Problem Set 1 Solutions

1 a) $f(x) = 1 + \frac{9}{x^2} + \frac{81}{x^4} + \frac{729}{x^6} + \ldots$

$$= \sum_{n=0}^{\infty} \left(\frac{9}{x^2}\right)^n = \frac{1}{1-\frac{9}{x^2}} \quad \text{for } \frac{9}{x^2} < 1$$

$$\Rightarrow x^2 > 9 \quad \text{or} \quad \boxed{|x| > 3}$$

b) $\sum_{n=1}^{\infty} \ln(1+\frac{1}{n}) = \sum_{n=1}^{\infty} a_n$

For large n limit look at

$$\frac{a_{n+1}}{a_n} = \frac{\ln(1+\frac{1}{n+1})}{\ln(1+\frac{1}{n})}, \quad \text{expand this in powers } \frac{1}{n} \text{ to } O(\tfrac{1}{n})^2$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots$$

$$\Rightarrow \ln(1+\tfrac{1}{n}) \cong \frac{1}{n} - \frac{1}{2}\left(\frac{1}{n}\right)^2 + \frac{1}{3}\left(\frac{1}{n}\right)^3$$

$$\ln(1+\tfrac{1}{n+1}) = \ln\left[1 + \frac{1}{n}\left(\frac{1}{1+1/n}\right)\right] \cong \ln\left[1 + \frac{1}{n}\left(1 - \frac{1}{n} + \frac{1}{n^2}\right)\right]$$

$$\cong \frac{1}{n}\left(1 - \frac{1}{n} + \frac{1}{n^2}\right) - \frac{1}{2}\frac{1}{n^2}\left(1 - \frac{1}{n} + \frac{1}{n^2}\right)^2 + \frac{1}{3}\frac{1}{(n)^3}$$

can set $(1-\frac{1}{n}+\frac{1}{n^2})$ = one

$$\cong \frac{1}{n} - \frac{3}{2}\frac{1}{n^2} + \left(1 + 1 + \frac{1}{3}\right)\frac{1}{(n)^3}$$

$$= \frac{1}{n} - \frac{3}{2}\frac{1}{n^2} + \frac{7}{3}\frac{1}{n^3}$$

$$\Rightarrow \frac{a_{n+1}}{a_n} \approx \frac{\frac{1}{n} - \frac{3}{2}\frac{1}{n^2} + \frac{7}{3}\frac{1}{n^3}}{\frac{1}{n} - \frac{1}{2}\frac{1}{n^2} + \frac{1}{3}\left(\frac{1}{n^3}\right)} \cong \left(1 - \frac{3}{2}\frac{1}{n} + \frac{7}{3}\frac{1}{n^2}\right)\left(1 + \frac{1}{2}\frac{1}{n} - \frac{1}{3}\frac{1}{n^2} + \frac{1}{4}\frac{1}{n^2}\right)$$

$$= 1 - \frac{1}{n} + \left(\frac{7}{3} - \frac{1}{3} + \frac{1}{4} - \frac{3}{4}\right)\frac{1}{n^2}$$

$$= 1 - \frac{1}{n} + \frac{3}{2}\frac{1}{n^2}$$

this diverges using the integral test developed in class

$$\Rightarrow \sum_{n} \ln\left(1+\frac{1}{n}\right) \longrightarrow \infty$$

2. $$f(x) = \sum_{n=0}^{\infty} n^2 x^{2n}$$

now $$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}$$ and $$\frac{d}{dx}\frac{1}{1-x^2} = \sum_{n=0}^{\infty} 2n\, x^{2n-1}$$

$$\Rightarrow \frac{1}{2}x \frac{d}{dx}\frac{1}{1-x^2} = \sum_{n=0}^{\infty} n\, x^{2n}$$

$$\Rightarrow \frac{1}{2}x \frac{d}{dx}\left(\frac{1}{2}x \frac{d}{dx}\frac{1}{1-x^2}\right) = \sum_{n=0}^{\infty} n^2 x^{2n}$$

↳ compute this

$$\frac{1}{2}x \frac{d}{dx}\left(\frac{1}{2}x\frac{d}{dx}\frac{1}{1-x^2}\right) = \frac{1}{2}x\frac{d}{dx}\left(x\frac{x}{(1-x^2)^2}\right)$$

$$= \frac{1}{2}x\frac{d}{dx}\left(\frac{x^2}{(1-x^2)^2}\right)$$

$$= \frac{1}{2}x\left[\frac{2x}{(1-x^2)^2} - \frac{2x^2(-2x)}{(1-x^2)^3}\right]$$

$$= \frac{x^2}{(1-x^3)^2} + \frac{2x^4}{(1-x^2)^3}$$

$$= \frac{1}{(1-x^2)^3}\left(x^2 - x^4 + 2x^4\right)$$

$$\sum_{n=0}^{\infty} n^2 x^{2n} = \frac{x^2(1+x^2)}{(1-x^2)^3}$$

↳ this converges for $|x|<1$

Exercise ?

a) $1+\frac{1}{4}-\frac{1}{16}-\frac{1}{64}+\frac{1}{256}+\frac{1}{1024}-- $

$$= 1-\frac{1}{16}+\frac{1}{256}-\cdots+\frac{1}{4}-\frac{1}{64}+\frac{1}{1024}-\cdots$$

$$= 1-\frac{1}{16}+\frac{1}{256}-\cdots+\frac{1}{4}\left(1-\frac{1}{16}+\frac{1}{256}-\cdots\right)$$

$$= \left(1+\frac{1}{4}\right)\left(1-\frac{1}{16}+\frac{1}{256}-+\cdots\right)$$

$$= \left(1+\frac{1}{4}\right)\frac{1}{1+\frac{1}{16}} = \frac{5}{4}\cdot\frac{16}{17} = \frac{20}{17}$$

b) $\frac{1}{1!}+\frac{8}{2!}+\frac{27}{3!}+\frac{64}{4!}+\cdots = \sum_{n=0}^{\infty}\frac{n^3}{n!}$

now consider $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$

then $x\frac{d}{dx}e^x = \sum_{n=0}^{\infty}\frac{nx^n}{n!}$ ← this converges for all x, including $x=1$

$$\Rightarrow x\frac{d}{dx}\left[x\frac{d}{dx}\left(x\frac{d}{dx}e^x\right)\right] = \sum_{n=0}^{\infty}\frac{n^3x^n}{n!}$$

↳ evaluate this and then set $x=1$

$x\frac{d}{dx}e^x = xe^x$; $x\frac{d}{dx}xe^x = x^2e^x+xe^x$

$$x\frac{d}{dx}(x^2e^x+xe^x) = 2x^2e^x+x^3e^x+x^2e^x+xe^x$$
$$= x^3e^x+3x^2e^x+xe^x$$

$$\Rightarrow x^3e^x+3x^2e^x+xe^x = \sum_{n=0}^{\infty}\frac{n^3x^n}{n!} \Rightarrow \boxed{\sum_{n=0}^{\infty}\frac{n^3}{n!} = 5e}$$