Practice solving text book problems for Math 332 Introduction to Partial differential equations Spring 2018 University Of Wisconsin, Milwaukee

Text book: Elementary Differential Equations and Boundary Value Problems. By Boyce and Di Prima

With interactive animations inside the PDF

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Contents

1 Chapter 10.1, Problem 9

Problem Either solve $y'' + 4y = \cos x$ with $y'(0) = 0, y'(\pi) = 0$ or show it has no solution.

Solution The homogeneous solution y_h can be easily found to be

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the basis solutions are

$$\begin{aligned} y_1 &= \cos 2x \\ y_2 &= \sin 2x \end{aligned}$$

And

$$\begin{aligned} y_1' &= -2 \sin 2x \\ y_2' &= 2 \cos 2x \end{aligned}$$

Hence

$$y_h'(x) = -2c_1 \sin 2x + 2c_2 \cos 2x$$

To find particular solution, let

$$y_p = A \cos x$$

The original ODE becomes

$$\begin{aligned} -A \cos x + 4A \cos x &= \cos x \\ 3A \cos x &= \cos x \\ A &= \frac{1}{3} \end{aligned}$$

Hence the full solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= -2c_1 \sin 2x + 2c_2 \cos 2x + \frac{1}{3} \cos x \end{aligned}$$

Therefore

$$y'(x) = -4c_1 \cos 2x - 4c_2 \sin 2x - \frac{1}{3} \sin x$$

First B.C. gives

$$\begin{aligned} y'(0) &= 0 = -4c_1 \\ c_1 &= 0 \end{aligned}$$

Therefore the solution now becomes $y(x) = 2c_2 \cos 2x + \frac{1}{3} \cos x$ and $y'(x) = -4c_2 \sin 2x - \frac{1}{3} \sin x$. The second B.C. gives

$$\begin{aligned} y'(\pi) &= 0 = -4c_2(0) \\ 0 &= -4c_2(0) \end{aligned}$$

Hence c_2 can be any value. Therefore, there is no unique solution. There are infinite number of solutions.

Final solution is

$$y(x) = 2c_2 \cos 2x + \frac{1}{3} \cos x$$

Since $2c_2$ is constant, we can rename it to A and write the above as

$$\boxed{y(x) = A\cos 2x + \tfrac{1}{3}\cos x}$$

To verify that there is no unique solution, we set up W where $y_1 = \cos 2x, y_2 = \sin 2x$, and y_1, y_2 as found above. These are the two basis solutions for the homogeneous ODE.

$$W = \begin{vmatrix} y_1'(0) & y_2'(0) \\ y_1'(\pi) & y_2'(\pi) \end{vmatrix} = \begin{vmatrix} 0 & 2 \\ 0 & 2 \end{vmatrix} = 0$$

Since $W = 0$, this implies there is no unique solution. Therefore the ODE can has no solution, or it can have an infinite number of solutions. In this case, as shown above, it has infinite number of solutions.

2 Chapter 10.1, Problem 12

Problem Either solve $x^2y'' + 3xy' + y = x^2$ with $y(1) = 0, y(e) = 0$ or show it has no solution.

Solution The homogeneous solution is first found. This is a Euler ODE. Let $y_h = x^r$, then $y_h' = rx^{r-1}, y_h'' = r(r-1)x^{r-2}$ and the homogeneous ODE becomes

$$\begin{aligned} r(r-1)x^r + 3rx^r + x^r &= 0 \\ r(r-1) + 3r + 1 &= 0 \\ r^2 - r + 3r + 1 &= 0 \\ r^2 + 2r + 1 &= 0 \\ (r+1)(r+1) &= 0 \end{aligned}$$

Hence double roots. Therefore the solution is

$$y_h = c_1\frac{1}{x} + c_2\frac{1}{x}\ln x$$

To find particular solution, let $y_p = c_1 + c_2x + c_3x^2$. Plugging this in original ODE gives

$$\begin{aligned} x^2(2c_3) + 3x(c_2 + 2c_3x) + \left(c_1 + c_2x + c_3x^2\right) &= x^2 \\ x^2(2c_3) + c_1 + x(3c_2 + c_2) + x^2(6c_3 + c_3) &= x^2 \end{aligned}$$

Comparing coefficients gives

$$\begin{aligned} c_1 &= 0 \\ 4c_2 &= 0 \\ 9c_3 &= 1 \end{aligned}$$

Hence solution is $c_2 = 0, c_1 = 0, c_3 = \frac{1}{9}$. Therefore $y_p = \frac{1}{9}x^2$ and the full solution is

$$\boxed{y(x) = c_1\tfrac{1}{x} + c_2\tfrac{1}{x}\ln x + \tfrac{1}{9}x^2} \tag{1}$$

Boundary conditions are now applied to find c_1, c_2. First BC gives

$$\begin{aligned} 0 &= c_1 + c_2\ln 1 + \frac{1}{9} \\ 0 &= c_1 + \frac{1}{9} \\ c_1 &= -\frac{1}{9} \end{aligned}$$

Second BC $y(e) = 0$ gives

$$\begin{aligned} 0 &= c_1 \frac{1}{e} + c_2 \frac{1}{e} \ln e + \frac{1}{9} e^2 \\ 0 &= -\frac{1}{9e} + c_2 \frac{1}{e} + \frac{1}{9} e^2 \\ c_2 &= \frac{1}{9} - \frac{1}{9} e^3 \\ &= \frac{1 - e^3}{9} \end{aligned}$$

Therefore the solution (1) becomes

$$\boxed{y(x) = -\frac{1}{9x} + \frac{x^2}{9} + \left(\frac{1-e^3}{9}\right) \frac{1}{x} \ln x}$$

Therefore solution exist and is unique. This is verified using W where now $y_1 = \frac{1}{x}, y_2 = \frac{1}{x} \ln x$. These are found above as the bases solutions for the homogeneous ODE.

$$W = \begin{vmatrix} y_1(1) & y_2(1) \\ y_1(e) & y_2(e) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \frac{1}{e} & \frac{1}{e} \end{vmatrix} = \frac{1}{e} \neq 0$$

This confirms that a unique solution exists.

3 Chapter 10.1, Problem 14

Problem Find eigenvalue and eigenfunction of $y'' + \lambda y = 0$ with $y(0) = 0, y'(\pi) = 0$.

Solution

Assuming the solution is $y = Ae^{rx}$, then the characteristic equation is

$$\begin{aligned} r^2 + \lambda &= 0 \\ r &= \pm\sqrt{-\lambda} \end{aligned}$$

Case $\lambda < 0$

In this case $-\lambda$ is positive and hence $\sqrt{-\lambda}$ is also positive. Let $\sqrt{-\lambda} = \mu$ where $\mu > 0$. Hence the roots are $\pm\mu$. This gives the solution

$$y = c_1 \cosh(\mu x) + c_2 \sinh(\mu x)$$

First BC gives

$$0 = c_1$$

Hence solution becomes

$$y(x) = c_2 \sinh(\mu x)$$

Second BC gives

$$\begin{aligned} y'(x) &= \mu c_2 \cosh(\mu x) \\ 0 &= \mu c_2 \cosh(\mu \pi) \end{aligned}$$

But $\cosh \mu\pi \neq 0$, hence only other choice is $c_2 = 0$, leading to trivial solution. Therefore $\lambda < 0$ is not eigenvalue.

Case $\lambda = 0$, then the homogenous solution is

$$y(x) = c_1 + c_2 x$$

First BC gives

$$0 = c_1$$

Hence solution becomes

$$y(x) = c_2 x$$

Second BC gives

$$\begin{aligned} y'(x) &= c_2 \\ 0 &= c_2 \end{aligned}$$

Leading to trivial solution. Therefore $\lambda = 0$ is not eigenvalue.

Case $\lambda > 0$, then the homogenous solution is

$$y(x) = c_1 \cos\left(\sqrt{\lambda} x\right) + c_2 \sin\left(\sqrt{\lambda} x\right)$$

First BC gives

$$0 = c_1$$

Hence solution becomes

$$y(x) = c_2 \sin\left(\sqrt{\lambda} x\right)$$

Second BC gives

$$\begin{aligned} y'(x) &= \sqrt{\lambda} c_2 \cos\left(\sqrt{\lambda} x\right) \\ 0 &= \sqrt{\lambda} c_2 \cos\left(\sqrt{\lambda} \pi\right) \end{aligned}$$

Non-trivial solution requires $\cos\left(\sqrt{\lambda}\pi\right) = 0$ or $\sqrt{\lambda}\pi = \frac{n\pi}{2}$ for $n = 1, 3, 5, \cdots$. Therefore

$$\begin{aligned} \sqrt{\lambda_n}\pi &= \frac{n\pi}{2} \\ \sqrt{\lambda_n} &= \frac{n}{2} \qquad n = 1, 3, 5, \cdots \end{aligned}$$

Hence the eigenvalues are

$$\lambda_n = \left(\frac{n}{2}\right)^2 \qquad n = 1, 3, 5, \cdots$$

And the corresponding eigenfunction is $\sin\left(\frac{n}{2}x\right)$ for $n = 1, 3, 5, \cdots$. The solution is

$$y(x) = \sum_{n=1,3,5,\cdots}^{\infty} c_n \sin\left(\frac{n}{2}x\right)$$

4 Chapter 10.1, Problem 20

Problem Find eigenvalue and eigenfunction of $x^2 y'' - x y' + \lambda y = 0$ with $y(1) = 0, y(L) = 0, L > 1$

Solution

This is Euler type ODE. Using standard substitution, ley $y = x^r$. The ODE now becomes

$$\begin{aligned} x^2 r(r-1)x^{r-2} - xrx^{r-1} + \lambda x^r &= 0 \\ r(r-1) - r + \lambda &= 0 \\ r^2 - 2r + \lambda &= 0 \end{aligned}$$

The above is called the characteristic equations. Its roots give the solution. The roots are

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4 - 4\lambda}}{2} = 1 \pm \sqrt{1 - \lambda}$$

case $1 - \lambda > 0$

Let $1 - \lambda = \mu^2$ for some real μ. Then the roots are $1 \pm \mu$ and hence the solution is

$$\begin{aligned} y &= c_1 x^{r_1} + c_2 x^{r_2} \\ &= c_1 x^{1+\mu} + c_2 x^{1-\mu} \\ &= x\left(c_1 x^\mu + c_2 \frac{1}{x^\mu}\right) \end{aligned}$$

At first BC $y(1) = 0$ the above gives

$$0 = c_1 + c_2$$

At second BC $y(L) = 0$

$$\begin{aligned} 0 &= L\left(c_1 L^\mu + c_2 \frac{1}{L^\mu}\right) \\ 0 &= c_1 L^\mu + c_2 \frac{1}{L^\mu} \\ 0 &= \frac{c_1 L^{2\mu} + c_2}{L^\mu} \end{aligned}$$

Hence

$$c_1 L^{2\mu} + c_2 = 0$$

But $c_2 = -c_1$, therefore

$$\begin{aligned} c_1 L^{2\mu} - c_1 &= 0 \\ c_1\left(L^{2\mu} - 1\right) &= 0 \end{aligned}$$

For arbitrary $L > 0$ the above can only be satisfied if $c_1 = 0$. This means both c_1, c_2 are zero. Hence $1 - \lambda > 0$ is not possible.

case $1 - \lambda = 0$

Hence the roots now are $r = 1$. Double root. We now in the case of double root the solution can be written as

$$\begin{aligned} y &= c_1 x^{r_1} + c_2 x^{r_1} \ln x \\ &= c_1 x + c_2 x \ln x \end{aligned}$$

At first BC $y(1) = 0$ the above gives

$$0 = c_1$$

Therefore the solution now becomes $y = c_2 x \ln x$. At second BC $y(L) = 0$

$$\begin{aligned} 0 &= c_2 L \ln L \\ 0 &= c_2 \ln L \end{aligned}$$

Since $L > 0$ then only possibility is that $c_2 = 0$.This means both c_1, c_2 are zero. Hence $1 - \lambda = 0$ is not possible.

case $1 - \lambda < 0$

Let $1 - \lambda = -\mu^2$ for some real μ. Then the roots are $1 \pm i\mu$ and hence the solution is

$$\begin{aligned} y &= c_1 x^{r_1} + c_2 x^{r_2} \\ &= c_1 x^{1+i\mu} + c_2 x^{1-i\mu} \\ &= x\left(c_1 x^{i\mu} + c_2 x^{-iu}\right) \end{aligned}$$

The above can be written as

$$\begin{aligned} y &= x\left(c_1 e^{\ln x^{i\mu}} + c_2 e^{\ln x^{-i\mu}}\right) \\ &= x\left(c_1 e^{i\mu \ln x} + c_2 e^{-i\mu \ln x}\right) \end{aligned}$$

Hence $c_1 e^{i\mu \ln x} + c_2 e^{-i\mu \ln x}$ can be written as $C_1 \cos(\mu \ln x) + C_2 \sin(\mu \ln x)$. This is done using Euler relation and the new constants C_1, C_2 are not the same as c_1, c_2. The solution becomes

$$y = x\left(C_1 \cos(\mu \ln x) + C_2 \sin(\mu \ln x)\right)$$

First BC $y(1) = 0$ the above becomes

$$\begin{aligned} 0 &= C_1 \cos(\mu \ln 1) + C_2 \sin(\mu \ln 1) \\ &= C_1 \end{aligned}$$

Therefore the solution is

$$y = xC_2 \sin(\mu \ln x) \tag{1}$$

For second BC $y(L) = 0$ the above becomes

$$\begin{aligned} 0 &= LC_2 \sin(\mu \ln L) \\ 0 &= C_2 \sin(\mu \ln L) \end{aligned}$$

Non-trivial solution requiressin $(\mu \ln L) = 0$ or $\mu \ln L = n\pi$ for $n = 1, 2, 3, \cdots$. This means

$$\mu = \frac{n\pi}{\ln L} \qquad n = 1, 2, 3, \cdots$$

But $1 - \lambda = -\mu^2$, or $\lambda = 1 + \mu^2$, therefore

$$\lambda_n = 1 + \left(\frac{n\pi}{\ln L}\right)^2 \qquad n = 1, 2, 3, \cdots \tag{2}$$

These are the eigenvalues. The corresponding eigenfunctions are from (1)

$$\begin{aligned} y_n(x) &= c_n x \sin(\mu_n \ln x) \\ &= c_n x \sin\left(\sqrt{\lambda_n - 1} \ln x\right) \\ &= c_n x \sin\left(\sqrt{1 + \left(\frac{n\pi}{\ln L}\right)^2 - 1} \ln x\right) \\ &= c_n x \sin\left(\sqrt{\left(\frac{n\pi}{\ln L}\right)^2} \ln x\right) \\ &= c_n x \sin\left(\frac{n\pi}{\ln L} \ln x\right) \qquad n = 1, 2, 3, \cdots \end{aligned}$$

Hence the solution is

$$y(x) = x \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{\ln L} \ln x\right)$$

5 Chapter 10.1, Problem 22

22. Consider a horizontal metal beam of length L subject to a vertical load $f(x)$ per unit length. The resulting vertical displacement in the beam $y(x)$ satisfies the differential equation

$$EI\frac{d^4y}{dx^4} = f(x),$$

where E is Young's modulus and I is the moment of inertia of the cross section about an axis through the centroid perpendicular to the xy-plane. Suppose that $f(x)/EI$ is a constant k. For each of the boundary conditions given below, solve for the displacement $y(x)$, and plot y versus x in the case that $L = 1$ and $k = -1$.

(a) Simply supported at both ends: $y(0) = y''(0) = y(L) = y''(L) = 0$

(b) Clamped at both ends: $y(0) = y'(0) = y(L) = y'(L) = 0$

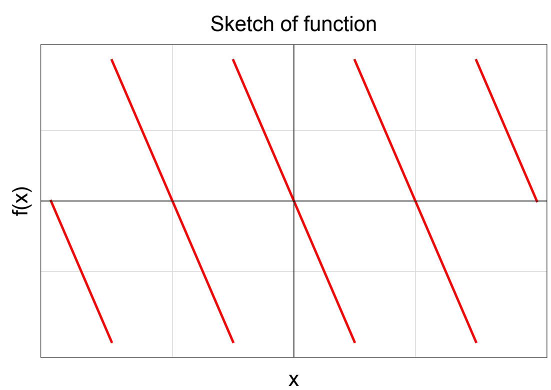
(c) Clamped at $x = 0$, free at $x = L$: $y(0) = y'(0) = y''(L) = y'''(L) = 0$

This is standard ODE with constant coefficients. Just integrating and substitutions.

6 Chapter 10.2, Problem 13 (With interactive animation)

Problem Sketch the graph of $f(x) = -x, -L \leq x < L$ where $f(x+2L) = f(x)$ and find the Fourier series of the function

Solution



This is an odd function. Only b_n needs to be evaluated.

$$b_n = \frac{1}{T/2}\int_{-T/2}^{T/2} f(x)\sin\left(n\frac{2\pi}{T}x\right)$$

T is the period of $f(x)$ which is $2L$. The above becomes

$$b_n = \frac{1}{L}\int_{-L}^{L} -x\sin\left(n\frac{\pi}{L}x\right)$$

Since x is odd and sin is odd then the product is even and the above simplifies to

$$b_n = \frac{-2}{L}\int_0^L x\sin\left(n\frac{\pi}{L}x\right) \tag{1}$$

Using integration by parts $\int udv = uv - \int vdu$ where $u = x, dv = \sin\left(n\frac{\pi}{L}x\right)$, therefore $du = 1$ and

$$v = -\frac{\cos\left(n\frac{\pi}{L}x\right)}{n\frac{\pi}{L}} = \frac{-L}{n\pi}\cos\left(n\frac{\pi}{L}x\right)$$

Integral (1) becomes

$$\begin{aligned}
b_n &= \frac{-2}{L}\left(\left[\frac{-L}{n\pi}x\cos\left(n\frac{\pi}{L}x\right)\right]_0^L - \int_0^L \frac{-L}{n\pi}\cos\left(n\frac{\pi}{L}x\right)dx\right)\\
&= \frac{-2}{L}\left(\left[\frac{-L^2}{n\pi}\cos(n\pi) - 0\right] + \frac{-L}{n\pi}\int_0^L \cos\left(n\frac{\pi}{L}x\right)dx\right)\\
&= \frac{-2}{L}\left(\frac{-L^2}{n\pi}\cos(n\pi) + \frac{-L}{n\pi}\frac{1}{n\frac{\pi}{L}}\left[\sin\left(n\frac{\pi}{L}x\right)\right]_0^L\right)\\
&= \frac{-2}{L}\left(\frac{-L^2}{n\pi}\cos(n\pi) + \frac{-L^2}{n^2\pi^2}\left[\sin(n\pi) - 0\right]\right)\\
&= \frac{-2}{L}\frac{-L^2}{n\pi}\cos(n\pi)\\
&= \frac{2L}{n\pi}\cos(n\pi)
\end{aligned}$$

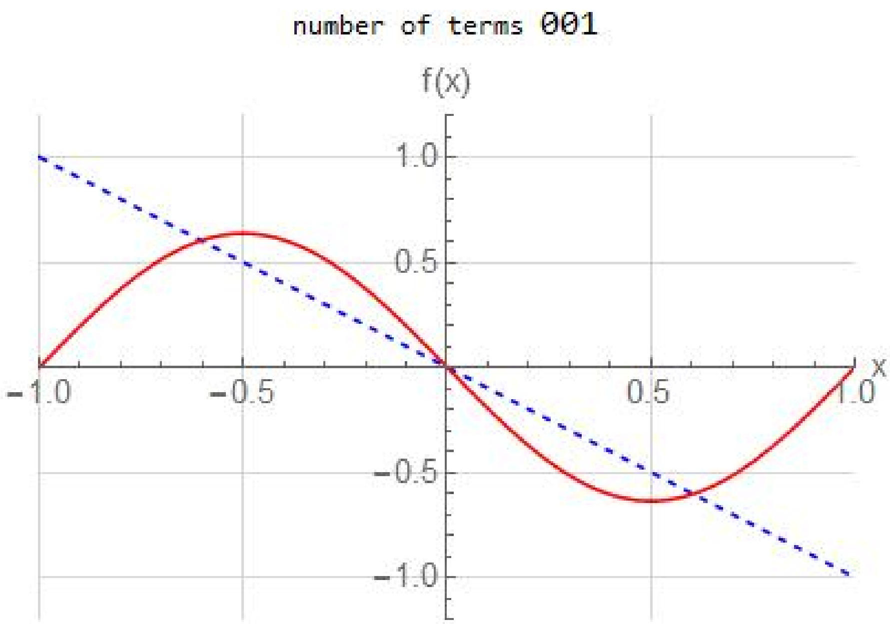
For $n = 1, 2, 3, \cdots$. Looking at few n values gives

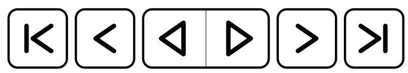
$$\begin{aligned}
b_n &= \frac{2L}{\pi}(-1), \frac{2L}{2\pi}, \frac{2L}{3\pi}(-1), \cdots\\
&= \frac{2L}{n\pi}(-1)^n
\end{aligned}$$

Therefore the Fourier series is

$$\begin{aligned}
f(x) &= \sum_{n=1}^{\infty}\frac{2L}{n\pi}(-1)^n\sin\left(\frac{n\pi}{L}x\right)\\
&= \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{n\pi}{L}x\right)
\end{aligned}$$

The following is an animation showing how the Fourier series converges to the function as more terms are added. This animation runs inside the PDF (need to use standard PDF reader to run the animation. Might not run inside Chrome or Firefox own browser PDF reader).



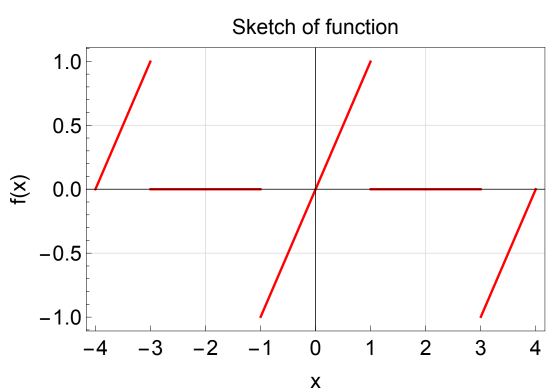


7 Chapter 10.2, Problem 18 (With interactive animation)

Problem Sketch the graph and find the Fourier series of the function

$$f(x) = \begin{cases} 0 & -2 \leq x \leq -1 \\ x & -1 < x < 1 \\ 0 & 1 \leq x < 2 \end{cases}$$

And $f(x+4) = f(x)$
Solution



$f(x)$ is an odd function. Therefore only b_n needs to be evaluated.

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi}{L} x\right)$$

$2L$ is the period of $f(x)$ which is 4. Hence $L = 2$. The above becomes

$$\begin{aligned}
b_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi}{2}x\right)\\
&= \frac{1}{2}\left(\int_{-2}^{-1} f(x)\sin\left(\frac{n\pi}{2}x\right) + \int_{-1}^{1} f(x)\sin\left(\frac{n\pi}{2}x\right) + \int_{1}^{2} f(x)\sin\left(\frac{n\pi}{2}x\right)\right)\\
&= \frac{1}{2}\int_{-1}^{1} f(x)\sin\left(\frac{n\pi}{2}x\right)\\
&= \frac{1}{2}\int_{-1}^{1} x\sin\left(\frac{n\pi}{2}x\right)
\end{aligned}$$

Since x is odd and sin is odd then the product is even and the above simplifies to

$$b_n = \int_{0}^{1} x\sin\left(\frac{n\pi}{2}x\right) \tag{1}$$

Using integration by parts $\int u dv = uv - \int v du$ where $u = x, dv = \sin\left(\frac{n\pi}{2}x\right)$, therefore $du = 1$ and

$$v = -\frac{\cos\left(\frac{n\pi}{2}x\right)}{\frac{n\pi}{2}} = \frac{-2}{n\pi}\cos\left(\frac{n\pi}{2}x\right)$$

Integral (1) becomes

$$\begin{aligned}
b_n &= \frac{-2}{n\pi}\left[x\cos\left(n\frac{\pi}{2}x\right)\right]_0^1 - \int_0^1 \frac{-2}{n\pi}\cos\left(n\frac{\pi}{2}x\right)dx\\
&= \frac{-2}{n\pi}\left[\cos\left(\frac{n}{2}\pi\right)\right] + \frac{2}{n\pi}\int_0^1 \cos\left(n\frac{\pi}{2}x\right)dx\\
&= \left(\frac{-2}{n\pi}\cos\left(\frac{n}{2}\pi\right) + \frac{2}{n\pi}\frac{1}{n\frac{\pi}{2}}\left[\sin\left(n\frac{\pi}{2}x\right)\right]_0^1\right)\\
&= \left(\frac{-2}{n\pi}\cos\left(\frac{n}{2}\pi\right) + \frac{4}{n^2\pi^2}\sin\left(\frac{n}{2}\pi\right)\right)
\end{aligned}$$

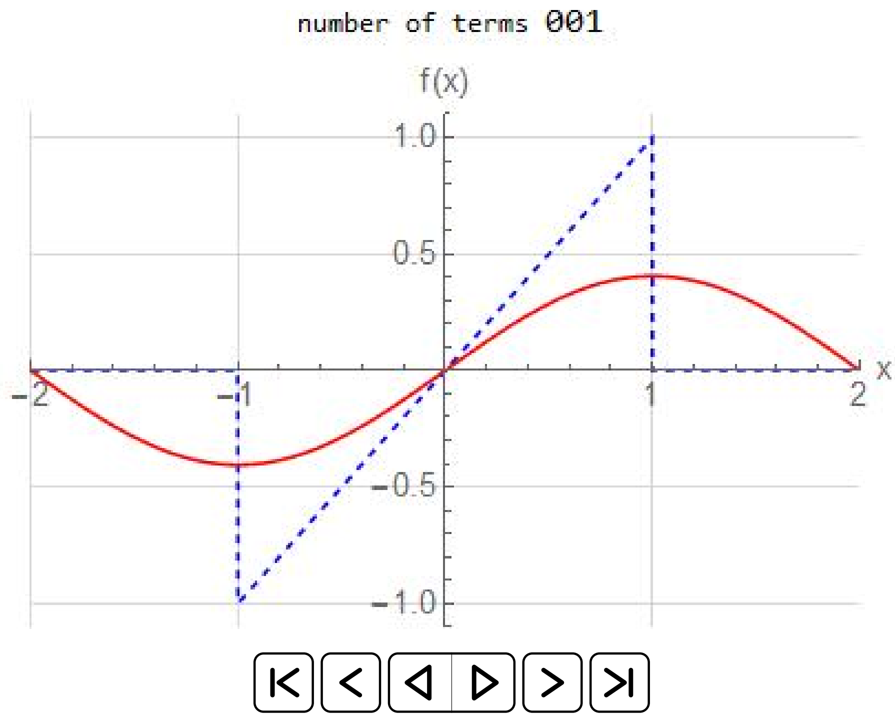
Therefore

$$b_n = \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n}{2}\pi\right) - \frac{2}{n\pi}\cos\left(\frac{n}{2}\pi\right) \qquad n = 1, 2, 3, \cdots$$

The Fourier series is

$$f(x) = \sum_{n=1}^{\infty}\left[\left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n}{2}\pi\right) - \frac{2}{n\pi}\cos\left(\frac{n}{2}\pi\right)\right]\sin\left(\frac{n\pi}{2}x\right)$$

The following is an animation showing how the Fourier series converges to the function as more terms are added. This animation runs inside the PDF (need to use standard PDF reader to run the animation. Might not run inside Chrome or Firefox own browser PDF reader).



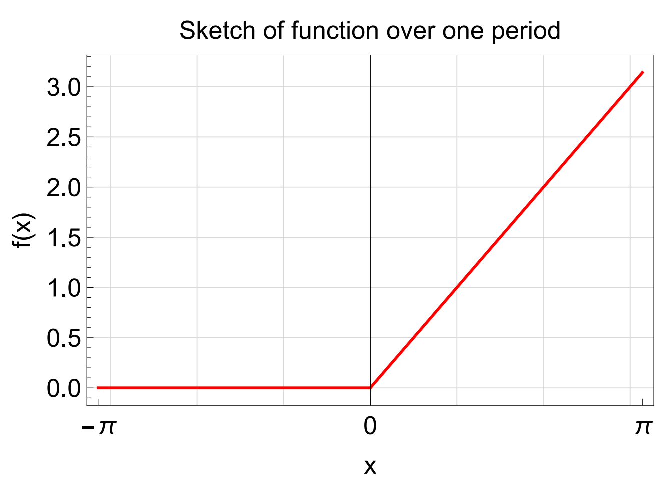
8 Chapter 10.3, Problem 2

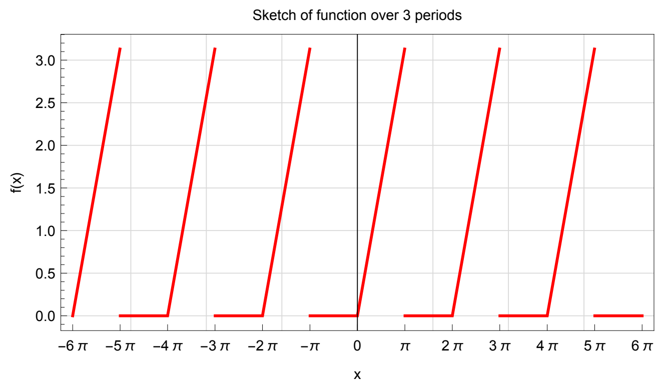
Problem Assume the function is periodically extended outside the original interval. (a) Find the Fourier series of the extended function. (b) Sketch the graph of the function to which the series converges for three periods.

$$f(x) = \begin{cases} 0 & -\pi \le x < 0 \\ x & 0 \le x < \pi \end{cases}$$

Solution

This is plot of the above function for one period, and then for 3 periods





8.1 part a

The calculation of the Fourier series will have a_n, b_n and will follow same methods as before. The period here is 2π.

8.2 part b

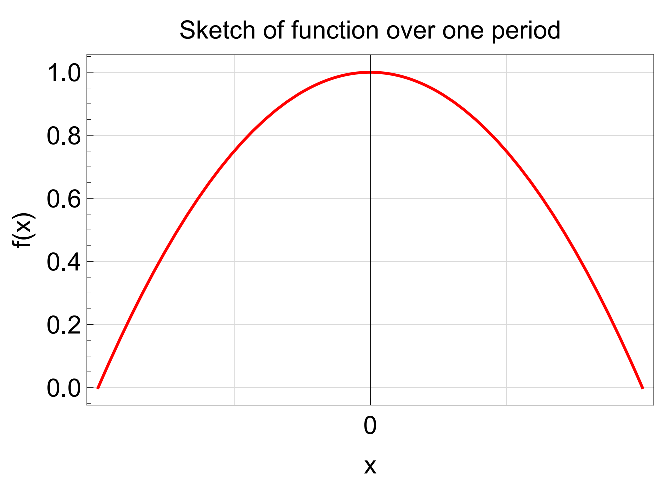
Since both $f(x)$ and $f'(x)$ are piecewise continuous, then the Fourier series will converge to the function $f(x)$. But at the points where $f(x)$ has jumps (such as at $x = \pm\pi$) the Fourier series will converge to the average value of $f(x)$ at these points.

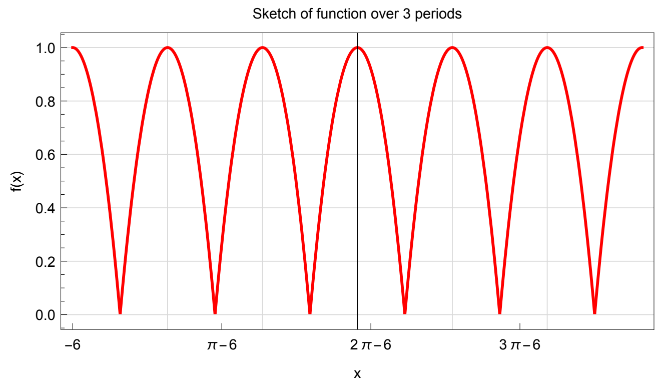
9 Chapter 10.3, Problem 4

Problem Assume the function is periodically extended outside the original interval. (a) Find the Fourier series of the extended function. (b) Sketch the graph of the function to which the series converges for three periods.

$$f(x) = 1 - x^2 \qquad -1 \leq x < 1$$

Solution
This is plot of the above function for one period, and then for 3 periods





9.1 part a

The calculation of the Fourier series will have only a_n since $f(x)$ is even, and will follow same methods as before. The period here is 2.

9.2 part b

Since both $f(x)$ and $f'(x)$ are piecewise continuous, then the Fourier series will converge to the function $f(x)$ for all x.

10 Chapter 10.4, Problem 17

Problem (a) Find the Fourier series of the given function (b) Sketch the graph of the function to which the series converges for three periods.

$$f(x) = 1 \qquad 0 \leq x \leq \pi$$

Use cosine series, with period 2π.

Solution

Extending this as even function gives

$$f_e(x) = 1 \qquad -\pi < x \leq \pi$$

Hence, since period is 2π, then $L = \pi$ now and

$$a_0 = \frac{1}{L}\int_{-L}^{L} f_e(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} dx = \frac{2}{\pi}\int_{0}^{\pi} dx = 2$$

And

$$a_n = \frac{1}{L}\int_{-L}^{L} f_e(x)\cos\left(\frac{n\pi}{L}x\right)dx = \frac{2}{\pi}\int_{0}^{\pi}\cos(nx)\,dx = \frac{2}{n\pi}\left(-\sin(nx)\right)_0^{\pi} = 0$$

Therefore the cosine extension Fourier series is

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) \\ &= \frac{a_0}{2} \\ &= 1 \end{aligned}$$

11 Chapter 10.4, Problem 18 (With interactive animation)

Problem (a) Find the Fourier series of the given function (b) Sketch the graph of the function to which the series converges for three periods.

$$f(x) = 1 \qquad 0 < x < \pi$$

Use sin series, with period 2π.

Solution

Extending this as odd function gives

$$f_o(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x \leq 0 \end{cases}$$

Hence, since period is 2π, then $L = \pi$ now and, since this is an odd function, only b_n terms will show up

$$\begin{aligned} b_n &= \frac{1}{L} \int_{-L}^{L} f_o(x) \sin\left(\frac{n\pi}{L} x\right) dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f_o(x) \sin(nx)\, dx \end{aligned}$$

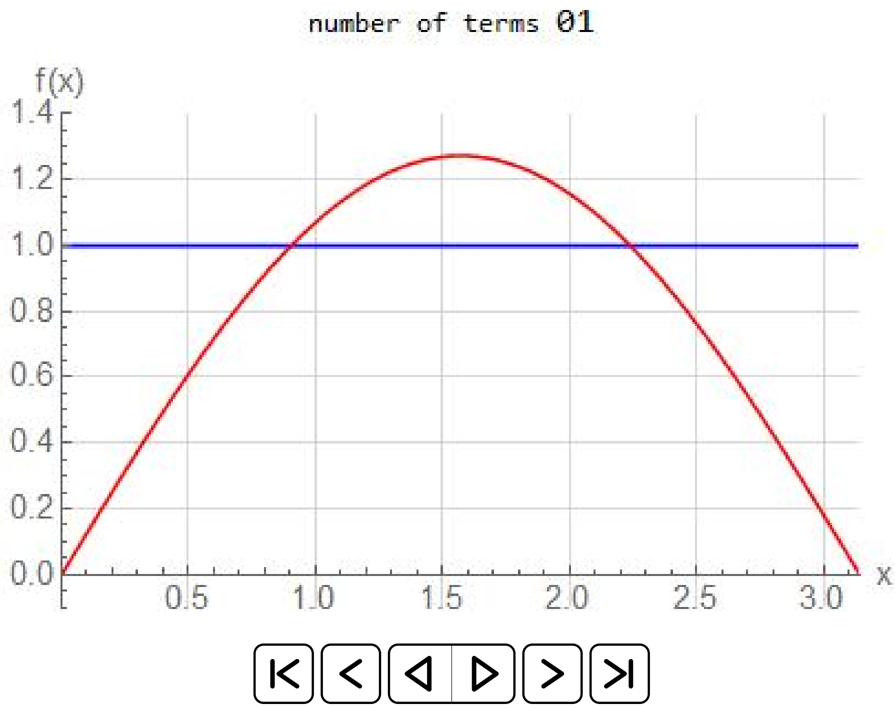
But now $f_o(x) \sin\left(\frac{n\pi}{L} x\right)$ is even, therefore the above simplifies to

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} f_o(x) \sin(nx)\, dx \\ &= \frac{2}{\pi} \int_0^{\pi} \sin(nx)\, dx \\ &= \frac{-2}{\pi} \left(\frac{\cos(nx)}{n}\right)_0^{\pi} \\ &= \frac{-2}{n\pi} (\cos(n\pi) - 1) \\ &= \frac{-2}{n\pi} (-1^n - 1) \end{aligned}$$

Therefore the sine extension Fourier series is

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin(nx) \\ &= \frac{-2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (-1^n - 1) \sin(nx) \end{aligned}$$

The following is an animation showing how the Fourier series converges to the function as more terms are added. This animation runs inside the PDF (need to use standard PDF reader to run the animation. Might not run inside Chrome or Firefox own browser PDF reader).



12 Chapter 10.5, Problem 7

Problem Find solution to $u_t = 100u_{xx}$ with $0 < x < 1, t > 0$ and boundary conditions $u(0,t) = u(1,t) = 0$ and initial conditions $u(x,0) = \sin(2\pi x) - \sin(5\pi x)$

Solution

The fundamental solution for this problem with homogenous B.C. was derived in earlier problem and it is given as

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n k t} \sin\left(\sqrt{\lambda_n} x\right)$$

Where in this problem $k = 100$ and $\lambda_n = (n\pi)^2, n = 1, 2, 3, \dots$. The c_n terms is the Fourier sine coefficients of the initial conditions. But the initial conditions is already expressed as sum of sine terms. Therefore the c_n coefficient can be read directly from $f(x)$, giving $c_2 = 1, c_5 = -1$. Therefore only two terms exist in the sum above, leading to the solution

$$\begin{aligned} u(x,t) &= c_2 e^{-(2\pi)^2(100)t} \sin(2\pi x) + c_5 e \,-^{(5\pi)^2(100)t} \sin(5\pi x) \\ &= e^{-400\pi^2 t} \sin(2\pi x) - e^{-25000\pi t} \sin(5\pi x) \end{aligned}$$

13 Chapter 10.5, Problem 10 (With interactive animation)

Problem Solve $u_t = u_{xx}$, with $0 < x < L$ and $L = 40cm$ and boundary conditions $u(0,t) = u(L,t) = 0^0$ with initial conditions

$$u(x,0) = \begin{cases} x & 0 \leq x < 20 \\ 40 - x & 20 \leq x \leq 40 \end{cases}$$

Solution

The fundamental solution for this problem with homogenous B.C. was derived in earlier problem and it is given as

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n k t} \sin\left(\sqrt{\lambda_n} x\right)$$

Where in this problem $k = 1$ and $\lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1, 2, 3, \ldots$and $L = 40$ cm. To find c_n, initial conditions are used. At $t = 0$

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\sqrt{\lambda_n} x\right)$$

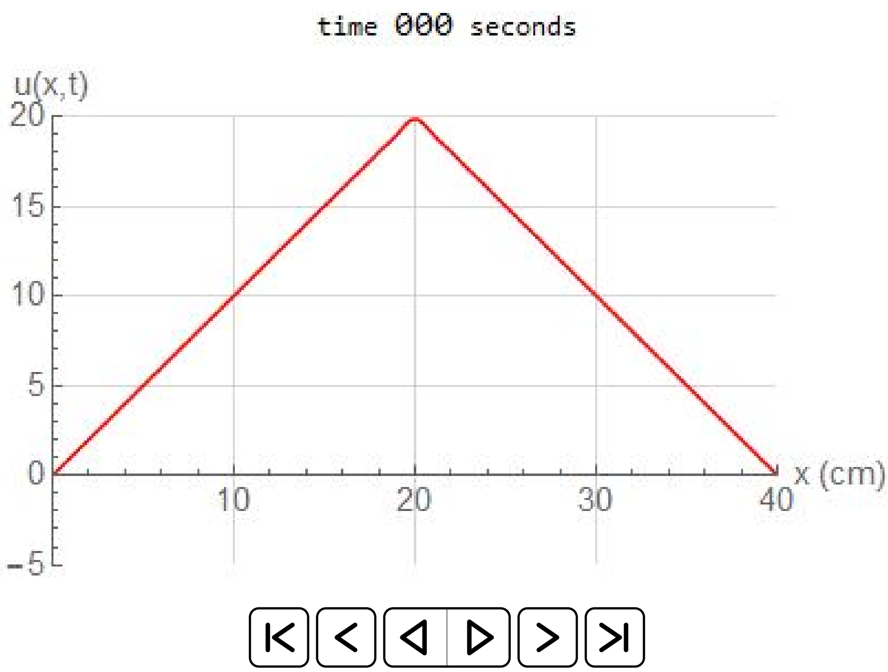
Applying orthogonality result in

$$\begin{aligned} c_n &= \frac{2}{L}\int_0^L f(x) \sin\left(\sqrt{\lambda_n} x\right) dx \\ &= \frac{2}{40}\left(\int_0^{20} x \sin\left(\sqrt{\lambda_n} x\right) dx + \int_{20}^{40} (40 - x) \sin\left(\sqrt{\lambda_n} x\right) dx\right) \\ &= \frac{2}{40}\left(\frac{3200}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)\right) \\ &= \frac{160}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \end{aligned}$$

Hence the solution is

$$u(x,t) = \frac{160}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi}{2}\right) e^{-\left(\frac{n\pi}{40}\right)^2 t} \sin\left(\frac{n\pi}{40} x\right)$$

The following is an animation of the above solution for 510 seconds. This runs inside the PDF (need to use standard PDF reader to run the animation. Might not run inside Chrome or Firefox own browser PDF reader).



14 Chapter 10.5, Problem 11 (With interactive animation)

Problem

Solve $u_t = u_{xx}$, with $0 < x < L$ and $L = 40cm$ and boundary conditions $u(0,t) = u(L,t) = 0^0$ with initial conditions

$$u(x,0) = \begin{cases} 0 & 0 \le x < 10 \\ 50 & 10 \le x \le 30 \\ 0 & 30 \le x \le 40 \end{cases}$$

Solution

The fundamental solution for this problem with homogenous B.C. was derived in earlier problem and it is given as

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n k t} \sin\left(\sqrt{\lambda_n} x\right)$$

Where in this problem $k = 1$ and $\lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1, 2, 3, \ldots$ and $L = 40$ cm. To find c_n, initial conditions are used. At $t = 0$

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\sqrt{\lambda_n} x\right)$$

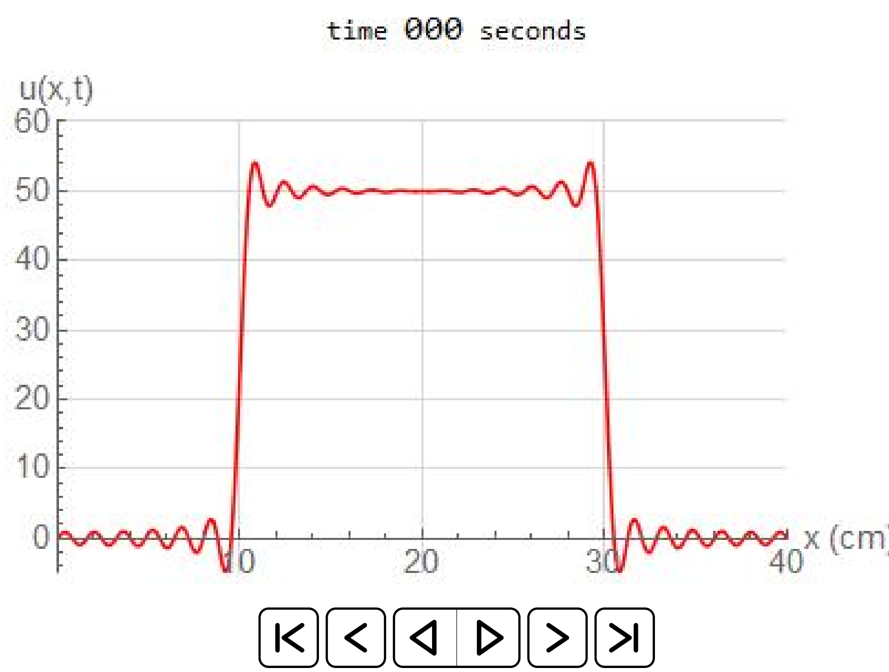
Applying orthogonality result in

$$\begin{aligned}
c_n &= \frac{2}{L}\int_0^L f(x) \sin\left(\sqrt{\lambda_n} x\right) dx \\
&= \frac{2}{40}\left(\int_0^{10} 0 \sin\left(\sqrt{\lambda_n} x\right) dx + \int_{10}^{30} 50 \sin\left(\sqrt{\lambda_n} x\right) dx + \int_{30}^{40} 0 \sin\left(\sqrt{\lambda_n} x\right) dx\right) \\
&= \frac{2}{40}\int_{10}^{30} 50 \sin\left(\sqrt{\lambda_n} x\right) dx \\
&= \frac{200}{n\pi} \sin\frac{n\pi}{4} \sin\frac{n\pi}{2}
\end{aligned}$$

Hence the solution is

$$u(x,t) = \frac{200}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{4}\right) \sin\left(\frac{n\pi}{2}\right) e^{-\left(\frac{n\pi}{40}\right)^2 t} \sin\left(\frac{n\pi}{40} x\right)$$

The following is an animation of the above solution for 510 seconds. This runs inside the PDF (need to use standard PDF reader to run the animation. Might not run inside Chrome or Firefox own browser PDF reader).



15 Chapter 10.6, Problem 5

Problem Find steady state solution that satisfies the given boundary conditions $u_t = \alpha^2 u_{xx}$ with $u(0,t) = 0, u_x(L,t) = 0$

solution at steady state

$$
\begin{aligned}
v''(x) &= 0 \\
v(0) &= 0 \\
v'(L) &= 0
\end{aligned}
$$

Solution to the above ODE is $v(x) = c_1 x + c_2$. At $x = 0$, this leads to $c_2 = 0$. Therefore the solution now becomes $v(x) = c_1 x$ and $v'(x) = c_1$. Second boundary condition implies $c_1 = 0$ as well. Therefore

$$v(x) = 0$$

is the steady state solution.

16 Chapter 10.6, Problem 7

Problem Find steady state solution that satisfies the given boundary conditions $u_t = \alpha^2 u_{xx}$ with $u_x(0,t) - u(0,t) = 0, u(L,t) = T$

solution at steady state

$$
\begin{aligned}
v''(x) &= 0 \\
v'(0) - v(0) &= 0 \\
v(L) &= T
\end{aligned}
$$

Solution to the above ODE is $v(x) = c_1 x + c_2$. At $x = 0$, this leads to $c_1 - c_2 = 0$. Second boundary condition implies $c_1 L + c_2 = 0$. Two equations in 2 unknowns

$$
\begin{aligned}
c_1 - c_2 &= 0 \\
c_1 L + c_2 &= T
\end{aligned}
$$

From first equation, $c_1 = c_2$. Second equation becomes $c_2(1 + L) = T$ or $c_2 = \frac{T}{1+L}$. Therefore the steady state solution

$$
\begin{aligned}
v(x) &= \frac{T}{1+L} x + \frac{T}{1+L} \\
&= \frac{T}{1+L}(1+x)
\end{aligned}
$$

17 Chapter 10.6, Problem 9 (With interactive animation)

Problem Let $L = 20$ cm, with initial temperature $25^0 C$, an initial conditions $u(0,x) = 0, u(L,0) = 60^0 C$. (a) Find $u(x,t)$. (b) Plot initial temperature distribution, final steady state solution and solution are two intermediate times on same axes. (c) Plot u vs. t for $x = 5, 10, 15$. (d) determine how much time has elapsed before the temperature at $x = 5$ cm comes and remains with 1% of the steady state value. Use $\alpha^2 = 0.86$

solution

$$
\begin{aligned}
u_t &= \alpha^2 u_{xx} \\
u(0,x) &= 0 \\
u(L,0) &= 60
\end{aligned}
$$

Let solution be $u(x,t) = w(x,t) + v(x)$ where $v(x)$ is solution to $v''(x) = 0$ with boundary conditions $v(0) = 0, v(L) = 60$. Hence the solution is

$$
v(x) = c_1 x + c_2
$$

At $x = 0$, this leads to $c_2 = 0$. Therefore solution is $v(x) = c_1 x$. At $x = L$, $60 = c_1 L$ or $c_1 = \frac{60}{L} = \frac{60}{20} = 3$. Therefore

$$
v(x) = 3x
$$

Hence the complete solution is

$$
u(x,t) = \left(\sum_{n=1}^{\infty} c_n e^{-\alpha^2 \lambda_n t} \sin\left(\sqrt{\lambda_n} x\right) \right) + 3x
$$

Where $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ for $n = 1, 2, 3, \cdots$. c_n is now found from initial conditions. At $t = 0$

$$
\begin{aligned}
25 &= \left(\sum_{n=1}^{\infty} c_n \sin\left(\sqrt{\lambda_n} x\right) \right) + 3x \\
25 - 3x &= \sum_{n=1}^{\infty} c_n \sin\left(\sqrt{\lambda_n} x\right)
\end{aligned}
$$

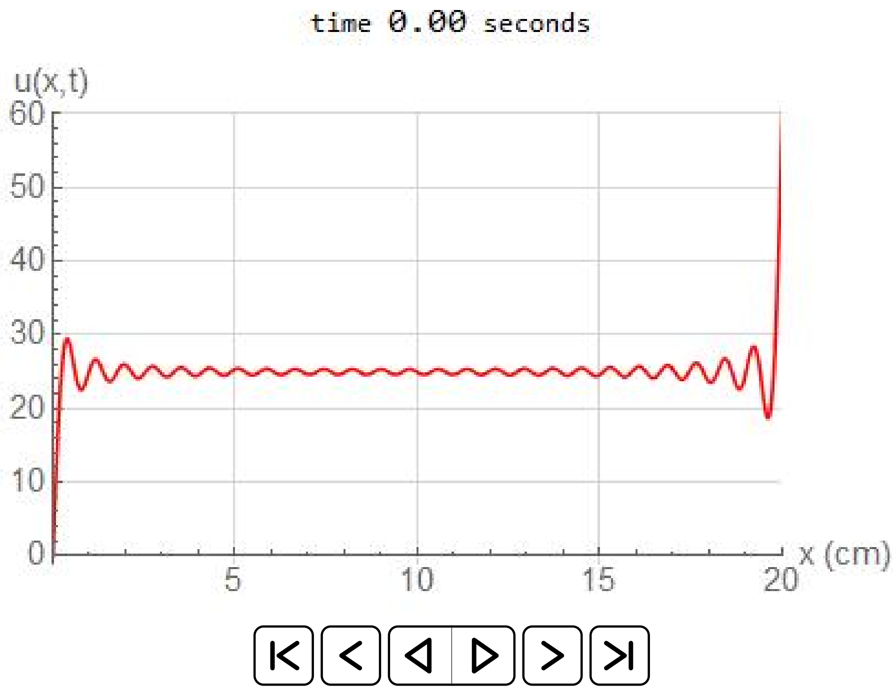
Applying orthogonality gives

$$
\begin{aligned}
\int_0^L (25 - 3x) \sin\left(\sqrt{\lambda_n} x\right) dx &= c_n \frac{L}{2} \\
c_n &= \frac{2}{L} \int_0^L (25 - 3x) \sin\left(\sqrt{\lambda_n} x\right) dx \\
&= \frac{2}{20} \int_0^L (25 - 3x) \sin\left(\sqrt{\lambda_n} x\right) dx
\end{aligned}
$$

Integrating gives $c_n = \frac{50+70(-1)^n}{n\pi}$. Therefore the solution is

$$
u(x,t) = \left(\sum_{n=1}^{\infty} \frac{50 + 70(-1)^n}{n\pi} e^{-\alpha^2 \lambda_n t} \sin\left(\sqrt{\lambda_n} x\right) \right) + 3x
$$

The following is an animation of the above solution for 20 seconds. This runs inside the PDF (need to use standard PDF reader to run the animation. Might not run inside Chrome or Firefox own browser PDF reader).



18 Chapter 10.6, Problem 10

10. (a) Let the ends of a copper rod 100 cm long be maintained at 0°C. Suppose that the center of the bar is heated to 100°C by an external heat source and that this situation is maintained until a steady state results. Find this steady state temperature distribution.

    (b) At a time $t = 0$ [after the steady state of part (a) has been reached], let the heat source be removed. At the same instant let the end $x = 0$ be placed in thermal contact with a reservoir at 20°C, while the other end remains at 0°C. Find the temperature as a function of position and time.

    (c) Plot u versus x for several values of t. Also plot u versus t for several values of x.

    (d) What limiting value does the temperature at the center of the rod approach after a long time? How much time must elapse before the center of the rod cools to within 1°C of its limiting value?

solution
To do.

19 Chapter 10.7, Problem 3 (With interactive animation)

Problem Consider elastic string of length L with ends held fixed. Let initial position $u(x,0) = f(x)$ and $u_t(x,0) = 0$. Let $L = 10, a = 1$. (a) Find $u(x,t)$. (b) Plot $u(x,t)$ vs x for $0 \leq x \leq 10$ and for several values of time between $t = 0$ and $t = 20$ (c) Plot $u(x,t)$ vs. t for $0 \leq t \leq 20$ and for several values of x (d) Construct an animation of the solution for at least one period. (e) Describe the motion of the string. Let $f(x) = \frac{8x(L-x)^2}{L^3}$

solution Since domain is finite, it is easier to use the series solution for wave equation than D'Alembert solution. This is given by

$$u(x,t)=\sum_{n=1}^{\infty}c_n\cos\left(\sqrt{\lambda_n}at\right)\sin\left(\sqrt{\lambda_n}x\right)$$

Where $\lambda_n=\left(\frac{n\pi}{L}\right)^2, n=1,2,3,\cdots$ and $c_n=\frac{2}{L}\int_0^L f(x)\sin\left(\sqrt{\lambda_n}x\right)dx$. Hence, since $a=1$ and $L=10$, the solution becomes

$$\begin{aligned}u(x,t)&=\sum_{n=1}^{\infty}c_n\cos\left(\frac{n\pi}{10}t\right)\sin\left(\frac{n\pi}{10}x\right)\\ c_n&=\frac{2}{10}\int_0^{10}\frac{8x(L-x)^2}{L^3}\sin\left(\frac{n\pi}{10}x\right)dx\\ &=\frac{2}{10}\int_0^{10}\frac{8x(10-x)^2}{10^3}\sin\left(\frac{n\pi}{10}x\right)dx\end{aligned}$$

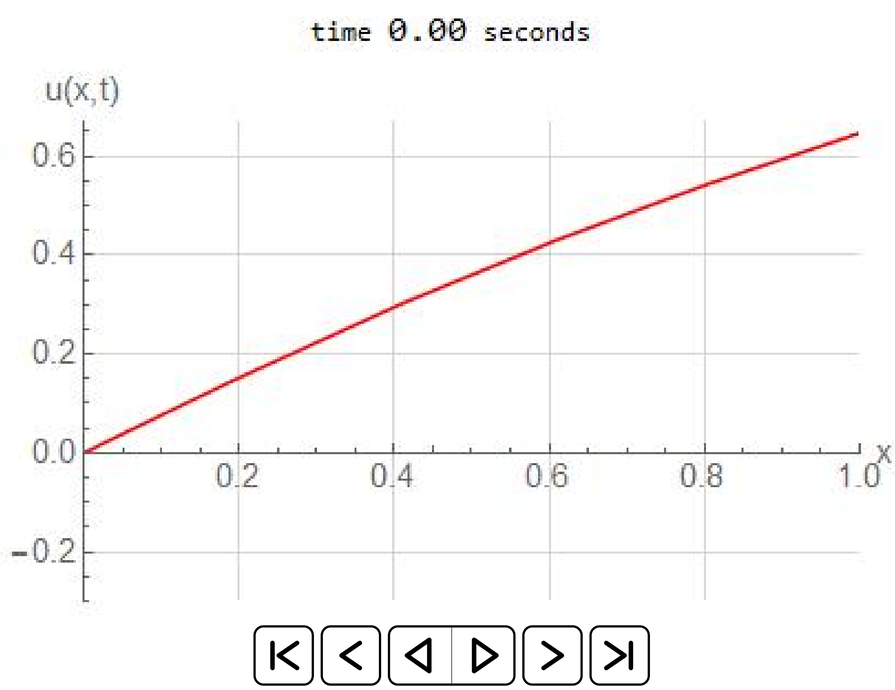
Integrating gives

$$c_n=\frac{32\left(2+(-1)^n\right)}{n^3\pi^3}$$

Hence solution is

$$u(x,t)=\frac{32}{\pi^3}\sum_{n=1}^{\infty}\frac{2+(-1)^n}{n^3}\cos\left(\frac{n\pi}{10}t\right)\sin\left(\frac{n\pi}{10}x\right)$$

The following is an animation of the above solution for 50 seconds. This runs inside the PDF (need to use standard PDF reader to run the animation. Might not run inside Chrome or Firefox own browser PDF reader).



20 Chapter 10.7, Problem 7 (With interactive animation)

Problem Consider elastic string of length L with ends held fixed. Let initial position $u(x,0)=0$ and $u_t(x,0)=g(x)$. Let $L=10$, $a=1$. (a) Find $u(x,t)$. (b) Plot $u(x,t)$ vs x for $0\le x\le 10$ and for several values of time between $t = 0$ and $t = 20$ (c) Plot $u(x,t)$ vs. t for $0 \leq t \leq 20$ and for several values of x (d) Construct an animation of the solution for at least one period. (e) Describe the motion of the string. Let $g(x) = \frac{8x(L-x)^2}{L^3}$

solution Since domain is finite, it is easier to use the series solution for wave equation than D'Alembert solution. The eigenvalue ODE is gives solution for $\lambda > 0$ as

$$X_n(x) = c_n \sin\left(\sqrt{\lambda_n} x\right)$$

Where $\lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1, 2, 3, \cdots$ The time solution is $T_n(t) = A_n \cos\left(\sqrt{\lambda_n} a t\right) + B_n \sin\left(\sqrt{\lambda_n} a t\right)$. At $t = 0$, this gives $0 = A_n$. Therefore $T_n(t) = B_n \sin\left(\sqrt{\lambda_n} a t\right)$. Hence the complete solution is

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} c_n T_n(t) X_n(x) \\ &= \sum_{n=1}^{\infty} c_n \sin\left(\sqrt{\lambda_n} a t\right) \sin\left(\sqrt{\lambda_n} x\right) \end{aligned}$$

To find c_n, time derivative of the above is taken giving

$$\frac{\partial}{\partial t} u(x,t) = \sum_{n=1}^{\infty} c_n \sqrt{\lambda_n} \cos\left(\sqrt{\lambda_n} a t\right) \sin\left(\sqrt{\lambda_n} x\right)$$

At $t = 0$ the above becomes

$$g(x) = \sum_{n=1}^{\infty} c_n \sqrt{\lambda_n} \sin\left(\sqrt{\lambda_n} x\right)$$

Applying orthogonality

$$\begin{aligned} \int_0^L g(x) \sin\left(\sqrt{\lambda_n} x\right) dx &= \sqrt{\lambda_n} c_n \frac{L}{2} \\ c_n &= \frac{2}{L\sqrt{\lambda_n}} \int_0^L g(x) \sin\left(\sqrt{\lambda_n} x\right) dx \end{aligned}$$

Hence since $g(x) = \frac{8x(L-x)^2}{L^3}$, $L = 10, a = 1$ the above becomes

$$c_n = \frac{2}{10\left(\frac{n\pi}{10}\right)} \int_0^{10} \frac{8x(10-x)^2}{10^3} \sin\left(\frac{n\pi}{10} x\right) dx$$

Integrating the above gives

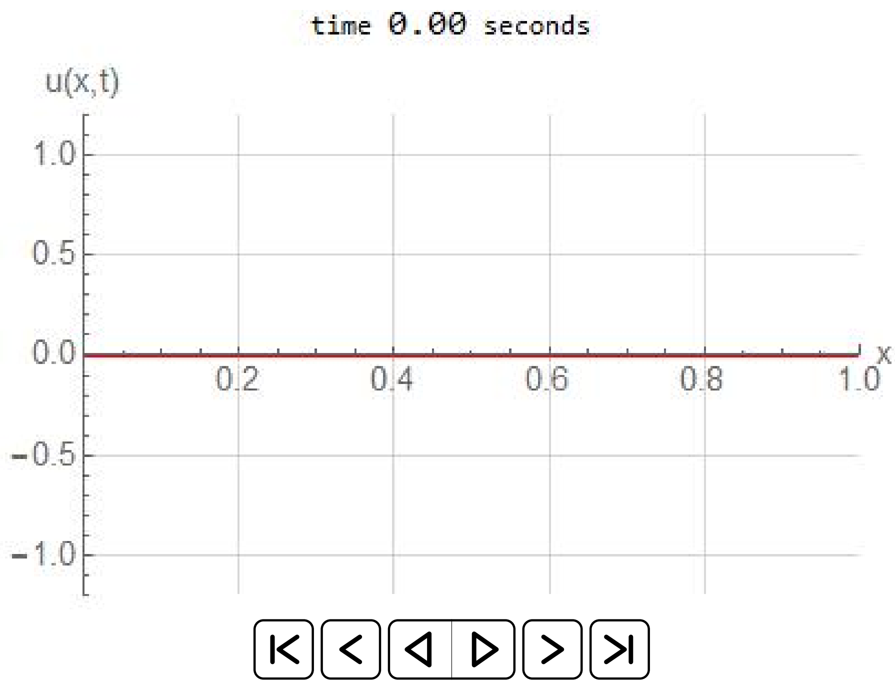
$$c_n = \frac{320\left(2 + (-1)^n\right)}{n^4 \pi^4}$$

Therefore the solution is

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \frac{320\left(2 + (-1)^n\right)}{n^4 \pi^4} T_n(t) X_n(x) \\ &= \frac{320}{\pi^4} \sum_{n=1}^{\infty} \frac{2 + (-1)^n}{n^4} \sin\left(\sqrt{\lambda_n} a t\right) \sin\left(\sqrt{\lambda_n} x\right) \end{aligned}$$

Where $\lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1, 2, 3, \cdots$

The following is an animation of the above solution for 40 seconds. This runs inside the PDF (need to use standard PDF reader to run the animation. Might not run inside Chrome or Firefox own browser PDF reader).



21 Chapter 10.7, Problem 9

9. If an elastic string is free at one end, the boundary condition to be satisfied there is that $u_x = 0$. Find the displacement $u(x,t)$ in an elastic string of length L, fixed at $x = 0$ and free at $x = L$, set in motion with no initial velocity from the initial position $u(x,0) = f(x)$, where f is a given function.
Hint: Show that the fundamental solutions for this problem, satisfying all conditions except the nonhomogeneous initial condition, are

$$u_n(x,t) = \sin \lambda_n x \cos \lambda_n at,$$

where $\lambda_n = (2n-1)\pi/(2L), n = 1, 2, \ldots$. Compare this problem with Problem 15 of Section 10.6; pay particular attention to the extension of the initial data out of the original interval $[0, L]$.

solution
The eigenvalue ODE is

$$X(x) = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)$$

Boundary condition at $x = 0$ gives

$$0 = A$$

Therefore the solution becomes $X(x) = B\sin\left(\sqrt{\lambda}x\right)$. And $X'(x) = B\sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right)$. Applying boundary conditions at $x = L$ gives

$$0 = B\sqrt{\lambda}\cos\left(\sqrt{\lambda}L\right)$$

Therefore

$$\sqrt{\lambda}L = \left\{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots\right\}$$

Hence

$$\sqrt{\lambda_n} = \frac{n\pi}{2L} \qquad n = 1, 3, 5, \cdots$$
$$\sqrt{\lambda_n} = \frac{(2n-1)\pi}{2L} \qquad n = 1, 2, 3, \cdots$$

Therefore

$$X_n(x) = c_n \sin\left(\frac{(2n-1)\pi}{2L}x\right)$$

And

$$T_n(t) = A_n \cos\left(\sqrt{\lambda_n}at\right) + B_n \sin\left(\sqrt{\lambda_n}at\right)$$
$$T_n'(t) = -A_n a\sqrt{\lambda_n} \sin\left(\sqrt{\lambda_n}at\right) + B_n a\sqrt{\lambda_n} \cos\left(\sqrt{\lambda_n}at\right)$$

Since initial velocity is zero, the above gives

$$0 = B_n a\sqrt{\lambda_n}$$

Which means $B_n = 0$. Hence

$$T_n(t) = A_n \cos\left(\sqrt{\lambda_n}at\right)$$

Therefore the complete solution becomes

$$u(x,t) = \sum_{n=1}^{\infty} c_n \cos\left(\frac{(2n-1)\pi}{2L}at\right) \sin\left(\frac{(2n-1)\pi}{2L}x\right)$$

c_n is found from initial position by applying orthogonality.

22 Chapter 10.7, Problem 10

10. Consider an elastic string of length L. The end $x = 0$ is held fixed, while the end $x = L$ is free; thus the boundary conditions are $u(0,t) = 0$ and $u_x(L,t) = 0$. The string is set in motion with no initial velocity from the initial position $u(x,0) = f(x)$, where

$$f(x) = \begin{cases} 1, & L/2 - 1 < x < L/2 + 1 \quad (L > 2), \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the displacement $u(x,t)$.
(b) With $L = 10$ and $a = 1$, plot u versus x for $0 \le x \le 10$ and for several values of t. Pay particular attention to values of t between 3 and 7. Observe how the initial disturbance is reflected at each end of the string.
(c) With $L = 10$ and $a = 1$, plot u versus t for several values of x.
(d) Construct an animation of the solution in time for at least one period.
(e) Describe the motion of the string in a few sentences.

Solution
Straight forward.

23 Chapter 10.8, Problem 3

3. (a) Find the solution $u(x,y)$ of Laplace's equation in the rectangle $0 < x < a, 0 < y < b$, that satisfies the boundary conditions

$$\begin{aligned} &u(0,y) = 0, \qquad && u(a,y) = f(y), \qquad && 0 < y < b, \\ &u(x,0) = h(x), && u(x,b) = 0, && 0 \le x \le a. \end{aligned}$$

Hint: Consider the possibility of adding the solutions of two problems, one with homogeneous boundary conditions except for $u(a,y) = f(y)$, and the other with homogeneous boundary conditions except for $u(x,0) = h(x)$.

(b) Find the solution if $h(x) = (x/a)^2$ and $f(y) = 1 - (y/b)$.

(c) Let $a = 2$ and $b = 2$. Plot the solution in several ways: u versus x, u versus y, u versus both x and y, and a contour plot.

Solution
To do.

24 Chapter 11.1, problem 12

Convert to form $(py')' + q(x)y = 0$

$$y'' - 2xy' + \lambda y = 0$$

Solution
Writing the ODE as $p(x)y'' + Q(x)y' + R(x)y = 0$, hence

$$\begin{aligned} p(x) &= 1 \\ Q(x) &= -2x \\ R(x) &= \lambda \end{aligned}$$

Then the new form is $(\mu(x)p(x)y')' + \mu(x)R(x)y = 0$, where

$$\begin{aligned} \mu(x) &= \frac{1}{p(x)} e^{\int^x \frac{Q(s)}{P(s)} ds} \\ &= e^{\int^x -2s ds} \\ &= e^{-x^2} \end{aligned}$$

Therefore the new form is

$$\left(e^{-x^2} y'\right)' + e^{-x^2} \lambda y = 0$$

25 Chapter 11.1, problem 13

Convert to form $(py')' + q(x)y = 0$

$$x^2 y'' + xy' + \left(x^2 - v^2\right) y = 0$$

Solution

Writing the ODE as $p(x)y'' + Q(x)y' + R(x)y = 0$, hence

$$\begin{aligned} p(x) &= x^2 \\ Q(x) &= x \\ R(x) &= \left(x^2 - v^2\right) \end{aligned}$$

The new form is $(\mu(x)p(x)y')' + \mu(x)R(x)y = 0$, where

$$\begin{aligned} \mu(x) &= \frac{1}{p(x)} e^{\int^x \frac{Q(s)}{P(s)} ds} \\ &= \frac{1}{x^2} e^{\int^x \frac{1}{s} ds} \\ &= \frac{1}{x^2} e^{|\ln x|} \\ &= \frac{1}{x^2} x \\ &= \frac{1}{x} \end{aligned}$$

Therefore the new form is

$$\left(\frac{1}{x}y'\right)' + \frac{1}{x}\left(x^2 - v^2\right) y = 0$$

26 Chapter 11.1, problem 18

18. Consider the boundary value problem

$$y'' + 4y' + (4 + 9\lambda)y = 0, \qquad y(0) = 0, \quad y'(L) = 0.$$

(a) Determine, at least approximately, the real eigenvalues and the corresponding eigenfunctions by proceeding as in Problem 17(a, b).
(b) Also solve the given problem directly (without introducing a new variable).
Hint: In part (a) be sure to pay attention to the boundary conditions as well as the differential equation.

Solution

26.1 part (a)

Let $y(x) = s(x)u(x)$. Then $y' = s'u + su'$ and $y'' = s''u + s'u' + s'u' + su'' = s''u + 2(s'u') + su''$. Therefore the original ODE becomes

$$s''u + 2(s'u') + su'' + 4(s'u + su') + (4 + 9\lambda)su = 0$$

Collecting terms in u gives

$$su'' + u'(2s' + 4s) + (s'' + 4s' + (4 + 9\lambda)s)u = 0$$

Making u' term vanish requires that $2s' + 4s$ or $s' + 2s = 0$. Hence $\frac{d}{dx}\left(se^{2x}\right) = 0$ or $s = e^{-2x}$. Hence $s' = -2e^{-2x}$, $s'' = 4e^{-2x}$. Substituting these into the above gives

$$\begin{aligned}
e^{-2x}u'' + \left(4e^{-2x} + 4\left(-2e^{-2x}\right) + (4+9\lambda)\,e^{-2x}\right) u &= 0\\
u'' + (4 + 4(-2) + (4+9\lambda))\,u &= 0\\
u'' + (4 - 8 + 4 + 9\lambda)\,u &= 0\\
u'' + 9\lambda u &= 0
\end{aligned}$$

Let $9\lambda = \hat{\lambda}$ so the above becomes

$$u'' + \hat{\lambda} u = 0$$

With boundary conditions $u(0) = \frac{y(0)}{s(0)} = 0$ and $u'(L) = \frac{y'(L)}{s'(L)} = 0$. This was solved before, the eigenfunctions of the above are

$$\begin{aligned}
\Phi_n(x) &= \sin\left(\sqrt{\hat{\lambda}_n}x\right)\\
\hat{\lambda}_n &= \left(\frac{n\pi}{2L}\right)^2 \qquad n = 1, 3, 5, \cdots
\end{aligned}$$

But $\hat{\lambda}_n = 9\lambda_n$, therefore the above becomes

$$\begin{aligned}
\Phi_n(x) &= \sin\left(3\sqrt{\lambda_n}x\right)\\
\lambda_n &= \frac{1}{9}\left(\frac{n\pi}{2L}\right)^2 \qquad n = 1, 2, 3, \cdots
\end{aligned}$$

Or

$$\Phi_n(x) = \sin\left(\frac{n\pi}{2L}x\right)$$

Now the eigenfunction is normalized

$$\begin{aligned}
\int_0^1 (k_n \Phi_n(x))^2\, dx &= 1\\
k_n^2 \int_0^1 \Phi_n(x)^2\, dx &= 1\\
k_n^2 \int_0^1 \sin^2\left(\frac{n\pi}{2L}x\right) dx &= 1\\
k_n^2 \frac{L}{2} &= 1\\
k_n &= \sqrt{\frac{2}{L}}
\end{aligned}$$

Hence

$$k_n = \sqrt{\frac{2}{L}}$$

And

$$\hat{\Phi}_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{2L}x\right)$$

Mapping back to $y(x) = s(x)\,u(s)$, and since $s(x) = e^{-2x}$ then the eigenfunction in y space is

$$\Phi_n(x) = e^{-2x}\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{2L}x\right) \qquad n = 1, 3, 5, \cdots$$

26.2 Part b

Now the ODE is solved directly. $y'' + 4y' + (4 + 9\lambda)\, y = 0$. The characteristic equation is

$$r^2 + 4r + (4 + 9\lambda) = 0$$

Hence roots are

$$\begin{aligned} r &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{16 - 4\,(4 + 9\lambda)}}{2} \\ &= \frac{-4 \pm \sqrt{16 - 16 - 36\lambda}}{2} = -2 \pm 3\sqrt{-\lambda} \end{aligned}$$

We know that $\lambda > 0$. So the roots are $r = -2 \pm i\sqrt{\lambda}$ and the solution is

$$y\,(x) = e^{-2x}\left(A \cos\left(\sqrt{\lambda}x\right) + B \sin\left(\sqrt{\lambda}x\right)\right)$$

Applying boundary conditions $y\,(0) = 0$ leads to $A = 0$. So the solution becomes

$$y\,(x) = e^{-2x} B \sin\left(\sqrt{\lambda}x\right)$$

Hence

$$y'\,(x) = -2e^{-2x} B \sin\left(\sqrt{\lambda}x\right) + e^{-2x} B \sqrt{\lambda} \cos\left(\sqrt{\lambda}x\right)$$

Applying second B.C. $y'\,(L) = 0$ the above becomes

$$\begin{aligned} 0 &= -2e^{-2L} B \sin\left(\sqrt{\lambda}L\right) + e^{-2L} B \sqrt{\lambda} \cos\left(\sqrt{\lambda}L\right) \\ &= B\left(-2 \sin\left(\sqrt{\lambda}L\right) + \sqrt{\lambda} \cos\left(\sqrt{\lambda}L\right)\right) \end{aligned}$$

Non-trivial solution requires that

$$\begin{aligned} -2 \sin\left(\sqrt{\lambda}L\right) + \sqrt{\lambda} \cos\left(\sqrt{\lambda}L\right) &= 0 \\ -2 \tan \sqrt{\lambda}L + \sqrt{\lambda} &= 0 \\ \tan \sqrt{\lambda}L &= \frac{1}{2}\sqrt{\lambda} \end{aligned}$$

Hence the direct method finds that the eigenvalues λ_n are the solutions to the above nonlinear equation and the corresponding eigenfunctions are $e^{-2x} \sin\left(\sqrt{\lambda_n}x\right)$.

27 Chapter 11.1, problem 19

Determine the real eigenvalues and eigenfunctions.

$$\begin{aligned} y'' + y' + \lambda\,(y' + y) &= 0 \\ y'\,(0) &= 0 \\ y\,(1) &= 0 \end{aligned}$$

Solution

Writing the ODE as

$$y'' + (1 + \lambda)\, y' + \lambda y = 0$$

Case $\lambda = 0$

$$y'' + y' = 0$$

The characteristic equation is

$$\begin{aligned} r^2 + r &= 0 \\ r(r+1) &= 0 \end{aligned}$$

The roots are $r = 0, -1$. Hence the solution is $y = c_1 + c_2 e^{-x}$. Hence $y' = -c_2 e^{-x}$. First BC gives $y'(0) = 0 \rightarrow 0 = -c_2$. Therefore the solution becomes $y = c_1$. Second BC gives $y(1) = 0 \rightarrow 0 = c_1$. Therefore trivial solution and $\lambda = 0$ is not eigenvalue.

Case $\lambda < 0$ Let $\lambda = -m^2$ for some real m. The ODE becomes

$$y'' + \left(1 - m^2\right) y' - m^2 y = 0$$

The characteristic equation is

$$r^2 + \left(1 - m^2\right) r - m^2 = 0$$

The roots are

$$\begin{aligned} r &= \frac{-\left(1 - m^2\right)}{2} \pm \frac{1}{2}\sqrt{\left(1 - m^2\right)^2 + 4m^2} \\ &= \frac{-\left(1 - m^2\right)}{2} \pm \frac{1}{2}\sqrt{1 + m^4 - 2m^2 + 4m^2} \\ &= \frac{-\left(1 - m^2\right)}{2} \pm \frac{1}{2}\sqrt{\left(1 + m^2\right)^2} \\ &= \frac{-\left(1 - m^2\right)}{2} \pm \frac{1}{2}\left(1 + m^2\right) \end{aligned}$$

Hence roots are $r_1 = \frac{-(1-m^2)}{2} + \frac{1}{2}\left(1 + m^2\right) = m^2$ and $r_2 = \frac{-(1-m^2)}{2} - \frac{1}{2}\left(1 + m^2\right) = -1$. Therefore the solution is

$$y = c_1 e^{m^2 x} + c_2 e^{-x}$$

Hence $y' = m^2 c_1 e^{m^2 x} - c_2 e^{-x}$. First BC gives $y'(0) = 0 \rightarrow 0 = m^2 c_1 - c_2$ or $c_2 = m^2 c_1$. Therefore the solution becomes

$$\begin{aligned} y &= c_1 e^{m^2 x} + m^2 c_1 e^{-x} \\ &= c_1 \left(e^{m^2 x} + m^2 e^{-x}\right) \end{aligned}$$

Second BC gives $y(1) = 0 \rightarrow 0 = c_1 \left(e^{m^2} + m^2 e^{-1}\right)$ therefore $c_1 = 0$ and trivial solution. Hence $\lambda < 0$ is not eigenvalue.

Case $\lambda > 0$ The characteristic equation is

$$r^2 + (1 + \lambda) r + \lambda = 0$$

The roots are

$$\begin{aligned} r &= \frac{-(1 + \lambda)}{2} \pm \frac{1}{2}\sqrt{(1 + \lambda)^2 - 4\lambda} \\ &= \frac{-(1 + \lambda)}{2} \pm \frac{1}{2}\sqrt{1 + \lambda^2 + 2\lambda - 4\lambda} \\ &= \frac{-(1 + \lambda)}{2} \pm \frac{1}{2}\sqrt{(1 - \lambda)^2} \\ &= \frac{-(1 + \lambda)}{2} \pm \frac{1}{2}(1 - \lambda) \end{aligned}$$

Hence roots are $r_1 = \frac{-1}{2}(1+\lambda) + \frac{1}{2}(1-\lambda) = -\lambda$ and $r_2 = \frac{-1}{2}(1+\lambda) - \frac{1}{2}(1-\lambda) = -1$. Therefore the solution is

$$y = c_1 e^{-\lambda x} + c_2 e^{-x}$$

Hence $y' = -\lambda c_1 e^{\lambda x} - c_2 e^{-x}$. First BC gives $y'(0) = 0 \rightarrow 0 = -\lambda c_1 - c_2$ or $c_2 = -\lambda c_1$. Therefore the solution becomes

$$\begin{aligned} y &= c_1 e^{-\lambda x} - \lambda c_1 e^{-x} \\ &= c_1 \left(e^{-\lambda x} - \lambda e^{-x}\right) \end{aligned}$$

Second BC gives $y(1) = 0 \rightarrow 0 = c_1\left(e^{-\lambda} - \lambda e^{-1}\right)$ For non-trivial solution, we need $e^{-\lambda} - \lambda e^{-1} = 0$. The solution to this is $\lambda = 1$.

When $\lambda = 1$ the eigenfunction is

$$y(x) = c_1\left(e^{-x} - e^{-x}\right) = 0$$

But eigenfunction can not be zero. Therefore there is eigenvalue when $\lambda > 0$. Hence for all cases, there is no eigenvalue with corresponding nonzero eigenfunction.

28 Chapter 11.1, problem 20

Determine the real eigenvalues and eigenfunctions.

$$\begin{aligned} x^2 y'' - \lambda(xy' - y) &= 0 \\ y(1) &= 0 \\ y(2) - y'(2) &= 0 \end{aligned}$$

Solution

This is a Euler ODE. $x^2 y'' - \lambda x y' + \lambda y = 0$. Let $y = x^r$, then $y' = r x^{r-1}, y'' = r(r-1)x^{r-2}$. The ODE becomes

$$\begin{aligned} x^2 r(r-1)x^{r-2} - \lambda x r x^{r-1} + \lambda x^r &= 0 \\ r(r-1)x^r - \lambda r x^r + \lambda x^r &= 0 \\ r(r-1) - \lambda r + \lambda &= 0 \end{aligned}$$

Case $\lambda = 0$

The characteristic equation becomes

$$r(r-1) = 0$$

The roots are $r = 0, r = 1$, hence the solution is

$$y = c_1 + c_2 x$$

At BC $y(1) = 0 \rightarrow 0 = c_1 + c_2$. Hence $c_1 = -c_2$ and the solution becomes $y = c_1 - c_1 x = c_1(1-x)$. Hence $y' = -c_1$. Second BC $y(2) - y'(2) = 0$ gives

$$\begin{aligned} 0 &= c_1(1-2) + c_1 \\ 0 &= -c_1 + c_1 \\ 0 &= 0 \end{aligned}$$

Therefore any c_1 will work. Giving a solution

$$y = c_1 (1 - x)$$

Therefore $\lambda = 0$ is an eigenvalue with eigenfunction $\Phi_0(x) = 1 - x$.

Case $\lambda < 0$ Let $\lambda = -m^2$. The characteristic equation becomes

$$\begin{aligned} r(r-1) + m^2 r - m^2 &= 0 \\ r^2 - r + m^2 r - m^2 &= 0 \\ r^2 + r\left(m^2 - 1\right) - m^2 &= 0 \end{aligned}$$

The roots are

$$\begin{aligned} r &= \frac{-\left(m^2-1\right)}{2} \pm \frac{1}{2}\sqrt{\left(m^2-1\right)^2 + 4m^2} \\ &= \frac{-\left(m^2-1\right)}{2} \pm \frac{1}{2}\sqrt{m^4 - 2m^2 + 1 + 4m^2} \\ &= \frac{-\left(m^2-1\right)}{2} \pm \frac{1}{2}\sqrt{\left(1+m^2\right)^2} \\ &= -\frac{1}{2}\left(m^2-1\right) \pm \frac{1}{2}\left(1+m^2\right) \end{aligned}$$

Roots are $r = -\frac{1}{2}\left(m^2-1\right) + \frac{1}{2}\left(1+m^2\right) = 1$ or $r = -\frac{1}{2}\left(m^2-1\right) - \frac{1}{2}\left(1+m^2\right) = -m^2$. Hence solution is

$$y = c_1 x + c_2 x^{-m^2}$$

At BC $y(1) = 0 \to 0 = c_2$. Therefore the solution is $y = c_1 x$ and $y' = c_1$. Second BC gives $y(2) - y'(2) = 0$ or

$$\begin{aligned} 0 &= 2c_1 - c_1 \\ 0 &= c_1 \end{aligned}$$

Hence trivial solution. So $\lambda < 0$ is not an eigenvalue.

Case $\lambda > 0$

The characteristic equation becomes

$$\begin{aligned} r^2 - r - \lambda r + \lambda &= 0 \\ r^2 - r(1+\lambda) + \lambda &= 0 \end{aligned}$$

The roots are

$$\begin{aligned} r &= \frac{1+\lambda}{2} \pm \frac{1}{2}\sqrt{(1+\lambda)^2 - 4\lambda} \\ &= \frac{1+\lambda}{2} \pm \frac{1}{2}\sqrt{1+\lambda^2 - 2\lambda} \\ &= \frac{1+\lambda}{2} \pm \frac{1}{2}\sqrt{(1-\lambda)^2} \\ &= \frac{1+\lambda}{2} \pm \frac{1}{2}(1-\lambda) \end{aligned}$$

Roots are $r = \frac{1}{2}(1+\lambda) + \frac{1}{2}(1-\lambda) = 1$ or $r = \frac{1}{2}(1+\lambda) - \frac{1}{2}(1-\lambda) = \lambda$. Hence solution is

$$y = c_1 x + c_2 x^{\lambda}$$

This is similar to the case above for $\lambda < 0$. Hence there is no eigenvalue for $\lambda > 0$.

29 Chapter 11.2, problem 1

Determine the normalized eigenfunction for

$$\begin{aligned} y'' + \lambda y &= 0 \\ y(0) &= 0 \\ y'(1) &= 0 \end{aligned} \tag{1}$$

<u>Solution</u>

The eigenfunction for the above problem can be easily found using chapter 10 methods to be

$$\Phi_n(x) = \sin\left(\sqrt{\lambda_n} x\right) \qquad n = 1, 3, 5, \cdots$$

Where

$$\lambda_n = \frac{n\pi}{2L} = \frac{n\pi}{2}$$

The normalized $\hat{\Phi}_n(x) = k_n \Phi_n(x)$. Where

$$\int_0^1 \hat{\Phi}_n^2(x)\, dx = 1$$

Hence solving the above for k_n gives

$$\begin{aligned} \int_0^1 (k_n \Phi_n(x))^2\, dx &= 1 \\ k_n^2 \int_0^1 \Phi_n^2(x)\, dx &= 1 \end{aligned}$$

But $\int_0^1 \Phi_n^2(x)\, dx = \int_0^1 \sin^2\left(\sqrt{\lambda_n} x\right) dx = \int_0^1 \sin^2\left(\frac{n\pi}{2} x\right) dx = \frac{1}{2}$. Hence the above becomes

$$\begin{aligned} k_n^2 \frac{1}{2} &= 1 \\ k_n &= \sqrt{2} \end{aligned}$$

Therefore

$$\begin{aligned} \hat{\Phi}_n(x) &= \sqrt{2}\Phi_n(x) \\ &= \sqrt{2}\sin\left(\frac{n\pi}{2} x\right) \qquad n = 1, 3, 5, \cdots \\ &= \left\{ \sqrt{2}\sin\left(\frac{\pi}{2} x\right), \sqrt{2}\sin\left(\frac{3\pi}{2} x\right), \sqrt{2}\sin\left(\frac{5\pi}{2} x\right), \cdots \right\} \end{aligned}$$

30 Chapter 11.2, problem 2

Determine the normalized eigenfunction for

$$\begin{aligned} y'' + \lambda y &= 0 \\ y'(0) &= 0 \\ y(1) &= 0 \end{aligned} \tag{1}$$

Solution

The eigenfunction for the above problem can be found using chapter 10 methods to be

$$\Phi_n(x) = \cos\left(\sqrt{\lambda_n} x\right) \qquad n = 1, 3, 5, \cdots$$

Where

$$\lambda_n = \frac{n\pi}{2L} = \frac{n\pi}{2}$$

The normalized $\hat{\Phi}_n(x) = k_n \Phi_n(x)$. Where

$$\int_0^1 \hat{\Phi}_n^2(x)\,dx = 1$$

Hence solving the above for k_n gives

$$\begin{aligned} \int_0^1 (k_n \Phi_n(x))^2\,dx &= 1 \\ k_n^2 \int_0^1 \Phi_n^2(x)\,dx &= 1 \end{aligned}$$

But $\int_0^1 \Phi_n^2(x)\,dx = \int_0^1 \cos^2\left(\sqrt{\lambda_n} x\right)\,dx = \int_0^1 \cos^2\left(\frac{n\pi}{2} x\right)\,dx = \frac{1}{2}$. Hence the above becomes

$$\begin{aligned} k_n^2 \frac{1}{2} &= 1 \\ k_n &= \sqrt{2} \end{aligned}$$

Therefore

$$\begin{aligned} \hat{\Phi}_n(x) &= \sqrt{2}\Phi_n(x) \\ &= \sqrt{2}\cos\left(\frac{n\pi}{2} x\right) \qquad n = 1, 3, 5, \cdots \\ &= \left\{\sqrt{2}\cos\left(\frac{\pi}{2} x\right), \sqrt{2}\cos\left(\frac{3\pi}{2} x\right), \sqrt{2}\cos\left(\frac{5\pi}{2} x\right), \cdots\right\} \end{aligned}$$

31 Chapter 11.2, problem 3

Determine the normalized eigenfunction for

$$\begin{aligned} y'' + \lambda y &= 0 \qquad (1) \\ y'(0) &= 0 \\ y'(1) &= 0 \end{aligned}$$

Solution

The eigenfunctions are first found. Let the solution be $y = Ae^{rx}$. This leads to the characteristic equation

$$\begin{aligned} r^2 + \lambda &= 0 \\ r &= \pm\sqrt{-\lambda} \end{aligned}$$

Case $\lambda < 0$

In this case $-\lambda$ is positive and hence $\sqrt{-\lambda}$ is also positive. Let $\sqrt{-\lambda} = \mu$ where $\mu > 0$. Hence the roots are $\pm\mu$. This gives the solution

$$
\begin{aligned}
y &= c_1 \cosh(\mu x) + c_2 \sinh(\mu x) \\
y' &= c_1 \mu \sinh(\mu x) + c_2 \mu \cosh(\mu x)
\end{aligned}
$$

First B.C. $y'(0) = 0$ gives

$$
\begin{aligned}
0 &= c_2 \mu \\
c_2 &= 0
\end{aligned}
$$

Hence solution becomes

$$y(x) = c_1 \cosh(\mu x)$$

Second B.C. $y'(1) = 0$ gives

$$0 = c_1 \mu \sinh(\mu)$$

But $\sinh(\mu)$ can not be zero since $\mu \neq 0$, hence $c_1 = 0$, Leading to trivial solution. Therefore $\lambda < 0$ is not eigenvalue.

Let $\lambda = 0$, The solution is

$$y(x) = c_1 + c_2 x$$

First B.C. $y'(0) = 0$ gives

$$0 = c_2$$

The solution becomes

$$y(x) = c_1$$

Second B.C. $y'(1) = 0$ gives

$$0 = 0$$

Therefore c_1 can be any value. Therefore $\lambda = 0$ is an eigenvalue and the corresponding eigenfunction is any constant, say 1.

Case $\lambda > 0$, The solution is

$$
\begin{aligned}
y(x) &= c_1 \cos\left(\sqrt{\lambda} x\right) + c_2 \sin\left(\sqrt{\lambda} x\right) \\
y'(x) &= -c_1 \sqrt{\lambda} \sin\left(\sqrt{\lambda} x\right) + c_2 \sqrt{\lambda} \cos\left(\sqrt{\lambda} x\right)
\end{aligned}
$$

First B.C. $y'(0) = 0$ gives

$$
\begin{aligned}
0 &= c_2 \sqrt{\lambda} \\
c_2 &= 0
\end{aligned}
$$

The solution becomes

$$y(x) = c_1 \cos\left(\sqrt{\lambda} x\right)$$

Second B.C. $y'(1) = 0$ gives

$$0 = -c_1 \sqrt{\lambda} \sin\left(\sqrt{\lambda}\right)$$

For non-trivial solution, we want $\sin\left(\sqrt{\lambda}\right) = 0$ or $\sqrt{\lambda} = n\pi$ for $n = 1, 2, 3, \cdots$ Therefore

$$\lambda_n = (n\pi)^2 \qquad n = 1, 2, 3, \cdots$$

And the corresponding eigenfunctions are

$$\Phi_n(x) = \cos\left(\sqrt{\lambda}x\right) \qquad n = 1, 2, 3, \cdots$$

Hence

$$\begin{aligned}\Phi_0(x) &= 1\\ \Phi_n(x) &= \cos\left(\sqrt{\lambda_n}x\right) \qquad n = 1, 2, 3, \cdots\end{aligned}$$

The normalized $\hat{\Phi}_0(x) = k_0\Phi_0(x)$. Where

$$\int_0^1 r(x)\,\hat{\Phi}_0^2(x)\,dx = 1$$

But $r(x) = 1$. Therefore solving the above for k_0 gives

$$\begin{aligned}\int_0^1 (k_0\Phi_0(x))^2\,dx &= 1\\ k_0^2\int_0^1 dx &= 1\\ k_0 &= 1\end{aligned}$$

And for $n = 1, 2, 3, \cdots$ we obtain

$$\begin{aligned}\int_0^1 \hat{\Phi}_n^2(x)\,dx &= 1\\ \int_0^1 (k_n\Phi_n(x))^2\,dx &= 1\\ k_n^2\int_0^1 \Phi_n^2(x)\,dx &= 1\\ k_n^2\int_0^1 \cos^2\left(\sqrt{n\pi x}\right)dx &= 1\end{aligned}$$

But $\int_0^1 \cos^2\left(\sqrt{n\pi x}\right) = \frac{1}{2}$. Hence the above becomes

$$\begin{aligned}k_n^2\frac{1}{2} &= 1\\ k_n &= \sqrt{2}\end{aligned}$$

Therefore

$$\hat{\Phi}_0(x) = 1$$

And for $n = 1, 2, 3, \cdots$

$$\begin{aligned}\hat{\Phi}_n(x) &= \sqrt{2}\Phi_n(x)\\ &= \sqrt{2}\cos(n\pi x)\\ &= \left\{\sqrt{2}\cos(\pi x), \sqrt{2}\cos(2\pi x), \sqrt{2}\cos(3\pi x), \cdots\right\}\end{aligned}$$

32 Chapter 11.2, problem 4

Determine the normalized eigenfunction for

$$\begin{aligned} y'' + \lambda y &= 0 \\ y'(0) &= 0 \\ y'(1) + y(1) &= 0 \end{aligned} \tag{1}$$

Solution

The eigenfunctions for the above problem are first found. Let the solution be $y = Ae^{rx}$. This leads to the characteristic equation

$$\begin{aligned} r^2 + \lambda &= 0 \\ r &= \pm\sqrt{-\lambda} \end{aligned}$$

Case $\lambda < 0$

In this case $-\lambda$ is positive and hence $\sqrt{-\lambda}$ is also positive. Let $\sqrt{-\lambda} = \mu$ where $\mu > 0$. Hence the roots are $\pm\mu$. This gives the solution

$$\begin{aligned} y &= c_1 \cosh(\mu x) + c_2 \sinh(\mu x) \\ y' &= c_1 \mu \sinh(\mu x) + c_2 \mu \cosh(\mu x) \end{aligned}$$

First B.C. $y'(0) = 0$ gives

$$\begin{aligned} 0 &= c_2 \mu \\ c_2 &= 0 \end{aligned}$$

Hence solution becomes

$$y(x) = c_1 \cosh(\mu x)$$

Second B.C. $y(1) + y'(1) = 0$ gives

$$0 = c_1 (\cosh(\mu) + \mu \sinh(\mu))$$

But $\sinh(\mu)$ can not be negative since its argument is positive here. And $\cosh \mu$ is always positive. In addition $\cosh(\mu) + \mu \sinh(\mu)$ can not be zero since $\sinh(\mu)$ can not be zero as $\mu \neq 0$ and $\cosh(\mu)$ is not zero. Therefore $c_1 = 0$, Leading to trivial solution. Therefore $\lambda < 0$ is not eigenvalue.

Case $\lambda = 0$, The solution is

$$y(x) = c_1 + c_2 x$$

First B.C. $y'(0) = 0$ gives

$$0 = c_2$$

The solution becomes

$$y(x) = c_1$$

Second B.C. $y(1) + y'(1) = 0$ gives

$$0 = c_1$$

This gives trivial solution. Therefore $\lambda = 0$ is not eigenvalue.

Case $\lambda > 0$, The solution is

$$\begin{aligned} y(x) &= c_1 \cos\left(\sqrt{\lambda} x\right) + c_2 \sin\left(\sqrt{\lambda} x\right) \\ y'(x) &= -c_1 \sqrt{\lambda} \sin\left(\sqrt{\lambda} x\right) + c_2 \sqrt{\lambda} \cos\left(\sqrt{\lambda} x\right) \end{aligned}$$

First B.C. $y'(0) = 0$ gives

$$\begin{aligned} 0 &= c_2\sqrt{\lambda} \\ c_2 &= 0 \end{aligned}$$

The solution becomes

$$y(x) = c_1 \cos\left(\sqrt{\lambda}x\right)$$

Second B.C. $y(1) + y'(1) = 0$ gives

$$\begin{aligned} 0 &= c_1 \cos\left(\sqrt{\lambda}\right) - c_1\sqrt{\lambda}\sin\left(\sqrt{\lambda}\right) \\ &= c_1\left(\cos\left(\sqrt{\lambda}\right) - \sqrt{\lambda}\sin\left(\sqrt{\lambda}\right)\right) \end{aligned}$$

For non-trivial solution the above implies

$$\cos\left(\sqrt{\lambda}\right) - \sqrt{\lambda}\sin\left(\sqrt{\lambda}\right) = 0 \tag{1}$$

Therefore the eigenvalues are the solution to the above nonlinear equation. And the corresponding eigenfunctions are

$$\Phi_n = \cos\left(\sqrt{\lambda_n}x\right) \qquad n = 1, 2, 3, \cdots$$

Where λ_n are the roots of equation (1).

The normalized $\hat{\Phi}_n = k_n\Phi_n$ eigenfunctions are now found.

$$\int_0^1 r(x)\,\hat{\Phi}_n^2 dx = 1$$

Since the weight function is $r(x) = 1$, then

$$\begin{aligned} \int_0^1 \hat{\Phi}_n^2 dx &= 1 \\ \int_0^1 k_n^2\Phi_n^2 dx &= 1 \\ k_n^2 \int_0^1 \Phi_n^2 dx &= 1 \\ k_n^2 \int_0^1 \cos^2\left(\sqrt{\lambda_n}x\right) dx &= 1 \end{aligned}$$

But $\int_0^1 \cos^2(ax)\,dx = \left(\frac{x}{2} + \frac{\sin 2ax}{4a}\right)_0^1 = \left(\frac{x}{2} + \frac{\sin\left(2\sqrt{\lambda_n}x\right)}{4\sqrt{\lambda_n}}\right)_0^1 = \left(\frac{1}{2} + \frac{\sin\left(2\sqrt{\lambda_n}\right)}{4\sqrt{\lambda_n}}\right) = \left(\frac{2\sqrt{\lambda_n}+\sin\left(2\sqrt{\lambda_n}\right)}{4\sqrt{\lambda_n}}\right)$.

Hence the above becomes

$$\begin{aligned} k_n^2 &= \frac{1}{\frac{2\sqrt{\lambda_n}+\sin\left(2\sqrt{\lambda_n}\right)}{4\sqrt{\lambda_n}}} \\ &= \frac{4\sqrt{\lambda_n}}{2\sqrt{\lambda_n} + \sin\left(2\sqrt{\lambda_n}\right)} \end{aligned}$$

But $\sin(2a) = 2\sin a\cos a$ and the above can be written as

$$k_n^2 = \frac{4\sqrt{\lambda_n}}{2\sqrt{\lambda_n} + 2\sin\left(\sqrt{\lambda_n}\right)\cos\sqrt{\lambda_n}}$$

But from (1) earlier, we found $\cos\left(\sqrt{\lambda}\right)-\sqrt{\lambda}\sin\left(\sqrt{\lambda}\right)=0$ or $\cos\left(\sqrt{\lambda}\right)=\sqrt{\lambda}\sin\left(\sqrt{\lambda}\right)$. Substituting this into the above gives

$$k_n^2=\frac{4\sqrt{\lambda_n}}{2\sqrt{\lambda_n}+2\sqrt{\lambda_n}\sin^2\left(\sqrt{\lambda_n}\right)}$$

And since $\lambda_n\neq 0$ the above simplifies to

$$\begin{aligned}k_n^2&=\frac{2}{1+\sin^2\left(\sqrt{\lambda_n}\right)}\\&=\frac{4}{4+\sin^2\left(\sqrt{\lambda_n}\right)}\end{aligned}$$

Therefore

$$k_n=\sqrt{\frac{2}{1+\sin^2\left(\sqrt{\lambda_n}\right)}}$$

Since there is no closed form solution to λ_n as it is a root of nonlinear equation $\sqrt{\lambda_n}\tan\left(\sqrt{\lambda_n}L\right)=1$.

Hence the normalized eigenfunctions are

$$\begin{aligned}\hat{\Phi}_n&=k_n\Phi_n\\&=\frac{\sqrt{2}}{\sqrt{1+\sin^2\left(\sqrt{\lambda_n}\right)}}\cos\left(\sqrt{\lambda_n}x\right)\end{aligned}$$

33 Chapter 11.2, problem 5

Determine the normalized eigenfunction for

$$\begin{aligned}y''-2y'+(1+\lambda)y&=0 \qquad (1)\\y(0)&=0\\y(1)&=0\end{aligned}$$

Solution

Let $y(x)=s(x)u(x)$. Then $y'=s'u+su'$ and $y''=s''u+s'u'+s'u'+su''=s''u+2(s'u')+su''$. Therefore the original ODE becomes

$$s''u+2(s'u')+su''-2(s'u+su')+(1+\lambda)su=0$$

Collecting terms in u the above becomes

$$su''+u'(2s'-2s)+u((1+\lambda)s+s''-2s')=0$$

To get rid of u' we therefore want $2s'-2s=0$ or $s'-s=0$. Hence the integrating factor is $I=e^{-x}$ and the solution is obtained from $\frac{d}{dx}(se^{-x})=0$ or $s=e^x$. Therefore, if $s=e^x$ then the original ODE becomes

$$\begin{aligned}e^xu''+u((1+\lambda)e^x+e^x-2e^x)&=0\\u''+u((1+\lambda)+1-2)&=0\\u''+u((1+\lambda)-1)&=0\\u''+\lambda u&=0\end{aligned}$$

With the boundary conditions $u(0) = \frac{y(0)}{s(0)} = \frac{y(0)}{e^0} = 0$ and $u(1) = \frac{y(1)}{s(1)} = 0$. Hence we need to find the eigenfunctions for

$$\begin{aligned} u'' + \lambda u &= 0 \\ u(0) &= 0 \\ u(1) &= 0 \end{aligned}$$

But this we did before. It has $\Phi_n(x) = \sin(n\pi x)$ for $n = 1, 2, \cdots$. And the normalized $\hat{\Phi}_n(x) = \sqrt{2}\sin(n\pi x)$. Mapping this normalized eigenfunction back to $y(x)$ using the transformation $y(x) = s(x)u(x)$ gives the normalized eigenfunction in y space as

$$\hat{\Phi}_n(x) = e^x\sqrt{2}\sin(n\pi x) \qquad n = 1, 2, 3, \cdots$$

34 Chapter 11.2, Example 1 redone. page 690

Here, example 1 is solved again, but without using normalization. Showing that one does not need to normalize the eigenfunctions as the book shows and will get same answer. Solve

$$y'' + 2y = -x \tag{1}$$

With boundary conditions $y(0) = 0, y(1) + y'(1) = 0$. Using the method of eigenfunction expansion without normalization.

Solution

The idea behind solving using eigenfunction expansion, is that

$$-(py')' + q(x)y(x) = \mu r(x)y(x) + f(x) \tag{1A}$$

Is solved using the eigenfunctions of the corresponding homogeneous eigenvalue ODE

$$-(py')' + q(x)y(x) = \lambda r(x)y(x) \tag{2A}$$

Where in (1A) μ is just a constant. And in (2A), λ is an eigenvalue. Writing (1) in same form as (1A) leads to

$$\begin{aligned} -(y')' - 2y &= x \\ -(y')' &= 2y + x \end{aligned} \tag{3A}$$

Therefore $\mu = 2$ and $r(x) = 1$. The corresponding homogeneous eigenvalue problem is

$$-(y')' = \lambda y(x)$$

Or

$$y'' + \lambda y(x) = 0$$

With boundary conditions $y(0) = 0, y(1) + y'(1) = 0$. The solution of the above is used to solve (3A), which is the original ODE. The solution to the above eigenvalue problem was done before. The result is that λ_n is the solution of nonlinear equation

$$\sin\left(\sqrt{\lambda}x\right) + \sqrt{\lambda}\cos\left(\sqrt{\lambda}x\right) = 0$$

Solving this numerically for the first 10 eigenvalues gives

$$\lambda_n = \{4.116, 24.139, 63.659, 122.889, 201.851, 300.55, 418.987, 557.162, 715.077, 892.73\}$$

And the eigenfunctions are

$$\Phi_n(x) = \sin\left(\sqrt{\lambda_n}x\right) \qquad n = 1, 2, 3, \cdots$$

Notice that the eigenfunction above is not normalized as in the text book. Now assuming that the solution of the original nonhomogeneous ODE (3A) is given by

$$y(x) = \sum_{n=1}^{\infty} b_n \Phi_n(x)$$

Where b_n is unknown as of now and substituting the above into (3A) gives

$$-\frac{d^2}{dx^2}\sum_{n=1}^{\infty} b_n \Phi_n(x) = 2\sum_{n=1}^{\infty} b_n \Phi_n(x) + \sum_{n=1}^{\infty} q_n \Phi_n(x)$$

Where $\sum_{n=1}^{\infty} q_n \Phi_n(x)$ is the eigenfunction expansion of the forcing terms $-x$. In this expression q_n is still not known. Now assuming that differentiation can be moved inside the summation above (this needs conditions which assumed valid here). The above equation now becomes

$$-\sum_{n=1}^{\infty} b_n \Phi_n''(x) - 2\sum_{n=1}^{\infty} b_n \Phi_n(x) = \sum_{n=1}^{\infty} q_n \Phi_n(x) \tag{1A}$$

q_n is now found. This is done by applying orthogonality as follows. Let $x = \sum_{n=1}^{\infty} q_n \Phi_n(x)$. Multiplying both sides by $\Phi_m(x)$ and integrating over the domain gives

$$\int_0^1 x\Phi_m(x)\,dx = \sum_{n=1}^{\infty} q_n \int_0^1 \Phi_n(x)\,\Phi_m(x)\,dx$$

$$\int_0^1 x\Phi_m(x)\,dx = q_n \int_0^1 \Phi_m^2(x)\,dx \tag{2}$$

Since $\Phi_n(x)$ is not normalized, one can not replace the integral by 1 as in the book. But since $\Phi_n(x) = \sin\left(\sqrt{\lambda_n}x\right)$, the integrals can be evaluated as follows. The right side of (2) is

$$\int_0^1 \sin^2\left(\sqrt{\lambda_n}x\right)\,dx = \frac{1}{2} - \frac{\sin\left(2\sqrt{\lambda_n}\right)}{4\sqrt{\lambda_n}} \tag{3}$$

And the left side of (2) is found by integration by parts

$$\int_0^1 x\Phi_m(x)\,dx = \int_0^1 x\sin\left(\sqrt{\lambda_n}x\right)\,dx$$

$$= \frac{\sin\sqrt{\lambda_n} - \sqrt{\lambda_n}\cos\sqrt{\lambda_n}}{\lambda_n} \tag{4}$$

Using (3) and (4) in (2) q_n is solved for giving

$$\frac{\sin\sqrt{\lambda_n} - \sqrt{\lambda_n}\cos\sqrt{\lambda_n}}{\lambda_n} = q_n\left(\frac{1}{2} - \frac{\sin\left(2\sqrt{\lambda_n}\right)}{4\sqrt{\lambda_n}}\right)$$

$$q_n = \frac{\sin\sqrt{\lambda_n} - \sqrt{\lambda_n}\cos\sqrt{\lambda_n}}{\lambda_n\left(\frac{1}{2} - \frac{\sin\left(2\sqrt{\lambda_n}\right)}{4\sqrt{\lambda_n}}\right)} \tag{5}$$

Now that q_n is known, b_n is found from (1A)

$$-\sum_{n=1}^{\infty} b_n \Phi_n''(x) - 2\sum_{n=1}^{\infty} b_n \Phi_n(x) = \sum_{n=1}^{\infty} q_n \Phi_n(x)$$

Since $\Phi_n(x) = \sin\left(\sqrt{\lambda_n} x\right)$ then $\Phi_n'(x) = \sqrt{\lambda_n}\cos\left(\sqrt{\lambda_n} x\right)$, $\Phi_n''(x) = -\lambda_n \sin\left(\sqrt{\lambda_n} x\right) = -\lambda_n \Phi_n(x)$ and the above simplifies to

$$\sum_{n=1}^{\infty} b_n \lambda_n \Phi_n(x) - 2\sum_{n=1}^{\infty} b_n \Phi_n(x) = \sum_{n=1}^{\infty} q_n \Phi_n(x)$$

Canceling summations and also $\Phi_n(x)$ since $\Phi_n(x) \neq 0$ the above simplifies to

$$\begin{aligned} b_n \lambda_n - 2b_n &= q_n \\ b_n &= \frac{q_n}{\lambda_n - 2} \end{aligned}$$

Hence the solution to the original ODE is

$$\begin{aligned} y(x) &= \sum_{n=1}^{\infty} b_n \Phi_n(x) \\ &= \sum_{n=1}^{\infty} \left(\frac{q_n}{\lambda_n - 2}\right) \sin\left(\sqrt{\lambda_n} x\right) \end{aligned}$$

Using the value found for q_n in (5), the above becomes

$$y(x) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n(\lambda_n - 2)} \frac{\sin\sqrt{\lambda_n} - \sqrt{\lambda_n}\cos\sqrt{\lambda_n}}{\frac{1}{2} - \frac{\sin\left(2\sqrt{\lambda_n}\right)}{4\sqrt{\lambda_n}}} \sin\left(\sqrt{\lambda_n} x\right) \tag{6}$$

The above is the solution, found without normalization. The book solution is

$$y(x) = 4\sum_{n=1}^{\infty} \frac{1}{\lambda_n(\lambda_n - 2)} \frac{1}{\left(1 + \cos^2\left(\sqrt{\lambda_n}\right)\right)} \sin\left(\sqrt{\lambda_n} x\right) \tag{7}$$

To show that (6) and (7) are actually the same, they are plotted against each others, using 10 terms in the sum, which is more than enough. The result shows identical plots.

Find eigenvalues numerically

```
ClearAll[y, z, x, λ]
eigenvalues = x /. NSolve[Sin[x] + x Cos[x] == 0 && 0 < x < 30, x];
z = eigenvalues^2
```

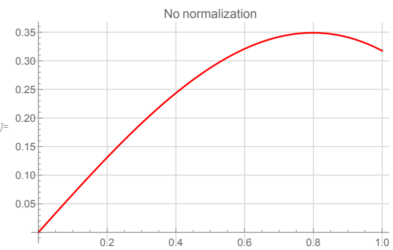
```
{4.11586, 24.1393, 63.6591, 122.889, 201.851, 300.55, 418.987, 557.162, 715.077, 892.73}
```

This is the solution without normalization

```
max = Length@z;

yApproxNoNormalization[x_] := Sum[λ = z[[n]]; 1/(λ (λ - 2)) ((Sin[√λ] - √λ Cos[√λ])/(1/2 - Sin[2 √λ]/(4 √λ))) Sin[√λ x], {n, 1, max}]
```

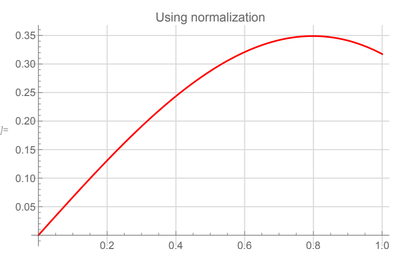
```
Plot[yApproxNoNormalization[x], {x, 0, 1}, GridLines → Automatic, GridLinesStyle → LightGray, PlotStyle → Red, PlotLabel → "No normalization"]
```



This is the solution using normalization (book solution)

```
yApproxBook[x_] := 4 Sum[λ = z[[n]]; Sin[√λ]/(λ (λ - 2) (1 + Cos[√λ]^2)) Sin[√λ x], {n, 1, max}];

Plot[yApproxBook[x], {x, 0, 1}, GridLines → Automatic, GridLinesStyle → LightGray, PlotStyle → Red, PlotLabel → "Using normalization"]
```



They also plotted against the solution found using standard methods, which is

$$y = \frac{\sin\left(\sqrt{2}x\right)}{\sin\left(\sqrt{2}\right) + \sqrt{2}\cos\left(\sqrt{2}\right)} - \frac{x}{2}$$

And both (6,7) matched exactly the above solution.

35 Chapter 11.2 Problem 14

Determine if the given boundary value problem is self-adjoint

$$\begin{aligned} y'' + y' + 2y &= 0 \\ y(0) &= 0 \\ y(1) &= 0 \end{aligned}$$

Solution

The ODE can be written as $(y' + y)' + 2y = 0$. Hence the operator is

$$L[y] = (y' + y)' + 2y$$

The ODE is self-adjoint if

$$\langle L[u], v\rangle = \langle u, L[v]\rangle$$

For any two functions u, v that satisfy the ODE. One way to proceed, is to start from the left side of the above equation and see if the right side can be arrived at. By definition

$$\begin{aligned}\langle L[u], v\rangle &= \int_0^1 L[u]\, v dx \\ &= \int_0^1 \left[(u'+u)' + 2u\right] v dx \\ &= \int_0^1 (u'+u)'\, v + uv dx \\ &= \int_0^1 \overbrace{(u'+u)'}^{dv} \overbrace{v}^{u} dx + \int_0^1 uv dx \end{aligned} \tag{1}$$

integration by parts of the above gives

$$\begin{aligned}\langle L[u], v\rangle &= \left[(u'+u)\, v\right]_0^1 - \int_0^1 (u'+u)\, v' dx + \int_0^1 uv dx \\ &= \left[(u'+u)\, v\right]_0^1 - \int_0^1 (u'v' + uv')\, dx + \int_0^1 uv dx \\ &= \left[(u'+u)\, v\right]_0^1 - \left(\int_0^1 u'v' dx + \int_0^1 uv' dx\right) + \int_0^1 uv dx\end{aligned}$$

Integrating by parts the term $\int_0^1 u'v'dx = [uv']_0^1 - \int_0^1 uv''dx$ the above becomes

$$\begin{aligned}\langle L[u], v\rangle &= \left[(u'+u)\, v\right]_0^1 - \left([uv']_0^1 - \int_0^1 uv''dx + \int_0^1 uv'dx\right) + \int_0^1 uv dx \\ &= \left[(u'+u)\, v - uv'\right]_0^1 - \left(-\int_0^1 uv''dx + \int_0^1 uv'dx\right) + \int_0^1 uv dx \\ &= \left[(u'+u)\, v - uv'\right]_0^1 + \int_0^1 uv''dx - \int_0^1 uv'dx + \int_0^1 uv dx \\ &= \left[(u'+u)\, v - uv'\right]_0^1 + \int_0^1 (v'' - v' + v)\, u dx\end{aligned}$$

The above can never be $\langle u, L[v]\rangle$ even if the boundary terms vanish, since $\int_0^1 (v'' - v' + v)\, udx \neq \int_0^1 (v'' + v' + v)\, udx$. There is a different sign in the operator obtained. Hence the ode is not self adjoint.

36 Chapter 11.2, Problem 15

Determine if the given boundary value problem is self-adjoint

$$\begin{aligned}\left(1+x^2\right) y'' + 2xy' + y &= 0 \\ y'(0) &= 0 \\ y(1) + 2y'(1) &= 0\end{aligned}$$

Solution

The ODE can be written as

$$\left(\left(1+x^2\right)y'\right)'+y=0$$

The operator

$$L\left[y\right]=\left(\left(1+x^2\right)y'\right)'+y$$

The ODE is self-adjoint if

$$\langle L\left[u\right],v\rangle=\langle u,L\left[v\right]\rangle$$

For any two functions u,v that satisfy the ODE. One way to proceed, is to start from the left side of the above equation and see if the right side can be arrived at. By definition

$$\begin{aligned}\langle L\left[u\right],v\rangle &= \int_0^1 L\left[u\right]v dx\\ &= \int_0^1 \left[\left(\left(1+x^2\right)u'\right)'+u\right]v dx\\ &= \int_0^1 \left(\left(1+x^2\right)u'\right)'v+uv dx\\ &= \int_0^1 \left(\left(1+x^2\right)u'\right)'v dx+\int_0^1 uv dx \end{aligned} \tag{1}$$

Starting with the first integral in (1) and using integration by parts

$$\int_0^1 \left(\left(1+x^2\right)u'\right)'v dx=\int_0^1 \overbrace{\left(\left(1+x^2\right)u'\right)'}^{dv}\overbrace{v}^{u} dx$$

By integration by parts, where $\int u dv=|uv|-\int v du$, the above becomes

$$\begin{aligned}\int_0^1 \left(\left(1+x^2\right)u'\right)'v dx &= \left[\left(1+x^2\right)u'v\right]_0^1-\int_0^1 \left(1+x^2\right)u'v' dx\\ &= \left[\left(1+x^2\right)u'v\right]_0^1-\int_0^1 \overbrace{\left(1+x^2\right)v'}^{u}\overbrace{u'}^{dv} dx\end{aligned}$$

Doing integration by parts again. But notice the choice of u and dv made above. This is important in order to get to the form needed. The above becomes

$$\begin{aligned}\int_0^1 \left(\left(1+x^2\right)u'\right)'v dx &= \left[\left(1+x^2\right)u'v\right]_0^1-\left(\left[u\left(1+x^2\right)v'\right]_0^1-\int_0^1 \left(\left(1+x^2\right)v'\right)'u dx\right)\\ &= \left[\left(1+x^2\right)u'v-u\left(1+x^2\right)v'\right]_0^1+\int_0^1 \left(\left(1+x^2\right)v'\right)'u dx\end{aligned}$$

Going back to (1) and adding the second integral which is left there gives

$$\begin{aligned}\langle L\left[u\right],v\rangle &= \left[\left(1+x^2\right)u'v-u\left(1+x^2\right)v'\right]_0^1+\int_0^1 \left(\left(1+x^2\right)v'\right)'u dx+\int_0^1 uv dx\\ &= \left[\left(1+x^2\right)u'v-u\left(1+x^2\right)v'\right]_0^1+\int_0^1 \left[\left(\left(1+x^2\right)v'\right)'+v\right]u dx\end{aligned}$$

But $\int_0^1 \left[\left(\left(1+x^2\right)v'\right)'+v\right]u dx=\langle u,L\left[v\right]\rangle$, hence the above becomes

$$\langle L\left[u\right],v\rangle=\left[\left(1+x^2\right)u'v-u\left(1+x^2\right)v'\right]_0^1+\langle u,L\left[v\right]\rangle \tag{2}$$

We are almost there. If the boundary terms above all go to zero, then it is self-adjoint. If the boundary terms do not vanish, then the problem is not self adjoint. Evaluating the boundary terms in (2)

$$\begin{aligned}\Delta &= \left[\left(1+x^2\right)u'v - u\left(1+x^2\right)v'\right]_0^1 \\ &= \left[2u'(1)\,v(1) - 2u(1)\,v'(1)\right] - \left[u'(0)\,v(0) - u(0)\,v'(0)\right]\end{aligned}$$

Since $u'(0) = 0$ and $v'(0) = 0$, from the given boundary conditions, then above simplifies to

$$\Delta = 2\left(u'(1)\,v(1) - u(1)\,v'(1)\right)$$

But $u(1) = -2u'(1)$ and $v(1) = -2v'(1)$, hence the above becomes

$$\begin{aligned}\Delta &= 2\left(u'(1)\left(-2v'(1)\right) - \left(-2u'(1)\right)v'(1)\right) \\ &= 4\left(-u'(1)\,v'(1) + u'(1)\,v'(1)\right) \\ &= 0\end{aligned}$$

Since the boundary terms Δ vanish, then from (2)

$$\langle L[u], v\rangle = \langle u, L[v]\rangle \tag{3}$$

Hence the ODE is self-adjoint.

37 Chapter 11.2, Problem 16

Determine if the given boundary value problem is self-adjoint

$$\begin{aligned}y'' + y &= \lambda y \\ y(0) - y'(1) &= 0 \\ y'(0) - y(1) &= 0\end{aligned}$$

Solution
The operator is

$$L[y] = y'' + y$$

The ODE is self-adjoint if

$$\langle L[u], v\rangle = \langle u, L[v]\rangle$$

For any two functions u, v that satisfy the ODE. One way to proceed, is to start from the left side of the above equation and see if the right side can be arrived at. By definition

$$\begin{aligned}\langle L[u], v\rangle &= \int_0^1 L[u]\,v\,dx \\ &= \int_0^1 \left(u'' + u\right) v\,dx \\ &= \int_0^1 u'' v\,dx + \int_0^1 uv\,dx \\ &= \int_0^1 \overbrace{u''}^{dv}\,\overbrace{v}^{u}\,dx + \int_0^1 uv\,dx \qquad (1)\end{aligned}$$

Integrating by parts

$$\langle L[u], v\rangle = [u'v]_0^1 - \int_0^1 \overbrace{u'}^{dv}\overbrace{v'}^{u} dx + \int_0^1 uvdx$$

Integrating by parts again

$$\begin{aligned}\langle L[u], v\rangle &= [u'v]_0^1 - \left([uv']_0^1 - \int_0^1 uv''dx\right) + \int_0^1 uvdx \\ &= [u'v - uv']_0^1 + \int_0^1 uv''dx + \int_0^1 uvdx \\ &= [u'v - uv']_0^1 + \int_0^1 (v'' + v)\, udx \\ &= [u'v - uv']_0^1 + \langle u, L[v]\rangle \end{aligned} \tag{2}$$

Hence if the boundary terms vanish, then it is self adjoint else it is not. Evaluating the boundary terms in (2)

$$\begin{aligned}\Delta &= [u'v - uv']_0^1 \\ &= [u'(1)v(1) - u(1)v'(1)] - [u'(0)v(0) - u(0)v'(0)]\end{aligned}$$

But $u'(1) = u(0)$ and $v'(1) = v(0)$ and $u'(0) = u(1)$ and $v'(0) = v(1)$ from the given boundary conditions. Substituting these into the above gives

$$\begin{aligned}\Delta &= [u(0)v(1) - u(1)v(0)] - [u(1)v(0) - u(0)v(1)] \\ &= 2u(1)v(0) \\ &\neq 0\end{aligned}$$

Since the boundary terms Δ do not vanish, then from (2)

$$\langle L[u], v\rangle \neq \langle u, L[v]\rangle$$

Hence the ODE is not self-adjoint.

38 Chapter 11.2, Problem 17

Determine if the given boundary value problem is self-adjoint

$$\begin{aligned}\left(1+x^2\right)y'' + 2xy' + y &= \lambda\left(1+x^2\right)y \\ y(0) - y'(1) &= 0 \\ y'(0) + 2y(1) &= 0\end{aligned}$$

Solution

The ode can be written as

$$\left(\left(1+x^2\right)y'\right)' + y = \lambda\left(1+x^2\right)y$$

Hence the operator is

$$L[y] = \left(\left(1+x^2\right)y'\right)' + y$$

The ODE is self-adjoint if

$$\langle L[u], v\rangle = \langle u, L[v]\rangle$$

For any two functions u, v that satisfy the ODE. One way to proceed, is to start from the left side of the above equation and see if the right side can be arrived at. By definition

$$\begin{aligned}
\langle L[u], v\rangle &= \int_0^1 L[u]\, v dx \\
&= \int_0^1 \left(\left(\left(1+x^2\right)u'\right)' + u\right) v dx \\
&= \int_0^1 \overbrace{\left(\left(1+x^2\right)u'\right)'}^{dv} \overbrace{v}^{u} dx + \int_0^1 uv dx
\end{aligned}$$

Integrating by parts

$$\begin{aligned}
\langle L[u], v\rangle &= \left[\left(1+x^2\right)u'v\right]_0^1 - \int_0^1 \left(1+x^2\right)u'v' dx + \int_0^1 uv dx \\
&= \left[\left(1+x^2\right)u'v\right]_0^1 - \int_0^1 \overbrace{\left(1+x^2\right)v'}^{u} \overbrace{u'}^{dv} dx + \int_0^1 uv dx
\end{aligned}$$

Integrating by parts

$$\begin{aligned}
\langle L[u], v\rangle &= \left[\left(1+x^2\right)u'v\right]_0^1 - \left(\left[u\left(1+x^2\right)v'\right]_0^1 - \int_0^1 \left(\left(1+x^2\right)v'\right)' u dx\right) + \int_0^1 uv dx \\
&= \left[\left(1+x^2\right)u'v - u\left(1+x^2\right)v'\right]_0^1 + \int_0^1 \left(\left(1+x^2\right)v'\right)' u dx + \int_0^1 uv dx \\
&= \left[\left(1+x^2\right)u'v - u\left(1+x^2\right)v'\right]_0^1 + \int_0^1 \left[\left(\left(1+x^2\right)v'\right)' + v\right] u dx \\
&= \left[\left(1+x^2\right)u'v - u\left(1+x^2\right)v'\right]_0^1 + \langle u, L[v]\rangle
\end{aligned}$$

Therefore, if the boundary terms vanish, then the ODE is self adjoint.

$$\Delta = \left[2u'(1)\, v(1) - 2u(1)\, v'(1)\right] - \left[u'(0)\, v(0) - u(0)\, v'(0)\right]$$

But $u'(1) = u(0)$ and $v'(1) = v(0)$ and $u'(0) = 2u(1)$ and $v'(0) = 2v(1)$, from the given boundary conditions. Substituting these in the above gives

$$\begin{aligned}
\Delta &= \left[2u(0)\, v(1) - 2u(1)\, v(0)\right] - \left[2u(1)\, v(0) - u(0)\, 2v(1)\right] \\
&= 2u(0)\, v(1) - 2u(1)\, v(0) - 2u(1)\, v(0) + u(0)\, 2v(1) \\
&= 4u(0)\, v(1) - 4u(1)\, v(0) \\
&= 0
\end{aligned}$$

Hence $\langle L[u], v\rangle = \langle u, L[v]\rangle$, therefore the ODE is self-adjoint.

39 Chapter 11.2, Problem 18

Determine if the given boundary value problem is self-adjoint

$$\begin{aligned}
y'' + \lambda y &= 0 \\
y(0) &= 0 \\
y(\pi) + y'(\pi) &= 0
\end{aligned}$$

Solution

The ode can be written as

$$y'' = -\lambda y$$

Hence $L[y] = y''$. The ODE is self-adjoint if

$$\langle L[u], v\rangle = \langle u, L[v]\rangle$$

For any two functions u, v that satisfy the ODE. One way to proceed, is to start from the left side of the above equation and see if the right side can be arrived at. By definition

$$\begin{aligned}\langle L[u], v\rangle &= \int_0^\pi L[u]\, v dx \\ &= \int_0^\pi u'' v dx\end{aligned}$$

Integrating by parts once

$$\langle L[u], v\rangle = [u'v]_0^\pi - \int_0^\pi u'v' dx$$

Integrating by parts again

$$\begin{aligned}\langle L[u], v\rangle &= [u'v]_0^\pi - \left([uv']_0^\pi - \int_0^\pi uv'' dx\right) \\ &= [u'v - uv']_0^\pi + \int_0^\pi uv'' dx \\ &= [u'v - uv']_0^\pi + \langle u, L[v]\rangle\end{aligned}$$

Now we will check if the boundary terms vanish or not.

$$\begin{aligned}\Delta &= [u'v - uv']_0^\pi \\ &= [u'(\pi) v(\pi) - u(\pi) v'(\pi)] - [u'(0) v(0) - u(0) v'(0)]\end{aligned}$$

Since $u(0) = 0, v(0) = 0$ then the above simplifies to

$$\Delta = u'(\pi) v(\pi) - u(\pi) v'(\pi)$$

But $u'(\pi) = -u(\pi)$ and $v'(\pi) = -v(\pi)$ the above becomes

$$\begin{aligned}\Delta &= -u(\pi) v(\pi) + u(\pi) v(\pi) \\ &= 0\end{aligned}$$

Hence $\langle L[u], v\rangle = \langle u, L[v]\rangle$ and the ODE is self adjoint.

40 Chapter 11.3, Problem 1

Solve by method of eigenfunction expansion

$$\begin{aligned}y'' + 2y &= -x \\ y(0) &= 0 \\ y(1) &= 0\end{aligned}$$

Solution

The corresponding homogeneous eigenvalue ODE is $y'' + \lambda y = 0$ with $y(0) = 0, y(1) = 0$. This was solved before.

$$\hat{\Phi}_n(x) = \sqrt{2}\sin\left(\sqrt{\lambda_n}x\right)$$
$$\lambda_n = (n\pi)^2 \qquad n = 1, 2, 3, \cdots$$

Hence eigenvalues are $\lambda_n = \left\{\pi^2, 4\pi^2, 9\pi^2, \cdots\right\}$. None of the eigenvalues is 2. Therefore the solution to the original ODE can be assumed to be

$$y = \sum_{n=1}^{\infty} b_n \hat{\Phi}_n(x) \tag{1}$$

Substituting this into the original ODE gives

$$\sum_{n=1}^{\infty} b_n \hat{\Phi}_n''(x) + 2\sum_{n=1}^{\infty} b_n \hat{\Phi}_n(x) = -x$$

Expanding $-x$ using same basis function as the solution gives

$$\sum_{n=1}^{\infty} b_n \hat{\Phi}_n''(x) + 2\sum_{n=1}^{\infty} b_n \hat{\Phi}_n(x) = \sum_{n=1}^{\infty} q_n \hat{\Phi}_n(x) \tag{2}$$

Where q_n is found by applying orthogonality on

$$\begin{aligned}
-x &= \sum_{n=1}^{\infty} q_n \hat{\Phi}_n(x) \\
-\int_0^1 x\hat{\Phi}_m(x)\,dx &= \sum_{n=1}^{\infty} q_n \int_0^1 \hat{\Phi}_n(x)\hat{\Phi}_m(x)\,dx \\
&= q_m \int_0^1 \hat{\Phi}_m^2(x)\,dx
\end{aligned}$$

Since normalized, $\int_0^1 \hat{\Phi}_m^2(x)\,dx = 1$ and the above simplifies to

$$-\int_0^1 x\hat{\Phi}_m(x)\,dx = q_m$$

But $\hat{\Phi}_m(x) = \sqrt{2}\sin(n\pi x)$ and the above becomes

$$-\sqrt{2}\int_0^1 x\sin(n\pi x)\,dx = q_n$$

Using $\int x\sin(ax)\,dx = \frac{\sin ax}{a^2} - \frac{x\cos ax}{a}$ the above gives

$$\begin{aligned}
-\sqrt{2}\left(\frac{\sin(n\pi x)}{(n\pi)^2} - \frac{x\cos(n\pi x)}{n\pi}\right)_0^1 &= q_n \\
-\sqrt{2}\left(\frac{\sin(n\pi)}{(n\pi)^2} - \frac{\cos(n\pi)}{n\pi}\right) &= q_n \\
\sqrt{2}\left(\frac{\cos(n\pi)}{n\pi}\right) &= q_n \\
\sqrt{2}\left(\frac{-1^n}{n\pi}\right) &= q_n
\end{aligned}$$

Now that q_n is found, then b_n can be solved for form (2) above giving

$$\sum_{n=1}^{\infty} b_n \hat{\Phi}_n''(x) + 2\sum_{n=1}^{\infty} b_n \hat{\Phi}_n(x) = \sum_{n=1}^{\infty} \sqrt{2}\left(\frac{-1^n}{4n\pi}\right)\hat{\Phi}_n(x) \tag{2A}$$

But $\hat{\Phi}_n''(x) = -\lambda_n \hat{\Phi}_n(x)$ since the eigenfunction satisfy the ode $y'' = -\lambda y$ and the above simplifies to

$$-\sum_{n=1}^{\infty} b_n \lambda_n \hat{\Phi}_n(x) + 2\sum_{n=1}^{\infty} b_n \hat{\Phi}_n(x) = \sum_{n=1}^{\infty} \sqrt{2}\left(\frac{-1^n}{4n\pi}\right)\hat{\Phi}_n(x)$$

Since $\hat{\Phi}_n(x) \neq 0$ the above simplifies to

$$-b_n\lambda_n + 2b_n = \sqrt{2}\left(\frac{-1^n}{n\pi}\right)$$

Therefore

$$\begin{aligned} b_n &= \frac{\sqrt{2}\left(\frac{-1^n}{n\pi}\right)}{2-\lambda_n} \\ &= \frac{\sqrt{2}(-1)^n}{\left(2-(n\pi)^2\right)n\pi} \end{aligned}$$

Therefore the solution from (1) is

$$y = \sum_{n=1}^{\infty} \frac{\sqrt{2}(-1)^n}{\left(2-(n\pi)^2\right)n\pi}\hat{\Phi}_n(x)$$

But $\hat{\Phi}_n(x) = \sqrt{2}\Phi_n(x) = \sqrt{2}\sin(n\pi x)$ and the above becomes

$$y = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{\left(2-(n\pi)^2\right)n\pi}\sin(n\pi x)$$

Or

$$y = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\left((n\pi)^2-2\right)n\pi}\sin(n\pi x)$$

41 Chapter 11.3, Problem 2

Solve by method of eigenfunction expansion

$$\begin{aligned} y'' + 2y &= -x \\ y(0) &= 0 \\ y'(1) &= 0 \end{aligned}$$

Solution

The corresponding homogeneous eigenvalue ODE is $y'' + \lambda y = 0$ with $y(0) = 0, y'(1) = 0$. This was solved before.

$$\begin{aligned} \Phi_n(x) &= \sin\left(\sqrt{\lambda_n}x\right) \\ \lambda_n &= \left(\frac{n\pi}{2}\right)^2 \qquad n = 1,3,5,\cdots \end{aligned}$$

Or, to keep the sum continuous, it can be written as

$$\lambda_n = \left((2n-1)\frac{\pi}{2}\right)^2 \qquad n = 1, 2, 3, \cdots$$

The normalized eigenfunctions weight k_n is found from solving $\int_0^1 k_n^2 \sin^2\left(\frac{n\pi}{2}x\right) dx = 1$ which results in $k_n = \sqrt{2}$

Hence

$$\hat{\Phi}_n(x) = \sqrt{2}\sin\left((2n-1)\frac{\pi}{2}x\right) \qquad n = 1, 2, 3, \cdots$$

The eigenvalues are $\lambda_n = \left\{\left(\frac{\pi}{2}\right)^2, 9\left(\frac{\pi}{2}\right)^2, 25\left(\frac{\pi}{2}\right)^2, \cdots\right\}$. None of the eigenvalues is 2. Therefore the solution to the original ODE can be assumed to be

$$y = \sum_{n=1}^{\infty} b_n \hat{\Phi}_n(x) \tag{1}$$

Substituting this into the original ODE gives

$$\sum_{n=1}^{\infty} b_n \hat{\Phi}_n''(x) + 2\sum_{n=1}^{\infty} b_n \hat{\Phi}_n(x) = -x$$

Expanding $-x$ using same basis function as the solution gives

$$\sum_{n=1}^{\infty} b_n \hat{\Phi}_n''(x) + 2\sum_{n=1}^{\infty} b_n \hat{\Phi}_n(x) = \sum_{n=1}^{\infty} q_n \hat{\Phi}_n(x) \tag{2}$$

Where q_n is found by applying orthogonality on

$$\begin{aligned}
-x &= \sum_{n=1}^{\infty} q_n \hat{\Phi}_n(x) \\
-\int_0^1 x\hat{\Phi}_m(x)\,dx &= \sum_{n=1}^{\infty} q_n \int_0^1 \hat{\Phi}_n(x)\hat{\Phi}_m(x)\,dx \\
&= q_m \int_0^1 \hat{\Phi}_m^2(x)\,dx
\end{aligned}$$

Since normalized, $\int_0^1 \hat{\Phi}_m^2(x)\,dx = 1$ and the above simplifies to

$$-\int_0^1 x\hat{\Phi}_m(x)\,dx = q_m$$

But $\hat{\Phi}_m(x) = \sqrt{2}\sin\left((2n-1)\frac{\pi}{2}x\right)$ and the above becomes

$$-\sqrt{2}\int_0^1 x\sin\left((2n-1)\frac{\pi}{2}x\right) dx = q_n$$

Using $\int x\sin(ax)\,dx = \frac{\sin ax}{a^2} - \frac{x\cos ax}{a}$ the above gives

$$\begin{aligned}
-\sqrt{2}\left(\frac{\sin\left((2n-1)\frac{\pi}{2}x\right)}{\left((2n-1)\frac{\pi}{2}\right)^2} - \frac{x\cos\left((2n-1)\frac{\pi}{2}x\right)}{(2n-1)\frac{\pi}{2}}\right)_0^1 &= q_n \\
-\sqrt{2}\left(\frac{\sin\left((2n-1)\frac{\pi}{2}x\right)}{\left((2n-1)\frac{\pi}{2}\right)^2} - \frac{\cos\left((2n-1)\frac{\pi}{2}x\right)}{(2n-1)\frac{\pi}{2}}\right)_0^1 &= q_n \\
-\sqrt{2}\left(\frac{\sin\left((2n-1)\frac{\pi}{2}\right)}{\left((2n-1)\frac{\pi}{2}\right)^2}\right) &= q_n
\end{aligned}$$

Using $\sin\left((2n-1)\frac{\pi}{2}\right) = -\cos(n\pi)$ which for $n = 1, 2, 3, \cdots$ can be written as $-(-1)^n$ or $(-1)^{n+1}$. The above simplifies to

$$
\begin{aligned}
-\sqrt{2}\left(\frac{(-1)^{n+1}}{\left((2n-1)\frac{\pi}{2}\right)^2}\right) &= q_n \\
\sqrt{2}\left(\frac{(-1)^{n}}{\left((2n-1)\frac{\pi}{2}\right)^2}\right) &= q_n
\end{aligned}
$$

Now that q_n is found, then b_n can be solved for form (2) above giving

$$
\sum_{n=1}^{\infty} b_n \hat{\Phi}_n''(x) + 2\sum_{n=1}^{\infty} b_n \hat{\Phi}_n(x) = \sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{2}}{\left((2n-1)\frac{\pi}{2}\right)^2} \hat{\Phi}_n(x) \tag{2A}
$$

But $\hat{\Phi}_n''(x) = -\lambda_n \hat{\Phi}_n(x)$ since the eigenfunction satisfy the ode $y'' = -\lambda y$ and the above simplifies to

$$
-\sum_{n=1}^{\infty} b_n \lambda_n \hat{\Phi}_n(x) + 2\sum_{n=1}^{\infty} b_n \hat{\Phi}_n(x) = \sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{2}}{\left((2n-1)\frac{\pi}{2}\right)^2} \hat{\Phi}_n(x)
$$

Since $\hat{\Phi}_n(x) \neq 0$ the above simplifies to

$$
-b_n\lambda_n + 2b_n = \frac{(-1)^n \sqrt{2}}{\left((2n-1)\frac{\pi}{2}\right)^2}
$$

Therefore

$$
\begin{aligned}
b_n &= \frac{\frac{(-1)^n\sqrt{2}}{\left((2n-1)\frac{\pi}{2}\right)^2}}{2-\lambda_n} \\
&= \frac{\frac{(-1)^n\sqrt{2}}{\left((2n-1)\frac{\pi}{2}\right)^2}}{\left(2-\left((2n-1)\frac{\pi}{2}\right)^2\right)} \\
&= \frac{(-1)^n \sqrt{2}}{\left(2-(2n-1)^2\left(\frac{\pi}{2}\right)^2\right)\left((2n-1)\frac{\pi}{2}\right)^2}
\end{aligned}
$$

Therefore the solution from (1) is

$$
y = \sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{2}}{\left(2-(2n-1)^2\left(\frac{\pi}{2}\right)^2\right)\left((2n-1)\frac{\pi}{2}\right)^2} \hat{\Phi}_n(x)
$$

But $\hat{\Phi}_n(x) = \sqrt{2}\Phi_n(x) = \sqrt{2}\sin\left((2n-1)\frac{\pi}{2}x\right)$ and the above becomes

$$
y = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\left((2n-1)^2\left(\frac{\pi}{2}\right)^2 - 2\right)\left((2n-1)\frac{\pi}{2}\right)^2} \sin\left((2n-1)\frac{\pi}{2}x\right)
$$

Since $(2n-1)\frac{\pi}{2} = \left(n-\frac{1}{2}\right)\pi$, the above can also be written as (to match back of book solution)

$$
y = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\left(\left(n-\frac{1}{2}\right)^2\pi^2 - 2\right)\left(\left(n-\frac{1}{2}\right)\pi\right)^2} \sin\left(\left(n-\frac{1}{2}\right)\pi x\right)
$$

42 Chapter 11.3, Problem 3

Solve by method of eigenfunction expansion

$$\begin{aligned} y'' + 2y &= -x \\ y'(0) &= 0 \\ y'(1) &= 0 \end{aligned}$$

Solution

The corresponding homogeneous eigenvalue ODE is $y'' + \lambda y = 0$ with $y'(0) = 0, y'(1) = 0$. This was solved above in Chapter 11.2, problem 3. The eigenvalues are

$$\begin{aligned} \lambda_n &= \left\{0, \pi^2, (2\pi)^2, (3\pi)^2, \cdots\right\} \\ &= (n\pi)^2 \qquad n = 0, 1, 2, \cdots \end{aligned}$$

The normalized eigenfunctions are

$$\hat{\Phi}_0(x) = 1$$

And for $n = 1, 2, 3, \cdots$

$$\begin{aligned} \hat{\Phi}_n(x) &= \sqrt{2}\Phi_n(x) \\ &= \sqrt{2}\cos(n\pi x) \\ &= \left\{\sqrt{2}\cos(\pi x), \sqrt{2}\cos(2\pi x), \sqrt{2}\cos(3\pi x), \cdots\right\} \end{aligned}$$

Since none of the eigenvalues is 2, the solution to the original ODE can be assumed to be

$$y = \sum_{n=0}^{\infty} b_n \hat{\Phi}_n(x) \tag{1}$$

Substituting this into the original ODE gives

$$\sum_{n=0}^{\infty} b_n \hat{\Phi}_n''(x) + 2\sum_{n=0}^{\infty} b_n \hat{\Phi}_n(x) = -x$$

Expanding $-x$ using same basis function as the solution gives

$$\sum_{n=0}^{\infty} b_n \hat{\Phi}_n''(x) + 2\sum_{n=0}^{\infty} b_n \hat{\Phi}_n(x) = \sum_{n=0}^{\infty} c_n \hat{\Phi}_n(x) \tag{2}$$

Where c_n is found by applying orthogonality on

$$\begin{aligned} -x &= \sum_{n=0}^{\infty} c_n \hat{\Phi}_n(x) \\ -\int_0^1 x\hat{\Phi}_m(x)\,dx &= \sum_{n=0}^{\infty} c_n \int_0^1 \hat{\Phi}_n(x)\hat{\Phi}_m(x)\,dx \\ &= c_m \int_0^1 \hat{\Phi}_m^2(x)\,dx \end{aligned}$$

Since normalized then $\int_0^1 \hat{\Phi}_m^2(x)\,dx = 1$ and the above simplifies to

$$-\int_0^1 x\hat{\Phi}_n(x)\,dx = c_n$$

For $n = 0$ the eigenfunction is $\hat{\Phi}_0(x) = 1$ and the above gives $c_0 = -\frac{1}{2}\left[x^2\right]_0^1 = -\frac{1}{2}$ and for $n > 0$ the eigenfunction is $\hat{\Phi}_n(x) = \sqrt{2}\cos(n\pi x)$ and the integrals becomes

$$-\sqrt{2}\int_0^1 x\cos(n\pi x)\,dx = c_n$$

Using $\int x\cos(ax)\,dx = \frac{\cos ax}{a^2} + \frac{x\sin ax}{a}$ the above gives

$$\begin{aligned}
c_n &= -\sqrt{2}\left(\frac{\cos(n\pi x)}{(n\pi)^2} + \frac{x\sin(n\pi x)}{n\pi}\right)_0^1 \\
&= -\sqrt{2}\left(\frac{\cos(n\pi)}{(n\pi)^2} + \frac{\sin(n\pi)}{n\pi} - \frac{1}{(n\pi)^2}\right) \\
&= -\sqrt{2}\left(\frac{\cos(n\pi)}{(n\pi)^2} - \frac{1}{(n\pi)^2}\right) \\
&= \frac{-\sqrt{2}}{(n\pi)^2}(\cos(n\pi) - 1) \qquad n = 1, 2, \cdots
\end{aligned}$$

When n is odd then $c_n = \frac{2\sqrt{2}}{(n\pi)^2}$ and when n is even it is zero. Now that q_n is found, then b_n can be solved for form (2) above giving

$$\sum_{n=0}^{\infty} b_n\hat{\Phi}_n''(x) + 2\sum_{n=0}^{\infty} b_n\hat{\Phi}_n(x) = \sum_{n=0}^{\infty} c_n\hat{\Phi}_n(x) \tag{2A}$$

But $\hat{\Phi}_n''(x) = -\lambda_n\hat{\Phi}_n(x)$ since the eigenfunction satisfies the ode $y'' = -\lambda y$ and the above simplifies to

$$-\sum_{n=0}^{\infty} b_n\lambda_n\hat{\Phi}_n(x) + 2\sum_{n=0}^{\infty} b_n\hat{\Phi}_n(x) = \sum_{n=0}^{\infty} c_n\hat{\Phi}_n(x)$$

Since $\hat{\Phi}_n(x) \neq 0$ the above simplifies to

$$\begin{aligned}
-b_n\lambda_n + 2b_n &= c_n \\
b_n &= \frac{c_n}{2 - \lambda_n}
\end{aligned}$$

Therefore the solution from (1) is

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} \frac{c_n}{2 - \lambda_n}\hat{\Phi}_n(x) \\
&= \frac{c_0}{2 - \lambda_0}\hat{\Phi}_0(x) + \sum_{n=1,3,5,\cdots}^{\infty} \frac{c_n}{2 - \lambda_n}\hat{\Phi}_n(x)
\end{aligned}$$

But $\lambda_0 = 0$, $c_0 = -\frac{1}{2}$and $\hat{\Phi}_0(x) = 1$, therefore the above becomes

$$\begin{aligned}
y(x) &= -\frac{1}{4} + \sum_{n=1,3,5,\cdots}^{\infty} \frac{\frac{2\sqrt{2}}{(n\pi)^2}}{2 - (n\pi)^2}\sqrt{2}\cos(n\pi x) \\
&= -\frac{1}{4} + \sum_{n=1,3,5,\cdots}^{\infty} \frac{2\sqrt{2}}{\left(2 - (n\pi)^2\right)(n\pi)^2}\sqrt{2}\cos(n\pi x) \\
&= -\frac{1}{4} - 4\sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{\left((n\pi)^2 - 2\right)(n\pi)^2}\cos(n\pi x)
\end{aligned}$$

To make the sum continuous, let $m = (2n - 1)$ and now m runs from $1, 2, 3, \cdots$ and above becomes

$$y(x) = -\frac{1}{4} - 4\sum_{n=1,3,5,\cdots}^{\infty} \frac{\cos((2n-1)\pi x)}{\left(((2n-1)\pi)^2 - 2\right)((2n-1)\pi)^2}$$

43 Chapter 11.3, Problem 10

Determine if there is any value of the constant a for which the ODE has a solution. Find the solution for each such value

$$\begin{aligned} y'' + \pi^2 y &= a + x \\ y(0) &= 0 \\ y(1) &= 0 \end{aligned}$$

Solution

The eigenvalues of the corresponding homogenous eigenvalue ODE $y'' + \lambda y = 0$ with same homogenous boundary conditions are $\lambda_n = (n\pi)^2$ for $n = 1, 2, \cdots$. Therefore one can see that λ_1 is eigenvalue in the original ODE $y'' + \pi^2 y = a + x$. This means there is a solution (which will be non unique) only if the forcing function is orthogonal to the specific eigenfunction $\Phi_1(x)$. Therefore the condition is

$$\begin{aligned} \int_0^1 f(x)\,\Phi_1(x)\,dx &= 0 \\ \int_0^1 (a + x)\sin(\pi x)\,dx &= 0 \\ \int_0^1 a\sin(\pi x)\,dx + \int_0^1 x\sin(\pi x)\,dx &= 0 \\ a\left(-\frac{\cos\pi x}{\pi}\right)_0^1 + \left[\frac{\sin\pi x}{\pi^2} - \frac{x\cos\pi x}{\pi}\right]_0^1 &= 0 \\ -\frac{a}{\pi}(\cos\pi - 1) + \left[\frac{\sin\pi}{\pi^2} - \frac{\cos\pi}{\pi}\right] &= 0 \\ -\frac{a}{\pi}(-1 - 1) + \left[-\frac{-1}{\pi}\right] &= 0 \\ \frac{2a}{\pi} + \frac{1}{\pi} &= 0 \end{aligned}$$

Hence

$$a = \frac{-1}{2}$$

Only when a is the above value, is there a solution. The original ODE is now solved using the direct method (meaning, not eigenfunction expansion) when $a = \frac{-1}{2}$ as follows. Solve

$$\begin{aligned} y'' + \pi^2 y &= -\frac{1}{2} + x \\ y(0) &= 0 \\ y(1) &= 0 \end{aligned}$$

The homogeneous solution is easily found to be $y_h = A\cos(\pi x) + B\sin(\pi x)$. Since the RHS is a polynomial, let the particular solution be $y_p = c_1 + c_2 x$. Then $y_p' = c_2$ and $y_p'' = 0$. Then

$$\begin{aligned} \pi^2(c_1 + c_2 x) &= -\frac{1}{2} + x \\ c_1\pi^2 + c_2\pi^2 x &= -\frac{1}{2} + x \end{aligned}$$

Therefore $c_2\pi^2 = 1$ or $c_2 = \frac{1}{\pi^2}$ and $c_1\pi^2 = -\frac{1}{2}$ or $c_1 = -\frac{1}{2\pi^2}$. Hence $y_p = -\frac{1}{2\pi^2} + \frac{1}{\pi^2}x$. The solution is

$$\begin{aligned} y &= y_h + y_p \\ &= A\cos(\pi x) + B\sin(\pi x) - \frac{1}{2\pi^2} + \frac{1}{\pi^2}x \end{aligned}$$

Applying boundary conditions, at $y(0) = 0$ the above becomes

$$\begin{aligned} 0 &= A - \frac{1}{2\pi^2} \\ A &= \frac{1}{2\pi^2} \end{aligned}$$

Hence the solution becomes

$$y(x) = \frac{1}{2\pi^2}\cos(\pi x) + B\sin(\pi x) - \frac{1}{2\pi^2} + \frac{1}{\pi^2}x$$

At $y(1) = 0$ the above gives

$$\begin{aligned} 0 &= \frac{1}{2\pi^2}\cos(\pi) + B\sin(\pi) - \frac{1}{2\pi^2} + \frac{1}{\pi^2} \\ 0 &= \frac{-1}{2\pi^2} - \frac{1}{2\pi^2} + \frac{1}{\pi^2} \\ 0 &= 0 \end{aligned}$$

Therefore B can be any value. Hence the final solution is

$$\boxed{y(x) = \frac{1}{2\pi^2}\cos(\pi x) + B\sin(\pi x) + \frac{1}{\pi^2}\left(x - \frac{1}{2}\right)}$$

The solution is not unique as expected. Any arbitrary value of B gives a solution.

44 Chapter 11.3, Problem 11

Determine if there is any value of the constant a for which the ODE has a solution. Find the solution for each such value

$$\begin{aligned} y'' + 4\pi^2 y &= a + x \\ y(0) &= 0 \\ y(1) &= 0 \end{aligned}$$

Solution

The eigenvalues of the corresponding homogenous eigenvalue ODE $y'' + \lambda y = 0$ with same homogenous boundary conditions are $\lambda_n = (n\pi)^2$ for $n = 1, 2, \cdots$. Therefore $\lambda_2 = 4\pi^2$ is eigenvalue in the original ODE $y'' + 4\pi^2 y = a + x$. This means there is a solution (which will be non unique) only if the forcing function is orthogonal to the eigenfunction $\Phi_2(x)$. Therefore the condition is

$$\begin{aligned}
\int_0^1 f(x)\Phi_2(x)\,dx &= 0\\
\int_0^1 (a+x)\sin(2\pi x)\,dx &= 0\\
\int_0^1 a\sin(2\pi x)\,dx + \int_0^1 x\sin(2\pi x)\,dx &= 0\\
a\left(-\frac{\cos 2\pi x}{2\pi}\right)_0^1 + \left[\frac{\sin(2\pi x)}{4\pi^2} - \frac{x\cos(2\pi x)}{2\pi}\right]_0^1 &= 0\\
-\frac{a}{2\pi}(\cos 2\pi - 1) + \left[\frac{\sin 2\pi}{4\pi^2} - \frac{\cos 2\pi}{2\pi}\right] &= 0\\
-\frac{a}{2\pi}(1-1) + \left[-\frac{1}{2\pi}\right] &= 0\\
-\frac{1}{2\pi} &= 0
\end{aligned}$$

But this is not possible. Hence there is no a which makes $\int_0^1 (a+x)\sin(2\pi x)\,dx = 0$. This means there is no solution for any a.

45 Chapter 11.3, Problem 12

Determine if there is any value of the constant a for which the ODE has a solution. Find the solution for each such value

$$\begin{aligned}
y'' + \pi^2 y &= a\\
y'(0) &= 0\\
y'(1) &= 0
\end{aligned}$$

Solution

The eigenvalues of the corresponding homogenous eigenvalue ODE $y'' + \lambda y = 0$ with same homogenous boundary conditions are $\lambda_0 = 0$ and $\lambda_n = (n\pi)^2$ for $n = 1, 2, \cdots$. Therefore $\lambda_1 = \pi^2$ is eigenvalue in the original ODE $y'' + \pi^2 y = a + x$. This means there is a solution (which will be non unique) only if the forcing function is orthogonal to $\Phi_1(x)$. The eigenfunctions in this case are $\Phi_n(x) = \cos(n\pi x)$. Therefore the condition is

$$\begin{aligned}
\int_0^1 f(x)\Phi_1(x)\,dx &= 0\\
\int_0^1 a\cos(\pi x)\,dx &= 0\\
a\left(\frac{\sin \pi x}{\pi}\right)_0^1 &= 0\\
\frac{a}{\pi}(0) &= 0
\end{aligned}$$

Hence any a will satisfy this. Therefore there is a solution for any a. The solution is

$$y = A\cos(\pi x) + B\sin(\pi x) + y_p$$

Since the RHS is a constant, let $y_p = k$. This leads to $\pi^2 k = a$ or $k = \frac{a}{\pi^2}$. Hence the solution is

$$y = A\cos(\pi x) + B\sin(\pi x) + \frac{a}{\pi^2}$$

Or

$$y'(x) = -\pi A\sin(\pi x) + B\pi\cos(\pi x)$$

At $y'(0) = 0$ the above becomes

$$0 = B\pi$$

Hence $B = 0$ and the solution now becomes

$$\begin{aligned} y &= A\cos(\pi x) + \frac{a}{\pi^2} \\ y' &= -A\pi\sin(\pi x) \end{aligned}$$

At $y(1) = 0$ the above becomes

$$\begin{aligned} 0 &= -A\pi\sin\pi \\ &= -A(0) \end{aligned}$$

Therefore A is arbitrary. Any A will give a solution. Hence the final solution is

$$\boxed{y = A\cos(\pi x) + \frac{a}{\pi^2}}$$

For any A and where a is the given a in the original ODE which can take in any value.

46 Chapter 11.3, Problem 13

Determine if there is any value of the constant a for which the ODE has a solution. Find the solution for each such value

$$\begin{aligned} y'' + \pi^2 y &= a - \cos\pi x \\ y(0) &= 0 \\ y(1) &= 0 \end{aligned}$$

Solution

The eigenvalues of the corresponding homogenous eigenvalue ODE $y'' + \lambda y = 0$ with same homogenous boundary conditions are $\lambda_0 = 0$ and $\lambda_n = (n\pi)^2$ for $n = 1, 2, \cdots$. Therefore $\lambda_1 = \pi^2$ is eigenvalue in the original ODE $y'' + \pi^2 y = a - \cos\pi x$. This means there is a solution (which will be non unique) only if the forcing function is orthogonal to $\Phi_1(x)$. The eigenfunctions in this case are $\Phi_n(x) = \sin(n\pi x)$. Therefore the condition is

$$\begin{aligned} \int_0^1 f(x)\Phi_1(x)\,dx &= 0 \\ \int_0^1 (a - \cos\pi x)\sin(\pi x)\,dx &= 0 \\ \int_0^1 a\sin(\pi x)\,dx - \int_0^1 \cos(\pi x)\sin(\pi x)\,dx &= 0 \end{aligned}$$

Using $\sin A\cos B = \frac{1}{2}(\sin(A-B)+\sin(A+B))$ then $\sin(\pi x)\cos(\pi x) = \frac{1}{2}(\sin(0)+\sin(2\pi x)) = \frac{1}{2}\sin(2\pi x)$ and the above becomes

$$
\begin{aligned}
\int_0^1 a\sin(\pi x)\,dx - \frac{1}{2}\int_0^1 \sin(2\pi x)\,dx &= 0\\
-\frac{a}{\pi}\left[\cos\pi x\right]_0^1 + \frac{1}{4\pi}\left[\cos(2\pi x)\right]_0^1 &= 0\\
-\frac{a}{\pi}(\cos\pi - 1) + \frac{1}{4\pi}(\cos(2\pi)-1) &= 0\\
\frac{2a}{\pi} &= 0
\end{aligned}
$$

Hence $a=0$. Therefore there is a solution only when $a=0$. The original ODE then becomes

$$y'' + \pi^2 y = -\cos\pi x$$

The homogenous solution is

$$y_h = A\cos(\pi x) + B\sin(\pi x)$$

Since the forcing function matches one of the basis solution, then the particular solution guess is multiplied by extra x. Therefore

$$
\begin{aligned}
y_p &= x\left(c_1\cos(\pi x) + c_2\sin(\pi x)\right)\\
y_p' &= c_1\cos(\pi x) + c_2\sin(\pi x) + x\left(-c_1\pi\sin(\pi x) + c_2\pi\cos(\pi x)\right)\\
y_p'' &= -c_1\pi\sin(\pi x) + c_2\pi\cos(\pi x) + \left(-c_1\pi\sin(\pi x) + c_2\pi\cos(\pi x)\right) + x\left(-c_1\pi^2\cos(\pi x) - c_2\pi^2\sin(\pi x)\right)\\
&= \sin(\pi x)\left(-2c_1\pi - c_2\pi^2 x\right) + \cos(\pi x)\left(2c_2\pi - c_1 x\pi^2\right)
\end{aligned}
$$

Substituting back into the ODE gives

$$
\begin{aligned}
\sin(\pi x)\left(-2c_1\pi - c_2\pi^2 x\right) + \cos(\pi x)\left(2c_2\pi - c_1 x\pi^2\right) + \pi^2\left(x\left(c_1\cos(\pi x) + c_2\sin(\pi x)\right)\right) &= -\cos\pi x\\
\sin(\pi x)\left(-2c_1\pi - c_2\pi^2 x + \pi^2 x c_2\right) + \cos(\pi x)\left(2c_2\pi - c_1 x\pi^2 + \pi^2 x c_1\right) &= -\cos\pi x\\
-2c_1\pi\sin(\pi x) + 2c_2\pi\cos(\pi x) &= -\cos\pi x
\end{aligned}
$$

Hence

$$
\begin{aligned}
-2c_1\pi &= 0\\
2c_2\pi &= -1
\end{aligned}
$$

Or

$$
\begin{aligned}
c_1 &= 0\\
c_2 &= -\frac{1}{2\pi}
\end{aligned}
$$

Therefore

$$y_p = -\frac{1}{2\pi}x\sin(\pi x)$$

And the general solution is

$$y(x) = A\cos(\pi x) + B\sin(\pi x) - \frac{1}{2\pi}x\sin(\pi x)$$

At $y(0)=0$ the above becomes

$$0 = A\cos(\pi x)$$

Hence $A=0$ and the solution now becomes

$$\boxed{y(x) = B\sin(\pi x) - \frac{1}{2\pi}x\sin(\pi x)}$$

One can stop here, since it is known that the solution is not unique and must contain an arbitrary constant. It is not possible to solve for B using the second boundary conditions.

47 Chapter 11.3, Problem 16

Show that the problem $y''+\pi^2 y = \pi^2 x, y(0) = 1, y(1) = 0$ has solution $y = c_1 \sin \pi x + c_2 \cos \pi x + x$ also show that the solution can not be obtained by splitting the problem as suggested in problem 15 since neither of the two subsidiary problems can be solve in this case.

Solution

To attempt to solve the problem by splitting, the solution is first assumed to be $y = u + v$ where u is the solution to $u'' + \pi^2 u = 0, u(0) = 1, u(1) = 0$ and v is the solution to $v'' + \pi^2 v = \pi^2 x, v(0) = 0, v(1) = 0$. Let us now try to solve the u ODE. The solution is

$$u(x) = A\cos \pi x + B \sin \pi x$$

Applying first BC $u(0) = 1$ gives $A = 1$. Hence the solution becomes $u = \cos \pi x + B \sin \pi x$. Applying second BC $u(1) = 0$ gives

$$\begin{aligned} 0 &= \cos \pi + B \sin \pi \\ 0 &= 1 + B \tan \pi \\ B &= \frac{-1}{\tan \pi} = \frac{-1}{0} \end{aligned}$$

Therefore there is no solution for u. Hence no solution is possible by splitting it was suggested in problem 15 for this problem. Now the problem is solved using the direct method. The homogeneous solution is

$$y_h = A\cos \pi x + B \sin \pi x$$

Since the forcing function $\pi^2 x$ is a polynomial, let y_p guess be $y_p = kx$substituting this back into the ODE gives $k = 1$. Hence the solution becomes

$$\begin{aligned} y &= y_h + y_p \\ &= A\cos \pi x + B \sin \pi x + x \end{aligned}$$

Applying first BC $y(0) = 1$ gives $1 = A$. Hence the solution now becomes $y = \cos \pi x + B \sin \pi x + x$. Applying second BC $y(1) = 0$ gives

$$\begin{aligned} 0 &= \cos \pi + B \sin \pi + 1 \\ 0 &= -1 + B \tan \pi + 1 \\ 0 &= B \tan \pi \\ 0 &= B(0) \end{aligned}$$

Therefore, any B will work. Hence the solution is not unique. Let $B = 1$. Therefore the final solution is

$$y = \cos \pi x + \sin \pi x + x$$

This is solution is not unique. This is also a solution $y = \cos \pi x + 3 \sin \pi x + x$ and also this $y = \cos \pi x + 100 \sin \pi x + x$ and also $y = \cos \pi x + x$ and so on.

48 Chapter 11.3, Problem 19 (With interactive animation)

Use eigenfunction expansion to solve

$$u_t = u_{xx} - x$$

With initial condition $u(x,0) = \sin\left(\frac{\pi x}{2}\right)$ and boundary conditions $u(0,t) = 0, u_x(1,t) = 0$

Solution

The homogenous PDE is first solved to find the eigenfunctions, and these are used to expand the non-homogenous term $-x$ in the PDE. By separation of variables, the spatial eigenvalue ODE is

$$\begin{aligned} X'' + \lambda X &= 0 \\ X(0) &= 0 \\ X'(1) &= 0 \end{aligned}$$

The eigenfunctions for this ODE are $\Phi_n(x) = \sin\left(\sqrt{\lambda_n} x\right)$ with $\lambda_n = \left(\frac{n\pi}{2}\right)^2$ for $n = 1, 3, 5, \cdots$. or $\lambda_n = (2n-1)^2 \left(\frac{\pi}{2}\right)^2$ for $n = 1, 2, 3, \cdots$. with now $\Phi_n(x) = \sin\left((2n-1)\frac{\pi}{2}x\right)$.

The normalized eigenfunctions are $\hat{\Phi}_n(x) = \sqrt{2}\sin\left(\sqrt{\lambda_n} x\right)$. Using these, the original PDE is now solved by assuming the solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\hat{\Phi}_n(x)$$

The coefficient $b_n(t)$ must be a function of time, since it includes all time contributions to the solution. Substituting the above back into the original PDE gives

$$\sum_{n=1}^{\infty} b_n'(t)\hat{\Phi}_n(x) = \frac{d^2}{dx^2}\sum_{n=1}^{\infty} b_n(t)\hat{\Phi}_n(x) + \sum_{n=1}^{\infty} c_n\hat{\Phi}_n(x)$$

Where $\sum_{n=1}^{\infty} c_n\hat{\Phi}_n(x)$ is the eigenfunction expansion of $-x$. Assuming term by term differentiation is allowed (can be shown to be justified here), the above becomes

$$\sum_{n=1}^{\infty} b_n'(t)\hat{\Phi}_n(x) = \sum_{n=1}^{\infty} b_n(t)\hat{\Phi}_n''(x) + \sum_{n=1}^{\infty} c_n\hat{\Phi}_n(x)$$

But $\hat{\Phi}_n''(x) = -\lambda_n\hat{\Phi}_n(x)$ then the above becomes

$$\sum_{n=1}^{\infty} \left(b_n'(t) + \lambda_n b_n(t)\right)\hat{\Phi}_n(x) = \sum_{n=1}^{\infty} c_n\hat{\Phi}_n(x) \tag{1}$$

Now c_n is found. Since $-x = \sum_{n=1}^{\infty} c_n\hat{\Phi}_n(x)$, then applying orthogonality gives

$$-\int_0^1 r(x)\,x\hat{\Phi}_m(x)\,dx = \sum_{n=1}^{\infty} c_n\int_0^1 r(x)\hat{\Phi}_n(x)\hat{\Phi}_m dx$$

But the weight $r(x) = 1$, hence the above simplifies to

$$-\int_0^1 x\hat{\Phi}_m(x)\,dx = c_n\int_0^1 \hat{\Phi}_m^2(x)\,dx$$

Since eigenfunctions are normalized, then $\int_0^1 r(x)\hat{\Phi}_m^2(x)\,dx = 1$ and the above reduces to

$$\begin{aligned} c_n &= -\int_0^1 x\hat{\Phi}_n(x)\,dx \\ &= -\int_0^1 x\sqrt{2}\sin\left((2n-1)\frac{\pi}{2}\right)dx \\ &= -\sqrt{2}\left[\frac{\sin\left((2n-1)\frac{\pi}{2}x\right)}{\left((2n-1)\frac{\pi}{2}\right)^2} - \frac{x\cos\left((2n-1)\frac{\pi}{2}x\right)}{(2n-1)\frac{\pi}{2}}\right]_0^1 \\ &= -\sqrt{2}\left[\frac{\sin\left((2n-1)\frac{\pi}{2}\right)}{\left((2n-1)\frac{\pi}{2}\right)^2} - \frac{\cos\left((2n-1)\frac{\pi}{2}\right)}{(2n-1)\frac{\pi}{2}}\right] \end{aligned}$$

But $\cos\left((2n-1)\frac{\pi}{2}\right) = 0$ for all n, and the above now simplifies to

$$\begin{aligned} c_n &= -\sqrt{2}\frac{\sin\left((2n-1)\frac{\pi}{2}\right)}{\left((2n-1)\frac{\pi}{2}\right)^2} \\ &= -4\sqrt{2}\frac{\sin\left((2n-1)\frac{\pi}{2}\right)}{\left((2n-1)\pi\right)^2} \end{aligned}$$

But $\sin\left((2n-1)\frac{\pi}{2}\right) = (-1)^{n-1}$ for $n = 1, 2, 3, \cdots$, hence the above becomes

$$\begin{aligned} c_n &= -4\sqrt{2}\frac{(-1)^{n-1}}{\left((2n-1)\pi\right)^2} \\ &= 4\sqrt{2}\frac{(-1)^{n}}{\left((2n-1)\pi\right)^2} \end{aligned}$$

Now that c_n is found, (1) is used to solve for $b_n(t)$

$$\sum_{n=1}^{\infty}\left(b_n'(t) + \lambda_n b_n(t)\right)\hat{\Phi}_n(x) = \sum_{n=1}^{\infty} c_n \hat{\Phi}_n(x)$$

The above simplifies to

$$b_n'(t) + \lambda_n b_n(t) = c_n$$

The integrating factor is $e^{\int \lambda_n dt} = e^{\lambda_n t}$, therefore $\frac{d}{dt}\left(b_n(t)e^{\lambda_n t}\right) = c_n e^{\lambda_n t}$. Integrating gives

$$\begin{aligned} b_n(t)e^{\lambda_n t} &= b(0) + c_n\int_0^t e^{\lambda_n s}ds \\ b_n(t) &= b(0)e^{-\lambda_n t} + c_n e^{-\lambda_n t}\int_0^t e^{\lambda_n s}ds \\ &= b(0)e^{-\lambda_n t} + c_n e^{-\lambda_n t}\frac{\left(e^{\lambda_n t}-1\right)}{\lambda_n} \\ &= b(0)e^{-\lambda_n t} + \frac{c_n}{\lambda_n}\left(1-e^{-\lambda_n t}\right) \end{aligned}$$

Therefore the solution becomes

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} b_n(t)\hat{\Phi}_n(x) \\ &= \sum_{n=1}^{\infty}\left(b(0)e^{-\lambda_n t} + \frac{c_n}{\lambda_n}\left(1-e^{-\lambda_n t}\right)\right)\hat{\Phi}_n(x) \qquad (2) \end{aligned}$$

At $t = 0$, the initial conditions is $u(x,0) = \sin\left(\frac{\pi x}{2}\right)$, therefore the above becomes

$$\begin{aligned} \sin\left(\frac{\pi x}{2}\right) &= \sum_{n=1}^{\infty}\left(b(0) + \frac{c_n}{\lambda_n}(1-1)\right)\hat{\Phi}_n(x) \\ &= \sum_{n=1}^{\infty} b(0)\hat{\Phi}_n(x) \\ &= \sum_{n=1}^{\infty} b(0)\sqrt{2}\sin\left((2n-1)\frac{\pi}{2}x\right) \end{aligned}$$

Hence only $n = 1$ gives a solution for $b(0)$, and therefore the above becomes

$$\sin\left(\frac{\pi x}{2}\right) = b(0)\sqrt{2}\sin\left(\frac{\pi}{2}x\right)$$

Or

$$\boxed{b(0) = \frac{1}{\sqrt{2}}}$$

Therefore the solution (2) now becomes

$$u(x,t) = \left(b(0)\,e^{-\lambda_1 t} + c_1 e^{-\lambda_1 t} \frac{\left(e^{\lambda_1 t} - 1\right)}{\lambda_1} \right) \hat{\Phi}_1(x) + \sum_{n=2}^{\infty} \frac{c_n}{\lambda_n} \left(1 - e^{-\lambda_n t}\right) \hat{\Phi}_n(x) \tag{3}$$

Where

$$\begin{aligned}
c_n &= 4\sqrt{2}\frac{(-1)^n}{((2n-1)\pi)^2} \\
b(0) &= \frac{1}{\sqrt{2}} \\
\lambda_n &= \left((2n-1)\frac{\pi}{2}\right)^2 \qquad n = 1,2,3,\cdots \\
\hat{\Phi}_n(x) &= \sqrt{2}\sin\left((2n-1)\frac{\pi}{2}x\right)
\end{aligned}$$

Hence the solution (3) becomes

$$\begin{aligned}
u(x,t) = &\left(\frac{1}{\sqrt{2}} e^{-\frac{\pi^2}{4}t} + c_1 e^{-\frac{\pi^2}{4}t} \frac{\left(e^{\frac{\pi^2}{4}t} - 1\right)}{\frac{\pi^2}{4}} \right) \sqrt{2}\sin\left(\frac{\pi}{2}x\right) \\
&+ \sum_{n=2}^{\infty} \frac{c_n}{(2n-1)^2 \frac{\pi^2}{4}} \left(1 - e^{-\left((2n-1)\frac{\pi}{2}\right)^2 t}\right) \sqrt{2}\sin\left((2n-1)\frac{\pi}{2}x\right)
\end{aligned}$$

To make it the same as back of the book solution, some more manipulation is needed.

$$\begin{aligned}
u(x,t) = &\, e^{-\frac{\pi^2}{4}t} \sin\left(\frac{\pi}{2}x\right) + 4\sqrt{2}\frac{c_1}{\pi^2} e^{-\frac{\pi^2}{4}t} \left(e^{\frac{\pi^2}{4}t} - 1\right) \sin\left(\frac{\pi}{2}x\right) \\
&+ \sqrt{2} \sum_{n=2}^{\infty} \frac{4c_n}{(2n-1)^2 \pi^2} e^{-\left((2n-1)\frac{\pi}{2}\right)^2 t} \left(e^{\left((2n-1)\frac{\pi}{2}\right)^2 t} - 1\right) \sin\left((2n-1)\frac{\pi}{2}x\right)
\end{aligned}$$

Or

$$\begin{aligned}
u(x,t) = &\, e^{-\frac{\pi^2}{4}t} \sin\left(\frac{\pi}{2}x\right) + 4\sqrt{2}\frac{c_1}{\pi^2} \left(1 - e^{-\frac{\pi^2}{4}t}\right) \sin\left(\frac{\pi}{2}x\right) \\
&+ \sqrt{2} \sum_{n=2}^{\infty} \frac{4c_n}{(2n-1)^2 \pi^2} \left(1 - e^{-\left((2n-1)\frac{\pi}{2}\right)^2 t}\right) \sin\left((2n-1)\frac{\pi}{2}x\right)
\end{aligned}$$

Or

$$\begin{aligned}
u(x,t) = &\, e^{-\frac{\pi^2}{4}t} \sin\left(\frac{\pi}{2}x\right) + 4\sqrt{2}\frac{c_1}{\pi^2} \sin\left(\frac{\pi}{2}x\right) - 4\sqrt{2}\frac{c_1}{\pi^2} e^{-\frac{\pi^2}{4}t} \sin\left(\frac{\pi}{2}x\right) \\
&+ \sqrt{2} \sum_{n=2}^{\infty} \frac{4c_n}{(2n-1)^2 \pi^2} \left(1 - e^{-\left((2n-1)\frac{\pi}{2}\right)^2 t}\right) \sin\left((2n-1)\frac{\pi}{2}x\right)
\end{aligned}$$

Or

$$\begin{aligned}
u(x,t) = &\, \sqrt{2}\left[4\frac{c_1}{\pi^2} + \left(\frac{1}{\sqrt{2}} - 4\frac{c_1}{\pi^2}\right) e^{-\frac{\pi^2}{4}t}\right] \sin\left(\frac{\pi}{2}x\right) \\
&+ \sqrt{2} \sum_{n=2}^{\infty} \frac{4c_n}{(2n-1)^2 \pi^2} \left(1 - e^{-\left((2n-1)\frac{\pi}{2}\right)^2 t}\right) \sin\left((2n-1)\frac{\pi}{2}x\right)
\end{aligned}$$

The back of the book uses $c_n = 4\sqrt{2}\frac{(-1)^{n+1}}{((2n-1)\pi)^2}$ instead of $c_n = 4\sqrt{2}\frac{(-1)^n}{((2n-1)\pi)^2}$ as was done in this solution. Therefore, changing c_n to be as the back of the book means flipping the sign of each c_n. (or multiplying by -1). Hence the solution becomes now the same as the back of the book

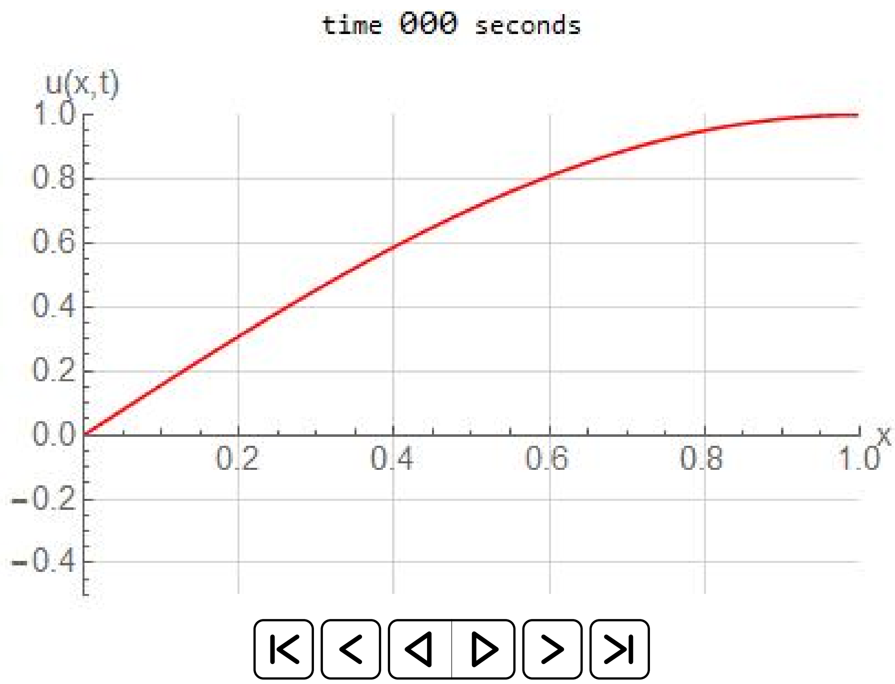
$$\begin{aligned} u(x,t) &= \sqrt{2}\left[-4\frac{c_1}{\pi^2} + \left(\frac{1}{\sqrt{2}} + 4\frac{c_1}{\pi^2}\right)e^{-\frac{\pi^2}{4}t}\right]\sin\left(\frac{\pi}{2}x\right) \\ &\quad - \sqrt{2}\sum_{n=2}^{\infty}\frac{4c_n}{(2n-1)^2\pi^2}\left(1 - e^{-\left((2n-1)\frac{\pi}{2}\right)^2 t}\right)\sin\left((2n-1)\frac{\pi}{2}x\right) \end{aligned}$$

Where in the above,

$$c_n = 4\sqrt{2}\frac{(-1)^{n+1}}{((2n-1)\pi)^2}$$

Both solutions are the same. The sign is either added to c_n or left outside. This completes the solution.

The following is an animation of the above solution for 1.8 seconds. This runs inside the PDF (need to use standard PDF reader to run the animation. Might not run inside Chrome or Firefox own browser PDF reader).



49 Chapter 11.3, Problem 20 (With interactive animation)

Use eigenfunction expansion to solve

$$u_t = u_{xx} + e^{-t}$$

With initial condition $u(x,0) = 1 - x$ and boundary conditions $u_x(0,t) = 0, u_x(1,t) + u(1,t) = 0$

Solution

The homogenous PDE is solved first to obtain the eigenfunctions. These are then used to expand the non-homogenous term e^{-t} in the PDE. By separation of variables, the spatial eigenvalue

ODE is

$$
\begin{aligned}
X'' + \lambda X &= 0 \\
X'(0) &= 0 \\
X'(1) + X(1) &= 0
\end{aligned}
$$

The eigenfunctions for this ODE were found earlier in problem 4, Chapter 11.2. They are

$$
\begin{aligned}
\hat{\Phi}_n &= k_n \Phi_n \\
&= \frac{\sqrt{2}}{\sqrt{1+\sin^2\left(\sqrt{\lambda_n}\right)}} \cos\left(\sqrt{\lambda_n} x\right)
\end{aligned}
$$

Where λ_n are the roots of

$$
\cos\left(\sqrt{\lambda_n}\right) - \sqrt{\lambda_n}\sin\left(\sqrt{\lambda_n}\right) = 0
$$

For $n = 1, 2, 3, \cdots$. Using these, the original PDE is now solved by assuming the solution is

$$
u(x,t) = \sum_{n=1}^{\infty} b_n(t)\hat{\Phi}_n(x)
$$

The coefficient $b_n(t)$ must be a function of time, since it includes all time contributions to the solution. Substituting the above back into the original PDE gives

$$
\sum_{n=1}^{\infty} b_n'(t)\hat{\Phi}_n(x) = \frac{d^2}{dx^2}\sum_{n=1}^{\infty} b_n(t)\hat{\Phi}_n(x) + \sum_{n=1}^{\infty} c_n(t)\hat{\Phi}_n(x)
$$

Where $\sum_{n=1}^{\infty} c_n(t)\hat{\Phi}_n(x)$ is the eigenfunction expansion of e^{-t}. In the above $c_n(t)$ is now a function of time, since the forcing function depends on time in this problem. Assuming term by term differentiation is allowed the above becomes

$$
\sum_{n=1}^{\infty} b_n'(t)\hat{\Phi}_n(x) = \sum_{n=1}^{\infty} b_n(t)\hat{\Phi}_n''(x) + \sum_{n=1}^{\infty} c_n(t)\hat{\Phi}_n(x)
$$

But $\hat{\Phi}_n''(x) = -\lambda_n\hat{\Phi}_n(x)$ therefore

$$
\sum_{n=1}^{\infty}\left(b_n'(t) + \lambda_n b_n(t)\right)\hat{\Phi}_n(x) = \sum_{n=1}^{\infty} c_n(t)\hat{\Phi}_n(x) \tag{1}
$$

Now $c_n(t)$ is found. Since $e^{-t} = \sum_{n=1}^{\infty} c_n(t)\hat{\Phi}_n(x)$, then applying orthogonality gives

$$
\int_0^1 r(x)e^{-t}\hat{\Phi}_m(x)\,dx = \sum_{n=1}^{\infty} c_n(t)\int_0^1 r(x)\hat{\Phi}_n(x)\hat{\Phi}_m dx
$$

But the weight $r(x) = 1$, hence the above simplifies to

$$
e^{-t}\int_0^1 \hat{\Phi}_m(x)\,dx = c_n(t)\int_0^1 \hat{\Phi}_m^2(x)\,dx
$$

Since eigenfunctions are normalized, then $\int_0^1 r(x)\hat{\Phi}_m^2(x)\,dx = 1$ and the above reduces to

$$
e^{-t}\int_0^1 \hat{\Phi}_m(x)\,dx = c_n(t)
$$

Hence

$$\begin{aligned} c_n(t) &= e^{-t}\int_0^1 k_n \cos\left(\sqrt{\lambda_n}x\right) dx \\ &= e^{-t}\frac{k_n}{\sqrt{\lambda_n}}\left[\sin\left(\sqrt{\lambda_n}x\right)\right]_0^1 \\ &= e^{-t}\frac{k_n}{\sqrt{\lambda_n}}\sin\left(\sqrt{\lambda_n}\right) \end{aligned} \tag{2}$$

To make it match the way the back of the book expressed the above, let us write

$$c_n(t) = e^{-t}c_n$$

Where now

$$c_n = \frac{k_n}{\sqrt{\lambda_n}}\sin\left(\sqrt{\lambda_n}\right)$$

This makes it easier to verify the final solution found here is the same as the back of the book.

Now that $c_n(t)$ is found, (1) is used to solve for $b_n(t)$

$$\sum_{n=1}^{\infty}\left(b_n'(t) + \lambda_n b_n(t)\right)\hat{\Phi}_n(x) = \sum_{n=1}^{\infty} e^{-t}c_n\hat{\Phi}_n(x)$$

The above simplifies to

$$b_n'(t) + \lambda_n b_n(t) = e^{-t}c_n$$

The integrating factor is $e^{\int \lambda_n dt} = e^{\lambda_n t}$, therefore $\frac{d}{dt}\left(b_n(t)e^{\lambda_n t}\right) = e^{-t}c_n e^{\lambda_n t}$. Integrating gives

$$\begin{aligned} b_n(t)e^{\lambda_n t} &= b(0) + c_n\int_0^t e^{-s}e^{\lambda_n s}ds \\ b_n(t) &= b(0)e^{-\lambda_n t} + c_n e^{-\lambda_n t}\int_0^t e^{(\lambda_n-1)s}ds \\ &= b(0)e^{-\lambda_n t} + c_n e^{-\lambda_n t}\frac{\left[e^{(\lambda_n-1)s}\right]_0^t}{\lambda_n - 1} \\ &= b(0)e^{-\lambda_n t} + c_n e^{-\lambda_n t}\frac{e^{(\lambda_n-1)t}-1}{\lambda_n - 1} \end{aligned} \tag{3}$$

Using the above in $u(x,t) = \sum_{n=1}^{\infty} b_n(t)\hat{\Phi}_n(x)$ gives the solution as

$$u(x,t) = \sum_{n=1}^{\infty}\left(b(0)e^{-\lambda_n t} + c_n e^{-\lambda_n t}\frac{e^{(\lambda_n-1)t}-1}{\lambda_n - 1}\right)\hat{\Phi}_n(x) \tag{4}$$

At $t = 0$, the above simplifies to

$$1 - x = \sum_{n=1}^{\infty} b(0)\hat{\Phi}_n(x)$$

Applying orthogonality gives

$$\begin{aligned} \int_0^1 r(x)(1-x)\hat{\Phi}_m(x)dx &= \sum_{n=1}^{\infty} b(0)\int_0^1 r(x)\hat{\Phi}_n(x)\hat{\Phi}_m(x)dx \\ \int_0^1 r(x)(1-x)\hat{\Phi}_m(x)dx &= b(0)\int_0^1 r(x)\hat{\Phi}_m^2(x)dx \end{aligned}$$

But $r(x)=1$ and $\int_0^1 r(x)\hat{\Phi}_m^2(x)\,dx=1$ therefore

$$\begin{aligned} b(0) &= \int_0^1 (1-x)\hat{\Phi}_n(x)\,dx \\ &= \int_0^1 \hat{\Phi}_n(x)\,dx - \int_0^1 x\hat{\Phi}_n(x)\,dx \\ &= k_n\left(\int_0^1 \Phi_n(x)\,dx - \int_0^1 x\Phi_n(x)\,dx\right) \end{aligned}$$

But $\Phi_n(x)=\cos\left(\sqrt{\lambda_n}x\right)$, hence the above becomes

$$\begin{aligned} b(0) &= k_n\left(\int_0^1 \cos\left(\sqrt{\lambda_n}x\right)dx - \int_0^1 x\cos\left(\sqrt{\lambda_n}x\right)dx\right) \\ &= k_n\left(\left[\frac{\sin\left(\sqrt{\lambda_n}x\right)}{\sqrt{\lambda_n}}\right]_0^1 - \left[\frac{\cos\left(\sqrt{\lambda_n}x\right)}{\lambda_n} + \frac{x\sin\left(\sqrt{\lambda_n}x\right)}{\sqrt{\lambda_n}}\right]_0^1\right) \\ &= k_n\left(\left[\frac{\sin\left(\sqrt{\lambda_n}\right)}{\sqrt{\lambda_n}}\right] - \left[\frac{\cos\left(\sqrt{\lambda_n}\right)}{\lambda_n} + \frac{\sin\left(\sqrt{\lambda_n}\right)}{\sqrt{\lambda_n}} - \frac{1}{\lambda_n}\right]\right) \\ &= k_n\left(\frac{\sin\left(\sqrt{\lambda_n}\right)}{\sqrt{\lambda_n}} - \frac{\cos\left(\sqrt{\lambda_n}\right)}{\lambda_n} - \frac{\sin\left(\sqrt{\lambda_n}\right)}{\sqrt{\lambda_n}} + \frac{1}{\lambda_n}\right) \\ &= \frac{k_n}{\lambda_n}\left(1-\cos\left(\sqrt{\lambda_n}\right)\right) \end{aligned}$$

Now that $b(0)$ is found, then the solution (4) becomes

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty}\left(\frac{k_n}{\lambda_n}\left(1-\cos\left(\sqrt{\lambda_n}\right)\right)e^{-\lambda_n t} + c_n e^{-\lambda_n t}\frac{e^{(\lambda_n-1)t}-1}{\lambda_n-1}\right)\hat{\Phi}_n(x) \\ &= \sum_{n=1}^{\infty}\left(\frac{k_n}{\lambda_n}\left(1-\cos\left(\sqrt{\lambda_n}\right)\right)e^{-\lambda_n t} + \frac{c_n}{\lambda_n-1}\left(e^{-t}-e^{-\lambda_n t}\right)\right)k_n\cos\left(\sqrt{\lambda_n}x\right) \end{aligned}$$

But $k_n=\frac{\sqrt{2}}{\sqrt{1+\sin^2\left(\sqrt{\lambda_n}\right)}}$, hence the above becomes

$$u(x,t) = \sqrt{2}\sum_{n=1}^{\infty}\left(\alpha_n e^{-\lambda_n t} + \frac{c_n}{\lambda_n-1}\left(e^{-t}-e^{-\lambda_n t}\right)\right)\frac{\cos\left(\sqrt{\lambda_n}x\right)}{\sqrt{1+\sin^2\left(\sqrt{\lambda_n}\right)}}$$

Where

$$\begin{aligned} \alpha_n &= \frac{k_n}{\lambda_n}\left(1-\cos\left(\sqrt{\lambda_n}\right)\right) \\ &= \frac{\sqrt{2}\left(1-\cos\left(\sqrt{\lambda_n}\right)\right)}{\lambda_n\sqrt{1+\sin^2\left(\sqrt{\lambda_n}\right)}} \end{aligned}$$

And

$$
\begin{aligned}
c_n &= \frac{k_n}{\sqrt{\lambda_n}} \sin\left(\sqrt{\lambda_n}\right) \\
&= \frac{\sqrt{2}}{\sqrt{\lambda_n}} \frac{\sin\left(\sqrt{\lambda_n}\right)}{\sqrt{1+\sin^2\left(\sqrt{\lambda_n}\right)}}
\end{aligned}
$$

The following is an animation of the above solution for 6 seconds. This runs inside the PDF (need to use standard PDF reader to run the animation. Might not run inside Chrome or Firefox own browser PDF reader).

50 Chapter 11.3, Problem 22 (With interactive animation)

Use eigenfunction expansion to solve

$$
u_t = u_{xx} + e^{-t}(1-x)
$$

With initial condition $u(x,0)=0$ and boundary conditions $u(0,t)=0, u_x(1,t)=0$

Solution

The homogenous PDE is solved first to obtain the eigenfunctions. These are then used to expand the non-homogenous term $e^{-t}(1-x)$ in the PDE. By separation of variables, the spatial eigenvalue ODE is

$$
\begin{aligned}
X'' + \lambda X &= 0 \\
X(0) &= 0 \\
X'(1) &= 0
\end{aligned}
$$

The eigenfunctions for this ODE were found earlier. They are

$$\begin{aligned}\hat{\Phi}_n &= k_n \Phi_n \\ &= \sqrt{2}\sin\left(\sqrt{\lambda_n} x\right)\end{aligned}$$

Where $\lambda_n = \left(\frac{n\pi}{2}\right)^2$ for $n = 1, 3, 5, \cdots$. Or

$$\begin{aligned}\hat{\Phi}_n &= \sqrt{2}\sin\left(\sqrt{\lambda_n} x\right) \\ \lambda_n &= \left((2n-1)\frac{\pi}{2}\right)^2 \qquad n = 1, 2, 3, \cdots\end{aligned}$$

The original PDE is now solved by assuming the solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\hat{\Phi}_n(x)$$

The coefficient $b_n(t)$ must be a function of time, since it includes all time contributions to the solution. Substituting the above back into the original PDE gives

$$\sum_{n=1}^{\infty} b_n'(t)\hat{\Phi}_n(x) = \frac{d^2}{dx^2}\sum_{n=1}^{\infty} b_n(t)\hat{\Phi}_n(x) + \sum_{n=1}^{\infty} c_n(t)\hat{\Phi}_n(x)$$

Where $\sum_{n=1}^{\infty} c_n(t)\hat{\Phi}_n(x)$ is the eigenfunction expansion of $e^{-t}(1-x)$. In the above $c_n(t)$ is now a function of time, since the forcing function depends on time in this problem. Assuming term by term differentiation is allowed the above becomes

$$\sum_{n=1}^{\infty} b_n'(t)\hat{\Phi}_n(x) = \sum_{n=1}^{\infty} b_n(t)\hat{\Phi}_n''(x) + \sum_{n=1}^{\infty} c_n(t)\hat{\Phi}_n(x)$$

But $\hat{\Phi}_n''(x) = -\lambda_n\hat{\Phi}_n(x)$ therefore

$$\sum_{n=1}^{\infty}\left(b_n'(t) + \lambda_n b_n(t)\right)\hat{\Phi}_n(x) = \sum_{n=1}^{\infty} c_n(t)\hat{\Phi}_n(x) \tag{1}$$

Now $c_n(t)$ is found. Since $e^{-t}(1-x) = \sum_{n=1}^{\infty} c_n(t)\hat{\Phi}_n(x)$, then applying orthogonality gives

$$\int_0^1 r(x)e^{-t}(1-x)\hat{\Phi}_m(x)\,dx = \sum_{n=1}^{\infty} c_n(t)\int_0^1 r(x)\hat{\Phi}_n(x)\hat{\Phi}_m dx$$

But the weight $r(x) = 1$, hence the above simplifies to

$$e^{-t}\int_0^1 (1-x)\hat{\Phi}_m(x)\,dx = c_n(t)\int_0^1 \hat{\Phi}_m^2(x)\,dx$$

Since eigenfunctions are normalized, then $\int_0^1 r(x)\hat{\Phi}_m^2(x)\,dx = 1$ and the above reduces to

$$e^{-t}\int_0^1 (1-x)\hat{\Phi}_m(x)\,dx = c_n(t)$$

Hence

$$
\begin{aligned}
c_n(t) &= e^{-t}\int_0^1 (1-x)\,k_n \sin\left(\sqrt{\lambda_n}x\right)dx \\
&= e^{-t}\sqrt{2}\left(\int_0^1 \sin\left(\sqrt{\lambda_n}x\right)dx - \int_0^1 x\sin\left(\sqrt{\lambda_n}x\right)dx\right) \\
&= e^{-t}\sqrt{2}\left(\left[\frac{-\cos\left(\sqrt{\lambda_n}x\right)}{\sqrt{\lambda_n}}\right]_0^1 - \left[\frac{\sin\sqrt{\lambda_n}x}{\lambda_n} - \frac{x\cos\sqrt{\lambda_n}x}{\sqrt{\lambda_n}}\right]_0^1\right) \\
&= e^{-t}\sqrt{2}\left(\left[\frac{-\cos\left(\sqrt{\lambda_n}\right)}{\sqrt{\lambda_n}} + \frac{1}{\sqrt{\lambda_n}}\right] - \left[\frac{\sin\sqrt{\lambda_n}}{\lambda_n} - \frac{\cos\sqrt{\lambda_n}}{\sqrt{\lambda_n}}\right]\right) \\
&= e^{-t}\sqrt{2}\left(\frac{-\cos\left(\sqrt{\lambda_n}\right)}{\sqrt{\lambda_n}} + \frac{1}{\sqrt{\lambda_n}} - \frac{\sin\sqrt{\lambda_n}}{\lambda_n} + \frac{\cos\sqrt{\lambda_n}}{\sqrt{\lambda_n}}\right) \\
&= e^{-t}\sqrt{2}\left(\frac{1}{\sqrt{\lambda_n}} - \frac{\sin\sqrt{\lambda_n}}{\lambda_n}\right) \\
&= e^{-t}\frac{\sqrt{2}}{\lambda_n}\left(\sqrt{\lambda_n} - \sin\sqrt{\lambda_n}\right) \qquad (2)
\end{aligned}
$$

But $\lambda_n = (2n-1)\frac{\pi}{2}$, therefore $\sin\left((2n-1)\frac{\pi}{2}\right) = \{1,-1,1,-1,\cdots\}$ or $(-1)^{n-1}$ and the above becomes

$$
\begin{aligned}
c_n(t) &= e^{-t}\frac{\sqrt{2}}{\lambda_n}\left(\sqrt{\lambda_n} - (-1)^{n-1}\right) \\
&= e^{-t}\frac{\sqrt{2}}{\lambda_n}\left(\sqrt{\lambda_n} + (-1)^{n}\right) \\
&= e^{-t}\frac{\sqrt{2}}{\lambda_n}\left(\sqrt{\lambda_n} + (-1)^{n}\right)
\end{aligned}
$$

To make it match the way the back of the book expressed the above, let us write

$$c_n(t) = e^{-t}c_n$$

Where

$$c_n = \frac{\sqrt{2}}{\lambda_n}\left(\sqrt{\lambda_n} + (-1)^n\right) \qquad (2A)$$

Now that $c_n(t)$ is found, (1) is used to solve for $b_n(t)$

$$\sum_{n=1}^{\infty}\left(b_n'(t) + \lambda_n b_n(t)\right)\hat{\Phi}_n(x) = \sum_{n=1}^{\infty} e^{-t}c_n\hat{\Phi}_n(x)$$

The above simplifies to

$$b_n'(t) + \lambda_n b_n(t) = e^{-t}c_n$$

The integrating factor is $e^{\int \lambda_n dt} = e^{\lambda_n t}$, therefore $\frac{d}{dt}\left(b_n(t)\, e^{\lambda_n t}\right) = e^{-t} c_n e^{\lambda_n t}$. Integrating gives

$$
\begin{aligned}
b_n(t)\, e^{\lambda_n t} &= b(0) + c_n \int_0^t e^{-s} e^{\lambda_n s} ds \\
b_n(t) &= b(0)\, e^{-\lambda_n t} + c_n e^{-\lambda_n t} \int_0^t e^{(\lambda_n - 1)s} ds \\
&= b(0)\, e^{-\lambda_n t} + c_n e^{-\lambda_n t} \frac{\left[e^{(\lambda_n-1)s}\right]_0^t}{\lambda_n - 1} \\
&= b(0)\, e^{-\lambda_n t} + c_n e^{-\lambda_n t} \frac{e^{(\lambda_n-1)t} - 1}{\lambda_n - 1} \qquad (3)
\end{aligned}
$$

Using the above in $u(x,t) = \sum_{n=1}^{\infty} b_n(t)\, \hat{\Phi}_n(x)$ gives the solution as

$$
u(x,t) = \sum_{n=1}^{\infty} \left(b(0)\, e^{-\lambda_n t} + c_n e^{-\lambda_n t} \frac{e^{(\lambda_n-1)t} - 1}{\lambda_n - 1} \right) \hat{\Phi}_n(x) \qquad (4)
$$

At $t = 0$, the initial conditions are zero, and above simplifies to

$$
0 = \sum_{n=1}^{\infty} b(0)\, \hat{\Phi}_n(x)
$$

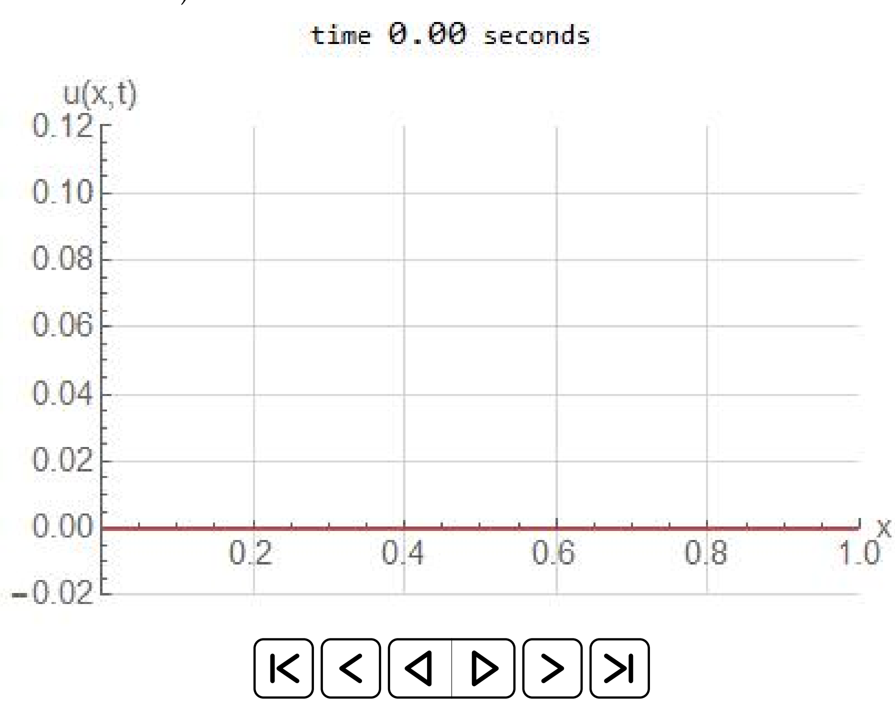
Which implies $b(0) = 0$. Now that $b(0)$ is found, then the solution (4) becomes

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} \frac{e^{(\lambda_n-1)t} - 1}{\lambda_n - 1} \hat{\Phi}_n(x) \\
&= \sqrt{2} \sum_{n=1}^{\infty} c_n \left(\frac{e^{-t} - e^{-\lambda_n t}}{\lambda_n - 1} \right) \sin\left(\sqrt{\lambda_n} x\right)
\end{aligned}
$$

Where $c_n = \frac{\sqrt{2}}{\lambda_n}\left(\sqrt{\lambda_n} + (-1)^n\right)$ and $\lambda_n = \left((2n-1)\frac{\pi}{2}\right)^2$. This completes the solution.

The solution was animated and verified it is correct against a numerical solution.

The following is an animation of the above solution for 5 seconds. This runs inside the PDF (need to use standard PDF reader to run the animation. Might not run inside Chrome or Firefox own browser PDF reader).



51 Chapter 11.3, Problem 24 (With interactive animation)

Solve

$$u_t = u_{xx} - 2$$

With initial condition $u(x,0) = x^2 - 2x + 2$ and boundary conditions $u(0,t) = 1, u(1,t) = 0$

Solution

Let

$$u(x,t) = w(x,t) + v(x)$$

where $v(x)$ is steady state solution which only needs to satisfy the non-homogenous boundary conditions and $w(x,t)$ is the transient solution which needs to satisfy the homogeneous boundary conditions.

At steady state, the PDE becomes an ODE

$$0 = v''(x) - 2$$

This has the solution

$$v(x) = c_1 + c_2 x + x^2$$

Where x^2 is the particular solution. From boundary conditions $v(0) = 1, v(1) = 0$, the solution becomes

$$v(x) = 1 - 2x + x^2$$

Hence $u(x,t) = w(x,t) + 1 - 2x + x^2$. Substituting this into the PDE $u_t = u_{xx} - 2$ results in

$$\begin{aligned} w_t &= w_{xx} + v''(x) - 2 \\ &= w_{xx} + 2 - 2 \\ &= w_{xx} \end{aligned}$$

Hence the PDE to solve is $w_t = w_{xx}$ with $w(0,t) = 0, w(1,t) = 0$. This heat PDE was solved before. Its solution is

$$w(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} \sin\left(\sqrt{\lambda_n} x\right) \tag{1}$$

Where $\lambda_n = (n\pi)^2$ for $n = 1,2,3,\cdots$. At $t = 0$, since $u(x,0) = w(x,0) + v(x)$ then $w(x,0) = u(x,0) - v(x)$ which gives

$$\begin{aligned} w(x,0) &= \left(x^2 - 2x + 2\right) - \left(1 - 2x + x^2\right) \\ &= 1 \end{aligned}$$

Hence at $t = 0$, (1) becomes

$$1 = \sum_{n=1}^{\infty} c_n \sin\left(\sqrt{\lambda_n} x\right) \tag{1A}$$

Applying orthogonality gives

$$\int_0^1 \sin\left(\sqrt{\lambda_n}x\right)\,dx = \frac{1}{2}c_n$$

$$\begin{aligned}
c_n &= 2\int_0^1 \sin\left(\sqrt{\lambda_n}x\right)\,dx \\
&= -2\left[\frac{\cos\left(\sqrt{\lambda_n}x\right)}{\sqrt{\lambda_n}}\right]_0^1 \\
&= \frac{-2}{\sqrt{\lambda_n}}\left[\cos\left(\sqrt{\lambda_n}\right)-1\right] \\
&= \frac{-2}{n\pi}\left[\cos(n\pi)-1\right]
\end{aligned}$$

For even n the above is zero. And for odd n the above becomes

$$c_n = \frac{4}{n\pi} \qquad n = 1, 3, 5, \cdots$$

Therefore from (1) the solution to $w(x,t)$ is

$$w(x,t) = \frac{4}{\pi}\sum_{n=1,3,5,\cdots}^{\infty}\frac{1}{n}e^{-\lambda_n t}\sin\left(\sqrt{\lambda_n}x\right)$$

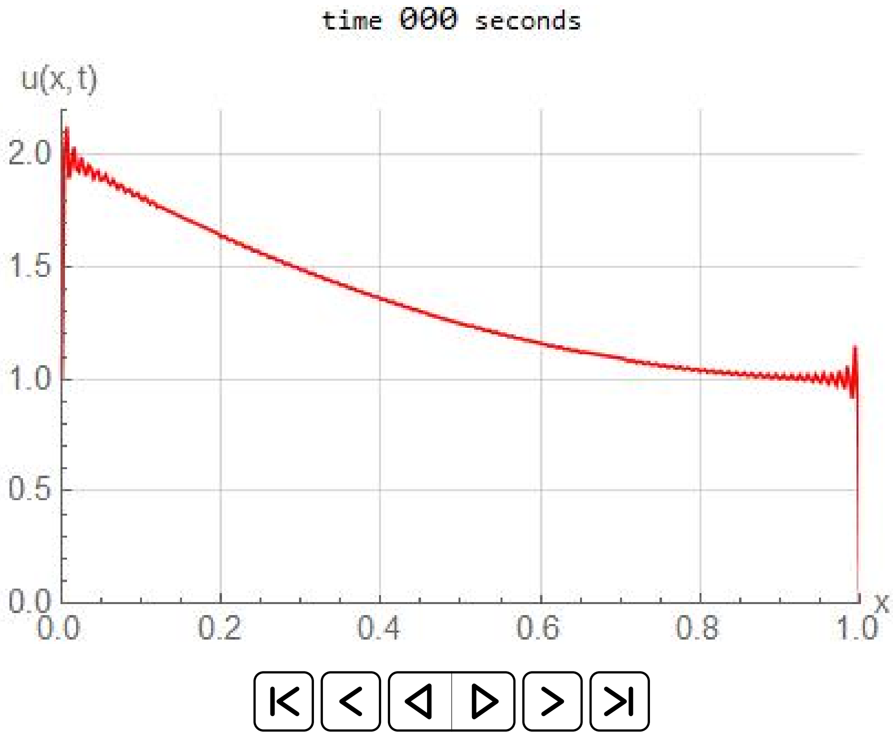
The above can also be written as

$$w(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}e^{-(2n-1)^2\pi^2 t}\sin((2n-1)\pi x)$$

Now, since $u(x,t) = w(x,t) + v(x)$, then the final solution is

$$u(x,t) = x^2 - 2x + 1 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}e^{-(2n-1)^2\pi^2 t}\sin((2n-1)\pi x)$$

The following is an animation of the above solution for half second. This runs inside the PDF (need to use standard PDF reader to run the animation. Might not run inside Chrome browser PDF reader).



52 Chapter 11.3, Problem 25 (With interactive animation)

Solve

$$u_t = u_{xx} - \pi^2 \cos \pi x$$

With initial condition $u(x,0) = \cos\left(\frac{3\pi}{2}x\right) - \cos(\pi x)$ and boundary conditions $u_x(0,t) = 0, u(1,t) = 1$

Solution

Let

$$u(x,t) = w(x,t) + v(x)$$

where $v(x)$ is steady state solution which only needs to satisfy the non-homogenous boundary conditions and $w(x,t)$ is the transient solution which needs to satisfy the homogeneous version of boundary conditions.

At steady state, the PDE becomes an ODE

$$0 = v''(x) - \pi^2 \cos \pi x$$

This ODE can be easily solved giving

$$v(x) = -\cos(\pi x)$$

Hence $u(x,t) = w(x,t) - \cos(\pi x)$. Substituting this into the PDE $u_t = u_{xx} - \pi^2 \cos \pi x$ results in

$$w_t = w_{xx} + v''(x) - \pi^2 \cos \pi x$$

But $v'(x) = \pi \sin(\pi x)$ and $v''(x) = \pi^2 \cos(\pi x)$. The above becomes

$$w_t = w_{xx}$$

With boundary conditions $w_x(0,t) = 0, w(1,t) = 0$. This was solved before. It has the solution

$$w(x,t) = \sum_{n=1,3,5,\cdots}^{\infty} c_n e^{-\lambda_n t} \cos\left(\sqrt{\lambda_n} x\right) \tag{1}$$

Where $\lambda_n = \left(\frac{n\pi}{2}\right)^2$ with $n = 1, 3, 5, \cdots$. At $t = 0$, from $u(x,0) = w(x,0) + v(x)$, then $w(x,0) = u(x,0) - v(x)$ or

$$\begin{aligned} w(x,0) &= \cos\left(\frac{3\pi}{2}x\right) - \cos(\pi x) + \cos(\pi x) \\ &= \cos\left(\frac{3\pi}{2}x\right) \end{aligned}$$

Therefore, from (1) and at $t = 0$ we obtain

$$\begin{aligned} w(x,0) &= \sum_{n=1,3,5,\cdots}^{\infty} c_n \cos\left(\sqrt{\lambda_n}x\right) \\ \cos\left(\frac{3\pi}{2}x\right) &= \sum_{n=1,3,5,\cdots}^{\infty} c_n \cos\left(\frac{n\pi}{2}x\right) \end{aligned}$$

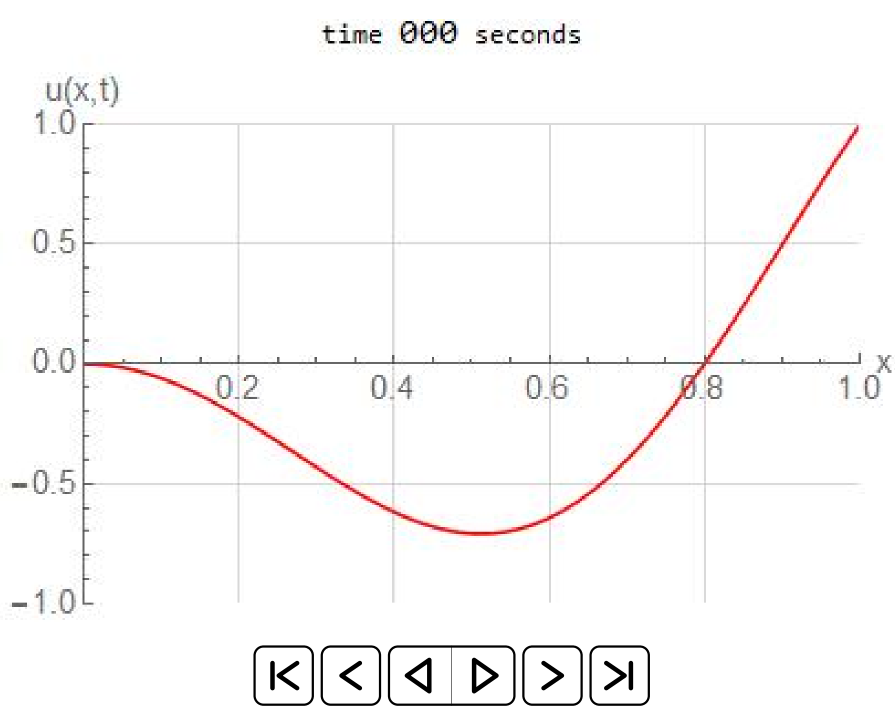
Therefore, only for $n = 3$ is there a solution. Therefore $c_3 = 1$. Hence (1) becomes

$$\begin{aligned} w(x,t) &= e^{-\lambda_3 t} \cos\left(\sqrt{\lambda_3}x\right) \\ &= e^{-\left(\frac{3\pi}{2}\right)^2 t} \cos\left(\frac{3\pi}{2}x\right) \end{aligned}$$

Therefore the final solution is

$$\begin{aligned} u(x,t) &= w(x,t) + v(x) \\ &= -\cos(\pi x) + e^{-\frac{9\pi^2}{4}t} \cos\left(\frac{3\pi}{2}x\right) \end{aligned}$$

The following is an animation of the above solution for half second. This runs inside the PDF (need to use standard PDF reader to run the animation. Might not run inside Chrome browser PDF reader).



53 Chapter 11.3, Problem 28

Part (a) Show that by method of variation of parameters that general solution to $y''(x) = -f(x)$ can be written as

$$y = c_1 + c_2 x - \int_0^x (x - s) f(s) \, ds$$

part (b). Let the solution required to satisfy boundary conditions $y(0) = 0, y(1) = 0$. Show that $c_1 = 0, c_2 = \int_0^1 (1 - x) f(s) \, ds$

part (c). Defining $G(x, s) = \begin{cases} s(1-x) & 0 \le s \le x \\ x(1-s) & x \le s \le 1 \end{cases}$ show that the solution can be written as $y(x) = \int_0^1 G(x, s) f(s) \, ds$

Solution

53.1 Part (a)

The solution is $y = y_h + y_p$. Where $y_h'' = 0$. This has the solution

$$y_h = c_1 + c_2 x$$

In this expression, the basis solutions are

$$y_1 = 1$$
$$y_2 = x.$$

The particular solution is now found using variation of parameters, where it is assumed that

$$y_p = y_1 u_1 + y_2 u_2 \tag{1}$$

And u_1, u_2 are two functions to be determined. Using the standard formulas for finding u_1, u_2 gives

$$u_1 = \int_0^x \frac{-y_2 F(s)}{W(s)} ds \tag{2}$$

Where in the above, $F(s)$ is the forcing function in the RHS of the original ODE which is $-f(x)$ here, and W is the Wronskian. The Wronskian is found as follows

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Substituting $y_1 = 1, y_2 = x$ in the above gives

$$W = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix} = 1$$

Therefore (2) becomes

$$\begin{aligned} u_1 &= \int_0^x -s(-f(s)) \, ds \\ &= \int_0^x s f(s) \, ds \end{aligned} \tag{3}$$

Similarly, u_2 is found using

$$
\begin{aligned}
u_2 &= \int_0^x \frac{y_1 F(s)}{W(s)} ds \\
&= \int_0^x -f(s)\,ds
\end{aligned} \tag{4}
$$

Using (3,4) in (1) gives the particular solution as

$$
\begin{aligned}
y_p &= y_1 \int_0^x s f(s)\,ds - y_2 \int_0^x f(s)\,ds \\
&= \int_0^x s f(s)\,ds - x \int_0^x f(s)\,ds \\
&= \int_0^x s f(s)\,ds - \int_0^x x f(s)\,ds \\
&= \int_0^x (s-x) f(s)\,ds \\
&= -\int_0^x (x-s) f(s)\,ds
\end{aligned}
$$

Now that particular solution is found, the complete solution is found from $y = y_h + y_p$ giving

$$
y = c_1 + c_2 x - \int_0^x (x-s) f(s)\,ds \tag{5}
$$

53.2 Part (b)

Using the BC $y(0) = 0$ on (5) gives

$$
\begin{aligned}
0 &= c_1 - \int_0^0 -s f(s)\,ds \\
c_1 &= 0
\end{aligned}
$$

Hence $c_1 = 0$ and the solution (5) now becomes

$$
y = c_2 x - \int_0^x (x-s) f(s)\,ds \tag{6}
$$

Using the second BC $y(1) = 0$ the above becomes

$$
\begin{aligned}
0 &= c_2 - \int_0^1 (1-s) f(s)\,ds \\
c_2 &= \int_0^1 (1-s) f(s)\,ds
\end{aligned}
$$

Hence the solution (6) now becomes

$$
\begin{aligned}
y &= x \int_0^1 (1-s) f(s)\,ds - \int_0^x (x-s) f(s)\,ds \\
&= \int_0^1 x(1-s) f(s)\,ds - \int_0^x (x-s) f(s)\,ds
\end{aligned}
$$

Writing $\int_0^1 x(1-s) f(s)\,ds = \int_0^x x(1-s) f(s)\,ds + \int_x^1 x(1-s) f(s)\,ds$ then the above becomes

$$y = \int_0^x x(1-s) f(s)\,ds + \int_x^1 x(1-s) f(s)\,ds - \int_0^x (x-s) f(s)\,ds$$

Combining the first and third integrals gives

$$\begin{aligned} y &= \int_0^x [x(1-s)-(x-s)] f(s)\,ds + \int_x^1 x(1-s) f(s)\,ds \\ &= \int_0^x [x - xs - x + s] f(s)\,ds + \int_x^1 x(1-s) f(s)\,ds \\ &= \int_0^x (-xs+s) f(s)\,ds + \int_x^1 x(1-s) f(s)\,ds \\ &= \int_0^x s(1-x) f(s)\,ds + \int_x^1 x(1-s) f(s)\,ds \end{aligned} \tag{7}$$

Which is the result required to show.

53.3 Part (c)

From part (b) above, the solution in (7) can be written as

$$y = \int_0^x G_L(x,s) f(s)\,ds + \int_x^1 G_R(x,s) f(s)\,ds \tag{8}$$

Where

$$G(x,s) = \begin{cases} G_L(x,x) \\ G_L(x,s) \end{cases} = \begin{cases} s(1-x) & 0 \le s \le x \\ x(1-s) & x \le s \le 1 \end{cases}$$

Hence (8) can be combined into one integral

$$y = \int_0^1 G(x,s) f(s)\,ds$$

54 Chapter 11.3, Problem 29

By using procedure in problem 28 show that solution to $y'' + y = -f(x), y(0) = 0, y(1) = 0$ is

$$y = \int_0^1 G(x,s) f(s)$$

Where

$$G(x,s) = \begin{cases} \frac{\sin(s)\sin(1-x)}{\sin(1)} & 0 \le s \le x \\ \frac{\sin(x)\sin(1-s)}{\sin(1)} & x \le s \le 1 \end{cases}$$

Solution

Let $y = y_h + y_p$. Where y_h is solution to $y_h'' + y_h = 0$. This has the solution $y_h = c_1 \cos x + c_2 \sin x$. Hence the bases solutions are

$$\begin{aligned} y_1 &= \cos x \\ y_2 &= \sin x \end{aligned}$$

And therefore the Wronskian is

$$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

Hence

$$u_1 = \int_0^x \frac{-y_2 F(s)}{W(s)} ds$$

Where in the above, $F(s)$ is the forcing function in the RHS of the original ODE which is $-f(x)$ here, and W is the Wronskian. Therefore

$$\begin{aligned} u_1 &= \int_0^x -\sin(s)(-f(s))\,ds \\ &= \int_0^x \sin(s) f(s)\,ds \end{aligned}$$

Similarly, u_2 is found using

$$\begin{aligned} u_2 &= \int_0^x \frac{y_1 F(s)}{W(s)} ds \\ &= \int_0^x \cos(s)(-f(s))\,ds \end{aligned}$$

Hence the particular solution is

$$\begin{aligned} y_p &= y_1 u_1 + y_2 u_2 \\ &= \cos(x)\int_0^x \sin(s) f(s)\,ds - \sin(x)\int_0^x \cos(s) f(s)\,ds \\ &= \int_0^x \cos(x)\sin(s) f(s)\,ds - \int_0^x \sin(x)\cos(s) f(s)\,ds \\ &= \int_0^x (\cos(x)\sin(s) - \sin(x)\cos(s)) f(s)\,ds \end{aligned}$$

Applying $(\sin A \cos B - \cos A \sin B) = \sin(A-B)$ to the integrand above, where $A = x, B = s$ gives

$$y_p = -\int_0^x \sin(x-s) f(s)\,ds$$

Therefore the solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos x + c_2 \sin x) - \int_0^x \sin(x-s) f(s)\,ds \qquad (1) \end{aligned}$$

Applying BC $y(0) = 0$ the above becomes

$$\begin{aligned} 0 &= c_1 - \int_0^0 \sin(-s) f(s)\,ds \\ c_1 &= 0 \end{aligned}$$

And the solution (1) simplifies to

$$y(x) = c_2 \sin x - \int_0^x \sin(x-s) f(s)\,ds \qquad (2)$$

Applying BC $y(1)=0$ the above becomes

$$y(x)=c_2\sin 1-\int_0^1 \sin(1-s)\,f(s)\,ds$$

Hence

$$c_2=\frac{1}{\sin 1}\int_0^1 \sin(1-s)\,f(s)\,ds$$

The solution in (2) now becomes

$$\begin{aligned}
y(x)&=\frac{\sin x}{\sin 1}\int_0^1 \sin(1-s)\,f(s)\,ds-\int_0^x \sin(x-s)\,f(s)\,ds\\
&=\frac{1}{\sin 1}\int_0^1 \sin x\sin(1-s)\,f(s)\,ds-\int_0^x \sin(x-s)\,f(s)\,ds
\end{aligned}$$

Writing $\int_0^1 \sin x\sin(1-s)\,f(s)\,ds=\int_0^x \sin x\sin(1-s)\,f(s)\,ds+\int_x^1 \sin x\sin(1-s)\,f(s)\,ds$ then the above becomes

$$\begin{aligned}
y(x)&=\frac{1}{\sin 1}\left(\int_0^x \sin x\sin(1-s)\,f(s)\,ds+\int_x^1 \sin x\sin(1-s)\,f(s)\,ds\right)-\int_0^x \sin(x-s)\,f(s)\,ds\\
&=\int_0^x \frac{\sin x\sin(1-s)}{\sin(1)}f(s)-\int_0^x \sin(x-s)\,f(s)\,ds+\int_x^1 \frac{\sin x\sin(1-s)}{\sin(1)}f(s)\,ds\\
&=\int_0^x \left[\frac{\sin x\sin(1-s)}{\sin(1)}-\sin(x-s)\right]f(s)\,ds+\int_x^1 \frac{\sin x\sin(1-s)}{\sin(1)}f(s)\,ds\\
&=\frac{1}{\sin(1)}\int_0^x (\sin x\sin(1-s)-\sin(1)\sin(x-s))\,f(s)\,ds+\int_x^1 \frac{\sin x\sin(1-s)}{\sin(1)}f(s)\,ds \qquad (3)
\end{aligned}$$

Using $\sin(A-B)=\sin A\cos B-\cos A\sin B$, where now $A=1,B=s$, then

$$\sin(1-s)=\sin 1\cos s-\cos 1\sin s$$

And also

$$\sin(x-s)=\sin x\cos s-\cos x\sin s$$

Using the above two relations in first integral of (3) which is $I=\int_0^x (\sin x\sin(1-s)-\sin(1)\sin(x-s))\,f(s)\,ds$ gives

$$\begin{aligned}
I&=\int_0^x (\sin x(\sin 1\cos s-\cos 1\sin s)-\sin 1(\sin x\cos s-\cos x\sin s))\,f(s)\,ds\\
&=\int_0^x (\sin x\sin 1\cos s-\sin x\cos 1\sin s-\sin 1\sin x\cos s+\sin 1\cos x\sin s)\,f(s)\,ds\\
&=\int_0^x (-\sin x\cos 1\sin s+\sin 1\cos x\sin s)\,f(s)\,ds\\
&=\int_0^x (\sin s(\sin 1\cos x-\sin x\cos 1))\,f(s)\,ds\\
&=\int_0^x (\sin s\sin(1-x))\,f(s)\,ds
\end{aligned}$$

Substituting the above result in (3) results in

$$y(x)=\int_0^x \frac{\sin s\sin(1-x)}{\sin 1}f(s)\,ds+\int_x^1 \frac{\sin x\sin(1-s)}{\sin(1)}f(s)\,ds \qquad (4)$$

Let

$$G(x,s)=\begin{cases}\frac{\sin(s)\sin(1-x)}{\sin(1)} & 0\leq s\leq x\\ \frac{\sin(x)\sin(1-s)}{\sin(1)} & x\leq s\leq 1\end{cases}$$

Then the solution (4) can be written as

$$y(x)=\int_0^1 G(x,s)\,f(s)\,ds$$

55 Chapter 11.3, Problem 31

By using procedure in problem 30 find Green function and express solution as definite integral for

$$-y''=f(x)$$
$$y'(0)=0$$
$$y(1)=0$$

Solution

The first step is to determine $y_1(x), y_2(x)$. These are the two fundamental solutions of $y''=0$. As the book says, to simplify the derivation, $y_1(x)$ is selected to be the solution that satisfies the boundary conditions at the left end of domain ($x=0$ in this problem) and $y_2(x)$ satisfies the boundary condition on the right end ($x=1$).

The homogeneous solution to $y''=0$ is

$$y_h(x)=c_1+c_2x$$

Therefore $y_1'(0)=0$. This gives $c_2=0$. Hence

$$y_1(x)=1$$

The second boundary conditions $y_2(1)=0$ gives $0=c_1+c_2$, or $c_1=-c_2$ and this leads to $y_2(x)=c_2(-1+x)$. Or

$$y_2(x)=x-1$$

Given y_1, y_2 found above, the next step is to determine the Wronskian as follows

$$W(x)=\begin{vmatrix}y_1 & y_2\\ y_1' & y_2'\end{vmatrix}=\begin{vmatrix}1 & x-1\\ 0 & 1\end{vmatrix}=1$$

Therefore, Green function is now computed using equation (iv) on page 701 of text book giving

$$G(x,s)=\frac{-1}{p(x)\,W(x)}\begin{cases}y_1(s)\,y_2(x) & 0\leq s\leq x\\ y_1(x)\,y_2(s) & x\leq s\leq 1\end{cases}$$

But $p(x)=1$ and $W(x)=1$, and using values found earlier for y_1, y_2, the above becomes

$$\begin{aligned}G(x,s)&=-1\begin{cases}(x-1) & 0\leq s\leq x\\ (s-1) & x\leq s\leq 1\end{cases}\\ &=\begin{cases}x-1 & 0\leq s\leq x\\ s-1 & x\leq s\leq 1\end{cases}\end{aligned}$$

Hence the solution is

$$y(x,s)=\int_0^1 G(x,s)\,f(s)\,ds \tag{1}$$

To verify this solution, it is compared to solution to same ODE using the direct method. Let $f(x) = x$. Hence the ODE is

$$
\begin{aligned}
-y'' &= x \\
y'(0) &= 0 \\
y(1) &= 0
\end{aligned}
$$

The solution found above in (1) can now be found as

$$
\begin{aligned}
y(x) &= \int_0^x G(x,s)\, s ds + \int_x^1 G(x,s)\, s ds \\
&= \int_0^x (1-x)\, s ds + \int_x^1 (1-s)\, s ds \\
&= \left(\frac{s^2}{2} - x\frac{s^2}{2}\right)_0^x + \left(\frac{s^2}{2} - \frac{s^3}{3}\right)_x^1 \\
&= \left(\frac{x^2}{2} - \frac{x^3}{2}\right) + \left(\left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\right) \\
&= \frac{1}{6} - \frac{1}{6}x^3 \qquad (2)
\end{aligned}
$$

Verification The solution is verified by solving the same problem using the direct method. The homogenous solution is $y_h = c_1 + c_2 x$. Since the forcing function is $-x$, let the particular solution be $y_p = kx^3, y_p' = 3kx^2, y'' = 6kx$. Therefore $6kx = -x$ or $k = \frac{-1}{6}$. Therefore the particular solution is $y_p = \frac{-1}{6}x^3$ and the general solution is

$$
y(x) = c_1 + c_2 x - \frac{1}{6}x^3
$$

Applying BC $y'(0) = 0$ gives

$$
c_2 = 0
$$

Hence the solution becomes $y(x) = c_1 - \frac{1}{6}x^3$. Applying BC $y(1) = 0$ gives $0 = c_1 - \frac{1}{6}$ or $c_1 = \frac{1}{6}$. Therefore the solution is

$$
y(x) = \frac{1}{6} - \frac{1}{6}x^3 \qquad (3)
$$

Which is the same answer found using Green function method. Of course in this case the direct method is much simpler and easier to find. The advantage of Green method, is that once the $G(x,s)$ is found, then for any new $f(x)$ only integration is needed to find the new solution, since $G(x,s)$ does not change when $f(x)$ changes. The direct method requires one to find the particular solution each time, and to determine the constants c_1, c_2 again from boundary conditions each time $f(x)$ changes since the particular solution changes when $f(x)$ changes. With Green function method, all the work in using $G(x,y)$ is done in the integration step only. The solution found using Green function already incorporated the boundary conditions in it.

56 Chapter 11.3, Problem 32

By using procedure in problem 30 find Green function and express solution as definite integral for

$$
\begin{aligned}
-y'' &= f(x) \\
y(0) &= 0 \\
y(1) + y'(1) &= 0
\end{aligned}
$$

Solution

The first step is to determine $y_1(x), y_2(x)$, where these are the fundamental solutions of $y'' = 0$ where $y_1(x)$ satisfies the boundary conditions at the left end of domain ($x = 0$) and $y_2(x)$ satisfies the boundary condition on the right end ($x = 1$).

Since the homogeneous solution to $y'' = 0$ is

$$y_h(x) = c_1 + c_2 x$$

Then $y_1(0) = 0$ gives $c_1 = 0$. Therefore

$$y_1(x) = x$$

And to satisfy $y_2(1) + y_2'(1) = 0$ then

$$\begin{aligned} 0 &= (c_1 + c_2) + c_2 \\ c_1 &= -2c_2 \end{aligned}$$

Therefore

$$\begin{aligned} y_2(x) &= -2c_2 + c_2 x \\ &= c_2(x-2) \end{aligned}$$

Hence

$$y_2(x) = x - 2$$

Now that y_1, y_2 are found, the next step is to find the Wronskian.

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x-2 \\ 1 & 1 \end{vmatrix} = x - (x-2) = 2$$

Therefore, Green function is, using equation (iv) on page 701 of text book

$$G(x,s) = \frac{-1}{p(x)W(x)} \begin{cases} y_1(s)\, y_2(x) & 0 \le s \le x \\ y_1(x)\, y_2(s) & x \le s \le 1 \end{cases}$$

But $p(x) = 1$ and $W(x) = 1$, and using values found earlier for y_1, y_2, then the above becomes

$$\begin{aligned} G(x,s) &= \frac{-1}{2} \begin{cases} s(x-2) & 0 \le s \le x \\ x(s-2) & x \le s \le 1 \end{cases} \\ &= \begin{cases} \frac{s(2-x)}{2} & 0 \le s \le x \\ \frac{x(2-s)}{2} & x \le s \le 1 \end{cases} \end{aligned}$$

And the solution is

$$y(x,s) = \int_0^1 G(x,s)\, f(s)\, ds \tag{1}$$

To verify this solution, it is compared to solution to same ODE using the direct method. Let $f(x) = x$. Hence the ODE is

$$\begin{aligned} -y'' &= x \\ y'(0) &= 0 \\ y(1) &= 0 \end{aligned}$$

The solution found above in (1) is now found as

$$\begin{aligned}
y(x) &= \int_0^x G(x,s)\,sds + \int_x^1 G(x,s)\,sds \\
&= \int_0^x \frac{s(2-x)}{2} sds + \int_x^1 \frac{x(2-s)}{2} sds \\
&= \frac{1}{2}\int_0^x \left(2s^2 - xs^2\right) ds + \frac{1}{2}\int_x^1 \left(2xs - xs^2\right) ds \\
&= \frac{1}{2}\left(\frac{2s^3}{3} - x\frac{s^3}{3}\right)_0^x + \frac{1}{2}\left(xs^2 - x\frac{s^3}{3}\right)_x^1 \\
&= \frac{1}{6}\left(2x^3 - x^4\right) + \frac{1}{2}\left(\left(x - \frac{x}{3}\right) - \left(x^3 - \frac{x^4}{3}\right)\right) \\
&= \frac{1}{6}\left(2x - x^3\right) \qquad (2)
\end{aligned}$$

Verification The solution is now verified by solving the same problem using the direct method. The homogenous solution is $y_h = c_1 + c_2 x$. Since the forcing function is $-x$, let the particular solution be $y_p = kx^3, y_p' = 3kx^2, y'' = 6kx$. Therefore $6kx = -x$ or $k = \frac{-1}{6}$. Therefore the particular solution is $y_p = \frac{-1}{6}x^3$ and the general solution is

$$y(x) = c_1 + c_2 x - \frac{1}{6}x^3$$

Applying BC $y(0) = 0$ gives

$$c_1 = 0$$

Hence the solution becomes

$$\begin{aligned}
y(x) &= c_2 x - \frac{1}{6}x^3 \\
y'(x) &= c_2 - \frac{1}{2}x^2
\end{aligned}$$

Applying BC $y(1) + y'(1) = 0$ gives

$$\begin{aligned}
0 &= \left(c_2 - \frac{1}{6}\right) + \left(c_2 - \frac{1}{2}\right) \\
0 &= 2c_2 - \frac{2}{3} \\
c_2 &= \frac{1}{3}
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y(x) &= \frac{1}{3}x - \frac{1}{6}x^3 \\
&= \frac{1}{6}\left(2x - x^3\right) \qquad (3)
\end{aligned}$$

Which is the same as (2) using Green function.

57 Chapter 11.3, Problem 33

By using procedure in problem 30 find Green function and express solution as definite integral for

$$-\left(y''+y\right)=f\left(x\right)$$
$$y'\left(0\right)=0$$
$$y\left(1\right)=0$$

Solution

The first step is to determine $y_1\left(x\right),y_2\left(x\right)$, where these are the fundamental solutions of $y''+y=0$ where $y_1\left(x\right)$ satisfies the boundary conditions at the left end of domain ($x=0$) and $y_2\left(x\right)$ satisfies the boundary condition on the right end ($x=1$).

Since the homogeneous solution to $y''+y=0$ is

$$y_h\left(x\right)=c_1\cos x+c_2\sin x$$

Then $y_1'=-c_1\sin x+c_2\cos x$ and $y_1'\left(0\right)=0$ leads to $c_2=0$, therefore

$$y_1\left(x\right)=\cos x$$

And to satisfy $y_2\left(1\right)=0$ then $0=c_1\cos 1+c_2\sin 1$, hence $c_2=-c_1\frac{\cos(1)}{\sin(1)}$. therefore

$$\begin{aligned}y_2\left(x\right)&=c_1\cos x-c_1\frac{\cos\left(1\right)}{\sin\left(1\right)}\sin x\\&=c_1\left(\cos x-\frac{\cos\left(1\right)}{\sin\left(1\right)}\sin x\right)\end{aligned}$$

Hence

$$y_2\left(x\right)=\cos x-\frac{\cos\left(1\right)}{\sin\left(1\right)}\sin x$$

Now that y_1,y_2 are found, the next step is to determine the Wronskian.

$$\begin{aligned}W\left(x\right)&=\begin{vmatrix}y_1&y_2\\y_1'&y_2'\end{vmatrix}\\&=\begin{vmatrix}\cos x&\left(\cos x-\frac{\cos(1)}{\sin(1)}\sin x\right)\\-\sin x&-\left(\sin x+\frac{\cos(1)}{\sin(1)}\cos x\right)\end{vmatrix}\\&=-\cos x\left(\sin x+\frac{\cos\left(1\right)}{\sin\left(1\right)}\cos x\right)+\sin x\left(\cos x-\frac{\cos\left(1\right)}{\sin\left(1\right)}\sin x\right)\\&=-\cos x\sin x-\frac{\cos\left(1\right)}{\sin\left(1\right)}\cos^2 x+\sin x\cos x-\frac{\cos\left(1\right)}{\sin\left(1\right)}\sin^2 x\\&=-\frac{\cos\left(1\right)}{\sin\left(1\right)}\left(\cos^2 x+\sin^2 x\right)\\&=-\frac{\cos\left(1\right)}{\sin\left(1\right)}\end{aligned}$$

Therefore, Green function is, using equation (iv) on page 701 of text book

$$G\left(x,s\right)=\frac{-1}{p\left(x\right)W\left(x\right)}\begin{cases}y_1\left(s\right)y_2\left(x\right)&0\leq s\leq x\\y_1\left(x\right)y_2\left(s\right)&x\leq s\leq 1\end{cases}$$

But $p(x) = 1$ and $W(x) = 1$, and using values found earlier for y_1, y_2, then the above becomes (using $p(x) = 1$)

$$
\begin{aligned}
G(x,s) &= \frac{-1}{-\frac{\cos(1)}{\sin(1)}} \begin{cases} \cos s\left(\cos x - \frac{\cos(1)}{\sin(1)}\sin x\right) & 0 \le s \le x \\ \cos x\left(\cos s - \frac{\cos(1)}{\sin(1)}\sin s\right) & x \le s \le 1 \end{cases} \\
&= \frac{\sin(1)}{\cos(1)} \begin{cases} \cos s\left(\cos x - \frac{\cos(1)}{\sin(1)}\sin x\right) & 0 \le s \le x \\ \cos x\left(\cos s - \frac{\cos(1)}{\sin(1)}\sin s\right) & x \le s \le 1 \end{cases} \\
&= \begin{cases} \frac{\cos s}{\cos(1)}\left(\sin(1)\cos x - \cos(1)\sin x\right) & 0 \le s \le x \\ \frac{\cos x}{\cos(1)}\left(\sin(1)\cos s - \cos(1)\sin s\right) & x \le s \le 1 \end{cases}
\end{aligned}
$$

Using $\sin A\cos B - \cos A\sin B = \sin(A-B)$ then $\sin(1)\cos x - \cos(1)\sin x = \sin(1-x)$ and $\sin(1)\cos s - \cos(1)\sin s = \sin(1-s)$ and the above becomes

$$
G(x,s) = \begin{cases} \frac{\cos s}{\cos(1)}\sin(1-x) & 0 \le s \le x \\ \frac{\cos x}{\cos(1)}\sin(1-s) & x \le s \le 1 \end{cases}
$$

And the solution is

$$
y(x,s) = \int_0^1 G(x,s) f(s)\,ds
$$

To verify this solution, it is compared to the solution to same ODE using the direct method. Let $f(x) = x$. Hence the ODE is

$$
\begin{aligned}
-\left(y'' + y\right) &= x \\
y'(0) &= 0 \\
y(1) &= 0
\end{aligned}
$$

The solution found above in (1) is now computed as

$$
\begin{aligned}
y(x) &= \int_0^x G(x,s)\,s ds + \int_x^1 G(x,s)\,s ds \\
&= \int_0^x \frac{\cos s}{\cos(1)}\sin(1-x)\,s ds + \int_x^1 \frac{\cos x}{\cos(1)}\sin(1-s)\,s ds \\
&= I_1 + I_2 \qquad (1)
\end{aligned}
$$

The first integral is

$$
\begin{aligned}
I_1 &= \frac{\sin(1-x)}{\cos(1)}\int_0^x s\cos s ds \\
&= \frac{\sin(1-x)}{\cos(1)}\left(\cos s + s\sin s\right)_0^x \\
&= \frac{\sin(1-x)}{\cos(1)}\left(\cos x + x\sin x - 1\right)
\end{aligned}
$$

The second integral is

$$
\begin{aligned}
I_2 &= \frac{\cos x}{\cos(1)}\int_x^1 s\sin(1-s)\,ds \\
&= \frac{\cos x}{\cos(1)}\left(s\cos(s-1) - \sin(s-1)\right)_x^1 \\
&= \frac{\cos x}{\cos(1)}\left(\left(\cos(1-1) - \sin(1-1)\right) - \left(x\cos(x-1) - \sin(x-1)\right)\right) \\
&= \frac{\cos x}{\cos(1)}\left(1 - \left(x\cos(x-1) - \sin(x-1)\right)\right)
\end{aligned}
$$

Hence (1) becomes

$$
\begin{aligned}
y(x) &= \frac{\sin(1-x)}{\cos(1)}(\cos x + x\sin x - 1) + \frac{\cos x}{\cos(1)}(1-(x\cos(x-1)-\sin(x-1)))\\
&= \frac{1}{\cos(1)}(\cos x\sin(1-x) + x\sin x\sin(1-x) - \sin(1-x) + \cos x - x\cos x\cos(x-1) - \cos x\sin(x-1))\\
&= \frac{1}{\cos(1)}(x\sin x\sin(1-x) - \sin(1-x) + \cos x - x\cos x\cos(x-1))\\
&= \frac{1}{\cos 1}(x(\sin x\sin(1-x) - \cos x\cos(x-1)) - \sin(1-x) + \cos x)
\end{aligned}
$$

But $\sin A\sin B - \cos A\cos B = -\cos(A+B)$, using this in the above, where now $x = A, B = (1-x)$ gives

$$
\begin{aligned}
y(x) &= \frac{1}{\cos 1}(x(-\cos(x+1-x)) - \sin(1-x) + \cos x)\\
&= \frac{1}{\cos 1}(-x\cos(1) - \sin(1-x) + \cos x)\\
&= \frac{\cos x}{\cos(1)} - \frac{\sin(1-x)}{\cos(1)} - x \qquad (2)
\end{aligned}
$$

<u>Verification</u> The solution is now verified by solving the same problem using the direct method. The homogenous solution to $y'' + y = 0$ is $y_h = c_1\cos x + c_2\sin x$. Since the forcing function is $-x$, let the particular solution be $y_p = k_1 x, y_p' = k_1, y'' = 0$. Therefore $k_1 x = -x$ or $k = -1$. Therefore the particular solution is $y_p = -x$ and the general solution is

$$y(x) = c_1\cos x + c_2\sin x - x$$

Now BC $y'(0) = 0$ is applied. $y'(x) = -c_1\sin x + c_2\cos x - 1$, therefore

$$
\begin{aligned}
0 &= c_2 - 1\\
c_2 &= 1
\end{aligned}
$$

Hence the solution becomes

$$y(x) = c_1\cos x + \sin x - x$$

Applying BC $y(1) = 0$ gives

$$
\begin{aligned}
0 &= c_1\cos(1) + \sin(1) - 1\\
c_1 &= \frac{1-\sin(1)}{\cos(1)}
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y(x) &= \frac{(1-\sin(1))}{\cos(1)}\cos x + \sin(x) - x\\
&= \frac{\cos x}{\cos(1)} + \frac{-\cos x\sin(1)}{\cos(1)} + \sin(x) - x\\
&= \frac{\cos x}{\cos(1)} + \frac{\sin(x)\cos(1) - \cos x\sin(1)}{\cos(1)} - x
\end{aligned}
$$

But $\sin(x)\cos(1) - \cos x\sin(1) = \sin(x-1) = -\sin(1-x)$, hence the above becomes

$$\boxed{y(x) = \frac{\cos x}{\cos(1)} - \frac{\sin(1-x)}{\cos(1)} - x} \qquad (3)$$

Which is the same solution in (2) found using Green function.

58 Chapter 11.3, Problem 34

By using procedure in problem 30 find Green function and express solution as definite integral for

$$
\begin{aligned}
-y'' &= f(x)\\
y(0) &= 0\\
y'(1) &= 0
\end{aligned}
$$

Solution

The first step is to determine $y_1(x), y_2(x)$, where these are the fundamental solutions of $y'' = 0$ where $y_1(x)$ satisfies the boundary conditions at the left end of domain ($x = 0$) and $y_2(x)$ satisfies the boundary condition on the right end ($x = 1$).

Since the homogeneous solution to $y'' = 0$ is

$$y_h(x) = c_1 + c_2 x$$

Then $y_1(0) = 0$ gives $c_1 = 0$. Therefore

$$y_1(x) = x$$

And to satisfy $y_2'(1) = 0$ then $0 = c_2$. and this leads to

$$y_2(x) = 1$$

Now that y_1, y_2 are found, the next step is to find the Wronskian.

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & 1 \\ 1 & 0 \end{vmatrix} = -1$$

Therefore, Green function is, using equation (iv) on page 701 of text book

$$G(x,s) = \frac{-1}{p(x)W(x)} \begin{cases} y_1(s)\, y_2(s) & 0 \le s \le x \\ y_1(x)\, y_2(x) & x \le s \le 1 \end{cases}$$

But $p(x) = 1$ and $W(x) = -1$, and using values found earlier for y_1, y_2, then the above becomes

$$G(x,s) = \begin{cases} s & 0 \le s \le x \\ x & x \le s \le 1 \end{cases}$$

And the solution is

$$y(x,s) = \int_0^1 G(x,s)\, f(s)\, ds \tag{1}$$

To verify this solution, it is now compared to the solution to same ODE using the direct method. Let $f(x) = x$. Hence the ODE now is

$$
\begin{aligned}
-y'' &= x\\
y(0) &= 0\\
y'(1) &= 0
\end{aligned}
$$

The solution found above in (1) is now computed as

$$\begin{aligned}
y(x) &= \int_0^x G(x,s)\, s ds + \int_x^1 G(x,s)\, s ds \\
&= \int_0^x (s)\, s ds + \int_x^1 (x)\, s ds \\
&= \left(\frac{s^3}{3}\right)_0^x + x\left(\frac{s^2}{2}\right)_x^1 \\
&= \frac{1}{3}x^3 + \frac{x}{2}\left(1-x^2\right) \\
&= \frac{1}{2}x - \frac{1}{6}x^3 \qquad (2)
\end{aligned}$$

Verification The above solution is now verified by solving the same problem using the direct method. The homogenous solution to $y'' = 0$ is $y_h = c_1 + c_2 x$. Since the forcing function is $-x$, the particular solution is $y_p = \frac{-1}{6}x^3$ and the general solution is

$$y(x) = c_1 + c_2 x - \frac{1}{6}x^3$$

BC $y(0) = 0$ gives $c_1 = 0$. The solution becomes $y(x) = c_2 x - \frac{1}{6}x^3$ and $y'(x) = c_2 - \frac{1}{2}x^2$. BC $y'(1) = 0$ gives

$$\begin{aligned}
0 &= c_2 - \frac{1}{2} \\
c_2 &= \frac{1}{2}
\end{aligned}$$

Hence the solution becomes

$$y(x) = \frac{1}{2}x - \frac{1}{6}x^3$$

Which is the same solution in (2) found using Green function.

59 Chapter 11.4, Problem 1

Find formal solution to

$$-(xy')' = \mu xy + f(x)$$

where y, y' bounded as $x \longrightarrow 0$ and $y(1) = 0$

Solution

The given ODE can be written as

$$-\frac{1}{x}(xy')' = \mu y + \frac{f(x)}{x} \qquad (1)$$

The corresponding homogeneous ODE

$$-\frac{1}{x}(xy')' = \lambda y \qquad (2)$$

Where $p = x, q = 0, r = x$. This was solved in the textbook at page 707. The fundamental solution is given by $y_n = \Phi_n(x) = J_0\left(\sqrt{\lambda_n} x\right)$ where the eigenvalues λ_n are the roots of $J_o\left(\sqrt{\lambda_n}\right) = 0$.

These eigenfunctions are not normalized. Therefore, the solution of the inhomogeneous ODE (1) can be now written as

$$y(x) = \sum_{n=1}^{\infty} b_n \Phi_n(x)$$

Using this in (1) gives

$$-\frac{1}{x}(xy')' = \mu \sum_{n=1}^{\infty} b_n \Phi_n(x) + \sum_{n=1}^{\infty} c_n \Phi_n(x)$$

But from (2), $-\frac{1}{x}(xy')'$ can be replaced by λy, so the above becomes

$$\sum_{n=1}^{\infty} \lambda_n b_n \Phi_n(x) = \mu \sum_{n=1}^{\infty} b_n \Phi_n(x) + \sum_{n=1}^{\infty} c_n \Phi_n(x) \tag{3}$$

Where

$$\sum_{n=1}^{\infty} c_n \Phi_n(x) = \frac{f(x)}{x}$$

c_n is now found by orthogonality. Multiplying both sides of the above by $r(x)\Phi_m(x)$, where the weight $r(x) = x$, and integrating gives

$$\begin{aligned}
\int_0^1 x \frac{f(x)}{x} \Phi_m(x)\,dx &= \sum_{n=1}^{\infty} c_n \int_0^1 x \Phi_n(x) \Phi_m(x)\,dx \\
\int_0^1 f(x) \Phi_m(x)\,dx &= \sum_{n=1}^{\infty} c_n \int_0^1 x \Phi_n(x) \Phi_m(x)\,dx
\end{aligned}$$

Due to orthogonality of the eigenfunctions, the above simplifies to

$$c_n = \frac{\int_0^1 f(x) \Phi_n(x)\,dx}{\int_0^1 x \Phi_n^2(x)\,dx} \tag{4}$$

Since $\Phi_n(x)$ is not normalized, $\int_0^1 x\Phi_n^2(x)\,dx$ can not be replaced by 1. The above is left as is. Substituting (4) in (3) and simplifying gives

$$\begin{aligned}
\lambda_n b_n &= \mu b_n + c_n \\
b_n &= \frac{c_n}{(\lambda_n - \mu)}
\end{aligned}$$

Where $\lambda_n \neq \mu$. Hence the formal solution $y = \sum_{n=1}^{\infty} b_n \Phi_n(x)$ can be written as

$$y(x) = \sum_{n=1}^{\infty} \frac{c_n}{(\lambda_n - \mu)} J_o\left(\sqrt{\lambda_n} x\right)$$

Using (4) in the above gives

$$y(x) = \sum_{n=1}^{\infty} \left(\frac{\int_0^1 f(x) \Phi_n(x)\,dx}{\int_0^1 x \Phi_n^2(x)\,dx} \right) \frac{J_o\left(\sqrt{\lambda_n} x\right)}{(\lambda_n - \mu)}$$

60 Chapter 11.4, Problem 2

Consider BVP

$$-(xy')' = \lambda xy$$

where y, y' bounded as $x \to 0$ and $y'(1) = 0$. (a) Show that $\lambda_0 = 0$ is eigenvalue corresponding to $\Phi_0 = 1$. If $\lambda > 0$ show formally that the eigenfunctions are given by $\Phi_n = J_0\left(\sqrt{\lambda_n}x\right)$ where $\sqrt{\lambda_n}$ is the n^{th} positive root in increasing order of $J_0'\left(\sqrt{\lambda_n}\right) = 0$. It is possible to show there are infinite sequence of such roots.

(b) Show that if $m = 0, 1, 2, \cdots$ then $\int_0^1 x\Phi_m(x)\Phi_n(x)\,dx = 0, m \neq n$.

(c) Find formal solution to nonhomogeneous problem $-(xy')' = \mu xy + f(x)$, where y, y' bounded as $x \to 0$ and $y'(1) = 0$, where f is given continuous function on $0 \leq x \leq 1$ and μ is not eigenvalue of the corresponding homogeneous ODE.

Solution

60.1 Part (a)

The given ODE can be written as

$$xy'' + y' + \lambda xy = 0 \tag{1}$$

Let $t = \sqrt{\lambda}x$, then $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\sqrt{\lambda}$ and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dt}\sqrt{\lambda}\right) = \sqrt{\lambda}\frac{d^2y}{dt^2}\frac{dt}{dx} = \sqrt{\lambda}\frac{d^2y}{dt^2}\sqrt{\lambda} = \lambda\frac{d^2y}{dt^2}$. Hence (1) becomes

$$\frac{t}{\sqrt{\lambda}}\lambda y''(t) + \sqrt{\lambda}y'(t) + \lambda\frac{t}{\sqrt{\lambda}}y(t) = 0$$
$$t\sqrt{\lambda}y''(t) + \sqrt{\lambda}y'(t) + \sqrt{\lambda}ty(t) = 0$$

Since problem says that $\lambda > 0$, then dividing by $\sqrt{\lambda}$ the above simplifies to

$$ty''(t) + y'(t) + ty(t) = 0$$

This is Bessel ODE of zero order. Its solution is $y(t) = c_1J_0(t) + c_2Y_0(t)$. Where $J_0(0) = 0$ and $\lim_{t\to 0} Y_0(t) \to \infty$. Hence a bounded solution requires that $c_2 = 0$. Therefore the solution becomes

$$y(t) = c_1J_0(t)$$

or in terms of x

$$y(x) = c_1J_0\left(\sqrt{\lambda}x\right)$$

To satisfy the second boundary condition, since $y'(x) = c_1J_0'\left(\sqrt{\lambda}x\right) = -c_1J_1\left(\sqrt{\lambda}x\right)$. Therefore the eigenvalues are roots of

$$J_1\left(\sqrt{\lambda}x\right) = 0$$

Plotting $J_1\left(\sqrt{\lambda}x\right)$ shows that the first roots are $\lambda = 0$. Numerically, the first few eigenvalues are

$$\lambda = \{0, 14.682, 49.2185, 103, 499, 177.532, \cdots\} \tag{2}$$

Hence the fundamental solution is $y(x) = J_0\left(\sqrt{\lambda_n}x\right)$ where λ_n is given by above. When $\lambda = 0, J_0(0) = 1$. Therefore the eigenfunction associated with $\lambda = 0$ is $\Phi_0(x) = 1$. Since there are infinite eigenvalues (2), there are infinite eigenfunctions $\Phi_n(x) = J_0\left(\sqrt{\lambda_n}x\right)$ where $n = 0, 1, 2, 3, \cdots$

60.2 Part (b)

Let $\Phi_n(x), \Phi_m(x)$ be any two eigenfunctions of $(xy')' + \lambda xy = 0$. Therefore each satisfies the ODE. Hence

$$\left(x\Phi_n'\right)' + \lambda_n x\Phi_n(x) = 0 \tag{3A}$$

$$\left(x\Phi_m'\right)' + \lambda_m x\Phi_m(x) = 0 \tag{3B}$$

Multiplying (3A) by Φ_m and (3B) by Φ_n and subtracting gives

$$\begin{aligned}\Phi_m\left(x\Phi_n'\right)' + \lambda_n x\Phi_m\Phi_n(x) - \Phi_n\left(x\Phi_m'\right)' - \lambda_m x\Phi_n\Phi_m(x) &= 0\\ \Phi_m\left(x\Phi_n'\right)' - \Phi_n\left(x\Phi_m'\right)' + (\lambda_n - \lambda_m)x\Phi_n\Phi_m(x) &= 0\end{aligned}$$

Integrating from $0\cdots 1$ gives

$$\int_0^1 \Phi_m\left(x\Phi_n'\right)' dx - \int_0^1 \Phi_n\left(x\Phi_m'\right)' dx + (\lambda_n - \lambda_m)\int_0^1 x\Phi_n\Phi_m(x)\,dx = 0 \tag{4}$$

Integrating $\int_0^1 \Phi_m\left(x\Phi_n'\right)' dx$ by parts gives

$$\int_0^1 \overbrace{\Phi_m}^{u}\overbrace{\left(x\Phi_n'\right)'}^{dv} dx = \left[\Phi_m x\Phi_n'\right]_0^1 - \int_0^1 \Phi_m'\left(x\Phi_n'\right) dx \tag{5A}$$

And similarly, Integrating $\int_0^1 \Phi_n\left(x\Phi_m'\right)' dx$ by parts gives

$$\int_0^1 \overbrace{\Phi_n}^{u}\overbrace{\left(x\Phi_m'\right)'}^{dv} dx = \left[\Phi_n x\Phi_m'\right]_0^1 - \int_0^1 \Phi_n'\left(x\Phi_m'\right) dx \tag{5B}$$

Substituting (5A,5B) back in (4) gives

$$\left[\Phi_m x\Phi_n'\right]_0^1 - \int_0^1 \Phi_m'\left(x\Phi_n'\right) dx - \left[\Phi_n x\Phi_m'\right]_0^1 + \int_0^1 \Phi_n'\left(x\Phi_m'\right) dx + (\lambda_n - \lambda_m)\int_0^1 x\Phi_n\Phi_m(x)\,dx = 0$$

The above simplifies to

$$\left[\Phi_m x\Phi_n' - \Phi_n x\Phi_m'\right]_0^1 + (\lambda_n - \lambda_m)\int_0^1 x\Phi_n\Phi_m(x)\,dx = 0 \tag{6}$$

The boundary terms above simplifies to

$$\left[\Phi_m x\Phi_n' - \Phi_n x\Phi_m'\right]_0^1 = \left[\Phi_m(1)\Phi_n'(1) - \Phi_n(1)\Phi_m'(1)\right]$$

But $\Phi_n'(1)$ and $\Phi_m'(1)$ are zero. This is because of the given boundary conditions $y'(1) = 0$. Hence $\left[\Phi_m x\Phi_n' - \Phi_n x\Phi_m'\right]_0^1 = 0$. Therefore (6) now simplifies to

$$(\lambda_n - \lambda_m)\int_0^1 x\Phi_n\Phi_m(x)\,dx = 0$$

But since $\lambda_n - \lambda_m \neq 0$, since these are different eigenvalues, then one concludes that

$$\int_0^1 x\Phi_n\Phi_m(x)\,dx = 0$$

Which is the result asked to show.

60.3 Part (c)

The problem to solve is written as

$$-\frac{1}{x}(xy')' = \mu y + \frac{f(x)}{x} \tag{A}$$

The solution to the corresponding homogeneous ODE $-\frac{1}{x}(xy')' = \lambda y$ was found in part (a). Using eigenfunction expansion, the solution of the nonhomogeneous ODE (A) can then be written as

$$y(x) = \sum_{n=0}^{\infty} b_n \Phi_n(x) \tag{7}$$

Where $\Phi_n(x) = J_0\left(\sqrt{\lambda_n} x\right), n = 0, 1, 2, \cdots$ and λ_n are roots of $-J_1\left(\sqrt{\lambda}\right) = 0$. Using (7) in $-(xy')' = \mu xy + f(x)$ gives

$$-\frac{1}{x}(xy')' = x\sum_{n=0}^{\infty} b_n \Phi_n(x) + \sum_{n=0}^{\infty} c_n \Phi_n(x)$$

But since $-\frac{1}{x}(xy')' = \lambda y$ from part (a), then the above becomes

$$\sum_{n=0}^{\infty} \lambda_n b_n \Phi_n(x) = \mu \sum_{n=0}^{\infty} b_n \Phi_n(x) + \sum_{n=0}^{\infty} c_n \Phi_n(x) \tag{8}$$

Where

$$\sum_{n=0}^{\infty} c_n \Phi_n(x) = \frac{f(x)}{x}$$

c_n is now found by orthogonality. Multiplying both sides of the above by $r(x)\Phi_m(x)$, where the weight $r(x) = x$, and integrating gives

$$\begin{aligned} \int_0^1 x \frac{f(x)}{x} \Phi_m(x)\,dx &= c_0 \int_0^1 x\Phi_0(x)\Phi_m(x)\,dx + \sum_{n=1}^{\infty} c_n \int_0^1 x\Phi_n(x)\Phi_m(x)\,dx \\ \int_0^1 f(x)\Phi_m(x)\,dx &= c_0 \int_0^1 x\Phi_0(x)\Phi_m(x)\,dx + \sum_{n=1}^{\infty} c_n \int_0^1 x\Phi_n(x)\Phi_m(x)\,dx \end{aligned} \tag{9}$$

For $m = 0$, the eigenfunction is $\Phi_0(x) = 1$, and the above becomes

$$\begin{aligned} \int_0^1 f(x)\,dx &= c_0 \int_0^1 x dx \\ &= c_0 \left[\frac{x^2}{2}\right]_0^1 = \frac{c_0}{2} \end{aligned}$$

Therefore

$$c_0 = 2\int_0^1 f(x)\,dx \tag{10}$$

For $m > 0$, (9) becomes

$$\int_0^1 f(x)\Phi_m(x)\,dx = \sum_{n=1}^{\infty} c_n \int_0^1 x\Phi_n(x)\Phi_m(x)\,dx$$

Due to orthogonality of the eigenfunctions from part (b) $\int_0^1 x\Phi_n(x)\Phi_m(x)\,dx = 0$ for $m \neq n$, and the above simplifies to

$$c_n = \frac{\int_0^1 f(x)\Phi_n(x)\,dx}{\int_0^1 x\Phi_n^2(x)\,dx} \tag{11}$$

Since $\Phi_n(x)$ is not normalized, $\int_0^1 x\Phi_n^2(x)\,dx$ can not be replaced by 1. The above is left as is. Substituting (10,11) in (8) and simplifying gives

$$\sum_{n=0}^{\infty}\lambda_n b_n\Phi_n(x)=\mu\sum_{n=0}^{\infty}b_n\Phi_n(x)+\sum_{n=0}^{\infty}c_n\Phi_n(x) \tag{12}$$

For $n=0$ only, and since $\lambda_n=0$ then (12) gives

$$0=\mu b_0\Phi_0(x)+c_0\Phi_0(x)$$

But $\Phi_0(x)=1$, hence

$$\begin{aligned}0&=\mu b_0+c_0\\ b_0&=-\frac{c_0}{\mu}\end{aligned}$$

For $n>0$, then (12) gives

$$\begin{aligned}\sum_{n=1}^{\infty}\lambda_n b_n\Phi_n(x)&=\mu\sum_{n=1}^{\infty}b_n\Phi_n(x)+\sum_{n=1}^{\infty}c_n\Phi_n(x)\\ \lambda_n b_n&=\mu b_n+c_n\\ b_n&=\frac{c_n}{(\lambda_n-\mu)}\end{aligned}$$

Where $\lambda_n\neq\mu$. Hence the formal solution $y=\sum_{n=0}^{\infty}b_n\Phi_n(x)$ can be written as

$$\begin{aligned}y(x)&=b_0\Phi_0(x)+\sum_{n=1}^{\infty}b_n\Phi_0(x)\\ &=-\frac{c_0}{\mu}+\sum_{n=1}^{\infty}\frac{c_n}{(\lambda_n-\mu)}J_o\left(\sqrt{\lambda_n}x\right)\\ &=-\frac{2}{\mu}\int_0^1 f(x)\,dx+\sum_{n=1}^{\infty}\frac{1}{(\lambda_n-\mu)}\frac{\int_0^1 f(x)J_o\left(\sqrt{\lambda_n}x\right)dx}{\int_0^1 xJ_o^2\left(\sqrt{\lambda_n}x\right)dx}J_o\left(\sqrt{\lambda_n}x\right)\end{aligned}$$

But $\int_0^1 xJ_o^2\left(\sqrt{\lambda_n}x\right)dx=\frac{1}{2}\left(J_o^2\left(\sqrt{\lambda_n}\right)+J_1^2\left(\sqrt{\lambda_n}\right)\right)$, hence the above becomes

$$y(x)=-\frac{2}{\mu}\int_0^1 f(x)\,dx+2\sum_{n=1}^{\infty}\frac{1}{(\lambda_n-\mu)}\frac{\int_0^1 f(x)J_o\left(\sqrt{\lambda_n}x\right)dx}{J_o^2\left(\sqrt{\lambda_n}\right)+J_1^2\left(\sqrt{\lambda_n}\right)}J_o\left(\sqrt{\lambda_n}x\right)$$

61 Chapter 11.4, Problem 3

Consider $-(xy')'+\frac{k^2}{x}y=\lambda xy$. with y,y' bounded as $x\rightarrow 0$ and $y(1)=0$, where k is positive integer. (a) using $t=\sqrt{\lambda}x$ show the ODE reduces to Bessel of order k. (b) show formally that the eigenvalues $\lambda_1,\lambda_2,\cdots$ of the given differential equation are the squares of positive zeros of $J_k\left(\sqrt{\lambda}\right)$ and that the corresponding eigenfunctions are $\Phi_n(x)=J_k\left(\sqrt{\lambda}x\right)$. It is possible to show there as infinite sequence of such zeros. (c) Show that the eigenfunctions $\Phi_n(x)$ satisfy the orthogonality relation

$$\int_0^1 x\Phi_m(x)\Phi_n(x)\,dx=0\qquad m\neq n$$

(d) Determine the coefficients of the formal series expansion $f(x) = \sum_{n=1}^{\infty} a_n \Phi_n(x)$. (e) Final formal solution of the nonhomogeneous problem

$$-(xy')' + \frac{k^2}{x} y = \mu x y + f(x)$$

With y, y' bounded as $x \to 0$ and $y(1) = 0$, where f is given continuous function on $0 \leq x \leq 1$ and μ is eigenvalue of the corresponding homogeneous problem.

Solution

61.1 part (a)

The ODE to solve is

$$-(xy')' + \frac{k^2}{x} y - \lambda x y = 0$$

Note: The problem seems to not have mentioned that $\lambda > 0$ here as well, as in the problem above it. This condition is needed to fully solve this problem with y, y' bounded as $x \to 0$ and $y(1) = 0$. The ODE can be written as

$$\begin{aligned} -xy'' - y' + y\left(\frac{k^2}{x} - \lambda x\right) &= 0 \\ xy'' + y' + y\left(\lambda x - \frac{k^2}{x}\right) &= 0 \end{aligned} \tag{1}$$

Let $t = \sqrt{\lambda} x$, then $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\sqrt{\lambda}$ and $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dt}\sqrt{\lambda}\right) = \sqrt{\lambda}\frac{d^2y}{dt^2}\frac{dt}{dx} = \sqrt{\lambda}\frac{d^2y}{dt^2}\sqrt{\lambda} = \lambda\frac{d^2y}{dt^2}$. Hence (1) becomes

$$\begin{aligned} \frac{t}{\sqrt{\lambda}} \lambda y''(t) + \sqrt{\lambda} y'(t) + y(t)\left(\lambda \frac{t}{\sqrt{\lambda}} - \frac{k^2}{t}\sqrt{\lambda}\right) &= 0 \\ t\sqrt{\lambda} y''(t) + \sqrt{\lambda} y'(t) + \sqrt{\lambda} y(t)\left(t - \frac{k^2}{t}\right) &= 0 \\ t^2 y'' + t y' + \left(t^2 - k^2\right) y &= 0 \end{aligned}$$

This is Bessel ODE of k order.

61.2 Part (b)

The solution to the above ODE is known to be

$$y(t) = c_1 J_k(t) + c_2 Y_k(t)$$

Where $J_k(0) = 0$ and $\lim_{t \to 0} Y_k(t) \to \infty$. Hence a bounded solution requires that $c_2 = 0$. Therefore the solution becomes

$$y(t) = c_1 J_k(t)$$

Or in terms of x

$$y(x) = c_1 J_k\left(\sqrt{\lambda} x\right)$$

To satisfy the second boundary condition $y(1) = 0$ gives

$$c_1 J_k\left(\sqrt{\lambda}\right) = 0$$

Non-trivial solution implies $J_k\left(\sqrt{\lambda}\right) = 0$. Therefore the eigenvalues are the square of positive roots of this equation. Even though there are negative and positive roots for $J_k\left(\sqrt{\lambda}\right) = 0$ but for real root, λ must be non-negative. It assumed $\lambda > 0$. There are infinite number of positive roots for $J_k\left(\sqrt{\lambda}\right) = 0$. Hence the eigenfunctions are

$$\Phi_n(x) = J_k\left(\sqrt{\lambda_n}x\right) \qquad n = 1, 2, 3, \cdots$$

Where λ_n are square of the all positive zeros of $J_k\left(\sqrt{\lambda}\right) = 0$.

61.3 Part (c)

Show that the eigenfunctions $\Phi_n(x)$ satisfy the orthogonality relation

$$\int_0^1 x\Phi_m(x)\Phi_n(x)\,dx = 0 \qquad m \neq n$$

Let $\Phi_n(x), \Phi_m(x)$ be any two eigenfunctions of $-(xy')' + \frac{k^2}{x}y = \lambda xy$ where now $\Phi_n(x) = J_k\left(\sqrt{\lambda_n}x\right)$ and $\Phi_m(x) = J_k\left(\sqrt{\lambda_m}x\right)$. Therefore each satisfies the ODE. Hence

$$-\left(x\Phi_n'\right)' + \frac{k^2}{x}\Phi_n(x) - \lambda_n x\Phi_n(x) = 0 \tag{3A}$$

$$-\left(x\Phi_m'\right)' + \frac{k^2}{x}\Phi_m(x) - \lambda_m x\Phi_m(x) = 0 \tag{3B}$$

Multiplying 3A by Φ_m and 3B by Φ_n and subtracting gives

$$-\Phi_m\left(x\Phi_n'\right)' + \Phi_m\frac{k^2}{x}\Phi_n(x) - \lambda_n x\Phi_m\Phi_n(x) - \left(-\left(\Phi_n x\Phi_m'\right)' + \frac{k^2}{x}\Phi_n\Phi_m(x) - \lambda_m x\Phi_n\Phi_m(x)\right) = 0$$

$$-\Phi_m\left(x\Phi_n'\right)' + \frac{k^2}{x}\Phi_m\Phi_n(x) - \lambda_n x\Phi_m\Phi_n(x) + \Phi_n\left(x\Phi_m'\right)' - \frac{k^2}{x}\Phi_n\Phi_m(x) + \lambda_m x\Phi_n\Phi_m(x) = 0$$

$$-\left(x\Phi_n'\right)' + \left(x\Phi_m'\right)' + (\lambda_m - \lambda_n)x\Phi_n\Phi_m(x) = 0$$

Integrating from $0\cdots 1$ gives

$$\int_0^1 \Phi_m\left(x\Phi_n'\right)'dx - \int_0^1 \Phi_n\left(x\Phi_m'\right)'dx + (\lambda_n - \lambda_m)\int_0^1 x\Phi_n\Phi_m(x)\,dx = 0 \tag{4}$$

Integrating $\int_0^1 \Phi_m\left(x\Phi_n'\right)'dx$ by parts gives

$$\int_0^1 \overbrace{\Phi_m}^{u}\overbrace{\left(x\Phi_n'\right)'}^{dv}dx = \left[\Phi_m x\Phi_n'\right]_0^1 - \int_0^1 \Phi_m'\left(x\Phi_n'\right)dx \tag{5A}$$

And similarly, Integrating $\int_0^1 \Phi_n\left(x\Phi_m'\right)'dx$ by parts gives

$$\int_0^1 \overbrace{\Phi_n}^{u}\overbrace{\left(x\Phi_m'\right)'}^{dv}dx = \left[\Phi_n x\Phi_m'\right]_0^1 - \int_0^1 \Phi_n'\left(x\Phi_m'\right)dx \tag{5B}$$

Substituting (5A,5B) back in (4) gives

$$\left[\Phi_m x\Phi_n'\right]_0^1 - \int_0^1 \Phi_m'\left(x\Phi_n'\right)dx - \left[\Phi_n x\Phi_m'\right]_0^1 + \int_0^1 \Phi_n'\left(x\Phi_m'\right)dx + (\lambda_n - \lambda_m)\int_0^1 x\Phi_n\Phi_m(x)\,dx = 0$$

The above simplifies to

$$\left[\Phi_m x\Phi_n' - \Phi_n x\Phi_m'\right]_0^1 + (\lambda_n - \lambda_m)\int_0^1 x\Phi_n\Phi_m(x)\,dx = 0 \tag{6}$$

Let $\Delta = \left[\Phi_m x\Phi_n' - \Phi_n x\Phi_m'\right]_0^1$, then the boundary terms above simplifies to

$$\Delta = \left[\Phi_m(1)\Phi_n'(1) - \Phi_n(1)\Phi_m'(1)\right] - \lim_{x\to 0}\left[x\Phi_m(x)\Phi_n'(x) - x\Phi_n(x)\Phi_m'(x)\right]$$

But $\Phi_n(1)$ and $\Phi_m(1)$ are zero. This is because of the given boundary conditions. Hence the above simplifies to

$$\left[\Phi_m x\Phi_n' - \Phi_n x\Phi_m'\right]_0^1 = -\lim_{x\to 0}\left(x\left(\Phi_m(x)\Phi_n'(x) - \Phi_n(x)\Phi_m'(x)\right)\right)$$

But since both $\Phi_m(x), \Phi_n(x), \Phi_n'(x), \Phi_m'(x)$ are bounded as $x \to 0$ then the above vanishes. This means the all the boundary terms are zero and (6) simplifies to

$$(\lambda_n - \lambda_m)\int_0^1 x\Phi_n\Phi_m(x)\,dx = 0$$

But since $\lambda_n - \lambda_m \neq 0$, since these are different eigenvalues, therefore

$$\int_0^1 x\Phi_n\Phi_m(x)\,dx = 0$$

Which is the result asked to show.

61.4 Part (d,e)

This is both parts combined. To solve $-(xy')' + \frac{k^2}{x}y = \mu xy + f(x)$, we start with dividing by x to get the ODE to the form

$$-\frac{1}{x}(xy')' + \frac{k^2}{x^2}y = \mu y + \frac{f(x)}{x} \tag{1}$$

The homogeneous ode $-\frac{1}{x}(xy')' + \frac{k^2}{x^2}y = \lambda y$ was solved in part (a,b). And since the problem says that $\lambda \neq \mu$, then the solution to the above nonhomogeneous ODE is

$$y(x) = \sum_{n=1}^{\infty} b_n\Phi_n(x) \tag{1}$$

Where $\Phi_n(x)$ are eigenfunctions of the homogeneous ODE found above to be

$$\Phi_n(x) = J_k\left(\sqrt{\lambda_n}x\right) \qquad n = 1, 2, 3, \cdots$$

Substituting (2) in RHS of (1) gives

$$-\frac{1}{x}(xy')' + \frac{k^2}{x^2}y = \mu\sum_{n=1}^{\infty} b_n\Phi_n(x) + \sum_{n=1}^{\infty} c_n\Phi_n(x)$$

Where $\sum_{n=1}^{\infty} c_n\Phi_n(x) = \frac{f(x)}{x}$. But $-\frac{1}{x}(xy')' + \frac{k^2}{x^2}y = \lambda y$ from part (a,b). Therefore the above becomes

$$\sum_{n=1}^{\infty}\lambda_n b_n\Phi_n(x) = \mu\sum_{n=1}^{\infty} b_n\Phi_n(x) + \sum_{n=1}^{\infty} c_n\Phi_n(x)$$

Or

$$\lambda_n b_n = \mu b_n + c_n$$
$$b_n = \frac{c_n}{\lambda_n - \mu}$$

What is left is to to find c_n (called a_n in this problem). Since $\sum_{n=1}^{\infty} c_n \Phi_n(x) = \frac{f(x)}{x}$, then applying orthogonality gives

$$c_n \int_0^1 r(x)\,\Phi_n^2(x)\,dx = \int_0^1 r(x)\,\frac{f(x)}{x}\Phi_n(x)\,dx$$

But $r(x) = x$, and the above becomes

$$c_n \int_0^1 x J_k^2\left(\sqrt{\lambda_n} x\right) dx = \int_0^1 f(x)\,J_k\left(\sqrt{\lambda_n} x\right) dx$$
$$c_n = \frac{\int_0^1 f(x)\,J_k\left(\sqrt{\lambda_n} x\right) dx}{\int_0^1 x J_k^2\left(\sqrt{\lambda_n} x\right) dx}$$

This complete the solution.

$$\begin{aligned} y(x) &= \sum_{n=1}^{\infty} b_n J_k\left(\sqrt{\lambda_n} x\right) \\ &= \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - \mu} J_k\left(\sqrt{\lambda_n} x\right) \\ &= \sum_{n=1}^{\infty} \frac{\int_0^1 f(x)\,J_k\left(\sqrt{\lambda_n} x\right) dx}{\int_0^1 x J_k^2\left(\sqrt{\lambda_n} x\right) dx} \frac{J_k\left(\sqrt{\lambda_n} x\right)}{\lambda_n - \mu} \end{aligned}$$

62 Chapter 11.4, Problem 4

Consider Legendre equation $-\left(\left(1-x^2\right) y'\right)' = \lambda y$ subject to boundary conditions $y(0) = 0$ with y, y' bounded as $x \to 1$ and $\Phi_1(x) = P_1(x)$, $\Phi_2(x) = P_3(x)$, $\Phi_n(x) = P_{2n-1}(x)$ corresponding to eigenvalues $\lambda_1 = 2, \lambda_2 = 4 \cdot 3, \cdots, \lambda_n = 2n(2n-1)$. (a) Show that the eigenfunctions $\Phi_n(x)$ satisfy the orthogonality relation

$$\int_0^1 \Phi_m(x)\,\Phi_n(x)\,dx = 0 \qquad m \neq n$$

(b) Final formal solution of the nonhomogeneous problem $-\left(\left(1-x^2\right) y'\right)' = \mu y + f(x)$ where $y(0) = 0$ with y, y' bounded as $x \to 1$ where $f(x)$ is continuous function on $0 \leq x \leq 1$ and μ is not eigenvalue of $-\left(\left(1-x^2\right) y'\right)' = \lambda y$

Solution

62.1 Part (a)

Let $\Phi_n(x), \Phi_m(x)$ be any two eigenfunctions of $-\left(\left(1-x^2\right) y'\right)' = \lambda y$ associated with eigenvalues λ_n, λ_m, where $\Phi_n(x) = P_n(x)$ and $\Phi_m(x) = P_m(x)$. Therefore each satisfies the ODE. Hence

$$\left(\left(1-x^2\right) \Phi_n'(x)\right)' + \lambda_n \Phi_n = 0 \tag{3A}$$
$$\left(\left(1-x^2\right) \Phi_m'(x)\right)' + \lambda_m \Phi_m = 0 \tag{3B}$$

Multiplying 3A by Φ_m and 3B by Φ_n and subtracting gives

$$\begin{aligned}\Phi_m\left(\left(1-x^2\right)\Phi_n'(x)\right)'+\lambda_n\Phi_m\Phi_n-\left(\Phi_n\left(\left(1-x^2\right)\Phi_m'(x)\right)'+\lambda_m\Phi_n\Phi_m\right)&=0\\ \Phi_m\left(\left(1-x^2\right)\Phi_n'(x)\right)'-\Phi_n\left(\left(1-x^2\right)\Phi_m'(x)\right)'+(\lambda_n-\lambda_m)\Phi_n\Phi_m&=0\end{aligned}$$

Integrating from $0\cdots1$ gives (all upper limits below show be $\lim_{\varepsilon\to0^-}\int_0^{1-\varepsilon}$ instead of $\int_0^1$ but to simplify notation, the latter is used and at the end, it is switched back to former.

$$\int_0^1\Phi_m\left(\left(1-x^2\right)\Phi_n'(x)\right)'dx-\int_0^1\Phi_n\left(\left(1-x^2\right)\Phi_m'(x)\right)'dx+(\lambda_n-\lambda_m)\int_0^1\Phi_n\Phi_m(x)\,dx=0 \quad (4)$$

The first integral in (4) $\int_0^1\overbrace{\Phi_m}^{u}\overbrace{\left(\left(1-x^2\right)\Phi_n'(x)\right)'}^{dv}dx$ is integrated by parts, giving

$$\begin{aligned}\int_0^1\Phi_m\left(\left(1-x^2\right)\Phi_n'(x)\right)'dx&=\left[\Phi_m\left(1-x^2\right)\Phi_n'(x)\right]_0^1-\int_0^1\Phi_m'\left(\left(1-x^2\right)\Phi_n'(x)\right)dx\\ &=\left[\Phi_m\left(1-x^2\right)\Phi_n'(x)\right]_0^1-\int_0^1\Phi_n'\left(\left(1-x^2\right)\Phi_m'(x)\right)dx \qquad (4A)\end{aligned}$$

Similarly, the second integral in (4) $\int_0^1\overbrace{\Phi_n}^{u}\overbrace{\left(\left(1-x^2\right)\Phi_m'(x)\right)'}^{dv}dx$ is integrated by parts, giving

$$\begin{aligned}\int_0^1\Phi_n\left(\left(1-x^2\right)\Phi_m'(x)\right)'dx&=\left[\Phi_n\left(1-x^2\right)\Phi_m'(x)\right]_0^1-\int_0^1\Phi_n'\left(\left(1-x^2\right)\Phi_m'(x)\right)dx\\ &=\left[\Phi_m\left(1-x^2\right)\Phi_n'(x)\right]_0^1-\int_0^1\Phi_n'\left(\left(1-x^2\right)\Phi_m'(x)\right)dx \qquad (4B)\end{aligned}$$

Substituting (4A) and (4B) back into (4) gives

$$\begin{aligned}&\left[\Phi_m\left(1-x^2\right)\Phi_n'(x)\right]_0^1-\int_0^1\Phi_n'\left(\left(1-x^2\right)\Phi_m'(x)\right)dx-\\ &\qquad\left(\left[\Phi_m\left(1-x^2\right)\Phi_n'(x)\right]_0^1-\int_0^1\Phi_n'\left(\left(1-x^2\right)\Phi_m'(x)\right)dx\right)\\ &\qquad\qquad+(\lambda_n-\lambda_m)\int_0^1\Phi_n\Phi_m(x)\,dx=0\end{aligned}$$

Terms cancel and the above reduces to

$$\begin{aligned}\left[\Phi_m\left(1-x^2\right)\Phi_n'(x)\right]_0^1-\left[\Phi_m\left(1-x^2\right)\Phi_n'(x)\right]_0^1+(\lambda_n-\lambda_m)\int_0^1\Phi_n\Phi_m(x)\,dx&=0\\ \left[\Phi_m\left(1-x^2\right)\Phi_n'(x)-\Phi_m\left(1-x^2\right)\Phi_n'(x)\right]_0^1+(\lambda_n-\lambda_m)\int_0^1\Phi_n\Phi_m(x)\,dx&=0 \qquad (5)\end{aligned}$$

Let $\Delta=\left[\Phi_m\left(1-x^2\right)\Phi_n'(x)-\Phi_m\left(1-x^2\right)\Phi_n'(x)\right]_0^1$. The boundary terms above are evaluated as follows

$$\Delta=\lim_{x\to1}\left[\Phi_m(x)\left(1-x^2\right)\Phi_n'(x)-\Phi_m(x)\left(1-x^2\right)\Phi_n'(x)\right]-\left(\Phi_m(0)\Phi_n'(0)-\Phi_m(0)\Phi_n'(0)\right)$$

Since $\Phi_m(0) = 0, \Phi_m(0) = 0$, the above simplifies to

$$\begin{aligned}\Delta &= \lim_{x\to 1}\left[\Phi_m(x)\left(1-x^2\right)\Phi_n'(x) - \Phi_m(x)\left(1-x^2\right)\Phi_n'(x)\right]\\ &= \lim_{x\to 1}\left(1-x^2\right)\left[\Phi_m(x)\Phi_n'(x) - \Phi_m(x)\Phi_n'(x)\right]\end{aligned}$$

Since $\Phi_m(x), \Phi_n'(x), \Phi_m(x)\Phi_n'(x)$ are all bounded as $x \to 1$ then the above goes to zero in the limit. Which means all boundary conditions term vanish. Hence (5) reduces to

$$(\lambda_n - \lambda_m)\int_0^1 \Phi_n\Phi_m(x)\,dx = 0$$

But since $\lambda_n - \lambda_m \neq 0$, since these are different eigenvalues, therefore

$$\int_0^1 \Phi_n\Phi_m(x)\,dx = 0$$

Which is the result asked to show.

62.2 Part (b)

Since $\lambda \neq \mu$, then the the solution to nonhomogeneous ODE is

$$y(x) = \sum_{n=1}^{\infty} b_n\Phi_n(x) \tag{1}$$

Where $\Phi_n(x)$ are eigenfunctions $\Phi_n(x) = P_{(2n-1)}(x)$. Substituting (1) in $-\left(\left(1-x^2\right)y'\right)' = \mu y + f(x)$ gives

$$-\left(\left(1-x^2\right)y'\right)' = \mu\sum_{n=1}^{\infty} b_n\Phi_n(x) + \sum_{n=1}^{\infty} c_n\Phi_n(x)$$

Where $\sum_{n=1}^{\infty} c_n\Phi_n(x) = f(x)$. But $-\left(\left(1-x^2\right)y'\right)' = \lambda y$, therefore the above becomes

$$\sum_{n=1}^{\infty} \lambda_n b_n\Phi_n(x) = \mu\sum_{n=1}^{\infty} b_n\Phi_n(x) + \sum_{n=1}^{\infty} c_n\Phi_n(x)$$

Or

$$\begin{aligned}\lambda_n b_n &= \mu b_n + c_n\\ b_n &= \frac{c_n}{\lambda_n - \mu}\end{aligned}$$

What is left is to to find c_n. Since $\sum_{n=1}^{\infty} c_n\Phi_n(x) = f(x)$, then applying orthogonality gives

$$c_n\int_0^1 r(x)\Phi_n^2(x)\,dx = \int_0^1 r(x)f(x)\Phi_n(x)\,dx$$

But $r(x) = 1$, and the above becomes

$$\begin{aligned}c_n\int_0^1 P_{(2n-1)}^2(x)\,dx &= \int_0^1 f(x)P_{(2n-1)}(x)\,dx\\ c_n &= \frac{\int_0^1 f(x)P_{(2n-1)}(x)\,dx}{\int_0^1 P_{(2n-1)}^2(x)\,dx}\end{aligned}$$

This complete the solution.

$$
\begin{aligned}
y(x) &= \sum_{n=1}^{\infty} b_n P_{(2n-1)}(x) \\
&= \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - \mu} P_{(2n-1)}(x) \\
&= \sum_{n=1}^{\infty} \frac{\int_0^1 f(x) P_{(2n-1)}(x)\,dx}{\int_0^1 P_{(2n-1)}^2(x)\,dx} \frac{P_{(2n-1)}(x)}{\lambda_n - \mu}
\end{aligned}
$$

63 Chapter 11.4, Problem 5

Equation $\left(1-x^2\right) y'' - xy' + \lambda y = 0$ is Chebyshev's equation. (a) show it can be written as

$$
-\left(\sqrt{1-x^2}y'\right)' = \frac{\lambda}{\sqrt{1-x^2}} y \qquad -1 < x < 1
$$

(b) consider boundary conditions y, y' bounded as $x \to -1$ and $x \to +1$. Show that the problem is self adjoint. (c) Show that

$$
\int_{-1}^{1} \frac{T_m(x) T_n(x)}{\sqrt{1-x^2}} dx = 0
$$

Where $T_n(x)$ are the eigenfunctions :$T_0(x) = 1, T_1(x) = x, T_2(x) = 1 - 2x^2, \cdots$ and eigenvalues are $\lambda_n = n^2$ for $n = 0, 1, 2, \cdots$

Solution

63.1 Part (a)

Writing the ODE $\left(1-x^2\right) y'' - xy' + \lambda y = 0$ as

$$
P(x) y'' + Q(x) y' + R(x) y = 0
$$

Where $P(x) = \left(1-x^2\right), Q(x) = -x, R(x) = \lambda$, then the integrating factor is

$$
\begin{aligned}
\mu &= \frac{1}{P} e^{\int \frac{Q(x)}{P(x)} dx} \\
&= \frac{1}{(1-x^2)} e^{\int \frac{-x}{\left(1-x^2\right)} dx}
\end{aligned}
$$

But $\int \frac{x}{(1-x^2)} dx = \frac{1}{2} \ln\left|1-x^2\right|$, therefore $e^{\frac{1}{2}\ln\left|1-x^2\right|} = \sqrt{1-x^2}$ and the above becomes $\mu = \frac{1}{\sqrt{1-x^2}}$. Hence the SL form is

$$
\begin{aligned}
(\mu P y')' + \mu R(x) y &= 0 \\
\left(\frac{1}{\sqrt{1-x^2}}\left(1-x^2\right) y'\right)' + \frac{1}{\sqrt{1-x^2}} \lambda y &= 0 \\
-\left(\sqrt{1-x^2}y'\right)' &= \frac{1}{\sqrt{1-x^2}} \lambda y
\end{aligned}
$$

63.2 Part (b)

A problem is self adjoint if

$$\langle L[u], v\rangle = \langle u, L[v]\rangle$$

Where u, v are any two arbitrary eigenfunctions of the ODE which therefore by definition satisfy the ODE and the boundary conditions as given. Starting with $\langle L[u], v\rangle$ and it is evaluated to see if it leads to $\langle u, L[v]\rangle$. The operator is defined as (from part (a)) as

$$L[y] = -\left(\sqrt{1-x^2}y'\right)' = \frac{1}{\sqrt{1-x^2}}\lambda y$$

Therefore

$$\langle L[u], v\rangle = \int_{-1}^{1} \overbrace{-\left(\sqrt{1-x^2}u'\right)'}^{dv} \overbrace{v}^{u} dx$$

Integrating by parts gives

$$\begin{aligned}
\langle L[u], v\rangle &= \left[-\left(\sqrt{1-x^2}u'\right)v\right]_{-1}^{1} - \int_{-1}^{1} -\left(\sqrt{1-x^2}u'\right)v'dx \\
&= \left[-\left(\sqrt{1-x^2}u'\right)v\right]_{-1}^{1} - \int_{-1}^{1} \overbrace{-\left(\sqrt{1-x^2}v'\right)}^{u} \overbrace{u'}^{dv} dx
\end{aligned}$$

Integrating by parts again gives

$$\begin{aligned}
\langle L[u], v\rangle &= \left[-\left(\sqrt{1-x^2}u'\right)v\right]_{-1}^{1} - \left(\left[-\left(\sqrt{1-x^2}v'\right)u\right]_{-1}^{1} - \int_{-1}^{1} -\left(\sqrt{1-x^2}v'\right)' udx\right) \\
&= \left[-\sqrt{1-x^2}u'v + \sqrt{1-x^2}v'u\right]_{-1}^{1} + \int_{-1}^{1} -\left(\sqrt{1-x^2}v'\right)' udx \\
&= \left[\sqrt{1-x^2}\left(v'u - u'v\right)\right]_{-1}^{1} + \langle u, L[v]\rangle
\end{aligned}$$

Therefore the ODE is self adjoint if the boundary terms vanish. Let $\Delta = \left[\sqrt{1-x^2}\left(v'u - u'v\right)\right]_{-1}^{1}$. Evaluating this gives

$$\Delta = \lim_{x\to 1}\sqrt{1-x^2}\left(v'(x)u(x) - u'(x)v(x)\right) - \lim_{x\to -1}\sqrt{1-x^2}\left(v'(x)u(x) - u'(x)v(x)\right)$$

But since u, u' are bounded as $x \to -1$ and $x \to +1$ and also v, v' are bounded as $x \to -1$ and $x \to +1$, then this shows that $\Delta \to 0$. Therefore

$$\langle L[u], v\rangle = \langle u, L[v]\rangle$$

Hence the ODE is self adjoint.

63.3 Part (c)

Since $T_n(x), T_m(x)$ are two eigenfunctions of $-\left(\sqrt{1-x^2}y'\right)' = \frac{1}{\sqrt{1-x^2}}\lambda y$ then each satisfies the ODE. Hence

$$\left(\sqrt{1-x^2}T_n'\right)' + \frac{1}{\sqrt{1-x^2}}\lambda_n T_n = 0 \tag{3A}$$

$$\left(\sqrt{1-x^2}T_m'\right)' + \frac{1}{\sqrt{1-x^2}}\lambda_m T_m = 0 \tag{3B}$$

Multiplying 3A by T_m and 3B by T_n and subtracting gives

$$T_m \left(\sqrt{1-x^2}T_n'\right)' + \frac{1}{\sqrt{1-x^2}}\lambda_n T_m T_n - \left(T_n \left(\sqrt{1-x^2}T_m'\right)' + \frac{1}{\sqrt{1-x^2}}\lambda_m T_n T_m\right) = 0$$

$$T_m \left(\sqrt{1-x^2}T_n'\right)' - T_n \left(\sqrt{1-x^2}T_m'\right)' + (\lambda_n - \lambda_m)\frac{1}{\sqrt{1-x^2}} T_m T_n = 0$$

Integrating from $-1 \cdots 1$ gives

$$\int_{-1}^{1} T_m \left(\sqrt{1-x^2}T_n'\right)' dx - \int_{-1}^{1} T_n \left(\sqrt{1-x^2}T_m'\right)' dx + (\lambda_n - \lambda_m)\int_{-1}^{1} \frac{T_m T_n}{\sqrt{1-x^2}} dx = 0 \tag{1}$$

Integrating by parts the first integral in (1) above gives

$$\int_{-1}^{1} T_m \left(\sqrt{1-x^2}T_n'\right)' dx = \left[T_m \sqrt{1-x^2}T_n'\right]_{-1}^{1} - \int_{-1}^{1} T_m' \left(\sqrt{1-x^2}T_n'\right) dx \tag{1A}$$

Integrating by parts the second integral in (1) gives

$$\int_{-1}^{1} T_n \left(\sqrt{1-x^2}T_m'\right)' dx = \left[T_n \sqrt{1-x^2}T_m'\right]_{-1}^{1} - \int_{-1}^{1} T_n' \left(\sqrt{1-x^2}T_m'\right) dx \tag{1B}$$

Substituting (1A) and (1B) back into (1) and simplifying gives

$$\left[T_m \sqrt{1-x^2}T_n'\right]_{-1}^{1} - \left[T_n \sqrt{1-x^2}T_m'\right]_{-1}^{1} + (\lambda_n - \lambda_m)\int_{-1}^{1} \frac{T_m T_n}{\sqrt{1-x^2}} dx = 0$$

$$\left[T_m \sqrt{1-x^2}T_n' - T_n \sqrt{1-x^2}T_m'\right]_{-1}^{1} + (\lambda_n - \lambda_m)\int_{-1}^{1} \frac{T_m T_n}{\sqrt{1-x^2}} dx = 0$$

$$\left[\sqrt{1-x^2}\left(T_m T_n' - T_n T_m'\right)\right]_{-1}^{1} + (\lambda_n - \lambda_m)\int_{-1}^{1} \frac{T_m T_n}{\sqrt{1-x^2}} dx = 0 \tag{1C}$$

Let $\Delta = \left[\sqrt{1-x^2}\left(T_m T_n' - T_n T_m'\right)\right]_{-1}^{1}$, then

$$\Delta = \lim_{x\to 1} \sqrt{1-x^2}\left(T_m(x) T_n'(x) - T_n(x) T_m'(x)\right) - \lim_{x\to -1} \sqrt{1-x^2}\left(T_m(x) T_n'(x) - T_n(x) T_m'(x)\right)$$

But since $T_n(x), T_m(x), T_n'(x), T_m'(x)$ are all bounded as $x \to -1$ and as $x \to +1$, then $\Delta \to 0$. Therefore (1C) becomes

$$(\lambda_n - \lambda_m)\int_{-1}^{1} \frac{T_m T_n}{\sqrt{1-x^2}} dx = 0$$

But since $\lambda_n \neq \lambda_m$, since $m \neq n$, then

$$\int_{-1}^{1} \frac{T_m T_n}{\sqrt{1-x^2}} dx = 0$$

Which is what we are asked to show.

64 Chapter 11.5, Problem 2 (With interactive animation)

Find displacement $u(r,t)$ in vibrating circular elastic membrane of radius 1 that satisfies the boundary conditions

$$u(1,t) = 0 \qquad t \geq 0$$

And initial conditions

$$u(r,0) = 0$$
$$u_t(r,0) = g(r)$$

For $0 \leq r \leq 1$, where $g(1) = 0$.

Solution

The wave equation is $u_{tt} = a^2 (u_{xx} + u_{yy})$. In polar coordinates this becomes

$$\frac{1}{a^2} u_{tt} = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$$

Due to circular symmetry, the above simplifies to

$$\frac{1}{a^2} u_{tt} = u_{rr} + \frac{1}{r} u_r$$

Applying separation of variables. Let $u = T(t) R(r)$. Substituting this in the above PDE gives

$$\frac{1}{a^2} T'' R = R'' T + \frac{1}{r} R' T$$

Dividing by RT results in

$$\frac{1}{a^2} \frac{T''}{T} = \frac{R''}{R} + \frac{1}{r} \frac{R'}{R} = -\lambda^2$$

Where λ is the sepration constant. For $\lambda > 0$ (it is known $\lambda = 0$ is not eigenvalue, as well as there are no negative eigenvalues.) The above gives two ODE

$$T'' + \lambda^2 a^2 T = 0$$

And

$$r R''(r) + R'(r) + \lambda^2 r R(r) = 0 \tag{1}$$

With the boundary conditions $R(1) = 0$ and to $R(0)$ is bounded. This comes from physics, since one expects the vibration not to blow up in the center of the membrane. The ODE (1) is now transformed to Bessel ODE using

$$\xi = \lambda r$$

Hence $\frac{dR}{dr} = \frac{dR}{d\xi}\frac{d\xi}{dr} = \lambda \frac{dR}{d\xi}$ and $\frac{d^2R}{dr^2} = \lambda^2 \frac{d^2R}{d\xi^2}$. Therefore (1) becomes

$$\frac{\xi}{\lambda} \lambda^2 R''(\xi) + \lambda R'(\xi) + \lambda^2 \frac{\xi}{\lambda} R(\xi) = 0$$

The above simplifies to

$$\xi R''(\xi) + R'(\xi) + \xi R(\xi) = 0$$

The above is Bessel ODE of order zero. Its solution is

$$R(\xi) = c_1 J_0(\xi) + c_2 Y_0(\xi)$$

Converting back to r the above becomes

$$R(r) = c_1 J_0(r\lambda) + c_2 Y_0(r\lambda)$$

Since $R(r)$ is bounded as $r \to 0$, then $c_2 = 0$ as $Y_0(r\lambda)$ blows up at $r = 0$. Therefore the radial solution becomes

$$R(r) = c_1 J_0(r\lambda)$$

At boundary conditions $R(1) = 0$ the above becomes

$$0 = c_1 J_0(\lambda)$$

Non trivial solution requires $J_0(\lambda) = 0$. Therefore the eigenvalues are the the positive roots of $J_0(\lambda) = 0$. The first few eigenvalues are $\lambda_1 = 5.78319, \lambda_2 = 30.4713, \lambda_3 = 74.887, \cdots$. Hence

$$R_n(r) = c_n J_0(\lambda_n r) \qquad n = 1, 2, 3, \cdots$$

Now the time ODE is

$$T'' + \lambda^2 a^2 T = 0$$

Since $\lambda > 0$ then the solution is

$$T_n(t) = A_n \cos(\lambda_n a t) + B_n \sin(\lambda_n a t)$$

Therefore the fundamental solution is

$$u_n(r, t) = T_n(t) R_n(r)$$

And by superposition, the general solution is

$$u(r, t) = \sum_{n=1}^{\infty} (A_n \cos(\lambda_n a t) + B_n \sin(\lambda_n a t)) J_0(\lambda_n r) \tag{1A}$$

Where the c_n is merged into A_n, B_n due to the product. At $t = 0$ and since $u(r, 0) = 0$, the above becomes

$$0 = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r)$$

Hence $A_n = 0$. The solution simplifies to

$$u(r, t) = \sum_{n=1}^{\infty} B_n \sin(\lambda_n a t) J_0(\lambda_n r)$$

Taking time derivative gives

$$u_t(r, t) = \sum_{n=1}^{\infty} B_n \lambda_n a \cos(\lambda_n a t) J_0(\lambda_n r)$$

At $t = 0$, and from initial conditions, the above becomes

$$g(r) = \sum_{n=1}^{\infty} B_n \lambda_n a J_0(\lambda_n r)$$

Applying orthogonality, and since the weight is r, therefore

$$\int_0^1 r g(r) J_0(\lambda_n r)\, dr = B_n \lambda_n a \int_0^1 r J_0^2(\lambda_n r)\, dr$$

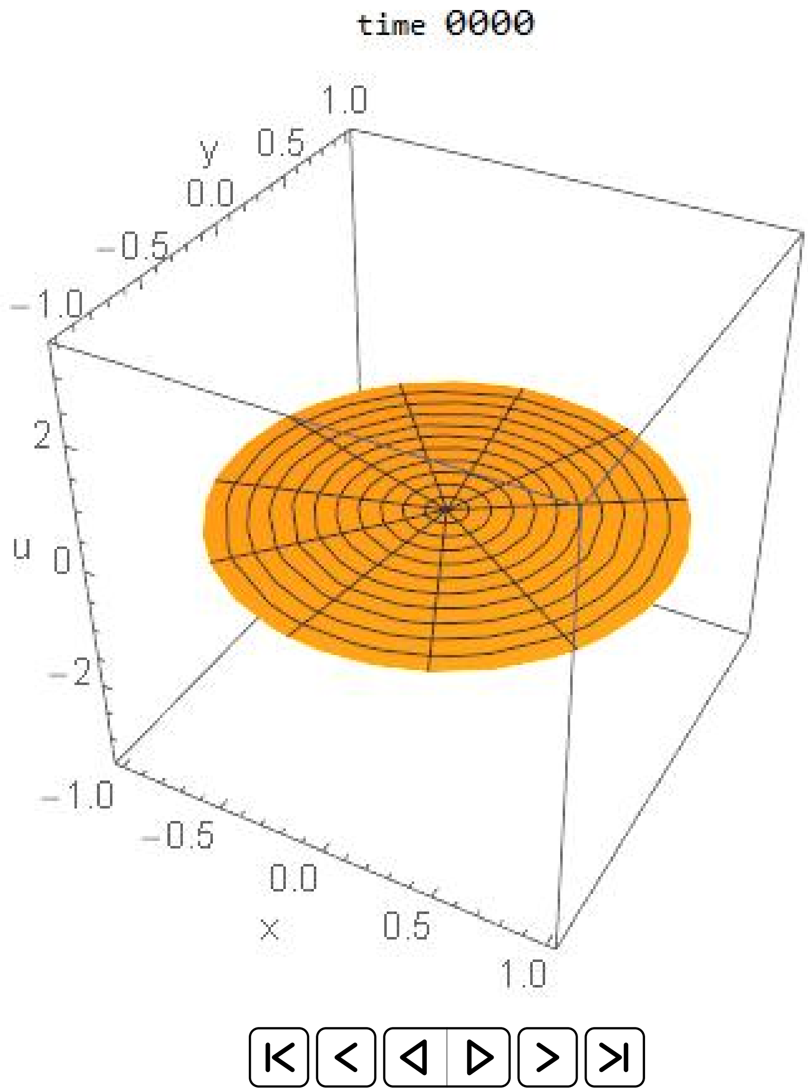
$$B_n = \frac{1}{\lambda_n a} \frac{\int_0^1 r g(r) J_0(\lambda_n r)\, dr}{\int_0^1 r J_0^2(\lambda_n r)\, dr} \tag{2}$$

Therefore the final solution is

$$u(r, t) = \sum_{n=1}^{\infty} B_n \sin(\lambda_n a t) J_0(\lambda_n r)$$

With B_n given by (2).

The following is an animation of the above solution. $a = 0.2$ and $g(r) = r$ was used. This runs inside the PDF (need to use standard PDF reader to run the animation. Might not run inside Chrome browser PDF reader).



65 Chapter 11.5, Problem 3 (With interactive animation)

Find displacement $u(r,t)$ in vibrating circular elastic membrane of radius 1 that satisfies the boundary conditions

$$u(1,t) = 0 \qquad t \geq 0$$

And initial conditions

$$u(r,0) = f(r)$$
$$u_t(r,0) = g(r)$$

For $0 \leq r \leq 1$, where $g(1) = 0$.

Solution

The same steps are used to reach the general solution as was done in the above problem. The difference is when initial conditions are used to determine the coefficients.

The general solution from the above problem was found to be

$$u(r,t) = \sum_{n=1}^{\infty} (A_n \cos(\lambda_n a t) + B_n \sin(\lambda_n a t)) J_0(\lambda_n r) \tag{1A}$$

At $t=0$

$$f(r)=\sum_{n=1}^{\infty}A_nJ_0(\lambda_n r)$$

Applying orthogonality, and since the weight is r results in

$$\int_0^1 rf(r)J_0(\lambda_n r)\,dr = A_n\int_0^1 rJ_0^2(\lambda_n r)\,dr$$

$$A_n=\frac{\int_0^1 rf(r)J_0(\lambda_n r)\,dr}{\int_0^1 rJ_0^2(\lambda_n r)\,dr} \tag{2}$$

Taking time derivative of the solution (1A)

$$u_t(r,t)=\sum_{n=1}^{\infty}-A_n\sqrt{\lambda_n}a\sin(\lambda_n at)+B_n\lambda_n a\cos(\lambda_n at)J_0(\lambda_n r)$$

At $t=0$, and from initial conditions, the above becomes

$$g(r)=\sum_{n=1}^{\infty}B_n\lambda_n aJ_0(\lambda_n r)$$

Applying orthogonality, and since the weight is r, therefore

$$\int_0^1 rg(r)J_0(\lambda_n r)\,dr = B_n\lambda_n a\int_0^1 rJ_0^2(\lambda_n r)\,dr$$

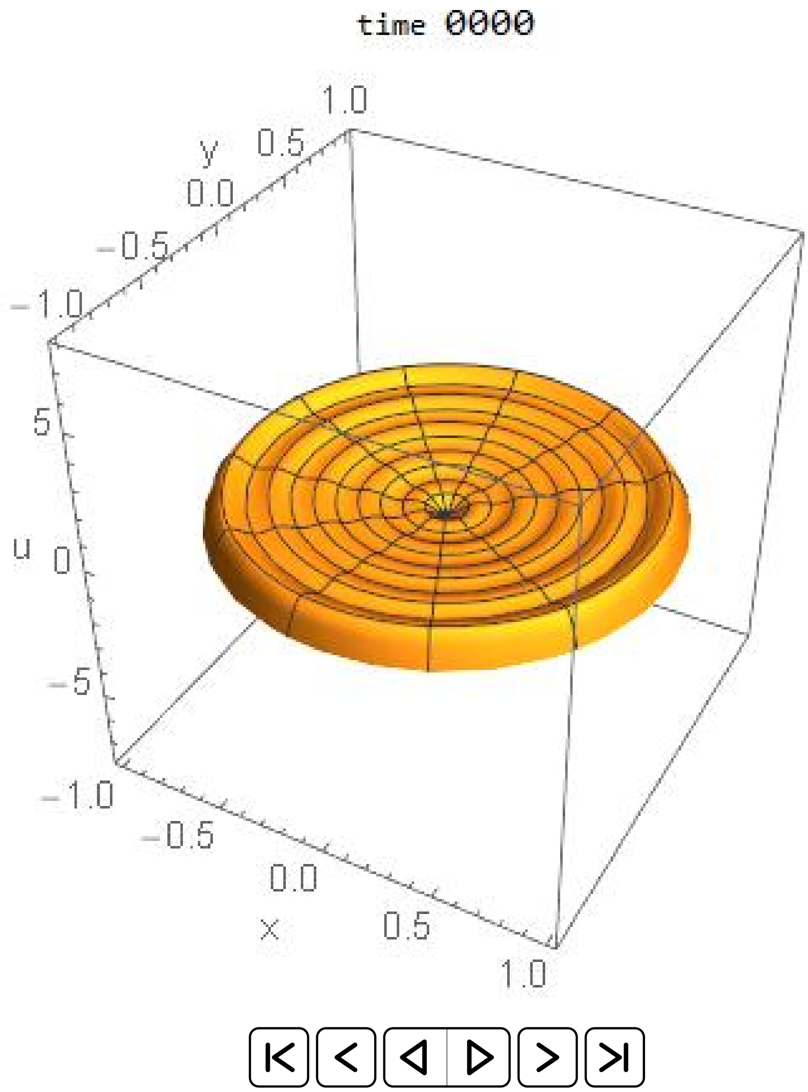
$$B_n=\frac{1}{\lambda_n a}\frac{\int_0^1 rg(r)J_0(\lambda_n r)\,dr}{\int_0^1 rJ_0^2(\lambda_n r)\,dr} \tag{3}$$

The two coefficients A_n, B_n are now found. Therefore the final solution is

$$u(r,t)=\sum_{n=1}^{\infty}\left(A_n\cos\left(\sqrt{\lambda_n}at\right)+B_n\sin\left(\sqrt{\lambda_n}at\right)\right)J_0\left(\sqrt{\lambda_n}r\right)$$

With A_n given by (2) and B_n given by (3)

The following is an animation of the above solution. $a=0.2$, $g(r)=r$ and $f(r)=1-r$ was used. This runs inside the PDF (need to use standard PDF reader to run the animation. Might not run inside Chrome browser PDF reader).



66 Chapter 11.5, Problem 4

The wave equation in polar coordinates is

$$\frac{1}{a^2}u_{tt} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

Show that if $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$ then R,Θ,T satisfy the ODE's

$$\begin{aligned} r^2R'' + rR' + \left(\lambda^2 r^2 - n^2\right)R &= 0 \\ \Theta'' + n^2\Theta &= 0 \\ T'' + \lambda^2 a^2 T &= 0 \end{aligned}$$

Solution

Let $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$. Substituting in the wave PDE gives

$$\frac{1}{a^2}T''R\Theta = R''T\Theta + \frac{1}{r}R'T\Theta + \frac{1}{r^2}\Theta''RT$$

dividing by $R\Theta T$ gives

$$\frac{1}{a^2}\frac{T''}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\lambda^2$$

Where λ is separation constant. The above now become

$$\frac{1}{a^2}\frac{T''}{T} = -\lambda^2 \tag{1}$$
$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\lambda^2$$

The second ODE above can now be written as

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} = -r^2\lambda^2$$
$$r^2\frac{R''}{R} + r\frac{R'}{R} + r^2\lambda^2 = -\frac{\Theta''}{\Theta} = n^2$$

Where n is the new separation constant (I do not like using n for this, but this is what the book did). The above now gives the ODE's

$$-\frac{\Theta''}{\Theta} = n^2 \tag{2}$$
$$r^2\frac{R''}{R} + r\frac{R'}{R} + r^2\lambda^2 = n^2 \tag{3}$$

Therefore (1,2,3) becomes

$$T'' + a^2\lambda^2 T = 0 \tag{1A}$$
$$\Theta'' + n^2\Theta = 0 \tag{2A}$$
$$r^2R'' + rR' + \left(r^2\lambda^2 - n^2\right) R = 0 \tag{3A}$$

Which is what the problem asked to show.

67 Chapter 11.5, Problem 5

In the circular cylindrical coordinates r, θ, z defined by

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Laplace equation is

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} = 0$$

(a) Show that if $u(r,\theta,t) = R(r)\Theta(\theta)Z(z)$ then R, Θ, Z satisfy the ODE's

$$r^2R'' + rR' + \left(\lambda^2 r^2 - n^2\right) R = 0$$
$$\Theta'' + n^2\Theta = 0$$
$$Z'' - \lambda^2 Z = 0$$

(b) Show that if $u(r,\theta,z)$ is independent of θ then the first equation in (a) becomes

$$r^2R'' + rR' + \lambda^2 r^2 R = 0$$

The second is omitted altogether and the third is unchanged.

Solution

67.1 Part (a)

Let $u(r,\theta,z) = R(r)\Theta(\theta)Z(z)$. Substituting in the wave PDE $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} = 0$ gives

$$R''\Theta Z + \frac{1}{r}R'\Theta Z + \frac{1}{r^2}\Theta'' RZ + Z''R\Theta = 0$$

dividing by $R\Theta Z$ gives

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\frac{Z''}{Z} = -\lambda^2$$

Where λ is separation constant. The above now become

$$Z'' - \lambda^2 Z = 0 \tag{1}$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\lambda^2$$

The second ODE above can now be written as

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} = -r^2\lambda^2$$

$$r^2\frac{R''}{R} + r\frac{R'}{R} + r^2\lambda^2 = -\frac{\Theta''}{\Theta} = n^2$$

Where n is the new separation constant. The above now gives the ODE's

$$-\frac{\Theta''}{\Theta} = n^2 \tag{2}$$

$$r^2\frac{R''}{R} + r\frac{R'}{R} + r^2\lambda^2 = n^2 \tag{3}$$

Therefore (1,2,3) becomes

$$Z'' - \lambda^2 Z = 0 \tag{1A}$$

$$\Theta'' + n^2\Theta = 0 \tag{2A}$$

$$r^2R'' + rR' + \left(r^2\lambda^2 - n^2\right)R = 0 \tag{3A}$$

67.2 Part (b)

When no dependency on θ then the ODE becomes $u_{rr} + \frac{1}{r}u_r + u_{zz} = 0$. Let $u(r,z) = R(r)Z(z)$. Substituting into the wave PDE

$$R''Z + \frac{1}{r}R'Z + Z''R = 0$$

dividing by RZ gives

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\frac{Z''}{Z} = -\lambda^2$$

The above gives

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\lambda^2$$

$$-\frac{Z''}{Z} = -\lambda^2$$

Or

$$R'' + \frac{1}{r}R' + \lambda^2 R = 0$$

$$Z'' - \lambda^2 Z = 0$$

68 Chapter 11.5, Problem 6

Find steady state solution in semi-infinite rod $0 < z < \infty, 0 \leq r \leq 1$ if the temperature is independent of θ and approaches zero as $z \to \infty$. Assume $u(r,z)$ satisfies boundary conditions

$$
\begin{aligned}
u(1,z) &= 0 \qquad z > 0 \\
u(r,0) &= f(r) \qquad 0 \leq r \leq 1
\end{aligned}
$$

Solution

The PDE is

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = 0$$

By separation of variables, as was done in problem 5 above, this gives

$$
\begin{aligned}
R'' + \frac{1}{r}R' + \lambda^2 R &= 0 \qquad (1) \\
R(1) &= 0 \\
\lim_{r\to 0} R(0) &\to \text{bounded}
\end{aligned}
$$

And

$$
\begin{aligned}
Z'' - \lambda^2 Z &= 0 \qquad (2) \\
Z(0) &= f(r) \\
\lim_{z\to\infty} Z(r) &\to 0
\end{aligned}
$$

The solution to (2) is known to be

$$R(r) = c_n J_0(\lambda_n r)$$

Where λ_n are the positive roots of $J_0(\lambda_n) = 0$. The solution to (2) is

$$Z(z) = A_n e^{\lambda_n z} + B_n e^{-\lambda_n z}$$

Since u goes to zero as $z \to \infty$, then this implies $A_n = 0$. Hence

$$Z(z) = B_n e^{-\lambda_n z}$$

Hence the overall solution becomes

$$u(r,z) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n z} J_0(\lambda_n r)$$

Where c_n is combined with B_n. To find B_n, using the final boundary condition $u(r,0) = f(r)$ gives

$$f(r) = \sum_{n=1}^{\infty} B_n J_0(\lambda_n r)$$

Applying orthogonality and using the weight of r gives

$$
\begin{aligned}
\int_0^1 r f(r) J_0(\lambda_n r)\, dr &= B_n \int_0^1 r J_0^2(\lambda_n r)\, dr \\
B_n &= \frac{\int_0^1 r f(r) J_0(\lambda_n r)\, dr}{\int_0^1 r J_0^2(\lambda_n r)\, dr}
\end{aligned}
$$

Hence the solution is now complete. It is given by

$$u(r,z) = \sum_{n=1}^{\infty} \frac{\int_0^1 r f(r) J_0(\lambda_n r)\, dr}{\int_0^1 r J_0^2(\lambda_n r)\, dr} e^{-\lambda_n z} J_0(\lambda_n r)$$

69 Chapter 11.5, Problem 7

7. The equation

$$v_{xx} + v_{yy} + k^2 v = 0$$

is a generalization of Laplace's equation and is sometimes called the Helmholtz[12] equation.

(a) In polar coordinates the Helmholtz equation is

$$v_{rr} + (1/r)v_r + (1/r^2)v_{\theta\theta} + k^2 v = 0.$$

If $v(r,\theta) = R(r)\Theta(\theta)$, show that R and Θ satisfy the ordinary differential equations

$$r^2 R'' + rR' + (k^2r^2 - \lambda^2)R = 0, \qquad \Theta'' + \lambda^2\Theta = 0.$$

(b) Consider the Helmholtz equation in the disk $r < c$. Find the solution that remains bounded at all points in the disk, that is periodic in θ with period 2π, and that satisfies the boundary condition $v(c,\theta) = f(\theta)$, where f is a given function on $0 \le \theta < 2\pi$.
Hint: The equation for R is a Bessel equation. See Problem 3 of Section 11.4.

Solution

69.1 Part (a)

Substituting $v(r,\theta) = R(r)\Theta(\theta)$ into the PDE gives

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}\Theta''R + k^2R\Theta = 0$$

Dividing by $R\Theta$ gives

$$\begin{aligned}\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + k^2 &= 0\\ r^2\frac{R''}{R} + r\frac{R'}{R} + r^2k^2 &= -\frac{\Theta''}{\Theta} = \lambda^2\end{aligned}$$

Where λ is the separation constant. This gives

$$r^2\frac{R''}{R} + r\frac{R'}{R} + r^2k^2 - \lambda^2 = 0$$

And

$$-\frac{\Theta''}{\Theta} = \lambda^2$$

Hence

$$\begin{aligned}r^2R'' + rR' + R\left(r^2k^2 - n^2\right) &= 0\\ \Theta'' + \lambda^2\Theta &= 0\end{aligned}$$

69.2 Part (b)

Starting with $\Theta'' + \lambda^2\Theta = 0$. The eigenvalue λ can not be negative. The following two cases are considered.

Case $\lambda = 0$

Solution is

$$\Theta(\theta) = c_1\theta + c_2$$

The boundary conditions are periodic with period 2π, meaning

$$\Theta(0) = \Theta(2\pi)$$
$$\Theta'(0) = \Theta'(2\pi)$$

Applying first BC gives

$$c_2 = c_1 2\pi + c_2 \tag{1}$$

Applying second BC gives

$$c_1 = c_1 \tag{2}$$

So c_1 can be any value. But to solve (1) c_1 must be zero. Hence first BC now gives

$$c_2 = c_2$$

Which means c_2 can be any value, say 1. Therefore $\lambda = 0$ is an eigenvalue with eigenfunction $\Phi_0(\theta) = 1$

Case $\lambda > 0$

The solution now is

$$\Theta(\theta) = A\cos(\lambda\theta) + B\sin(\lambda\theta)$$

The boundary conditions are periodic with period 2π, meaning

$$\Theta(0) = \Theta(2\pi)$$
$$\Theta'(0) = \Theta'(2\pi)$$

Applying the above boundary conditions gives

$$A = A\cos(\lambda 2\pi) + B\sin(\lambda 2\pi)$$
$$B\lambda = A\lambda\sin(\lambda 2\pi) + B\lambda\cos(\lambda 2\pi)$$

This means λ must be an integer $n = 1, 2, \cdots$ for the above relations be satisfied. Since only when n is an integer, the above gives $A = A$ and $B\lambda = B\lambda$. Hence the eigenfunction in this case is

$$\Phi_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta) \qquad n = 1, 2, \cdots$$

Now that the eigenvalues are found, the solution to the R ODE is found. Summary of the above result: The eigenvalues are $n = 0$ with eigenfunction $\Phi_0(\theta) = 1$ and $n = 1, 2, 3, \cdots$ with eigenfunction $\Phi_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta)$.

Case $\lambda = n = 0$

In this case, the R ODE above $r^2R'' + rR' + R\left(r^2k^2 - \lambda^2\right) = 0$ reduces to

$$r^2R'' + rR' + Rr^2k^2 = 0$$

let

$$t = rk$$

Therefore $R'(r) = R'(t)\,k$ and $R''(r) = R''(t)\,k^2$. Substituting these in the above ODE gives

$$\begin{aligned}\frac{t^2}{k^2}k^2 R''(t) + \frac{t}{k}kR'(r) + R\frac{t^2}{k^2}k^2 &= 0\\ t^2 R''(t) + tR'(t) + t^2 R(t) &= 0\end{aligned}$$

This is now Bessel ODE of order zero. Its solution is

$$R_0(t) = A_0 J_0(t) + B_0 Y_0(t)$$

Converting back to r, the above becomes

$$R_0(r) = A_0 J_0(rk) + B_0 Y_0(rk)$$

Since R is bounded at $r = 0$, this implies $B_0 = 0$, since $Y_0(rk)$ blows up at $r = 0$. Hence

$$R_0(r) = A_0 J_0(rk)$$

This is the solution for eigenvalue $n = 0$.

Case $\lambda = n > 0$

The Bessel PDE now has the form $r^2 R''(r) + rR'(r) + \left(r^2k^2 - n^2\right) R(r) = 0$. To convert the ODE to standard Bessel form let

$$t = rk$$

Therefore $R'(r) = R'(t)\,k$ and $R''(r) = R''(t)\,k^2$. Substituting these in the above ODE gives

$$\begin{aligned}\frac{t^2}{k^2}k^2 R''(t) + \frac{t}{k}kR'(r) + R\left(\frac{t^2}{k^2}k^2 - n^2\right) &= 0\\ t^2 R''(t) + tR'(t) + R(t)\left(t^2 - n^2\right) &= 0\end{aligned}$$

This is now Bessel ODE of order n. Its solution is

$$R_n(t) = A_n J_n(t) + B_n Y_n(t)$$

Converting back to r, the above becomes

$$R_n(r) = A_n J_n(rk) + B_n Y_n(rk)$$

Since R is bounded at $r = 0$, this implies $B_n = 0$, since $Y_n(rk)$ blows up at $r = 0$. Hence $R(r) = A_n J_n(rk)$. This is the solution for eigenvalue $n > 0$.

Hence the fundamental solution is

$$\begin{aligned}v_0(r,\theta) &= \Phi_0(\theta)\,R_0(r)\\ &= A_0 J_0(rk)\end{aligned}$$

Since $\Phi_0(\theta) = 1$ and

$$\begin{aligned}v_n(r,\theta) &= \Phi_n(\theta)\,R_n(r)\\ &= (A_n \cos(n\theta) + B_n \sin(n\theta))\,J_n(rk)\end{aligned}$$

Where the constants are combined. Therefore the general solution becomes

$$v(r,\theta) = A_0 J_0(rk) + \sum_{n=1}^{\infty} (A_n \cos(n\theta) + B_n \sin(n\theta))\,J_n(rk) \tag{3}$$

Constants A_0, A_n, B_n are found from boundary conditions. At $r = c$, $u(c,\theta) = f(\theta)$ and the above becomes

$$f(\theta) = A_0 J_0(ck) + \sum_{n=1}^{\infty} (A_n \cos(n\theta) + B_n \sin(n\theta)) J_n(ck)$$

For $n = 0$ only and applying orthogonality

$$\begin{aligned}\int_0^{2\pi} f(\theta)\,d\theta &= \int_0^{2\pi} A_0 J_0(ck)\,d\theta \\ \int_0^{2\pi} f(\theta)\,d\theta &= A_0 J_0(ck) \int_0^{2\pi} d\theta \\ &= 2\pi A_0 J_0(ck)\end{aligned}$$

Hence

$$A_0 = \frac{\int_0^{2\pi} f(\theta)\,d\theta}{2\pi J_0(ck)}$$

And for $n > 0$

$$\begin{aligned}\int_0^{2\pi} f(\theta)\sin(m\theta)\,d\theta &= \sum_{n=1}^{\infty} \int_0^{2\pi} (A_n \cos(n\theta) + B_n \sin(n\theta)) \sin(m\theta) J_n(ck)\,d\theta \\ &= \sum_{n=1}^{\infty} J_n(ck) A_n \int_0^{2\pi} \cos(n\theta)\sin(m\theta)\,d\theta + B_n \sum_{n=1}^{\infty} J_n(ck) \int_0^{2\pi} \sin(n\theta)\sin(m\theta)\,d\theta\end{aligned}$$

But $\int_0^{2\pi} \cos(n\theta)\sin(m\theta)\,d\theta = 0$ for all n, m and the above now is solved for B_n

$$\begin{aligned}\int_0^{2\pi} f(\theta)\sin(m\theta)\,d\theta &= B_n \sum_{n=1}^{\infty} J_n(ck) \int_0^{2\pi} \sin(n\theta)\sin(m\theta)\,d\theta \\ &= B_m J_m(ck) \int_0^{2\pi} \sin^2(m\theta)\,d\theta \\ &= B_m J_m(ck)\,\pi\end{aligned}$$

Hence

$$B_n = \frac{\int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta}{\pi J_n(ck)}$$

Similarly, to find A_n

$$\begin{aligned}\int_0^{2\pi} f(\theta)\cos(m\theta)\,d\theta &= \sum_{n=1}^{\infty} \int_0^{2\pi} (A_n \cos(n\theta) + B_n \sin(n\theta)) \cos(m\theta) J_n(ck)\,d\theta \\ &= \sum_{n=1}^{\infty} J_n(ck) A_n \int_0^{2\pi} \cos(n\theta)\cos(m\theta)\,d\theta + B_n \sum_{n=1}^{\infty} J_n(ck) \int_0^{2\pi} \sin(n\theta)\cos(m\theta)\,d\theta\end{aligned}$$

But $\int_0^{2\pi} \sin(n\theta)\cos(m\theta)\,d\theta = 0$ for all n, m and the above now is solved for A_n

$$\begin{aligned}\int_0^{2\pi} f(\theta)\sin(m\theta)\,d\theta &= A_n \sum_{n=1}^{\infty} J_n(ck) \int_0^{2\pi} \cos(n\theta)\cos(m\theta)\,d\theta \\ &= A_m J_m(ck) \int_0^{2\pi} \cos^2(m\theta)\,d\theta \\ &= A_m J_m(ck)\,\pi\end{aligned}$$

Hence

$$A_n = \frac{\int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta}{\pi J_n(ck)}$$

The complete solution from (3) becomes

$$\begin{aligned}
v(r,\theta) &= A_0 J_0(rk) + \sum_{n=1}^{\infty}\left(A_n\cos(n\theta) + B_n\sin(n\theta)\right)J_n(rk)\\
A_0 &= \frac{\int_0^{2\pi} f(\theta)\,d\theta}{2\pi J_0(ck)}\\
B_n &= \frac{\int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta}{\pi J_n(ck)}\\
A_n &= \frac{\int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta}{\pi J_n(ck)}
\end{aligned}$$

70 Chapter 11.5, Problem 8

8. Consider the flow of heat in a cylinder $0 \le r < 1, 0 \le \theta < 2\pi, -\infty < z < \infty$ of radius 1 and of infinite length. Let the surface of the cylinder be held at temperature zero, and let the initial temperature distribution be a function of the radial variable r only. Then the temperature u is a function of r and t only and satisfies the heat conduction equation

$$\alpha^2[u_{rr} + (1/r)u_r] = u_t, \qquad 0 < r < 1, \quad t > 0,$$

and the following initial and boundary conditions:

$$u(r,0) = f(r), \qquad 0 \le r \le 1,$$
$$u(1,t) = 0, \qquad t > 0.$$

Show that

$$u(r,t) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r)e^{-\alpha^2\lambda_n^2 t},$$

where $J_0(\lambda_n) = 0$. Find a formula for c_n.

<u>Solution</u>

Let $u(r,t) = R(r)T(t)$. Substituting into the PDE gives

$$\frac{1}{\alpha^2}T'R = R''T + \frac{1}{r}R'T$$

Dividing by RT gives

$$\frac{1}{\alpha^2}\frac{T'}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\lambda^2$$

Where λ is the separation constant. This gives the ODE

$$
\begin{aligned}
R'' + \frac{1}{r}R + \lambda^2 R &= 0 \\
rR'' + R + \lambda^2 rR &= 0 \\
(rR')' + \lambda^2 rR &= 0
\end{aligned} \tag{1}
$$

With BC

$$
\begin{aligned}
R(1) &= 0 \\
\lim_{r\to 0} R(r) &\to \text{bounded}
\end{aligned}
$$

And

$$
T' + \alpha^2\lambda^2 R = 0 \tag{2}
$$

ODE (1) is Sturm Liouville ODE where $p = r, q = 0$ and the weight is r. The eigenvalue can not be negative. Two cases to consider.

Case $\lambda = 0$

The ODE becomes $(rR')' = 0$ which has solution $rR' = c_1$ or $r\frac{dR}{dr} = c_1$ or $dR = \frac{c_1}{r}dr$. Integrating gives

$$
R(r) = c_1 \ln r + c_2
$$

Since R is bounded at $r = 0$, then $c_1 = 0$. The solution becomes $R(r) = c_2$. Since $R(1) = 0$ then $c_2 = 0$. Hence trivial solution. Therefore $\lambda = 0$ is not an eigenvalue.

Case $\lambda > 0$

The ODE now becomes $rR''(r) + R(r) + \lambda^2 rR(r) = 0$. Let $t = \lambda r$. Hence $R'(r) = \lambda R'(t)$ and $R''(r) = \lambda^2 R''(t)$ and the ODE becomes

$$
\begin{aligned}
\frac{t}{\lambda}\lambda^2 R''(t) + \lambda R'(t) + \lambda^2\frac{t}{\lambda}R(t) &= 0 \\
t\lambda R''(t) + \lambda R'(t) + \lambda tR(t) &= 0 \\
tR''(t) + R'(t) + tR(t) &= 0
\end{aligned}
$$

This is Bessel ODE of order zero. Its solution is

$$
R(t) = c_1 J_o(t) + c_2 Y_o(t)
$$

Converting back to r

$$
R(r) = c_1 J_o(\lambda r) + c_2 Y_o(\lambda r)
$$

Since R is bounded at $r = 0$ then $c_2 = 0$ and the solution becomes

$$
R(r) = c_1 J_o(\lambda r)
$$

Since $R(1) = 0$ then

$$
0 = c_1 J_o(\lambda)
$$

For nontrivial solution, $J_o(\lambda) = 0$. This gives the eigenvalues as the positive roots of $J_o(\lambda) = 0$. Hence the solution is

$$
R_n(r) = c_n J_o(\lambda_n r)
$$

Where λ_n are roots of $J_o(\lambda) = 0$ for $n = 1, 2, 3, \cdots$. The Time ODE (2) has solution

$$
T_n(t) = A_n e^{-\lambda_n^2 \alpha^2 t}
$$

Hence the final solution is

$$u(r,t)=\sum_{n=1}^{\infty}c_n e^{-\lambda_n^2\alpha^2 t}J_0(\lambda_n r)$$

Where constants A_n, c_n are combined into c_n. c_n is now found from initial conditions. At $t=0$ the above becomes

$$u(r,0)=f(r)=\sum_{n=1}^{\infty}c_n J_0(\lambda_n r)$$

The weight is r, since the R ODE in S.L. form is $(rR')'+\lambda^2 rR=0$. Therefore, applying orthogonality gives

$$\begin{aligned}\int_0^1 r f(r) J_0(\lambda_n r)\,dr &= c_n\int_0^1 r J_0^2(\lambda_n r)\,dr\\ c_n &= \frac{\int_0^1 r f(r) J_0(\lambda_n r)\,dr}{\int_0^1 r J_0^2(\lambda_n r)\,dr}\end{aligned}$$

This completes the solution.

$$u(r,t)=\sum_{n=1}^{\infty}\frac{\int_0^1 r f(r) J_0(\lambda_n r)\,dr}{\int_0^1 r J_0^2(\lambda_n r)\,dr}e^{-\lambda_n^2\alpha^2 t}J_0(\lambda_n r)$$

71 Chapter 11.5, Problem 9

9. In the spherical coordinates ρ, θ, ϕ ($\rho > 0, 0 \leq \theta < 2\pi, 0 \leq \phi \leq \pi$) defined by the equations

$$x = \rho \cos\theta \sin\phi, \qquad y = \rho \sin\theta \sin\phi, \qquad z = \rho\cos\phi,$$

Laplace's equation is

$$\rho^2 u_{\rho\rho} + 2\rho u_\rho + (\csc^2\phi) u_{\theta\theta} + u_{\phi\phi} + (\cot\phi) u_\phi = 0.$$

(a) Show that if $u(\rho,\theta,\phi) = \mathrm{P}(\rho)\Theta(\theta)\Phi(\phi)$, then P, Θ, and Φ satisfy ordinary differential equations of the form

$$\rho^2 \mathrm{P}'' + 2\rho \mathrm{P}' - \mu^2 \mathrm{P} = 0,$$
$$\Theta'' + \lambda^2\Theta = 0,$$
$$(\sin^2\phi)\Phi'' + (\sin\phi\cos\phi)\Phi' + (\mu^2\sin^2\phi - \lambda^2)\Phi = 0.$$

The first of these equations is of the Euler type, while the third is related to Legendre's equation.

(b) Show that if $u(\rho,\theta,\phi)$ is independent of θ, then the first equation in part (a) is unchanged, the second is omitted, and the third becomes

$$(\sin^2\phi)\Phi'' + (\sin\phi\cos\phi)\Phi' + (\mu^2\sin^2\phi)\Phi = 0.$$

(c) Show that if a new independent variable is defined by $s = \cos\phi$, then the equation for Φ in part (b) becomes

$$(1-s^2)\frac{d^2\Phi}{ds^2} - 2s\frac{d\Phi}{ds} + \mu^2\Phi = 0, \qquad -1 \leq s \leq 1.$$

Note that this is Legendre's equation.

Solution

71.1 Part (a)

Let

$$u(\rho,\theta,\phi) = P(\rho)\Theta(\theta)\Phi(\phi)$$

Substituting the above in the Laplace PDE given results in

$$\rho^2 P''\Theta\Phi + 2\rho P'\Theta\Phi + \left(\csc^2\phi\right)\Theta'' P\Phi + \Phi'' P\Theta + \cot(\phi)\Phi' P\Theta = 0$$

Dividing by $P\Theta\Phi$ gives

$$\rho^2\frac{P''}{P} + 2\rho\frac{P'}{P} + \left(\csc^2\phi\right)\frac{\Theta''}{\Theta} + \frac{\Phi''}{\Phi} + \cot(\phi)\frac{\Phi'}{\Phi} = 0$$
$$\rho^2\frac{P''}{P} + 2\rho\frac{P'}{P} = -\left(\csc^2\phi\right)\frac{\Theta''}{\Theta} - \frac{\Phi''}{\Phi} - \cot(\phi)\frac{\Phi'}{\Phi} = \mu^2$$

Where μ is the first separation constant. The above gives

$$\rho^2 \frac{P''}{P} + 2\rho \frac{P'}{P} = \mu^2$$
$$-\left(\csc^2 \phi\right) \frac{\Theta''}{\Theta} - \frac{\Phi''}{\Phi} - \cot(\phi) \frac{\Phi'}{\Phi} - \mu^2 = 0$$

The first ODE above becomes

$$\rho^2 P'' + 2\rho P' - P\mu^2 = 0$$

And the second equation is now separated again into two additional ODE's as follows

$$-\frac{\Theta''}{\Theta} - \frac{1}{\csc^2 \phi}\frac{\Phi''}{\Phi} - \frac{\cot \phi}{\csc^2 \phi}\frac{\Phi'}{\Phi} - \frac{\mu^2}{\csc^2 \phi} = 0$$
$$\frac{1}{\csc^2 \phi}\frac{\Phi''}{\Phi} + \frac{\cot(\phi)}{\csc^2 \phi}\frac{\Phi'}{\Phi} + \frac{\mu^2}{\csc^2 \phi} = -\frac{\Theta''}{\Theta} = \lambda^2$$

Where λ is the second separation constant. The above gives the following two ODE's

$$\Theta'' + \lambda^2 \Theta = 0$$

And, since $\csc^2 \phi = \frac{1}{\sin^2 \phi}$ and $\cot(\phi) = \frac{1}{\tan \phi}$, the third ODE is

$$\sin^2 \phi \frac{\Phi''}{\Phi} + \frac{\sin^2 \phi}{\tan \phi}\frac{\Phi'}{\Phi} + \mu^2 \sin^2 \phi = \lambda^2$$
$$\sin^2 \phi \frac{\Phi''}{\Phi} + \sin \phi \cos \phi \frac{\Phi'}{\Phi} + \mu^2 \sin^2 \phi = \lambda^2$$
$$\left(\sin^2 \phi\right) \Phi'' + (\sin \phi \cos \phi) \Phi' + \left(\mu^2 \sin^2 \phi - \lambda^2\right) \Phi = 0$$

71.2 Part (b)

If u is independent of θ then the PDE simplifies to

$$\rho^2 u_{\rho\rho} + 2\rho u_\rho + u_{\phi\phi} + \cot \phi u_\phi = 0 \tag{1}$$

Let

$$u(\rho, \phi) = P(\rho) \Phi(\phi)$$

Substituting the above in the Laplace PDE (1) results in

$$\rho^2 P'' \Phi + 2\rho P' \Phi + \Phi'' P + \cot(\phi) \Phi' P = 0$$

Dividing by $P\Phi$ gives

$$\rho^2 \frac{P''}{P} + 2\rho \frac{P'}{P} + \frac{\Phi''}{\Phi} + \cot(\phi) \frac{\Phi'}{\Phi} = 0$$
$$\rho^2 \frac{P''}{P} + 2\rho \frac{P'}{P} = -\frac{\Phi''}{\Phi} - \cot(\phi) \frac{\Phi'}{\Phi} = \mu^2$$

Where μ is the first separation constant. The above gives

$$\rho^2 \frac{P''}{P} + 2\rho \frac{P'}{P} = \mu^2$$
$$-\frac{\Phi''}{\Phi} - \cot(\phi) \frac{\Phi'}{\Phi} - \mu^2 = 0$$

The first ODE above becomes

$$\rho^2 P'' + 2\rho P' - \mu^2 P = 0$$

And the second ODE becomes

$$\begin{aligned}
-\Phi'' - \cot(\phi)\,\Phi' - \mu^2\Phi &= 0\\
\Phi'' + \frac{1}{\tan\phi}\Phi' + \mu^2\Phi &= 0\\
\Phi'' + \frac{\cos\phi}{\sin\phi}\Phi' + \mu^2\Phi &= 0\\
(\sin\phi)\,\Phi'' + (\cos\phi)\,\Phi' + \left(\mu^2\sin\phi\right)\Phi &= 0
\end{aligned}$$

Multiplying again by $\sin\phi$ to get it to the form needed gives

$$\sin^2\phi\Phi'' + (\sin\phi\cos\phi)\,\Phi' + \left(\mu^2\sin^2\phi\right)\Phi = 0 \tag{2}$$

Therefore the first PDE in $P(\rho)$, the second ODE in $\Theta(\theta)$ is now eliminated, and the third ODE changes to the above.

71.3 Part (c)

The equation for Φ found in part (b) is

$$\sin^2\phi\frac{d^2\Phi}{d\phi^2} + (\sin\phi\cos\phi)\,\frac{d\Phi}{d\phi} + \left(\mu^2\sin^2\phi\right)\Phi = 0 \tag{1}$$

Let $s = \cos\phi$, then

$$\begin{aligned}
\frac{d\Phi}{d\phi} &= \frac{d\Phi}{ds}\frac{ds}{d\phi}\\
&= \frac{d\Phi}{ds}(-\sin\phi) \qquad (2)
\end{aligned}$$

And

$$\begin{aligned}
\frac{d^2\Phi}{d\phi^2} &= \frac{d}{d\phi}\left(\frac{d\Phi}{d\phi}\right)\\
&= \frac{d}{d\phi}\left(\frac{d\Phi}{ds}(-\sin\phi)\right)\\
&= \frac{d^2\Phi}{ds^2}\left(\sin^2\phi\right) - \frac{d\Phi}{ds}(\cos\phi) \qquad (3)
\end{aligned}$$

Substituting (2,3) into (1) gives

$$\sin^2\phi\left(\frac{d^2\Phi}{ds^2}\left(\sin^2\phi\right) - \frac{d\Phi}{ds}(\cos\phi)\right) + (\sin\phi\cos\phi)\left(\frac{d\Phi}{ds}(-\sin\phi)\right) + \left(\mu^2\sin^2\phi\right)\Phi = 0$$

Dividing by $\sin^2\phi$ gives

$$\begin{aligned}
\frac{d^2\Phi}{ds^2}\sin^2\phi - \frac{d\Phi}{ds}\cos\phi - \cos\phi\frac{d\Phi}{ds} + \mu^2\Phi &= 0\\
\frac{d^2\Phi}{ds^2}\sin^2\phi - 2\frac{d\Phi}{ds}\cos\phi + \mu^2\Phi &= 0
\end{aligned}$$

But $\cos\phi = s$ and $\sin^2\phi = 1 - \cos^2\phi = 1 - s^2$, therefore the above reduces to

$$\left(1 - s^2\right)\frac{d^2\Phi}{ds^2} - 2s\frac{d\Phi}{ds} + \mu^2\Phi = 0$$

Which is Legendre's equation.

72 Chapter 11.5, Problem 10

10. Find the steady state temperature $u(\rho, \phi)$ in a sphere of unit radius if the temperature is independent of θ and satisfies the boundary condition

$$u(1, \phi) = f(\phi), \qquad 0 \le \phi \le \pi.$$

Hint: Refer to Problem 9 and to Problems 22 through 29 of Section 5.3. Use the fact that the only solutions of Legendre's equation that are finite at both ± 1 are the Legendre polynomials.

Solution
TO DO