

HW 3, Physics 501
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Nasser M. Abbasi

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1 Problem set

HW: MATH 601

problems

15.2 - 15.5 Eds 9, 10

Sept 24, 201
Tuesday

[1] Consider the power series below. Give the center and find the radius of convergence for each.

a) $\sum_{n=1}^{\infty} n(z+i\sqrt{2})^n$ b) $\sum_{n=0}^{\infty} \left(\frac{a}{b}\right)^n (z-\pi i)^n$ c) $\sum_{n=0}^{\infty} \frac{(3n)!}{2^n (n!)^3} z^n$

d) $\sum_{n=0}^{\infty} \frac{1}{(1+i)^n} (z+2-i)^n$

[2] Find the radius of convergence using both:

① The formula $R = \frac{1}{\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|}$ (Hadamard)

② The termwise differentiation/integration properties of power series

for the series: a) $\sum_{n=1}^{\infty} \frac{6^n}{n} (z-i)^n$; b) $\sum_{n=0}^{\infty} \frac{3^n (n+1)4^n}{5^n} z^{2n}$

[3] Show that $\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1)z^n$

a) using the Cauchy product

b) differentiating a suitable series

[4] If $f(z)$ is an even function (i.e. $f(-z) = f(z)$) where $f(z) = \sum_{n=0}^{\infty} a_n z^n$, show that $a_n = 0$ when n is odd.

If $f(z)$ is odd function (i.e. $f(-z) = -f(z)$) show that $a_n = 0$ for n even.

[5] Develop the functions below in a Maclaurin series and determine the radius of convergence R for each

a) $\cos(2z^2)$

b) $\frac{z+2}{1-z^2}$

6) Develop

a) $f(z) = \frac{1}{z}$ in a Taylor series with center $z_0 = i$

b) $g(z) = e^z$ ———— $z_0 = a$.

What is the radius of convergence for each?

7) Show that $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} z^n$ converges uniformly in $|z| \leq 3$

8) Where does the series $\sum_{n=1}^{\infty} \left(\frac{n+2}{5^{n-3}}\right)^n z^n$ converge uniformly?

2 Problem 1

Consider the power series below. Given the center and find radius of convergence for each.

1. $\sum_{n=1}^{\infty} n (z + i\sqrt{2})^n$

2. $\sum_{n=1}^{\infty} \left(\frac{a}{b}\right)^n (z - i\pi)^n$

3. $\sum_{n=0}^{\infty} \frac{(3n)!}{2^n (n!)^3} z^n$

4. $\sum_{n=0}^{\infty} \frac{1}{(1+i)^n} (z + 2 - i)^n$

Solution

1) Comparing to form $\sum_{n=1}^{\infty} a_n (z - z_0)^n$ then center is $-i\sqrt{2}$. Now,

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)}{n} \right| \\ &= 1 \end{aligned}$$

Hence $R = 1$

2) Comparing to form $\sum_{n=1}^{\infty} a_n (z - z_0)^n$ then center is $i\pi$. Now,

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{\left(\frac{a}{b}\right)^{n+1}}{\left(\frac{a}{b}\right)^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \left(\frac{a}{b}\right) \right| \\ &= \frac{a}{b} \end{aligned}$$

Hence $R = \frac{b}{a}$

3) Comparing $\sum_{n=0}^{\infty} \frac{(3n)!}{2^n (n!)^3} z^n$ to $\sum_{n=1}^{\infty} a_n (z - z_0)^n$ then center is 0. Now

$$\begin{aligned}
L &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= \lim_{n \rightarrow \infty} \left| \frac{\frac{(3(n+1))!}{2^{n+1}((n+1)!)^3}}{\frac{(3n)!}{2^n(n!)^3}} \right| \\
&= \lim_{n \rightarrow \infty} \left| \frac{(3(n+1))! 2^n (n!)^3}{(3n)! 2^{n+1} ((n+1)!)^3} \right| \\
&= \frac{1}{2} \lim_{n \rightarrow \infty} \left| \frac{(3n+3)! (n!)^3}{(3n)! ((n+1)!)^3} \right| \\
&= \frac{1}{2} \lim_{n \rightarrow \infty} \left| \frac{(3n+1)(3n+2)(3n+1)(3n)! (n!)^3}{(3n)! ((n+1)!)^3} \right| \\
&= \frac{1}{2} \lim_{n \rightarrow \infty} \left| \frac{(3n+1)(3n+2)(3n+1)(n!)^3}{((n+1)!)^3} \right| \\
&= \frac{1}{2} \lim_{n \rightarrow \infty} \left| \frac{(3n+1)(3n+2)(3n+1)(n!)^3}{((n+1)n!)^3} \right| \\
&= \frac{1}{2} \lim_{n \rightarrow \infty} \left| \frac{(3n+1)(3n+2)(3n+1)(n!)^3}{(n+1)^3 (n!)^3} \right| \\
&= \frac{1}{2} \lim_{n \rightarrow \infty} \left| \frac{(3n+1)(3n+2)(3n+1)}{(n+1)^3} \right| \\
&= \frac{1}{2} \lim_{n \rightarrow \infty} \left| \frac{(3n+1)(3n+2)(3n+1)}{(n+1)^3} \right|
\end{aligned}$$

Hence the above becomes

$$\begin{aligned}
L &= \frac{1}{2} \lim_{n \rightarrow \infty} \left| \frac{27n^3 + 36n^2 + 15n + 2}{n^3 + 3n^2 + 3n + 1} \right| \\
&= \frac{1}{2} \lim_{n \rightarrow \infty} \left| \frac{27 + 36\frac{1}{n} + 15\frac{1}{n^2} + \frac{2}{n^3}}{1 + 3\frac{1}{n} + 3\frac{1}{n^2} + \frac{1}{n^3}} \right| \\
&= \frac{27}{2}
\end{aligned}$$

Hence $R = \frac{2}{27}$.

4) Comparing $\sum_{n=0}^{\infty} \frac{1}{(1+i)^n} (z - (-2+i))^n$ to $\sum_{n=1}^{\infty} a_n (z - z_0)^n$ shows that center is $z_0 = -2 + i$. Now

$$\begin{aligned}
L &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= \lim_{n \rightarrow \infty} \left| \frac{\frac{1}{(1+i)^{n+1}}}{\frac{1}{(1+i)^n}} \right| \\
&= \lim_{n \rightarrow \infty} \left| \frac{(1+i)^n}{(1+i)^{n+1}} \right| \\
&= \lim_{n \rightarrow \infty} \left| \frac{1}{1+i} \right| \\
&= \left| \frac{1}{1+i} \right| \\
&= \frac{1}{\sqrt{2}}
\end{aligned}$$

Hence $R = \sqrt{2}$

3 Problem 2

Find radius of convergence using both 1) $R = \frac{1}{L}$ where $L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$ and 2) the termwise differentiation/integration properties of power series. Do this for

1. $\sum_{n=1}^{\infty} \frac{6^n}{n} (z-i)^n$
2. $\sum_{n=0}^{\infty} \frac{3^{n(n+1)n}}{5^n} z^{2n}$

Solution

1) First method. The center is i . And

$$\begin{aligned}
L &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= \lim_{n \rightarrow \infty} \left| \frac{\frac{6^{n+1}}{n+1}}{\frac{6^n}{n}} \right| \\
&= \lim_{n \rightarrow \infty} \left| \frac{(6^{n+1}) n}{6^n (n+1)} \right| \\
&= \lim_{n \rightarrow \infty} \left| \frac{6n}{n+1} \right| \\
&= 6 \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \right| \\
&= 6
\end{aligned}$$

Hence $R = \frac{1}{6}$.

Second method: Taking termwise differentiation gives

$$\begin{aligned}
f'(z) &= \sum_{n=1}^{\infty} \frac{6^n}{n} n (z-i)^{n-1} \\
&= 6 \sum_{n=1}^{\infty} 6^{n-1} (z-i)^{n-1}
\end{aligned}$$

Changing the indexing gives

$$\begin{aligned} f'(z) &= 6 \sum_{n=0}^{\infty} 6^n (z-i)^n \\ &= 6 \sum_{n=0}^{\infty} (6(z-i))^n \end{aligned}$$

Comparing to Binomial series $\sum_{n=0}^{\infty} r^n$, the above is $6 \frac{1}{1-r}$ where $r = 6(z-i)$. Hence this converges for $|r| < 1$ or $|6(z-i)| < 1$ or $|z-i| < \frac{1}{6}$ and diverges for $|z-i| > \frac{1}{6}$. Since termwise differentiated series has same radius of convergence, then $R = \frac{1}{6}$ as using first method.

2) First method. Comparing $\sum_{n=0}^{\infty} \frac{3^n(n+1)n}{5^n} (z^2)^n$ to $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ then center is zero. And
 TODO

4 Problem 3

Show that $\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1)z^n$, using (a) the Cauchy product. (b) By differentiating a suitable series.
Solution (a)

$$\begin{aligned} \frac{1}{(1-z)^2} &= \frac{1}{(1-z)} \frac{1}{(1-z)} \\ &= (1+z+z^2+z^3+\dots) (1+z+z^2+z^3+\dots) \\ &= (1+z+z^2+z^3+\dots) + z(1+z+z^2+z^3+\dots) + z^2(1+z+z^2+z^3+\dots) + \dots \\ &= (1+z+z^2+z^3+\dots) + (z+z^2+z^3+\dots) + (z^2+z^3+z^4+\dots) + \dots \\ &= 1+2z+3z^2+4z^3+\dots \quad |z| < 1 \end{aligned}$$

But $\sum_{n=0}^{\infty} (n+1)z^n = 1+2z+3z^2+4z^3+\dots$. Hence the same.

Solution (b) Observing that

$$(n+1)z^n = \frac{d}{dz} z^{n+1}$$

Then

$$\begin{aligned} \sum_{n=0}^{\infty} (n+1)z^n &= \sum_{n=0}^{\infty} \frac{d}{dz} z^{n+1} \\ &= \frac{d}{dz} \sum_{n=0}^{\infty} z^{n+1} \\ &= \frac{d}{dz} \sum_{n=1}^{\infty} z^n \\ &= \frac{d}{dz} (z+z^2+z^3+\dots) \\ &= \frac{d}{dz} (z(1+z+z^2+\dots)) \\ &= \frac{d}{dz} \left(\frac{z}{1-z} \right) \end{aligned}$$

But $\frac{d}{dz} \frac{A(z)}{B(z)} = \frac{A'B-AB'}{B^2}$, hence the above becomes, where $A = z, B = 1-z$

$$\begin{aligned}\sum_{n=0}^{\infty} (n+1)z^n &= \frac{(1-z) - z(-1)}{(1-z)^2} \\ &= \frac{1-z+z}{(1-z)^2} \\ &= \frac{1}{(1-z)^2}\end{aligned}$$

5 Problem 4

If $f(z)$ is an even function, where $f(z) = \sum_{n=0}^{\infty} a_n z^n$, show that $a_n = 0$ when n is odd. And if $f(z)$ is odd function, show that $a_n = 0$ when n is even.

Solution

If $f(z)$ is even, then $f(-z) = f(z)$. Therefore

$$\begin{aligned}\sum_{n=0}^{\infty} a_n (-z)^n &= \sum_{n=0}^{\infty} a_n z^n \\ a_0 - a_1 z + a_2 z^2 - a_3 z^3 + a_4 z^4 - \dots &= a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \dots\end{aligned}$$

Since power series is unique, then we must have $a_1 = -a_1$ which means $a_1 = 0$, the same for $a_3 = -a_3$, which gives $a_3 = 0$ and so on for all odd a_n .

If $f(z)$ is odd, then $f(-z) = -f(z)$. Therefore

$$\begin{aligned}\sum_{n=0}^{\infty} a_n (-z)^n &= -\sum_{n=0}^{\infty} a_n z^n \\ a_0 - a_1 z + a_2 z^2 - a_3 z^3 + a_4 z^4 - \dots &= -(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \dots) \\ &= -a_0 - a_1 z - a_2 z^2 - a_3 z^3 - a_4 z^4 + \dots\end{aligned}$$

Since power series is unique, then we must have $a_0 = -a_0$ which means $a_0 = 0$, the same for $a_2 = -a_2$, which gives $a_2 = 0$ and so on for all even a_n .

6 Problem 5

Develop the functions below in Maclaurin's series and determine the radius of convergence R for each.

(a) $\cos(2z^2)$, (b) $\frac{z+2}{1-z^2}$

Solution (a)

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Replacing $x = 2z^2$ gives

$$\begin{aligned}
\cos(2z^2) &= 1 - \frac{(2z^2)^2}{2!} + \frac{(2z^2)^4}{4!} - \frac{(2z^2)^6}{6!} + \dots \\
&= 1 - \frac{2^2 z^4}{2!} + \frac{2^4 z^8}{4!} - \frac{2^6 z^{12}}{6!} + \dots \\
&= 1 - \frac{4z^4}{2!} + \frac{4^2 z^8}{4!} - \frac{4^3 z^{12}}{6!} + \dots \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{4^n z^{2n}}{(2n)!} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{(4z^2)^n}{(2n)!}
\end{aligned}$$

Hence

$$\begin{aligned}
L &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \\
&= \lim_{n \rightarrow \infty} \left| \frac{\frac{(4z^2)^{n+1}}{(2(n+1))!}}{\frac{(4z^2)^n}{(2n)!}} \right| \\
&= \lim_{n \rightarrow \infty} \left| \frac{(4z^2)^{n+1} (2n)!}{(4z^2)^n (2(n+1))!} \right| \\
&= \lim_{n \rightarrow \infty} \left| \frac{4z^2 (2n)!}{(2n+2)!} \right| \\
&= \lim_{n \rightarrow \infty} \left| \frac{4z^2 (2n)!}{(2n+2)(2n+1)(2n)!} \right| \\
&= \lim_{n \rightarrow \infty} \left| \frac{4z^2}{(2n+2)(2n+1)} \right| \\
&= \lim_{n \rightarrow \infty} \left| \frac{4z^2}{4n^2 + 6n + 2} \right| \\
&= \lim_{n \rightarrow \infty} \left| \frac{\frac{4}{n^2} z^2}{4 + 6\frac{1}{n} + \frac{2}{n}} \right| \\
&= |z^2| \lim_{n \rightarrow \infty} \left| \frac{0}{4} \right| \\
&= 0
\end{aligned}$$

Hence $R = \frac{1}{L} = \infty$

(b) $\frac{z+2}{1-z^2}$. Apply partial fractions. Obtain two binomial series and combine.

7 Problem 6

Develop (a) $f(z) = \frac{1}{z}$ in Taylor series around $z_0 = i$. (b) $g(z) = e^z$ around $z_0 = a$. What is radius of convergence?

Solution (a)

$$f(z) = f(i) + (z-i)f'(i) + \frac{(z-i)^2 f''(i)}{2!} + \frac{(z-i)^3 f'''(i)}{3!} + \dots$$

But $f'(z) = -\frac{1}{z^2}$, $f''(z) = \frac{2}{z^3}$, $f'''(z) = -\frac{2(3)}{z^4}$, \dots , hence the above becomes

$$\begin{aligned} f(z) &= \frac{1}{i} - (z-i) \frac{1}{i^2} + \frac{(z-i)^2}{2!} \frac{2}{i^3} + \frac{(z-i)^3}{3!} \left(-\frac{2(3)}{i^4} \right) + \dots \\ &= -i + (z-i) + 2i \frac{(z-i)^2}{2!} - 2(3) \frac{(z-i)^3}{3!} + \dots \\ &= -i + (z-i) + i(z-i)^2 - (z-i)^3 + \dots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{i^{n+1}} (z-i)^n \end{aligned}$$

Hence this convergence for $|z-i| < 1$.

Solution (b)

$$g(z) = g(a) + (z-a)g'(a) + \frac{(z-a)^2}{2!} g''(a) + \frac{(z-a)^3}{3!} g'''(a) + \dots$$

But $g'(z) = e^z$, $g''(z) = e^z$, $g'''(z) = e^z$, \dots , hence the above becomes

$$\begin{aligned} g(z) &= e^a + (z-a)e^a + \frac{(z-a)^2}{2!} e^a + \frac{(z-a)^3}{3!} e^a + \dots \\ &= e^a \left(1 + (z-a) + \frac{(z-a)^2}{2!} + \frac{(z-a)^3}{3!} + \dots \right) \\ &= e^a \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!} \end{aligned}$$

Where $L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} \right| = \lim_{n \rightarrow \infty} \left| \frac{n!}{(1+n)!} \right| = \lim_{n \rightarrow \infty} \frac{n!}{n!(1+n)} = \lim_{n \rightarrow \infty} \frac{1}{1+n} = 0$.

Hence $R = \frac{1}{L} = \infty$. Converges everywhere.

8 Problem 7

Show that $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} z^n$ converges uniformly in $|z| \leq 3$

Solution:

To find if it converges uniformly for $|z| \leq 3$, we need to find R , the radius of converges using normal

method, then it $R > 3$, then it will converge uniformly for $|z| \leq 3$.

$$\begin{aligned}
 L &= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| \\
 &= \lim_{n \rightarrow \infty} \left| \frac{\frac{((n+1)!)^2}{(2(n+1))!}}{\frac{(n!)^2}{(2n)!}} \right| \\
 &= \lim_{n \rightarrow \infty} \left| \frac{((n+1)!)^2 (2n)!}{(n!)^2 (2(n+1))!} \right| \\
 &= \lim_{n \rightarrow \infty} \left| \frac{((n+1)n!)^2 (2n)!}{(n!)^2 (2(n+1))!} \right| \\
 &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2 (2n)!}{(2n+2)!} \right| \\
 &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2 (2n)!}{(2n+2)(2n+1)(2n)!} \right| \\
 &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)^2}{(2n+2)(2n+1)} \right| \\
 &= \lim_{n \rightarrow \infty} \left| \frac{n^2 + 2n + 1}{4n^2 + 6n + 2} \right| \\
 &= \lim_{n \rightarrow \infty} \left| \frac{1 + \frac{2}{n} + \frac{1}{n^2}}{4 + \frac{6}{n} + \frac{2}{n^2}} \right| \\
 &= \frac{1}{4}
 \end{aligned}$$

Hence Radius of convergence $R = 4$. Since $3 < 4$, then it converges uniformly for $R < 3$.

9 Problem 8

Where does $\sum_{n=1}^{\infty} \left(\frac{n+2}{5n-3}\right)^n z^n$ converges uniformly?

Solution We first find R . Since the series of the form $\sum_{n=1}^{\infty} A^n z^n$ then it is easier to use

$$\begin{aligned}
 L &= \lim_{n \rightarrow \infty} \sqrt[n]{|A^n|} \\
 &= \lim_{n \rightarrow \infty} \sqrt[n]{|A^n|} \\
 &= \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{n+2}{5n-3} \right|^n} \\
 &= \lim_{n \rightarrow \infty} \sqrt[n]{\left| \frac{1 + \frac{2}{n}}{5 - \frac{3}{n}} \right|^n} \\
 &= \frac{1}{5}
 \end{aligned}$$

Hence $R = 5$. Therefore it converges uniformly for $|z| \leq r < 5$

10 key solution

-- Solutions:

$$\text{II} \quad \textcircled{a} \quad \sum_{n=1}^{\infty} n (z + i\sqrt{2})^n = \sum_{n=1}^{\infty} n (z - (-i\sqrt{2}))^n \quad a_n = n$$

$$\text{center } z_0 = -i\sqrt{2} \quad R = \frac{1}{L^*} = 1$$

$$L^* = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{n+1}{n} = 1$$

$$\textcircled{d} \quad \sum_{n=0}^{\infty} \frac{1}{(1+i)^n} (z - (i-2))^n \quad a_n = \frac{1}{(1+i)^n} \quad z_0 = i-2$$

$$L^* = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(1+i)^n}{(1+i)^{n+1}} \right| = \frac{1}{|1+i|} = \frac{1}{\sqrt{2}} ; R = \sqrt{2}$$

$$\textcircled{c} \quad \sum_{n=0}^{\infty} \frac{(3n)!}{2^n (n!)^3} z^n \quad \text{center } z_0 = 0 \quad a_n = \frac{(3n)!}{2^n (n!)^3}$$

$$L^* = \lim_{n \rightarrow \infty} \frac{\frac{(3n+3)!}{2^{n+1} (n+1)!^3}}{\frac{(3n)!}{2^n (n!)^3}} = \lim_{n \rightarrow \infty} \frac{\cancel{3n} (3n+1)(3n+2)(3n+3) \cancel{2^n} (n!)^3}{\cancel{2^n} 3n \cdot 2^{n+1} [(n+1)!]^3}$$

$$= \lim_{n \rightarrow \infty} \frac{(3n+1)(3n+2)(3n+3)}{2 \cdot (n+1)(n+1)(n+1)} = \frac{3^3}{2} = \frac{27}{2}$$

$$R = \frac{1}{L^*} = \frac{2}{27}$$

$$\textcircled{b} \quad L^* = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{\left(\frac{a}{b}\right)^{n+1}}{\left(\frac{a}{b}\right)^n} = \frac{a}{b} ; R = \frac{1}{L^*} = \frac{b}{a}$$

2

(a) $\sum_{n=1}^{\infty} \frac{6^n}{n} (z-i)^n$ $a_n = \frac{6^n}{n}$

1) $L^x = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{6^{n+1}}{6^n} \frac{n}{n+1} = 6 \cdot \lim_{n \rightarrow \infty} \frac{n}{n+1} = 6$; $R = \frac{1}{6}$

2) $\sum_{n=1}^{\infty} \frac{6^n}{n} (z-i)^n$ termwise differentiated gives

$$6 \sum_{n=1}^{\infty} 6^{n-1} (z-i)^{n-1} = 6 \sum_{m=0}^{\infty} 6^m (z-i)^m = 6 \cdot \frac{1}{1-6(z-i)}$$

for $|6(z-i)| < 1$ or $|z-i| < \frac{1}{6}$
convergent

for $|z-i| > \frac{1}{6}$ divergent.

Termwise differentiated series has same rad. of conv.

(b) $\sum_{n=0}^{\infty} \frac{3^n (n+1)n}{5^n} z^{2n}$ $R = \frac{1}{6}$

1) the coefficient of series are

$a_0 = 0$ $a_1 = 0$

$a_2 = \frac{3^1 (1+1)1}{5^1}$ $a_3 = 0$

$a_4 = \frac{3^2 (2+1)2}{5^2}$ $a_5 = 0$

\vdots \vdots

$\lim_{n \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right|$ or $\lim_{n \rightarrow \infty} \sqrt[k]{|a_k|}$ does not exist.

z^k k odd: $\lim_{n \rightarrow \infty} \sqrt[k]{|a_k|} = 0$

k even: $\lim_{n \rightarrow \infty} \sqrt[k]{|a_k|} = \lim_{n \rightarrow \infty} \sqrt[2n]{\frac{3^n (n+1)n}{5^n}} = \lim_{n \rightarrow \infty} \sqrt[2]{\sqrt[n]{\frac{3^n}{5^n} (n+1)n}} =$

$\left(\lim_{n \rightarrow \infty} \frac{3}{5} \sqrt[n]{n} \sqrt[n]{n+1} \right)^{1/2} = \sqrt{3/5}$

$\sqrt[k]{|a_k|}$ has two cluster points 0 and $\sqrt{3/5}$

$\lim_{n \rightarrow \infty} \sqrt[k]{|a_k|} = \sqrt{3/5}$ $R = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[k]{|a_k|}} = \sqrt{5/3}$

2) cont.

2) Let $v = z^2$ new complex variable
in terms of v the series can be written as

$$\sum_{n=0}^{\infty} \frac{3^n (n+1)n}{5^n} v^n = v \sum_{n=0}^{\infty} \frac{3^n}{5^n} (n+1)n v^{n-1} \quad R_v$$

everywhere convergent
integrate termwise w.r.t. v
(same radius of conv)

$$\sum_{n=0}^{\infty} \frac{3^n}{5^n} (n+1) v^n \quad R_v$$

integrate w.r.t. v
(same R)

$$\sum_{n=0}^{\infty} \frac{3^n}{5^n} v^{n+1} = v \sum_{n=0}^{\infty} \frac{3^n}{5^n} v^n$$

convergent geom series
everywhere

$$\sum_{n=0}^{\infty} \frac{3^n}{5^n} v^n \quad \text{convergent when}$$

$$|\frac{3}{5}v| < 1 \quad |v| < \frac{5}{3}$$

$$R_v = \frac{5}{3}$$

thus
$$\sum_{n=0}^{\infty} \frac{3^n}{5^n} (n+1)n z^{2n} = \sum_{n=0}^{\infty} \frac{3^n}{5^n} (n+1)n \cdot v^n$$

converges when $|z^2| = |v| < R_v = \frac{5}{3}$

or $|z| < \frac{\sqrt{5}}{3} \quad |z| < \sqrt{\frac{5}{3}} \quad \left(R = \sqrt{\frac{5}{3}} \right)$

3) a)
$$\frac{1}{(1-z)^2} = \frac{1}{1-z} \cdot \frac{1}{1-z} = \left(\sum_{n=0}^{\infty} z^n \right) \cdot \left(\sum_{n=0}^{\infty} z^n \right) \quad \text{if } |z| < 1$$

$$= (1+z+z^2+\dots)(1+z+z^2+\dots) = \sum_{n=0}^{\infty} (a_n b_n + a_n b_{n-1} + \dots + a_{n-1} b_n + a_{n-1} b_{n-1}) z^n$$

$a_n = 1 \quad b_n = 1$
 $\begin{matrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{matrix}$

$$= \sum_{n=0}^{\infty} (n+1) z^n \quad (\text{if } |z| < 1)$$

$$b) \frac{1}{(1-z)^2} = \frac{d}{dz} \frac{1}{1-z} = \frac{d}{dz} \left(\sum_{n=0}^{\infty} z^n \right) \quad \text{if } |z| < 1$$

termwise derivative of series converges to derivative:

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} n z^{n-1} = \sum_{n=1}^{\infty} n z^{n-1} = \sum_{l=0}^{\infty} \underline{(l+1)} z^l$$

$l = n-1$
 $n = l+1$

4) Let $f(z)$ be even: $f(-z) = f(z)$ for all z in $|z| < R$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ Show that $a_n = 0$ when n is odd.

$$f(-z) = \sum_{n=0}^{\infty} a_n (-z)^n = \sum_{n=0}^{\infty} a_n (-1)^n z^n$$

$$0 = f(z) - f(-z) = \sum_{n=0}^{\infty} a_n z^n - \sum_{n=0}^{\infty} a_n (-1)^n z^n = \sum_{n=0}^{\infty} \underbrace{a_n (1 - (-1)^n)}_{\substack{0 \text{ if } n \text{ is odd} \\ 2 \text{ if } n \text{ is even}}} z^n$$

thus

$$0 = \sum_{n=0}^{\infty} 0 z^n$$

for all z

$$0 = 2a_0 + 2a_2 z^2 + 2a_4 z^4 + \dots$$

for $|z| < R$

By theorem 2 in $|z| < R$ / where both series converge / the coefficients of the two series have to coincide.

Thus $2a_0 = 0, 2a_2 = 0, 2a_4 = 0, \dots$

or $a_0 = 0, a_2 = 0, a_4 = 0, \dots$

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$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots \quad \text{for any } z \in \mathbb{C}$$

(a)

$$\cos(2z^2) = 1 - \frac{(2z^2)^2}{2!} + \frac{(2z^2)^4}{4!} - \frac{(2z^2)^6}{6!} + \dots$$

$$= 1 - \frac{2^2 z^4}{2!} + \frac{2^4 z^8}{4!} - \frac{2^6 z^{12}}{6!} + \dots \quad \text{for any } 2z^2 \in \mathbb{C}$$

or $z \in \mathbb{C}$

$$= \sum_{n=0}^{\infty} \frac{4^n z^{4n}}{2n!} \cdot (-1)^n$$

radius of conv. $R = \infty$

(b)

$$\frac{z+2}{1-z^2} = \frac{z+2}{(1-z)(z+1)} = \frac{A}{1-z} + \frac{B}{z+1} = \frac{Az+A-Bz+B}{(1-z)(z+1)}$$

$$\Rightarrow A-B=1 \quad \Rightarrow A=3/2$$

$$A+B=2$$

$$B=1/2$$

$$= \frac{3}{2} \frac{1}{1-z} + \frac{1}{2} \frac{1}{z+1}$$

$$\frac{3}{2} \frac{1}{1-z} = \frac{3}{2} [1 + z + z^2 + z^3 + \dots] \quad \text{for } |z| < 1$$

$$\frac{1}{2} \frac{1}{1+z} = \frac{1}{2} \left[\frac{1}{1-(-z)} \right] = \frac{1}{2} [1 - z + z^2 - z^3 + \dots] \quad \text{if } |-z|=|z| < 1$$

$$\frac{z+2}{1-z^2} = \left[\frac{3}{2} + \frac{1}{2} \right] + \left(\frac{3}{2} - \frac{1}{2} \right) z + \left(\frac{3}{2} + \frac{1}{2} \right) z^2 + \left(\frac{3}{2} - \frac{1}{2} \right) z^3 + \dots$$

$$= 2 + z + 2z^2 + z^3 + 2z^4 + z^5$$

($R=1$)

$$\sum_{h=0}^{\infty} \left(\frac{3}{2} + \frac{1}{2} (-1)^h \right) z^h$$

if $|z| < 1$

6

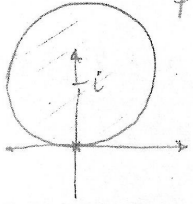
a

$$f(z) = \frac{1}{z} \quad z_0 = i$$

$$f'(z) = -\frac{1}{z^2}, \quad f''(z) = \frac{2 \cdot 1}{z^3},$$

$$f'''(z) = -\frac{3 \cdot 2 \cdot 1}{z^4}$$

$$f^{(n)}(z) = \frac{(-1)^n n!}{z^{n+1}}$$



$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n =$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(i)^{n+1}} (z-i)^n = -i + (z-i) - i(z-i)^2 + (z-i)^3 - i(z-i)^4 + (z-i)^5 - \dots$$

$$|z-i| < 1$$

b

$$f(z) = e^z \quad z_0 = a$$

$$f^{(n)} = e^z$$

$$f(z) = \sum_{n=0}^{\infty} \frac{e^a}{n!} (z-a)^n = e^a \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!}$$

$$\sqrt[n]{\frac{e^a}{n!}} \rightarrow 0$$

$$R = \infty$$

$$= e^a + e^a(z-a) + \frac{e^a}{2!}(z-a)^2 + \frac{e^a}{3!}(z-a)^3 + \dots$$

converges for all z

Power series $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} z^n$ is uniformly convergent in $|z| < r < R$ where R is radius of convergence

7

We need to show that $R > 3$.

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)! \cdot (n+1)!}{(2n+2)!} \cdot \frac{2n!}{n! \cdot n!} = \frac{(n+1)^2}{(2n+1)(2n+2)} \rightarrow \frac{1}{4} = L^*$$

$$R = \frac{1}{L^*} = 4$$

8

$$\sum_{n=1}^{\infty} \left[\frac{n+2}{5n-3} \right]^n z^n$$

a_n $z_0 = 0$

$$R = \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{|a_n|}} = \lim_{n \rightarrow \infty} \frac{1}{\frac{n+2}{5n-3}} = 5$$

Series converges uniformly in any disk $|z| < r < 5$

1 a) Verify that $y(x) = -\sin x + ax^2 + bx + c$ for any a, b, c constants is a solution to $y''' = \cos x$

b) Show that $y(x) = \tan(x+c)$ solves $y' = 1+y^2$ for any constant c

2 Show that the given function solves the specified initial value problem

a) $y(x) = ce^{x/2}$ $\begin{cases} y' = \frac{1}{2}y \\ y(2) = 2 \end{cases}$

b) $y(x) = ce^{-x^2}$ $\begin{cases} y' + 2xy = 0 \\ y(1) = 1/e \end{cases}$

3 $y = cx - c^2$ is a solution to the ODE $(y')^2 - xy' + y = 0$ for any c const.

Find a singular solution to the ODE (not given by $y = cx - c^2$) by rewriting the ODE using the quadratic formula.

4 Find all solutions of the following differential equations.

a) $yy' + 25x = 0$

b) $y' = ky^2$

c) $xy' = x + y$ (Hint $u = y/x$)

5 Solve the IVPs:

a) $\begin{cases} y' = 1 + 4y^2 \\ y(0) = 0 \end{cases}$

b) $\begin{cases} y' = -x/y \\ y(1) = \sqrt{3} \end{cases}$

c) $\begin{cases} e^x y' = 2(x+1)y^2 \\ y(0) = 1/6 \end{cases}$