MATH 601 HW #5 Solutions

Laplace tr.

1

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^1 \underset{u'}{e^{-st}}\cdot \underset{v}{t}\,dt + \int_1^2 e^{-st}\,dt =$$

$$= -\frac{1}{s}e^{-st}\cdot t\Big|_{t=0}^{1} + \frac{1}{s}\int_0^1 e^{-st}\,dt - \frac{1}{s}e^{-st}\Big|_1^2$$

$$-\frac{1}{s}e^{-s} + 0 + \frac{1}{s}\int_0^1 e^{-st}\,dt - \frac{1}{s}e^{-2t} + \cancel{\frac{1}{s}e^{-s}} =$$

$$-\frac{1}{s}e^{-2t} + \frac{1}{s}\left[-\frac{1}{s}e^{-st}\Big|_0^1\right] = -\frac{1}{s^2}e^{-s} + \frac{1}{s^2} - \frac{1}{s}e^{-2t}$$

$$\frac{1}{s^2}(1-e^{-s}) - \frac{1}{s}e^{-2t}$$

2

$$e^{-\alpha t}\cos\beta t = \mathcal{L}(\cos\beta t)(s+\alpha) = \frac{s+\alpha}{(s+\alpha)^2+\beta^2}$$

3

$$\mathcal{L}^{-1}\left(\frac{4}{s^2+2s-3}\right) = \mathcal{L}^{-1}\left(\frac{1}{s-1}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+3}\right) = e^{t} - e^{-3t}$$

$$\frac{4}{s^2+2s-3} = \frac{4}{(s-1)(s+3)} = \frac{A}{s-1} + \frac{B}{s+3} = \frac{As+3A+Bs-B}{(s-1)(s-3)}$$

$$\parallel$$

$$\frac{1}{s-1} - \frac{1}{s+3} \qquad \Rightarrow \begin{cases} A+B=0 & \Rightarrow A=-B \\ 3A-B=4 & \Rightarrow 4A=4 \Rightarrow A=1 \end{cases}$$

$$\Rightarrow B=-1$$

4

$$\mathcal{L}^{-1}\left(\sum_{k=1}^{5}\frac{a_k}{s+k^2}\right) = \sum_{k=1}^{5}\frac{a_k}{k}\mathcal{L}^{-1}\left(\frac{k}{s+k^2}\right) = \sum_{k=1}^{5}\frac{a_k}{k}\sin kt =$$

$$= a_1\sin t + \frac{a_2}{2}\sin 2t + \frac{a_3}{3}\sin 3t + \frac{a_4}{4}\sin 4t + \frac{a_5}{5}\sin 5t$$

$\boxed{5}$ (IVP) $\begin{cases} y' + 3y = 10 \sin t \\ y(0) = 0 \end{cases}$ $\qquad \mathcal{L}(y') = sY - y(0) = sY$

Transforming the equation

$$sY + 3Y = 10 \frac{1}{s^2+1} \Rightarrow Y(s) = \frac{10}{(s+3)(s^2+1)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+1} = \frac{As^2 + A + Bs^2 + Cs + 3Bs + 3C}{(s+3)(s^2+1)}$$

$$\begin{cases} A+B=0 & \Rightarrow A=-B \\ C+3B=0 & C=3A \\ A+3C=10 & \frac{C}{3}+3C=10 \end{cases} \Leftarrow$$

$10C = 30$

$$\boxed{\begin{matrix} C=3 \\ A=1 \\ B=-1 \end{matrix}}$$

Thus $\quad Y(s) = \frac{1}{s+3} - \frac{s}{s^2+1} + \frac{3}{s^2+1}$

$\mathcal{L}^{-1} \Downarrow$

$$\underline{y(t) = e^{-3t} - \cos t + 3 \sin t}$$

Indeed solution

$$\begin{cases} y' + 3y = \left[\cancel{-3e^{-3t}} + \sin t + \cancel{3\cos t}\right] + \left[\cancel{3y^{-3t}} - \cancel{3\cos t} + 9 \sin t\right] = 10 \sin t \\ y(0) = 1 - 1 + 0 = 0 \end{cases}$$

$\boxed{6}$ $\begin{cases} y'' + y = 2\cos t \\ y(0) = 6 \\ y'(0) = 0 \end{cases} \Rightarrow \mathcal{L}$ $\qquad s^2 Y - s\,y(0) - y'(0) + Y = 2 \frac{s}{s^2+1}$

$$Y = \frac{2s}{(s^2+1)(s^2+1)} + \frac{6s}{s^2+1} =$$

$$\Rightarrow y(t) = 2 \overbrace{\mathcal{L}^{-1}\left(\frac{s}{s^2+1} \cdot \frac{1}{s^2+1}\right)}^{\cos t * \sin t} + 6 \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right)$$

$$= 2 \cdot \int_0^t \cos\tau \sin(t-\tau)\, d\tau + 6 \cos t = \underline{t \sin t + 6 \cos t}$$

since

$$\cos t * \sin t = \int_0^t \cos\tau \sin(t-\tau)\, d\tau =$$

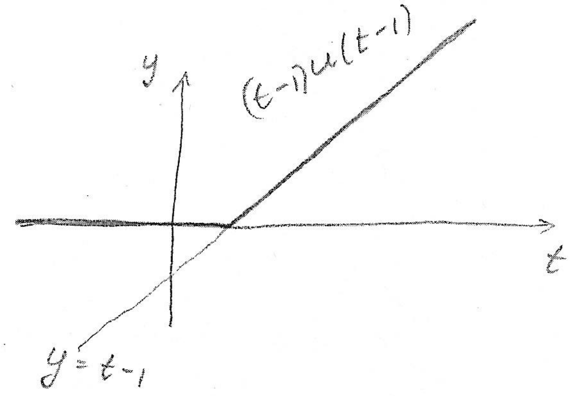
$$= \int_0^t \cos\tau\,(\sin t \cos\tau - \cos t \sin\tau)\, d\tau =$$

$$= \sin t \int_0^t \underbrace{\underbrace{\cos^2\tau}_{\frac{1+\cos 2\tau}{2}}\, d\tau}_{\frac{\tau}{2} + \frac{\sin 2\tau}{4}\Big|_0^t} - \frac{1}{2}\cos t \underbrace{\int_0^t 2\sin\tau\cos\tau\, d\tau}_{\sin^2\tau\Big|_0^t} =$$

$$= \frac{t}{2}\sin t + \cancel{\frac{\sin^2 t \cos t}{2}} - \cancel{\frac{1}{2}\cos t \sin^2 t} = \frac{t}{2}\sin t$$

7

$$\mathcal{L}\left[(t-1)\,u(t-1)\right]$$



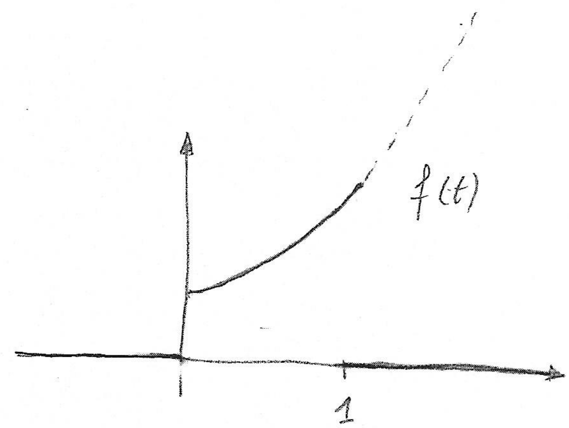
$$= e^{-s}\mathcal{L}(t-1) =$$

$$e^{-s}\left[\mathcal{L}(t)-\mathcal{L}(1)\right] =$$

$$\underline{e^{-s}\left[\frac{1}{s^2}+\frac{1}{s}\right]}$$

8

$$f(t)=\begin{cases} e^t & \text{if } t\in(0,1) \\ 0 & \text{otherwise}\end{cases}$$



$$f(t) = e^t u(t) - e^t u(t-1)$$

$$\mathcal{L}(f(t)) = \underbrace{\mathcal{L}(e^t u(t))}_{\mathcal{L}(e^t)} - \underbrace{\mathcal{L}(e^t u(t-1))}_{\mathcal{L}(g(t-1)\,u(t-1)) = e^{-s}\mathcal{L}(g(t))} =$$

where $g(t) = e^{(t+1)} = e\cdot e^t$

$$\underline{\mathcal{L}(f(t))} = \frac{1}{s-1} - e^{-s}\mathcal{L}(e\cdot e^t) = \frac{1}{s-1} - e^{-s}\cdot e\cdot\frac{1}{s-1} = \underline{\frac{1-e^{1-s}}{s-1}}$$

9

$$\mathcal{L}^{-1}\left(3\,\frac{1-e^{-\pi s}}{s^2+9}\right) = \underbrace{\mathcal{L}^{-1}\left(\frac{3}{s^2+9}\right)}_{\sin 3t} - \underbrace{\mathcal{L}^{-1}\left[e^{-\pi s}\underbrace{\frac{3}{s^2+9}}_{F(s)}\right]}_{\sin 3(t-\pi)\cdot u(t-\pi)} =$$

$F(s)$
$f(t) = \sin 3t$

$$= \sin 3t - \underbrace{\sin 3(t-\pi)}_{\sin 3t\,\underbrace{\cos(3\pi)}_{-1} - \cos 3t\,\cancel{\sin 3\pi}^{\,0}}\,u(t-\pi) = \sin 3t + \sin 3t\; u(t-\pi)$$

Thus

$$\mathcal{L}^{-1}\left(3\frac{1-e^{-\pi s}}{s^2+9}\right) = \begin{cases} \sin 3t & \text{if } t<\pi \\ 2\sin 3t & \text{if } t\geq\pi \end{cases}$$

$\boxed{10}$

$$\begin{cases} y''+3y'+2y=g(t)=\begin{cases} 4t & \text{if } t\in(0,1) \\ 8 & \text{if } t>1 \end{cases} \\ y(0)=y'(0)=0 \end{cases}$$

$g(t) = 4t(1-u(t-1)) + 8u(t-1)$ if $t\geq 0$

use Laplace transform:

$$\mathcal{L}(y''+3y'+2y) = \mathcal{L}(g(t))$$

$$[s^2Y - s^2\cancel{y(0)} - s\cancel{y'(0)}] + 3[sY - \cancel{y(0)}] + 2Y = \mathcal{L}(4t(1-u(t-1)) + \mathcal{L}[8u(t-1)]$$

$$Y(s^2+3s+2) = \mathcal{L}(4t) - \mathcal{L}(4t\,u(t-1)) + 8\mathcal{L}(u(t-1))$$

$$= \frac{4}{s^2} - 4\frac{s+1}{s^2}e^{-s} + \frac{8e^{-s}}{s}$$

$\mathcal{L}(t\,u(t-1)) = \mathcal{L}((t+1-1)u(t-1))$

$= e^{-s}\mathcal{L}(t+1) = e^{-s}\left(\frac{1}{s^2}+\frac{1}{s}\right)$

$$\Rightarrow Y = \frac{4}{s^2(s+1)(s+2)} + e^{-s}\frac{4s-4}{s^2(s+1)(s+2)}$$

$$= \frac{4}{s^2(s+1)(s+2)} + e^{-s}\frac{4}{s(s+1)(s+2)} - e^{-s}\frac{4}{s^2(s+1)(s+2)}$$

Use Partial fraction decomposition.

$$\frac{4}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} = \frac{As^2+3As+2A+Bs^2+2Bs+Cs^2+Cs}{s(s+1)(s+2)}$$

$$\left.\begin{aligned} A+B+C&=0 \\ 3A+2B+C&=0 \end{aligned}\right\} \qquad \begin{aligned} B+C&=-2 \\ 2B+C&=-6 \end{aligned} \qquad \Big| \qquad \underline{B=-4 \quad C=2}$$

$$2A=4 \qquad \underline{A=2}$$

$$\frac{4}{s(s+1)(s+2)} = \frac{2}{s} - \frac{4}{s+1} + \frac{2}{s+2}$$

$$\frac{4}{s^2(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s+2} = \frac{-3}{s} + \frac{2}{s^2} + \frac{4}{s+1} - \frac{1}{s+2}$$

$$= \frac{As^3+3As^2+2As + Bs^2+3Bs+2B + Cs^3+2Cs^2 + Ds^3+Ds^2}{s^2(s+1)(s+2)}$$

$$\begin{aligned} A+C+D&=0 \\ 3A+B+2C+D&=0 \\ 2A+3B&=0 \\ 2B&=4 \quad \Rightarrow \underline{B=2} \end{aligned} \qquad \begin{aligned} A+C+D&=0 \\ 3A+2C+D&=-2 \\ \underline{2A}&=-6 \\ A&=-3 \end{aligned} \qquad \Big| \qquad \begin{aligned} C+D&=3 \\ 2C+D&=-2+9=7 \\ &\Downarrow \\ C&=7-3=4 \\ D&=-1 \end{aligned}$$

$$A=-3,\ B=2,\ C=-4,\ D=1$$

$$Y = -\frac{3}{s} + \frac{2}{s^2} + \frac{4}{s+1} - \frac{1}{s+2} + e^{-s}\frac{2}{s} - e^{-s}\frac{4}{s+1} + e^{-s}\frac{2}{s+2}$$

$$+ e^{-s}\frac{3}{s} - e^{-s}\frac{2}{s^2} - e^{-s}\frac{4}{s+1} + e^{-s}\frac{1}{s+2}$$

$$= -\frac{3}{s} + \frac{2}{s^2} + \frac{4}{s+1} - \frac{1}{s+2} + e^{-s}\frac{5}{s} - e^{-s}\frac{2}{s^2} - e^{-s}\frac{8}{s+1} + e^{-s}\frac{3}{s+2}$$

$$\underline{y(t) = -3+2t+4e^{-t} - e^{-2t} + 5u(t-1) + 2(t-1)u(t-1) - 8e^{(t-1)}u(t-1)}$$

$$\underline{+3e^{2(t-1)}u(t-1)}$$

11

a)
$$\mathcal{L}(t e^{-t}\cos t) = -\frac{d}{ds}\mathcal{L}(e^{-t}\cos t) = -\frac{d}{ds}\mathcal{L}(\cos t)(s+1)$$

$$= -\frac{d}{ds}\left[\frac{s+1}{(s+1)^2+1}\right] =$$

$$-\left[\frac{(s+1)^2+1-(s+1)\cdot 2(s+1)}{[(s+1)^2+1]^2}\right] =$$

$$-\frac{(s+1)^2-2(s+1)^2+1}{[(s+1)^2+1]^2} = \underline{\frac{(s+1)^2-1}{[(s+1)^2+1]^2}}$$

b)
$$\mathcal{L}^{-1}\left[\frac{s}{(s^2-9)^2}\right] = \mathcal{L}^{-1}\left(\frac{d}{ds}\left[\frac{1}{2(9-s^2)}\right]\right) = -t\,\mathcal{L}^{-1}\left(-\frac{1}{2}\,\frac{1}{(s-3)(s+3)}\right)$$

$$\left[\frac{1}{2(9-s^2)}\right]' = \frac{1}{2}\left[-\frac{-2s}{(9-s^2)^2}\right] = \frac{s}{(s^2-9)^2}$$

$$= \frac{t}{2}\mathcal{L}^{-1}\left(\frac{A}{s-3}+\frac{B}{s+3}\right)$$

$$\frac{1}{(s-3)(s+3)} = \frac{As+3A+Bs-3B}{(s-3)(s+3)}$$

$A+B=0$

$3A-3B=1$

$6A=1$ $\quad A=1/6$

$B=-1/6$

$$= \frac{t}{2}\left(\frac{1}{6}e^{3t}-\frac{1}{6}e^{-3t}\right) =$$

$$\underline{\frac{t}{12}\left(e^{3t}-e^{-3t}\right)}$$

$\boxed{12}$ compute

a) $$1 * \sin \omega t = \int_0^t 1 \sin \omega(\tau - t)\, d\tau = \int_0^t \sin \omega t \cos \omega \tau - \cos \omega t \sin \omega \tau \, d\tau$$

$$= \sin \omega t \int_0^t \cos \omega \tau \, d\tau - \cos \omega t \int_0^t \sin \omega \tau \, d\tau =$$

$$\sin \omega t \left[\frac{\sin \omega \tau}{\omega} \Big|_0^t \right] - \cos \omega t \left[- \frac{\cos \omega \tau}{\omega} \Big|_0^t \right] =$$

$$\frac{\sin^2 \omega t}{\omega} + \frac{\cos^2 \omega t}{\omega} - \frac{\cos \omega t}{\omega} = \frac{1 - \cos \omega t}{\omega}$$

b) $$e^t * e^{-t} = \int_0^t e^t e^{-(\tau - t)} d\tau = e^{2t} \int_0^t e^{-\tau} d\tau =$$

$$= e^{2t} \left[-e^{-\tau} \Big|_0^t \right] = e^{2t} \left[1 - e^t \right] = e^{2t} - e^{3t}$$

$\boxed{13}$ $$\begin{cases} y'' + 4y = \begin{cases} 1 & \text{if } t \in (0,1) \\ 0 & \text{if } t > 0 \end{cases} = (1 - u(t-1)) & \text{when } t \geq 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

Laplace tr. $$s^2 Y - s\, y(0) - y'(0) - 4Y = \mathcal{L}(1) - \mathcal{L}(u(t-1))$$

$$= \frac{1}{s} - \frac{e^{-s}}{s}$$

$$\Rightarrow Y = \frac{1}{s^2 - 4} \left[s + \frac{1}{s} - \frac{e^{-s}}{s} \right]$$

$$= \frac{s}{(s-2)(s+2)} + \frac{1}{s(s-2)(s+2)} - e^{-s} \frac{1}{s(s-2)(s+2)}$$

$$= \frac{1}{2} \frac{1}{s-2} + \frac{1}{2} \frac{1}{s+2} - \frac{1}{4} \frac{1}{s} + \frac{1}{8} \frac{1}{s-2} + \frac{1}{8} \frac{1}{s+2} + \frac{1}{4} e^{-s} \frac{1}{s} - \frac{1}{8} e^{-s} \frac{1}{s-2} - \frac{1}{8} e^{-s} \frac{1}{s+2} = ^*$$

$$\frac{s}{(s-2)(s+2)} = \frac{A}{s-2} + \frac{B}{s+2} = \frac{As+2A+Bs-2B}{(s-2)(s+2)} = \frac{1}{2}\left[\frac{1}{s-2} + \frac{1}{s+2}\right]$$

$A+B=1 \qquad 2A=1 \qquad A=B=\frac{1}{2}$

$2(A-B)=0 \Rightarrow A=B$

$$\frac{1}{s(s-2)(s+2)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+2} = \frac{As^2-4A+Bs^2+2Bs+Cs^2-2Cs}{s(s-2)(s+2)}$$

$$= -\frac{1}{4}\frac{1}{s} + \frac{1}{8}\frac{1}{s-2} + \frac{1}{8}\frac{1}{s+2}$$

$A+B+C=0 \qquad 2B=\frac{1}{4} \qquad B=C=\frac{1}{8}$

$2(B-C)=0 \qquad B=C$

$-4A=1 \qquad A=-\frac{1}{4}$

$$Y = -\frac{1}{4}\frac{1}{s} + \frac{5}{8}\frac{1}{s-2} + \frac{5}{8}\frac{1}{s+2} + \frac{1}{4}e^{-s}\frac{1}{s} - \frac{1}{8}e^{-s}\frac{1}{s-2} - \frac{1}{8}e^{-s}\frac{1}{s+2}$$

$$y(t) = -\frac{1}{4} + \frac{5}{8}e^{2t} + \frac{5}{8}e^{-2t} + \frac{1}{4}u(t-1) - \frac{1}{8}e^{2(t-1)}u(t-1) - \frac{1}{8}e^{-2(t-1)}u(t-1)$$

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a) $y_1' = 6y_1 + 9y_2 \qquad y_1(0) = -3$

$y_2' = y_1 + 6y_2 \qquad y_2(0) = -3$

$$sY_1 - y_1(0) = 6Y_1 - 9Y_2$$
$$sY_2 - y_2(0) = Y_1 + 6Y_2$$
$$\Rightarrow \begin{bmatrix} 6-s & -9 \\ 1 & 6-s \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$Y_1 = \frac{1}{(6-s)^2+9}\left[3(6-s)+27\right]$$

$$Y_2 = \frac{1}{(6-s)^2+9}\left[3(6-s)-3\right]$$

$$Y_1 = -3\,\frac{s-6}{(s-6)^2+9} + 9\,\frac{3}{(s-6)^2+9}$$

$$y_1(t) = -3e^{6t}\mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right) + 9e^{6t}\mathcal{L}^{-1}\left(\frac{3}{s^2+9}\right) =$$

$$e^{6t}(-3\cos 3t + 9\sin 3t)$$

$$Y_2 = -3\,\frac{s-6}{(s-6)^2+9} - \frac{3}{(6-s)^2+9}$$

$$y_2(t) = -3e^{6t}\mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right) - e^{6t}\mathcal{L}^{-1}\left(\frac{3}{s^2+9}\right)$$

$$e^{6t}(-3\cos 3t - \sin 3t)$$

b) $y_1'' + y_2 = -5\cos 2t \qquad y_1(0) = y_1'(0) = 1$

$y_2'' + y_1 = 5\cos 2t \qquad y_2(0) = -y_2'(0) = -1$

$\mathcal{L}$
$$\begin{cases} s^2Y_1 - sy_1(0) - y_1'(0) + Y_2 = -5\,\dfrac{s}{s^2+4} \\ s^2Y_2 - sy_2(0) - y_2'(0) + Y_1 = 5\,\dfrac{s}{s^2+4} \end{cases}$$

$$\begin{bmatrix} s^2 & 1 \\ 1 & s^2 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} -5\frac{s}{s^2+4} + s + 1 \\ 5\frac{s}{s^2+4} + s - 1 \end{bmatrix}$$

$$Y_1 = \frac{1}{s^4-1}\left[s^2\left[-5\frac{s}{s^2+4} + s + 1\right] - \left[5\frac{s}{s^2+4} + s - 1\right]\right] =$$

$$= -5\frac{s^2+1}{s^4-1} \cdot \frac{s}{s^2+4} + \frac{s^2-1}{s^4-1}s + \frac{s^2+1}{s^4-1} = \frac{-5s}{(s-1)(s+1)(s^2+4)}$$

$$+ \frac{s}{s^2+1} + \frac{1}{(s-1)(s+1)} =$$

$$Y_1 = \frac{1}{2}\frac{1}{s-1} + \frac{5}{2}\frac{1}{s+1} - 3\frac{s}{s^2+4} + \frac{2}{s^2+4} + \frac{s}{s^2+1} + \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}$$

$$\frac{A}{s-1} + \frac{B}{s+1} + \frac{Cs+D}{s^2+4} = \frac{As^3 + As^2 + 4As + 4A + Bs^3 - Bs^2 + 4Bs - 4B + Cs^3 - Cs + Ds^2 - D}{(s-1)(s+1)(s^2+4)}$$

s^3: $A+B+C=0$

s^2: $A-B+D=0$

s: $4A-4B-C=-5$

1: $4A-4B-D=0$

$A+B+D=5$

$A-B+D=0$

$4A-4B-D=-10$

$\Rightarrow D-C=-5$

$C=D-5$

$2B=5$

$A+D=5/2$

$4A-D=0$

$$\boxed{B=5/2 \quad A=\tfrac{1}{2} \quad D=2 \quad C=-3}$$

$$= \frac{-5s}{(s-1)(s+1)(s^2+4)}$$

$$\frac{1}{(s-1)(s+1)} = \frac{A}{s-1} + \frac{B}{s+1} = \frac{As+A+Bs-B}{(s-1)(s+1)}$$

$A+B=0$

$A-B=1$

$$\boxed{A=\tfrac{1}{2} \quad B=-\tfrac{1}{2}}$$

$$y_1 = e^t + 2e^{-t} - 3\cos 2t + \sin 2t + \cos t$$

$$Y_2 = \frac{1}{s^4-1}\left[s^2\left[5\frac{s}{s^2+4} + s - 1\right] + \left[5\frac{s}{s^2+4} - s - 1\right]\right]$$

$$= 5\,\frac{s^2+1}{s^4-1}\,\frac{s}{s^2+4} + \frac{s^2-1}{s^4-1}\,s - \frac{s^2+1}{s^4-1} = \frac{5s}{(s-1)(s+1)(s^2+4}$$

$$+ \frac{s}{s^2+1} - \frac{1}{(s-1)(s+1)}$$

$$Y_2 = -\frac{1}{2}\,\frac{1}{s-1} - \frac{5}{2}\,\frac{1}{s+1} + 3\,\frac{s}{s^2+4} - \frac{2}{s^2+4} + \frac{s}{s^2+1} - \frac{1}{2}\,\frac{1}{s-1} + \frac{1}{2}\,\frac{1}{s+1}$$

$$-\frac{1}{s-1} - 2\,\frac{1}{s+1} + 3\,\frac{s}{s^2+4} - \frac{2}{s^2+4} + \frac{s}{s^2+1}$$

$$y_2(t) = -e^{t} - 2e^{-t} + 3\cos 2t - \sin 2t + \cos t$$