Homework 4, Math 322

1. Solve the the partial differential equation

$$u_t = u_{xx} + xt, \quad 0 \le x \le 1, t \ge 0$$

with boundary conditions

$$u(0,t) = u(1,t) = 0$$

and initial condition

$$u(x,0) = \sin \pi x.$$

Solution: The eigenvalue problem $-y'' = \lambda y$, $y(0) = y(1) = 0$ has eigenvalues $\lambda_n = n^2\pi^2$ and normalized eigenfunctions $\hat{\phi}_n(x) = \sqrt{2}\sin(nx)$, $n = 1, 2, \ldots$ Therefore, we are looking for a solution in the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\hat{\phi}_n(x).$$

We know from the textbook, Section 11.3, that $b_n(t)$ is determined by

$$b_n'(t) + \lambda_n b_n(t) = \gamma_n(t), \quad b_n(0) = B_n,$$

where

$$\gamma_n(t) = \int_0^1 xt\hat{\phi}_n(x)\,dx$$

and the sequence B_n is determined by

$$u(x,0) = \sum_{n=1}^{\infty} B_n\hat{\phi}_m(x).$$

Therefore, in our problem, $B_1 = \frac{1}{\sqrt{2}}$ and $B_n = 0$ for $n \ge 2$. Moreover,

$$\gamma_n(t) = t\sqrt{2}\int_0^1 x\sin(n\pi x)\,dx = t\sqrt{2}(-1)^{n+1}\frac{1}{n\pi}.$$

Solving the differential equation for $b_n(t)$ we find

$$b_1(t) = \frac{\sqrt{2}}{\pi^5}\left(\pi^2 t - 1 + e^{-\pi^2 t}\left(\frac{1}{2}\pi^5 + 1\right)\right).$$

and, for $n \ge 2$,

$$b_n(t) = \frac{\sqrt{2}(-1)^{n+1}}{\pi^5 n^5}\left(\pi^2 n^2 t - 1 + e^{-n^2\pi^2 t}\right).$$

The solution is

$$
\begin{aligned}
u(x,t) =& \frac{2}{\pi^5}\left(\pi^2 t - 1 + e^{-\pi^2 t}\left(\frac{1}{2}\pi^5 + 1\right)\right)\sin \pi x \\
&+ \sum_{n=2}^{\infty} \frac{2(-1)^{n+1}}{\pi^5 n^5}\left(\pi^2 n^2 t - 1 + e^{-n^2\pi^2 t}\right)\sin(n\pi x) \\
=& e^{-\pi^2 t}\sin \pi x + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{\pi^5 n^5}\left(\pi^2 n^2 t - 1 + e^{-n^2\pi^2 t}\right)\sin(n\pi x).
\end{aligned}
$$

2. Show that

$$
(\lambda-\mu)\int_0^1 xJ_0(\sqrt{\lambda}x)J_0(\sqrt{\mu}x)\,dx = \sqrt{\mu}J_0'(\sqrt{\mu})J_0(\sqrt{\lambda}) - \sqrt{\lambda}J_0'(\sqrt{\lambda})J_0(\sqrt{\mu})
$$

for $\lambda, \mu > 0$.

Solution: Let $y = J_0(\sqrt{\lambda}x)$ and $z = J_0(\sqrt{\mu}x)$. Then

$$
-(xy')' = \lambda xy, \quad -(xz')' = \mu xz.
$$

We multiply the first equation by z, the second by y, subtract and integrate, to find

$$
\int_0^1 \left((xz')'y - (xy')'z\right)\,dx = (\lambda-\mu)\int_0^1 xyz\,dx.
$$

Integration by parts gives

$$
xz'y - xy'z\big|_0^1 = (\lambda-\mu)\int_0^1 xyz\,dx
$$

which is the desired identity.

3. By letting $\mu \to \lambda$ in the formula of Problem 2, derive a formula for $\int_0^1 xJ_0(\sqrt{\lambda}x)^2\,dx$. Then show that the normalized eigenfunctions of the eigenvalue problem in Section 11.4 are

$$
\hat{\phi}_n(x) = \frac{\sqrt{2}J_0(j_n x)}{|J_0'(j_n)|},
$$

where $0 < j_1 < j_2 < j_3 < \dots$ denote the positive zeros of J_0.

Solution: We divide the identity from problem 2 by $\lambda-\mu$, and let $\mu \to \lambda$. Using L'Hospital's rule we find

$$
\begin{aligned}
&\int_0^1 xJ_0(\sqrt{\lambda}x)^2\,dx = \lim_{\mu\to\lambda}\frac{\sqrt{\mu}J_0'(\sqrt{\mu})J_0(\sqrt{\lambda}) - \sqrt{\lambda}J_0'(\sqrt{\lambda})J_0(\sqrt{\mu})}{\lambda-\mu} \\
&= -\lim_{\mu\to\lambda}\frac{1}{2}\mu^{-1/2}J_0'(\sqrt{\mu})J_0(\sqrt{\lambda}) + \frac{1}{2}J_0''(\sqrt{\mu})J_0(\sqrt{\lambda}) - \sqrt{\lambda}J_0'(\sqrt{\lambda})\frac{1}{2}\mu^{-1/2}J_0'(\sqrt{\mu}) \\
&= -\frac{1}{2}\lambda^{-1/2}J_0'(\sqrt{\lambda})J_0(\sqrt{\lambda}) - \frac{1}{2}J_0''(\sqrt{\lambda})J_0(\sqrt{\lambda}) + \frac{1}{2}J_0'(\sqrt{\lambda})^2.
\end{aligned}
$$

Now we use

$$
\sqrt{\lambda}J_0''(\sqrt{\lambda}) + J_0'(\sqrt{\lambda}) + \sqrt{\lambda}J_0(\sqrt{\lambda}) = 0.
$$

Then we obtain

$$\int_0^1 x J_0(\sqrt{\lambda}x)^2\,dx = \frac{1}{2}J_0(\sqrt{\lambda})^2 + \frac{1}{2}J_0'(\sqrt{\lambda})^2.$$

If $\lambda = j_n^2$ then this formula simplifies to

$$\int_0^1 x J_0(j_n x)^2\,dx = \frac{1}{2}J_0'(j_n)^2.$$

The normalized eigenfunctions are

$$\frac{J_0(j_n r)}{\left(\int_0^1 x J_0(j_n x)^2\,dx\right)^{1/2}} = \frac{\sqrt{2}J_0(j_n r)}{|J_0'(j_n)|}.$$

4. Solve the inhomogeneous differential equation

$$-((1-x^2)y')' = y + x^3, \quad -1 < x < 1$$

with boundary condition

$$y(x), y'(x) \text{ bounded as } x \to -1^+ \text{ and } x \to 1^-.$$

Solution We use the method from Section 11.3. It is stated for regular Sturm-Liouville problems but it works equally well for our singular Sturm-Liouville problem. We look for the solution in the form

$$y = \sum_{n=0}^{\infty} b_n P_n(x).$$

The b_n satisfy

$$b_n = \frac{c_n}{\lambda_n - \mu}.$$

The sequence c_n is determined by

$$x^3 = \sum_{n=0}^{\infty} c_n P_n(x).$$

Since $P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$ and $P_1(x) = x$, we find

$$c_3 = \frac{2}{5}, \quad c_1 = \frac{3}{5},$$

and all other $c_n = 0$. The eigenvalues are $\lambda_n = n(n+1)$ and $\mu = 1$. Therefore,

$$y = \frac{c_1}{2-1}P_1(x) + \frac{c_3}{12-1}P_3(x) = \frac{3}{5}x + \frac{2}{5}\frac{1}{11}\left(\frac{5}{2}x^3 - \frac{3}{2}x\right) = \frac{6}{11}x + \frac{1}{11}x^3.$$