HW 2, Math 601
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Contents

1 Problem set

PROBLEM SET 15.1

1–10 SEQUENCES

Are the following sequences $z_1, z_2, \cdots, z_n, \cdots$ bounded? Convergent? Find their limit points. (Show the details of your work.)

1. $z_n = (-1)^n + i/2^n$
2. $z_n = e^{-n\pi i/4}$
3. $z_n = (-1)^n/(n + i)$
4. $z_n = (1 + i)^n$
5. $z_n = \text{Ln}\,((2 + i)^n)$
6. $z_n = (3 + 4i)^n/n!$
7. $z_n = \sin(n\pi/4) + i^n$
8. $z_n = \left[(1 + 3i)/\sqrt{10}\right]^n$
9. $z_n = (0.9 + 0.1i)^{2n}$
10. $z_n = (5 + 5i)^{-n}$

11. Illustrate Theorem 1 by an example of your own.

12. **(Uniqueness of limit)** Show that if a sequence converges, its limit is unique.

13. **(Addition)** If $z_1, z_2, \cdots$ converges with the limit l and $z_1^*, z_2^*, \cdots$ converges with the limit l^*, show that $z_1 + z_1^*, z_2 + z_2^*, \cdots$ converges with the limit $l + l^*$.

14. **(Multiplication)** Show that under the assumptions of Prob. 13 the sequence $z_1z_1^*, z_2z_2^*, \cdots$ converges with the limit ll^*.

15. **(Boundedness)** Show that a complex sequence is bounded if and only if the two corresponding sequences of the real parts and of the imaginary parts are bounded.

16–24 SERIES

Are the following series convergent or divergent? (Give a reason.)

16. $\displaystyle\sum_{n=0}^{\infty} \frac{(10 - 15i)^n}{n!}$

17. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(1 + 2i)^{2n+1}}{(2n + 1)!}$

18. $\displaystyle\sum_{n=0}^{\infty} \frac{i^n}{n^2 - 2i}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

20. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{i^n}{n}$

22. $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^3}{(3n)!}(1 + i)^n$

23. $\displaystyle\sum_{n=0}^{\infty} \frac{n - i}{3n + 2i}$

24. $\displaystyle\sum_{n=1}^{\infty} n^2\left(\frac{i}{3}\right)^n$

25. What is the difference between (7) and just stating $|z_{n+1}/z_n| < 1$?

26. Illustrate Theorem 2 by an example of your choice.

27. For what n do we obtain the term of greatest absolute value of the series in Example 4? About how big is it? First guess, then calculate it by the Stirling formula in Sec. 24.4.

28. Give another example showing that Theorem 7 is more general than Theorem 8.

29. **CAS PROJECT. Sequences and Series. (a)** Write a program for graphing complex **sequences.** Apply it to sequences of your choice that have interesting "geometrical" properties (e.g., lying on an ellipse, spiraling toward its limit, etc.).

(b) Write a program for computing and graphing numeric values of the first n partial sums of a **series** of complex numbers. Use the program to experiment with the rapidity of convergence of series of your choice.

30. **TEAM PROJECT. Series. (a) Absolute convergence.** Show that if a series converges absolutely, it is convergent.

(b) Write a short report on the basic concepts and properties of series of numbers, explaining in each case whether or not they carry over from real series (discussed in calculus) to complex series, with reasons given.

PROBLEM SET 15.2

1. **(Powers missing)** Show that if $\Sigma\, a_n z^n$ has radius of convergence R (assumed finite), then $\Sigma\, a_n z^{2n}$ has radius of convergence $\sqrt{R}$. Give examples.
2. **(Convergence behavior)** Illustrate the facts shown by Examples 1–3 by further examples of your own.

3–18 RADIUS OF CONVERGENCE

Find the center and the radius of convergence of the following power series. (Show the details.)

3. $\sum_{n=1}^{\infty} \frac{(z+i)^n}{n^2}$

4. $\sum_{n=0}^{\infty} \frac{n^n}{n!}(z+2i)^n$

5. $\sum_{n=0}^{\infty} \frac{n!}{n^n}(z+1)^n$

6. $\sum_{n=0}^{\infty} \frac{2^{100n}}{n!} z^n$

7. $\sum_{n=0}^{\infty} \left(\frac{a}{b}\right)^n z^n$

8. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} z^{2n}$

9. $\sum_{n=0}^{\infty} (n-i)^n z^n$

10. $\sum_{n=0}^{\infty} \frac{(2z)^{2n}}{(2n)!}$

11. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n$

12. $\sum_{n=0}^{\infty} \frac{4^n}{(1+i)^n}(z-5)^n$

13. $\sum_{n=2}^{\infty} n(n-1)(z-3+2i)^n$

14. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$

15. $\sum_{n=0}^{\infty} 2^n (z-i)^{4n}$

16. $\sum_{n=0}^{\infty} \left(\frac{2+3i}{5-i}\right)^n (z-\pi)^n$

17. $\sum_{n=0}^{\infty} \frac{n^4}{2^n} z^{2n}$

18. $\sum_{n=0}^{\infty} \frac{(4n)!}{2^n (n!)^4}(z+\pi i)^n$

2 Problem 2, section 15.1

Is sequence $z_n = e^{-\frac{n\pi i}{4}}$ bounded? convergent? Find their limit points.

Solution

Sequence is bounded, since each element has modulus 1. It does not converge, since sequence repeats. $2\pi = \frac{n\pi}{4}$, hence $n = 8$. So only 8 elements are unique. Each of these is limit point. These are roots of $\sqrt[8]{1}$.

3 Problem 6, section 15.1

Is sequence $z_n = \frac{(3+4i)^n}{n!}$ bounded? convergent? Find their limit points.
Solution

$$z_n = \frac{\left(re^{i\theta_0}\right)^n}{n!}$$

But $r = 5$ and $\theta_0 = \arctan\left(\frac{4}{3}\right)$. The above becomes

$$\begin{aligned} z_n &= \frac{5^n e^{in\theta_0}}{n!} \\ &= \frac{5^n}{n!} e^{in\theta_0} \end{aligned}$$

Since modulus of $e^{in\theta_0} = 1$, then we just need to look at $\frac{5^n}{n!}$ to see if it is bounded or not. $\lim_{n\to\infty} \frac{5^n}{n!} = 0$. So it is bounded. Since n^{th} term goes to zero as $n \longrightarrow \infty$ it converges. The terms are $\frac{5^n}{n!} (\cos n\theta_0 + i \sin n\theta_0)$. It converges to zero, since $\lim_{n\to\infty} \frac{5^n}{n!} = 0$.

4 Problem 13, section 15.1

If $z_1, z_2, \cdots$ converges to L, and $\bar{z}_1, \bar{z}_2, \cdots$ converges to $\bar{L}$, show that $z_1 + \bar{z}_1, z_2 + \bar{z}_2, \cdots$ converges to $L + \bar{L}$
Solution
This problem seems to be based on the idea that if sequence is convergent to L, then for any ε no matter how small we can find an n, such that $|z_n - L| < \varepsilon$. So let us pick

$$\begin{aligned} |z_n - L| &< \frac{1}{2}\varepsilon \\ \left|\bar{z}_n - \bar{L}\right| &< \frac{1}{2}\varepsilon \end{aligned}$$

Where in the above, we did the same for the other sequence. Now by triangle inequality $|A + B| \leq |A| + |B|$, where now we treat A as $(z_n - L)$ and B as $\left(\bar{z}_n - \bar{L}\right)$, we have

$$\begin{aligned} \left|(z_n - L) + \left(\bar{z}_n - \bar{L}\right)\right| &\leq |z_n - L| + \left|\bar{z}_n - \bar{L}\right| \\ \left|(z_n + \bar{z}_n) - \left(L + \bar{L}\right)\right| &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon \end{aligned}$$

The above is $\left|(z_n + \bar{z}_n) - \left(L + \bar{L}\right)\right| < \varepsilon$. But this is the definition of a limit. It says that $(z_n + \bar{z}_n)$ has limit $L + \bar{L}$, which is what we are asked to show.

5 Problem 18, section 15.1

Are the following series convergent or divergent? Give a reason. $\sum_{n=0}^{\infty} \frac{i^n}{n^2 - 2i}$
Solution
The numerator has modulus 1. So we just need to consider $\sum_{n=0}^{\infty} \frac{1}{|n^2-2i|}$. Since $\frac{1}{n^2}$ converges and since $\left|n^2 - 2i\right| > n^2$ (vectors, Argand diagram), then $\frac{1}{|n^2-2i|} < \frac{1}{n^2}$, therefore it converges. We could also use the ratio test, but this is simpler.

6 Problem 19, section 15.1

Are the following series convergent or divergent? Give a reason. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$
Solution
Since terms are $\frac{1}{n^{\alpha}}$ where $|\alpha| < 1$, since $\alpha = \frac{1}{2}$ here. Then we know it is divergent. It series becomes convergent for $\alpha > 1$. To show this, we can try the ratio test. But this gives the limit of 1, so ratio test is inconclusive. Using the integral test is best here. (notice that only upper limit is needed in this test, no need to use lower limit). We can use the integral test because the terms $\frac{1}{\sqrt{n}}$ are monotonically decreasing.

$$\begin{aligned}
\lim_{N\to\infty} \int^{N} \frac{1}{x^{\frac{1}{2}}} dx &= \lim_{N\to\infty} \left(2\sqrt{x}\right)^{N} \\
&= \lim_{N\to\infty} 2\sqrt{N} \\
&= \infty
\end{aligned}$$

Hence diverges.

7 Problem 24, section 15.1

Are the following series convergent or divergent? Give a reason. $\sum_{n=1}^{\infty} n^2 \left(\frac{i}{3}\right)^n$
Solution
Trying ratio test gives

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(n+1)^2}{n^2} \frac{\left(\frac{i}{3}\right)^{n+1}}{\left(\frac{i}{3}\right)^{n}}\right| \\
&= \lim_{n\to\infty} \left|\frac{(n+1)^2}{n^2}\right| \left|\frac{\left(\frac{i}{3}\right)^{n+1}}{\left(\frac{i}{3}\right)^{n}}\right| \\
&= \lim_{n\to\infty} \left|\frac{\left(\frac{i}{3}\right)^{n+1}}{\left(\frac{i}{3}\right)^{n}}\right| \\
&= \lim_{n\to\infty} \left|\frac{i^{n+1}3^n}{i^n 3^{n+1}}\right| \\
&= \lim_{n\to\infty} \left|\frac{i}{3}\right| \\
&= \frac{1}{3}
\end{aligned}$$

Since limit is smaller than 1, then converges.

8 Problem 7, section 15.2

Find center and radius of convergence of series $\sum_{n=0}^{\infty} \left(\frac{a}{b}\right)^n z^n$
Solution
For these type of problem, always compare it to standard form $\sum_{n=0}^{\infty} A_n (z - z_0)^n$. Where z_0 is the center of disk. So we see that here z_0 is the origin. Now to find R (the radius of convergence), it is given by the

inverse of $L = \lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right|$. Therefore we start by finding L

$$\begin{aligned} L &= \lim_{n\to\infty}\left|\frac{\left(\frac{a}{b}\right)^{n+1}}{\left(\frac{a}{b}\right)^{n}}\right| \\ &= \lim_{n\to\infty}\left|\frac{a^{n+1}b^n}{a^n b^{+1}}\right| \\ &= \lim_{n\to\infty}\left|\frac{a}{b}\right| \\ &= \left|\frac{a}{b}\right| \end{aligned}$$

Hence $R = \left|\frac{b}{a}\right|$

9 Problem 9, section 15.2

Find center and radius of convergence of series $\sum_{n=0}^{\infty}(n-i)^n z^n$
Solution
The center is $z_0 = 0$ by comparing to $\sum_{n=0}^{\infty}A_n(z-z_0)^n$. To find L

$$\begin{aligned} L &= \lim_{n\to\infty}\left|\frac{(n-i)^{n+1}}{(n-i)^n}\right| \\ &= 1 \end{aligned}$$

Hence $R = 1$.

10 Problem 11, section 15.2

Find center and radius of convergence of series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}z^n$
Solution
The center is $z_0 = 0$ by comparing to $\sum_{n=0}^{\infty}A_n(z-z_0)^n$. To find L

$$\begin{aligned} L &= \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}}{n+2}}{\frac{(-1)^{n+1}}{n}}\right| \\ &= \lim_{n\to\infty}\left|\frac{(-1)^{n+2}\,n}{(-1)^n\,(n+2)}\right| \\ &= \lim_{n\to\infty}\left|\frac{n}{(n+2)}\right| \\ &= 1 \end{aligned}$$

Hence $R = 1$.

11 Problem 12, section 15.2

Find center and radius of convergence of series $\sum_{n=1}^{\infty}\frac{4^n}{(1+i)^n}(z-5)^n$
Solution

The center is $z_0 = 5$ by comparing to $\sum_{n=0}^{\infty} A_n (z - z_0)^n$. To find L

$$\begin{aligned}
L &= \lim_{n\to\infty} \left| \frac{\frac{4^{n+1}}{(1+i)^{n+1}}}{\frac{4^n}{(1+i)^n}} \right| \\
&= \lim_{n\to\infty} \left| \frac{4^{n+1}(1+i)^n}{4^n (1+i)^{n+1}} \right| \\
&= \lim_{n\to\infty} \left| \frac{4(1+i)^n}{(1+i)^{n+1}} \right| \\
&= \lim_{n\to\infty} \left| \frac{4}{1+i} \right| \\
&= \lim_{n\to\infty} \frac{4}{|1+i|} \\
&= \lim_{n\to\infty} \frac{4}{\sqrt{2}}
\end{aligned}$$

Hence

$$R = \frac{\sqrt{2}}{4}$$

12 Problem 18, section 15.2

Find center and radius of convergence of series $\sum_{n=1}^{\infty} \frac{(4n)!}{2^n (n!)^4} (z + \pi i)^n$

<u>Solution</u>

The center is $z_0 = -\pi i$ by comparing to $\sum_{n=0}^{\infty} A_n (z - z_0)^n$. To find L

$$\begin{aligned}
L &= \lim_{n\to\infty} \left| \frac{\frac{(4(n+1))!}{2^{(n+1)}((n+1)!)^4}}{\frac{(4n)!}{2^n (n!)^4}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(4(n+1))! 2^n (n!)^4}{(4n)! 2^{(n+1)} ((n+1)!)^4} \right| \\
&= \frac{1}{2} \lim_{n\to\infty} \left| \frac{(4(n+1))! (n!)^4}{(4n)! ((n+1)!)^4} \right| \\
&= \frac{1}{2} \lim_{n\to\infty} \left| \frac{(4n+4)! (n!)^4}{(4n)! ((n+1)!)^4} \right| \\
&= \frac{1}{2} \lim_{n\to\infty} \left| \frac{(4n+4)(4n+3)(4n+2)(4n+1)(4n)! (n!)^4}{(4n)! ((n+1)!)^4} \right| \\
&= \frac{1}{2} \lim_{n\to\infty} \left| \frac{(4n+4)(4n+3)(4n+2)(4n+1)(n!)^4}{((n+1)!)^4} \right| \\
&= \frac{1}{2} \lim_{n\to\infty} \left| \frac{(4n+4)(4n+3)(4n+2)(4n+1)(n!)^4}{((n+1) n!)^4} \right| \\
&= \frac{1}{2} \lim_{n\to\infty} \left| \frac{(4n+4)(4n+3)(4n+2)(4n+1)}{(n+1)^4} \right| \\
&= \frac{1}{2} \lim_{n\to\infty} \left| \frac{256n^4 + 640n^3 + 560n^2 + 200n + 24}{n^4 + 4n^3 + 6n^2 + 4n + 1} \right|
\end{aligned}$$

Hence

$$\begin{aligned} L &= \frac{1}{2} \lim_{n\to\infty} \left| \frac{256 + 640\frac{1}{n} + 560\frac{1}{n^2} + 200\frac{1}{n^3} + \frac{24}{n^4}}{1 + 4\frac{1}{n} + 6\frac{1}{n^2} + 4\frac{1}{n^3} + \frac{1}{n^4}} \right| \\ &= \frac{1}{2}(256) \\ &= 128 \end{aligned}$$

Hence

$$R = \frac{1}{128}$$

13 key solution