Solving d'Alembert and Clairaut first order ode's

Nasser M. Abbasi

December 31, 2024 Compiled on December 31, 2024 at 12:45 Noon

Contents

1 top level algorithm

This is the top level algorithm

---

function SOLVE_FIRST_ORDER_ODE_NONLINEAR_P($F(x, y, p)$)

Where $p = y'$ and the ode is non-linear in p. An example is $x(y')^2 - yy' = -1$ and $y = x\left(y' + a\sqrt{1 + (y')^2}\right)$

if degree of p an integer in $F(x, y, p)$ **then**

As an example $p^2 x + yp + y = 0$ and it is possible to find the roots (i.e. solve for p) then let the roots be p_i and each generated ode is solved as a first order ode which is now linear in each in y_i'. So we need to solve $y_i' = f(x, y)$ for each root.

else if we can solve for x from $F(x, y, p)$ **then**

This is currently not implemented.

Let $x = \phi(y, p)$ then differentiating w.r.t. y gives

$$\begin{aligned} \frac{dx}{dy} &= \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial p}\frac{dp}{dy} \\ \frac{1}{p} &= \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial p}\frac{dp}{dy} \qquad (1) \end{aligned}$$

Solving (1) for p from the above and substituting the result in $x = \phi(y, p)$ gives the solution.

else

CALL clairaut_dAlembert_solver($F(x, y, p)$)

end if

end function

Algorithm below is Clairaut dAlembert solver algorithm

---

function CLAIRAUT_DALEMBERT_SOLVER($F(x, y, p)$)

Solve for y and write the ode as (where $p = y'$)

$$y = xf(p) + g(p) \qquad (1)$$

where $f(p) \neq 0$

if $f(p) = p$ **then** ▷ Example $y = xp + g(p)$

if $g(p) = 0$ **then** ▷ Example $y = xp$

return as this is neither Clairaut nor d'Alembert.

else if $g(p)$ is linear in p **then** ▷ Example $y = xp + p$

return as this is neither Clairaut nor d'Alembert.

else ▷ Example $y = xp + p^2$ or $y = xp + \sin(p)$

This is a Clairaut ode. Taking the derivative of (1) w.r.t. x gives

$$\begin{aligned} p &= \frac{d}{dx}(xp + g) \\ p &= \left(p + x\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\ p &= p + (x + g')\frac{dp}{dx} \\ 0 &= (x + g')\frac{dp}{dx} \end{aligned}$$

where g' is the derivative of $g(p)$ w.r.t. p. The general solution is

$$\frac{dp}{dx} = 0 \qquad p = c_1$$

where c_1 is constant. Substituting $p = c_1$ the in (1) gives the general solution y_g The singular solution y_s is now found from solving the ode $(x + g'(p)) = 0$ for p and substituting the solution p_i back in (1).

return y_g, y_s

end if

else
    CALL dalembert_solver($F(x,y,p)$)
  end if
end function

Algorithm below is just the dAlembert solver algorithm

function DALEMBERT_SOLVER($F(x,y,p)$)
  Write the ode as (where $p = y'$)

$$y = xf(p) + g(p) \tag{1}$$

  where $f(p) \neq 0$. Note that We get here when $f(p) \neq p$ else it is Clairaut.
  if $g(p) = 0$ **then** ▷ Example $y = xf(p)$
    $f(p)$ must be nonlinear in p but can not be the special case $p^{\frac{1}{n}}$ where n integer because then it is separable.
    if $f(p) = p^{\frac{1}{n}}$ and $n \in \mathbb{Z}$ **then** ▷ Ex. $y = x(y')^{\frac{1}{2}}$
      return as this is not dAlmbert ode.
    end if
  else
    In this case any form of $f(p)$ is OK even $f(p) = p^{\frac{1}{n}}$ with n integer except ofcourse $f(p) = p$ since this would have made it Clairaut and not dAlembert. Example is $y = xf(p) + p$ is dAlembert.
    if $g(p)$ is constant and does not depend on p **then** ▷ Ex. $y = xf(p) + 1$
      return as this is not dAlmbert ode.
    else
      if $g(p) = f(p)$ **then**
        if $g(p), f(p)$ have the form $p^{\frac{1}{n}}$ with n integer **then** ▷ Ex. $y = xp^{\frac{1}{2}} + p^{\frac{1}{2}}$
          return as this is not dAlmbert ode.
        else ▷ Ex. $y = xp^{\frac{2}{3}} + p^{\frac{2}{3}}$ or $y = xp^2 + p^2$
          This is dAlmbert ode.
        end if
      end if
    end if
  end if
  When we get here then (1) is dAlmbert ode. Note that all the above cases $f(p), g(p)$ can not be function of x in any case. Now we solve (1) using dAlmbert algorithm. Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xf + g) \\
p &= \left(f + xf' \frac{dp}{dx}\right) + \left(g' \frac{dp}{dx}\right)
\end{aligned}$$

  where f' means $\frac{df}{dp}$ and g' means $\frac{dg}{dp}$. The above becomes

$$\begin{aligned}
p &= f + (xf' + g') \frac{dp}{dx} \\
p - f &= (xf' + g') \frac{dp}{dx} \qquad (2)
\end{aligned}$$

  The singular solution is given when $\frac{dp}{dx} = 0$ above. Hence

$$p - f = 0$$

  Solving the above for p and substituting the result back in (1) gives the singular solution y_s. The general solution y_g is found by solving the ode in (2) for p and substituting the result in (1). there are two cases to consider.
  if ode (2) is separable or linear in p as is **then**
    Solve (2) for p directly and substitute the solution in (1). This gives the general solution y_g.
  else

Inverting (2) first gives

$$\frac{dx}{dp} = \frac{xf' + g'}{p - f}$$

Which makes it linear ode in x. This is solved for $x(p)$ as function of p. Let

$$x = h(p) + c_1 \tag{3}$$

be the solution. Now two possible cases exist

if able to isolate p from (3) **then**

Substitute p in (1). This gives the general solution y_g.

else

Solve for p from (1) and substitute the result in (3). This gives an implicit solution for y_g instead of explicit one.

end if

end if

end function

2 Solved examples

2.1 Algorithm diagram

The following is the flow chart.

Figure 1: Algorithm for solving first order ode with nonlinear y'

2.2 Solved examples

#	original ode	$y = xf(p) + g(p)$	$f(p)$	$g(p)$	type
1	$x(y')^2 - yy' = -1$	$y = xp + \frac{1}{p}$	p	$\frac{1}{p}$	Clairaut
2	$y = xy' - (y')^2$	$y = xp - p^2$	p	$-p^2$	Clairaut
3	$y = xy' - \frac{1}{4}(y')^2$	$y = xp - \frac{1}{4}p^2$	p	$-\frac{1}{4}p^2$	Clairaut
4	$y = x(y')^2$	$y = xp^2$	p^2	0	d’Alembe
5	$y = x + (y')^2$	$y = x + p^2$	1	p^2	d’Alembe
6	$(y')^2 - 1 - x - y = 0$	$y = -x + (p^2 - 1)$	-1	$(p^2 - 1)$	d’Alembe
7	$yy' - (y')^2 = x$	$y = \frac{1}{p}x + p$	$\frac{1}{p}$	p	d’Alembe
8	$y = x(y')^2 + (y')^2$	$y = xp^2 + p^2$	p^2	p^2	d’Alembe
9	$y = \frac{x}{a}y' + \frac{b}{ay'}$	$y = \frac{x}{a}p + \frac{b}{a}\frac{1}{p}$	$\frac{p}{a}$	$\frac{b}{a}\frac{1}{p}$	d’Alembe
10	$y = x\left(y' + a\sqrt{1 + (y')^2}\right)$	$y = x\left(p + a\sqrt{1 + p^2}\right)$	$p + a\sqrt{1 + p^2}$	0	d’Alembe
11	$y = x + (y')^2\left(1 - \frac{2}{3}y'\right)$	$y = x + p^2\left(1 - \frac{2}{3}p\right)$	1	$p^2\left(1 - \frac{2}{3}p\right)$	d’Alembe
12	$y = 2x - \frac{1}{2}\ln\left(\frac{(y')^2}{y'-1}\right)$	$y = 2x - \frac{1}{2}\ln\left(\frac{p^2}{p-1}\right)$	2	$-\frac{1}{2}\ln\left(\frac{p^2}{p-1}\right)$	d’Alembe
13	$(y')^2 - x(y')^2 + y(1 + y') - xy' = 0$	$y = \frac{xp + xp^2 - p^2}{p+1} = xp - \frac{p^2}{p+1}$	p	$-\frac{p^2}{p+1}$	Clairaut
14	$x(y')^2 + (x - y)\,y' + 1 - y = 0$	$y = xp + \frac{1}{1+p}$	p	$\frac{1}{1+p}$	Clairaut
15	$xyy' = y^2 + x\sqrt{4x^2 + y^2}$	$y = \operatorname{RootOf}(h(p))\,x$	$\operatorname{RootOf}(h(p))$	0	d’Alembe
16	$\ln(\cos y') + y'\tan y' = y$	$y = \ln(\cos p) + p\tan p$	0	$\ln(\cos p) + p\tan p$	d’Alembe
17	$x(y')^2 - 2yy' + 4x = 0$	$y = x\left(\frac{1}{2}p + \frac{2}{p}\right)$	$\frac{1}{2}p + \frac{2}{p}$	0	d’Alembe
18	$x - yy' = a(y')^2$	$y = \frac{x}{p} - ap$	$\frac{1}{p}$	$-ap$	d’Alembe
19	$y = xF(p) + G(p)$	$y = xF(p) + G(p)$	$F(p)$	$G(p)$	d’Alembe
20	$y' = -\frac{x}{2} - 1 + \frac{1}{2}\sqrt{x^2 + 4x + 4y}$	$y = xp + (1 + 2p + p^2)$	p	$1 + 2p + p^2$	Clairaut
21	$\frac{y'y}{1+\frac{1}{2}\sqrt{1+(y')^2}} = -x$	$y = -x\left(\frac{2+\sqrt{1+p^2}}{2p}\right)$	$-\left(\frac{2+\sqrt{1+p^2}}{2p}\right)$	0	d’Alembe
22	$x(y')^3 = yy' + 1$	$y = xp^2 - \frac{1}{p}$	p^2	$-\frac{1}{p}$	d’Alembe
23	$(y')^2 - 2yy' = 2x$	$y = -x\frac{1}{p} + \frac{1}{2}p$	$-\frac{1}{p}$	$\frac{1}{2}p$	d’Alembe
24	$xy' - y = \sqrt{x^2 - y^2}$	$y = x\left(\frac{p}{2} \pm \frac{1}{2}\sqrt{2 - p^2}\right)$	$\frac{p}{2} \pm \frac{1}{2}\sqrt{2 - p^2}$	0	d’Alembe

2.2.1 Example 1

$x(y')^2 - yy' = -1$, is put in normal form (by replacing y' with p) and solving for y gives

$$
\begin{aligned}
y &= xp + \frac{1}{p} \qquad (1)\\
&= xf(p) + g(p)
\end{aligned}
$$

Where $f(p) = p$ and $g(p) = \frac{1}{p}$. Since $f(p) = p$ then this is Clairaut ode. Taking derivative of the above w.r.t. x gives

$$
\begin{aligned}
p &= \frac{d}{dx}(xp + g(p))\\
p &= p + (x + g'(p))\frac{dp}{dx}\\
0 &= (x + g'(p))\frac{dp}{dx}
\end{aligned}
$$

The general solution is given by

$$
\begin{aligned}
\frac{dp}{dx} &= 0\\
p &= c_1
\end{aligned}
$$

Substituting this in (1) gives the general solution

$$y = c_1 x + \frac{1}{c_1}$$

The term $(x + g'(p)) = 0$ is used to find singular solutions.

$$\begin{aligned} x + g'(p) &= x + \frac{d}{dp}\frac{1}{p} \\ &= x - \frac{1}{p^2} \end{aligned}$$

Hence $x - \frac{1}{p^2} = 0$ or $p = \pm\frac{1}{\sqrt{x}}$. Substituting these back in (1) gives

$$\begin{aligned} y_1(x) &= xp + \frac{1}{p} \\ &= x\frac{1}{\sqrt{x}} + \sqrt{x} \\ &= 2\sqrt{x} \qquad (3) \\ y_2(x) &= -x\sqrt{\frac{1}{x}} - \sqrt{x} \\ &= -2\sqrt{x} \qquad (4) \end{aligned}$$

Eq. (2) is the general solution and (3,4) are the singular solutions.

Another method to find the singular solutions if it exists is called the p-discriminant. This is used only for first order ode with nonlinear in y'. We set up the following two equations

$$\begin{aligned} F(x, y, y') &= 0 \\ \frac{\partial F(x, y, y')}{\partial y'} &= 0 \end{aligned}$$

We eliminate y' and obtain $G(x, y) = 0$ equation. This is the singular solution. But we still have to check if it satisfies the ode and also if it is true singular solution curve. More on this later. Let us now just find the singular solution found above but using the p-discriminant method. The above two equations are

$$\begin{aligned} y - xy' - \frac{1}{y'} &= 0 \\ -x + \frac{1}{(y')^2} &= 0 \end{aligned}$$

Second equation gives $(y')^2 = \frac{1}{x}$. Hence $y' = \pm\sqrt{\frac{1}{x}}$. Hence the first equation now gives (starting with positive root)

$$y - x\sqrt{\frac{1}{x}} - \frac{1}{\sqrt{\frac{1}{x}}} = 0$$

$$\begin{aligned} y &= x\sqrt{\frac{1}{x}} + \frac{1}{\sqrt{\frac{1}{x}}} \\ &= \frac{x\sqrt{\frac{1}{x}}\sqrt{\frac{1}{x}} + 1}{\sqrt{\frac{1}{x}}} \\ &= 2\sqrt{x} \end{aligned}$$

And for the second root $y' = -\sqrt{\frac{1}{x}}$ we obtain $y = -2\sqrt{x}$. We see these are the same singular solutions obtained earlier.

2.2.2 Example 2

$y = xy' - (y')^2$ is put in normal form (by replacing y' with p) and solving for y gives

$$\begin{aligned} y &= xp - p^2 \\ &= xf(p) + g(p) \end{aligned} \tag{1}$$

Where $f(p) = p$ and $g(p) = -p^2$. Since $f(p) = p$ then this is Clairaut ode. Taking derivative of the above w.r.t. x gives

$$\begin{aligned} p &= \frac{d}{dx}(xp + g(p)) \\ p &= p + (x + g'(p))\frac{dp}{dx} \\ 0 &= (x + g'(p))\frac{dp}{dx} \end{aligned}$$

The general solution is given by

$$\begin{aligned} \frac{dp}{dx} &= 0 \\ p &= c_1 \end{aligned}$$

Substituting this in (1) gives the general solution

$$y = c_1 x - c_1^2$$

The term $(x + g'(p)) = 0$ is used to find singular solutions.

$$\begin{aligned} x + g'(p) &= x + \frac{d}{dp}(-p^2) \\ &= x + 2p \end{aligned}$$

Hence $x + 2p = 0$ or $p = \frac{x}{2}$. Substituting this back in (1) gives

$$\begin{aligned} y(x) &= \frac{x^2}{2} - \frac{x^2}{4} \\ &= \frac{x^2}{4} \end{aligned} \tag{3}$$

Eq. (2) is the general solution and (3) is the singular solution.

Another method to find the singular solutions if it exists is called the p-discriminant. This is used only for first order ode with nonlinear in y'. We set up the following two equations

$$\begin{aligned} F(x, y, y') &= 0 \\ \frac{\partial F(x, y, y')}{\partial y'} &= 0 \end{aligned}$$

We eliminate y' and obtain $G(x, y) = 0$ equation. This is the singular solution. But we still have to check if it satisfies the ode and also if it is true singular solution curve. More on this later. Let us now just find the singular solution found above but using the p-discriminant method. The above two equations are

$$\begin{aligned} y - xy' + (y')^2 &= 0 \\ -x + 2y' &= 0 \end{aligned}$$

Second equation gives $y' = \frac{x}{2}$. Hence the first equation now gives the singular solution as

$$\begin{aligned} y - x\left(\frac{x}{2}\right) + \left(\frac{x}{2}\right)^2 &= 0 \\ y &= \frac{x^2}{2} - \frac{x^2}{4} \\ &= \frac{1}{4}x^2 \end{aligned}$$

Which is the same obtained earlier.

2.2.3 Example 3

$y = xy' - \frac{1}{4}(y')^2$ is put in normal form (by replacing y' with p) and solving for y gives

$$\begin{aligned} y &= xp - \frac{1}{4}p^2 \\ &= xf(p) + g(p) \end{aligned} \tag{1}$$

Where $f(p) = p$ and $g(p) = -\frac{1}{4}p^2$. Since $f(p) = p$ then this is Clairaut ode. Taking derivative of the above w.r.t. x gives

$$\begin{aligned} p &= \frac{d}{dx}(xp + g(p)) \\ p &= p + (x + g'(p))\frac{dp}{dx} \\ 0 &= (x + g'(p))\frac{dp}{dx} \end{aligned}$$

The general solution is given by

$$\begin{aligned} \frac{dp}{dx} &= 0 \\ p &= c_1 \end{aligned}$$

Substituting this in (1) gives the general solution

$$y = c_1 x - \frac{1}{4}c_1^2$$

The term $(x + g'(p)) = 0$ is used to find singular solutions.

$$\begin{aligned} x + g'(p) &= x + \frac{d}{dp}\left(-\frac{1}{4}p^2\right) \\ &= x - \frac{1}{2}p \end{aligned}$$

Hence $x - \frac{1}{2}p = 0$ or $p = 2x$. Substituting this back in (1) gives

$$\begin{aligned} y(x) &= 2x^2 - x^2 \\ &= x^2 \end{aligned} \tag{3}$$

Eq. (2) is the general solution and (3) is the singular solution.

Another method to find the singular solutions if it exists is called the p-discriminant. This is used only for first order ode with nonlinear in y'. We set up the following two equations

$$\begin{aligned} F(x, y, y') &= 0 \\ \frac{\partial F(x, y, y')}{\partial y'} &= 0 \end{aligned}$$

We eliminate y' and obtain $G(x, y) = 0$ equation. This is the singular solution. But we still have to check if it satisfies the ode and also if it is true singular solution curve. More on this later. Let us now just find the singular solution found above but using the p-discriminant method. The above two equations are

$$\begin{aligned} y - xy' + \frac{1}{4}(y')^2 &= 0 \\ -x + \frac{1}{2}y' &= 0 \end{aligned}$$

Second equation gives $y' = 2x$. Hence the first equation now gives the singular solution as

$$\begin{aligned} y - 2x^2 + \frac{1}{4}(4x^2) &= 0 \\ y - x^2 &= 0 \\ y &= x^2 \end{aligned}$$

Which is the same obtained earlier.

2.2.4 Example 4

$y = x(y')^2$ is put in normal form (by replacing y' with p) and solving for y gives

$$\begin{aligned} y &= xp^2 \\ &= xf(p) \end{aligned} \tag{1}$$

This is the case when $f(p) = p^2$ and $g(p) = 0$. Since $f(p) \neq p$ then this is d'Almbert ode.

Writing $f \equiv f(p)$ and $g \equiv g(p)$ to make notation simpler but remembering that f is function of $p(x)$ which in turn is function of x. Same for $g(p)$.

$$y = xf$$

Taking derivative of the above w.r.t. x gives

$$\begin{aligned} p &= \frac{d}{dx}(xf) \\ p &= f + xf'\frac{dp}{dx} \\ p - f &= xf'\frac{dp}{dx} \end{aligned}$$

Since $f = p^2$ then the above becomes

$$p - p^2 = 2xp\frac{dp}{dx} \tag{2}$$

The singular solution is given when $\frac{dp}{dx} = 0$ or $p - p^2 = 0$. This gives $p = 0$ or $p = 1$. Substituting these values of p in (1) gives singular solutions

$$y_{s1} = 0 \tag{3}$$

$$y_{s2} = x \tag{4}$$

General solution is found when $\frac{dp}{dx} \neq 0$. Eq(2) is a first order ode in p. Now we could either solve ode (2) directly as it is for $p(x)$, or do an inversion and solve for $x(p)$. If the ode is linear as is in p then no need to do inversion. Since (2) is separable as is, no need to do an inversion. The solution to (2) is

$$\begin{aligned} p_1 &= 0 \\ p_2 &= 1 + \frac{c_1}{\sqrt{x}} \end{aligned}$$

For each p, there is a general solution. Substituting each of the above in (1) gives

$$\begin{aligned} y_1(x) &= 0 \\ y_2(x) &= x\left(1 + \frac{c_1}{\sqrt{x}}\right)^2 \end{aligned}$$

Hence the final solutions are

$$\begin{aligned} y &= x \qquad \text{(singular)} \\ y &= 0 \\ y &= x\left(1 + \frac{c_1}{\sqrt{x}}\right)^2 \end{aligned}$$

But $y = x$ can be obtained from the general solution when $c_1 = 0$. Hence it is removed. Therefore the final solutions are

$$y = 0 \tag{6}$$

$$y = x\left(1 + \frac{c_1}{\sqrt{x}}\right)^2 \tag{7}$$

What will happen if we had done an inversion to $x(p)$? Let us find out. ode(5) now becomes

$$
\begin{aligned}
\frac{p-p^2}{p}\frac{dx}{dp} &= 2x \\
\frac{dx}{2x} &= \frac{p}{p-p^2}dp
\end{aligned}
$$

This is also separable in x. Solving this for x gives

$$
x = \frac{c_1}{(p-1)^2}
$$

Solving for p from the above gives

$$
\begin{aligned}
p_1 &= \frac{x+\sqrt{xc_1}}{x} \\
p_2 &= \frac{x-\sqrt{xc_1}}{x}
\end{aligned}
$$

Substituting each of the above in (1) gives

$$
\begin{aligned}
y_1 &= x\left(\frac{x+\sqrt{xc_1}}{x}\right)^2 \\
&= \frac{\left(x+\sqrt{xc_1}\right)^2}{x} \\
y_2 &= x\left(\frac{x-\sqrt{xc_1}}{x}\right)^2 \\
&= \frac{\left(x-\sqrt{xc_1}\right)^2}{x}
\end{aligned}
$$

Now we see that singular solution $y = x$ can be obtained from the above general solutions from $c_1 = 0$. But $y = 0$ can not. Hence the final solutions are

$$
\begin{aligned}
y &= 0 \qquad \text{(singular)} && (8) \\
y &= \frac{\left(x+\sqrt{xc_1}\right)^2}{x} && (9) \\
y &= \frac{\left(x-\sqrt{xc_1}\right)^2}{x} && (10)
\end{aligned}
$$

All solutions (6,7,8,9,10) are correct and verified. Maple gives the solutions given in (8,9,10) and not those in (6,7).

Another method to find the singular solutions if it exists is called the p-discriminant. This is used only for first order ode with nonlinear in y'. We set up the following two equations

$$
\begin{aligned}
F(x,y,y') &= 0 \\
\frac{\partial F(x,y,y')}{\partial y'} &= 0
\end{aligned}
$$

We eliminate y' and obtain $G(x,y) = 0$ equation. This is the singular solution. But we still have to check if it satisfies the ode and also if it is true singular solution curve. More on this later. Let us now just find the singular solution found above but using the p-discriminant method. The above two equations are

$$
\begin{aligned}
y - x(y')^2 &= 0 \\
-2xy' &= 0
\end{aligned}
$$

Second equation gives $y' = 0$. Hence the first equation now gives the singular solution as

$$
y = 0
$$

Which is the same obtained earlier.

2.2.5 Example 5

$y = x + (y')^2$ is put in normal form (by replacing y' with p) which gives

$$\begin{aligned} y &= x + p^2 \\ &= xf + g \end{aligned} \tag{1}$$

Hence $f(p) = 1, g(p) = p^2$. Since $f(p) \neq p$ then this is d'Almbert ode. Taking derivative w.r.t. x gives

$$\begin{aligned} p &= \left(f + xf'\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\ p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Using $f = 1, g = p^2$ the above simplifies to

$$p - 1 = 2p\frac{dp}{dx} \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in (2) which results in $p - f = 0$ or $p - 1 = 0$. Hence $p = 1$. Substituting these values of p in (1) gives singular solution as

$$y = x + 1 \tag{3}$$

General solution is found when $\frac{dp}{dx} \neq 0$. Eq (2A) is a first order ode in p. Now we could either solve ode (2) directly as it is for $p(x)$, or do an inversion and solve for $x(p)$. Since (2) is separable as is, no need to do an inversion. Solving (2) for p gives

$$p = \text{LambertW}\left(c_1 e^{\frac{x}{2}-1}\right) + 1$$

Substituting this in (1) gives the general solution

$$y(x) = x + \left(\text{LambertW}\left(c_1 e^{\frac{x}{2}-1}\right) + 1\right)^2 \tag{4}$$

Note however that when $c_1 = 0$ then the general solution becomes $y(x) = x + 1$. Hence (3) is a particular solution and not a singular solution. (4) is the only solution.

2.2.6 Example 6

$(y')^2 - 1 - x - y = 0$ is put in normal form (by replacing y' with p) which gives

$$\begin{aligned} y &= -x + (p^2 - 1) \\ &= xf + g \end{aligned} \tag{1}$$

Hence $f = -1, g(p) = (p^2 - 1)$. Since $f(p) \neq p$ then this is d'Almbert ode. Taking derivative w.r.t. x gives

$$\begin{aligned} p &= \left(f + xf'\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\ p &= f + (xf' + g')\frac{dp}{dx} \\ p - f &= (xf' + g')\frac{dp}{dx} \end{aligned} \tag{2}$$

Using $f = -1, g = (p^2 - 1)$ the above simplifies to

$$p + 1 = 2p\frac{dp}{dx} \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ which results in $p = -1$. Substituting this in (1) gives singular solution as

$$y(x) = -x \tag{3}$$

The general solution is found by finding p from (2A). No need here to do the inversion as (2) is separable already. Solving (2) gives

$$\begin{aligned}
p &= -\operatorname{LambertW}\left(-e^{-\frac{x}{2}-1+\frac{c_2}{2}}\right) - 1 \\
&= -\operatorname{LambertW}\left(-c_1 e^{-\frac{x}{2}-1}\right) - 1
\end{aligned}$$

Substituting the above in (1) gives the general solution

$$\begin{aligned}
y(x) &= -x + \left(p^2 - 1\right) \\
y(x) &= -x + \left(-\operatorname{LambertW}\left(-c_1 e^{-\frac{x}{2}-1}\right) - 1\right)^2 - 1
\end{aligned} \tag{4}$$

Note however that when $c_1 = 0$ then the general solution becomes $y(x) = -x$. Hence (3) is a particular solution and not a singular solution. Solution (4) is therefore the only solution.

2.2.7 Example 7

$yy' - (y')^2 = x$ is put in normal form (by replacing y' with p) which gives

$$\begin{aligned}
y &= \frac{x + p^2}{p} \\
&= \frac{1}{p}x + p \\
&= xf + g
\end{aligned} \tag{1}$$

Hence $f = \frac{1}{p}, g(p) = p$. Taking derivative w.r.t. x gives

$$\begin{aligned}
p &= \left(f + xf'\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= f + (xf' + g')\frac{dp}{dx} \\
p - f &= (xf' + g')\frac{dp}{dx}
\end{aligned}$$

Using $f = \frac{1}{p}, g = p$. Since $f(p) \neq p$ then this is d'Almbert ode. the above simplifies to

$$p - \frac{1}{p} = \left(-\frac{x}{p^2} + 1\right)\frac{dp}{dx} \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ in (2) which results in $Q(p) = 0$ or $p - 1 = 0$ or $p = 1$. Substituting these values in (1) gives the solutions

$$y_1(x) = x + 1 \tag{3}$$

The general solution is found by finding p from (2A). Since (2A) is not linear and not separable in p, then inversion is needed. Writing (2) as

$$\begin{aligned}
\frac{dx}{dp} &= \frac{1 - \frac{x}{p^2}}{p - \frac{1}{p}} \\
&= \frac{1}{p - p^3}\left(x - p^2\right)
\end{aligned}$$

Hence

$$\frac{dx}{dp} + \frac{x}{p(p^2-1)} = \frac{p^2}{p(p^2-1)}$$

This is now linear ODE in $x(p)$. The solution is

$$
\begin{aligned}
x &= \frac{p\sqrt{(p-1)(1+p)}\ln\left(p+\sqrt{p^2-1}\right)}{(1+p)(p-1)} + c_1 \frac{p}{\sqrt{(1+p)(p-1)}} \\
&= \frac{p\sqrt{p^2-1}\ln\left(p+\sqrt{p^2-1}\right)}{p^2-1} + c_1 \frac{p}{\sqrt{p^2-1}} \qquad (4)
\end{aligned}
$$

Now we need to eliminate p from (1,4). From (1) since $y=\frac{1}{p}x+p$ then solving for p gives

$$
\begin{aligned}
p_1 &= \frac{y}{2} + \frac{1}{2}\sqrt{y^2-4x} \\
p_2 &= \frac{y}{2} - \frac{1}{2}\sqrt{y^2-4x}
\end{aligned}
$$

Substituting each p_i in (4) gives the general solution (implicit) in $y(x)$. First solution is

$$
x = \frac{\left(\frac{y}{2}+\frac{1}{2}\sqrt{y^2-4x}\right)\sqrt{\left(\frac{y}{2}+\frac{1}{2}\sqrt{y^2-4x}\right)^2-1}\ln\left(\frac{y}{2}+\frac{1}{2}\sqrt{y^2-4x}+\sqrt{\left(\frac{y}{2}+\frac{1}{2}\sqrt{y^2-4x}\right)^2-1}\right)}{\left(\frac{y}{2}+\frac{1}{2}\sqrt{y^2-4x}\right)^2-1} + c_1 \frac{}{\sqrt{\left(\frac{y}{2}\right.}}
$$

And second solution is

$$
x = \frac{\left(\frac{y}{2}-\frac{1}{2}\sqrt{y^2-4x}\right)\sqrt{\left(\frac{y}{2}-\frac{1}{2}\sqrt{y^2-4x}\right)^2-1}\ln\left(\frac{y}{2}-\frac{1}{2}\sqrt{y^2-4x}+\sqrt{\left(\frac{y}{2}-\frac{1}{2}\sqrt{y^2-4x}\right)^2-1}\right)}{\left(\frac{y}{2}-\frac{1}{2}\sqrt{y^2-4x}\right)^2-1} + c_1 \frac{}{\sqrt{\left(\frac{y}{2}\right.}}
$$

2.2.8 Example 8

$y=x(y')^2+(y')^2$ is put in normal form (by replacing y' with p) which gives

$$
\begin{aligned}
y &= xp^2+p^2 \qquad (1)\\
&= xf+g
\end{aligned}
$$

where $f=p^2, g=p^2$. Since $f(p)\neq p$ then this is d'Almbert ode. Taking derivative and simplifying gives

$$
\begin{aligned}
p &= \left(f+xf'\frac{dp}{dx}\right)+\left(g'\frac{dp}{dx}\right) \\
p &= f+(xf'+g')\frac{dp}{dx} \\
p-f &= (xf'+g')\frac{dp}{dx}
\end{aligned}
$$

Using values for f,g the above simplifies to

$$
p-p^2 = (2xp+2p)\frac{dp}{dx} \qquad (2A)
$$

The singular solution is found by setting $\frac{dp}{dx}=0$ which results in $p=0$ or $p=1$. Substituting these values in (1) gives the singular solutions

$$
\begin{aligned}
y_1(x) &= 0 \qquad (3)\\
y_2(x) &= x+1 \qquad (4)
\end{aligned}
$$

The general solution is found by finding p from (2A). Since (2A) is not linear in p, then inversion is needed. Writing (A2) as

$$
\frac{p(1-p)}{2p(x+1)} = \frac{dp}{dx}
$$

Inverting gives

$$
\begin{aligned}
\frac{dx}{dp} &= \frac{2(x+1)}{(1-p)} \\
\frac{dx}{dp} - x\frac{2}{(1-p)} &= \frac{2}{(1-p)}
\end{aligned}
$$

This is now linear $x(p)$. The solution is

$$
x = \frac{C^2}{(p-1)^2} - 1
$$

Solving for p gives

$$
\begin{aligned}
\frac{C^2}{(p-1)^2} &= x+1 \\
(p-1)^2 &= \frac{C^2}{x+1} \\
(p-1) &= \pm\frac{C}{\sqrt{x+1}} \\
p &= 1 \pm \frac{C}{\sqrt{x+1}}
\end{aligned}
$$

Substituting the above in (1) gives the general solutions

$$
y = (x+1)\,p^2
$$

Therefore

$$
\begin{aligned}
y(x) &= (x+1)\left(1+\frac{C}{\sqrt{x+1}}\right)^2 \\
y(x) &= (x+1)\left(1-\frac{C}{\sqrt{x+1}}\right)^2
\end{aligned}
$$

The solution $y_1(x) = 0$ found earlier can not be obtained from the above general solution hence it is singular solution. But $y_2(x) = x+1$ can be obtained from the general solution when $C = 0$. Hence there are only three solutions, they are

$$
\begin{aligned}
y_1(x) &= 0 \\
y_2(x) &= (x+1)\left(1+\frac{C}{\sqrt{x+1}}\right)^2 \\
y_3(x) &= (x+1)\left(1-\frac{C}{\sqrt{x+1}}\right)^2
\end{aligned}
$$

2.2.9 Example 9

$y = \frac{x}{a}y' + \frac{b}{ay'}$ is put in normal form (by replacing y' with p) which gives

$$
\begin{aligned}
y &= \frac{x}{a}p + \frac{b}{a}p^{-1} \qquad (1)\\
&= xf + g
\end{aligned}
$$

Where $f = \frac{p}{a}, g = \frac{b}{a}p^{-1}$. Since $f(p) \neq p$ then this is d'Almbert ode. Taking derivative w.r.t. x gives

$$
\begin{aligned}
p &= \left(f + xf'\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= f + (xf' + g')\frac{dp}{dx} \\
p - f &= (xf' + g')\frac{dp}{dx}
\end{aligned}
$$

Using values for f, g the above simplifies to

$$p-\frac{p}{a}=\left(\frac{x}{a}-\frac{b}{a}p^{-2}\right)\frac{dp}{dx} \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx}=0$ which results in $p=0$. Substituting this in (1) does not generate any solutions due to division by zero. Hence no singular solution exist.

The general solution is found by finding p from (2A). Since (2A) is not linear in p, then inversion is needed. Writing (2A) as

$$\frac{p\left(1-\frac{1}{a}\right)}{\frac{x}{a}-\frac{b}{a}p^{-2}}=\frac{dp}{dx}$$

Since this is nonlinear, then inversion is needed

$$\begin{aligned}\frac{dx}{dp}&=\frac{\frac{x}{a}-\frac{b}{a}p^{-2}}{p\left(1-\frac{1}{a}\right)}\\ \frac{dx}{dp}-x\frac{1}{p\left(a-1\right)}&=-\frac{b}{a}\frac{1}{p^{3}\left(1-\frac{1}{a}\right)}\end{aligned}$$

This is now linear ode in $x(p)$. The solution is

$$x=\frac{b}{(2a-1)p^2}+C_1p^{\frac{1}{a-1}} \tag{3}$$

There are now two choices to take. The first is by solving for p from the above in terms of x and then substituting the result in (1) to obtain explicit solution for $y(x)$, and the second choice is by solving for p algebraically from (1) and substituting the result in (3). The second choice is easier in this case but gives an implicit solution. Solving for p from (1) gives

$$\begin{aligned}p_1&=\frac{ay+\sqrt{a^2y^2-4xb}}{2x}\\ p_1&=\frac{ay-\sqrt{a^2y^2-4xb}}{2x}\end{aligned}$$

Substituting each one of these solutions back in (3) gives two implicit solutions

$$\begin{aligned}x&=\frac{b}{(2a-1)\left(\frac{ay+\sqrt{a^2y^2-4xb}}{2x}\right)^2}+C_1\left(\frac{ay+\sqrt{a^2y^2-4xb}}{2x}\right)^{\frac{1}{a-1}}\\ x&=\frac{b}{(2a-1)\left(\frac{ay-\sqrt{a^2y^2-4xb}}{2x}\right)^2}+C_1\left(\frac{ay-\sqrt{a^2y^2-4xb}}{2x}\right)^{\frac{1}{a-1}}\end{aligned}$$

2.2.10 Example 10

$y=xy'+ax\sqrt{1+(y')^2}$ is put in normal form (by replacing y' with p) which gives

$$\begin{aligned}y&=x\left(p+a\sqrt{1+p^2}\right) &(1)\\ &=xf\end{aligned}$$

where $f=p+a\sqrt{1+p^2}, g=0$. Since $f(p)\neq p$ then this is d'Almbert ode. Taking derivative and simplifying gives

$$\begin{aligned}p&=\left(f+xf'\frac{dp}{dx}\right)\\ p-f&=xf'\frac{dp}{dx}\end{aligned}$$

Using values for f, g the above simplifies to

$$-a\sqrt{1+p^2} = x\left(1+\frac{ap}{\sqrt{1+p^2}}\right)\frac{dp}{dx} \tag{2A}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ which results in $-a\sqrt{1+p^2} = 0$. This gives no real solution for p. Hence no singular solution exists.

The general solution is when $\frac{dp}{dx} \neq 0$ in (2A). Since (2A) is nonlinear, inversion is needed.

$$\begin{aligned}
\frac{-a\sqrt{1+p^2}}{x+\frac{1}{2}x\frac{2ap}{\sqrt{1+p^2}}} &= \frac{dp}{dx} \\
\frac{dx}{dp} &= \frac{x\left(1+\frac{1}{2}\frac{2ap}{\sqrt{1+p^2}}\right)}{-a\sqrt{1+p^2}} \\
\frac{dx}{x} &= \frac{1+\frac{1}{2}\frac{2ap}{\sqrt{1+p^2}}}{-a\sqrt{1+p^2}}dp \\
\frac{dx}{x} &= \frac{\sqrt{1+p^2}+\frac{1}{2}2ap}{-a\left(1+p^2\right)}dp \\
\frac{dx}{x} &= \left(-\frac{1}{a\sqrt{1+p^2}}-\frac{p}{(1+p^2)}\right)dp
\end{aligned}$$

Integrating gives

$$\ln x(p) = -\frac{1}{2}\ln\left(p^2+1\right)-\frac{1}{a}\operatorname{arcsinh}(p)$$

Therefore

$$x = c_1\frac{-e^{-\frac{1}{a}(\operatorname{arcsinh}(p))}}{\sqrt{p^2+1}} \tag{3}$$

There are now two choices to take. The first is by solving for p from the above in terms of x and substituting the result in (1) to obtain explicit solution for $y(x)$, and the second choice is by solving for p algebraically from (1) and substituting the result in (3). The second choice is easier in this case but gives an implicit solution. Solving for p from (1) gives

$$\begin{aligned}
p_1 &= -\frac{1}{x}\frac{ay+\sqrt{-a^2x^2+x^2+y^2}a-y}{a^2-1} \\
p_2 &= \frac{1}{x}\frac{-ay+\sqrt{-a^2x^2+x^2+y^2}a-y}{a^2-1}
\end{aligned}$$

Substituting each one of these solutions back in (3) gives two implicit solutions

$$\begin{aligned}
x &= c_1\frac{-e^{-\frac{1}{a}\left(\operatorname{arcsinh}\left(-\frac{1}{x}\frac{ay+\sqrt{-a^2x^2+x^2+y^2}a-y}{a^2-1}\right)\right)}}{\sqrt{\left(-\frac{1}{x}\frac{ay+\sqrt{-a^2x^2+x^2+y^2}a-y}{a^2-1}\right)^2+1}} \\
x &= c_1\frac{-e^{-\frac{1}{a}\left(\operatorname{arcsinh}\left(\frac{1}{x}\frac{-ay+\sqrt{-a^2x^2+x^2+y^2}a-y}{a^2-1}\right)\right)}}{\sqrt{\left(\frac{1}{x}\frac{-ay+\sqrt{-a^2x^2+x^2+y^2}a-y}{a^2-1}\right)^2+1}}
\end{aligned}$$

2.2.11 Example 11

$$
\begin{aligned}
y &= x + (y')^2\left(1-\frac{2}{3}y'\right)\\
&= x + p^2\left(1-\frac{2}{3}p\right)
\end{aligned}
$$

Where $f = 1, g = p^2\left(1-\frac{2}{3}p\right)$. Since $f(p) \neq p$ then this is d'Almbert ode. Taking derivative w.r.t. x gives

$$
\begin{aligned}
p &= \left(f + xf'\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right)\\
p &= f + (xf' + g')\frac{dp}{dx}\\
p - f &= (xf' + g')\frac{dp}{dx}
\end{aligned}
$$

Using values for f, g the above simplifies to

$$
p - 1 = \left(2p - 2p^2\right)\frac{dp}{dx} \tag{2A}
$$

The singular solution is when $\frac{dp}{dx} = 0$ which results in $p = 1$. Substituting this in (1) gives

$$
\begin{aligned}
y &= x - \left(1-\frac{2}{3}\right)\\
&= x + \frac{1}{3}
\end{aligned}
$$

The general solution is when $\frac{dp}{dx} \neq 0$. Then (2A) is now separable. Solving for p gives

$$
\begin{aligned}
p &= -\sqrt{c_1 - x}\\
p &= \sqrt{c_1 - x}
\end{aligned}
$$

Substituting each one of the above solutions of p in (1) gives

$$
\begin{aligned}
y_1 &= x + \left(p^2 - \frac{2}{3}p^3\right)\\
&= x + \left(\left(-\sqrt{c_1 - x}\right)^2 - \frac{2}{3}\left(-\sqrt{c_1 - x}\right)^3\right)\\
&= x + \left(c_1 - x + \frac{2}{3}(c_1 - x)^{\frac{3}{2}}\right)\\
&= c_1 + \frac{2}{3}(c_1 - x)^{\frac{3}{2}}
\end{aligned}
$$

And

$$
\begin{aligned}
y_2 &= x + \left(p^2 - \frac{2}{3}p^3\right)\\
&= x + \left(\left(\sqrt{c_1 - x}\right)^2 - \frac{2}{3}\left(\sqrt{c_1 - x}\right)^3\right)\\
&= x + \left(c_1 - x - \frac{2}{3}(c_1 - x)^{\frac{3}{2}}\right)\\
&= c_1 - \frac{2}{3}(c_1 - x)^{\frac{3}{2}}
\end{aligned}
$$

Therefore the solutions are

$$
\begin{aligned}
y &= x + \frac{1}{3}\\
y &= c_1 + \frac{2}{3}(c_1 - x)^{\frac{3}{2}}\\
y &= c_1 - \frac{2}{3}(c_1 - x)^{\frac{3}{2}}
\end{aligned}
$$

2.2.12 Example 12

$$\begin{aligned}
(y')^2 &= e^{4x-2y}(y'-1) \\
\ln (y')^2 &= (4x-2y) + \ln (y'-1) \\
4x-2y &= \ln (y')^2 - \ln (y'-1) \\
4x-2y &= \ln \frac{(y')^2}{y'-1} \\
2y &= 4x - \ln \frac{(y')^2}{y'-1} \\
y &= 2x - \frac{1}{2}\ln \left(\frac{(y')^2}{y'-1}\right) \\
&= 2x - \frac{1}{2}\ln \left(\frac{p^2}{p-1}\right) \\
&= xf + g
\end{aligned}$$

Where $f = 2, g = -\frac{1}{2}\ln\left(\frac{p^2}{p-1}\right)$. Since $f(p) \neq p$ then this is d'Almbert ode. Taking derivative w.r.t. x gives

$$\begin{aligned}
p &= \left(f + xf'\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= f + (xf' + g')\frac{dp}{dx} \\
p - f &= (xf' + g')\frac{dp}{dx}
\end{aligned}$$

Using values for f, g the above simplifies to

$$p - 2 = \left(\frac{2-p}{2p^2 - 2p}\right)\frac{dp}{dx} \tag{2A}$$

The singular solution is when $\frac{dp}{dx} = 0$ which gives $p = 2$. From (1) this gives

$$y = 2x - \frac{1}{2}\ln 4$$

The general solution is when $\frac{dp}{dx} \neq 0$. Then (2) becomes

$$\begin{aligned}
\frac{dp}{dx} &= (p-2)\left(\frac{2p^2 - 2p}{2-p}\right) \\
&= 2p(1-p)
\end{aligned}$$

is now separable. Solving for p gives

$$p = \frac{1}{1 + ce^{-2x}}$$

Substituting the above solutions of p in (1) gives

$$\begin{aligned}
y &= 2x - \frac{1}{2}\ln\left(\frac{\left(\frac{1}{1+ce^{-2x}}\right)^2}{\frac{1}{1+ce^{-2x}} - 1}\right) \\
&= 2x - \frac{1}{2}\ln\left(\frac{-e^{4x}}{c\left(c + e^{2x}\right)}\right)
\end{aligned}$$

2.2.13 Example 13

$$
\begin{aligned}
y &= \frac{xy' + x(y')^2 - (y')^2}{y' + 1} \\
&= \frac{xp + xp^2 - p^2}{p + 1} \\
&= xp - \frac{p^2}{p + 1} \\
&= xf + g
\end{aligned} \tag{1}
$$

Where $f = p$ and $g = -\frac{p^2}{p+1}$. Since $f(p) = p$ then this is Clairaut ode. Taking derivative of the above w.r.t. x gives

$$
\begin{aligned}
p &= \frac{d}{dx}(xp + g(p)) \\
p &= p + (x + g'(p)) \frac{dp}{dx} \\
0 &= (x + g'(p)) \frac{dp}{dx}
\end{aligned}
$$

The general solution is given by

$$
\begin{aligned}
\frac{dp}{dx} &= 0 \\
p &= c_1
\end{aligned}
$$

Substituting this in (1) gives the general solution

$$
y = xc_1 - \frac{c_1^2}{c_1 + 1}
$$

The term $(x + g'(p)) = 0$ is used to find singular solutions.

$$
\begin{aligned}
x + g'(p) &= x + \frac{d}{dp} \frac{1}{p} \\
&= x - \frac{1}{p^2}
\end{aligned}
$$

Hence $x - \frac{1}{p^2} = 0$ or $p = \pm \frac{1}{\sqrt{x}}$. Substituting these back in (1) gives

$$
\begin{aligned}
y_1(x) &= xp + \frac{1}{p} \\
&= x\frac{1}{\sqrt{x}} + \sqrt{x} \\
&= 2\sqrt{x}
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
y_2(x) &= -x\sqrt{\frac{1}{x}} - \sqrt{x} \\
&= -2\sqrt{x}
\end{aligned} \tag{4}
$$

Eq. (2) is the general solution and (3,4) are the singular solutions.

2.2.14 Example 14

$$
\begin{aligned}
x(y')^2 + (x - y)\, y' + 1 - y &= 0 \\
x(y')^2 + xy' - yy' + 1 - y &= 0 \\
y(-y' - 1) + x(y')^2 + xy' + 1 &= 0
\end{aligned}
$$

Solving for y

$$
\begin{aligned}
y &= \frac{-x(y')^2 - xy' - 1}{-y' - 1}\\
&= \frac{-xp^2 - xp - 1}{-p - 1}\\
&= \frac{xp^2 + xp + 1}{p + 1}\\
&= x\left(\frac{p^2 + p}{p + 1}\right) + \frac{1}{1 + p}\\
&= xp + \frac{1}{1 + p}\\
&= xf + g \qquad (1)
\end{aligned}
$$

Where $f = p$ and $g = \frac{1}{1+p}$. Since $f(p) = p$ then this is Clairaut ode. Taking derivative of the above w.r.t. x gives

$$
\begin{aligned}
p &= \frac{d}{dx}(xp + g(p))\\
p &= p + (x + g'(p))\frac{dp}{dx}\\
0 &= (x + g'(p))\frac{dp}{dx}
\end{aligned}
$$

The general solution is given by

$$
\begin{aligned}
\frac{dp}{dx} &= 0\\
p &= c_1
\end{aligned}
$$

Substituting this in (1) gives the general solution

$$
y = c_1 x + \frac{1}{c_1 + 1} \qquad (4)
$$

The term $(x + g'(p)) = 0$ is used to find singular solutions. But

$$
\begin{aligned}
x + g'(p) &= x + \frac{d}{dp}\left(\frac{1}{1 + p}\right)\\
&= x - \frac{1}{(p + 1)^2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
x - \frac{1}{(p + 1)^2} &= 0\\
x(p + 1)^2 - 1 &= 0\\
(p + 1)^2 &= \frac{1}{x}\\
p + 1 &= \pm\frac{1}{\sqrt{x}}\\
p &= \pm\frac{1}{\sqrt{x}} - 1
\end{aligned}
$$

Substituting these values into (1) gives

$$
\begin{aligned}
y_1 &= xp_1 + \frac{1}{1 + p_1}\\
&= x\left(\frac{1}{\sqrt{x}} - 1\right) + \frac{1}{1 + \left(\frac{1}{\sqrt{x}} - 1\right)}\\
&= \frac{x}{\sqrt{x}} - x + \sqrt{x}\\
&= \frac{x\sqrt{x}}{x} - x + \sqrt{x}\\
&= 2\sqrt{x} - x \qquad (5)
\end{aligned}
$$

And substituting p_2 into (1) gives

$$
\begin{aligned}
y_1 &= xp_1 + \frac{1}{1+p_1} \\
&= x\left(-\frac{1}{\sqrt{x}} - 1\right) + \frac{1}{1+\left(-\frac{1}{\sqrt{x}} - 1\right)} \\
&= -\frac{x}{\sqrt{x}} - x - \sqrt{x} \\
&= \frac{-x\sqrt{x}}{x} - x - \sqrt{x} \\
&= -2\sqrt{x} - x \qquad (6)
\end{aligned}
$$

There are 3 solutions given in (4,5,6). One is general and two are singular.

2.2.15 Example 15

$$xyy' = y^2 + x\sqrt{4x^2+y^2}$$

Solving for y gives

$$
\begin{aligned}
y &= \operatorname{RootOf}\left(\_z^4 - 4 + \left(p^2-1\right)\_z^2 - 2\_z^3 p\right) x \\
y &= xf + g
\end{aligned}
$$

Where $f = \operatorname{RootOf}\left(\_z^4 - 4 + \left(p^2-1\right)\_z^2 - 2\_z^3 p\right)$ and $g = 0$. Since $f(p) \neq p$ then this is d'Almbert ode. Taking derivative of the above w.r.t. x gives

$$
\begin{aligned}
p &= \left(f + xf'\frac{dp}{dx}\right) + \left(g'\frac{dp}{dx}\right) \\
p &= f + xf'\frac{dp}{dx} \\
p - f &= xf'\frac{dp}{dx}
\end{aligned}
$$

Using values for f the above simplifies to

$$p - \operatorname{RootOf}\left(\_z^4 - 4 + \left(p^2-1\right)\_z^2 - 2\_z^3 p\right) = \left(x\frac{d}{dp}\operatorname{RootOf}\left(\_z^4 - 4 + \left(p^2-1\right)\_z^2 - 2\_z^3 p\right)\right)\frac{dp}{dx} \qquad \text{(2A)}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ which results in $p = \operatorname{RootOf}\left(\_z^4 - 4 + \left(p^2-1\right)\_z^2 - 2\_$ Substituting this in (1) does not generate any real solutions (only 2 complex ones) hence will not be used.

The general solution is found by finding p from (2A). Since (2A) is not linear in p, then inversion is needed. Writing (2A) as

$$
\begin{aligned}
\frac{dx}{dp} &= \frac{xf}{p-f} \\
\frac{1}{x}dx &= \frac{f}{p-f}dp
\end{aligned}
$$

Due to complexity of result, one now needs to obtain explicit result for RootOf which makes the computation very complicated. So this is not practical to solve by hand. Will stop here. It is much easier to solve this ode as a homogeneous ode instead which gives the solution as

$$-\frac{\sqrt{4x^2+y^2}}{x} + \ln(x) = c_1$$

2.2.16 Example 16

$$\ln(\cos y') + y' \tan y' = y$$

Solving for y gives

$$\begin{align*} y &= \ln(\cos p) + p\tan p \tag{1} \\ y &= xf + g \\ &= g \tag{1A} \end{align*}$$

Where $f = 0$ and $g(p) = \ln(\cos p) + p\tan p$. *Important note*: This ode has $f = 0$ which is strictly speaking is not of the form $y = xf(p) + g(p)$. But Maple says this is dAlembert. This is why it is included. I should make special case dAlmbert classification to handle this special case.

Taking derivative of (1A) w.r.t. x gives

$$\begin{align*} p &= \frac{dg}{dp}\frac{dp}{dx} \\ p &= \left(-\frac{\sin p}{\cos p} + \tan p + p\left(1+\tan^2 p\right)\right)\frac{dp}{dx} \\ p &= \left(-\tan p + \tan p + p\left(1+\tan^2 p\right)\right)\frac{dp}{dx} \\ p &= p\left(1+\tan^2 p\right)\frac{dp}{dx} \\ 1 &= \left(1+\tan^2 p\right)\frac{dp}{dx} \tag{1} \end{align*}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ which does not result in solution.

The general solution is found by finding p from (2). Since (2) is not linear in p, then <u>inversion is needed</u>. Writing (1) as

$$\begin{align*} \frac{dx}{dp} &= 1 + \tan^2 p \\ dx &= \left(1+\tan^2 p\right) dp \end{align*}$$

Integrating gives

$$\begin{align*} x &= \tan p + c \\ p &= \arctan(x - c) \end{align*}$$

Substituting the above in (1) gives the solution

$$\begin{align*} y &= \ln(\cos p) + p\tan p \\ &= \ln(\cos(\arctan(x-c))) + (\arctan(x-c))\tan(\arctan(x-c)) \\ &= \ln(\cos(\arctan(x-c))) + (x-c)\arctan(x-c) \end{align*}$$

This ode also have solution $y = 0$.

2.2.17 Example 17

$$x(y')^2 - 2yy' + 4x = 0$$

Solving for y gives

$$\begin{align*} y &= x\left(\frac{1}{2}y' + 2\frac{1}{y'}\right) \tag{1} \\ &= x\left(\frac{1}{2}p + 2\frac{1}{p}\right) \\ y &= xf \end{align*}$$

where $f = \frac{1}{2}p + 2\frac{1}{p}, g = 0$. Since $f(p) \neq p$ then this is d'Almbert ode. Taking derivative and simplifying gives

$$
\begin{aligned}
p &= \left(f + xf'\frac{dp}{dx}\right) \\
p - f &= xf'\frac{dp}{dx}
\end{aligned}
$$

Using values for f, g the above simplifies to

$$
\begin{aligned}
p - \frac{1}{2}p - 2\frac{1}{p} &= x\left(\frac{1}{2} - \frac{2}{p^2}\right)\frac{dp}{dx} \\
\frac{1}{2}p - \frac{2}{p} &= x\left(\frac{1}{2} - \frac{2}{p^2}\right)\frac{dp}{dx} \qquad \text{(2A)}
\end{aligned}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ which results in $\frac{1}{2}p - \frac{2}{p} = 0$ or $\frac{1}{2}p^2 - 2 = 0$ or $p^2 = 4$ or $p = \pm 2$. Hence $y = \pm 2x$ are the singular solutions.

The general solution is when $\frac{dp}{dx} \neq 0$ in (2A). Since (2A) is nonlinear, <u>inversion is needed</u>. General solution can be shown to be

$$
y = -\frac{1}{2}\left(-\frac{x^2}{c_1^2} - 4\right)c_1 \qquad (3)
$$

Will now show a more general method to find singular solution that works for any first order ode. This requires finding the general solution above first. Let the general solution be

$$
\begin{aligned}
\Phi(x, y, c) &= 0 \\
&= y + \frac{1}{2}\left(-\frac{x^2}{c_1^2} - 4\right)c_1
\end{aligned}
$$

The ode is

$$
\begin{aligned}
F(x, y, y') &= 0 \\
&= x(y')^2 - 2yy' + 4x
\end{aligned}
$$

First we find the p-discriminant curve. This is found by eliminating y' from

$$
\begin{aligned}
F &= 0 \\
\frac{\partial F}{\partial y'} &= 0
\end{aligned}
$$

Or

$$
\begin{aligned}
x(y')^2 - 2yy' + 4x &= 0 \\
2xy' - 2y &= 0
\end{aligned}
$$

Second equation gives $y' = \frac{y}{x}$. Substituting into first equation gives $x\left(\frac{y}{x}\right)^2 - 2y\left(\frac{y}{x}\right) + 4x = 0$ or $\frac{y^2}{x} - 2\frac{y^2}{x} + 4x = 0$ or $y = \pm 2x$. These are the candidate singular solutions

$$
y_s = \pm 2x
$$

Next, we verify these satisfy the ode itself. We see both do. Next we have to check that for an arbitrary point x_0 the following two equations are satisfied

$$
\begin{aligned}
y_g(x_0) &= y_s(x_0) \\
y_g'(x_0) &= y_s'(x_0)
\end{aligned}
$$

Where $y_g(x)$ is the general solution obtained above in (3). Starting with $y_s = 2x$ the above two equations now become

$$
\begin{aligned}
-\frac{1}{2}\left(-\frac{x_0^2}{c_1^2}-4\right)c_1 &= 2x_0 \\
-\frac{1}{2}\left(-\frac{2x_0}{c_1^2}\right)c_1 &= 2
\end{aligned}
$$

Or

$$
\begin{aligned}
\frac{x_0^2}{2c_1}+2c_1 &= 2x_0 \\
\frac{x_0}{c_1} &= 2
\end{aligned}
$$

Second equation gives $c_1 = \frac{x_0}{2}$. Using this in first equation gives

$$
\begin{aligned}
\frac{x_0^2}{2\frac{x_0}{2}}+2\left(\frac{x_0}{2}\right) &= 2x_0 \\
x_0+x_0 &= 2x_0 \\
2x_0 &= 2x_0
\end{aligned}
$$

Which shows it is satisfied. Hence this shows that $y_s = 2x$ is indeed a singular solution. Now we have to do the same for second $y_s = -2x$. Hence the steps of this method are the following

1. Find y_s using p-discriminant method by eliminating y' from $F = 0$ and $\frac{\partial F}{\partial y'} = 0$.
2. Verify that each y_s found satisfies the ode.
3. Find general solution to the ode $y_g(x)$.
4. Verify that the two equations $y_g(x_0) = y_s(x_0)$ and $y_g'(x_0) = y_s'(x_0)$ are satisfied at an arbitrary point x_0. If so, then y_s is singular solution. (envelope of the family of curves of the general solution).

2.2.18 Example 18

$$
x - yy' = a(y')^2
$$

Solving for y gives

$$
\begin{aligned}
-yp &= -x+ap^2 \\
-y &= -\frac{x}{p}+ap \\
y &= \frac{x}{p}-ap \qquad (1)\\
y &= xf(p)+g(p)
\end{aligned}
$$

Where $f = \frac{1}{p}, g = -ap$. Since $f(p) \neq p$ then this is d'Almbert ode. Taking derivative and simplifying gives

$$
\begin{aligned}
p &= \frac{d}{dx}(xf(p)+g(p)) \\
&= f(p)+xf'(p)\frac{dp}{dx}+g'(p)\frac{dp}{dx}
\end{aligned}
$$

But $f(p) = \frac{1}{p}, f'(p) = \frac{-1}{p^2}, g'(p) = -a$ and the above becomes

$$
\begin{aligned}
p &= \frac{1}{p}-\frac{x}{p^2}\frac{dp}{dx}-a\frac{dp}{dx} \\
p-\frac{1}{p} &= \left(-\frac{x}{p^2}-a\right)\frac{dp}{dx} \qquad (2)
\end{aligned}
$$

The singular solution is found by setting $\frac{dp}{dx}=0$ which results in $p=\pm 1$. Hence $y'=\pm 1$ or $y=\pm x$ but these do not satisfy the ode, hence no singular solutions exist.

The general solution is when $\frac{dp}{dx}\neq 0$ in (2). This gives the ode

$$\begin{aligned}\frac{dp}{dx}&=\frac{p-\frac{1}{p}}{-\frac{x}{p^2}-a}\\&=\frac{p-p^3}{ap^2+x}\end{aligned}$$

But this is non-linear. Hence inversion is needed. This becomes

$$\frac{dx}{dp}=\frac{-x(p)-ap^2}{p^3-p}$$

Which is now linear in $x(p)$. The solution is

$$x=\frac{-pa\sqrt{(p-1)(p+1)}\ln\left(p+\sqrt{p^2-1}\right)}{(p-1)(p+1)}+\frac{pc_1}{\sqrt{p-1}\sqrt{p+1}}\tag{3}$$

From (1) $y=\frac{x}{p}-ap$, hence

$$\begin{aligned}p_1&=\frac{1}{2}\frac{-y+\sqrt{4ax+y^2}}{a}\\p_2&=-\frac{1}{2}\frac{y+\sqrt{4ax+y^2}}{a}\end{aligned}$$

Plugging p_1 into (3) gives one solution and Plugging p_2 into (3) gives the second solution.

2.2.19 Example 19

$$y=xf(p)+g(p)$$

This problem is meant to show what to do when we are unable to solve explicitly for $x(p)$ when doing inversion. Taking derivative the above becomes

$$\begin{aligned}p&=\frac{d}{dx}(xf(p)+g(p))\\&=f(p)+xf'(p)\frac{dp}{dx}+g'(p)\frac{dp}{dx}\\p-f(p)&=(xf'(p)+g'(p))\frac{dp}{dx}\\\frac{dp}{dx}&=\frac{p-f(p)}{(xf'(p)+g'(p))}\end{aligned}$$

Inversion is needed. Hence gives

$$\begin{aligned}\frac{dx(p)}{dp}&=\frac{(x(p)f'(p)+g'(p))}{p-f(p)}\\\frac{dx}{dp}&=\frac{xf'}{p-f}+\frac{g'}{p-f}\end{aligned}$$

This is now linear in x.

$$\frac{dx}{dp}-\frac{xf'}{p-f}=\frac{g'}{p-f}$$

Integrating factor is $\mu=e^{\int\frac{f'(p)}{p-f}dp}$. Hence the above becomes

$$\begin{aligned}\frac{d}{dp}(x\mu)&=\mu\frac{g'}{p-f}\\x\mu&=\int\mu\frac{g'}{p-f}dp+c_1\\x&=\frac{1}{\mu}\int\mu\frac{g'}{p-f}dp+c_1\mu\end{aligned}\tag{1}$$

Now we solve for p from $y = xf(p)+g(p)$ and plug-in the result into the above. To show how this work, lets apply the earlier problem to the above which was to solve $x - yy' = a(y')^2$. From that problem we found that

$$
\begin{aligned}
p_1 &= \frac{1}{2}\frac{-y+\sqrt{4ax+y^2}}{a} \\
p_2 &= -\frac{1}{2}\frac{y+\sqrt{4ax+y^2}}{a}
\end{aligned}
$$

And we had $f = \frac{1}{p}, g = -ap$. Using these value we now find

$$
\begin{aligned}
\mu &= e^{\int \frac{f'(p)}{p-f} dp} \\
&= e^{\int \frac{-\frac{1}{p^2}}{p-\frac{1}{p}} dp} \\
&= \frac{p}{\sqrt{p^2-1}}
\end{aligned}
$$

Hence

$$
\begin{aligned}
x &= \frac{\sqrt{p^2-1}}{p}\int \frac{p}{\sqrt{p^2-1}}\frac{-a}{p-\frac{1}{p}} dp + c_1\frac{p}{\sqrt{p^2-1}} \\
&= -\frac{a\sqrt{p^2-1}}{p}\int \frac{p^2}{(p^2-1)^{\frac{3}{2}}} dp + c_1\frac{p}{\sqrt{p^2-1}} \\
&= -\frac{a\sqrt{p^2-1}}{p}\left(-\frac{p}{\sqrt{p^2-1}} + \ln\left(p+\sqrt{p^2-1}\right)\right) + c_1\frac{p}{\sqrt{p^2-1}} \\
&= a - \frac{a\sqrt{p^2-1}}{p}\ln\left(p+\sqrt{p^2-1}\right) + c_1\frac{p}{\sqrt{p^2-1}}
\end{aligned}
$$

Substituting each one of the above value for p in (2) gives the two solutions. For example, using $p_1 = \frac{1}{2}\frac{-y+\sqrt{4ax+y^2}}{a}$ gives

$$
x = a - \frac{a\sqrt{\left(\frac{1}{2}\frac{-y+\sqrt{4ax+y^2}}{a}\right)^2 - 1}}{\frac{1}{2}\frac{-y+\sqrt{4ax+y^2}}{a}} \ln\left(\frac{1}{2}\frac{-y+\sqrt{4ax+y^2}}{a} + \sqrt{\left(\frac{1}{2}\frac{-y+\sqrt{4ax+y^2}}{a}\right)^2 - 1}\right) + c_1\frac{\frac{1}{2}\frac{-y+}{}}{\sqrt{\left(\frac{1}{2}\frac{-y+\sqrt{}}{}\right.}}
$$

And same for the other p_2.

In the above example it was possible to evaluate the integrals in p, then replace p by its solution from the original ode. What if this was not possible? Let say we have integral

$$
\int ap^2 dp
$$

And for some reason we are not able to the integration. In this case we first replace the above with

$$
\int^p a\tau^2 d\tau
$$

And only now replace p with its solution as the upper limit.

2.2.20 Example 20

$$
y' = -\frac{x}{2} - 1 + \frac{1}{2}\sqrt{x^2+4x+4y}
$$

Solving for y gives

$$
\begin{aligned}
y &= xp + \left(1+2p+p^2\right) \qquad (1) \\
y &= xf + g
\end{aligned}
$$

Hence $f = p, g = (1 + 2p + p^2)$. Since $f = p$ then this is Clairaut. Taking derivative of the above w.r.t. x gives

$$\begin{aligned} y' &= f + x\frac{df}{dp}\frac{dp}{dx} + \frac{dg}{dp}\frac{dp}{dx} \\ p &= f + \frac{dp}{dx}\left(x\frac{df}{dp} + \frac{dg}{dp}\right) \end{aligned}$$

But $\frac{df}{dp} = 1, \frac{dg}{dp} = 2 + 2p$. The above becomes

$$p - f = \frac{dp}{dx}(x + 2 + 2p)$$

But $f = p$. The above simplifies to

$$0 = \frac{dp}{dx}(x + 2 + 2p) \tag{2}$$

The general solution is when $\frac{dp}{dx} = 0$. Hence $p = c_1$. Substituting this into (1) gives

$$y = xc_1 + \left(1 + 2c_1 + c_1^2\right)$$

The singular solution is when $\frac{dp}{dx} \neq 0$ in (2) which gives

$$\begin{aligned} x + 2 + 2p &= 0 \\ p &= \frac{-x - 2}{2} \end{aligned}$$

Substituting this in (1) gives

$$\begin{aligned} y &= x\left(\frac{-x-2}{2}\right) + \left(1 + 2\left(\frac{-x-2}{2}\right) + \left(\frac{-x-2}{2}\right)^2\right) \\ &= -\frac{1}{4}x(x+4) \\ &= -\frac{1}{4}x^2 - x \end{aligned}$$

Checking this solution against the ode shows it is verifies the ode. Hence there are two solutions, one general and one singular

$$y = \begin{cases} xc_1 + 1 + 2c_1 + c_1^2 \\ -\frac{1}{4}x^2 - x \end{cases}$$

2.2.21 Example 21

$$\frac{y'y}{1 + \frac{1}{2}\sqrt{1 + (y')^2}} = -x$$

Let $y' = p$ and rearranging gives

$$\begin{aligned} py &= -x\left(1 + \frac{1}{2}\sqrt{1+p^2}\right) \\ y &= -x\left(\frac{1}{p} + \frac{1}{2p}\sqrt{1+p^2}\right) \\ &= -x\left(\frac{2}{2p} + \frac{1}{2p}\sqrt{1+p^2}\right) \\ &= -x\left(\frac{2 + \sqrt{1+p^2}}{2p}\right) \\ &= xf + g \end{aligned} \tag{1}$$

Hence

$$
\begin{aligned}
f &= -\frac{2+\sqrt{1+p^2}}{2p} \\
g &= 0
\end{aligned}
$$

Since $f(p) \neq p$ then this is d'Albert ode. Taking derivative of (1) w.r.t. x gives

$$
\begin{aligned}
p &= \frac{d}{dx}(xf(p)+g(p)) \\
&= f(p) + xf'(p)\frac{dp}{dx} + g'(p)\frac{dp}{dx}
\end{aligned}
$$

But $f(p) = -\frac{2+\sqrt{1+p^2}}{2p}, f'(p) = \frac{-1}{p^2}, g = 0, g' = 0$ and the above becomes

$$
\begin{aligned}
p &= -\frac{2+\sqrt{1+p^2}}{2p} + x\left(-\frac{1}{2\sqrt{1+p^2}} - \frac{-2-\sqrt{1+p^2}}{2p^2}\right)\frac{dp}{dx} \\
p + \frac{2+\sqrt{1+p^2}}{2p} &= x\left(-\frac{1}{2\sqrt{1+p^2}} - \frac{-2-\sqrt{1+p^2}}{2p^2}\right)\frac{dp}{dx} \qquad (2)
\end{aligned}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ which results in $p + \frac{2+\sqrt{1+p^2}}{2p} = 0$. Hence $p = \pm i$ or $y' = \pm i$ or $y = \pm ix$. But these do not satisfy the ode, hence no singular solutions exist.

The general solution is when $\frac{dp}{dx} \neq 0$ in (2). This gives the ode

$$
\begin{aligned}
\frac{dp}{dx} &= \frac{1}{x}\frac{\left(p + \frac{2+\sqrt{1+p^2}}{2p}\right)}{\left(-\frac{1}{2\sqrt{1+p^2}} - \frac{-2-\sqrt{1+p^2}}{2p^2}\right)} \\
&= \frac{1}{x}(p^3+p)
\end{aligned}
$$

But this is non-linear in p. Hence inversion is needed. This becomes

$$
\begin{aligned}
\frac{dx}{dp} &= x\frac{\left(-\frac{1}{2\sqrt{1+p^2}} - \frac{-2-\sqrt{1+p^2}}{2p^2}\right)}{\left(p + \frac{2+\sqrt{1+p^2}}{2p}\right)} \\
\frac{dx}{dp} &= \frac{x}{p^3+p} \\
\frac{dx}{dp} - \frac{1}{p+p^3}x &= 0
\end{aligned}
$$

Which is now linear in $x(p)$. The solution is

$$
x = \frac{p}{\sqrt{1+p^2}}c_1 \qquad (3)
$$

We now need to eliminate p. We have two equations to do that, (1) and (3). Here they are side by side

$$
\begin{aligned}
y &= -x\left(\frac{2+\sqrt{1+p^2}}{2p}\right) \qquad (1) \\
x &= \frac{p}{\sqrt{1+p^2}}c_1 \qquad (3)
\end{aligned}
$$

We can either solve for p from (1) and plugin in the value found into (3). Or we can solve for p from (3) and plugin the value found in (1). Using CAS we can just use the solve command. For an example, using Maple it gives

```
eq1:=y=-x*(  (2+sqrt(1+p^2))/(2*p));
eq2:=x=p/sqrt(1+p^2)*_C1
sol:=solve([eq1,eq2],[p,y],'allsolutions');
[[p = x*RootOf((c__1^2 - x^2)*_Z^2 - 1), y = -(RootOf((c__1^2 - x^2)*_Z^2 - 1)*c__1 +
```

Now we can use allvalues

```
map(X->allvalues(X),sol)
[[p = x*sqrt(1/(c__1^2 - x^2)), y = -(sqrt(1/(c__1^2 - x^2))*c__1 + 2)/(2*sqrt(1/(c__1
[p = -x*sqrt(1/(c__1^2 - x^2)), y = (-sqrt(1/(c__1^2 - x^2))*c__1 + 2)/(2*sqrt(1/(c__1
```

Hence the solutions are

$$
\begin{aligned}
y_1 &= -\frac{\sqrt{\frac{1}{c_1^2-x^2}}\,c_1 + 2}{2\sqrt{\frac{1}{c_1^2-x^2}}} \\
y_2 &= -\frac{-\sqrt{\frac{1}{c_1^2-x^2}}\,c_1 + 2}{2\sqrt{\frac{1}{c_1^2-x^2}}}
\end{aligned}
$$

These are verified valid solutions to the ode (had to use assuming positive)

2.2.22 Example 22

$$
x(y')^3 = yy' + 1
$$

Let $y' = p$ and rearranging gives

$$
\begin{aligned}
xp^3 &= yp + 1 \\
y &= \frac{xp^3 - 1}{p} \\
&= xp^2 - \frac{1}{p} \\
&= xf + g \qquad (1)
\end{aligned}
$$

Hence

$$
\begin{aligned}
f &= p^2 \\
g &= -\frac{1}{p}
\end{aligned}
$$

Since $f(p) \neq p$ then this is d'Almbert ode. Taking derivative of (1) w.r.t. x gives

$$
\begin{aligned}
p &= \frac{d}{dx}(xf(p) + g(p)) \\
&= f(p) + xf'(p)\frac{dp}{dx} + g'(p)\frac{dp}{dx} \\
&= f(p) + (xf' + g')\frac{dp}{dx}
\end{aligned}
$$

But $f(p) = p^2, f'(p) = 2p, g = -\frac{1}{p}, g' = \frac{1}{p^2}$ and the above becomes

$$
\begin{aligned}
p &= p^2 + \left(2xp + \frac{1}{p^2}\right)\frac{dp}{dx} \\
p - p^2 &= \left(2xp + \frac{1}{p^2}\right)\frac{dp}{dx} \qquad (2)
\end{aligned}
$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ which results in $p - p^2 = 0$. Hence $p = 0$ or $p = 1$. Substituting $p = 0$ in (1) gives $1/0$ error. Hence this is not valid solution. Substituting $p = 1$ in (1) gives $y = x - 1$ which verifies the ode. Hence this is valid singular solution.

The general solution is when $\frac{dp}{dx} \neq 0$ in (2). This gives the ode

$$
\frac{dp}{dx} = \frac{p^3(1-p)}{2xp^3 + 1}
$$

But this is non-linear in p. Hence inversion is needed. This becomes

$$\frac{dx}{dp}=\frac{2xp^3+1}{p^3\left(1-p\right)}$$

Which is now linear in $x(p)$. The solution is

$$x=\frac{2c_1p^2+2p-1}{2p^2\left(p-1\right)^2} \tag{3}$$

We now need to eliminate p. We have two equations to do that, (1) and (3). Here they are side by side

$$y=xp^2-\frac{1}{p} \tag{1}$$
$$x=\frac{2c_1p^2+2p-1}{2p^2\left(p-1\right)^2} \tag{3}$$

We can either solve for p from (1) and plugin in the value found into (3). Or we can solve for p from (3) and plugin the value found in (1). Using CAS we can just use the solve command. For an example, using Maple it gives

```
eq1:=y=x*p^2-1/p;
eq2:=x= (2*_C1*p^2+2*p-1)/(2*p^2*(p-1)^2);
solve({eq1,eq2},{y,p})
```

Whch gives

```
{p = RootOf(1 + 2*x*_Z^4 - 4*x*_Z^3 + (-2*c__1 + 2*x)*_Z^2 - 2*_Z),
y = (x*RootOf(1 + 2*x*_Z^4 - 4*x*_Z^3 + (-2*c__1 + 2*x)*_Z^2 - 2*_Z)^3 - 1)/RootOf(1 +
```

Hence the general solution is

$$y=\frac{x\operatorname{RootOf}\left(1+2xZ^4-4xZ^3+\left(-2c_1+2x\right)Z^2-2Z\right)^3-1}{\operatorname{RootOf}\left(1+2xZ^4-4xZ^3+\left(-2c_1+2x\right)Z^2-2Z\right)}$$

And the singular solution is

$$y=x-1$$

2.2.23 Example 23

$$\left(y'\right)^2-2yy'=2x$$

Let $y'=p$ and rearranging gives

$$\begin{aligned}
p^2-2yp&=2x\\
y&=\frac{p^2-2x}{2p}\\
&=-x\frac{1}{p}+\frac{1}{2}p\\
&=xf+g
\end{aligned} \tag{1}$$

Hence

$$\begin{aligned}
f&=-\frac{1}{p}\\
g&=\frac{1}{2}p
\end{aligned}$$

Since $f(p) \neq p$ then this is d'Almbert ode. Taking derivative of (1) w.r.t. x gives

$$\begin{aligned}
p &= \frac{d}{dx}(xf(p) + g(p)) \\
&= f(p) + xf'(p)\frac{dp}{dx} + g'(p)\frac{dp}{dx} \\
&= f(p) + (xf' + g')\frac{dp}{dx}
\end{aligned}$$

But $f(p) = -\frac{1}{p}, f'(p) = \frac{1}{p^2}, g = \frac{1}{2}p, g' = \frac{1}{2}$ and the above becomes

$$\begin{aligned}
p &= -\frac{1}{p} + \left(\frac{x}{p^2} + \frac{1}{2}\right)\frac{dp}{dx} \\
p + \frac{1}{p} &= \left(\frac{x}{p^2} + \frac{1}{2}\right)\frac{dp}{dx} \qquad (2)
\end{aligned}$$

The singular solution is found by setting $\frac{dp}{dx} = 0$ which results in $p^2 + 1 = 0$. Hence $p = \pm i$ But these do not verify the ode. Hence no singular solutions exist.

The general solution is when $\frac{dp}{dx} \neq 0$ in (2). This gives the ode

$$\frac{dp}{dx} = \frac{(p^2 + 1)\,2p}{2x + p^2}$$

But this is non-linear in p. Hence inversion is needed. This becomes

$$\frac{dx}{dp} = \frac{2x + p^2}{(p^2 + 1)\,2p}$$

Which is now linear in $x(p)$. The solution is

$$x = \frac{\left(\frac{1}{2}\operatorname{arcsinh}(p) + c_1\right)p}{\sqrt{p^2 - 1}} \qquad (3)$$

We now need to eliminate p. We have two equations to do that, (1) and (3). Here they are side by side

$$\begin{aligned}
y &= -x\frac{1}{p} + \frac{1}{2}p \qquad (1) \\
x &= \frac{\left(\frac{1}{2}\operatorname{arcsinh}(p) + c_1\right)p}{\sqrt{p^2 - 1}} \qquad (3)
\end{aligned}$$

We can either solve for p from (1) and plugin in the value found into (3). Or we can solve for p from (3) and plugin the value found in (1). In this case it is easier to solve for p from (1) which gives

$$\begin{aligned}
p_1 &= y + \sqrt{2x + y^2} \\
p_2 &= y - \sqrt{2x + y^2}
\end{aligned}$$

Substituting each of these into (3) gives these two general solutions

$$\begin{aligned}
x &= \frac{\left(\frac{1}{2}\operatorname{arcsinh}\left(y + \sqrt{2x + y^2}\right) + c_1\right)\left(y + \sqrt{2x + y^2}\right)}{\sqrt{\left(y + \sqrt{2x + y^2}\right)^2 - 1}} \\
x &= \frac{\left(\frac{1}{2}\operatorname{arcsinh}\left(y - \sqrt{2x + y^2}\right) + c_1\right)\left(y - \sqrt{2x + y^2}\right)}{\sqrt{\left(y - \sqrt{2x + y^2}\right)^2 - 1}}
\end{aligned}$$

2.2.24 Example 24

$$xy' - y = \sqrt{x^2 - y^2}$$

Let $y' = p$ and rearranging gives

$$xp - y = \sqrt{x^2 - y^2}$$

Solving for y gives two solutions

$$\begin{aligned} y &= x\left(\frac{p}{2} + \frac{1}{2}\sqrt{2 - p^2}\right) \qquad (1)\\ y &= x\left(\frac{p}{2} - \frac{1}{2}\sqrt{2 - p^2}\right) \end{aligned}$$

We will here solve the first one above. The second one will have similar solution. Comparing the above to $y = xf(p) + g(p)$ shows that

$$\begin{aligned} f &= \frac{p}{2} + \frac{1}{2}\sqrt{2 - p^2} \qquad (2)\\ g &= 0 \end{aligned}$$

Since $f(p) \neq p$ then this is d'Almbert ode. Taking derivative of (2) w.r.t. x gives

$$\begin{aligned} p &= \frac{d}{dx}(xf(p)) \\ &= f(p) + xf'(p)\frac{dp}{dx} \\ &= \left(\frac{p}{2} + \frac{1}{2}\sqrt{2 - p^2}\right) + x\left(\frac{1}{2} - \frac{p}{2\sqrt{2 - p^2}}\right)\frac{dp}{dx} \\ p - \left(\frac{p}{2} + \frac{1}{2}\sqrt{2 - p^2}\right) &= x\left(\frac{1}{2} - \frac{p}{2\sqrt{2 - p^2}}\right)\frac{dp}{dx} \qquad (3) \end{aligned}$$

Singular solution is when $\frac{dp}{dx} = 0$ which results in

$$\begin{aligned} p - \left(\frac{p}{2} + \frac{1}{2}\sqrt{2 - p^2}\right) &= 0 \\ \frac{p}{2} - \frac{1}{2}\sqrt{2 - p^2} &= 0 \end{aligned}$$

Hence $p = 1$. Substituting this in (2) gives singular solution

$$\begin{aligned} y &= x\left(\frac{1}{2} + \frac{1}{2}\sqrt{2 - 1}\right) \\ &= x \end{aligned}$$

To find general solution, we need to solve (3) for p. EQ (3) becomes

$$\begin{aligned} \frac{dp}{dx} &= \frac{\frac{p}{2} - \frac{1}{2}\sqrt{2 - p^2}}{\frac{x}{2} - \frac{xp}{2\sqrt{2-p^2}}} \\ &= -\frac{1}{x}\sqrt{2 - p^2} \end{aligned}$$

This is separable ode.

$$\begin{aligned} \frac{-dp}{\sqrt{2 - p^2}} &= \frac{1}{x}dx \\ -\arcsin\left(\frac{\sqrt{2}}{2}p\right) &= \ln x + c_1 \end{aligned}$$

Substituting this into (1) gives

$$\begin{aligned} y &= x\left(\frac{p}{2} + \frac{1}{2}\sqrt{2 - p^2}\right) \\ &= x\left(\frac{-\frac{2}{\sqrt{2}}\sin(\ln x + c_1)}{2} + \frac{1}{2}\sqrt{2 - \left(-\frac{2}{\sqrt{2}}\sin(\ln x + c_1)\right)^2}\right) \\ &= x\left(\frac{-\sin(\ln x + c_1)}{\sqrt{2}} + \frac{1}{2}\sqrt{2 - 2\sin^2(\ln x + c_1)}\right) \end{aligned}$$

2.2.25 Extra example

This ode is an example where y does not appear explicitly in the ode so not possible to directly solve for y. It is given here to show possible problems with this method.

$$y' = \sqrt{1+x+y} \tag{1A}$$

This ode is squared to first solve for y which gives

$$(y')^2 = 1+x+y \tag{2A}$$

However, here care is needed. To get back to original ode (1A) then (2A) means two possible equations

$$y' = \pm\sqrt{1+x+y}$$

Hence the solutions obtained using (2A) can be the solution to one of these

$$y' = +\sqrt{1+x+y} \tag{B1}$$
$$y' = -\sqrt{1+x+y} \tag{B2}$$

Therefore the solution obtained by squaring both sides of (1A), which is done in order to solve for y, must be checked to see if it satisfies the original ode, else it will be extraneous solution resulting from squaring both sides of the ode.

Starting from (2A), in normal form (by replacing y' with p) it becomes

$$\begin{aligned} y &= -x - 1 + p^2 \\ &= xf + g \end{aligned} \tag{1}$$

Where $f = -1, g = -1 + p^2$. Taking derivative w.r.t. x gives

$$\begin{aligned} p &= f + (xf' + g')\frac{dp}{dx} \\ p + 1 &= 2p\frac{dp}{dx} \end{aligned} \tag{2}$$

Since $\frac{\partial \phi}{\partial x} = -1 \neq p$ then this is d'Alembert ode. The singular solution is found by setting $\frac{dp}{dx} = 0$ which results in $p = -1$. Substituting this in (1) gives the singular solution

$$y(x) = -x \tag{3}$$

But this solution does not satisfy the ode, hence it is extraneous. The general solution is found by finding p from (2). Since (2) is nonlinear, then it is inverted which gives

$$\begin{aligned} \frac{p+1}{2p} &= \frac{dp}{dx} \\ \frac{dx}{dp} &= \frac{2p}{p+1} \end{aligned}$$

Which is linear in x. Solving gives

$$x = 2p - 2\ln(p+1) + c_1 \tag{4}$$

Instead of inverting this to find p in terms of x, p is found from (1) which gives

$$\begin{aligned} y + x + 1 &= p^2 \\ p &= \pm\sqrt{y+x+1} \end{aligned}$$

Substituting these solutions in (4) gives implicit solutions as

$$\begin{aligned} x &= 2\sqrt{y+x+1} - 2\ln\left(1 + \sqrt{y+x+1}\right) + c_1 \\ x &= -2\sqrt{y+x+1} - 2\ln\left(1 - \sqrt{y+x+1}\right) + c_1 \end{aligned}$$

But only the first one above satisfies the ode. The second is extraneous. Therefore the final solution is

$$x = 2\sqrt{y + x + 1} - 2\ln\left(1 + \sqrt{y + x + 1}\right) + c_1$$

And no singular solutions exist. If instead of doing the above, p was found from (4) using inversion, then it will be

$$p = -\operatorname{LambertW}\left(-c_1 e^{\frac{-x}{2}-1}\right) - 1$$

Substituting this in (1) gives

$$y = -x - 1 + \left(-\operatorname{LambertW}\left(-c_1 e^{\frac{-x}{2}-1}\right) - 1\right)^2$$

But this general solution does not satisfy the original ode. In general, it is best to avoid squaring both side of the ode in order to solve for y as this can generate extraneous solutions. Only use this method if the original ode is already given in the form where y shows explicitly.

2.3 references

1. An elementary treatise on differential equations. By Abraham Cohen. 1906.
2. Applied differential equations, N Curle. 1972
3. Ordinary differential equations, LB Jones. 1976.
4. Elementary differential equations, William Martin, Eric Reissner. second edition. 1961.
5. Differentialgleichungen, by E. Kamke, page 30.
6. Differential and integral calculus by N. Piskunov, Vol II