Problem Set 2 Solutions

Exercise 1:

i) $\log(1+\sqrt{3}i)$; now $1+\sqrt{3}i = re^{i\theta}$

$\Rightarrow r = 2$ and $\theta = \frac{\pi}{3} + 2n\pi$

or $1+\sqrt{3}i = 2e^{i(\frac{\pi}{3}+2n\pi)}$

$$\Rightarrow \boxed{\log(1+\sqrt{3}i) = \ln 2 + i\left(\frac{\pi}{3} + 2n\pi\right)} \quad n = 0, \pm 1, \pm 2, \text{etc}$$

ii) $(1+i\sqrt{3})^{2i}$; again $1+\sqrt{3}i = 2e^{i(\frac{\pi}{3}+2n\pi)}$

$$\Rightarrow (1+i\sqrt{3})^{2i} = e^{2i\left[\ln 2 + i\frac{\pi}{3} + in2\pi\right]}$$

$$= e^{i\ln 4}\, e^{-\frac{2\pi}{3}}\, e^{-4n\pi}$$

$$\Rightarrow (1+i\sqrt{3})^{2i} = \left[\cos(\ln 4) + i\sin(\ln 4)\right] e^{-\frac{2\pi}{3}}\, e^{-4n\pi}$$

Exercise 2:

i) $u(x,y) = 3x^2y - y^3$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 6xy$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 3x^2 - 3y^2$$

$$\Rightarrow \frac{\partial v}{\partial y} = 6xy \Rightarrow v(x,y) = 3y^2x + g(x)$$

$$\frac{\partial v}{\partial x} = 3y^2 + g'(x) = -(3x^2 - 3y^2)$$

$\Rightarrow g'(x) = -3x^2 \Rightarrow g = -x^3 + c$ [can set c=0]

$$\Rightarrow v(x,y) = 3y^2x - x^3 \Rightarrow \boxed{f(z) = -i(x+iy)^3}$$

ii) $u(x,y) = \dfrac{y}{x^2+y^2}$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{-2xy}{(x^2+y^2)^2} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial v}{\partial y} = \frac{-2xy}{(x^2+y^2)^2} \Rightarrow v(x,y) = \frac{+x}{x^2+y^2} + g(x)$$

$$\Rightarrow \frac{\partial v}{\partial x} = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + g'(x)$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2} + g'(x) = \frac{-(x^2-y^2)}{(x^2+y^2)^2}$$

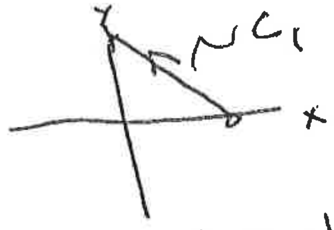
$\Rightarrow g'(x) = 0$, can choose $g = 0$

$$\Rightarrow v = \frac{x}{x^2+y^2} \Rightarrow f = \frac{y+ix}{x^2+y^2} = \frac{i(x-iy)}{x^2+y^2}$$

or $\boxed{f = \dfrac{i(x-iy)}{(x+iy)(x-iy)} = \dfrac{i}{x+iy} = \dfrac{i}{z}}$

Exercise 3

i) $\int_{C_1} |z|^2 \, dz$ where



$= \int_{C_1} (x^2+y^2)(dx+i\,dy)$ now $x = 1-y$ along C_1

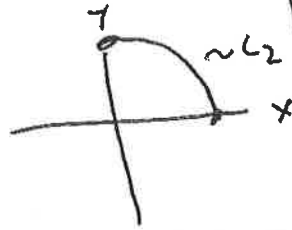
$\Rightarrow y = 1-x \Rightarrow dy = -dx$

$= \int_1^0 (1-i)\,dx\,(x^2+(1-x)^2)$

$= (i-1)\int_0^1 dx\,[2x^2-2x+1] = (i-1)\left[\frac{2}{3}x^3 - x^2 + x\right]_0^1$

$= \boxed{(i-1)\frac{2}{3}}$

$\int_{C_2} |z|^2 \, dz$ where



$= \int_{C_2} (x^2+y^2)\,dz$ now $z = e^{i\theta}$ $dz = ie^{i\theta}\,d\theta$

$= \int_0^{\pi/2} i\,d\theta\, e^{i\theta} = \frac{i}{i}\, e^{i\theta}\Big|_0^{\pi/2} = e^{i\pi/2} - 1 = \boxed{i-1}$

ii) $\int_{C_1} \frac{1}{z^2}\,dz = (1-i)\int_1^0 \frac{dx}{[x+i(1-x)]^2} = (i-1)\int_0^1 \frac{dx}{[(1-i)x+i]^2}$

$= \frac{i-1}{(i-1)^2}\int_0^1 \frac{dx}{[x+\frac{i}{1-i}]^2} = \frac{1}{i-1}\int_0^1 \frac{dx}{[x+\frac{i-1}{2}]^2}$

$= \frac{-1}{i-1}\,\frac{1}{x+\frac{i-1}{2}}\Big|_0^1 = \frac{1}{i-1}\left[\frac{1}{\frac{i-1}{2}} - \frac{1}{1+\frac{i-1}{2}}\right]$

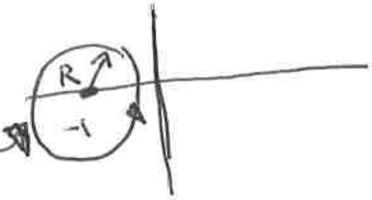
$= \frac{1}{i-1}\left(\frac{2}{i-1} - \frac{2}{1+i}\right) = \frac{2}{i-1}\,\frac{(1+i-(i-1))}{-2} = \frac{-2}{i-1} = \frac{2}{1-i} = \frac{2(1+i)}{2} = 1+i$

$\int_{C_2} \frac{1}{z^2}\,dz = \int_0^{\pi/2} i e^{-i\theta}\,d\theta = -e^{-i\theta}\Big|_0^{\pi/2} = 1 - e^{-i\pi/2} = 1+i$ same result

- notice $|z|^2$ was not analytic while $\frac{1}{z^2}$ was along C_1 and C_2 and $|z|^2$ gave different answers while $\frac{1}{z^2}$ gave the same answer.

Exercise 4

$$\oint_C \frac{dz}{(z+1)(z+2)}$$



this is C

use

$$f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0}\, dz$$

let $f(z) = \dfrac{1}{z+2}$, then $f(z)$ is analytic on and inside C

$$\Rightarrow \oint \frac{dz\, f(z)}{z - z_0} = \oint_C \frac{dz\, f(z)}{z-(-1)} = 2\pi i f(-1) = 2\pi i$$

$$\therefore \oint_C \frac{dz}{(z+1)(z+2)} = 2\pi i$$

- For $R > 1$, and $|z| = R$, we no longer have $\dfrac{1}{z+2}$ analytic everywhere inside the contour C, we cannot use Cauchy integral formula.

Exercise 5

$$\oint_C e^{z^2}\left(\frac{1}{z^2}-\frac{1}{z^3}\right)dz = I$$

$\llcorner$ unit circle about origin

now $$e^{z^2} = \sum_{n=0}^{\infty} \frac{z^{2n}}{n!}$$

$$\Rightarrow I = \oint_C \sum_{n=0}^{\infty} \frac{z^{2n}}{n!}\left(\frac{1}{z^2}-\frac{1}{z^3}\right)dz = \sum_{n=0}^{\infty} \frac{1}{n!} \oint_C \left(z^{2n-2} - z^{2n-3}\right) dz$$

now we know that $\oint_C z^n = 2\pi i \, \delta_{n,-1}$

$$\Rightarrow \quad I = \sum_{n=0}^{\infty} \frac{1}{n!} 2\pi i \left[\delta_{2n-2,-1} - \delta_{2n-3,-1}\right]$$

$\hookrightarrow n = \frac{1}{2}$ (not possible) $\quad \hookrightarrow n = 1$

$$\boxed{I = 2\pi i}$$