Math 320 (Smith): Practice Exam 1

1. (16 points - 10 min.) For

$$\frac{dy}{dt} = (e^{y+2} - 1)(e^y - 1)(y - 2), \quad y(0) = y_o, \quad -\infty < y_o < \infty \tag{1}$$

(a) (14 points) Sketch, roughly, a direction field and classify all critical points.
(b) (2 points) Determine (from your sketch), the asymptotic behavior of the solution for $y_o = -1$, $t \to \infty$.

2. (21 points - 15-20 min.) Solve (15 points)

$$y' - y^3 x \exp(x^2) = 0 \tag{2}$$

for $y(0) = -2$. **Give the range of validity of the solution** (6 points).

3. (24 points - 15-20 min) Write the following systems as $\mathbf{Ax} = \mathbf{b}$ and determine for what values of k the system has (i) a unique solution, (ii) no solution, and (iii) infinitely many solutions.
(a) (12 points)

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 4 \\ 2x_1 + 3x_2 - x_3 &= k \\ -2x_1 + x_2 - 3x_3 &= 2 \end{aligned}$$

(b) (12 points)

$$\begin{aligned} x_1 + 3x_3 &= 8 \\ -x_1 + kx_2 - x_3 &= 4 \\ 3x_1 + x_2 + 10x_3 &= 0 \end{aligned}$$

4. (39 points - 20-25 min) Given

$$\frac{dy}{dx} = y + \exp(x), \quad y(0) = 2. \tag{4}$$

(a) (15 points) Find the exact solution and state the region of validity of the exact solution.
(b) (8 points) Use one step of the Forward Euler method with step h to find an approximation for $y(x_o + h)$.
(c) (8 points) Use one step of the Improved Euler method with step h to find an approximation for $y(x_o + h)$.
(d) (8 points) Compare the Taylor series expansions for $y(x_o + h)$ using (i) the exact solution, (ii) the Forward Euler approximation and (iii) the Improved Euler approximation. Explain what these Taylor series expansions tell us about the truncation error of the Forward Euler and Improved Euler methods.