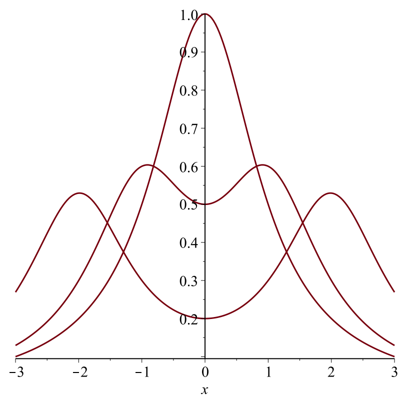
Homework 2, Math 322

1. Solve the wave equation $u_{tt} = u_{xx}$ for $-\infty < x < \infty$, $t \geq 0$ with initial conditions

$$u(x,0) = \frac{1}{1+x^2}, \quad u_t(x,0) = 0.$$

Plot the solutions $u(x,t)$ for $t = 0$, $t = 1$, $t = 2$.
Solution: The d'Alembert solution is

$$u(x,t) = \frac{1}{2}\left(f(x+t) + f(x-t)\right) = \frac{1}{2}\left(\frac{1}{(1+(x-t)^2} + \frac{1}{1+(x-t)^2}\right).$$



2. Apply the method of separation of variables to the damped wave equation $u_{tt} + 2u_t = u_{xx}$, $u(0,t) = u(\pi,t) = 0$, $u(x,0) = f(x)$, $u_t(x,0) = 0$. Determine the first term in the solution $u(x,t) = \sum_{n=1}^{\infty} \cdots$
Solution: By separation of variables we obtain the solutions

$$y_n(x,t) = \sin(nx)T_n(t),$$

where T_n is the solution of

$$T_n'' + 2T_n' + n^2 T_n = 0, \quad T_n(0) = 1, T_n'(0) = 0.$$

Thus

$$T_1(t) = (1+t)e^{-t}$$

and, for $n \geq 2$,

$$T_n(t) = e^{-t}\frac{\sin(t\sqrt{n^2-1})}{\sqrt{n^2-1}} + e^{-t}\cos(t\sqrt{n^2-1}).$$

By superposition, we obtain the solution

$$y(x,t) = \sum_{n=1}^{\infty} b_n T_n(t)\sin(nx)$$

where b_n are the Fourier sine coefficients of $f(x)$. The first term in the solution formula is

$$\frac{2}{\pi}\left(\int_0^\pi f(s)\sin s\,ds\right)e^{-t}(t+1)\sin x.$$

3. Solve the Dirichlet problem $u_{xx}+u_{yy}=0$ on the square $0\le x,y\le 1$ if $u(0,y)=u(x,0)=u(x,1)=0$ and $u(1,y)=y(1-y)$. Find an approximate value for $u(\frac{1}{2},\frac{1}{2})$.

Solution: The Fourier sine series for $y(1-y)$, $0\le y\le 1$, is

$$y(1-y)=\frac{8}{\pi^3}\sum_{n=1,n\text{ odd}}^{\infty}\frac{1}{n^3}\sin(n\pi y).$$

According to Section 10.8, the solution of the Dirichley problem is

$$u(x,y)=\frac{8}{\pi^3}\sum_{n=1,n\text{ odd}}^{\infty}\frac{1}{n^3}\frac{\sinh n\pi x}{\sinh n\pi}\sin(n\pi y).$$

Then

$$u(\tfrac{1}{2},\tfrac{1}{2})=\frac{8}{\pi^3}\sum_{n=1,n\text{ odd}}^{\infty}\frac{1}{n^3}\frac{\sinh n\pi\frac{1}{2}}{\sinh n\pi}(-1)^{(n-1)/2}.$$

Taking two terms of the series, we find

$$u(\tfrac{1}{2},\tfrac{1}{2})\approx 0.05132\ldots$$

4. Solve the Dirichlet problem

$$u_{xx}+u_{yy}=0\quad\text{if } x^2+y^2<1,$$
$$u(x,y)=xy^2\quad\text{if } x^2+y^2=1.$$

Express the solution $u(x,y)$ in terms of x,y.

Solution: We use the terminating Fourier series

$$\cos\theta\sin^2\theta=\cos\theta-\cos^3\theta=\cos\theta-\left(\frac{3}{4}\cos\theta+\frac{1}{4}\cos(3\theta)\right)=\frac{1}{4}\cos\theta-\frac{1}{4}\cos(3\theta).$$

Then from Section 10.8

$$v(r,\theta)=u(r\cos\theta,r\sin\theta)=\frac{1}{4}r\cos\theta-\frac{1}{4}r^3\cos(3\theta),$$

or

$$v(r,\theta)=\frac{1}{4}r\cos\theta-\frac{1}{4}r^3(-3\cos\theta+4\cos^3\theta).$$

Then

$$u(x,y)=\frac{1}{4}x+\frac{3}{4}x(x^2+y^2)-x^3=\frac{1}{4}x-\frac{1}{4}x^3+\frac{3}{4}xy^2.$$