Cheatsheet. Math 322. Intro to PDE. Nasser M ABBASI

Fourier Series

$$F(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi}{L}x\right)+b_n\sin\left(\frac{n\pi}{L}x\right)$$

. note $f(x)$ assume to have $2L$ Period.

$$a_0=\frac{1}{L}\int_{-L}^{L} f(x)\,dx,\quad a_n=\frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx,\quad b_n=\frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dX$$

Fourier Convergence Theorem

IF $f(x)$ is periodic with period $2L$, and integrable over $-L \cdots L$ and if f and f' are Piecewise Continuous on $-L\le x<L$, Then its Fourier Series converges to the average value of $f(x)$ at each point.

Solutions to heat PDE on Rod.

$$u_t=\alpha^2 u_{xx},\quad 0<x<L,\ t>0$$

$u=0$ —— $u=0$, $f(x)$ @ $u(x,0)$, L

$$u(x,t)=\sum_{n=1}^{\infty} C_n e^{-\lambda_n\alpha^2 t}\sin(\sqrt{\lambda_n}x);\quad \lambda_n=\left(\frac{n\pi}{L}\right)^2,\ n=1,2,\cdots$$

$\frac{\partial u}{\partial x}=0$ —— $\frac{\partial u}{\partial x}=0$

$$u(x,t)=A_0+\sum_{n=1}^{\infty} C_n e^{-\lambda_n\alpha^2 t}\cos(\sqrt{\lambda_n}x);\quad \lambda_n=\left(\frac{n\pi}{L}\right)^2,\ n=1,2,\cdots$$

$\frac{du}{dx}=0$ —— $u=0$

$$u(x,t)=\sum_{n=1,3,5\cdots}^{\infty} C_n e^{-\lambda_n\alpha^2 t}\cos(\sqrt{\lambda_n}x);\quad \lambda_n=\left(\frac{n\pi}{2L}\right)^2\quad n=1,3,5?$$

$u=0$ —— $\frac{\partial u}{\partial x}=0$

$$u(x,t)=\sum_{n=1,3,5,\cdots}^{\infty} C_n e^{-\lambda_n\alpha^2 t}\sin(\sqrt{\lambda_n}x);\quad \lambda_n=\left(\frac{n\pi}{2L}\right)^2,\ n=1,3,5$$

in all above Cases $C_n=\frac{2}{L}\int_0^L f(x)\Phi_n(x)\,dx$. Where $\Phi_n(x)=\sin\sqrt{\lambda_n}x$ or $\cos\sqrt{\lambda_n}x$

IF problem has $u=0$ —— $u=0$, $f(x)$, $-L/2$, 0, $L/2$ Then solution is

$$u(x,t)=\sum_{n=1}^{\infty} C_n e^{-\lambda_n\alpha^2 t}\sin\left(\sqrt{\lambda_n}\left(x+\frac{L}{2}\right)\right);\quad \lambda_n=\left(\frac{n\pi}{L}\right)^2$$

with $C_n=\frac{1}{L}\int_{-L/2}^{L/2} f\left(x+\frac{L}{2}\right)\sin\left(\sqrt{\lambda_n}\left(x+\frac{L}{2}\right)\right)dx$

B.C. nonhomogeneous. Solve $u(x,t)=w(x,t)+r(x)$, where $r(x)$ is steady state solution, that satisfies the nonhomogeneous B.C. and $w(x,t)$ is standard one.

IF given point source $u=0$ $\underset{0}{\bullet}$——$\overset{Q}{\underset{L/2}{\downarrow}}$——$\underset{L}{\bullet}$ $u=0$

Then Solve. 2 problems: $u=0$ $\underset{L/2}{\bullet\!\!-\!\!-\!\!-\!\!\bullet}$ $u=Q$ using steady state $u_1=w_1+r_1$

and Then solve $u=Q$ $\underset{L/2}{\bullet}$——$\underset{L}{\bullet}$ $u=0$ also using steady state $u_2=w_2+r_2$

other sources

IF source do not depend on time, use steady state trick.

$$T_1 \bullet\!\!\underset{L}{-\!\!-\!\!-\!\!-\!\!-}\!\!\bullet T_2 \Rightarrow r(x)=T_1+(T_2-T_1)\frac{x}{L}$$ ← this function satisfies B.C.

IF there is source. $u=0$ $\overset{Q(x)=K}{\bullet\!\!-\!\!-\!\!-\!\!-\!\!-\!\!\bullet}$ $u=0$ Then $r(x)=\frac{KL}{2}x-\frac{Kx^2}{2}$.

This comes from solving $r''=K$ and finding the particular solution.

Wave PDE $u_{tt}=a^2u_{xx}$. bounded domain. on string

Series solution: both ends fixed

$$u(x,t)=\sum_{n=1}^{\infty}C_n\cos(\sqrt{\lambda_n}\,at)\sin(\sqrt{\lambda_n}\,x)$$ case initial Velocity zero

$$=\sum_{n=1}^{\infty}C_n\sin(\sqrt{\lambda_n}\,at)\sin(\sqrt{\lambda_n}\,x)$$ case initial position zero

$\lambda_n=\left(\frac{n\pi}{L}\right)^2,\quad n=1,2,\ldots$

in First case: $C_n=\frac{2}{L}\int_0^L f(x)\sin(\sqrt{\lambda_n}\,x)\,dx$.

in second case $C_n=\frac{2}{L\sqrt{\lambda_n}}\int_0^L g(x)\sin\sqrt{\lambda_n}\,x\,dx$

Left end Fixed, right end free (i.e. $\frac{\partial u}{\partial x}=0$ at $x=L$) initial Velocity zero.

$$u(x,t)=\sum_{n=1,3,5}^{\infty}C_n\cos(\sqrt{\lambda_n}\,at)\sin(\sqrt{\lambda_n}\,x);\quad \lambda_n=\left(\frac{n\pi}{2L}\right)^2,$$ $n=1,3,5,\ldots$

using d'Almbert Solution:

$$u(x,t)=\frac{1}{2}\left(h(x+at)+h(x-at)\right)+\frac{1}{2a}\int_{x-at}^{x+at}g(s)\,ds$$

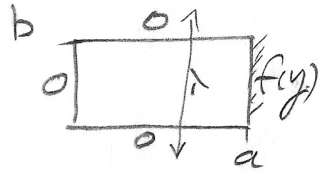
where $h(x)$ is the odd extension of $f(x)$

where $f(x)$ is original position on $0<x<L$. and $h(x)$ is periodic with period $2L$

sheat sheet #2. Math 322. UWM. by Nasser M. Abbasi

Laplace PDE $\quad \nabla^2 u = 0 \quad$ or $\quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

on square

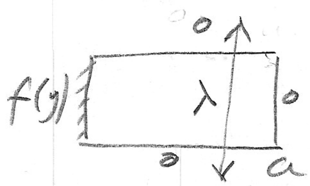


$\Phi_n(y) = \sin(\sqrt{\lambda_n}\, y)$, $\lambda_n = \left(\frac{n\pi}{b}\right)^2 \Rightarrow \Phi_n(y) = \sin\left(\frac{n\pi}{b} y\right)$

$u(x,y) = \sum_{n=1} C_n \sinh(\sqrt{\lambda_n}\, x)\, \Phi_n(y)$

$C_n \sinh(\sqrt{\lambda_n}\, a) = \frac{2}{b}\int_0^b f(y)\, \Phi_n(y)\, dy.$

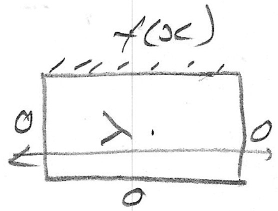
$$u(x,y) = \sum_{n=1}^{\infty} \frac{2}{b} \frac{\sinh(\sqrt{\lambda_n}\, x)}{\sinh(\sqrt{\lambda_n}\, a)} \left(\int_0^b f(y)\, \Phi_n(y)\, dy\right) \Phi_n(y).$$



$\Phi_n(y) = \sin(\sqrt{\lambda_n}\, y)$, $\lambda_n = \left(\frac{n\pi}{b}\right)^2$

$u(x,y) = \sum_{n=1}^{\infty} C_n \sinh(\sqrt{\lambda_n}\,(a-x))\, \Phi_n(y).$

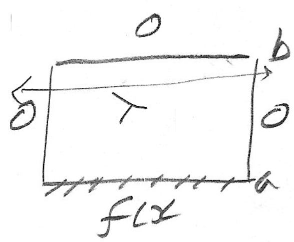
where $C_n \sinh(\sqrt{\lambda_n}\,(a-x)) = \frac{2}{b}\int_0^b f(y)\, \Phi_n(y)\, dy.$



$\Phi_n(x) = \sin(\sqrt{\lambda_n}\, x)$, $\lambda_n = \left(\frac{n\pi}{a}\right)^2$

$u(x,y) = \sum_{n=1}^{\infty} C_n \sinh(\sqrt{\lambda_n}\, y)\, \Phi_n(x).$

where $C_n \sinh(\sqrt{\lambda_n}\, b) = \frac{2}{a}\int_0^a f(x)\, \Phi_n(x)\, dx$

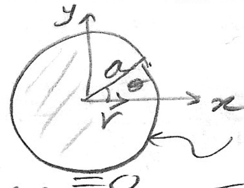


$\Phi_n(x) = \sin(\sqrt{\lambda_n}\, x)$, $\lambda_n = \left(\frac{n\pi}{a}\right)^2$

$u(x,y) = \sum_{n=1}^{\infty} C_n \sinh(\sqrt{\lambda_n}\,(b-y))\, \Phi_n(x)$

where $C_n \sinh(\sqrt{\lambda_n}\,(b-y)) = \frac{2}{a}\int_0^a f(x)\, \Phi_n(x)\, dx$

Laplace inside disk



$f(\theta)$

$\lambda_n = n^2, \quad n = 1, 2, \dots$ also $n=0$

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

Solution is $\quad u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (C_n \cos n\theta + K_n \sin n\theta)$

at origin, u = average of $f(\theta)$.

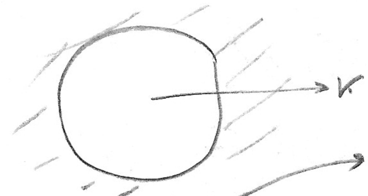
$$\boxed{A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta} \quad \text{and} \quad C_n a^n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta$$

$$K_n a^n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta$$

Same as heat PDE in steady state

Functions $r^n \cos n\theta$, $r^n \sin n\theta$ are called spherical harmonics.

outside disk



only difference

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} \boxed{r^{-n}} (C_n \cos n\theta + K_n \sin n\theta)$$

exterior harmonics: $r^{-n}\cos n\theta$, $r^{-n}\sin n\theta$.

to convert solution in (r,θ) back to (x,y) use

$$r^n \cos n\theta = \sum_{\substack{k=0 \\ \text{even}}}^{n} \frac{n!}{k!(n-k)!} x^{n-k} (-1)^{\frac{k}{2}} y^k$$

$$r^n \sin n\theta = \sum_{\substack{k=1 \\ \text{odd}}}^{n} \frac{n!}{k!(n-k)!} x^{n-k} (-1)^{\frac{k-1}{2}} y^k$$

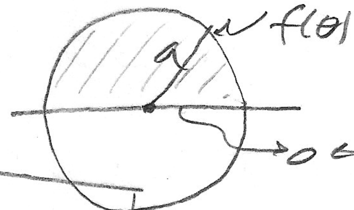
For exterior use $\quad r^{-n}\cos n\theta = \frac{r^n \cos n\theta}{r^{2n}} = \frac{r^n \cos n\theta}{(x^2+y^2)^n}$

$$r^{-n}\sin n\theta = \frac{r^n \sin n\theta}{r^{2n}} = \frac{r^n \sin n\theta}{(x^2+y^2)^n}$$

Cheat sheet #3. Math 322. UWM. Nasser M Abbasi

Laplace on semi-circle

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$



$$\begin{cases} u(r,0) = 0 \\ u(r,\pi) = 0 \end{cases}$$

$u(a,\theta) = f(\theta)$

$0 < \theta < \pi$

Solution is $\boxed{u(r,\theta) = \sum_{n=1}^{\infty} r^n c_n \sin n\theta}$

To find c_n. at $r=a$, $f(\theta) = \sum_{n=1}^{\infty} a^n c_n \sin n\theta$.

by orthogonality $\int_0^{\pi} f(\theta) \sin n\theta = a^n c_n \cdot \frac{\pi}{2} \Rightarrow c_n = \frac{2}{\pi a^n} \int_0^{\pi} f(\theta) \sin n\theta \, d\theta$

integration table.

$$\int x \cos ax \, dx = \frac{\cos ax}{a^2} + \frac{x \sin ax}{a}$$

$$\int x \sin ax \, dx = \frac{\sin ax}{a^2} - \frac{x \cos ax}{a}$$

$$\int x^2 \sin ax \, dx = \frac{2x \sin ax}{a^2} + \left(\frac{2}{a^3} - \frac{x^2}{a}\right) \cos ax$$

$$\int x^2 \cos ax \, dx = \frac{2x \cos ax}{a^2} + \left(\frac{x^2}{a} - \frac{2}{a^3}\right) \sin ax$$

$$\int \sin^2 ax \, dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$$

$$\int \cos^2 ax \, dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$\int \sin ax \cos ax \, dx = \frac{\sin^2 ax}{2a}$$

$$\int \sin ax \cos bx \, dx = -\frac{\cos(a-b)x}{2(a-b)} - \frac{\cos(a+b)x}{2(a+b)}$$

$$\int \frac{dx}{ax+b} dx = \frac{1}{a} \ln(ax+b)$$

$$\int \frac{x}{ax+b} dx = \frac{x}{a} - \frac{b}{a^2} \ln(ax+b)$$

$$\int (x-b)\sin ax\,dx = \int x\sin ax - b\int \sin ax$$
$$= \frac{\sin ax}{a^2} - \frac{x\cos ax}{a} + \frac{b}{a}\cos ax$$

$$\int (x+b)\sin ax\,dx = \int x\sin ax + b\int \sin ax\,dx$$
$$= \frac{\sin ax}{a^2} - \frac{x\cos ax}{a} - \frac{b}{a}\cos ax.$$

<u>trig identities</u>

$e^{i\theta} = \cos\theta + i\sin\theta$. $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$, $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$

$x+iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$\sinh x = \dfrac{e^{x}-e^{-x}}{2}$, $\cosh x = \dfrac{e^{x}+e^{-x}}{2}$,

$\cos^2 x + \sin^2 x = 1$; $\cosh^2 x - \sinh^2 x = 1$

$\sin(A\pm B) = \sin A\cos B \pm \cos A\sin B$

$\cos(A\pm B) = \cos A\cos B \mp \sin A\sin B$

$\frac{d}{dx}\sin u = \cos u\frac{du}{dx}$

$\frac{d}{dx}\cos u = -\sin u\frac{du}{dx}$

$\frac{d}{dx}\tan u = \sec^2 u\frac{du}{dx}$

$\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

$\frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx}$

$\sin 2A = 2\sin A\cos A$

$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$

$\sin 3A = 3\sin A - 4\sin^3 A$

$\cos 3A = 4\cos^3 A - 3\cos A.$

$\sin^2 A = \frac{1}{2} - \frac{1}{2}\cos 2A.$

$\cos^2 A = \frac{1}{2} + \frac{1}{2}\cos 2A$

$\sin^3 A = \frac{3}{4}\sin A - \frac{1}{4}\sin 3A$

$\cos^3 A = \frac{3}{4}\cos A + \frac{1}{4}\cos 3A$

$\sin^4 A = \frac{3}{8} - \frac{1}{2}\cos 2A + \frac{1}{8}\cos 4A$

$\cos^4 A = \frac{3}{8} + \frac{1}{2}\cos 2A + \frac{1}{8}\cos 4A$

$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A+B}{2}$

$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

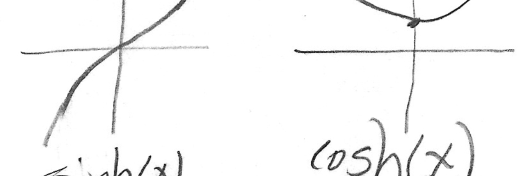
$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

$\cos A - \cos B = 2\sin\frac{A+B}{2}\sin\frac{B-A}{2}$

$\sin A\sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$

$\cos A\cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))$

$\sin A\cos B = \frac{1}{2}(\sin(A-B) + \sin(A+B))$



sinh(x) cosh(x)

Cheat sheet #4. math 322. Nasser M ABBASI.

$d(uv) = u\,dv + v\,du$.

$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$

$z = f(x,y)$. $dz = df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$.

integration by parts: $\int u\,dv = uv - \int v\,du$

$\int u^n du = \frac{u^{n+1}}{n+1}$, $n \neq -1$. $\int \frac{1}{u}du = \ln u$, if $u > 0$

$= \ln|u|$ otherwise.

$\int a^u du = \frac{a^u}{\ln a}$. $a>0$, $a \neq 1$

$\int \sec^2 u\,du = \tan u$

$\int \tan^2 du = \tan u - u$

Fourier Series related integrals

$\int_0^L \sin^2\left(\frac{\pi}{L}x\right) dx = \frac{L}{2}$ same with $\cos^2\left(\frac{\pi}{L}x\right)$

period is $\underline{\underline{2L}}$

$\int_{-L}^{L} \sin^2\left(\frac{\pi}{L}x\right) dx = L$

period is $\underline{\underline{2L}}$

$\int_{-L}^{L} \sin\left(\frac{\pi}{L}x\right)\cos\left(\frac{\pi}{L}x\right) dx = 0$

$\int_{-\pi}^{\pi} \sin^2(x)\,dx = \pi$

period 2π

$\int_0^{\pi} \sin^2(x)\,dx = \frac{\pi}{2}$

$$\int e^{ax}\,dx = \frac{e^{ax}}{a}$$

$$\int x e^{ax}\,dx = \frac{e^{ax}}{a}\left(x - \frac{1}{a}\right)$$

$$\int x^2 e^{ax}\,dx = \frac{e^{ax}}{a}\left(x^2 - \frac{2}{a}x + \frac{2}{a^2}\right)$$

$$\int e^{ax}\sin bx = \frac{e^{ax}}{(a^2+b^2)}\left(a\sin bx - b\cos bx\right)$$

$$\int e^{ax}\cos bx = \frac{e^{ax}}{(a^2+b^2)}\left(a\cos bx + b\sin ax\right)$$

$$\int e^{ax}\ln x = \frac{e^{ax}}{a}\ln x - \frac{1}{a}\int \frac{e^{ax}}{x}\,dx$$

$$\int \ln x\,dx = x\ln x - x$$

$$\int x\ln x\,dx = \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right)$$

$$\int \frac{1}{x}\ln x = \frac{1}{2}\ln^2 x$$

Cheat sheet #5

to convert $ay'' + by' + (c + \lambda d)y = 0$ to

$(py')' - qy + \lambda r y = 0$ do

$\boxed{\mu(x) = e^{\int \frac{b}{a} dx}}$ then

$$\boxed{\begin{aligned} p &= \mu \\ q &= -\mu \frac{c}{a} \\ r &= \frac{\mu}{a} d \end{aligned}}$$

to convert ODE to exact form:

$a_0 y'' + a_1 y' + a_2 y = 0 \Rightarrow (a_0 y' + (a_1 - a_0')y)'$

check if these are the same. if yes, Then exact.

$P(x) y'' + Q(x) y' + R(x) y = 0$

This is book formula but for non eigenvalue ODE.

$\Rightarrow (\mu(x) P y')' + \mu(x) R(x) y = 0$

where $\mu(x) = \frac{1}{P} e^{\int^x \frac{Q(s)}{P(s)} ds}$

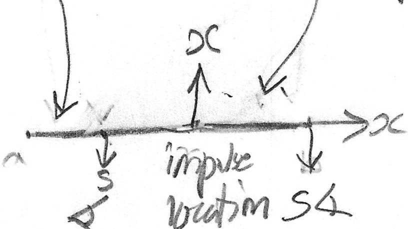
Green Function

$$G(x,s) = \frac{-1}{PW} \begin{cases} y_1(s)\, y_2(x) \\ y_1(x)\, y_2(s) \end{cases}$$

x ↙ impulse, s ↘ measure at

$s \le x$ — left side ~~right side~~

$x < s$ — ~~left side~~ right side



where $y_1(x)$ satisfies BC at (a)

$y_2(x)$ satisfies BC at (b)

$y_1(x), y_2(x)$ are solutions to $L[y] = 0$

$y_1(a) + y_1'(a) = 0$

$y_2(b) + y_2'(b) = 0$

$y = \int_0^1 G(x,s) f(s)\, ds$

so first find y_1, y_2

$G(X, S)$: $X \rightarrow$ impulse location, $S \rightarrow$ response measured at.

Bessel ODE $r^2R''+rR'+(r^2-n^2)R=0$

The order is $\boxed{n}$ for order zero. The above becomes

$$R''+\frac{1}{r}R'+R=0 \quad \text{or} \quad r^2R''+rR'+r^2R=0$$

if we have $R''+\frac{1}{r}R'+\lambda^2 R=0$. to convert to Bessel use

$\boxed{t=r\lambda}$ Then $\frac{dR}{dr}=\frac{dR}{dt}\frac{dt}{dr}=\lambda\frac{dR}{dt}$; $\frac{d^2R}{dr^2}=\frac{d^2R}{dt^2}\lambda^2$.

so ODE becomes $R''(t)+\frac{1}{t}\lambda^2 R'(t)+\lambda^2 R=0$ or

$$\boxed{R''(t)+\frac{1}{t}R'(t)+R=0}$$

Legendre equation $(1-x^2)y''-2xy+n(n+1)=0$

For integer n, solutions are $P_n(x), P_m(x)$.

even orders, odd orders.

Recurrence $(n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x)$.

orthogonality $\int_{-1}^{1} P_n(x)P_m(x)\,dx=\begin{cases}0 & m\neq n\\ \frac{2}{2n+1} & m=n\end{cases}$

$$J_n'=J_{n-1}-\frac{n+1}{x}J_n$$

$$J_n'=-J_{n+1}+\frac{n}{x}J_n$$