Homework 3, Math 322

1. Find the eigenvalues and normalized eigenfunctions of the regular Sturm-Liouville problem

$$y'' + \lambda y = 0, \quad y(0) - y'(0) = 0,\ y(\pi) - y'(\pi) = 0.$$

Solution: Let $\lambda = -\omega^2$ with $\omega > 0$, and $y(x) = c_1 e^{\omega x} + c_2 e^{-\omega x}$. Then the boundary conditions give

$$(c_1 + c_2) - \omega(c_1 - c_2) = 0, \quad (c_1 e^{\omega\pi} + c_2 e^{-\omega\pi}) - (c_1 \omega e^{\omega\pi} - c_2 \omega e^{-\omega\pi}) = 0.$$

In order to get a nontrivial solution c_1, c_2 we need

$$\begin{vmatrix} 1-\omega & 1+\omega \\ (1-\omega)e^{\omega\pi} & (1+\omega)e^{-\omega\pi} \end{vmatrix} = (1-\omega^2)(e^{-\omega\pi} - e^{\omega\pi}) = 0.$$

The only solution $\omega > 0$ is $\omega = 1$. Then $c_2 = 0$. Therefore, $\lambda = -1$ is an eigenvalue and $\phi_0(x) = k_0 e^x$ is a corresponding eigenfunction.
If $\lambda = 0$ then $y(x) = c_1 + c_2 x$. The boundary conditions give

$$c_1 - c_2 = 0, \quad c_1 + c_2\pi - c_2 = 0.$$

It follows that $c_1 = c_2 = 0$, so $\lambda = 0$ is not an eigenvalue.
Let $\lambda = \omega^2$, $\omega > 0$, and $y(x) = c_1 \cos \omega x + c_2 \sin \omega x$. The boundary conditions give

$$c_1 - \omega c_2 = 0, \quad (c_1 \cos \omega\pi + c_2 \sin \omega\pi) - (-c_1 \omega \sin \omega\pi + c_2 \omega \cos \omega\pi) = 0.$$

In order to get a nontrivial solution we need

$$\begin{vmatrix} 1 & -\omega \\ \cos \omega\pi + \omega \sin \omega\pi & \sin \omega\pi - \omega \cos \omega\pi \end{vmatrix} = (1+\omega^2)\sin \omega\pi.$$

The solutions $\omega > 0$ are $\omega = n = 1, 2, \dots$. Then $c_1 = nc_2$. Therefore, we found the eigenvalues $\lambda_n = n^2$ with corresponding eigenfunctions $\phi_n(x) = k_n(n \cos nx + \sin nx)$. We calculate

$$1 = k_0^2 \int_0^\pi (e^x)^2\, dx = k_0^2 \frac{1}{2}(e^{2\pi} - 1),$$

$$1 = k_n^2 \int_0^\pi (n \cos nx + \sin nx)^2\, dx = k_n^2 (1+n^2)\frac{\pi}{2},$$

and find the normalized eigenfunctions

$$\hat{\phi}_0(x) = \frac{\sqrt{2}}{\sqrt{e^{2\pi} - 1}}\, e^x,$$

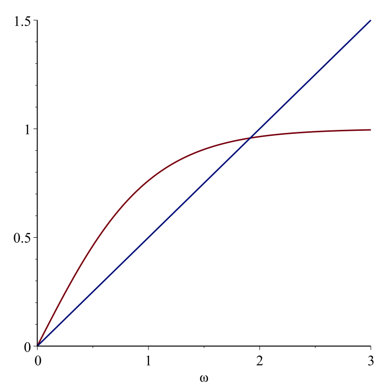
$$\hat{\phi}_n(x) = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{1+n^2}}(n \cos nx + \sin nx), \quad n = 1, 2, \dots$$

2. Expand the function $f(x) = 1$ in a series of eigenfunctions of problem 1.
Solution: For general $f(x)$ the expansion is

$$f(x) = \sum_{n=0}^{\infty} c_n \hat{\phi}_n(x),$$

where
$$c_n = \int_0^\pi f(t)\hat{\phi}_n t)\,dt.$$
If $f(x) = 1$ then
$$c_0 = k_0 \int_0^\pi e^t\,dt = k_0(e^\pi - 1),$$
$$c_n = k_n \int_0^\pi (n\cos nt + \sin nt)\,dt = k_n \frac{1 + (-1)^{n+1}}{n}.$$
Therefore,
$$1 = \frac{2}{e^\pi + 1}\,e^x + \frac{4}{\pi} \sum_{n\geq 1 \text{ odd}} \frac{1}{n(1+n^2)}(n\cos nx + \sin nx).$$
3. Consider the regular Sturm-Liouville problem
$$y'' + \lambda y = 0, \quad y(0) = 0, \quad 2y(1) - y'(1) = 0.$$
Show that this problem has exactly one negative eigenvalue and compute it numerically.

Solution: We set $\lambda = -\omega^2$ with $\omega > 0$. The condition $y(0) = 0$ gives $y(x) = c\sinh\omega x$. The boundary condition at $x = 1$ shows that λ is an eigenvalue if and only if
$$2\sinh\omega = \omega\cosh\omega$$
or
$$\tanh\omega = \frac{1}{2}\omega.$$
The function $\tanh\omega$ is concave for $\omega > 0$ so it is clear from the picture that there is exactly one positive solution $\omega = 1.9150\ldots$ The negative eigenvalue is $\lambda = -3.66725\ldots$



4. Solve the inhomogeneous boundary value problem

$$-y'' = \mu y + 1, \quad y(0) - y'(0) = 0,\ y(\pi) - y'(\pi) = 0$$

for $\mu = 0$ and $\mu = 1$ by the method of Section 11.3.
Solution: If μ is not an eigenvalue, then the solution is

$$y(x) = \sum_{n=0}^{\infty} \frac{c_n}{\lambda_n - \mu} \hat{\phi}_n(x).$$

Therefore, if $\mu = 0$ the solution is

$$y(x) = -\frac{2}{e^{\pi} + 1}\, e^x + \frac{4}{\pi} \sum_{n \geq 1 \text{ odd}} \frac{1}{n^3(1 + n^2)} (n \cos nx + \sin nx).$$

$\mu = 1$ agrees with the eigenvalue λ_1. There exists a solution only if $c_1 = 0$. But in our example, $c_1 \neq 0$ so there is no solution.