HW 7 (4th HW on ODE's), Math 601
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Contents

1 problems description

MATH 601 HW #4 HW # 7. 4th ODE

[1] Find the general solution of

a) $16y'' - 8y' + 5y = 0$ b) $y'' + 4y' + (4 - \omega^2)y = 0$

[2] Find the solution of the ODE that satisfy the given conditions:

a) $y'' + 0.4y' + 0.29y = 0$ b) $y'' + 2y' + 2y = 0$

$y(0) = 1$; $y'(0) = -1.2$ $y(0) = 1$, $y(\frac{\pi}{2}) = 0$

[3] Can you make a harmonic oscillator

$$my'' + ky = 0$$

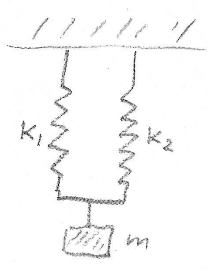
move faster by giving the body greater initial push?

[4] What are the frequencies of vibration of a 5 kg mass vibrating in vacuum

a) on a spring with $k_1 = 20 \frac{N}{m}$ spring constant

b) on a spring with $k_2 = 45 \frac{N}{m}$ spring constant

c) on the two springs connected parallelly?



2 Key solution

Math 601 HW #4 Solution

1 a) $16y'' - 8y' + 5y = 0$ characteristic eq. $16r^2 - 8r + 5 = 0$

$$r = \frac{8 \pm \sqrt{64 - 5 \cdot 64}}{2 \cdot 16} = \frac{1}{4} \pm \frac{i}{2}$$

$$y(x) = e^{x/4}\left(c_1 \cos \frac{x}{2} + c_2 \sin x/2\right)$$

b) $y'' + 4y' + (4 - \omega^2)y = 0$ char eq $r^2 + 4r + 4 - \omega^2 = 0$

$\omega > 0$

$$r = \frac{-4 \pm \sqrt{16 - 16 + 4\omega^2}}{2} = -2 \pm \omega$$

solution $y(x) = e^{-2x}(c_1 \cos \omega x + c_2 \sin \omega x)$

or $\omega = 0$ $y(x) = c_1 e^{2x} + c_2 x e^{2x}$

2 a) $\begin{cases} y'' + 0.4y' + 0.29y = 0 \\ y(0) = 1, \ y'(0) = 1.2 \end{cases}$

$$r^2 + \frac{4}{10} r + \frac{29}{100} = 0$$

$$r = \frac{-0.4 \pm \sqrt{0.16 - 1.16}}{2} = -0.2 \pm i/2$$

$$y(x) = e^{-\frac{2x}{10}}\left(c_1 \cos \frac{x}{2} + c_2 \sin \frac{x}{2}\right)$$

$$y'(x) = -\frac{2}{10} e^{-\frac{2x}{10}}\left(c_1 \cos \frac{x}{2} + c_2 \sin \frac{x}{2}\right) + \frac{1}{2} e^{-\frac{2x}{10}}\left(-c_1 \sin \frac{x}{2} + c_2 \cos \frac{x}{2}\right)$$

$y(0) = c_1 = 1$

$y'(0) = -\frac{2}{10} c_1 + \frac{1}{2} c_2 = -\frac{2}{10} + \frac{c_2}{2} = -1.2 \Rightarrow c_2 = -2$

$$y(x) = e^{-\frac{2x}{10}}\left(\cos \frac{x}{2} - 2 \cdot \sin x/2\right)$$

b) $\begin{cases} y'' + 2y' + 2y = 0 \\ y(0) = 1, \ y(\pi/2) = 0 \end{cases}$ $r^2 + 2r + 2 = 0$ $r = \frac{-2 \pm \sqrt{4 - 8}}{2} = -1 \pm i$

$y(x) = e^{-x}(c_1 \cos x + c_2 \sin x)$

$y(0) = c_1 = 1$

$y(\pi/2) = e^{-\pi/2}(\cos \pi/2 + c_2 \sin \pi/2) = c_2 e^{-\pi/2} = 0 \Rightarrow c_2 = 0$

$$y(x) = e^{-x} \cdot \cos x$$

3

$$\begin{cases} my'' + ky = 0 \\ y(0) = y_0,\ y'(0) = v_0 \end{cases}$$

$mr^2 = -k \qquad r = \pm i\sqrt{\frac{k}{m}}$

$$y(x) = c_1 \cos\sqrt{\tfrac{k}{m}}\,x + c_2 \sin\sqrt{\tfrac{k}{m}}\,x$$

$$y'(x) = \sqrt{\tfrac{k}{m}}\left[c_1\left(-\sin\sqrt{\tfrac{k}{m}}\,x\right) + c_2 \cos\sqrt{\tfrac{k}{m}}\,x\right]$$

$y(0) = c_1 = y_0$

$y'(0) = \sqrt{\frac{k}{m}}\, c_2 = v_0 \Rightarrow c_2 = v_0\sqrt{\frac{m}{k}}$

$$y(x) = y_0 \cos\sqrt{\tfrac{k}{m}}\,x + v_0\sqrt{\tfrac{m}{k}} \sin\sqrt{\tfrac{k}{m}}\,x$$

$$y'(x) = \underbrace{-\sqrt{\tfrac{k}{m}}\,y_0}_{A} \sin\sqrt{\tfrac{k}{m}}\,x + \underbrace{v_0}_{B} \cos\sqrt{\tfrac{k}{m}}\,x = \sqrt{\tfrac{k}{m}y_0^2 + v_0^2}\,\cos\left(\sqrt{\tfrac{k}{m}}\,x - \delta\right)$$

velocity

$$\delta = \tan^{-1}\left(-\frac{y_0}{v_0}\sqrt{\frac{k}{m}}\right)$$

The magnitude of the velocity $\sqrt{\frac{k}{m}y_0^2 + v_0^2}$ increases as v_0 increases.

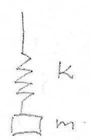
The period of the oscillation is not affected by changing v_0.

4

$my'' = -ky - cy'$ k spring coefficient

$my'' + ky = 0$ $c = 0$ damping coefficient

in vacuum no air resistance



Solution as in **3** is

$$y(x) = c_1 \cos\sqrt{\tfrac{k}{m}}\,x + c_2 \sin\sqrt{\tfrac{k}{m}}\,x$$

a) $k_1 = 20\ \frac{kg}{sec^2}$, $m = 5$, $y(x)$ has period

$$T = 2\pi\sqrt{\frac{m}{k_1}} = \frac{2\pi}{2} = \pi \text{ sec}$$

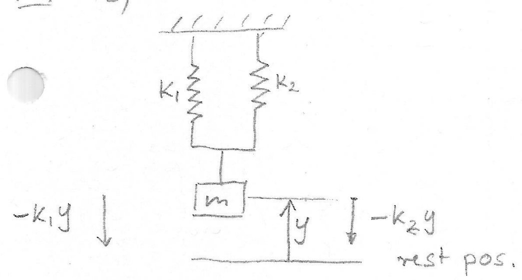
and frequency $\sqrt{\frac{k_1}{m}}\frac{1}{2\pi} = \frac{1}{\pi}\ \frac{1}{sec}$

b) $k_2 = 45\ \frac{kg}{sec^2}$, $m = 5$, $y(x)$ has period

$$T = 2\pi\sqrt{\frac{m}{k_2}} = \frac{2\pi}{3} \text{ sec}$$

oscillations per second → and frequency $\frac{3}{2\pi}\ \frac{1}{sec}$

[4] c,



spring force from spring 1, from spring 2

$$my'' = -k_1 y - k_2 y = -(k_1 + k_2) y$$

$$my'' + ky = 0 \qquad k = k_1 + k_2$$

$$= 65 \ \frac{kg}{sec^2}$$

has period $2\pi\sqrt{\frac{m}{k}} = \frac{2\pi}{\sqrt{11}}$ sec,

frequency $\frac{\sqrt{13}}{2\pi} \ \frac{1}{sec}$