Quizz 3

Math 332
Introduction to Partial Differential Equations

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Contents

1 Problem 1

Problem Find the eigenvalues and normalized eigenfunctions of the RSL problem

$$\begin{aligned} y'' + \lambda y &= 0 \\ y(0) - y'(0) &= 0 \\ y(\pi) - y'(\pi) &= 0 \end{aligned} \tag{1}$$

solution

The characteristic equation for $y'' + \lambda y = 0$ is given by $r^2 + \lambda = 0$. Hence the roots are

$$r = \pm\sqrt{-\lambda}$$

There are 3 cases to consider.

case $\lambda = 0$ This implies that $r = 0$ is a double root. The solution becomes

$$\begin{aligned} y &= c_1 + c_2 x \\ y' &= c_2 \end{aligned}$$

The first boundary conditions $y(0) - y'(0) = 0$ gives $c_1 - c_2 = 0$ or $c_1 = c_2$. The above solution now becomes

$$\begin{aligned} y &= c_1(1 + x) \\ y' &= c_1 \end{aligned}$$

The second boundary conditions $y(\pi) - y'(\pi) = 0$ gives $c_1(1 + \pi) - c_1 = 0$ or $\pi = 0$. Which is not possible. Therefore $\lambda = 0$ is not an eigenvalue.

case $\lambda < 0$ Let $\lambda = -\omega^2$ for some real ω. Hence the roots now are $r = \pm\sqrt{\omega^2} = \pm\omega$. Therefore the solution is

$$y = c_1 e^{\omega x} + c_2 e^{-\omega x}$$

Since the exponents are real, the solution can be written in terms of hyperbolic trigonometric functions as

$$\begin{aligned} y &= c_1 \cosh \omega x + c_2 \sinh \omega x \\ y' &= c_1 \omega \sinh \omega x + c_2 \omega \cosh \omega x \end{aligned}$$

The first boundary conditions $y(0) - y'(0) = 0$ gives $0 = c_1 - c_2\omega$ or $c_1 = c_2\omega$. Therefore the above solution becomes

$$\begin{aligned} y &= c_2 \omega \cosh \omega x + c_2 \sinh \omega x \\ &= c_2(\omega \cosh \omega x + \sinh \omega x) \end{aligned} \tag{2}$$

Hence

$$y' = c_2\left(\omega^2 \sinh \omega x + \omega \cosh \omega x\right)$$

The second boundary conditions $y(\pi) - y'(\pi) = 0$ gives

$$\begin{aligned} 0 &= c_2(\omega \cosh \omega\pi + \sinh \omega\pi) - c_2\left(\omega^2 \sinh \omega\pi + \omega \cosh \omega\pi\right) \\ &= c_2\left(\omega \cosh \omega\pi + \sinh \omega\pi - \omega^2 \sinh \omega\pi - \omega \cosh \omega\pi\right) \\ &= c_2\left(\sinh \omega\pi - \omega^2 \sinh \omega\pi\right) \\ &= c_2\left(1 - \omega^2\right) \sinh \omega\pi \end{aligned}$$

Non-trivial solution implies either $\left(1 - \omega^2\right) = 0$ or $\sinh \omega\pi = 0$. But $\sinh \omega\pi = 0$ only when its argument is zero. But $\omega \neq 0$ in this case. The other option is that $\left(1 - \omega^2\right) = 0$. This implies $\omega^2 = 1$

or, since $\lambda = -\omega^2$, that $\lambda = -1$. Hence $\lambda = -1$ is an eigenvalue. Therefore the solution from (2) above becomes

$$\begin{aligned} y(x) &= c_2 \cosh x + c_2 \sinh x \\ &= c_2 (\cosh x + \sinh x) \end{aligned}$$

But $e^x = \cosh x + \sinh x$, hence the solution can be written as

$$y = c_2 e^x$$

The eigenfunction in this case is therefore

$$\Phi_{-1}(x) = e^x$$

To obtain the normalized eigenfunction, let $\hat{\Phi}_{-1}(x) = k_{-1}\Phi_{-1}(x)$. The normalization factor k_{-1} is found by setting $\int_0^\pi \left(r(x)\hat{\Phi}_{-1}(x)\right)^2 dx = 1$. But the weight $r(x) = 1$ in this problem from looking at the Sturm Liouville form given. Therefore solving

$$\begin{aligned} \int_0^\pi \hat{\Phi}_{-1}^2(x)\,dx &= 1 \\ \int_0^\pi (k_{-1}e^x)^2\,dx &= 1 \\ k_{-1}^2 \int_0^\pi e^{2x}\,dx &= 1 \\ k_{-1}^2 \left(\frac{e^{2x}}{2}\right)_0^\pi &= 1 \\ \frac{k_{-1}^2}{2}\left(e^{2\pi} - 1\right) &= 1 \end{aligned}$$

Therefore

$$k_{-1} = \frac{\sqrt{2}}{\sqrt{e^{2\pi} - 1}}$$

Hence the normalized eigenfunction is

$$\hat{\Phi}_{-1}(x) = \left(\frac{\sqrt{2}}{\sqrt{e^{2\pi} - 1}}\right) e^x$$

<u>case $\lambda > 1$</u> Since λ is positive, then the roots are $r = \pm\sqrt{-\lambda} = \pm i\sqrt{\lambda}$. This gives the solution

$$y = c_1 e^{i\sqrt{\lambda}x} + c_2 e^{-i\sqrt{\lambda}x}$$

Since the exponents are complex, the above solution can be written in terms of the circular trigonometric functions as

$$\begin{aligned} y &= c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\sqrt{\lambda}x \\ y' &= -c_1\sqrt{\lambda}\sin\left(\sqrt{\lambda}x\right) + c_2\sqrt{\lambda}\cos\sqrt{\lambda}x \end{aligned}$$

The first boundary conditions $y(0) - y'(0) = 0$ gives $0 = c_1 - c_2\sqrt{\lambda}$ or $c_1 = c_2\sqrt{\lambda}$. The above solution becomes

$$\begin{aligned} y &= c_2\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right) + c_2 \sin\sqrt{\lambda}x \\ &= c_2\left(\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right) + \sin\sqrt{\lambda}x\right) \end{aligned} \tag{3}$$

Therefore

$$y' = c_2\left(-\lambda\sin\left(\sqrt{\lambda}x\right) + \sqrt{\lambda}\cos\sqrt{\lambda}x\right)$$

Applying second boundary condition $y(\pi) - y'(\pi) = 0$ to the above gives

$$\begin{aligned} 0 &= c_2\left(\sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right) + \sin\left(\sqrt{\lambda}\pi\right)\right) - c_2\left(-\lambda\sin\left(\sqrt{\lambda}\pi\right) + \sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right)\right) \\ &= c_2\left(\sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right) + \sin\left(\sqrt{\lambda}\pi\right) + \lambda\sin\left(\sqrt{\lambda}\pi\right) - \sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right)\right) \\ &= c_2\left(\sin\left(\sqrt{\lambda}\pi\right) + \lambda\sin\left(\sqrt{\lambda}\pi\right)\right) \\ &= c(1+\lambda)\sin\left(\sqrt{\lambda}\pi\right) \end{aligned}$$

For non-trivial solution, either $1+\lambda=0$ or $\sin\left(\sqrt{\lambda}\pi\right)=0$. But $1+\lambda=0$ implies $\lambda=-1$. But it is assumed that λ is positive. The other possibility is that $\sin\left(\sqrt{\lambda}\pi\right)=0$ which implies

$$\sqrt{\lambda}\pi = n\pi \qquad n=1,2,3,\cdots$$

Or

$$\lambda_n = n^2 \qquad 1,2,3,\cdots$$

The corresponding solution from (3) becomes

$$y_n(x) = c_n\left(n\cos(nx)+\sin(nx)\right)$$

Therefore the eigenfunctions are

$$\Phi_n(x) = n\cos(nx)+\sin(nx)$$

To obtain the normalized eigenfunctions, as was done above, $\int_0^\pi \left(r(x)\hat{\Phi}_n(x)\right)^2 dx = 1$ is solved for k_n giving

$$\begin{aligned}
\int_0^\pi \left(k_n\Phi_n(x)\right)^2 dx &= 1\\
k_n^2\int_0^\pi \left(n\cos(nx)+\sin(nx)\right)^2 dx &= 1\\
k_n^2\int_0^\pi \left(n^2\cos^2(nx)+\sin^2(nx)+2n\cos(nx)\sin(nx)\right) dx &= 1\\
n^2\int_0^\pi \cos^2(nx)\,dx+\int_0^\pi \sin^2(nx)\,dx+2n\int_0^\pi \cos(nx)\sin(nx)\,dx &= \frac{1}{k_n^2} \qquad (4)
\end{aligned}$$

But $\int_0^\pi \cos^2(nx)\,dx=\frac{\pi}{2}$ and $\int_0^\pi \sin^2(nx)\,dx=\frac{\pi}{2}$ and for the last integral above

$$\begin{aligned}
\int_0^\pi \cos(nx)\sin(nx)\,dx &= \int_0^\pi \frac{1}{2}\sin(2nx)\,dx\\
&= \frac{1}{2}\left(\frac{-\cos(2nx)}{2n}\right)_0^\pi\\
&= \frac{-1}{4n}\left(\cos(2nx)\right)_0^\pi\\
&= \frac{-1}{4n}\left(\cos(2n\pi)-1\right)
\end{aligned}$$

But $\cos(2n\pi)=1$ because $n=1,2,3,\cdots$. Therefore the above simplifies to $\int_0^\pi \cos(nx)\sin(nx)\,dx=0$. Using these results in (4) gives

$$k_n^2\left(n^2\frac{\pi}{2}+\frac{\pi}{2}\right)=1$$

Or

$$k_n = \frac{\sqrt{2}}{\sqrt{\pi\left(1+n^2\right)}}$$

The normalized eigenfunctions are therefore

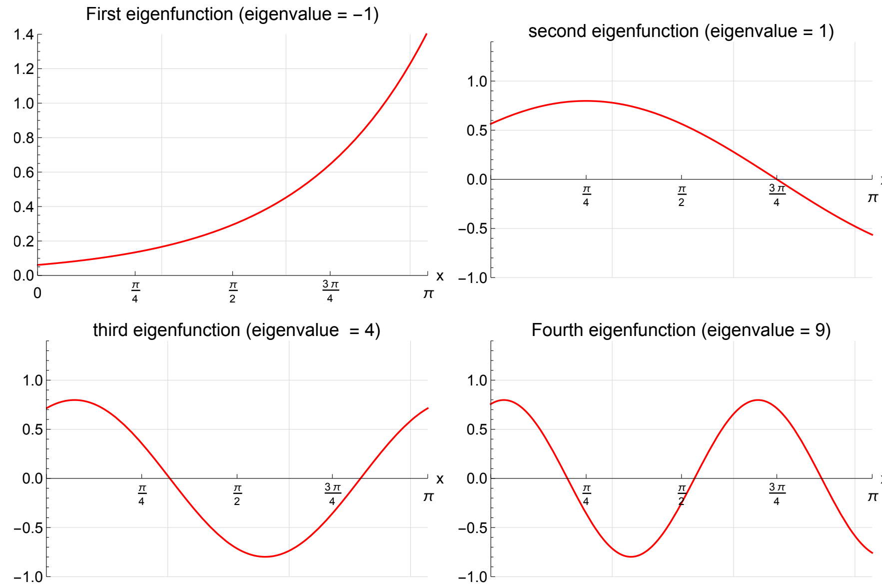
$$\hat{\Phi}_n(x) = \frac{\sqrt{2}}{\sqrt{\pi\left(1+n^2\right)}}\left(n\cos(nx)+\sin(nx)\right) \qquad n=1,2,3,\cdots$$

In summary

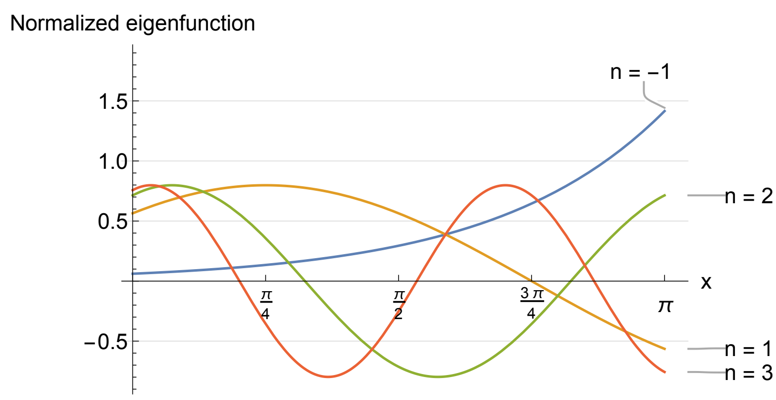
$\lambda=-1$ is eigenvalue with corresponding normalized eigenfunction $\hat{\Phi}_{-1}(x)=\left(\frac{\sqrt{2}}{\sqrt{e^{2\pi}-1}}\right)e^x$

$\lambda_n=n^2$ for $n=1,2,\cdots$ with corresponding normalized eigenfunctions $\hat{\Phi}_n(x)=\frac{\sqrt{2}}{\sqrt{\pi(1+n^2)}}\left(n\cos(nx)+\sin(nx)\right)$.

The normalized eigenfunctions $\hat{\Phi}_{-1},\hat{\Phi}_1,\hat{\Phi}_2,\hat{\Phi}_3$ are plotted next to each others below



The normalized eigenfunctions $\hat{\Phi}_{-1}, \hat{\Phi}_1, \hat{\Phi}_2, \hat{\Phi}_3$ are plotted on the same plot below as well for illustration.



Some observations: The first eigenfunction $\hat{\Phi}_{-1}(x)$ has no root in $[0, \pi]$, the second eigenfunction $\hat{\Phi}_1$ has one root in $[0, \pi]$ and the third eigefunction has two roots in $[0, \pi]$ and so on. This is what is to be expected. The n^{th} ordered eigenfunction will have $(n-1)$ number of roots (or x axis crossings) inside the domain.

2 Problem 2

Problem Expand $f(x) = 1$ in a series of eigenfunctions of problem 1

solution

Let

$$f(x) = b_{-1}\hat{\Phi}_{-1}(x) + \sum_{n=1}^{\infty} b_n \hat{\Phi}_n(x) \tag{1}$$

The goal is to determine $b_{-1}, b_1, b_2, \cdots$. This is done by applying orthogonality. Multiplying both sides of (1) by $r(x)\hat{\Phi}_{-1}(x)$ and integrating over the domain gives

$$\int_0^{\pi} r(x) f(x) \hat{\Phi}_{-1}(x)\,dx = \int_0^{\pi} b_{-1} r(x) \hat{\Phi}_{-1}^2(x)\,dx + \sum_{n=1}^{\infty} b_n \int_0^{\pi} r(x) \hat{\Phi}_{-1}(x) \hat{\Phi}_n(x)\,dx$$

But $r(x) = 1$ and due to orthogonality of eigenfunctions, all terms in the sum are zero. The above simplifies to

$$\int_0^{\pi} f(x) \hat{\Phi}_{-1}(x)\,dx = b_{-1} \int_0^{\pi} \hat{\Phi}_{-1}^2(x)\,dx$$

But $f(x) = 1$ and $\int_0^{\pi} \hat{\Phi}_{-1}^2(x)\,dx = 1$ since normalized eigenfunctions. Hence the above becomes

$$b_{-1} = \int_0^{\pi} \hat{\Phi}_{-1}(x)\,dx$$

From problem one, $\hat{\Phi}_{-1}(x) = \left(\frac{\sqrt{2}}{\sqrt{e^{2\pi}-1}}\right) e^x$, therefore the above becomes

$$\begin{aligned} b_{-1} &= \frac{\sqrt{2}}{\sqrt{e^{2\pi}-1}} \int_0^{\pi} e^x dx \\ &= \frac{\sqrt{2}}{\sqrt{e^{2\pi}-1}} \left[e^x\right]_0^{\pi} \\ &= \frac{\sqrt{2}\left(e^{\pi}-1\right)}{\sqrt{e^{2\pi}-1}} \end{aligned}$$

Going back to equation (1), but now the equation is multiplied by $r(x)\hat{\Phi}_m(x)$ for $m > 0$ and integrated using $r(x) = 1$ and $f(x) = 1$ giving

$$\int_0^{\pi} \hat{\Phi}_m(x)\, dx = \int_0^{\pi} b_{-1}\hat{\Phi}_{-1}(x)\hat{\Phi}_m(x)\, dx + \sum_{n=1}^{\infty} b_n \int_0^{\pi} \hat{\Phi}_n(x)\hat{\Phi}_m(x)\, dx$$

Due to orthogonality of eigenfunctions, the above simplifies to

$$\int_0^{\pi} \hat{\Phi}_m(x)\, dx = b_m \int_0^{\pi} \hat{\Phi}_m^2(x)\, dx$$

But $\int_0^{\pi} \hat{\Phi}_m^2(x)\, dx = 1$, therefore the above becomes

$$b_n = \int_0^{\pi} \hat{\Phi}_n(x)\, dx$$

From problem one, using $\hat{\Phi}_n(x) = \frac{\sqrt{2}}{\sqrt{\pi(1+n^2)}}\left(n\cos(nx) + \sin(nx)\right)$ the above becomes

$$\begin{aligned} b_n &= \frac{\sqrt{2}}{\sqrt{\pi(1+n^2)}} \int_0^{\pi} \left(n\cos(nx) + \sin(nx)\right) dx \\ &= \frac{\sqrt{2}}{\sqrt{\pi(1+n^2)}} \left(\int_0^{\pi} n\cos(nx)\, dx + \int_0^{\pi} \sin(nx)\, dx\right) \\ &= \frac{\sqrt{2}}{\sqrt{\pi(1+n^2)}} \left(n\left[\frac{\sin(nx)}{n}\right]_0^{\pi} - \left[\frac{\cos(nx)}{n}\right]_0^{\pi}\right) \\ &= \frac{\sqrt{2}}{\sqrt{\pi(1+n^2)}} \left(\sin(n\pi) - \frac{1}{n}\left[\cos(n\pi) - 1\right]\right) \end{aligned}$$

But $\sin(n\pi) = 0$ since n is integer and $\cos(n\pi) = (-1)^n$. The above becomes

$$\begin{aligned} b_n &= \frac{\sqrt{2}}{\sqrt{\pi(1+n^2)}} \left(-\frac{1}{n}\left[-1^n - 1\right]\right) \\ &= \frac{\sqrt{2}}{n\sqrt{\pi(1+n^2)}} \left((-1)^{n+1} + 1\right) \end{aligned}$$

For $n = 1, 3, 5, \cdots$ the above simplifies to

$$b_n = \frac{2\sqrt{2}}{n\sqrt{\pi(1+n^2)}}$$

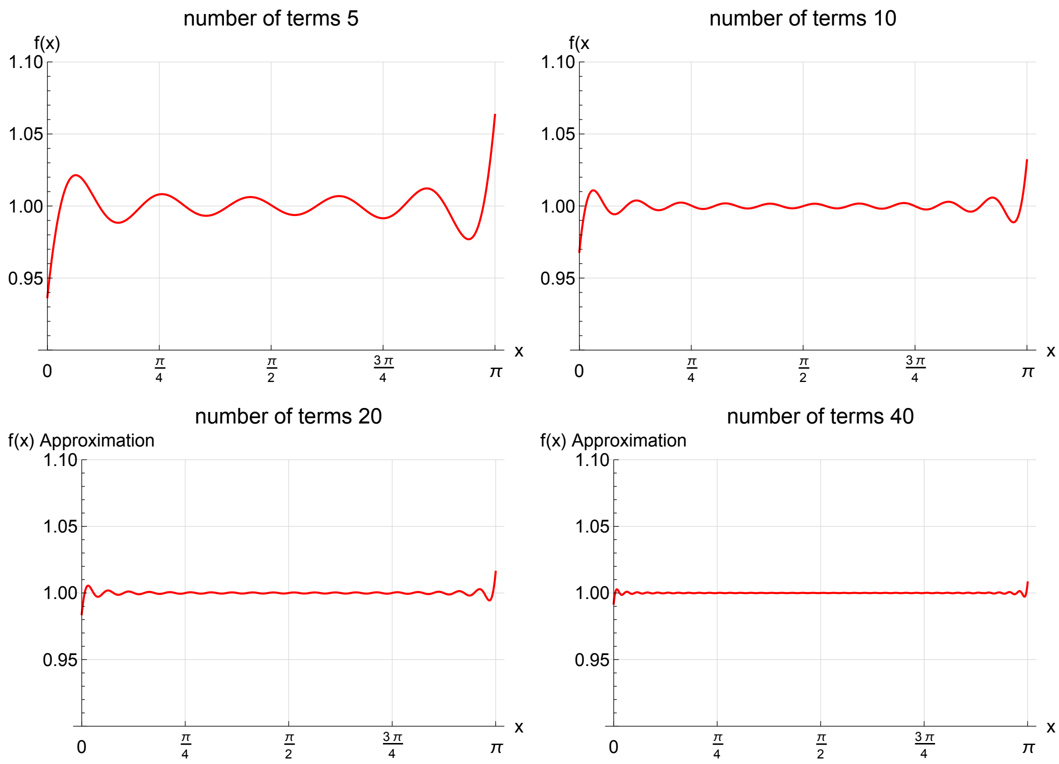
And for $n = 2, 4, 6, \cdots$ gives $b_n = 0$. Therefore the expansion (1) becomes

$$\begin{aligned} f(x) &= \frac{\sqrt{2}\left(e^{\pi}-1\right)}{\sqrt{e^{2\pi}-1}}\hat{\Phi}_{-1}(x) + \sum_{n=1,3,5,\cdots}^{\infty} \frac{2\sqrt{2}}{n\sqrt{\pi(1+n^2)}}\hat{\Phi}_n(x) \\ 1 &= \frac{\sqrt{2}\left(e^{\pi}-1\right)}{\sqrt{e^{2\pi}-1}}\left(\frac{\sqrt{2}}{\sqrt{e^{2\pi}-1}}\right) e^x + \sum_{n=1,3,5,\cdots}^{\infty} \frac{2\sqrt{2}}{n\sqrt{\pi(1+n^2)}} \frac{\sqrt{2}}{\sqrt{\pi(1+n^2)}}\left(n\cos(nx) + \sin(nx)\right) \\ 1 &= \frac{2\left(e^{\pi}-1\right)}{e^{2\pi}-1} e^x + \frac{4}{\pi}\sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n(1+n^2)}\left(n\cos(nx) + \sin(nx)\right) \end{aligned}$$

The above can also be written as

$$1 = \frac{2\left(e^{\pi}-1\right)}{e^{2\pi}-1} e^x + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)\left(1+(2n-1)^2\right)}\left((2n-1)\cos((2n-1)x) + \sin((2n-1)x)\right)$$

To verify the above result, it is plotted for increasing number of n and compared to $f(x) = 1$ to see how well it converges.



Some observations: As more terms are added, the series approximation approaches $f(x) = 1$ more. The convergence is more rapid in the internal of the domain than near the edges. Near the edges at $x = 0$ and $x = 1$, more terms are needed to get better approximation. More oscillation is seen near the edges. This is due to Gibbs phenomenon. Converges is of the order of $O\left(\frac{1}{n^2}\right)$ and the converges is to the mean of $f(x)$.

3 Problem 3

Problem Consider the regular SL problem

$$\begin{aligned} y'' + \lambda y &= 0 \\ y(0) &= 0 \\ 2y(1) - y'(1) &= 0 \end{aligned} \tag{1}$$

Show that the problem has exactly one negative eigenvalue and compute numerically.

solution

The characteristic equation is $r^2 + \lambda = 0$. Therefore the roots are $r = \pm\sqrt{-\lambda}$. There are 3 cases to consider. This problem is asking only for the negative eigenvalues. Therefore only the case $\lambda < 0$ is considered.

Let $\lambda = -\omega^2$ for some real constant. The roots are $r = \pm\sqrt{\omega^2} = \pm\omega$. The solution becomes

$$y = c_1 e^{\omega x} + c_2 e^{-\omega x}$$

Since the exponents are real, the solution can be written in terms of hyperbolic trigonometric functions

$$y = c_1 \cosh \omega x + c_2 \sinh \omega x$$

The first boundary conditions $y(0) = 0$ gives $0 = c_1$. The solution becomes

$$\begin{aligned} y &= c_2 \sinh \omega x \\ y' &= c_2 \omega \cosh \omega x \end{aligned} \tag{2}$$

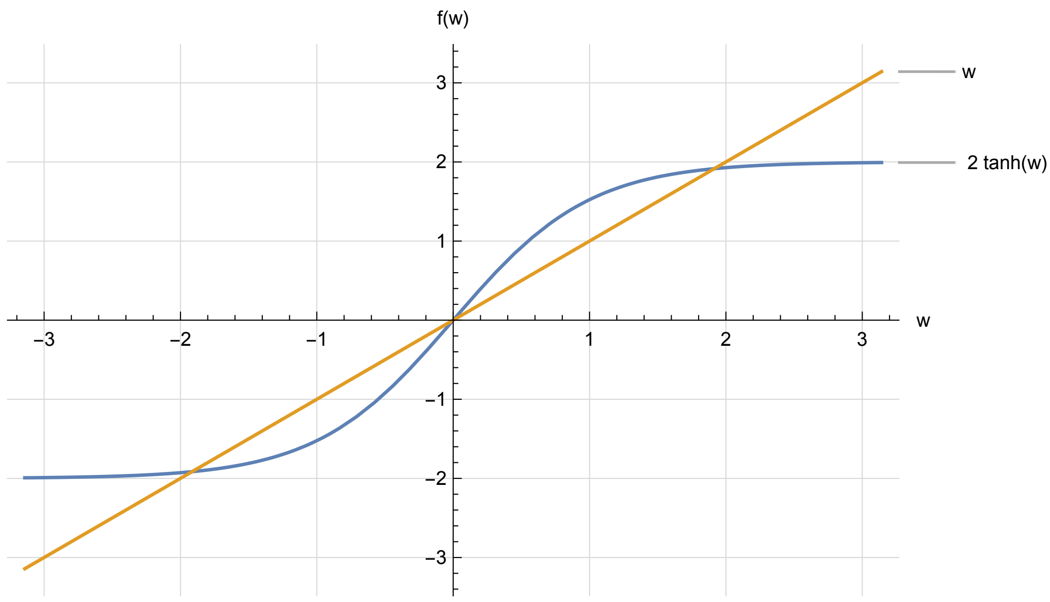
Applying the second boundary conditions $2y(1) - y'(\pi) = 0$ gives

$$\begin{aligned} 0 &= 2c_2 \sinh \omega - c_2 \omega \cosh \omega \\ &= c_2 (2 \sinh \omega - \omega \cosh \omega) \end{aligned}$$

Non trivial solution requires that

$$\begin{aligned} 2 \sinh \omega - \omega \cosh \omega &= 0 \\ 2 \tanh \omega &= \omega \end{aligned}$$

The above equation needs to be solved numerically to find its real roots ω. One root is $\omega = 0$, but this implies $\lambda = 0$. To find if there are other real roots, the function $2\tanh\omega$ and ω were plotted and where they intersect is located. Root finding was then used to obtain the exact numerical value of the roots. The plot below shows that near $\omega = \pm 2$ there is an intersection. There are no other roots since the line $f(\omega) = \omega$ will keep increasing/decreasing and will not intersect $f(\omega) = 2\tanh\omega$ any more after these two roots.



Numerical root finding was used to find the roots near points of intersections. It shows that the exact value of $\omega = \pm 1.91501$. Since $\lambda = -\omega^2$, therefore

$$\boxed{\lambda = -3.66726}$$

Is the only negative eigenvalue.

4 Problem 4

Problem Solve the inhomogeneous B.V.P.

$$\begin{aligned} -y'' &= \mu y + 1 \qquad (1) \\ y(0) - y'(0) &= 0 \\ y(\pi) - y'(\pi) &= 0 \end{aligned}$$

for $\mu = 0$, $\mu = 1$ by methods of section 11.3

5 Part (a)

$$\begin{aligned} -y'' - \mu y &= 1 \\ y'' + \mu y &= -1 \end{aligned}$$

Using chapter 11.3 method, first the eigenfunctions for the corresponding homogenous ODE $y'' + \mu y = 0$ are found for the same boundary conditions. In problem one, it was found that $\lambda = -1$ is eigenvalue with corresponding normalized eigenfunction $\hat{\Phi}_{-1}(x) = \left(\frac{\sqrt{2}}{\sqrt{e^{2\pi}-1}}\right) e^x$ and $\lambda_n = n^2$ for $n = 1, 2, \cdots$ with corresponding normalized eigenfunctions $\hat{\Phi}_n(x) = \frac{\sqrt{2}}{\sqrt{\pi(1+n^2)}}(n\cos(nx) + \sin(nx))$. Since $\lambda = 0$ is not an eigenvalue of the corresponding homogeneous B.V.P., then there is a solution which is by eigenfunction expansion is given by

$$y = b_{-1}\hat{\Phi}_{-1}(x) + \sum_{n=1}^{\infty} b_n \hat{\Phi}_n(x) \qquad (1)$$

Substituting this back into the original ODE gives

$$\left(b_{-1}\hat{\Phi}''_{-1}(x) + \sum_{n=1}^{\infty} b_n \hat{\Phi}''_n(x)\right) + \mu\left(b_{-1}\hat{\Phi}_{-1}(x) + \sum_{n=1}^{\infty} b_n \hat{\Phi}_n(x)\right) = c_{-1}\hat{\Phi}_{-1}(x) + \sum_{n=1}^{\infty} c_n \hat{\Phi}_n(x)$$

Where $-1 = c_{-1}\hat{\Phi}_{-1}(x) + \sum_{n=1}^{\infty} c_n \hat{\Phi}_n(x)$ is the eigenfunction expansion of -1. Since $\mu = 0$, and $\hat{\Phi}_n''(x) = -\lambda_n \hat{\Phi}_n(x)$, the above simplifies to

$$-\lambda_{-1} b_{-1}\hat{\Phi}_{-1}(x) - \sum_{n=1}^{\infty} b_n \lambda_n \hat{\Phi}_n(x) = c_{-1}\hat{\Phi}_{-1}(x) + \sum_{n=1}^{\infty} c_n \hat{\Phi}_n(x)$$

Therefore, equating coefficients gives

$$\begin{aligned} -\lambda_{-1} b_{-1} &= c_{-1} \\ -b_n \lambda_n &= c_n \end{aligned}$$

Or

$$\begin{aligned} b_{-1} &= -\frac{c_{-1}}{\lambda_{-1}} \\ b_n &= -\frac{c_n}{\lambda_n} \end{aligned} \tag{2}$$

What is left is to find c_{-1}, c_n. These are found by applying orthogonality since

$$-1 = c_{-1}\hat{\Phi}_{-1}(x) + \sum_{n=1}^{\infty} c_n \hat{\Phi}_n(x)$$

This was done in problem 2. The difference is the minus sign. Therefore the result from problem 2 is used but c_{-1}, c_n from problem 2 are now multiplied by -1 giving

$$\begin{aligned} c_{-1} &= -\frac{\sqrt{2}\left(e^{\pi} - 1\right)}{\sqrt{e^{2\pi} - 1}} \\ c_n &= -\frac{2\sqrt{2}}{n\sqrt{\pi\left(1 + n^2\right)}} \qquad n = 1, 3, 5, \cdots \end{aligned}$$

Now that c_{-1}, c_n are found, using equation (2) b_{-1}, b_n are can now be found

$$\begin{aligned} b_{-1} &= \frac{\frac{\sqrt{2}(e^{\pi}-1)}{\sqrt{e^{2\pi}-1}}}{(-1)} = -\frac{\sqrt{2}\left(e^{\pi} - 1\right)}{\sqrt{e^{2\pi} - 1}} \\ b_n &= \frac{\frac{2\sqrt{2}}{n\sqrt{\pi(1+n^2)}}}{n^2} = \frac{2\sqrt{2}}{n^3\sqrt{\pi\left(1 + n^2\right)}} \qquad n = 1, 3, 5, \cdots \end{aligned}$$

Hence the solution (1) becomes

$$\begin{aligned} y &= b_{-1}\hat{\Phi}_{-1}(x) + \sum_{n=1}^{\infty} b_n \hat{\Phi}_n(x) \\ &= -\frac{\sqrt{2}\left(e^{\pi} - 1\right)}{\sqrt{e^{2\pi} - 1}}\hat{\Phi}_{-1}(x) + \sum_{n=1,3,5,\cdots}^{\infty} \frac{2\sqrt{2}}{n^3\sqrt{\pi\left(1 + n^2\right)}}\hat{\Phi}_n(x) \\ &= -\frac{\sqrt{2}\left(e^{\pi} - 1\right)}{\sqrt{e^{2\pi} - 1}}\left(\frac{\sqrt{2}}{\sqrt{e^{2\pi} - 1}}\right)e^x + \sum_{n=1,3,5,\cdots}^{\infty} \frac{2\sqrt{2}}{n^3\sqrt{\pi\left(1 + n^2\right)}}\frac{\sqrt{2}}{\sqrt{\pi\left(1 + n^2\right)}}\left(n\cos(nx) + \sin(nx)\right) \\ &= -\frac{2\left(e^{\pi} - 1\right)}{e^{2\pi} - 1}e^x + \frac{4}{\pi}\sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n^3\left(1 + n^2\right)}\left(n\cos(nx) + \sin(nx)\right) \end{aligned} \tag{2A}$$

The above can also be also be written as

$$y(x) = -\frac{2\left(e^{\pi} - 1\right)}{e^{2\pi} - 1}e^x + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3\left(1 + (2n-1)^2\right)}\left((2n-1)\cos((2n-1)x) + \sin((2n-1)x)\right) \tag{2A}$$

To verify the above solution, it was plotted against the solution of $y'' = -1$ found using the direct method to see if they match. The solution using the direct method is found as follows: The homogenous solution is $y_h = c_1 + c_2 x$. Let $y_p = kx^2, y_p' = 2kx, y_p'' = 2k$. Substituting these back into $y'' = -1$ gives $2k = -1$ or $k = -\frac{1}{2}$. Hence $y_p = -\frac{x^2}{2}$ and the solution becomes

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 + c_2 x - \frac{x^2}{2} \end{aligned}$$

Boundary conditions are now applied to determine c_1, c_2. From above, $y'(x) = c_2 - x$. Applying $y(0) - y'(0) = 0$ gives

$$\begin{aligned} 0 &= c_1 - c_2 \\ c_2 &= c_1 \end{aligned}$$

Therefore the solution becomes

$$
\begin{aligned}
y(x) &= c_1(1+x) - \frac{x^2}{2} \\
y'(x) &= c_1 - x
\end{aligned}
$$

Applying second BC $y(\pi) - y'(\pi) = 0$ gives

$$
\begin{aligned}
0 &= c_1(1+\pi) - \frac{\pi^2}{2} - c_1 + \pi \\
0 &= c_1(1+\pi-1) - \frac{\pi^2}{2} + \pi \\
c_1 &= \frac{\frac{\pi^2}{2} - \pi}{\pi} \\
&= \frac{\pi}{2} - 1
\end{aligned}
$$

Therefore, the solution, using direct method is

$$
\begin{aligned}
y(x) &= \left(\frac{\pi}{2} - 1\right)(1+x) - \frac{x^2}{2} \\
&= \frac{\pi}{2} + \frac{\pi}{2}x - 1 - x - \frac{x^2}{2}
\end{aligned}
$$

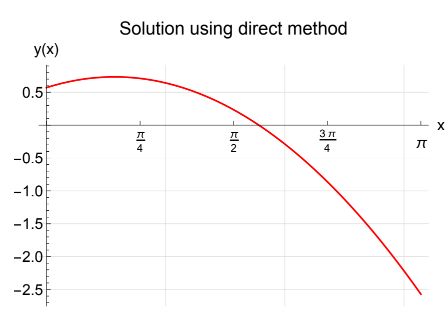
Or

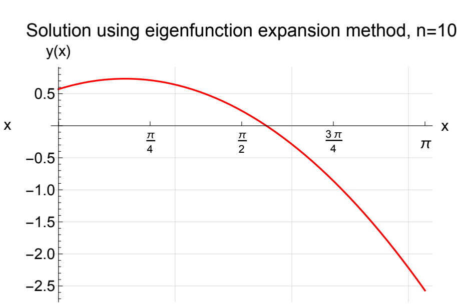
$$
\boxed{y(x) = -\tfrac{x^2}{2} + x\left(\tfrac{\pi}{2} - 1\right) - 1 + \tfrac{\pi}{2}} \tag{3}
$$

What the above says, is that if (2A) solution is correct, it will converge to solution (3) as more terms are added. In other words

$$
-\frac{x^2}{2} + x\left(\frac{\pi}{2} - 1\right) - 1 + \frac{\pi}{2} \approx -\frac{2(e^{\pi}-1)}{e^{2\pi}-1}e^{x} + \frac{4}{\pi}\sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n^3(1+n^2)}\left(n\cos(nx) + \sin(nx)\right)
$$

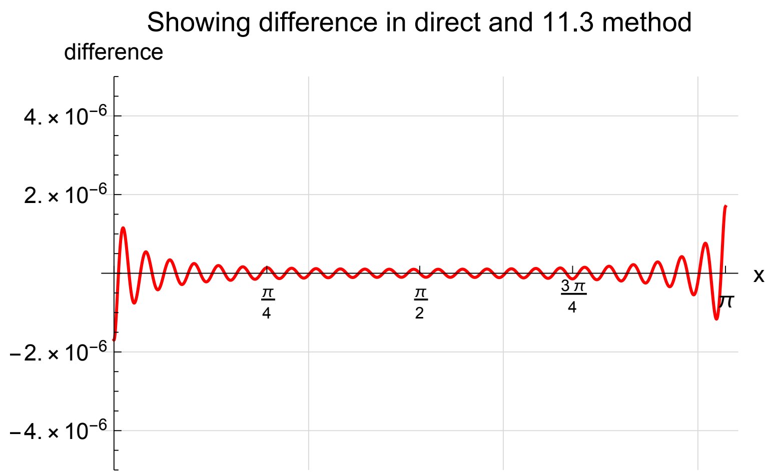
To verify this, the solution from both the direct and the series method were plotted next to each other. Using only $n = 10$ in the sum shows that the plots are identical.





Then the difference between these two solution was plotted. A maximum of $n = 50$ is used in the sum. The plot shows the difference is almost zero in the internal region and near the edges of the domain the difference of order 10^{-7}. This is expected due to Gibbs phenomenon. Adding more terms made the difference smaller. The converges is of order $O\left(\frac{1}{n^2}\right)$.

```
mySol[max_, x_] := -(2 (Exp[Pi] - 1))/(Exp[2 Pi] - 1) Exp[x] + 4/Pi Sum[1/(n^3 (1 + n^2)) (n Cos[n x] + Sin[n x]), {n, 1, max, 2}]
direct[x_] := -x^2/2 + x (Pi/2 - 1) - 1 + Pi/2;
```



6 Part (b)

Now the same process as in part (a) is repeated for $\mu = 1$

$$\begin{aligned} -y'' - \mu y &= 1 \\ y'' + \mu y &= -1 \end{aligned}$$

Using 11.3 method, first the eigenfunctions for the corresponding homogenous ODE $y'' + \mu y = 0$ are found for the same boundary conditions. In problem one, it was found that $\lambda = -1$ is eigenvalue with corresponding normalized eigenfunction $\hat{\Phi}_{-1}(x) = \left(\frac{\sqrt{2}}{\sqrt{e^{2\pi}-1}}\right) e^x$ and $\lambda_n = n^2$ for $n = 1, 2, \cdots$ with corresponding normalized eigenfunctions $\hat{\Phi}_n(x) = \frac{\sqrt{2}}{\sqrt{\pi(1+n^2)}}(n\cos(nx) + \sin(nx))$. Therefore $\lambda = 1$ is an eigenvalue that corresponds to $\mu = 1$. In this case, a solution will exist (and will not be unique) only if the forcing function -1 is orthogonal to $\hat{\Phi}_1(x)$. This is verified as follows. Since $r(x) = 1$, and $n = 1$, then

$$\begin{aligned} \int_0^\pi (-1)\, r(x)\, \hat{\Phi}_1(x)\, dx &= -\int_0^\pi \frac{\sqrt{2}}{\sqrt{\pi(1+n^2)}}(n\cos(nx) + \sin(nx))\, dx \\ &= -\int_0^\pi \frac{\sqrt{2}}{\sqrt{\pi(1+1)}}(\cos(x) + \sin(x))\, dx \\ &= \frac{-\sqrt{2}}{\sqrt{2\pi}} \int_0^\pi \cos(x) + \sin(x)\, dx \\ &= \frac{-1}{\sqrt{\pi}}\left((\sin x)_0^\pi - (\cos x)_0^\pi\right) \\ &= \frac{-1}{\sqrt{\pi}}(0 - (-1 - 1)) \\ &= \frac{-2}{\sqrt{\pi}} \end{aligned}$$

Which is not zero. This means there is no solution.