# HW 7, Physics 501
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# Contents

# 1 Problem 1

Exercise 1: Consider Hermite's differential equation valid for $(-\infty < x < \infty)$:

$$y'' - 2xy' + 2ny = 0 \tag{1}$$

i) Assume the existence of a generating function $g(x,t) = \sum_{n=0}^{\infty} H_n(x)t^n/n!$. Differentiate $g(x,t)$ with respect to $x$ and use the recurrence relation $H_n'(x) = 2nH_{n-1}(x)$to develop a first order differential equation for $g(x,t)$.
ii) Integrate this equation with respect to $x$ holding $t$ fixed.
iii) Use the relationships $H_{2n}(0) = (-1)^n(2n)!/n!$ and $H_{2n+1}(0) = 0$ to evaluate $g(0,t)$ and show $g(x,t) = \exp(-t^2 + 2tx)$.
iv) Use the generating function to find the recurrence relation $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$.
v) By integrating the product $e^{-x^2}g(x,s)g(x,t)$ over all $x$, show

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x)H_n(x)dx = 2^n n!\sqrt{\pi}\delta_{nm}. \tag{2}$$

Figure 1: Problem statement

Solution

$$y'' - 2xy' + 2ny = 0 \qquad -\infty < x < \infty$$

## 1.1 Part 1

$$g(x,t) = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$$

Differentiating w.r.t $x$, and assuming term by term differentiation is allowed, gives

$$\frac{\partial g(x,t)}{\partial x} = \sum_{n=0}^{\infty} H_n'(x) \frac{t^n}{n!}$$

Using $H_n'(x) = 2nH_{n-1}(x)$ in the above results in

$$\frac{\partial g(x,t)}{\partial x} = \sum_{n=0}^{\infty} 2nH_{n-1}(x) \frac{t^n}{n!}$$

But for $n = 0$, the first term is zero, so the sum can start from 1 and give the same result

$$\frac{\partial g(x,t)}{\partial x} = \sum_{n=1}^{\infty} 2nH_{n-1}(x)\frac{t^n}{n!}$$

Now, decreasing the summation index by 1 and increasing the $n$ inside the sum by 1 gives

$$\begin{aligned}\frac{\partial g(x,t)}{\partial x} &= \sum_{n=0}^{\infty} 2(n+1)H_n(x)\frac{t^{n+1}}{(n+1)!}\\ &= \sum_{n=0}^{\infty} 2(n+1)H_n(x)\frac{t^{n+1}}{(n+1)n!}\\ &= \sum_{n=0}^{\infty} 2H_n(x)\frac{t^{n+1}}{n!}\\ &= \sum_{n=0}^{\infty} 2t\left(H_n(x)\frac{t^n}{n!}\right)\\ &= 2t\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}\end{aligned}$$

But $\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = g(x,t)$ and the above reduces to

$$\frac{\partial g(x,t)}{\partial x} = 2tg(x,t)$$

The problem says it is supposed to be a first order differential equation and not a first order partial differential equation. Therefore, by assuming $x$ to be a fixed parameter instead of another independent variable, the above can now be written as

$$\frac{d}{dx}g(x,t) - 2tg(x,t) = 0$$

## 1.2 Part 2

From the solution found in part (1)

$$\begin{aligned}\frac{\frac{d}{dx}g(x,t)}{g(x,t)} &= 2t\\ \frac{dg(x,t)}{g(x,t)} &= 2tdx\end{aligned}$$

Integrating both sides gives

$$\begin{aligned}\int \frac{dg(x,t)}{g(x,t)} &= \int 2tdx\\ \ln|g(x,t)| &= 2tx + C\\ g(x,t) &= e^{2tx+C}\\ g(x,t) &= C_1e^{2tx}\end{aligned}$$

Where $C_1 = e^C$ a new constant. Let $g(0,t) = g_0$ then the above shows that $C_1 = g_0$ and the above can now be written as

$$g(x,t) = g(0,t)\, e^{2tx}$$

## 1.3 Part 3

Using the given definition of $g(x,t) = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$ and when $x = 0$ then

$$\begin{aligned} g(0,t) &= \sum_{n=0}^{\infty} H_n(0) \frac{t^n}{n!} \\ &= H_0(0) + H_1(0) + \sum_{n=2}^{\infty} H_n(0) \frac{t^n}{n!} \end{aligned}$$

But $H_0(x) = 1$, hence $H_0(0) = 1$ and $H_1(x) = 2x$, hence $H_1(0) = 0$ and the above becomes

$$g(0,t) = 1 + \sum_{n=2}^{\infty} H_n(0) \frac{t^n}{n!}$$

For the remaining series, it can be written as sum of even and odd terms

$$g(0,t) = 1 + \sum_{n=2,4,6,\cdots}^{\infty} H_n(0) \frac{t^n}{n!} + \sum_{n=3,5,7,\cdots}^{\infty} H_n(0) \frac{t^n}{n!}$$

Or, equivalently

$$g(0,t) = 1 + \sum_{n=1,2,3,\cdots}^{\infty} H_{2n}(0) \frac{t^{2n}}{(2n)!} + \sum_{n=1,2,3,\cdots}^{\infty} H_{2n+1}(0) \frac{t^{2n+1}}{(2n+1)!}$$

But using the hint given that $H_{2n+1}(0) = 0$ and $H_{2n}(0) = \frac{(-1)^n (2n)!}{n!}$ the above simplifies to

$$\begin{aligned} g(0,t) &= 1 + \sum_{n=1,2,3,\cdots}^{\infty} \frac{(-1)^n (2n)!}{n!} \frac{t^{2n}}{(2n)!} \\ &= 1 + \sum_{n=1,2,3,\cdots}^{\infty} (-1)^n \frac{t^{2n}}{n!} \end{aligned}$$

But since $(-1)^n \frac{t^{2n}}{n!} = 1$ when $n = 0$, then the above sum can be made to start as zero and it simplifies to

$$g(0,t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!}$$

Therefore the solution $g(x,t) = g(0,t)\, e^{tx}$ found in part (2) becomes

$$g(x,t) = \left( \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!} \right) e^{2tx} \tag{1}$$

Now the sum $\sum_{n=0}^{\infty}(-1)^n \frac{t^{2n}}{n!} = 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots$ and comparing this sum to standard series of $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$, then this shows that when $z = -t^2$ and series for $e^{-t^2}$ becomes

$$\begin{aligned} e^{-t^2} &= 1 + \left(-t^2\right) + \frac{\left(-t^2\right)^2}{2!} + \frac{\left(-t^2\right)^3}{3!} + \frac{\left(-t^2\right)^4}{4!} \cdots \\ &= 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \frac{t^8}{4!} \cdots \end{aligned}$$

Hence

$$\sum_{n=0}^{\infty}(-1)^n \frac{t^{2n}}{n!} = e^{-t^2}$$

Substituting this into (1) gives

$$\begin{aligned} g(x,t) &= e^{-t^2} e^{2tx} \\ &= e^{2tx-t^2} \end{aligned}$$

## 1.4 Part 4

Since $g(x,t) = e^{2tx-t^2}$ from part (3), then

$$\begin{aligned} \frac{\partial}{\partial t} g(x,t) &= (2x - 2t)\, e^{2tx-t^2} \\ &= (2x-2t)\, g(x,t) \end{aligned}$$

But $g(x,t) = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$, therefore the above can be written as

$$\begin{aligned} \frac{\partial}{\partial t} g(x,t) &= (2x-2t) \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \\ &= 2x \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} - 2t \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \\ &= 2x \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} - 2 \sum_{n=0}^{\infty} H_n(x) \frac{t^{n+1}}{n!} \\ &= 2x \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} - 2 \sum_{n=1}^{\infty} H_{n-1}(x) \frac{t^n}{(n-1)!} \\ &= 2x \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} - 2 \sum_{n=1}^{\infty} nH_{n-1}(x) \frac{t^n}{n(n-1)!} \\ &= 2x \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} - 2 \sum_{n=1}^{\infty} nH_{n-1}(x) \frac{t^n}{n!} \qquad (1) \end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{\partial}{\partial t} g(x,t) &= \frac{\partial}{\partial t} \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} nH_n(x) \frac{t^{n-1}}{n!} \end{aligned}$$

Since at $n=0$ the sum is zero, then it can be started from $n=1$ without changing the result

$$\begin{aligned}\frac{\partial}{\partial t}g(x,t) &= \sum_{n=1}^{\infty} nH_n(x)\frac{t^{n-1}}{n!}\\ &= \sum_{n=0}^{\infty}(n+1)H_{n+1}(x)\frac{t^n}{(n+1)!}\\ &= \sum_{n=0}^{\infty}(n+1)H_{n+1}(x)\frac{t^n}{(n+1)n!}\\ &= \sum_{n=0}^{\infty}H_{n+1}(x)\frac{t^n}{n!} \qquad (2)\end{aligned}$$

Equating (1) and (2) gives

$$\sum_{n=0}^{\infty}H_{n+1}(x)\frac{t^n}{n!} = 2x\sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!} - 2\sum_{n=1}^{\infty}nH_{n-1}(x)\frac{t^n}{n!}$$

But $\sum_{n=1}^{\infty} nH_{n-1}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} nH_{n-1}(x)\frac{t^n}{n!}$ because at $n=0$ it is zero, so it does not affect the result to start the sum from zero, and now the above can be written as

$$\sum_{n=0}^{\infty}H_{n+1}(x)\frac{t^n}{n!} = 2x\sum_{n=0}^{\infty}H_n(x)\frac{t^n}{n!} - 2\sum_{n=0}^{\infty}nH_{n-1}(x)\frac{t^n}{n!}$$

Now since all the sums start from $n=0$ then the above means the same as

$$H_{n+1}(x)\frac{t^n}{n!} = 2xH_n(x)\frac{t^n}{n!} - 2nH_{n-1}(x)\frac{t^n}{n!}$$

Canceling $\frac{t^n}{n!}$ from each term gives

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

Which is the result required to show.

## 1.5 Part 5

The problem is asking to show that

$$\int_{-\infty}^{\infty} e^{-x^2}H_m(x)H_n(x)\,dx = \begin{cases} 0 & n\neq m \\ 2^n n!\sqrt{\pi} & n=m \end{cases}$$

The first part below will show the case for $n\neq m$ and the second part part will show the case for $n=m$

case $n \neq m$ This is shown by using the differential equation directly. I found this method easier and more direct. Before starting, the ODE $y'' - 2xy' + 2ny = 0$ is rewritten as

$$e^{x^2}\frac{d}{dx}\left(e^{-x^2}y'\right) + 2ny = 0 \qquad (1)$$

The above form is exactly the same as the original ODE as can be seen by expanding it. Now, Let $H_n(x)$ be one solution to (1) and let $H_m(x)$ be another solution to (1) which results in the following two ODE's

$$e^{x^2}\frac{d}{dx}\left(e^{-x^2}H_n'\right)+2nH_n=0 \tag{1A}$$

$$e^{x^2}\frac{d}{dx}\left(e^{-x^2}H_m'\right)+2mH_m=0 \tag{2A}$$

Multiplying (1A) by $H_m$ and (2A) by $H_n$ and subtracting gives

$$\begin{aligned}
H_m\left(e^{x^2}\frac{d}{dx}\left(e^{-x^2}H_n'\right)+2nH_n\right)-H_n\left(e^{x^2}\frac{d}{dx}\left(e^{-x^2}H_m'\right)+2mH_m\right)&=0\\
\left(H_me^{x^2}\frac{d}{dx}\left(e^{-x^2}H_n'\right)+2nH_nH_m\right)-\left(H_ne^{x^2}\frac{d}{dx}\left(e^{-x^2}H_m'\right)+2mH_nH_m\right)&=0\\
H_me^{x^2}\frac{d}{dx}\left(e^{-x^2}H_n'\right)-H_ne^{x^2}\frac{d}{dx}\left(e^{-x^2}H_m'\right)+2(n-m)H_nH_m&=0\\
H_m\frac{d}{dx}\left(e^{-x^2}H_n'\right)-H_n\frac{d}{dx}\left(e^{-x^2}H_m'\right)+2(n-m)H_nH_me^{-x^2}&=0 \qquad (3)
\end{aligned}$$

But

$$H_m\frac{d}{dx}\left(e^{-x^2}H_n'\right)=\frac{d}{dx}\left(e^{-x^2}H_n'H_m\right)-e^{-x^2}H_n'H_m'$$

And

$$H_n\frac{d}{dx}\left(e^{-x^2}H_m'\right)=\frac{d}{dx}\left(e^{-x^2}H_m'H_n\right)-e^{-x^2}H_m'H_n'$$

Therefore

$$\begin{aligned}
H_m\frac{d}{dx}\left(e^{-x^2}H_n'\right)-H_n\frac{d}{dx}\left(e^{-x^2}H_m'\right)&=\left(\frac{d}{dx}\left(e^{-x^2}H_n'H_m\right)-e^{-x^2}H_n'H_m'\right)-\left(\frac{d}{dx}\left(e^{-x^2}H_m'H_n\right)-e^{-x^2}H_m'H_n'\right)\\
&=\frac{d}{dx}\left(e^{-x^2}H_n'H_m\right)-\frac{d}{dx}\left(e^{-x^2}H_m'H_n\right)\\
&=\frac{d}{dx}\left(e^{-x^2}\left(H_n'H_m-H_m'H_n\right)\right)
\end{aligned}$$

Substituting the above relation back into (3) gives

$$\frac{d}{dx}\left(e^{-x^2}\left(H_n'H_m-H_m'H_n\right)\right)+2(n-m)H_nH_me^{-x^2}=0$$

Integrating gives

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{d}{dx}\left(e^{-x^2}\left(H_n'H_m-H_m'H_n\right)\right)dx+\int_{-\infty}^{\infty}2(n-m)H_nH_me^{-x^2}dx&=0\\
\int_{-\infty}^{\infty}d\left(e^{-x^2}\left(H_n'H_m-H_m'H_n\right)\right)+2(n-m)\int_{-\infty}^{\infty}H_nH_me^{-x^2}dx&=0\\
\left[e^{-x^2}\left(H_n'H_m-H_m'H_n\right)\right]_{-\infty}^{\infty}+2(n-m)\int_{-\infty}^{\infty}H_nH_me^{-x^2}dx&=0
\end{aligned}$$

But $\lim_{x\to\pm\infty} e^{-x^2} \longrightarrow 0$ so the first term above vanishes and the above becomes

$$2(n-m)\int_{-\infty}^{\infty} H_n H_m e^{-x^2}\,dx = 0$$

Since this is the case where $n \neq m$ then the above shows that

$$\int_{-\infty}^{\infty} H_n H_m e^{-x^2}\,dx = 0 \qquad n \neq m$$

Now the case $n = m$ is proofed. When $H_n = H_m$ then the integral becomes $\int_{-\infty}^{\infty} H_n H_n e^{-x^2}\,dx$. Using the known Rodrigues formula for Hermite polynomials, given by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

Then applying the above the above to one of the $H_n(x)$ in the integral $\int_{-\infty}^{\infty} H_n H_n e^{-x^2}\,dx$, gives

$$\begin{aligned}
\int_{-\infty}^{\infty} H_n H_n e^{-x^2}\,dx &= \int_{-\infty}^{\infty} \left( (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \right) H_n e^{-x^2}\,dx \\
&= (-1)^n \int_{-\infty}^{\infty} \left( \frac{d^n}{dx^n} e^{-x^2} \right) H_n\,dx
\end{aligned}$$

Now integration by parts is carried out. $\int u\,dv = uv - \int v\,du$. Let $u = H_n$ and let $dv = \frac{d^n}{dx^n} e^{-x^2}$, therefore $du = H_n'(x) = 2nH_{n-1}(x)$ and $v = \frac{d^{n-1}}{dx^{n-1}} e^{-x^2}$, therefore

$$\int_{-\infty}^{\infty} H_n H_n e^{-x^2}\,dx = (-1)^n \left( \left[ H_n(x) \frac{d^{n-1}}{dx^{n-1}} e^{-x^2} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \left( \frac{d^{n-1}}{dx^{n-1}} e^{-x^2} \right) 2nH_{n-1}(x)\,dx \right)$$

But $\left[ H_n(x) \frac{d^{n-1}}{dx^{n-1}} e^{-x^2} \right]_{-\infty}^{\infty} \longrightarrow 0$ as $x \longrightarrow \pm\infty$ because each derivative of $\frac{d^{n-1}}{dx^{n-1}} e^{-x^2}$ produces a term with $e^{-x^2}$ which vanishes at both ends of the real line. Hence the above integral now becomes

$$\int_{-\infty}^{\infty} H_n H_n e^{-x^2}\,dx = (-1)^n \left( -2n \int_{-\infty}^{\infty} \left( \frac{d^{n-1}}{dx^{n-1}} e^{-x^2} \right) H_{n-1}(x)\,dx \right)$$

Now the process is repeated, doing one more integration by parts. This results in

$$\int_{-\infty}^{\infty} H_n H_n e^{-x^2}\,dx = (-1)^n \left( -2n \left( -2(n-1) \int_{-\infty}^{\infty} \left( \frac{d^{n-2}}{dx^{n-2}} e^{-x^2} \right) H_{n-2}(x)\,dx \right) \right)$$

And again

$$\int_{-\infty}^{\infty} H_n H_n e^{-x^2}\,dx = (-1)^n \left( -2n \left( -2(n-1) \left( -2(n-2) \int_{-\infty}^{\infty} \left( \frac{d^{n-3}}{dx^{n-3}} e^{-x^2} \right) H_{n-3}(x)\,dx \right) \right) \right)$$

This process continues $n$ times. After $n$ integrations by parts, the above becomes

$$\begin{aligned}
\int_{-\infty}^{\infty} H_n H_n e^{-x^2}\,dx &= (-1)^n \left( -2n \left( -2(n-1) \left( -2(n-2) \left( \cdots \left( \int_{-\infty}^{\infty} e^{-x^2} H_0(x)\,dx \right) \right) \right) \right) \right) \\
&= (-1)^n (-2)^n n! \int_{-\infty}^{\infty} e^{-x^2} H_0(x)\,dx \\
&= 2^n n! \int_{-\infty}^{\infty} e^{-x^2} H_0(x)\,dx
\end{aligned}$$

But $H_0(x) = 1$, therefore the above becomes

$$\int_{-\infty}^{\infty} H_n H_n e^{-x^2} dx = 2^n n! \int_{-\infty}^{\infty} e^{-x^2} dx$$

But

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-x^2} &= 2\int_{0}^{\infty} e^{-x^2} \\ &= 2\frac{\sqrt{\pi}}{2} \\ &= \sqrt{\pi}\end{aligned}$$

Therefore

$$\int_{-\infty}^{\infty} H_n H_n e^{-x^2} dx = 2^n n! \sqrt{\pi}$$

This completes the case for $n = m$. Hence

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x) dx = \begin{cases} 0 & n \neq m \\ 2^n n! \sqrt{\pi} & n = m \end{cases}$$

Which is what the problem asked to show.

# 2 Problem 2

Exercise 2: a) Consider the differential equation for $0 < r < \infty$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\right)y(r) = 0 \tag{3}$$

where $n = 0, 1, 2, 3, \ldots$ Find two independent solutions, one which vanishes as $r \to 0$ and the other that vanishes as $r \to \infty$. Hint let $x = \ln r$.

b) Given the result of part a), find the solution to the differential equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\right)y(r) = \frac{1}{r}\delta(r - r') \tag{4}$$

with the boundary condition that the solution vanishes as $r \to 0$ and $r \to \infty$.

Figure 2: Problem statement

Solution

## 2.1 Part (a)

$$y''(r) + \frac{1}{r}y'(r) - \frac{n^2}{r^2}y(r) = 0 \qquad 0 < r < \infty$$

Or

$$r^2 y''(r) + r y'(r) - n^2 y(r) = 0$$

case $n = 0$

The ode becomes $r^2 y''(r) + r y'(r) = 0$. Let $z = y'$ and it becomes $r^2 z'(r) + r z(r) = 0$ or $z'(r) + \frac{1}{r} z(r) = 0$. This is linear in $z(r)$. Integrating factor is $I = e^{\int \frac{1}{r} dr} = r$. Multiplying the ode by $I$ it becomes exact differential $\frac{d}{dr}(zr) = 0$ or $d(zr) = 0$, hence $z = \frac{c_1}{r}$ where $c_1$ is constant of integration. Therefore

$$y'(r) = \frac{c_1}{r}$$

Integrating again gives

$$y(r) = \frac{c_1}{\ln r} + c_2$$

Since $\lim_{r \to 0}$ the solution is bounded, then $c_1$ must be zero. Therefore $0 = c_2$ and this implies $c_2 = 0$ also. Therefore when $n = 0$ the solution is

$$y(r) = 0$$

Case $n \neq 0$

Since powers of $r$ is the same as order of derivative in each term, this is an Euler ODE. It is solved by assuming $y = r^{\alpha}$. Hence $y' = \alpha r^{\alpha-1}, y'' = \alpha(\alpha-1) r^{\alpha-2}$. Substituting these into the above ODE gives

$$
\begin{aligned}
r^2 \alpha(\alpha-1) r^{\alpha-2} + r\alpha r^{\alpha-1} - n^2 r^{\alpha} &= 0 \\
\alpha(\alpha-1) r^{\alpha} + \alpha r^{\alpha} - n^2 r^{\alpha} &= 0 \\
r^{\alpha}\left(\alpha(\alpha-1) + \alpha - n^2\right) &= 0
\end{aligned}
$$

Assuming non-trivial solution $r^{\alpha} \neq 0$, then the indicial equation is

$$
\begin{aligned}
\alpha(\alpha-1) + \alpha - n^2 &= 0 \\
\alpha^2 &= n^2 \\
\alpha &= \pm n
\end{aligned}
$$

Hence one solution is

$$y_1(r) = r^n$$

And second solution is

$$y_2(r) = r^{-n}$$

And the general solution is linear combination of these solutions

$$y(r) = c_1 r^n + c_2 r^{-n}$$

The above shows that $\lim_{r\to 0} y_1(r) = 0$ and $\lim_{r\to\infty} y_2(r) = 0$.

## 2.2 Part (b)

Short version of the solution

To simplify the notations, $r_0$ is used instead of $r'$ in all the following.

$$y''(r) + \frac{1}{r} y'(r) - \frac{n^2}{r^2} y(r) = \frac{1}{r}\delta(r - r_0) \qquad 0 < r < \infty$$

Multiplying both sides by $r$ the above becomes

$$r y''(r) + y'(r) - \frac{n^2}{r} y(r) = \delta(r - r_0) \tag{1}$$

But the two solutions[1] to the homogeneous ODE $r y''(r) + y'(r) - \frac{n^2}{r} y(r) = 0$ were found in part (a). These are

$$
\begin{aligned}
y_1(r) &= r^n \qquad \text{(1A)} \\
y_2(r) &= r^{-n}
\end{aligned}
$$

[1] All the following is for $n \neq 0$, since for $n = 0$, only trivial solution exist

The Green function is the solution to

$$rG(r,r_0)+G(r,r_0)-\frac{n^2}{r}G(r,r_0)=\delta(r-r_0) \tag{1B}$$
$$\lim_{r\rightarrow 0}G(r,r_0)=0$$
$$\lim_{r\rightarrow \infty}G(r,r_0)=0$$

Which is given by (Using class notes, Lecture December 5, 2018) as

$$G(r,r_0)=\frac{1}{C}\begin{cases} y_1(r)\,y_2(r_0) & 0<r<r_0\\ y_1(r_0)\,y_2(r) & r_0<r<\infty\end{cases} \tag{2}$$

Note, I used $\frac{+1}{C}$ and not $\frac{-1}{C}$ as in class notes, since I am using $L=-\left((py')'-qy\right)$ as the operator and not $L=+\left((py')'+qy\right)$. Now $C$ is given by

$$C=p(r_0)\left(y_1(r_0)\,y_2'(r_0)-y_1'(r_0)\,y_2(r_0)\right)$$

Where from (1A) we see that

$$y_1(r_0)=r_0^n$$
$$y_2'(r_0)=-nr_0^{-n-1}$$
$$y_1'(r_0)=nr_0^{n-1}$$
$$y_2(r_0)=r_0^{-n}$$

Therefore $C$ becomes

$$\begin{aligned}C&=p(r_0)\left(-nr_0^{-n-1}r_0^n-nr_0^{n-1}r_0^{-n}\right)\\&=2nr_0^{-1}p(r_0)\end{aligned}$$

We just need now to find $p(r_0)$. This comes from Sturm Liouville form. We need to convert the ODE $r^2y''(r)+ry'(r)-n^2y(r)=0$ to Sturm Liouville. Writing this ODE as $ay''+by'+(c+\lambda)y=0$ where $a=r^2,b=r,c=0,\lambda=-n^2$, therefore

$$p=e^{\int\frac{b}{a}dr}=e^{\int\frac{r}{r^2}dr}=r$$
$$q=-p\frac{c}{a}=0$$
$$\rho=\frac{p}{a}=\frac{r}{r^2}=\frac{1}{r}$$

Hence the SL form is $(py')'-qy+\lambda\rho y=0$. Hence the SL form is $(py')'-qy+\lambda\rho y=0$ or

$$(ry')'-\frac{1}{r}n^2y=0 \tag{2A}$$

Hence the operator is $L[y]=-\left(\frac{d}{dr}\left(r\frac{d}{dr}\right)\right)[y]$ and in standard form it becomes $L[y]+\frac{1}{r}n^2y=0$.

The above shows that $p(r_0) = r_0$. Therefore

$$C = 2n$$

Hence Green function is now found from (2) as, for $n \neq 0$

$$G(r, r_0) = \frac{1}{2n} \begin{cases} r^n r_0^{-n} & 0 < r < r_0 \\ r_0^n r^{-n} & r_0 < r < \infty \end{cases}$$

Since $f(r)$ in the original ODE is zero, there is nothing to convolve with. i.e. $y(r) = \int_0^\infty G(r, r_0) f(r_0) \, dr_0$ here is not needed since there is no $f(r)$. Therefore the above is the final solution.

Extended solution

This solution shows derivation of (2) above. It can be considered as an appendix. The Green function is the solution to

$$rG(r, r_0) + G(r, r_0) - \frac{n^2}{r} G(r, r_0) = \delta(r - r_0) \tag{1B}$$
$$\lim_{r \to 0} G(r, r_0) = 0$$
$$\lim_{r \to \infty} G(r, r_0) = 0$$

In (1B), $r_0$ is the location of the impulse and $r$ is the location of the observed response due to this impulse. The solution to the above ODE is now broken to two regions

$$G(r, r_0) = \begin{cases} A_1 y_1(r) + A_2 y_2(r) & 0 < r < r_0 \\ B_1 y_1(r) + B_1 y_2(r) & r_0 < r < \infty \end{cases} \tag{2}$$

Where $y_1(r), y_2(r)$ are the solution to $ry''(r) + y'(r) - \frac{n^2}{r} y(r) = 0$ and these were found in part (a) to be $y_1(r) = r^n, y_2(r) = r^{-n}$ and $A_1, A_2, B_1, B_2$ needs to be determined. Hence (2) becomes

$$G(r, r_0) = \begin{cases} A_1 r^n + A_2 r^{-n} & 0 < r < r_0 \\ B_1 r^n + B_2 r^{-n} & r_0 < r < \infty \end{cases} \tag{3}$$

The left boundary condition $\lim_{r \to 0} G(r, r_0) = 0$ implies $A_2 = 0$ and the right boundary condition $\lim_{r \to \infty} G(r, r_0) = 0$ implies $B_1 = 0$. This is needed to keep the solution bounded. Hence (3) simplifies to

$$G(r, r_0) = \begin{cases} A_1 r^n & 0 < r < r_0 \\ B_2 r^{-n} & r_0 < r < \infty \end{cases} \tag{4}$$

To determine the remaining two constants $A_1, B_2$, two additional conditions are needed. The first is that $G(r, r_0)$ is continuous at $r = r_0$ which implies

$$A_1 r_0^n = B_2 r_0^{-n} \tag{5}$$

The second condition is the jump in the derivative of $G(r, r_0)$ given by

$$\left. \frac{d}{dr} G(r, r_0) \right|_{r > r_0} - \left. \frac{d}{dr} G(r, r_0) \right|_{r < r_0} = \frac{-1}{p(r_0)}$$

Where $p(r_0)$ comes from the Sturm Liouville form of the homogeneous ODE. This was found above as $p(r_0) = r_0$. Hence the above condition becomes

$$\left.\frac{d}{dr}G(r,r_0)\right|_{r>r_0} - \left.\frac{d}{dr}G(r,r_0)\right|_{r<r_0} = \frac{-1}{r_0}$$

Equation (4) shows that $\left.\frac{d}{dr}G(r,r_0)\right|_{r>r_0} = -nB_2r_0^{-n-1}$ and that $\left.\frac{d}{dr}G(r,r_0)\right|_{r<r_0} = nA_1r_0^{n-1}$. Using these in the above gives the second equation needed

$$-nB_2r_0^{-n-1} - nA_1r_0^{n-1} = \frac{-1}{r_0} \tag{6}$$

Solving (5,6) for $A_1, B_1$: From (5) $A_1 = B_2r_0^{-2n}$. Substituting this in (6) gives

$$\begin{aligned}
-nB_2r_0^{-n-1} - n\left(B_2r_0^{-2n}\right)r^{n-1} &= \frac{-1}{r_0}\\
-nB_2r^{-n-1} - nB_2r^{-n-1} &= \frac{-1}{r_0}\\
-2nB_2r_0^{-n-1} &= -r_0^{-1}\\
B_2 &= \frac{-r_0^{-1}}{-2nr_0^{-n-1}}\\
&= \frac{1}{2n}r_0^n
\end{aligned}$$

But since $A_1 = B_2r_0^{-2n}$, then

$$\begin{aligned}
A_1 &= \frac{1}{2n}r_0^nr_0^{-2n}\\
&= \frac{1}{2n}r_0^{-n}
\end{aligned}$$

Therefore the solution (4), which is the Green function, becomes, for $n \neq 0$

$$G(r,r_0) = \begin{cases} \frac{1}{2n}r_0^{-n}r^n & 0 < r < r_0\\ \frac{1}{2n}r_0^nr^{-n} & r_0 < r < \infty \end{cases} \tag{7}$$