Problem Set 6 Solutions

Exercise 1 : $xy'' - (c-x)y' - ay = 0$

- this has a regular singularity at $x=0$
- let $y = x^k \sum_{n=0}^{\infty} d_n x^n$, then

$$\sum_{n=0}^{\infty} d_n (n+k)(n+k-1) x^{n+k-1} + \sum_{n=0}^{\infty} d_n (n+k)(c-x) x^{n+k-1}$$

$$- a \sum_{n=0}^{\infty} d_n x^{n+k} = 0$$

or $$\sum_{n=0}^{\infty} d_n \left\{ [(n+k)(n+k-1) + c(n+k)] x^{n+k-1} - [n+k+a] x^{n+k} \right\} = 0$$

the x^{k-1} term gives the indicial equation

$$k(k-1) + ck = 0$$

$$\Rightarrow \boxed{k=0} \quad \text{or} \quad \boxed{k = -c+1}$$

- next let us find the recurrence relation

$$d_{n+1} \left[(n+k+1)(n+k) + c(n+k+1) \right] - d_n (n+k+a) = 0$$

or $$d_{n+1} = \frac{n+k+a}{(n+k+1)(n+k+c)} d_n$$

$$\Rightarrow \quad d_n = \frac{(n+k+a-1)!\; k!\; (k-1+c)!}{(k+a-1)!\; (n+k)!\; (n+k-1+c)!}$$

For $k=0$, we get

$$d_n = \frac{(n+a-1)!\,(c-1)!}{(a-1)!\,n!\,(n+c-1)!}$$

or $$y = d_0\left[1+\frac{a}{c}x+\frac{(a+1)a}{(c+1)c}\frac{1}{2}x^2+\cdots\right]$$

check this for convergence

as $n \to \infty$

$$\frac{d_{n+1}\,x^{n+1}}{d_n\,x^n} = \frac{n+a+2}{(n+1)(n+c)}x \approx \frac{x}{n} \Rightarrow \text{converges for all } x$$

second solution $k=-c+1$

$$d_n = \frac{(n+a-c)!\,(1-c)!\,c!}{(a-c)!\,(n-c+1)!\,n!}d_0 \quad \text{or}$$

$$y = a_0\left[1+\frac{a-c+1}{-c}x+\frac{(a-c+1)(a-c+2)}{-c(c+1)2}x^2\cdots\right.$$

$$\frac{d_{n+1}\,x^{n+1}}{d_n\,x^n} \approx \frac{x}{n}$$, also converges for all x

Exercise 2 :

- $\mathcal{L} u(x) + \lambda \rho(x) u(x) = 0$
- show $\mathcal{L}$ is Hermitian:

$$\int_a^b u^* \mathcal{L} v \, dx = \int_a^b u^* \left[p \frac{d^2}{dx^2} + p' \frac{d}{dx} - q(x) \right] v \, dx$$

$$= \cancelto{0}{\left[u^* p \frac{d}{dx} v \right]_a^b} - \int_a^b \frac{dv}{dx} \frac{d}{dx}(u^* p) \, dx + \int_a^b u^* p' \frac{dv}{dx} \, dx - \int_a^b u^* q v \, dx$$

$$= - \int_a^b \frac{dv}{dx} \frac{du^*}{dx} p \, dx - \cancel{\int_a^b \frac{dv}{dx} u^* p' \, dx} + \cancel{\int_a^b u^* p' \frac{dv}{dx} \, dx} - \int_a^b u^* q V \, dx$$

$$= \cancelto{0}{\left[-v \frac{d}{dx}\left(p \frac{du^*}{dx} \right) \right]_a^b} + \int_a^b v \frac{d}{dx} \left(p \frac{du^*}{dx} \right) dx - \int_a^b u^* q v \, dx$$

$$= \left[\int_a^b v^* \mathcal{L} u \, dx \right]^* \Rightarrow \text{Hermitian}$$

- now let u_i and u_j be eigen functions with eigenvalues λ_i and λ_j:

$\mathcal{L} u_i = -\lambda_i \rho u_i$ and $\mathcal{L} u_j = -\lambda_j \rho u_j$

$$\Rightarrow \int_a^b u_j^* \mathcal{L} u_i \, dx = -\lambda_i \int_a^b u_j^* u_i \rho \, dx$$

and $\int_a^b u_i^* \mathcal{L} u_j \, dx = -\lambda_j \int_a^b u_i^* u_j \rho \, dx$ now since $\mathcal{L}$ is Hermitian $\int_a^b u_j^* \mathcal{L} u_i \, dx = \left[\int_a^b u_i^* \mathcal{L} u_j \, dx \right]^*$

$$\therefore (\lambda_i - \lambda_j^*) \int_a^b u_i^*(x) u_j(x) \rho(x) \, dx = 0$$

$\Rightarrow$ when $\lambda_i \neq \lambda_j$ then u_i and u_j are orthogonal $0 = \int_a^b u_i^*(x) u_j(x) \rho(x) dx$

Exercise 3

i) $y'' + \frac{1-\alpha^2}{4x^2} y = 0$, let $y = x^p$

$\Rightarrow \quad p(p-1) x^{p-2} + \frac{(1-\alpha^2)}{4} x^{p-2} = 0$

$\Rightarrow \quad p^2 - p + \frac{1-\alpha}{2}\frac{1+\alpha}{2} = 0$

check that $p_\pm = \frac{1\pm\alpha}{2}$ is a solution

$\left(\frac{1\pm\alpha}{2}\right)^2 - \left(\frac{1\pm\alpha}{2}\right) + \frac{1-\alpha}{2}\frac{1+\alpha}{2} \overset{?}{=} 0$

$\left(\frac{1\pm\alpha}{2}\right)\left[\frac{1\pm\alpha}{2} - 1 + \frac{1\mp\alpha}{2}\right] \overset{?}{=} 0$

↳ this is indeed ∅

$\Rightarrow \quad y = x^{\frac{1\pm\alpha}{2}}$ are solutions

ii) For $\alpha = 0$ $\quad y_0 = \sqrt{x}$

now $y'' + \frac{1-\alpha^2}{4x^2} y = 0$ has Sturm-Liouville form with $p(x) = 1 \Rightarrow W = c$, c is a constant

$\Rightarrow \quad y_{20}' y_0 - y_{20} y_0' = c$

or $\quad y_0^2 \left(\frac{y_{20}}{y_0}\right)' = c \quad \Rightarrow \quad y_{20} = y_0 \int \frac{c}{y_0^2} dx'$

or $\quad y_{20} = \sqrt{x}\, c \int \frac{dx'}{x'} \quad$ or $\quad \boxed{y_{20} = c\sqrt{x} \ln x}$

iii) want $\lim_{\alpha \to 0} \dfrac{x^{\frac{1+\alpha}{2}} - x^{\frac{1-\alpha}{2}}}{\alpha}$, use L'Hopital's rule

$$= \lim_{\alpha \to 0} \frac{d}{d\alpha} x^{\frac{1+\alpha}{2}} \Big|_{\alpha=0} - \frac{d}{d\alpha} x^{\frac{1-\alpha}{2}} \Big|_{\alpha=0}$$

$$= \lim_{\alpha \to 0} \sqrt{x} \left[\frac{d}{d\alpha} x^{\frac{\alpha}{2}} - \frac{d}{d\alpha} x^{-\frac{\alpha}{2}} \right] \Big|_{\alpha=0}$$

now $\dfrac{d}{d\alpha} x^{\frac{\alpha}{2}} \Big|_{\alpha=0} = \dfrac{d}{d\alpha} e^{\frac{\alpha}{2} \ln x} \Big|_{\alpha=0} = \dfrac{1}{2} \ln x$

$$\Rightarrow \lim_{\alpha \to 0} \frac{x^{\frac{1+\alpha}{2}} - x^{\frac{1-\alpha}{2}}}{\alpha} = \frac{\sqrt{x}}{2}(\ln x + \ln x) = \sqrt{x} \ln x = \tau_c$$

as required.