + \frac{1}{63}x^4 + \cdots
\end{aligned}$$

Now we find y_{p_2}. EQ. (1A) now is

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)\, c_n x^{n+r} - \sum_{n=0}^{\infty} (n+r)\, c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} - \sum_{n=2}^{\infty} c_{n-2} x^{n+r} = x$$

For $n = 0$

$$\begin{aligned}
2(n+r)(n+r-1)\, c_n x^{n+r} - (n+r)\, c_n x^{n+r} + c_n x^{n+r} &= x \\
2r(r-1)\, c_0 x^r - r c_0 x^r + c_0 x^r &= x \\
(2r(r-1) - r + 1)\, c_0 x^r &= x \\
\left(2r^2 - 3r + 1\right) c_0 x^r &= x
\end{aligned}$$

For balance we need $r = 1$. This results in

$$\begin{aligned}
\left(2r^2 - 3r + 1\right) c_0 &= 1 \\
(2 - 3 + 1)\, c_0 &= 1 \\
0 c_0 &= 1
\end{aligned}$$

Not possible. We see why there is no series solution. It is not possible to solve for y_{p2}.

4 Frobenius series where indicial equation roots are complex conjugate

4.1 Example 1. homogeneous ode $x^2 y'' + x^2 y' + y = 0$

$$x^2 y'' + x^2 y' + y = 0 \tag{1}$$

With expansion around $x = 0$. This is a regular singular ODE. Let

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Where r is to be determined. It is the root of the indicial equation. Therefore

$$\begin{aligned}
y'(x) &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \\
y''(x) &= \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-2}
\end{aligned}$$

Substituting the above in (1) gives

$$\begin{aligned}
x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-2} + x^2 \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\
\sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r+1} + \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \qquad \text{(1A)}
\end{aligned}$$

Here, we need to make all powers on x the same, without making the sums start below zero. This can be done by adjusting the middle term above as follows

$$\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r+1} = \sum_{n=1}^{\infty} (n+r-1)\, a_{n-1} x^{n+r}$$

And now Eq (1A) becomes

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r} + \sum_{n=1}^{\infty} (n+r-1)\, a_{n-1} x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0 \tag{3A}$$

$n = 0$ gives the indicial equation

$$\begin{aligned}
(n+r)(n+r-1)\, a_0 x^r + a_0 x^r &= 0 \\
(r)(r-1)\, a_0 x^r + (r)\, a_0 x^r &= 0 \\
(r(r-1)\, a_0 + a_0)\, x^r &= 0 \\
(r(r-1) + 1)\, a_0 x^r &= 0 \qquad \text{(3B)}
\end{aligned}$$

EQ (3B) is used to solve for r. Since $a_0 \neq 0$ then (3B) gives

$$\begin{aligned}
(r)(r-1) + 1 &= 0 \\
r^2 - r + 1 &= 0
\end{aligned}$$

The roots are

$$r_1 = \frac{1}{2} + \frac{1}{2} i\sqrt{3}$$
$$r_2 = \frac{1}{2} - \frac{1}{2} i\sqrt{3}$$

The roots will always be complex conjugate of each others (since second order ode) and the real part will always be equal. Let the roots be

$$r_{1,2} = \alpha \pm i\beta$$

When this happens, the solution is given similar to the case when the roots differ by non integer, except now the solution and the coefficients will be complex. Let the solution be

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

Now $y_1(x)$ is solved for. The solution is

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n$$

Starting with Eq (2A) which was derived above gives

$$\sum_{n=0}^{\infty} (n+r_1)(n+r_1-1) a_n x^{n+r_1} + \sum_{n=1}^{\infty} (n+r_1-1) a_{n-1} x^{n+r_1} + \sum_{n=0}^{\infty} a_n x^{n+r_1} = 0$$

The case of $n = 0$ is skipped since this was used to find the roots and $a_0 \neq 0$. $n \geq 1$ gives the recursion equation

$$\begin{aligned} (n+r_1)(n+r_1-1) a_n + (n+r_1-1) a_{n-1} + a_n &= 0 \\ (n+r_1)(n+r_1-1) a_n + a_n &= -(n+r_1-1) a_{n-1} \\ a_n &= \frac{-(n+r_1-1) a_{n-1}}{(n+r_1)(n+r_1-1)+1} \end{aligned} \tag{3}$$

For $n = 1$ Eq. (3) gives

$$a_1 = \frac{-r_1 a_{n-1}}{(1+r_1)(r_1)+1}$$

But $r_1 = \alpha \pm i\beta = \frac{1}{2} + \frac{1}{2} i\sqrt{3}$. and $a_0 = 1$. Hence the above becomes

$$a_1 = \frac{-\left(\frac{1}{2} + \frac{1}{2} i\sqrt{3}\right)}{\left(1 + \frac{1}{2} + \frac{1}{2} i\sqrt{3}\right)\left(\frac{1}{2} + \frac{1}{2} i\sqrt{3}\right) + 1} = \frac{-1}{2}$$

For $n = 2$ Eq. (3) gives

$$a_2 = \frac{-(1+r_1) a_1}{(2+r_1)(1+r_1)+1} = \frac{-(1+r_1)\left(\frac{-1}{2}\right)}{(2+r_1)(1+r_1)+1} = \frac{-\left(1 + \frac{1}{2} + \frac{1}{2} i\sqrt{3}\right)\left(\frac{-1}{2}\right)}{\left(2 + \frac{1}{2} + \frac{1}{2} i\sqrt{3}\right)\left(1 + \frac{1}{2} + \frac{1}{2} i\sqrt{3}\right) + 1} = \frac{9}{56} - \frac{1}{56} i\sqrt{3}$$

For $n = 3$ Eq. (3) gives

$$a_3 = \frac{-(3+r_1-1) a_2}{(3+r_1)(2+r_1)+1} = \frac{-(3+r_1-1)\left(\frac{9}{56} - \frac{1}{56} i\sqrt{3}\right)}{(3+r_1)(2+r_1)+1} = \frac{-\left(2 + \frac{1}{2} + \frac{1}{2} i\sqrt{3}\right)\left(\frac{9}{56} - \frac{1}{56} i\sqrt{3}\right)}{\left(3 + \frac{1}{2} + \frac{1}{2} i\sqrt{3}\right)\left(2 + \frac{1}{2} + \frac{1}{2} i\sqrt{3}\right) + 1} = \frac{1}{112} i\sqrt{3} - \frac{13}{336}$$

And so on. Hence the first solution is

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n = x^{\frac{1}{2} + \frac{1}{2} i\sqrt{3}} \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right) \tag{4}$$

but

$$x^{\frac{1}{2} + \frac{1}{2} i\sqrt{3}} = x^{\frac{1}{2}} x^{\frac{1}{2} i\sqrt{3}} = x^{\frac{1}{2}} e^{\ln\left(x^{\frac{1}{2} i\sqrt{3}}\right)} = x^{\frac{1}{2}} e^{i \ln\left(x^{\frac{\sqrt{3}}{2}}\right)} = x^{\frac{1}{2}} \left(\cos\left(\ln x^{\frac{\sqrt{3}}{2}}\right) + i \sin\left(\ln x^{\frac{\sqrt{3}}{2}}\right)\right)$$

Substituting the above in (4) and using values found for a_n gives

$$y_1(x) = x^{\frac{1}{2}} \left(\cos\left(\ln x^{\frac{\sqrt{3}}{2}}\right) + i \sin\left(\ln x^{\frac{\sqrt{3}}{2}}\right)\right) \left(1 - \frac{1}{2} x + \left(\frac{9}{56} - \frac{1}{56} i\sqrt{3}\right) x^2 + \left(-\frac{13}{336} + \frac{1}{112} i\sqrt{3}\right) x^3 + \cdots\right) \tag{5}$$

Since the roots are complex conjugate of each others, then the second solution is

$$y_2(x) = x^{\frac{1}{2}} \left(\cos\left(\ln x^{\frac{\sqrt{3}}{2}}\right) - i \sin\left(\ln x^{\frac{\sqrt{3}}{2}}\right)\right) \left(1 - \frac{1}{2} x + \left(\frac{9}{56} + \frac{1}{56} i\sqrt{3}\right) x^2 + \left(-\frac{13}{336} - \frac{1}{112} i\sqrt{3}\right) x^3 + \cdots\right) \tag{6}$$

The final solution is therefore

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

5 Ordinary point for second order ode

5.1 Example 1. $y'' = \frac{1}{x}$

$$y'' = \frac{1}{x} \tag{1}$$

With expansion around $x = 0$. This is an ordinary point for the ode itself. Let

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Therefore

$$\begin{aligned} y'(x) &= \sum_{n=0}^{\infty} n a_n x^{n-1} \\ &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y''(x) &= \sum_{n=1}^{\infty} n(n-1)\, a_n x^{n-2} \\ &= \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} \end{aligned}$$

Hence the ode becomes

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = \frac{1}{x} \tag{2}$$

The solution is given by $y = y_h + y_p$, where y_h is solution to

$$\sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2} = 0$$

Recursive equation is

$$n(n-1)\, a_n = 0 \qquad n \geq 2$$

Hence all $a_n = 0$ for $n \geq 2$, therefore

$$\begin{aligned} y_h &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x \end{aligned}$$

Now we need to find y_p. From (2), and now we replace a_n by c_n

$$\sum_{n=2}^{\infty} n(n-1)\, c_n x^{n-2} = \frac{1}{x}$$

$n = 2$ gives

$$2c_2 x^0 = x^{-1}$$

Hence for balance $0 = -1$ which is not possible. Hence no y_p exist using series method. Solution exist by direct integration.

6 Examples using Frobenius series on first order ode

6.1 Example 1. $xy' + y = 0$

$$xy' + y = 0 \tag{1}$$

With expansion around $x = 0$. Since x is regular singular point, then let

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\ y' &= \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} \end{aligned}$$

Then (1) becomes

$$x \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0 \tag{2}$$

$$\sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0 \tag{3A}$$

All x in sums start from same index, so there is no adjustments needed. For $n=0$, EQ (3) gives

$$\begin{aligned} ra_0x^r + a_0x^r &= 0 \\ (r+1)\,a_0x^r &= 0 \end{aligned}$$

But $a_0 \neq 0$, hence $r+1=0$ or

$$r = -1$$

Therefore (3) becomes

$$\sum_{n=0}^{\infty} (n-1)\,a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n-1} = 0 \tag{3B}$$

For $n>0$ (because $n=0$ was used to find r) EQ (4) gives the recursive relation

$$\begin{aligned} (n-1)\,a_n + a_n &= 0 \\ na_n &= 0 \end{aligned} \tag{3C}$$

We see that for all $n>0$ then $a_n=0$. Hence solution is for $n=0$ only, which is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+r} \\ &= a_0 x^{-1} \\ &= \frac{a_0}{x} \end{aligned}$$

6.2 Example 2. $xy' + y = x$

$$xy' + y = x \tag{1}$$

This is same as example 1, but with nonzero on RHS. The solution is $y = y_h + y_p$. Where y_h was found above as

$$y_h = \frac{a_0}{x}$$

To find y_p assume it has power series

$$y_p = \sum_{n=0}^{\infty} c_n x^{n+r}$$

Substituting the above into the ode and simplifying as we did in first example gives EQ. (3) in the first example, but now with x on RHS

$$\sum_{n=0}^{\infty} (n+r)\,c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} = x \tag{3A}$$

For $n=0$, EQ. (3A) gives

$$\begin{aligned} rc_0x^r + c_0x^r &= x \\ (rc_0 + c_0)\,x^r &= x \end{aligned}$$

For balance we need $r=1$. Hence $rc_0 + c_0 = 1$ or $c_0 = \frac{1}{2}$. EQ. (3A) becomes

$$\sum_{n=0}^{\infty} (n+1)\,c_n x^{n+1} + \sum_{n=0}^{\infty} c_n x^{n+1} = x \tag{3B}$$

For $n>0$ we obtain recursive relation (since no balance now, as all x powers on left side do not match the right side)

$$\begin{aligned} (n+1)\,c_n + c_n &= 0 \\ (n+2)\,c_n &= 0 \end{aligned} \tag{4}$$

This show that $c_n = 0$ for $n>0$. Therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+r} \\ &= \sum_{n=0}^{\infty} c_n x^{n+1} \\ &= c_0 x \\ &= \frac{1}{2}x \end{aligned}$$

Therefore

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{a_0}{x} + \frac{1}{2}x \end{aligned}$$

6.3 Example 3. $xy' + y = 1$

$$xy' + y = 1 \tag{1}$$

This is same as example 1, but with nonzero on RHS. The solution is $y = y_h + y_p$. Where y_h was found above as

$$y_h = \frac{a_0}{x}$$

To find y_p assume it has power series

$$y_p = \sum_{n=0}^{\infty} c_n x^{n+r}$$

Substituting the above into the ode and simplifying as we did in first example gives EQ. (3) in the first example, but now with x on RHS

$$\sum_{n=0}^{\infty} (n+r)\, c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} = 1 \tag{3A}$$

For $n = 0$

$$\begin{aligned} rc_0 x^r + c_0 x^r &= x^0 \\ (rc_0 + c_0)\, x^r &= x^0 \end{aligned}$$

For balance $r = 0$. And $rc_0 + c_0 = 1$ or $c_0 = 1$. Hence (3A) becomes

$$\sum_{n=0}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 1 \tag{3B}$$

Now that we used $n = 0$ to find r by matching against the RHS which is 1, from now on, for all $n > 0$ we will use (3B) to solve for all other c_n. From now on, we just need to solve with RHS zero, since there can be no more matches for any x^n on the right. The recursive relation for $n > 0$ is

$$\begin{aligned} n c_n x^n + c_n x^n &= 0 \\ (n+1)\, c_n x^n &= 0 \end{aligned} \tag{3C}$$

We see that this implies $(n+1)\, c_n = 0$ for all $n > 0$, or $c_n = 0$. Therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^{n+r} \\ &= \sum_{n=0}^{\infty} c_n x^n \\ &= c_0 \\ &= 1 \end{aligned}$$

Therefore

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{a_0}{x} + 1 \end{aligned}$$

6.4 Example 4. $xy' + y = k$

$$xy' + y = k \tag{1}$$

This is similar to above example. Carrying out same steps shows that $c_0 = k$ and all other $c_n = 0$. Hence $y_p = k$. The solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{a_0}{x} + k \end{aligned}$$

6.5 Example 5. $xy' + y = \sin x$

$$xy' + y = \sin x \tag{1}$$

This is same as example 1, but with nonzero on RHS. The solution is $y = y_h + y_p$. Where y_h was found above as

$$y_h = \frac{a_0}{x}$$

To find y_p we assume it has power series

$$y_p = \sum_{n=0}^{\infty} c_n x^{n+r}$$

Substituting the above into the ode and simplifying as we did in first example gives EQ. (3) in the first example, but now with $\sin x$ on RHS

$$\begin{aligned}
\sum_{n=0}^{\infty} (n+r) c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} &= \sin x \\
\sum_{n=0}^{\infty} (n+r) c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} &= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \cdots
\end{aligned}$$

We have to find y_p for each term at a time. Starting with x

$$\sum_{n=0}^{\infty} (n+r) c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} = x \tag{3A}$$

For $n = 0$ the above becomes

$$\begin{aligned}
r c_0 x^r + c_0 x^r &= x \\
(r+1) c_0 x^r &= x
\end{aligned}$$

Balance gives $r = 1$ and $(r+1) c_0 = 1$ or $c_0 = \frac{1}{2}$.

Now that we used $n = 0$ to find r by matching against the RHS which is x, from now on, for all $n > 0$ we will use (3A) to solve for all other c_n. From now on, we just need to solve with RHS zero, since there can be no more matches for any x on the right. Hence (3A) becomes

$$\sum_{n=0}^{\infty} (n+1) c_n x^{n+1} + \sum_{n=0}^{\infty} c_n x^{n+1} = x \tag{3B}$$

The recursive relation for $n > 0$ is

$$(n+2) c_n x^{n+1} = 0 \tag{3C}$$

We see from (3C) for all $n > 0$, that $(n+2) c_n = 0$ which implies $c_n = 0$ for all $n > 0$. Hence y_{p_1} that corresponds to x is

$$\begin{aligned}
y_{p_1} &= \sum_{n=0}^{\infty} c_n x^{n+r} = \sum_{n=0}^{\infty} c_n x^{n+1} \\
&= c_0 x \\
&= \frac{1}{2}x
\end{aligned}$$

Now we repeat the above for the second term in the sin expansion. (3A) now becomes the following, with x replace by $-\frac{1}{6}x^3$ on the right side

$$\sum_{n=0}^{\infty} (n+r) c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} = -\frac{1}{6}x^3 \tag{3A}$$

For $n = 0$

$$\begin{aligned}
r c_0 x^r + c_0 x^r &= -\frac{1}{6}x^3 \\
(r+1) c_0 x^r &= -\frac{1}{6}x^3
\end{aligned}$$

Balance gives $r = 3$ and $(r+1) c_0 = -\frac{1}{6}$, hence $c_0 = \frac{-1}{6(4)} = -\frac{1}{24}$. (3A) becomes

$$\sum_{n=0}^{\infty} (n+3) c_n x^{n+3} + \sum_{n=0}^{\infty} c_n x^{n+3} = -\frac{1}{6}x^3 \tag{3B}$$

Now that we used $n = 0$ to find r by matching against the RHS which is $-\frac{1}{6}x^3$, from now on, for all $n > 0$ we will use (3B) to solve for all other c_n. From now on, we just need to solve with RHS zero, since there can be no more matches for any x^3 on the right. Hence (3B) becomes for $n > 0$

$$(n+4) c_n x^{n+3} = 0 \tag{3C}$$

We see that for all $n > 0$ that $(n+4) c_n = 0$ or $c_n = 0$ for all $n > 0$. Hence y_{p_2} that corresponds to

$-\frac{1}{6}x^3$ is

$$\begin{aligned} y_{p_2} &= \sum_{n=0}^{\infty} c_n x^{n+r} = \sum_{n=0}^{\infty} c_n x^{n+3} \\ &= c_0 x^3 \\ &= -\frac{1}{24}x^3 \end{aligned}$$

Now we repeat the above for the third term in the sin expansion. (3A) becomes

$$\sum_{n=0}^{\infty}(n+r)\,c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} = \frac{1}{120}x^5 \tag{3A}$$

For $n=0$

$$\begin{aligned} r c_0 x^r + c_0 x^r &= \frac{1}{120}x^5 \\ (r+1)\,c_0 x^r &= \frac{1}{120}x^5 \end{aligned}$$

Balance gives $r=5$ and $(r+1)\,c_0 = \frac{1}{120}$, hence $c_0 = \frac{1}{120(6)} = \frac{1}{720}$. (3A) becomes

$$\sum_{n=0}^{\infty}(n+5)\,c_n x^{n+5} + \sum_{n=0}^{\infty} c_n x^{n+5} = \frac{1}{120}x^5 \tag{3B}$$

Now that we used $n=0$ to find r by matching against the RHS which is $\frac{1}{120}x^5$, from now on, for all $n>0$ we will use (3B) to solve for all other c_n. From now on, we just need to solve with RHS zero, since there can be no more matches for any x^5 on the right. Hence (3B) becomes for $n>0$

$$(n+6)\,c_n x^{n+5} = 0 \tag{3C}$$

We see that for all $n>0$ that $(n+6)\,c_n = 0$ or $c_n = 0$ for all $n>0$. Hence y_{p_2} that corresponds to $\frac{1}{120}x^5$ is

$$\begin{aligned} y_{p_3} &= \sum_{n=0}^{\infty} c_n x^{n+r} = \sum_{n=0}^{\infty} c_n x^{n+5} \\ &= c_0 x^5 \\ &= \frac{1}{720}x^5 \end{aligned}$$

And so on. Now we add all the y_p found above, which gives

$$\begin{aligned} y_p &= y_{p_1} + y_{p_2} + y_{p_3} + \cdots \\ &= \frac{1}{2}x - \frac{1}{24}x^3 + \frac{1}{720}x^5 - \cdots \end{aligned}$$

We found y_p. The solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{a_0}{x} + \left(\frac{1}{2}x - \frac{1}{24}x^3 + \frac{1}{720}x^5 + \cdots\right) \end{aligned}$$

6.6 Example 6. $xy' + y = x + x^3 + 2x^4$

$$xy' + y = x + x^3 + 2x^4 \tag{1}$$

This is same as example 1, but with nonzero on RHS. The solution is $y = y_h + y_p$. Where y_h was found above as

$$y_h = \frac{a_0}{x}$$

To find y_p we assume it has power series

$$y_p = \sum_{n=0}^{\infty} c_n x^{n+r}$$

Substituting the above into the ode and simplifying as we did in first example gives EQ. (3) in the first example, but now with $\sin x$ on RHS

$$\sum_{n=0}^{\infty}(n+r)\,c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} = x + x^3 + 2x^4$$

We have to find y_p for each term at a time. Starting with x

$$\sum_{n=0}^{\infty}(n+r)\,c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} = x \tag{3A}$$

For $n=0$ the above becomes

$$\begin{aligned} rc_0x^r + c_0x^r &= x \\ (r+1)\,c_0x^r &= x \end{aligned}$$

Balance gives $r=1$ and $(r+1)\,c_0 = 1$ or $c_0 = \frac{1}{2}$. Hence (3A) becomes

$$\sum_{n=0}^{\infty}(n+1)\,c_nx^{n+1} + \sum_{n=0}^{\infty}c_nx^{n+1} = x \tag{3B}$$

Now that we used $n=0$ to find r by matching against the RHS which is x, from now on, for all $n>0$ we will use (3B) to solve for all other c_n. From now on, we just need to solve with RHS zero, since there can be no more matches for any x on the right. Hence (3B) becomes for $n>0$

$$(n+2)\,c_nx^{n+1} = 0 \tag{3C}$$

We see that for all $n>0$ then $(n+2)\,c_n = 0$ or $c_n = 0$ for all $n>0$. Hence y_{p_1} that corresponds to x is

$$\begin{aligned} y_{p_1} &= \sum_{n=0}^{\infty}c_nx^{n+r} = \sum_{n=0}^{\infty}c_nx^{n+1} \\ &= c_0x \\ &= \frac{1}{2}x \end{aligned}$$

Now we repeat the above for the second term x^3. Starting with (3A) again

$$\sum_{n=0}^{\infty}(n+r)\,c_nx^{n+r} + \sum_{n=0}^{\infty}c_nx^{n+r} = x^3 \tag{3A}$$

For $n=0$

$$\begin{aligned} rc_0x^r + c_0x^r &= x^3 \\ (r+1)\,c_0x^r &= x^3 \end{aligned}$$

Hence $r=3$ and $(r+1)\,c_0 = 1$ or $c_0 = \frac{1}{4}$. Then (3A) becomes

$$\sum_{n=0}^{\infty}(n+3)\,c_nx^{n+3} + \sum_{n=0}^{\infty}c_nx^{n+3} = x^3 \tag{3B}$$

Now that we used $n=0$ to find r by matching against the RHS which is x^3, from now on, for all $n>0$ we will use (3B) to solve for all other c_n. From now on, we just need to solve with RHS zero, since there can be no more matches for any x^3 on the right. Hence (3B) becomes for $n>0$

$$(n+4)\,c_nx^{n+3} = 0 \tag{3C}$$

We see that for all $n>0$ then $(n+4)\,c_n = 0$ or $c_n = 0$ for all $n>0$. Hence y_{p_2} that corresponds to x^3 is

$$\begin{aligned} y_{p_1} &= \sum_{n=0}^{\infty}c_nx^{n+r} = \sum_{n=0}^{\infty}c_nx^{n+3} \\ &= c_0x^3 \\ &= \frac{1}{4}x^3 \end{aligned}$$

Now we repeat the above for the third and final term $2x^4$. Starting with (3A) again

$$\sum_{n=0}^{\infty}(n+r)\,c_nx^{n+r} + \sum_{n=0}^{\infty}c_nx^{n+r} = 2x^4 \tag{3A}$$

For $n=0$

$$\begin{aligned} rc_0x^r + c_0x^r &= 2x^4 \\ (r+1)\,c_0x^r &= 2x^4 \end{aligned}$$

Hence $r=4$ and $(r+1)\,c_0 = 2$ or $c_0 = \frac{2}{5}$. Then (3A) becomes

$$\sum_{n=0}^{\infty}(n+4)\,c_nx^{n+4} + \sum_{n=0}^{\infty}c_nx^{n+4} = 2x^4 \tag{3B}$$

Now that we used $n=0$ to find r by matching against the RHS which is $2x^4$, from now on, for all $n>0$ we will use (3B) to solve for all other c_n. From now on, we just need to solve with RHS zero, since there can be no more matches for any x^4 on the right. Hence (3B) becomes for $n>0$

$$(n+5)\,c_nx^{n+4} = 0 \tag{3C}$$

We see that for all $n>0$ then $(n+5)\,c_n = 0$ or $c_n = 0$ for all $n>0$. Hence y_{p_3} that corresponds to

$2x^4$ is

$$\begin{aligned} y_{p_3} &= \sum_{n=0}^{\infty} c_n x^{n+r} = \sum_{n=0}^{\infty} c_n x^{n+4} \\ &= c_0 x^4 \\ &= \frac{2}{5} x^4 \end{aligned}$$

Therefore

$$\begin{aligned} y_p &= y_{p_1} + y_{p_2} + y_{p_3} \\ &= \frac{1}{2}x + \frac{1}{4}x^3 + \frac{2}{5}x^5 \end{aligned}$$

We found y_p. Hence the solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{a_0}{x} + \frac{1}{2}x + \frac{1}{4}x^3 + \frac{2}{5}x^4 \end{aligned}$$

6.7 Example 7. $xy' + y = \frac{1}{x}$

$$xy' + y = \frac{1}{x} \tag{1}$$

This is same as example 1, but with nonzero on RHS. The solution is $y = y_h + y_p$. Where y_h was found above as

$$y_h = \frac{a_0}{x}$$

This is same as example 1, but with nonzero on RHS. The solution is $y = y_h + y_p$. Where y_h was found above as

$$y_h = \frac{a_0}{x}$$

To find y_p we assume it has power series

$$y_p = \sum_{n=0}^{\infty} c_n x^{n+r}$$

Substituting the above into the ode and simplifying as we did in first example gives EQ. 3 in the first example, but now with $\sin x$ on RHS

$$\sum_{n=0}^{\infty} (n+r) c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} = x^{-1}$$

For $n = 0$ the above becomes

$$\begin{aligned} r c_0 x^r + c_0 x^r &= x^{-1} \\ (r+1) c_0 x^r &= x^{-1} \end{aligned}$$

Hence $r = -1$ and $(r+1) c_0 = 1$ which gives $0 c_0 = 1$. So not possible to solve for c_0. Since we can not find c_0, can not find y_p. This is an example where there is *no series solution.* This ode of course can be easily solved directly which gives solution $y = \frac{c_1}{x} + \frac{1}{x} \ln x$, but not using series method.

6.8 Example 9. $xy' + y = \frac{1}{x^2}$

$$xy' + y = x^{-2} \tag{1}$$

This is same as example 1, but with nonzero on RHS. The solution is $y = y_h + y_p$. Where y_h was found above as

$$y_h = \frac{a_0}{x}$$

To find y_p we assume it has power series

$$y_p = \sum_{n=0}^{\infty} c_n x^{n+r}$$

Substituting the above into the ode and simplifying as we did in first example gives EQ. 3 in the above example, but now with x^{-2} on RHS

$$\sum_{n=0}^{\infty} (n+r) c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} = x^{-2} \tag{3A}$$

For $n = 0$

$$\begin{aligned} r c_0 x^r + c_0 x^r &= x^{-2} \\ (r+1) c_0 x^r &= x^{-2} \end{aligned}$$

For balance $r=-2$. And $(r+1)\,c_0=1$ or $c_0=-1$. Hence (3A) becomes

$$\sum_{n=0}^{\infty}(n-2)\,c_n x^{n-2}+\sum_{n=0}^{\infty}c_n x^{n-2}=x^{-2} \tag{3B}$$

The recursive relation for $n>0$ is

$$\begin{aligned}(n-2)\,c_n x^{n-2}+c_n x^{n-2}&=0\\ (n-1)\,c_n x^{n-2}&=0\end{aligned} \tag{3C}$$

We see that for all $n>0$ then $(n-1)\,c_n=0$. or $c_n=0$ for all $n>0$. Hence y_p that corresponds to x^{-2} is

$$\begin{aligned}y_p&=\sum_{n=0}^{\infty}c_n x^{n+r}\\ &=\sum_{n=0}^{\infty}c_n x^{n-2}\\ &=c_0 x^{-2}\\ &=-x^{-2}\end{aligned}$$

Hence the solution is

$$\begin{aligned}y&=y_h+y_p\\ &=\frac{a_0}{x}-\frac{1}{x^2}\end{aligned}$$

6.9 Example 10. $xy'+y=3+x$

$$xy'+y=3+x \tag{1}$$

This is same as example 1, but with nonzero on RHS. The solution is $y=y_h+y_p$. Where y_h was found above as

$$y_h=\frac{a_0}{x}$$

To find y_p we assume it has power series

$$y_p=\sum_{n=0}^{\infty}c_n x^{n+r}$$

Substituting the above into the ode and simplifying as we did in first example gives EQ. 3 in the above example, but now with $3+x$ on RHS

$$\sum_{n=0}^{\infty}(n+r)\,c_n x^{n+r}+\sum_{n=0}^{\infty}c_n x^{n+r}=3+x \tag{3A}$$

Because RHS has more than one term, we have to solve for each term on its own at a time. Looking at first term the above becomes

$$\sum_{n=0}^{\infty}(n+r)\,c_n x^{n+r}+\sum_{n=0}^{\infty}c_n x^{n+r}=3 \tag{3A}$$

For $n=0$

$$\begin{aligned}rc_0x^r+c_0x^r&=3\\ (r+1)\,c_0x^r&=3\end{aligned}$$

For balance $r=0$. And $(r+1)\,c_0=3$ or $c_0=3$. Hence (3A) becomes

$$\sum_{n=0}^{\infty}nc_n x^{n}+\sum_{n=0}^{\infty}c_n x^{n}=3 \tag{3B}$$

The recursive relation for $n>0$ is

$$\begin{aligned}nc_nx^n+c_nx^n&=0\\ (n+1)\,c_nx^n&=0\end{aligned} \tag{3C}$$

We see that for all $n>0$ then $(n+1)\,c_n=0$ or $c_n=0$ for all $n>0$. Hence y_{p_1} that corresponds to 3 is

$$\begin{aligned}y_{p_1}&=\sum_{n=0}^{\infty}c_n x^{n+r}\\ &=\sum_{n=0}^{\infty}c_n x^{n}\\ &=3\end{aligned}$$

Now we go back and process the second term on the right side, (3A) now becomes

$$\sum_{n=0}^{\infty}(n+r)\,c_n x^{n+r}+\sum_{n=0}^{\infty}c_n x^{n+r}=x \tag{3A}$$

For $n=0$

$$\begin{aligned} rc_0x^r+c_0x^r&=x\\ (r+1)\,c_0x^r&=x \end{aligned}$$

For balance $r=1$. And $(r+1)\,c_0=1$ or $c_0=\frac{1}{2}$. Hence (3A) becomes

$$\sum_{n=0}^{\infty}(n+1)\,c_n x^{n+1}+\sum_{n=0}^{\infty}c_n x^{n+1}=x \tag{3B}$$

The recursive relation for $n>0$ is

$$\begin{aligned} (n+1)\,c_nx^{n+1}+c_nx^{n+1}&=0\\ (n+2)\,c_nx^{n+1}&=0 \end{aligned} \tag{3C}$$

We see that for all $n>0$ then $(n+2)\,c_n=0$ or $c_n=0$ for all $n>0$. Hence y_{p_2} that corresponds to x is

$$\begin{aligned} y_{p_1}&=\sum_{n=0}^{\infty}c_nx^{n+r}\\ &=\sum_{n=0}^{\infty}c_nx^{n+1}\\ &=\frac{1}{2}x \end{aligned}$$

Hence the particular solution solution is

$$\begin{aligned} y_p&=y_{p_1}+y_{p_2}\\ &=3+\frac{1}{2}x \end{aligned}$$

And the complete solution is

$$\begin{aligned} y&=y_h+y_p\\ &=\frac{a_0}{x}+3+\frac{1}{2}x \end{aligned}$$

6.10 Example 11. $xy'+y=\frac{1}{x^3}$

$$xy'+y=x^{-3} \tag{1}$$

This is same as example 1, but with nonzero on RHS. The solution is $y=y_h+y_p$. Where y_h was found above as

$$y_h=\frac{a_0}{x}$$

To find y_p we assume it has power series

$$y_p=\sum_{n=0}^{\infty}c_nx^{n+r}$$

Substituting the above into the ode and simplifying as we did in first example gives EQ. 3 in the first example, but now with x^{-3} on RHS

$$\sum_{n=0}^{\infty}(n+r)\,c_n x^{n+r}+\sum_{n=0}^{\infty}c_n x^{n+r}=x^{-3} \tag{3A}$$

For $n=0$

$$\begin{aligned} rc_0x^r+c_0x^r&=x^{-3}\\ (r+1)\,c_0x^r&=x^{-3} \end{aligned}$$

For balance $r=-3$. And $(r+1)\,c_0=1$ or $c_0=-\frac{1}{2}$. Hence (3A) becomes

$$\sum_{n=0}^{\infty}(n-3)\,c_n x^{n-3}+\sum_{n=0}^{\infty}c_n x^{n-3}=x^{-3} \tag{3B}$$

The recursive relation for $n>0$ is

$$\begin{aligned} (n-3)\,c_nx^{n-3}+c_nx^{n-3}&=0\\ (n-2)\,c_nx^{n-3}&=0 \end{aligned} \tag{3C}$$

We see that for all $n>0$ then $(n-2)\,c_n=0$ or $c_n=0$ for all $n>0$. Hence y_{p_3} that corresponds to x^{-3} is

$$
\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^{n+r} = \sum_{n=0}^{\infty} c_n x^{n-3} \\
&= c_0 x^{-3} \\
&= -\frac{1}{2} x^{-3}
\end{aligned}
$$

Hence the solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \frac{a_0}{x} - \frac{1}{2x^3}
\end{aligned}
$$

6.11 Example 11. $y' + xy = x^2$

$$
y' + xy = x^2 \tag{1}
$$

Looking first at $y' + xy = 0$. Expansion around $x = 0$. This is an ordinary point. Hence standard power series will be used. let $y = \sum_{n=0}^{\infty} a_n x^n$. Substituting into the ode gives

$$
\begin{aligned}
\sum_{n=1}^{\infty} n a_n x^{n-1} + x \sum_{n=0}^{\infty} a_n x^n &= 0 \\
\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} &= 0
\end{aligned}
$$

We always want to power on x be the same in each sum and be x^n. Adjusting gives

$$
\sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 0 \tag{3A}
$$

For $n = 0$ we have

$$
a_1 x^0 = x^0
$$

Hence $a_1 = 1$. The recurrence relation is for $n > 0$. From (3A) we have

$$
\begin{aligned}
(n+1)\, a_{n+1} x^n + a_{n-1} x^n &= 0 \\
((n+1)\, a_{n+1} + a_{n-1})\, x^n &= 0 \\
(n+1)\, a_{n+1} + a_{n-1} &= 0
\end{aligned}
$$

For $n = 1$

$$
\begin{aligned}
2a_2 + a_0 &= 0 \\
a_2 &= -\frac{a_0}{2}
\end{aligned}
$$

For $n = 2$

$$
\begin{aligned}
3a_3 + a_1 &= 0 \\
a_3 &= 0
\end{aligned}
$$

For $n = 3$

$$
\begin{aligned}
4a_4 + a_2 &= 0 \\
a_4 &= -\frac{a_2}{4} \\
&= \frac{a_0}{8}
\end{aligned}
$$

And so on. Hence

$$
\begin{aligned}
y_h &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \\
&= a_0 - \frac{a_0}{2} x^2 + \frac{a_0}{8} x^4 - \frac{a_0}{48} x^6 - \cdots \\
&= a_0 \left(1 - \frac{1}{2} x^2 + \frac{1}{8} x^4 - \frac{1}{48} x^6 - \cdots \right)
\end{aligned}
$$

Now we find y_p. Let

$$
y = y_h + y_p
$$

Let us start by trying the undetermined coefficients method. We will see this will fail. Let $y_p =$

$c_2x^2 + c_1x + c_0$. Substituting this into the ode gives

$$\begin{aligned}
\left(c_2x^2 + c_1x + c_0\right)' + x\left(c_2x^2 + c_1x + c_0\right) &= x^2 \\
2c_2x + c_1 + c_2x^3 + c_1x^2 + c_0x &= x^2 \\
c_1 + x(2c_2 + c_0) + x^2(c_1) + x^3(c_2) &= x^2
\end{aligned}$$

Hence $c_1 = 1, c_2 = 0, c_1 = 0$. We see this did not work. We get both $c_1 = 0$ and $c_1 = 1$. What went wrong? The problem is that we used undetermined coefficients on an ode with non constant coefficients. And this is a no-no. Undetermined coefficients can sometimes work on ode with non constant coefficients but by chance as we see in earlier examples.

In solving an ode not the series method and if the ode have non constant coefficients, we should also not use undetermined coefficient but use the variation of parameters method which works on constant and non constant coefficients. We can not use variation of parameters here, since we are solving using series and do not have basis functions for integration.

So what to do now? How to find y_p? We use the same balance equation method as was done in earlier examples for singular point. This example was added here to show that undetermined coefficients can fail when finding y_p even on ordinary point. So we should always use balance method in series solution to find y_p as was done in all the earlier examples.

Assuming y_p is

$$\begin{aligned}
y_p &= \sum_{n=0}^{\infty} c_n x^n \\
y_p' &= \sum_{n=0}^{\infty} n c_n x^{n-1}
\end{aligned}$$

Substituting this into the ode gives

$$\begin{aligned}
\sum_{n=0}^{\infty} n c_n x^{n-1} + x \sum_{n=0}^{\infty} c_n x^n &= x^2 \\
\sum_{n=0}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^{n+1} &= x^2 \\
\sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^{n+1} &= x^2 \\
\sum_{n=0}^{\infty} (n+1)\, c_{n+1} x^n + \sum_{n=1}^{\infty} c_{n-1} x^n &= x^2
\end{aligned}$$

And the above is what will be used to obtain the c_n and y_p. For $n = 0$

$$c_1x^0 = x^2$$

No balance. Hence $c_1 = 0$. For $n = 1$

$$\begin{aligned}
2c_2x + c_0x &= x^2 \\
(2c_2 + c_0)\, x &= x^2
\end{aligned}$$

No balance. Hence $2c_2 + c_0 = 0$. Here, we are free to choose $c_0 = 0$. Therefore $c_2 = 0$. For $n = 2$

$$\begin{aligned}
3c_3x^2 + c_1x^2 &= x^2 \\
(3c_3 + c_1)\, x^2 &= x^2
\end{aligned}$$

Balance exist. Hence $3c_3 + c_1 = 1$. But $c_1 = 0$, which gives $c_3 = \frac{1}{3}$. For $n = 3$

$$4c_4x^3 + c_2x^3 = x^2$$

No balance. Hence $4c_4 + c_2 = 0$ But $c_2 = 0$ then $c_4 = 0$. For $n = 4$

$$5c_5x^4 + c_3x^4 = x^2$$

No balance. Hence $5c_5 + c_3 = 0$ or $c_5 = -\frac{1}{15}$. For $n = 5$

$$6c_6x^6 + c_4x^6 = x^2$$

No balance. Hence $6c_6 + c_4 = 0$. But $c_4 = 0$, hence $c_6 = 0$. For $n = 6$

$$7c_7x^6 + c_5x^6 = x^2$$

No balance. Hence $7c_7 + c_5 = 0$. But $c_5 = -\frac{1}{15}$, therefore $c_7 = \frac{1}{105}$ and so on. Therefore

$$\begin{aligned} y_p &= \sum_{n=0}^{\infty} c_n x^n \\ &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \\ &= 0 + 0 + 0 + \frac{1}{3}x^3 + 0 - \frac{1}{15}x^5 + 0 + \frac{1}{105}x^7 - \cdots \\ &= \frac{1}{3}x^3 - \frac{1}{15}x^5 + \frac{1}{105}x^7 - \frac{1}{945}x^9 + \cdots \end{aligned}$$

The final solution is

$$\begin{aligned} y &= y_h + y_p \\ &= a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 - \cdots\right) + \left(\frac{1}{3}x^3 - \frac{1}{15}x^5 + \frac{1}{105}x^7 - \frac{1}{945}x^9 + \cdots\right) \end{aligned}$$

In this example, we choose $c_0 = 0$. This was arbitrary choice.