Problem Set 4

Exercise 1

$$\ln \frac{1+x}{1-x} = \ln(1+x) - \ln(1-x)$$

$$= \left\{ x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \cdots \right\} - \left\{ -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} \cdots \right\}$$

$$= 2x + \frac{2x^3}{3} + \frac{2x^5}{5}$$

$$= 2\left\{ x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots \right\}$$

ratio test $\quad \frac{a_{n+1}}{a_n} = \frac{x^{2n+1}}{2n+1} \frac{2n-1}{x^{2n-1}} \overset{n\to\infty}{\sim} x^2$ $\quad$ converges for $|x| < 1$

$$I = 2 \int_0^1 \left\{ 1 + \frac{x^2}{3} + \frac{x^4}{5} + \cdots \right\} dx$$

$$= 2\left\{ 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots \right\}$$

Recall $\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

$$= \frac{1}{2} I + \frac{1}{4}\left\{ \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right\} = \frac{1}{2} I + \frac{1}{4}\zeta(2)$$

$$\left(\left\{ \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \right\} + \left\{ \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots \right\} \right.$$

$$\Rightarrow \quad I = \frac{3}{2}\zeta(2) = \frac{3}{2}\frac{\pi^2}{6} = \frac{\pi^2}{4}.$$

Exercise 2: $I(x) = \int_0^\infty e^{x f(t)}\,dt$ with $f(t) = t - \frac{e^t}{x}$

$f'(t) = 1 - \frac{e^t}{x} \longrightarrow 0 = f'(t_0) = 1 - \frac{e^{t_0}}{x} \longrightarrow t_0 = \ln x$

← maximum

$\therefore$
$f(t_0) = \ln x - 1$
$f'(t_0) = 0$
$f''(t_0) = -\frac{e^{t_0}}{x} = -1$

$\therefore$ $f(t) \approx \ln x - 1 - \frac{1}{2}(t - \ln x)^2$ near maximum

$$\therefore \quad I(x) \approx \int_0^\infty e^{x\ln x - x}\, e^{-\frac{1}{2}x(t-\ln x)^2}\,dt$$

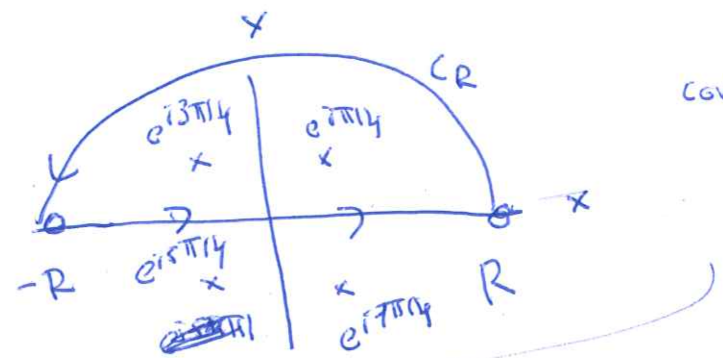
$$\approx x^x e^{-x} \int_{-\infty}^{\infty} e^{-\frac{1}{2}x(t-\ln x)^2}\,dt$$

← sharply peaked about $t = \ln x$

$$= x^x e^{-x} \sqrt{\frac{2\pi}{x}}$$

$$\longrightarrow \quad I(x) \approx \sqrt{2\pi}\, x^{x-\frac{1}{2}} e^{-x} \text{ for large } x$$

Exercise 3 a) $\int_0^\infty \frac{dx}{x^4+1} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^4+1}$



consider $z_0^4 = -1$ has poles

for $z_0 = e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4}, e^{i7\pi/4}$

$\Rightarrow$ $e^{i\pi/4}$ and $e^{i3\pi/4}$ are inside the contour

. find the residues use:

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)} = \frac{1}{4} z_0^{-3} \quad \Rightarrow \quad \operatorname*{Res}_{z=e^{i\pi/4}} f(z) = \frac{1}{4} e^{-i3\pi/4}$$

$$\text{and} \quad \operatorname*{Res}_{z=e^{i3\pi/4}} f(z) = \frac{1}{4} e^{-i9\pi/4}$$

$$\therefore \int_{-R}^{R} \frac{dx}{x^4+1} + \int_{C_R} f(z)\,dz = 2\pi i \sum \text{Res}$$

$$= 2\pi i \frac{1}{4}\left(e^{-i3\pi/4} + e^{-i7\pi/4}\right)$$

$$= \frac{\pi}{2i}\left(-e^{i\pi/4} + e^{-i\pi/4}\right) = \pi \sin\frac{\pi}{4} = \frac{\pi}{\sqrt{2}}$$

Also $\left|\int_{C_R} f(z)\,dz\right| \le \frac{\pi R 1}{R^4 - 1} \to 0$ as $R \to \infty$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}$$

$$\Rightarrow \int_0^\infty \frac{dx}{x^4+1} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$$

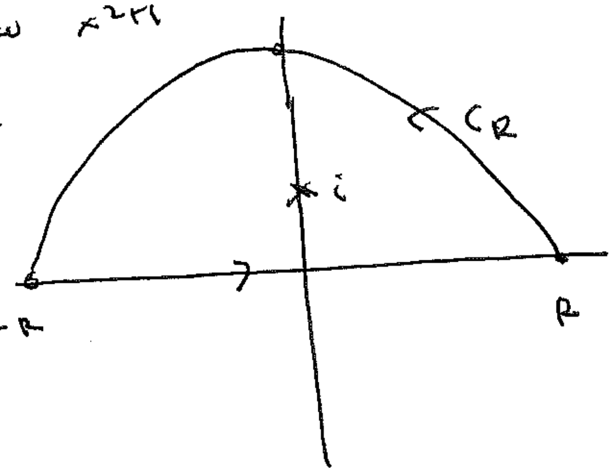
b) $\int_0^\infty \frac{\cos ax}{x^2+1}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\,dx \quad a \geq 0$

$= \frac{1}{2}\int_{-\infty}^{\infty}\left(\frac{\cos ax}{x^2+1} + i\,\frac{\sin ax}{x^2+1}\right)dx$

↑ odd function, so Cauchy P.V. vanishes

$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,dx$

now choose contour



$f(z) = \frac{e^{iaz}}{z^2+1}$

$f(z) = \frac{e^{iaz}}{(z+i)(z-i)}$

└ choose UHP to close contour since $e^{iaz} = e^{-ay}e^{iax}$ ($e^{-ay} < 1$ for $a \geq 0$)

. the contour contains one pole

$\operatorname{Res}_{z=i} f(z) = \frac{e^{-a}}{2i}$ also on C_R $|f(z)| \leq \frac{|e^{iaz}|}{R^2-1} = \frac{e^{-ay}}{R^2-1} \leq \frac{1}{R^2-1}$

$\Rightarrow \left|\int_{C_R} f(z)\,dz\right| \leq \frac{\pi R}{R^2-1} \to 0$ as $R \to \infty$

$\therefore \int_0^\infty \frac{\cos ax}{x^2+1}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,dx = \frac{1}{2}\,2\pi i\left(\frac{e^{-a}}{2i}\right) = \frac{\pi}{2}e^{-a}$ for $a \geq 0$

Exercise 4

a) $$I = \int_0^{2\pi} \frac{d\theta}{1+a\cos\theta} = \int_C \frac{1}{1+a\frac{z+z^{-1}}{2}} \frac{dz}{iz} = \int_C \frac{2/ai}{z^2+\frac{2}{a}z+1}\, dz$$

roots of denominator are $z_\pm = \frac{1}{a}\left(-1 \pm \sqrt{1-a^2}\right)$

note $z_+ z_- = 1 \Rightarrow z_+ = \frac{1}{z_-}$ also $|z_-| = \frac{1}{|a|}\left(1+\sqrt{1-a^2}\right) > 1$ ($|a|<1$)

$\Rightarrow |z_+| < 1 \rightarrow z_+$ is the pole within the unit circle

residue at z_+ is $\frac{2/ai}{z_+ - z_-} = \frac{2/ai}{\frac{2}{a}\sqrt{1-a^2}} = \frac{1}{i}\frac{1}{\sqrt{1-a^2}}$

$$\therefore \quad I = 2\pi i\left(\frac{1}{i}\frac{1}{\sqrt{1-a^2}}\right) = \frac{2\pi}{\sqrt{1-a^2}}$$

b) $$I = \int_0^{\pi} \cos^{2n}(\theta)\, d\theta = \frac{1}{2}\int_0^{2\pi} \sin^{2n}\theta\, d\theta$$

$$= \frac{1}{2}\int_C \left(\frac{z+z^{-1}}{2}\right)^{2n} \frac{dz}{iz} = \frac{1}{2i}\frac{1}{2^{2n}}\int_C \frac{1}{z}\left(z+z^{-1}\right)^{2n} dz$$

use binomial expansion

$$= \frac{1}{2i}\frac{1}{2^{2n}}\int_C \frac{1}{z}\left\{ z^{2n} + z^{2n-2} + \cdots + \binom{2n}{n} z^0 + \cdots + z^{-2n} \right\} dz$$

↑ this is the only term that gives a pole

$$= \frac{1}{2i}\frac{1}{2^{2n}}\, 2\pi i \binom{2n}{n}$$

$$= \frac{\pi}{2^{2n}}\binom{2n}{n} = \frac{\pi}{2^{2n}}\frac{(2n)!}{(n!)^2}$$