HW 4, Physics 501
Fall 2018
University Of Wisconsin, Milwaukee

Nasser M. Abbasi

November 22, 2018 Compiled on November 22, 2018 at 2:04am

Contents

1 Problem 1

Using series expansion evaluate the integral $I = \int_0^1 \ln\left(\frac{1+x}{1-x}\right)\frac{dx}{x}$

Solution

We first need to find the Taylor series for $\ln\left(\frac{1+x}{1-x}\right)$ expanded around $x = 0$. Since

$$\begin{aligned}\ln\left(\frac{1+x}{1-x}\right) &= \ln\left((1+x)\left(\frac{1}{1-x}\right)\right)\\ &= \ln(1+x) + \ln\left(\frac{1}{1-x}\right)\\ &= \ln(1+x) - \ln(1-x)\end{aligned} \tag{1}$$

Looking at $\ln(1+x)$, where now $f(x) = \ln(1+x)$, then we see that $f'(x) = \frac{1}{1+x}, f''(x) = \frac{-1}{(1+x)^2}, f'''(x) = \frac{2}{(1+x)^3}, f^{(4)}(x) = -\frac{2\cdot 3}{(1+x)^4}, \cdots$, therefore

$$\begin{aligned}\ln(1+x) &= f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(4)}(0) + \cdots\\ &= 0 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\end{aligned} \tag{2}$$

Similarly for $\ln(1-x)$, where now $f'(x) = \frac{-1}{1-x}, f''(x) = \frac{-1}{(1-x)^2}, f'''(x) = \frac{-2}{(1-x)^3}, f^{(4)}(x) = -\frac{2\cdot 3}{(1-x)^4}, \cdots$, therefore

$$\begin{aligned}\ln(1-x) &= f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(4)}(0) + \cdots\\ &= 0 - x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \cdots\end{aligned} \tag{3}$$

Using (2,3) in (1) gives the series expansion for $\ln\left(\frac{1+x}{1-x}\right)$ as

$$\begin{aligned}\ln\left(\frac{1+x}{1-x}\right) &= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right)\\ &= 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \frac{2}{7}x^7 + \cdots\end{aligned} \tag{4}$$

Using (4) in the integral given results in

$$\begin{aligned}I &= \int_0^1 \left(2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \frac{2}{7}x^7 + \cdots\right)\frac{dx}{x}\\ &= \int_0^1 \left(2 + \frac{2}{3}x^2 + \frac{2}{5}x^4 + \frac{2}{7}x^6 + \cdots\right)dx\\ &= \left[2x + \frac{2}{3}\frac{x^3}{3} + \frac{2}{5}\frac{x^5}{5} + \frac{2}{7}\frac{x^7}{7} + \cdots\right]_0^1\end{aligned}$$

Which simplifies to

$$
\begin{aligned}
I &= 2 + \frac{2}{3}\frac{1}{3} + \frac{2}{5}\frac{1}{5} + \frac{2}{7}\frac{1}{7} + \cdots \\
&= 2 + \frac{2}{3^2} + \frac{2}{5^2} + \frac{2}{7^2} + \frac{2}{9^2} + \cdots \\
&= 2\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots\right) \\
&= 2\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \qquad (5)
\end{aligned}
$$

The following are two methods to obtain closed form sum for $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$. The first method is based on writing

$$
\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(2n)^2} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \qquad (6)
$$

Where the sum on the left is broken into odd and even terms on the right, as in

$$
1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \left(\frac{1}{2^2} + \frac{1}{4^2} + \cdots\right) + \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)
$$

But, from lecture Sept. 12, 2018, we showed in class that

$$
\sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6} \qquad (7)
$$

(This is called the Basel problem, and the above closed form sum was first given by Euler in 1734). Now using (7) into (6) results in

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} &= \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} \\
&= \frac{3}{4}\left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right) \\
&= \frac{3}{4}\left(\frac{\pi^2}{6}\right) \\
&= \frac{\pi^2}{8}
\end{aligned}
$$

Another way to obtained closed form sum for $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$ is to use Fourier series. Considering the Fourier series for the following periodic function

$$
f(x) = \begin{cases} -x & -\pi < x < 0 \\ 0 & 0 \leq x \leq \pi \end{cases}
$$

Using

$$
f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nx) + \sum_{n=1}^{\infty} B_n \sin(nx)
$$

Therefore

$$A_0 = \frac{1}{\pi}\int_{-\pi}^{0} -x dx = \frac{-1}{\pi}\left(\frac{x^2}{2}\right)_{-\pi}^{0} = \frac{-1}{2\pi}\left(x^2\right)_{-\pi}^{0} = \frac{-1}{2\pi}\left(-\pi^2\right) = \frac{1}{2}\pi$$

And

$$\begin{aligned} A_n &= \frac{-1}{\pi}\int_{-\pi}^{0} x\cos(nx)\,dx = \frac{1+(-1)^{n+1}}{n^2} \\ B_n &= \frac{-1}{\pi}\int_{-\pi}^{0} x\sin(nx)\,dx = \frac{(-1)^{n+1}}{n}\pi \end{aligned}$$

Hence the Fourier series for $f(x)$ is

$$\begin{aligned} f(x) &= \frac{\pi}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1+(-1)^{n+1}}{n^2}\cos(nx) - \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n}\pi(\sin nx) \\ &= \frac{\pi}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1+(-1)^{n+1}}{n^2}\cos(nx) - \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx) \end{aligned}$$

Evaluating the above at $x = 0$ then all the sin terms vanish and we obtain

$$\begin{aligned} 0 &= \frac{\pi}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1+(-1)^{n+1}}{n^2} \\ &= \frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right) \\ &= \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} \end{aligned}$$

Therefore

$$\begin{aligned} \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} &= \frac{\pi}{4} \\ \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} &= \frac{\pi^2}{8} \end{aligned}$$

Now that we found closed form sum for $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$, we can find the value of the integral. Since $I = 2\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$, then

$$\begin{aligned} \int_0^1 \ln\left(\frac{1+x}{1-x}\right)\frac{dx}{x} &= 2\left(\frac{\pi^2}{8}\right) \\ &= \frac{\pi^2}{4} \end{aligned}$$

2 Problem 2

Let $I(x) = \int_0^\infty e^{xf(t)}dt$ with $f(t) = t - \frac{e^t}{x}$, find a large x approximation for this integral.

Solution

$$
\begin{aligned}
I &= \int_0^\infty \exp(xf(t))\,dt \\
&= \int_0^\infty \exp\left(x\left(t - \frac{e^t}{x}\right)\right)dt \\
&= \int_0^\infty \exp\left(xt - e^t\right)dt \\
&= \int_0^\infty \exp(F(t))\,dt
\end{aligned} \tag{1}
$$

Where $F(t) = xt - e^t$. We need to find saddle point where $F(t)$ is maximum. Hence

$$
\begin{aligned}
\frac{d}{dt}F(t) &= 0 \\
x - e^t &= 0 \\
e^t &= x \\
t_0 &= \ln(x)
\end{aligned}
$$

Where t_0 is location of t where $F(t)$ is maximum. We called this in class t_{peak}. We now expand $F(t)$ around t_0 using Taylor series

$$
F(t) = F(t_0) + F'(t_0)(t - t_0) + \frac{1}{2}F''(t_0)(t - t_0)^2 + \cdots \tag{2}
$$

But

$$
\begin{aligned}
F(t_0) &= x\ln(x) - e^{\ln x} \\
&= x\ln x - x
\end{aligned}
$$

And $F'(t) = x - e^t$, hence as expected $F'(t_0) = 0$. And $F''(t) = -e^t$, therefore $F''(t_0) = -e^{\ln x} = -x$. We see also that $F''(t_0) < 0$, which means the saddle point was a maximum and not a minimum (since x is positive). Using these in (2) gives

$$
\begin{aligned}
F(t) &\approx (x\ln x - x) + \frac{1}{2}(-x)(t - \ln x)^2 \\
&= x\ln x - x - \frac{1}{2}x(t - \ln x)^2
\end{aligned}
$$

Substituting the above into (1) gives

$$\begin{aligned}
I &= \int_0^\infty \exp\left(x\ln x - x - \frac{1}{2}x\left(t-\ln x\right)^2\right)dt \\
&= \int_0^\infty \exp\left(x\ln x\right)\exp\left(-x\right)\exp\left(-\frac{1}{2}x\left(t-\ln x\right)^2\right)dt \\
&= \exp\left(x\ln x\right)\exp\left(-x\right)\int_0^\infty \exp\left(-\frac{1}{2}x\left(t-\ln x\right)^2\right)dt \\
&= x^x e^{-x}\int_0^\infty e^{-\frac{1}{2}x(t-\ln x)^2}dt \qquad (3)
\end{aligned}$$

Now, since the peak value where $F(t)$ occurs is on the positive real axis, because $t_0 = \ln(x)$, therefore $x > 1$ to have a maximum, and assuming a narrow peak, then all the contribution to the integral comes from x close to the peak location, so we can change $\int_0^\infty e^{-\frac{1}{2}x(t-\ln x)^2}dt$ to $\int_{-\infty}^\infty e^{-\frac{1}{2}x(t-\ln x)^2}dt$ without affecting the final result. Therefore (3) becomes

$$I = x^x e^{-x}\int_{-\infty}^\infty e^{-\frac{1}{2}x(t-\ln x)^2}dt \qquad (4)$$

Now comparing $\int_{-\infty}^\infty e^{-\frac{1}{2}x(t-\ln x)^2}dt$ to the Gaussian integral $\int_{-\infty}^\infty e^{-a(t-b)^2}dt = \sqrt{\frac{\pi}{a}}$, shows that $a = \frac{x}{2}$ for our case. Hence

$$\int_{-\infty}^\infty e^{-\frac{1}{2}x(t-\ln x)^2}dt = \sqrt{\frac{2\pi}{x}}$$

Therefore (4) becomes

$$I \approx x^x e^{-x}\sqrt{\frac{2\pi}{x}}$$

For large x.

3 Problem 3

Evaluate the following integrals with aid of residue theorem $a \geq 0$. (a) $\int_0^\infty \frac{1}{x^4+1} dx$ (b) $\int_0^\infty \frac{\cos(ax)}{x^2+1} dx$

3.1 Part (a)

Since the integrand is even, then

$$I = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{x^4+1} dx$$

Now we consider the following contour

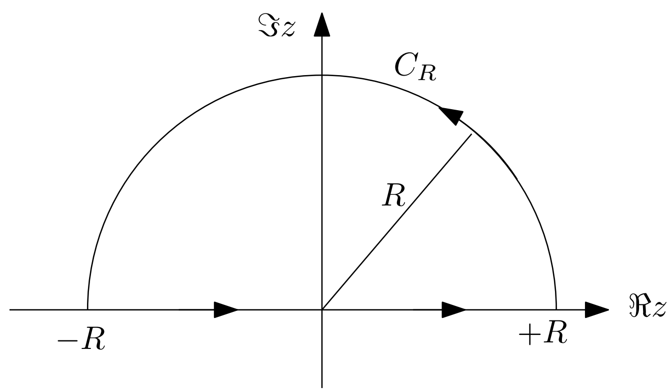

Figure 1: contour used for problem 3

Therefore

$$\oint_C f(z)\,dz = \left(\lim_{R\to\infty} \int_{-R}^{0} f(x)\,dx + \lim_{\tilde{R}\to\infty} \int_0^{\tilde{R}} f(x)\,dx \right) + \lim_{R\to\infty} \int_{C_R} f(z)\,dz$$

Using Cauchy principal value the integral above can be written as

$$\begin{aligned} \oint_C f(z)\,dz &= \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx + \lim_{R\to\infty} \int_{C_R} f(z)\,d \\ &= 2\pi i \sum \text{Residue} \end{aligned}$$

Where $\sum$ Residue is sum of residues of $\frac{1}{z^4+1}$ for poles that are inside the contour C. Therefore the above becomes

$$\begin{aligned} \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx &= 2\pi i \sum \text{Residue} - \lim_{R\to\infty} \int_{C_R} f(z)\,dz \\ \int_{-\infty}^{\infty} \frac{1}{x^4+1} dx &= 2\pi i \sum \text{Residue} - \lim_{R\to\infty} \int_{C_R} \frac{1}{z^4+1} dz \end{aligned} \tag{1}$$

Now we will show that $\lim_{R\to\infty}\int_{C_R}\frac{1}{z^4+1}dz=0$. Since

$$\left|\int_{C_R}\frac{1}{z^4+1}dz\right|\leq ML$$
$$=|f(z)|_{\max}(\pi R) \tag{2}$$

But

$$f(z)=\frac{1}{(z^2-i)(z^2+i)}$$

Hence, and since $z=R\,e^{i\theta}$ then

$$|f(z)|_{\max}\leq\frac{1}{|z^2-i|_{\min}|z^2+i|_{\min}}$$

But but inverse triangle inequality $\left|z^2-i\right|\geq|z|^2+1$ and $\left|z^2+i\right|\geq|z|^2-1$, and since $|z|=R$ then the above becomes

$$|f(z)|_{\max}\leq\frac{1}{(R^2+1)(R^2-1)}$$
$$=\frac{1}{R^4-1}$$

Therefore (2) becomes

$$\left|\int_{C_R}\frac{1}{z^4+1}dz\right|\leq\frac{\pi R}{R^4-1}$$

Then it is clear that as $R\longrightarrow\infty$ the above goes to zero since $\lim_{R\to\infty}\frac{\pi R}{R^4-1}=\lim_{R\to\infty}\frac{\frac{\pi}{R^3}}{1-\frac{1}{R^4}}=\frac{0}{1}=0$. Then (1) now simplifies to

$$\int_{-\infty}^{\infty}\frac{1}{x^4+1}dx=2\pi i\sum\text{Residue} \tag{2A}$$

We just now need to find the residues of $\frac{1}{z^4+1}$ located in upper half plane. The zeros of the denominator $z^4+1=0$ are at $z=-1^{\frac{1}{4}}=\left(e^{i\pi}\right)^{\frac{1}{4}}$, then the first zero is at $e^{i\frac{\pi}{4}}$, and the second zero at $e^{i\left(\frac{\pi}{4}+\frac{\pi}{2}\right)}=e^{i\left(\frac{3}{4}\pi\right)}$ and the third zero at $e^{i\left(\frac{3}{4}\pi+\frac{\pi}{2}\right)}=e^{i\left(\frac{5}{4}\pi\right)}$ and the fourth zero at $e^{i\left(\frac{5}{4}\pi+\frac{\pi}{2}\right)}=e^{i\frac{7}{4}\pi}$. Hence poles are at

$$z_1=e^{i\frac{\pi}{4}}$$
$$z_2=e^{i\frac{3}{4}\pi}$$
$$z_3=e^{i\frac{5}{4}\pi}$$
$$z_4=e^{i\frac{7}{4}\pi}$$

Out of these only the first two are in upper half plane z_1 and z_1. Hence

$$\text{Residue}(z_1)=\lim_{z\to z_1}(z-z_1)f(z)$$
$$=\lim_{z\to z_1}(z-z_1)\frac{1}{z^4-1}$$

Applying L'Hopitals

$$\begin{aligned}\text{Residue}\left(z_1\right) &= \lim_{z\longrightarrow z_1} \frac{1}{4z^3}\\ &= \frac{1}{4\left(e^{i\frac{\pi}{4}}\right)^3}\\ &= \frac{1}{4e^{i\frac{3\pi}{4}}}\end{aligned}$$

Similarly for the other residue

$$\begin{aligned}\text{Residue}\left(z_2\right) &= \lim_{z\longrightarrow z_2} \left(z - z_2\right) f\left(z\right)\\ &= \lim_{z\longrightarrow z_1} \left(z - z_2\right) \frac{1}{z^4 - 1}\end{aligned}$$

Applying L'Hopitals

$$\begin{aligned}\text{Residue}\left(z_2\right) &= \lim_{z\longrightarrow z_2} \frac{1}{4z^3}\\ &= \frac{1}{4\left(e^{i\frac{3}{4}\pi}\right)^3}\\ &= \frac{1}{4e^{i\frac{9\pi}{4}}}\\ &= \frac{1}{4e^{i\frac{\pi}{4}}}\end{aligned}$$

Hence (2A) becomes

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{x^4 + 1} dx &= 2\pi i \left(\frac{1}{4e^{i\frac{3\pi}{4}}} + \frac{1}{4e^{i\frac{\pi}{4}}}\right)\\ &= 2\pi i \left(\frac{\sqrt{2}}{4i}\right)\\ &= \frac{1}{2}\sqrt{2}\pi\end{aligned}$$

But $\int_0^\infty \frac{1}{x^4+1} dx = \frac{1}{2}\int_{-\infty}^\infty \frac{1}{x^4+1} dx$, therefore

$$\begin{aligned}\int_0^{\infty} \frac{1}{x^4 + 1} dx &= \frac{\sqrt{2}}{4}\pi\\ &= \frac{2}{4\sqrt{2}}\pi\\ &= \frac{\pi}{2\sqrt{2}}\end{aligned}$$

3.2 Part (b)

Since the integrand is even, then

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+1} dx$$

We will evaluate $\int_{-\infty}^{\infty} \frac{e^{iaz}}{x^2+1} dx$ and at the end take the real part of the answer. Considering the following contour

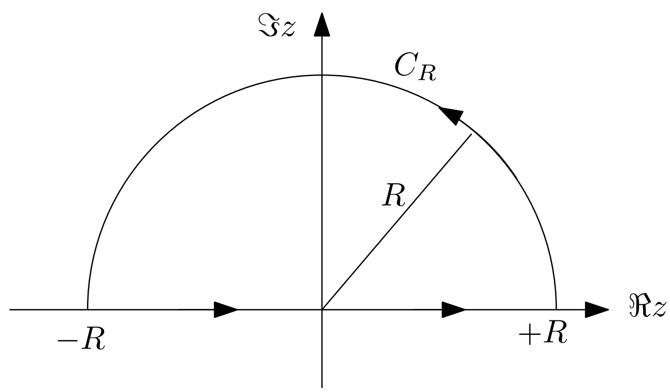

Figure 2: contour used for part b

Then

$$\oint_C f(z)\, dz = \left(\lim_{R\to\infty} \int_{-R}^{0} f(x)\, dx + \lim_{\tilde{R}\to\infty} \int_{0}^{\tilde{R}} f(x)\, dx \right) + \lim_{R\to\infty} \int_{C_R} f(z)\, dz$$

Using Cauchy principal value the integral above can be written as

$$\begin{aligned} \oint_C f(z)\, dz &= \lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx + \lim_{R\to\infty} \int_{C_R} f(z)\, d \\ &= 2\pi i \sum \text{Residue} \end{aligned}$$

Where $\sum$ Residue is sum of residues of $\frac{e^{iaz}}{x^2+1}$ for poles that are inside the contour C. Therefore the above becomes

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx = 2\pi i \sum \text{Residue} - \lim_{R\to\infty} \int_{C_R} f(z)\, dz$$

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1} dx = 2\pi i \sum \text{Residue} - \lim_{R\to\infty} \int_{C_R} \frac{e^{iaz}}{z^2+1} dz \tag{1}$$

Now we will show that $\lim_{R\to\infty} \int_{C_R} \frac{e^{iaz}}{z^2+1} dz = 0$. Since

$$\begin{aligned} \left| \int_{C_R} \frac{e^{iaz}}{z^2+1} dz \right| &\leq ML \\ &= |f(z)|_{\max} (\pi R) \end{aligned} \tag{2}$$

But

$$\begin{aligned}
f(z) &= \frac{e^{iaz}}{(z-i)(z+i)}\\
&= \frac{e^{ia(x+iy)}}{(z-i)(z+i)}\\
&= \frac{e^{iax-ay}}{(z-i)(z+i)}\\
&= \frac{e^{iax}e^{-ay}}{(z-i)(z+i)}
\end{aligned}$$

Hence

$$\begin{aligned}
|f(z)|_{\max} &= \frac{\left|e^{iaz}\right|_{\max}\left|e^{-ay}\right|_{\max}}{|z-i|_{\min}|z+i|_{\min}}\\
&= \frac{\left|e^{-ay}\right|_{\max}}{(R+1)(R-1)}\\
&= \frac{\left|e^{-ay}\right|_{\max}}{R^2-1}
\end{aligned}$$

Since $a>0$ and since in upper half $y>0$ then $\left|e^{-ay}\right|_{\max} = \left|e^{-aR}\right|_{\max} = 1$. Jordan inequality was not needed here, since there is no extra x in the numerator of the integrand in this problem. The above now reduces to

$$|f(z)|_{\max} = \frac{1}{R^2-1}$$

Equation (2) becomes

$$\left|\int_{C_R} \frac{e^{iaz}}{z^2+1}dz\right| \leq \frac{\pi R}{R^2-1}$$

$R\to\infty$ the above goes to zero since $\lim_{R\to\infty}\frac{\pi R}{R^2-1} = \lim_{R\to\infty}\frac{\frac{\pi}{R^2}}{1-\frac{1}{R^2}} = \frac{0}{1} = 0$. Equation (1) now simplifies to

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{x^4+1}dx = 2\pi i\sum \text{Residue}$$

We just now need to find the residues of $\frac{1}{z^2+1}$ that are located in upper half plane. The zeros of the denominator $z^2+1=0$ are at $z=\pm i$, hence poles are at

$$\begin{aligned}
z_1 &= i\\
z_2 &= -i
\end{aligned}$$

Only z_1 is in upper half plane. Therefore

$$\begin{aligned}
\text{Residue}\left(z_1\right) &= \lim_{z\to z_1}\left(z-z_1\right) f\left(z\right) \\
&= \lim_{z\to z_1}\left(z-z_1\right)\frac{e^{iaz}}{\left(z-z_1\right)\left(z-z_2\right)} \\
&= \lim_{z\to z_1}\frac{e^{iaz}}{\left(z-z_2\right)} \\
&= \frac{e^{ia(i)}}{\left(i+i\right)} \\
&= \frac{e^{-a}}{2i}
\end{aligned}$$

Since $\int_{-\infty}^{\infty}\frac{e^{ax}}{x^4+1}dx = 2\pi i\sum\text{Residue}$ then

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{e^{iax}}{x^4+1}dx &= 2\pi i\left(\frac{e^{-a}}{2i}\right) \\
&= \pi e^{-a}
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_{0}^{\infty}\frac{e^{iax}}{x^4+1}dx &= \frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{ax}}{x^4+1}dx \\
&= \frac{\pi}{2}e^{-a}
\end{aligned}$$

But real part of the above is

$$\int_{0}^{\infty}\frac{\cos\left(ax\right)}{x^4+1}dx = \frac{\pi}{2}e^{-a}$$

4 Problem 4

Using residues evaluate(a) $\int_0^{2\pi} \frac{1}{1+a\cos\theta} d\theta$ for $|a| < 1$ (b) $\int_0^{\pi} (\cos(\theta))^{2n} d\theta$ for n integer.

4.1 Part (a)

Using contour which is anti-clockwise over the unit circle

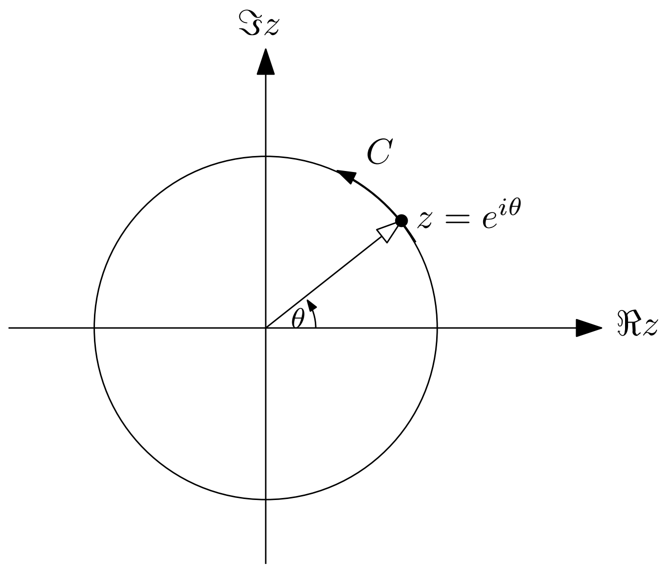

Figure 3: contour used for problem 4

Let $z = e^{i\theta}$, hence $dz = d\theta i e^{i\theta} = d\theta iz$. Using $\cos\theta = \frac{z+z^{-1}}{2}$ then the integral can be written in complex domain as

$$\begin{aligned}
\oint_C \frac{\frac{1}{iz} dz}{1 + a\frac{z+z^{-1}}{2}} &= \frac{2}{i}\oint_C \frac{\frac{1}{z} dz}{2 + a\left(z + \frac{1}{z}\right)} \\
&= \frac{2}{i}\oint_C \frac{dz}{2z + az^2 + a} \\
&= \frac{2}{ai}\oint_C \frac{dz}{z^2 + \frac{2}{a}z + 1} \\
&= \frac{2}{ai}\oint_C \frac{dz}{(z - z_1)(z - z_2)}
\end{aligned}$$

Where z_1, z_2 are roots of $z^2 + \frac{2}{a}z + 1 = 0$ which are found to be (using the quadratic formula) as

$$
\begin{aligned}
z_1 &= \frac{-1-\sqrt{1-a^2}}{a}\\
z_2 &= \frac{-1+\sqrt{1-a^2}}{a}
\end{aligned}
$$

Since $|a| < 1$ then only z_2 will be inside the unit disk for all a values. Therefore

$$
\begin{aligned}
\frac{2}{ai}\oint_C \frac{dz}{(z-z_1)(z-z_2)} &= \left(\frac{2}{ai}\right) 2\pi i \,\text{Residue}\,(z_2)\\
&= \frac{4}{a}\pi \,\text{Residue}\,(z_2) \qquad (1)
\end{aligned}
$$

Now we will find the Residue (z_2) where in this case $f(z) = \frac{1}{(z-z_1)(z-z_2)}$. Hence

$$
\begin{aligned}
\text{Residue}\,(z_2) &= \lim_{z\to z_2} (z-z_2) f(z)\\
&= \lim_{z\to z_2} (z-z_2)\frac{1}{(z-z_1)(z-z_2)}\\
&= \lim_{z\to z_2} \frac{1}{(z-z_1)}\\
&= \frac{1}{\left(\frac{-1+\sqrt{1-a^2}}{a}\right) - \left(\frac{-1-\sqrt{1-a^2}}{a}\right)}\\
&= \frac{a}{2\sqrt{1-a^2}}
\end{aligned}
$$

Using the above result in (1) gives

$$
\begin{aligned}
\int_0^{2\pi} \frac{1}{1+a\cos\theta} d\theta &= \left(\frac{4}{a}\pi\right)\frac{a}{2\sqrt{1-a^2}}\\
&= \frac{2\pi}{\sqrt{1-a^2}} \qquad a \neq 1
\end{aligned}
$$

Using Maple, verified that the above result is correct.

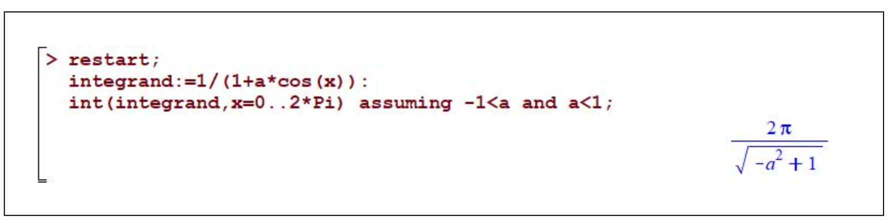

Figure 4: Verification using Maple

4.2 Part (b)

Since integrand is even, then $\int_0^{\pi} (\cos(\theta))^{2n} d\theta = \frac{1}{2}\int_0^{2\pi} (\cos(\theta))^{2n} d\theta$. Using same contour as in part (a), and letting $z = e^{i\theta}$, hence $dz = d\theta i e^{i\theta} = d\theta i z$ and using $\cos\theta = \frac{z+z^{-1}}{2}$ then the integral can be written in complex domain as

$$\begin{aligned}
\int_0^{2\pi} (\cos(\theta))^{2n}\, d\theta &= \oint_C \left(\frac{z+\frac{1}{z}}{2}\right)^{2n} \frac{dz}{iz} \\
&= \frac{1}{i}\oint_C \frac{\left(z+\frac{1}{z}\right)^{2n}}{2^{2n}} \frac{dz}{z} \\
&= \frac{1}{4^n i}\oint_C \left(z+\frac{1}{z}\right)^{2n} \frac{dz}{z} \\
&= \frac{1}{4^n i}\oint_C \left(\frac{z^2+1}{z}\right)^{2n} \frac{dz}{z} \\
&= \frac{1}{4^n i}\oint_C \frac{\left(z^2+1\right)^{2n}}{z^{2n}} \frac{dz}{z} \\
&= \frac{1}{4^n i}\oint_C \frac{\left(z^2+1\right)^{2n}}{z^{2n+1}} dz
\end{aligned}$$

Considering $f(z) = \frac{\left(z^2+1\right)^{2n}}{z^{2n+1}}$, this has a pole at $z = 0$ of order $m = 2n + 1$. Therefore

$$\frac{1}{4^n i}\oint_C \frac{\left(z^2+1\right)^{2n}}{z^{2n+1}} dz = \left(\frac{1}{4^n i}\right) 2\pi i \,\text{Residue}\,(z=0) \tag{1}$$

So we now need to find residue of $f(z)$ at $z = 0$ but for pole of order $m = 2n + 1$. Using the formula for finding residue for pole of order m gives

$$\text{Residue}\,(z_0 = 0) = \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}} \frac{(z-z_0)^m f(z)}{(m-1)!}$$

But $m = 2n + 1$, and $z_0 = 0$, hence the above becomes

$$\begin{aligned}
\text{Residue}\,(0) &= \lim_{z\to 0} \frac{d^{2n}}{dz^{2n}} \frac{z^{2n+1}}{(2n)!} \frac{\left(z^2+1\right)^{2n}}{z^{2n+1}} \\
&= \frac{1}{(2n)!} \lim_{z\to 0} \left(\frac{d^{2n}}{dz^{2n}} \left(z^2+1\right)^{2n}\right)
\end{aligned}$$

Equation (1) becomes

$$\int_0^{2\pi} (\cos(\theta))^{2n}\, d\theta = \left(\frac{1}{4^n}\right) 2\pi \left(\frac{1}{(2n)!} \lim_{z\to 0} \left(\frac{d^{2n}}{dz^{2n}} \left(z^2+1\right)^{2n}\right)\right)$$

Therefore

$$\begin{aligned}
\int_0^{\pi} (\cos(\theta))^{2n}\, d\theta &= \frac{1}{2}\left(\frac{1}{4^n}\right) 2\pi \left(\frac{1}{(2n)!} \lim_{z\to 0} \left(\frac{d^{2n}}{dz^{2n}} \left(z^2+1\right)^{2n}\right)\right) \\
&= \frac{1}{4^n} \frac{\pi}{(2n)!} \lim_{z\to 0} \left(\frac{d^{2n}}{dz^{2n}} \left(z^2+1\right)^{2n}\right)
\end{aligned}$$

Will now try to obtained closed form solution. Trying for different n values in order to see the pattern. From few lectures ago, we learned also that

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^n}\sqrt{\pi}$$

Now will generate a table to see the pattern

n	$\frac{1}{4^n}\frac{\pi}{(2n)!}\lim_{z\to 0}\left(\frac{d^{2n}}{dz^{2n}}\left(z^2+1\right)^{2n}\right)$	result of integral	$\Gamma\left(n+\frac{1}{2}\right)$
1	$\frac{1}{4}\frac{\pi}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}\left(z^2+1\right)^2$	$\frac{\pi}{2}$	$\Gamma\left(1+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2}$
2	$\frac{1}{4^2}\frac{\pi}{4!}\lim_{z\to 0}\frac{d^4}{dz^4}\left(z^2+1\right)^4$	$\frac{3\pi}{8}$	$\Gamma\left(2+\frac{1}{2}\right)=\frac{3\sqrt{\pi}}{4}$
3	$\frac{1}{4^3}\frac{\pi}{6!}\lim_{z\to 0}\frac{d^6}{dz^6}\left(z^2+1\right)^6$	$\frac{5\pi}{16}$	$\Gamma\left(3+\frac{1}{2}\right)=\frac{15\sqrt{\pi}}{8}$
4	$\frac{1}{4^4}\frac{\pi}{8!}\lim_{z\to 0}\frac{d^8}{dz^8}\left(z^2+1\right)^8$	$\frac{35\pi}{128}$	$\Gamma\left(4+\frac{1}{2}\right)=\frac{105\sqrt{\pi}}{16}$
5	$\frac{1}{4^5}\frac{\pi}{10!}\lim_{z\to 0}\frac{d^{10}}{dz^{10}}\left(z^2+1\right)^{10}$	$\frac{63\pi}{256}$	$\Gamma\left(5+\frac{1}{2}\right)=\frac{945\sqrt{\pi}}{32}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Based on the above, we see that $I=\frac{\sqrt{\pi}\Gamma\left(n+\frac{1}{2}\right)}{n!}$, which is verified as follows

n	result of integral	$\Gamma\left(n+\frac{1}{2}\right)$	$\frac{\sqrt{\pi}\Gamma\left(n+\frac{1}{2}\right)}{n!}$
1	$\frac{\pi}{2}$	$\Gamma\left(1+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2}$	$\frac{\sqrt{\pi}\left(\frac{\sqrt{\pi}}{2}\right)}{1}=\frac{1}{2}\pi$
2	$\frac{3\pi}{8}$	$\Gamma\left(2+\frac{1}{2}\right)=\frac{3\sqrt{\pi}}{4}$	$\frac{\sqrt{\pi}\left(\frac{3\sqrt{\pi}}{4}\right)}{2!}=\frac{3}{8}\pi$
3	$\frac{5\pi}{16}$	$\Gamma\left(3+\frac{1}{2}\right)=\frac{15\sqrt{\pi}}{8}$	$\frac{\sqrt{\pi}\left(\frac{15\sqrt{\pi}}{8}\right)}{3!}=\frac{15\pi}{(6)(8)}=\frac{15\pi}{48}=\frac{3}{16}\pi$
4	$\frac{35\pi}{128}$	$\Gamma\left(4+\frac{1}{2}\right)=\frac{105\sqrt{\pi}}{16}$	$\frac{\sqrt{\pi}\left(\frac{105\sqrt{\pi}}{16}\right)}{4!}=\frac{\sqrt{\pi}(105\sqrt{\pi})}{(24)(16)}=\frac{105\pi}{384}=\frac{35}{128}\pi$
5	$\frac{63\pi}{256}$	$\Gamma\left(5+\frac{1}{2}\right)=\frac{945\sqrt{\pi}}{32}$	$\frac{\sqrt{\pi}\left(\frac{945\sqrt{\pi}}{32}\right)}{5!}=\frac{945\pi}{(120)(32)}=\frac{945\pi}{3840}=\frac{63}{256}\pi$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Therefore

$$\int_0^{\pi}(\cos(\theta))^{2n}\,d\theta=\frac{\sqrt{\pi}\Gamma\left(n+\frac{1}{2}\right)}{n!}$$

Tried to do pole/zero cancellation on the integrand of $\oint_C\frac{\left(z^2+1\right)^{2n}}{z^{2n+1}}dz$ in order to find a simpler method than the above but was not able to. The above result was verified using the computer

In[ ]:=
```
Assuming[Element[n, Integers] && n > 0, Integrate[Cos[x]^(2 n), {x, 0, π}]];
TraditionalForm[%]
```

Out[ ]//TraditionalForm=

$$\frac{\sqrt{\pi}\ \Gamma\left(n+\frac{1}{2}\right)}{n!}$$

Figure 5: Verification using Mathematica