Math 320 (Smith): Practice Exam 1

1. The autonomous ODE given by

$$\frac{dP(t)}{dt} = -(bP^2(t) - aP(t) + h), \quad a > 0, \;\; b > 0, \;\; h > 0 \tag{1}$$

models a logistic population with harvesting, for example, the population of fish in a lake from which h fish per year are removed by fishing.

(a) Consider $a = 6$ and $b = 1$. How does the number of critical points depend on the parameter h? What are the values of h that yield real-valued critical point(s)?

(b) Consider $a = 6$, $b = 1$ and $h = 7$. Find and classify the critical points. Make a (rough) sketch of the direction field.

(c) For $a = 6$, $b = 1$, $h = 7$, and starting from the initial condition $P(0) = 3$, find the limiting behavior for large time $t \to \infty$.

2. The following augmented coefficient matrix results from elementary row operations on a 3×3 system of linear algebraic equations $\mathbf{Ax} = \mathbf{b}$.

$$\begin{bmatrix} -1 & 1 & 1 & 2 \\ 0 & 5 & -k & 4 \\ 0 & 0 & k & p+3 \end{bmatrix} \tag{2}$$

Consider 2 different values of the parameter p: (a) $p = -3$, and (b) $p = -2$.

Determine for what values of k the system has (i) a unique solution, (ii) no solution, and (iii) infinitely many solutions.

FOR PART (a) ONLY when $p = -3$: Find all solutions in cases (i) and/or (iii), and write the solution $\mathbf{x}$ in vector form.

3. Given

$$\frac{dy}{dx} = -\frac{y(x)}{(x-1)} + \frac{\exp(-x)}{(x-1)}, \quad y(0) = 2. \tag{3}$$

(a) Find the exact solution. For what values of x is the solution defined?

(b) Use one step of the Forward Euler method with step size h to find an approximation for $y(h)$.

4. (20 points) Consider the initial value problem

$$\frac{dy}{dx} = -\frac{5}{2}x^4y^3, \quad y(0) = -1. \tag{4}$$

(a) Find $y(x)$ explicitly. For what values of x is the solution defined?

(b) Use one step of the Modified Euler (Improved Euler, RK2) method with step size h to find an approximation for $y(h)$.

5. (5 points) TRUE or FALSE: The initial value problem

$$\frac{dy}{dt} = (y-1)^{3/2}, \quad y(1) = 2 \tag{5}$$

is guaranteed to have a unique solution in a subrange of $-\infty < t < \infty$.