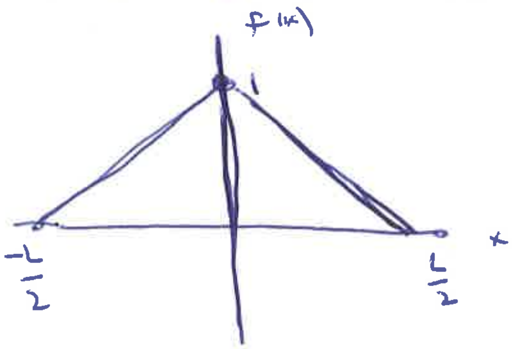
Problem Set 5 Solution

Exercise 1:

i) $f(x) = 1 - 2|x|/L$ for $-L/2 \le x \le L/2$



For a Fourier series

$$A_0 = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\,dx$$

$$A_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\cos\left(\frac{2\pi n x}{L}\right)dx$$

$$B_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\sin\left(\frac{2\pi n x}{L}\right)dx$$

• By symmetry $B_n = 0$

$$A_0 = \frac{4}{L}\int_0^{L/2}\left(1 - \frac{2x}{L}\right)dx = \frac{4}{L}\left[x - \frac{x^2}{L}\right]\Big|_0^{L/2} = 1$$

$$A_n = \frac{4}{L}\int_0^{L/2}\left(1 - \frac{2x}{L}\right)\cos\left(\frac{2n\pi x}{L}\right)dx$$

$$= \frac{4}{L}\frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)\Big|_0^{L/2} - \frac{8}{L^2}\left(\frac{L}{2n\pi}\right)^2\left[\cos\left(\frac{2\pi n x}{L}\right) + \left(\frac{2\pi n x}{L}\right)\sin\left(\frac{2\pi n x}{L}\right)\right]\Big|_0^{L/2}$$

$$= -\frac{2}{n^2\pi^2}\left[(-1)^n - 1\right] \Rightarrow A_n = \frac{4}{\pi^2 n^2}\ n\ \text{odd},\quad A_n = 0,\ n\ \text{even}$$

$$\therefore f(x) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{2\pi n x}{L}\right)$$

b) $f(x) = e^x \quad$ For $-L/2 \le x \le L/2$

$$A_0 = \frac{2}{L}\int_{-L/2}^{L/2} e^x dx = \frac{2}{L} e^x \Big|_{-L/2}^{L/2} = \frac{2}{L}\left(e^{L/2} - e^{-L/2}\right) = 2\,\frac{\sinh(L/2)}{L/2}$$

$$A_n = \frac{2}{L}\int_{-L/2}^{L/2} e^x \cos\left(\frac{2n\pi x}{L}\right) dx = \frac{2}{L}\left[e^x \frac{\cos\left(\frac{2n\pi x}{L}\right) + \left(\frac{2n\pi}{L}\right)\sin\left(\frac{2n\pi x}{L}\right)}{1+\left(\frac{2n\pi}{L}\right)^2}\right]_{-L/2}^{L/2}$$

$$= 2\,\frac{\sinh(L/2)}{L/2}\,\frac{(-1)^n}{1+\left(\frac{2n\pi}{L}\right)^2}$$

$$B_n = \frac{2}{L}\int_{-L/2}^{L/2} e^x \sin\left(\frac{2n\pi x}{L}\right) dx = \frac{2}{L}\left[e^x \frac{\sin\left(\frac{2n\pi x}{L}\right) - \left(\frac{2n\pi}{L}\right)\cos\left(\frac{2n\pi x}{L}\right)}{1+\left(\frac{2n\pi}{L}\right)^2}\right]_{-L/2}^{L/2}$$

$$= -2\,\frac{\sinh(L/2)}{L/2}\,\frac{(-1)^n}{1+\left(\frac{2n\pi}{L}\right)^2}\left(\frac{2n\pi}{L}\right)$$

$$\therefore\ f(x) = \frac{\sinh(L/2)}{L/2}\left\{1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{1+\left(\frac{2n\pi}{L}\right)^2}\left[\cos\left(\frac{2n\pi x}{L}\right) - \left(\frac{2n\pi}{L}\right)\sin\left(\frac{2n\pi x}{L}\right)\right]\right\}$$

Exercise 2 :

i) $2x^3 y' = 1 + \sqrt{1+4x^2 y}$, the form of this equation suggests $x^2 y$ is dimensionless

$\Rightarrow$ let $v = x^2 y$ or $y = \frac{v}{x^2}$

$\Rightarrow y' = \frac{v'}{x^2} - 2\frac{v}{x^3}$

$\Rightarrow 2x v' - 4v = 1 + \sqrt{1+4v}$

$\Rightarrow 2x\,dv = (1 + 4v + \sqrt{1+4v})\,dx$ ~ separable

$\frac{dx}{x} = \frac{2dv}{1+4v+\sqrt{1+4v}}$ let $u^2 = 1+4v$
$2u\,du = 4dv$
$\Rightarrow 2dv = u\,du$

$\therefore \frac{dx}{x} = \frac{u\,du}{u^2+u} = \frac{du}{u+1}$

$\Rightarrow \ln x + c = \ln(u+1)$

$\Rightarrow u = 2Ax - 1$ where $2A = e^c$

$\therefore 1+4v = (2Ax-1)^2 = 4A^2x^2 - 4Ax + 1$

$\Rightarrow v = A^2 x^2 - Ax$

$\Rightarrow \boxed{y = \frac{v}{x^2} = A^2 - \frac{A}{x}}$

ii) $e^x \sin y - 2y\sin x + (y^2 + e^x\cos y + 2\cos x)\, y' = 0$

or $dx\,(e^x \sin y - 2y\sin x) + dy\,(y^2 + e^x\cos y + 2\cos x) = 0$

check if this is exact : $df = A\,dx + B\,dy = 0$

then $\frac{\partial A}{\partial y} \stackrel{?}{=} \frac{\partial B}{\partial x}$ or $\frac{\partial}{\partial y}(e^x\sin y - 2y\sin x) \stackrel{?}{=} \frac{\partial}{\partial x}(y^2 + e^x\cos y + 2\cos x)$

$\Rightarrow\ e^x\cos y - 2\sin x \stackrel{?}{=} e^x\cos y - 2\sin x$ ✓

$\Rightarrow\ \frac{\partial f}{\partial x} = e^x\sin y - 2y\sin x \qquad \frac{\partial f}{\partial y} = y^2 + e^x\cos y + 2\cos x$

$\Rightarrow\ f = e^x\sin y + 2y\cos x + g(y)$

and $\frac{\partial f}{\partial y} = e^x\cos y + 2\cos x + \frac{dg}{dy} = y^2 + e^x\cos y + 2\cos x$

$\Rightarrow\ \frac{dg}{dy} = y^2 \Rightarrow\ g = \frac{y^3}{3} + \text{cnst}$

$\Rightarrow\ \boxed{f = e^x\sin y + 2y\cos x + \frac{y^3}{3} + \text{cnst} = 0}$

iii) $y' + y\cos x = \frac{1}{2}\sin 2x$

for this problem, let us introduce an integrating factor $\lambda(x) = \exp\int^x \cos x'\,dx' = e^{\sin x}$

then $e^{\sin x}\left[y' + y\cos x\right] = e^{\sin x}\sin x\cos x$

$\frac{d}{dx}\left(y\, e^{\sin x}\right) = e^{\sin x}\sin x\cos x$

$$\Rightarrow \quad y e^{\sin x} = \int^{x} e^{\sin x'} \sin x' \cos x' \, dx'$$

let $\quad u = \sin x' \,, \quad du = \cos x' \, dx'$

$$\therefore \quad y e^{\sin x} = \int^{\sin x} u\, e^{u} du = \left[u e^{u} - e^{u} + c \right]^{\sin x}$$

or $\quad y e^{\sin x} = \sin x \, e^{\sin x} - e^{\sin x} + c$

$$\therefore \quad \boxed{y = \sin x - 1 + c\, e^{-\sin x}}$$

Exercise 3

i) $y''' - 4y'' - 4y' + 16 = 8\sin x$

· let $p = y'$

$\Rightarrow \quad p'' - 4p' - 4(p-4) = 8\sin x$

· let $\ell = p - 4$ (so $y' = \ell + 4$)

$\Rightarrow \quad \ell'' - 4\ell' - 4\ell = 8\sin x$

solve the homogeneous equation $\ell'' - 4\ell' - 4\ell = 0$

· try $\ell = e^{mt}$ $\Rightarrow$ $m^2 - 4m - 4 = 0$

$\Rightarrow \quad m_\pm = \frac{4 \pm \sqrt{16+16}}{2} = 2(1 \pm \sqrt{2})$

$\Rightarrow \quad \ell = a e^{m_+ x} + b e^{m_- x}$

· Find a particular solution

$\ell_p'' - 4\ell_p' - 4\ell_p = 8\sin x$

try $\ell_p = \alpha \cos x + \beta \sin x$

$\Rightarrow \; -(\alpha\cos x + \beta\sin x) - 4(-\alpha\sin x + \beta\cos x) - 4(\alpha\cos x + \beta\sin x) = 8\sin x$

$\Rightarrow \quad -\alpha - 4\beta - 4\alpha = 0 \quad (\cos x)$

$-\beta + 4\alpha - 4\beta = 8$

$\Rightarrow \quad \alpha = \frac{32}{41}$ and $\beta = -\frac{40}{41}$

$\Rightarrow \quad \ell_p = \frac{32}{41}\cos x - \frac{40}{41}\sin x$

$\Rightarrow \quad \ell = \cancel{a e^{m_+ x}}\; a e^{m_+ x} + b e^{m_- x} + \frac{32}{41}\cos x - \frac{40}{41}\sin x \; ; \; y' = \ell + 4$

$\Rightarrow \quad \boxed{y = \text{cnst} + 4x + \frac{a}{m_+} e^{m_+ x} + \frac{b}{m_-} e^{m_- x} + \frac{32}{41}\sin x + \frac{40}{41}\cos x}$

iii) $a^2 y'^2 = (1+y'^2)^3$

$\Rightarrow$ $y'^6 + 3y'^4 + 3y'^2 + 1 = a^2 y'^2$

or $y'^6 + 3y'^4 + (3-a^2)y'^2 + 1 = 0$

this is a polynomial in y'^2, for which we can find three roots u_i:

$$u_1 = \frac{P}{6} + \frac{2a^2}{P} - 1$$

$$u_2 = -\frac{P}{12} - \frac{a^2}{P} - 1 + \frac{\sqrt{3}I}{2}\left(\frac{P}{6} - \frac{2a^2}{P}\right)$$

$$u_3 = -\frac{P}{12} - \frac{a^2}{P} - 1 - \frac{\sqrt{3}I}{2}\left(\frac{P}{6} - \frac{2a^2}{P}\right)$$

where $P = \left[a^2\left(-108 + 12\sqrt{81-12a^2}\right)\right]^{1/3}$

$\Rightarrow$ $y'^2 = u_i$ or $y' = \pm\sqrt{u_i}$

$\Rightarrow$ $\boxed{y = \pm\sqrt{u_i}\,x + c}$ is the solution