-- Solutions:

$\boxed{1}$ (a) $\sum_{n=1}^{\infty} n\,(z+i\sqrt{2})^n = \sum_{n=1}^{\infty} n\,(z-(-i\sqrt{2}))^n$ $\qquad a_n = n$

center $z_0 = -i\sqrt{2}$ $\qquad R = \frac{1}{L^*} = 1$

$$L^* = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n+1}{n} = 1$$

(d) $\sum_{n=0}^{\infty} \frac{1}{(1+i)^n}\left(z-(i-2)\right)^n$ $\qquad a_n = \frac{1}{(1+i)^n}$ $\qquad z_0 = i-2$

$$L^* = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim\left|\frac{(1+i)^n}{(1+i)^{n+1}}\right| = \frac{1}{|1+i|} = \frac{1}{\sqrt{2}}\ ; \quad R = \sqrt{2}$$

(c) $\sum_{n=0}^{\infty} \frac{(3n)!}{2^n (n!)^3} z^n$ $\qquad$ center $z_0 = 0$ $\qquad a_n = \frac{(3n)!}{2^n (n!)^3}$

$$L^* = \lim_{n\to\infty} \frac{(3n+3)!}{2^{n+1}(n+1)!^3} \Big/ \frac{3n!}{2^n (n!)^3} = \lim \frac{\cancel{1\cdots 3n}\,(3n+1)(3n+2)(3n+3)\cancel{2^n}(n!)^3}{\cancel{1\cdots 3n}\ 2^{n+1}\left[(n+1)!\right]^3}$$

$$= \lim_{n\to\infty}\frac{(3n+1)(3n+2)(3n+3)}{2\cdot(n+1)(n+1)(n+1)} = \frac{3^3}{2} = \frac{27}{2}$$

$$R = \frac{1}{L^*} = \frac{2}{27}$$

(b) $$L^* = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim\frac{\left(\frac{a}{b}\right)^{n+1}}{\left(\frac{a}{b}\right)^n} = \frac{a}{b}\ ; \quad R = \frac{1}{L^*} = \frac{b}{a}$$

$\boxed{2}$ (a) $\sum_{n=1}^{\infty} \frac{6^n}{n}(z-i)^n \qquad a_n = 6^n/n$

1, $L^* = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{6^{n+1}}{6^n} \frac{n}{n+1} = 6 \cdot \lim_{n\to\infty} \frac{n}{n+1} = 6 \; ; \; R = \frac{1}{6}$

2, $\sum_{n=1}^{\infty} \frac{6^n}{n}(z-i)^n$ termwise differentiated gives

$$6\sum_{n=1}^{\infty} 6^{n-1}(z-i)^{n-1} = 6\sum_{m=0}^{\infty} 6^m (z-i)^m = 6 \cdot \frac{1}{1-6(z-i)}$$

for $|6(z-i)|<1$ or $|z-i| < \frac{1}{6}$ convergent

for $|z-i| > \frac{1}{6}$ divergent.

Termwise differentiated series has same rad. of conv. $R = \frac{1}{6}$

(b) $\sum_{n=0}^{\infty} \frac{3^n(n+1)n}{5^n} z^{2n}$

1, the coefficient of series are

$a_0 = 0 \qquad a_1 = 0$

$a_2 = \frac{3^1(1+1)1}{5^1} \qquad a_3 = 0$

$a_4 = \frac{3^2(2+1)2}{5^2} \qquad a_5 = 0$

$\vdots \qquad \vdots$

$\lim_{n\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$ or $\lim_{n\to\infty} \sqrt[k]{|a_k|}$ does not exist.

z^k $\quad$ k odd: $\lim_{n\to\infty} \sqrt[k]{|a_k|} = 0$

k even: $\lim_{n\to\infty} \sqrt[k]{|a_k|} = \lim \sqrt[2n]{\frac{3^n(n+1)n}{5^n}} = \lim \sqrt[2]{\sqrt[n]{\frac{3^n}{5^n}(n+1)n}} =$

k = 2n

$$\left(\lim_{n\to\infty} \frac{3}{5} \sqrt[n]{n} \sqrt[n]{n+1}\right)^{1/2} = \sqrt{3/5}$$

$\sqrt[k]{|a_k|}$ has two cluster points 0 and $\sqrt{3/5}$

$\overline{\lim} \sqrt[k]{|a_k|} = \sqrt{3/5} \qquad R = \frac{1}{\overline{\lim} \sqrt[k]{|a_k|}} = \sqrt{5/3}$

$\boxed{2}$ cont.

2) Let $v = z^2$ new complex variable

in terms of v the series can be written as

$$\sum_{n=0}^{\infty} \frac{3^n (n+1)n}{5^n} v^n = v \underbrace{\sum_{n=0}^{\infty} \frac{3^n}{5^n}(n+1)n\, v^{n-1}}_{\text{integrate termwise w.r.t. } v \text{ (same radius of conv)}} \qquad R_v$$

everywhere convergent

$$\sum_{n=0}^{\infty} \frac{3^n}{5^n}(n+1) v^n \qquad R_v$$

integrate w.r.t. v (same R)

$$\sum_{n=0}^{\infty} \frac{3^n}{5^n} v^{n+1} = v \underbrace{\sum_{n=0}^{\infty} \frac{3^n}{5^n} v^n}_{\text{geom series}}$$

convergent everywhere

$\sum_{n=0}^{\infty} \frac{3^n}{5^n} v^n$ convergent when

$\left|\frac{3}{5} v\right| < 1 \quad |v| < \frac{5}{3}$

$$R_v = \frac{5}{3}$$

thus $\sum_{n=0}^{\infty} \frac{3^n}{5^n}(n+1)n\, z^{2n} = \sum_{n=0}^{\infty} \frac{3^n}{5^n}(n+1)n \cdot v^n$

converges when $|z^2| = |v| < R_v = \frac{5}{3}$

or $|z|^2 < \frac{5}{3} \quad |z| < \sqrt{5/3}$ $\quad \boxed{R = \sqrt{\frac{5}{3}}}$

$\boxed{3}$ a) $\frac{1}{(1-z)^2} = \frac{1}{1-z} \cdot \frac{1}{1-z} = \left(\sum_{n=0}^{\infty} z^n\right) \cdot \left(\sum_{n=0}^{\infty} z^n\right)$ if $|z| < 1$

$= \underbrace{(1+z+z^2+\dots)}_{a_n = 1} \underbrace{(1+z+z^2+\dots)}_{b_n = 1} = \sum_{n=0}^{\infty} (\underset{1}{a_0 b_n} + \underset{1}{a_1 b_{n-1}} + \dots + \underset{1}{a_{n-1} b_1} + \underset{1}{a_n b_0}) z^n$

$= \sum_{0}^{\infty} (n+1) z^n \quad$ (if $|z| < 1$)

b) $$\frac{1}{(1-z)^2} = \frac{d}{dz}\frac{1}{1-z} = \frac{d}{dz}\left(\sum_{n=0}^{\infty} z^n\right) \qquad \text{if } |z|<1$$

termwise derivative of series converges to derivative:

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} n z^{n-1} = \sum_{n=1}^{\infty} n z^{n-1} = \underline{\sum_{\ell=0}^{\infty} (\ell+1) z^{\ell}}$$

$$\ell = n-1$$
$$n = \ell+1$$

$\boxed{4}$ Let $f(z)$ be even: $f(-z) = f(z)$ for all z. in $|z|<R$ where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ Show that $a_n = 0$ when n is odd.

$$f(-z) = \sum_{n=0}^{\infty} a_n (-z)^n = \sum_{n=0}^{\infty} a_n (-1)^n z^n$$

$$0 = f(z) - f(-z) = \sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{\infty} a_n (-1)^n z^n = \sum_{n=0}^{\infty} a_n \underbrace{(1-(-1)^n)}_{\substack{0 \text{ if } n \text{ is odd} \\ 2 \text{ if } n \text{ is even}}} z^n$$

thus

$$0 = \sum_{n=0}^{\infty} 0 z^n \qquad\qquad 0 = 2a_0 + 2a_2 z^2 + 2a_4 z^4 + \ldots$$

for all z $\qquad\qquad$ for $|z| < R$

By theorem 2 in $|z|<R$ / where both series converge / the coefficients of the two series have to coincide.

Thus $2a_0 = 0$, $2a_2 = 0$, $2a_4 = 0, \ldots$

or $a_0 = 0$, $a_2 = 0$, $a_4 = 0, \ldots$

$\boxed{5}$ $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots$ for any $z \in \mathbb{C}$

ⓐ $\cos(2z^2) = 1 - \frac{(2z^2)^2}{2!} + \frac{(2z^2)^4}{4!} - \frac{(2z^2)^6}{6!} + \dots$

$= 1 - \frac{2^2 z^4}{2!} + \frac{2^4 z^8}{4!} - \frac{2^6 z^{12}}{6!} + \dots$ for any $2z^2 \in \mathbb{C}$ or $z \in \mathbb{C}$

$= \sum_{n=0}^{\infty} \frac{4^n z^{4n}}{2n!} \cdot (-1)^n$ radius of conv. $R = \infty$

ⓑ $\frac{z+2}{1-z^2} = \frac{z+2}{(1-z)(z+1)} = \frac{A}{1-z} + \frac{B}{z+1} = \frac{Az + A - Bz + B}{(1-z)(z+1)}$

$\Rightarrow$ $A - B = 1$, $A + B = 2$ $\Rightarrow$ $A = 3/2$, $B = 1/2$ $= \frac{3}{2}\frac{1}{1-z} + \frac{1}{2}\frac{1}{z+1}$

$\frac{3}{2}\frac{1}{1-z} = \frac{3}{2}\left[1 + z + z^2 + z^3 + \dots\right]$ for $|z| < 1$

$\frac{1}{2}\frac{1}{1+z} = \frac{1}{2}\left[\frac{1}{1-(-z)}\right] = \frac{1}{2}\left[1 - z + z^2 - z^3 + \dots\right]$ if $|-z| = |z| <$

$\frac{z+2}{1-z^2} = \left[\frac{3}{2} + \frac{1}{2}\right] + \left(\frac{3}{2} - \frac{1}{2}\right)z + \left(\frac{3}{2} + \frac{1}{2}\right)z^2 + \left(\frac{3}{2} - \frac{1}{2}\right)z^3 + \dots$

$= 2 + z + 2z^2 + z^3 + 2z^4 + z^5$ $(R = 1)$

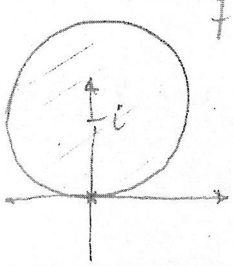
$\sum_{n=0}^{\infty} \left(\frac{3}{2} + \frac{1}{2}(-1)^n\right) z^n$ if $|z| < 1$

$\boxed{6}$

(a) $f(z) = \frac{1}{z}$ $\quad z_0 = i$ $\qquad f'(z) = -\frac{1}{z^2}, \; f''(z) = \frac{2\cdot 1}{z^3},$

$f'''(z) = -\frac{3\cdot 2\cdot 1}{z^4} \qquad f^{(n)}(z) = \frac{(-1)^n \, n!}{z^{n+1}}$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n =$$



$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(i)^{n+1}} (z-i)^n = -i + (z-i) - i(z-i)^2 + (z-i)^3$$

$$-i(z-i)^4 + (z-i)^5 - \dots$$

$|z - i| < 1$

(b) $f(z) = e^z \qquad z_0 = a$

$f^{(n)} = e^z$

$$f(z) = \sum_{n=0}^{\infty} \frac{e^a}{n!}(z-a)^n = e^a \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!}$$

$\sqrt[n]{\frac{e^a}{n!}} \to 0$

$$= e^a + e^a(z-a) + \frac{e^a}{2!}(z-a)^2 + \frac{e^a}{3!}(z-a)^3 + \dots$$

$R = \infty$

converges for all z

Power series $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} z^n$ is uniformly convergent in

$\boxed{7}$ $|z| < r < R$ where R is radius of convergence

we need to show that $R > 3$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)! \cdot (n+1)!}{(2n+2)!} \cdot \frac{2n!}{n! \cdot n!} = \frac{(n+1)^2}{(2n+1)(2n+2)} \longrightarrow \frac{1}{4} = L^*$$

$$R = \frac{1}{L^*} = 4$$

$\boxed{8}$ $\sum_{n=1}^{\infty} \underbrace{\left[\frac{n+2}{5n-3}\right]^n}_{a_n} z^n$ $\quad z_0 = 0$

$$R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}} = \lim_{n\to\infty} \frac{1}{\frac{n+2}{5n-3}} = 5$$

Series converges uniformly in any disk $|z| < r < 5$

HW 3 MATH 601

1 a) Verify that $y(x) = -\sin x + ax^2 + bx + c$ for any a, b, c constants is a solution to $y''' = \cos x$

b) Show that $y(x) = \tan(x+c)$ solves $y' = 1 + y^2$ for any constant c

2 Show that the given function solves the specified initial value problem

a) $y(x) = Ce^{x/2}$ $\quad \begin{cases} y' = \frac{1}{2}y \\ y(2) = 2 \end{cases}$

b) $y(x) = Ce^{-x^2}$ $\quad \begin{cases} y' + 2xy = 0 \\ y(1) = 1/e \end{cases}$

3 $y = Cx - C^2$ is a solution to the ODE $(y')^2 - xy' + y = 0$ for any C const.
Find a singular solution to the ODE (not given by $y = Cx - C^2$) by rewriting the ODE using the quadratic formula.

4 Find all solutions of the following differential equations.

a) $yy' + 25x = 0$ $\qquad$ b) $y' = ky^2$

c) $xy' = x + y$ (Hint $u = y/x$)

5 Solve the IVPs:

a) $\begin{cases} y' = 1 + 4y^2 \\ y(0) = 0 \end{cases}$ $\qquad$ b $\begin{cases} y' = -x/y \\ y(1) = \sqrt{3} \end{cases}$

c) $\begin{cases} e^x y' = 2(x+1)y^2 \\ y(0) = 1/6 \end{cases}$