Key solution.

Math 601

$\boxed{13.1}$ $\boxed{1}$ By definition

$$i^1 = i$$
$$i^2 = i\cdot i = -1$$
$$i^3 = i^2\cdot i = -i$$
$$i^4 = i^2\cdot i^2 = (-1)(-1) = 1$$

the sequence $\{i, -1, -i, 1\}$ repeats.

$$i^5 = i^4\cdot i = i$$
$$i^6 = i^4\cdot i^2 = -1$$
$$i^7 = i^4\cdot i^3 = -i$$
$$i^8 = i^4\cdot i^4 = 1$$
$$\vdots$$

$\frac{1}{i} = -i$ since $1 = -i\cdot i$

$$\frac{1}{i^2} = \frac{1}{-1} = -1$$
$$1/i^3 = \frac{1}{i^2}\cdot\frac{1}{i} = -1\cdot(-i) = i$$
$$1/i^4 = \frac{1}{1} = 1$$

$\{-i, -1, i, 1\}$ repeats

$$\frac{1}{i^5} = \frac{1}{i^4}\cdot\frac{1}{i} = -i$$
$$\frac{1}{i^6} = \frac{1}{i^4}\cdot\frac{1}{i^2} = -1$$
$$\frac{1}{i^7} = \frac{1}{i^4}\cdot\frac{1}{i^3} = i$$
$$\frac{1}{i^8} = \frac{1}{i^4}\cdot\frac{1}{i^4} = 1$$

$\boxed{7}$ $z_1 = 2+3i$, $z_2 = 4-5i$

$$(5z_1+3z_2)^2 = (5(2+3i)+3(4-5i))^2 =$$
$$= (10+\cancel{15i}+12-\cancel{15i})^2 = 22^2 = 484$$

$\boxed{10}$ $z_2 = 4-5i$ $\qquad (\mathrm{Re}\, z_2)^2 = 4^2 = 16$

$$z_2^2 = (4-5i)(4-5i) \Rightarrow \mathrm{Re}(z_2^2) = -9$$
$$= 16-25-2\cdot 4\cdot 5i$$
$$= -9-40i$$

$\boxed{12}$
$$\frac{\bar{z}_1}{\bar{z}_2} = \frac{2-3i}{4+5i} = \frac{\bar{z}_1 z_2}{\bar{z}_2 z_2} = \frac{\bar{z}_1 z_2}{|z_2|^2} = \frac{-7-22i}{41} = -\frac{7}{41} - \frac{22}{41}i$$

$$|z_2|^2 = 4^2+5^2 = 41$$
$$\bar{z}_1 z_2 = (2-3i)(4-5i) = 8-15-(10+12)i = -7-22i$$

We know that $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2} = -\frac{7}{41} - \frac{22}{41}i$

$\boxed{13.2}$ $\boxed{8}$ polar form of $z = \dfrac{2+3i}{5+4i} = re^{i\theta} = r(\cos\theta + i\sin\theta)$

with $r = \sqrt{z \cdot \bar{z}}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$(z = x + iy)$

$$z = \frac{2+3i}{5+4i} = \frac{(2+3i)(5-4i)}{(5+4i)(5-4i)} = \frac{10+12+(15-8)i}{25+16} = \frac{22}{41} + \frac{7}{41}i = x + y \cdot i$$

thus $r = \sqrt{\left(\frac{22}{41}\right)^2 + \left(\frac{7}{41}\right)^2}$

and $\theta = \tan^{-1}\dfrac{7/41}{22/41} = \tan^{-1}\left(\frac{7}{22}\right)$

$\boxed{11}$

$z = x + iy$ $\quad z = 4 + 3i$ $\quad -\pi < \operatorname{Arg} z \leq \pi$ and $\tan \operatorname{Arg} z = \dfrac{y}{x} = \dfrac{3}{4}$

$\bar{z} = x - iy$ $\quad \bar{z} = 4 - 3i$

$\operatorname{Arg} z = \tan^{-1}\frac{3}{4} = \boxed{0.64}$

$$\tan^{-1}\frac{-y}{x} = \operatorname{Arg}\bar{z} = -\operatorname{Arg} z = \tan^{-1}\frac{y}{x}$$

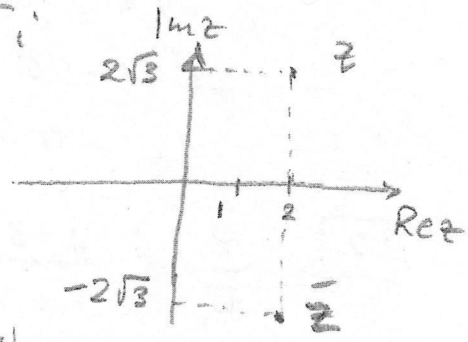
(and $\tan^{-1}$ is odd function)

$\boxed{18}$

$z = 4 \cdot \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = \frac{4}{2} + i\,\frac{4\sqrt{3}}{2} = 2 + 2\sqrt{3}\,i$

$\bar{z} = 4 \cdot \left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right) = 2 - 2\sqrt{3}\,i$

$\cos\frac{\pi}{3} = \frac{1}{2}$, $\quad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

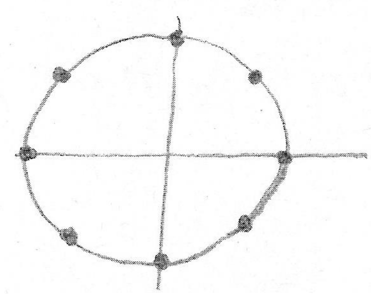


$\boxed{22}$ $\quad \sqrt[8]{1} = \sqrt[8]{r}\left(\cos\dfrac{\theta + 2k\pi}{8} + i\sin\dfrac{\theta + 2k\pi}{8}\right) =$

$k = 0, 1, 2, \ldots, 7$

$r = |1| = 1$

$\theta = \operatorname{Arg} 1 = 0$ $\quad = \cos\dfrac{k\pi}{4} + i\sin\dfrac{k\pi}{4}$



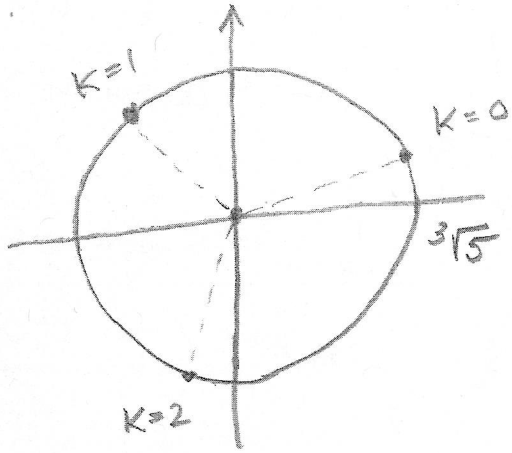
13.2 24 $\sqrt[3]{3+4i} = r^{1/3}\left(\cos\frac{\theta+2k\pi}{3} + i\sin\frac{\theta+2k\pi}{3}\right)$

$r = |3+4i| = \sqrt{9+16} = 5$

$\theta = \tan^{-1}\frac{4}{3} \approx 0.9273$

$= \sqrt[3]{5}\left[\cos\left(\frac{\theta}{3}+\frac{2\pi}{3}k\right) + i\sin\left(\frac{\theta}{3}+\frac{2\pi}{3}k\right)\right]$

$k = 0, 1, 2$



the three roots.

35 Show that $|z_1+z_2| \le |z_1| + |z_2|$

or equivalently

$|z_1+z_2|^2 \le |z_1|^2 + 2|z_1||z_2| + |z_2|^2$

Let $z_1 = x_1 + y_1 i$

$z_2 = x_2 + y_2 i$

$|z_1| = \sqrt{x_1^2+y_1^2}$

$|z_2| = \sqrt{x_2^2+y_2^2}$

$z_1+z_2 = x_1+x_2 + i(y_1+y_2)$

$|z_1+z_2|^2 = (x_1+x_2)^2 + (y_1+y_2)^2 = x_1^2+2x_1x_2+x_2^2 + y_1^2+2y_1y_2+y_2^2$

thus we need to show:

$\cancel{x_1^2} + 2x_1x_2 + \cancel{x_2^2} + \cancel{y_1^2} + 2y_1y_2 + \cancel{y_2^2} \le$

$\le \cancel{x_1^2} + \cancel{y_1^2} + \cancel{x_2^2} + \cancel{y_2^2} + 2\sqrt{x_1^2+y_1^2}\cdot\sqrt{x_2^2+y_2^2}$

or $x_1x_2 + y_1y_2 \le \sqrt{x_1^2+y_1^2}\sqrt{x_2^2+y_2^2}$

(Note: RHS non-negative) to show this, square each sides

$\cancel{x_1^2x_2^2} + 2x_1x_2y_1y_2 + \cancel{y_1^2y_2^2} \overset{?}{\le} (x_1^2+y_1^2)(x_2^2+y_2^2)$

$= \cancel{x_1^2x_2^2} + x_1^2y_2^2 + x_2^2y_1^2 + \cancel{y_1^2y_2^2}$

or

$0 \le x_1^2y_2^2 + x_2^2y_1^2 - 2x_1x_2y_1y_2 =$

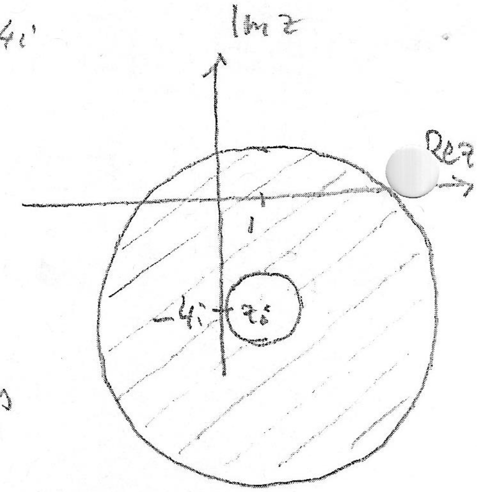
$= (x_1y_2 - x_2y_1)^2$ which is true for any $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

$\boxed{13.3}$ $\boxed{2}$ $1 \le |z-(1-4i)| \le 5$ take $z_0 = 1-4i$

$|z-z_0| = 1$: circle of radius 1 around z_0

$|z-z_0| = 5$ — " — radius 5

annulus



with $xy \Rightarrow y = \frac{1}{x}$ easier!

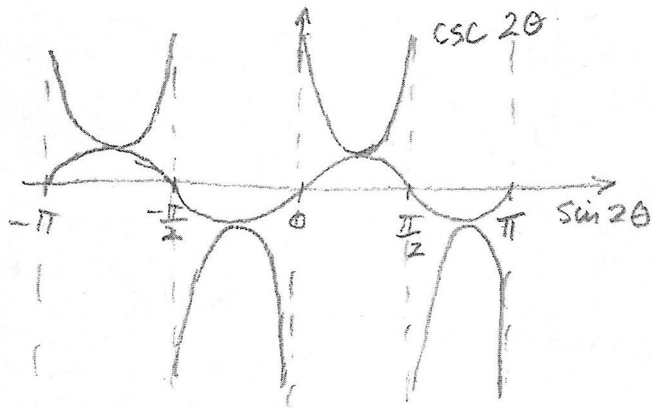
$\boxed{5}$ $\operatorname{Im} z^2 = 2$ $z = r\cdot e^{i\theta}$ $z^2 = r^2\cdot e^{i2\theta} =$

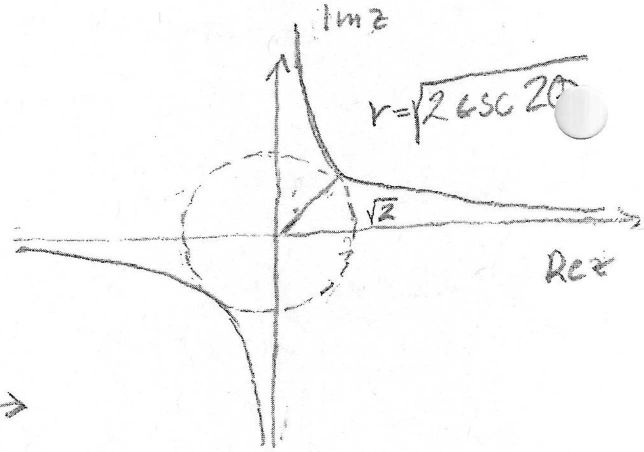
$r^2 \cos 2\theta + i\, r^2 \sin 2\theta$

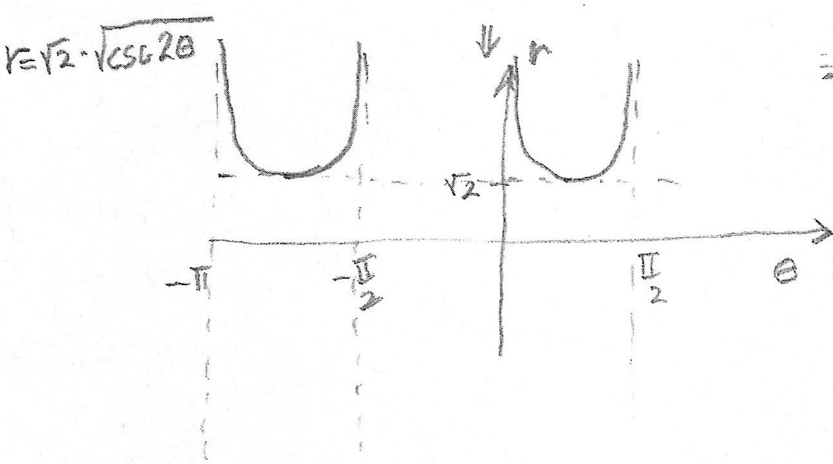
$\operatorname{Im} z^2 = r^2 \sin 2\theta$

$= 2$

$\Rightarrow r^2 = 2\csc 2\theta$





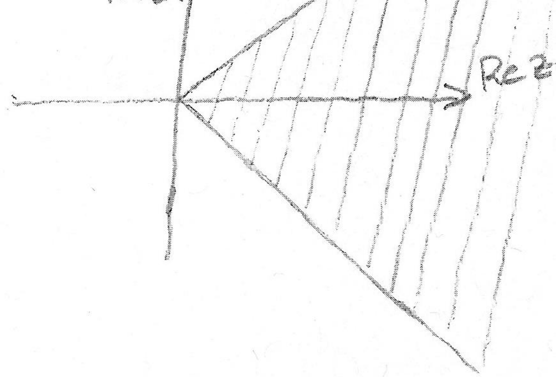


$\boxed{8}$ $|\operatorname{Arg} z| \le \frac{1}{4}\pi \Rightarrow -\frac{\pi}{4} \le \operatorname{Arg} z \le \frac{\pi}{4}$

$z = r\cdot e^{\operatorname{Arg} z}$

with

$-\pi < \operatorname{Arg} z \le \pi$



$\boxed{13.5}$ $\boxed{2}$ $e^{3+\pi i} = e^3 e^{\pi i} = e^3(\underbrace{\cos\pi}_{-1} + i\underbrace{\sin\pi}_{0}) = -e^3$

$\boxed{12}$ $z = r\cdot e^{i\theta} \Rightarrow z^{-1} = r^{-1}\cdot e^{-i\theta} = \frac{1}{r}(\cos(-\theta) + i\sin(-\theta)) =$

$= \frac{1}{r}(\cos\theta - i\sin\theta)$

$r = |z|$

$\theta = \arg z$

if $z = x+iy$

$|z| = \sqrt{x^2+y^2}$

$\arg z = \tan^{-1}\frac{y}{x}$

$e^{\frac{1}{z}} = e^{\frac{1}{r}\cos\theta - \frac{1}{r}\sin\theta\, i} =$

$= e^{\frac{1}{r}\cos\theta}\left(\cos(-\frac{1}{r}\sin\theta) + i\sin(-\frac{1}{r}\sin\theta)\right) =$

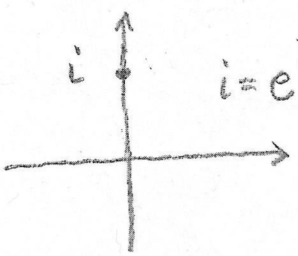
$= e^{\frac{1}{r}\cos\theta}\cdot\cos(\frac{1}{r}\sin\theta)$

$- i\, e^{\frac{1}{r}\cos\theta}\cos(\frac{1}{r}\sin\theta)$

Thus $\mathrm{Re}\, e^{\frac{1}{z}} = e^{1/r}\cos\theta\cdot\cos(\frac{1}{r}\sin\theta)$

$\mathrm{Im}\, e^{1/z} = e^{\frac{1}{r}\cos\theta}\cdot\cos(\frac{1}{r}\sin\theta)$

$\boxed{13}$ $\sqrt{i}$:



$i = e^{\frac{\pi}{2}i} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$|i| = 1$

$\mathrm{Arg}\, i = \frac{\pi}{2}$

thus $\sqrt{i} = \sqrt{1}\cdot\left(\cos\frac{\pi/2 + k\cdot 2\pi}{2} + i\sin\frac{\pi/2 + k\cdot 2\pi}{2}\right)$

with $k=0, k=1$

1, • either $\sqrt{i} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$
(k=0)

2, • or $\sqrt{i} = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$
k=1

i.e $\sqrt{i} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

or $\sqrt{i} = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

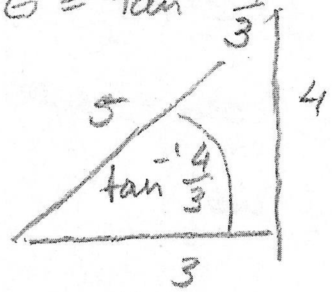
$\boxed{13.5}$ $\boxed{16}$

$$3+4i = r\cdot e^{i\theta} = 5\cdot\left(\cos\left(\tan^{-1}\frac{4}{3}\right) + i\sin\left(\tan^{-1}\frac{4}{3}\right)\right)$$

$$= 5\left(\frac{3}{5} + i\frac{4}{5}\right)$$

$r = \sqrt{3^2+4^2} = 5$

$\theta = \tan^{-1}\frac{4}{3}$



$\Rightarrow$ $\cos\left(\tan^{-1}\frac{4}{3}\right) = \frac{3}{5}$

$\sin\left(\tan^{-1}\frac{4}{3}\right) = \frac{4}{5}$

$\boxed{20}$ $e^z = e^{|z|}\cdot e^{\text{Arg}\, z\, i}$ $\Rightarrow |e^z| = |e^r\cdot e^{i\theta}| =$

$r = |z|$

$\theta = \text{Arg}\, z$

$\underbrace{|e^r|}_{>0}\cdot\underbrace{|\cos\theta + i\sin\theta|}_{=1}$

thus $|e^{|z|}| = 0$

so $e^{|z|} = 0$ has no solution.

* typo $(4-3i)$

$\boxed{21}$ $e^z = \underbrace{3-4i}_{x+yi}$ $\Rightarrow \text{Ln}\, e^z = z = \text{Ln}(3-4i) =$

$= \ln|3-4i| + i\,\text{Arg}\, z =$

$|3-4i| = 5$

$= \ln 5 - i\tan^{-1}\frac{4}{3}$

$\text{Arg}\, z = \tan^{-1}-\frac{4}{3}$

$\boxed{13.6}$ $\boxed{8}$ Compute as $u+iv$: $\sin(1+i)$

$z = 1+i$

$iz = -1+i$

$-iz = 1-i$

$|iz| = |-iz| = |z| = \sqrt{2}$

$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

$e^{iz} = e^{-1+i} = \frac{1}{e}(\cos 1 + i\sin 1)$

$e^{-iz} = e^{1-i} = e(\cos 1 - i\sin 1)$

$$\sin(1+i) = \frac{e^{iz} - e^{-iz}}{2i} =$$

$$= \frac{-i}{2}\left(\cos 1 \cdot \left(\frac{1}{e} - e\right) + i\sin 1 \cdot \left(\frac{1}{e} + e\right)\right)$$

$$= \underbrace{\frac{\sin 1}{2}\left(\frac{1}{e} + e\right)}_{u} - i\underbrace{\frac{\cos 1}{2}\left(\frac{1}{e} - e\right)}_{v} = u + iv$$

$\boxed{23}$ $$\cos(-z) = \frac{1}{2}\left(e^{i(-z)} + e^{-i(-z)}\right) = \frac{1}{2}\left(e^{-iz} + e^{iz}\right) = \cos z$$

$$\sin(-z) = \frac{1}{2i}\left(e^{i(-z)} - e^{-i(-z)}\right) = \frac{1}{2i}\left(e^{-iz} - e^{iz}\right) =$$

$$= -\frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = -\sin z$$

thus $\cos(z)$ is even

$\sin(z)$ is odd

$\boxed{13.7}$ $\boxed{14}$ See $\boxed{13.5}$ $\boxed{21}$

$\boxed{19}$ Solve $\ln z = 0.3 + 0.7i \Rightarrow z = e^{\ln z} = e^{\frac{3}{10} + \frac{7}{10}i} =$

$$= e^{3/10}\left(\cos\frac{7}{10} + i\sin\frac{7}{10}\right) =$$

$$e^{0.3}\cos 0.7 + i\, e^{3/10}\sin 0.7$$

$\boxed{13.7}$ $\boxed{25}$ and $\boxed{29}$

$$(1+i)^{(1-i)} = e^{\ln(1+i)^{(1-i)}} = e^{(1-i)\ln(1+i)}$$

$\ln(1+i) = \ln\sqrt{2} + i\frac{\pi}{4}$ thus $(1-i)(\ln\sqrt{2} + i\frac{\pi}{4}) =$

$\ln\sqrt{2} + \frac{\pi}{4} + i(\frac{\pi}{4} - \ln\sqrt{2})$

$|1+i| = \sqrt{2}$

$\mathrm{Arg}(1+i) = \tan^{-1} 1 = \frac{\pi}{4}$

So

$$(1+i)^{1-i} = e^{\ln\sqrt{2}+\frac{\pi}{4}} \cdot \cos\left(\frac{\pi}{4} - \ln\sqrt{2}\right) + i\, e^{\ln\sqrt{2}+\frac{\pi}{4}} \cdot \sin\left(\frac{\pi}{4} - \ln\sqrt{2}\right)$$

consider

$z^{\bar{z}}$ and $\bar{z}^{z}$

① ②

$$\overline{z^{\bar{z}}} = \overline{e^{\bar{z}\cdot\ln z}} = e^{\bar{\bar{z}}\cdot\overline{\ln z}} = e^{z} e^{\ln\bar{z}} = e^{z\cdot\ln\bar{z}} = \bar{z}^{z}$$

$\boxed{\text{note}}$ $\overline{e^{z}} = \overline{e^{x}(\cos y + i\sin y)} = e^{x}(\cos y - i\sin y) = e^{\bar{z}}$

$z = x+iy$

$\boxed{\text{also}}$ $\overline{\mathrm{Ln}\, z} = \overline{\ln|z| + i\,\mathrm{Arg}\, z}$

$= \ln|z| - i\,\mathrm{Arg}\, z$

$= \ln|z| + i\,\mathrm{Arg}\, \bar{z}$

$= \mathrm{Ln}\, \bar{z}$

thus (as $\overline{1+i} = 1-i$)

taking $z = 1+i$

we have that

$$\overline{(1+i)^{1-i}} = (1-i)^{1+i}$$