Problem Set 3 Solutions

Exercise 1

a) $|z| = \sqrt{x^2+y^2} \Rightarrow u(x,y) = \sqrt{x^2+y^2}$

$v(x,y) = 0$

now $\dfrac{\partial u}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2}} \neq \dfrac{\partial v}{\partial y} = 0$ (unless $x=0$)

$\dfrac{\partial u}{\partial y} = \dfrac{y}{\sqrt{x^2+y^2}} \neq \dfrac{-\partial v}{\partial y} = 0$ (unless $y=0$)

$\Rightarrow |z|$ is only possibly analytic at $(x,y) = (0,0)$ everywhere else it is not analytic. However at $x=y=0$ the derivatives $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ do not exist at this point

$\Rightarrow |z|$ is nowhere analytic

b) $\operatorname{Re} z = x \Rightarrow u = x \quad \dfrac{\partial u}{\partial x} = 1, \dfrac{\partial u}{\partial y} = 0$

$\dfrac{\partial u}{\partial x} = 1 \neq \dfrac{\partial v}{\partial y} = 0$ for all $x,y \Rightarrow \operatorname{Re} z$ is nowhere analytic

c) $e^{\sin z}$, need to find u and v

$e^{\sin z} = e^{\sin(x+iy)} = e^{[\sin x \cos(iy) + \sin(iy) \cos x]}$

now $\cos(iy) = \cosh(y)$

$\sin(iy) = i\sinh(y)$

$e^{\sin z} = e^{\sin x \cosh(y) + i \sinh(y) \cos x}$

$$\Rightarrow \quad u(x,y) = \cos(\sinh y \cos x)\, e^{\sin x \cosh y}$$

$$v(x,y) = \sin(\sinh y \cos x)\, e^{\sin x \cosh y}$$

now $$\frac{\partial u}{\partial x} = +\sin(\sinh y \cos x) \sinh y \sin x\, e^{\sin x \cosh y} + \cos(\sinh y \cos x) \cos x \cosh y\, e^{\cosh y \sin x}$$

$$\frac{\partial u}{\partial y} = -\sin(\sinh y \cos x) \cos x \cosh y\, e^{\sin x \cosh y} + \cos(\sinh y \cos x) \sin x \sinh y\, e^{\sin x \cosh y}$$

$$\frac{\partial v}{\partial x} = -\cos(\sinh y \cos x) \sinh y \sin x\, e^{\sin x \cosh y} + \sin(\sinh y \cos x) \cos x \cosh y\, e^{\sin x \cosh y}$$

$$\frac{\partial v}{\partial y} = \cos(\sinh y \cos x) \cosh y \cos x\, e^{\sin x \cosh y} + \sin(\sinh y \cos x) \sin x \sinh y\, e^{\sin x \cosh y}$$

then $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

$\Rightarrow$ $e^{\sin z}$ is everywhere analytic.

Exercise 2

a) $$\frac{z+3}{z-2} = \frac{-(z+3)}{2(1-\frac{z}{2})} = -\left(\frac{z}{2}+\frac{3}{2}\right)\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+ \cdots\right)$$

$$= -\frac{3}{2} - \frac{5}{2}\frac{z}{2} - \frac{5}{2}\left(\frac{z}{2}\right)^2 - \frac{5}{2}\left(\frac{z}{2}\right)^3 - \cdots$$

which is valid for $|z|<2$

now consider series for $\frac{1}{z}$

$$\frac{z+3}{z-2} = \frac{1+\frac{3}{z}}{1-\frac{2}{z}} = \left(1+\frac{3}{z}\right)\left(1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+\cdots\right)$$

$$= 1 + 5\frac{1}{z} + 5\cdot\frac{2}{z^2} + 5\cdot\frac{4}{z^3} + 5\cdot\frac{8}{z^4} + \cdots$$

this is valid for $|z|>2$

b) $$\frac{z}{(z+1)(z-3)} = \frac{a}{z+1} + \frac{b}{z-3} = \frac{1/4}{z+1} + \frac{3/4}{z-3}$$

For $|z|<1$

$$\frac{1/4}{z+1} = \frac{1}{4}\sum_{n=0}^{\infty}(-z)^n$$ which converges for $|z|<1$

similarly we have

$$\frac{3/4}{z-3} = -\frac{1}{4}\frac{1}{1-\frac{z}{3}} = -\frac{1}{4}\sum_{n=0}^{\infty}\frac{z^n}{3^n}$$ which converges for $|z|<3$

$$\Rightarrow \frac{z}{(z+1)(z-3)} = \frac{1}{4}\sum_{n=0}^{\infty}(-z)^n - \frac{1}{4}\sum_{n=0}^{\infty}\frac{z^n}{3^n}$$

$$= \frac{1}{4}\sum_{n=1}^{\infty}\left((-1)^n - \frac{1}{3^n}\right)z^n \qquad \text{for } |z|<1$$

ii) for $1<|z|<3$

we can still use $\frac{3/4}{z-3} = -\frac{1}{4}\sum_{n=0}^{\infty}\frac{z^n}{3^n}$ $(|z|<3)$

look at $\frac{1/4}{z+1} = \frac{1}{4z}\frac{1}{1+\frac{1}{z}} = \frac{1}{4z}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^n} = \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}}$

valid for $|z|>1$

$$\Rightarrow \frac{z}{(z+1)(z-3)} = \frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^n} - \frac{1}{4}\sum_{n=0}^{\infty}\frac{z^n}{3^n}$$

which converges for $1<|z|<3$

iii) For $|z|>3$

we have $\frac{1/4}{z+1} = \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}}$ which converges for $|z|>1$

consider $\frac{3/4}{z-3} = \frac{3}{4z}\frac{1}{1-\frac{3}{z}} = \frac{3}{4z}\sum_{n=0}^{\infty}\frac{3^n}{z^n} = \frac{3}{4}\sum_{n=0}^{\infty}\frac{3^n}{z^{n+1}}$

which converges for $|z|>3$

$$\Rightarrow \frac{z}{(z+1)(z-3)} = \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}} + \frac{3}{4}\sum_{n=0}^{\infty}\frac{3^n}{z^{n+1}} = \frac{1}{4}\sum_{n=1}^{\infty}\left[(-1)^{n+1}+3^n\right]\frac{1}{z^n}$$

which converges for $|z|>3$

Exercise 3

a) $$\int_C \frac{e^{-2z}}{z^2}\,dz = \int_C \frac{\sum_{n=0}^{\infty} \frac{(-2z)^n}{n!}}{z^2}\,dz = \int_C \left[\frac{1}{z^2} - \frac{2z}{z^2} + \frac{4z}{2!\,z^2} - \frac{8z^3}{3!\,z^2} + \cdots\right] dz$$

$$= \int_C \left[\frac{1}{z^2} - \frac{2}{z} + \frac{2}{\cancel{4}} - \frac{4z}{3} + \cdots\right] dz$$

only term that gives non-zero integral

$$= 2\pi i\,(-2) = -4\pi i$$

b) $$\int_C z e^{\frac{1}{z}}\,dz = \int_C z\left[1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \cdots\right] dz$$

$$= \int_C \left[z + 1 + \frac{1}{2z} + \frac{1}{6z^2} + \cdots\right] dz$$

only term that gives non-zero integral

$$= 2\pi i \cdot \frac{1}{2} = \pi i$$

c) $$\int_C \frac{z+2}{z^2 - z/2} = \int_C \frac{z+2}{z(z-\frac{1}{2})}\,dz$$

this has poles at $z = 1/2$ and $z = 0$

$$\operatorname*{Res}_{z=1/2} f(z) = \frac{\frac{1}{2}+2}{\frac{1}{2}} = 5 \qquad \operatorname*{Res}_{z=0} f(z) = \frac{2}{-1/2} = -4$$

$$\Rightarrow \int_C \frac{z+2}{z(z-1/2)} = 2\pi i\,(5-4) = 2\pi i$$