HW 8, Math 320, Spring 2017

Nasser M. Abbasi (Discussion section 383, 8:50 AM - 9:40 AM Monday)

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Contents

0.1 Section 4.1 problem 7 (page 237)

problem Determine if u, v are linearly dependent or not

$$\bar{u} = (2,2)$$
$$\bar{v} = (2,-2)$$

solution

Two vectors $\bar{u}, \bar{v}$ are L.D if there exist scalars a, b, not both zero such that

$$a\bar{u} + b\bar{v} = \bar{0}$$
$$a\begin{pmatrix}2\\2\end{pmatrix} + b\begin{pmatrix}2\\-2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$
$$\begin{pmatrix}2 & 2\\2 & -2\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

The above is now is in $Ax = 0$ format. The determinant of A is $|A| = -4 - 4 = -8$. Since $|A| \neq 0$, then a unique exist. Since $\bar{0}$ vector is is always solution to $Ax = 0$, and so it is the only solution here (since solution is unique). This means that only $a = 0, b = 0$ satisfy $a\bar{u} + b\bar{v} = \bar{0}$. Therefore, $\bar{u}, \bar{v}$ are linearly independent.

0.2 Section 4.1 problem 12

problem Express w as linear combination of u, v.

$$\bar{u} = (4,1)$$
$$\bar{v} = (-2,-1)$$
$$\bar{w} = (2,-2)$$

solution

Need to find scalars a, b such that $a\bar{u} + b\bar{v} = \bar{w}$, hence

$$a\begin{pmatrix}4\\1\end{pmatrix} + b\begin{pmatrix}-2\\-1\end{pmatrix} = \begin{pmatrix}2\\-2\end{pmatrix}$$
$$\begin{pmatrix}4 & -2\\1 & -1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}2\\-2\end{pmatrix}$$

Applying Gaussian elimination

$$\begin{pmatrix}4 & -2 & 2\\1 & -1 & -2\end{pmatrix} \xrightarrow{R_2 = R_2 - \frac{1}{4}R_1} \begin{pmatrix}4 & -2 & 2\\0 & -\frac{1}{2} & -\frac{5}{2}\end{pmatrix}$$

Hence, from last equation

$$-\frac{1}{2}b = -\frac{5}{2}$$
$$b = 5$$

From first equation

$$4a - 2b = 2$$
$$4a = 2(5) + 2$$
$$a = 3$$

Therefore

$$5\bar{u} - 3\bar{v} = \bar{w}$$

0.3 Section 4.1 problem 18

problem Apply theorem 4 (that is calculate a determinant) to determine whether the given vectors $\bar{u}, \bar{v}, \bar{w}$ are L.D. or L.I.

$$\bar{u} = (1,1,0)$$
$$\bar{v} = (4,3,1)$$
$$\bar{w} = (3,-2,-4)$$

solution

Let a, b, c be scalars, such that $a\bar{u} + b\bar{v} + c\bar{w} = \bar{0}$. The goal now is to determine a, b, c and see they are are all zero or not. Setting up $A\bar{x} = \bar{0}$ system gives

$$a\begin{pmatrix}1\\1\\0\end{pmatrix} + b\begin{pmatrix}4\\3\\1\end{pmatrix} + c\begin{pmatrix}3\\-2\\-4\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1 & 4 & 3\\1 & 3 & -2\\0 & 1 & -4\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Now the $|A|$ is found. Subtracting row one from second row first, gives

$$\begin{pmatrix}1 & 4 & 3\\0 & -1 & -5\\0 & 1 & -4\end{pmatrix}$$

Performing cofactor expansion on the first column gives

$$\begin{aligned}|A| &= a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}\\&= a_{11}A_{11}\\&= a_{11}(-1)^{1+1}M_{11}\\&= 1 \times M_{11}\\&= M_{11}\\&= \begin{vmatrix}-1 & -5\\1 & -4\end{vmatrix}\\&= 4 + 5\\&= 9\end{aligned}$$

Since $|A|$ is not zero, then solution of $Ax = 0$ is unique. Hence only solution is

$$\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Which implies that $\bar{u}, \bar{v}, \bar{w}$ are linearly independent.

0.4 Section 4.1 problem 24

problem Use the method of example 3 to determine whether the given vectors $\bar{u}, \bar{v}, \bar{w}$ are L.D. or L.I. If they are L.D. then find scalars a, b, c not all zero such that $a\bar{u} + b\bar{v} + c\bar{w} = 0$

$$\begin{aligned}\bar{u} &= (1, 4, 5)\\\bar{v} &= (4, 2, 5)\\\bar{w} &= (-3, 3, -1)\end{aligned}$$

solution

Let a, b, c be scalars, such that $a\bar{u} + b\bar{v} + c\bar{w} = \bar{0}$. Setting up $A\bar{x} = \bar{0}$ system gives

$$a\begin{pmatrix}1\\4\\5\end{pmatrix} + b\begin{pmatrix}4\\2\\5\end{pmatrix} + c\begin{pmatrix}-3\\3\\-1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1 & 4 & -3\\4 & 2 & 3\\5 & 5 & -1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Applying Gaussian elimination, $R_2 = R_2 - 4R_1$ gives

$$\begin{pmatrix}1 & 4 & -3\\0 & -14 & 15\\5 & 5 & -1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$R_3 = R_3 - 5R_1$ gives

$$\begin{pmatrix} 1 & 4 & -3 \\ 0 & -14 & 15 \\ 0 & -15 & 14 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_3 = R_3 - \frac{15}{14}R_2$ gives

$$\begin{pmatrix} 1 & 4 & -3 \\ 0 & -14 & 15 \\ 0 & 0 & -\frac{29}{14} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Since the final Echelon form has no zero pivot, therefore $|A| \neq 0$. This means the solution is unique. Hence

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Which implies $\bar{u}, \bar{v}, \bar{w}$ are linearly independent.

0.5 Section 4.1 problem 28

problem Express vector t as linear combination of vectors u, v, w

$$\bar{t} = (7,7,7)$$
$$\bar{u} = (2,5,3)$$
$$\bar{v} = (4,1,-1)$$
$$\bar{w} = (1,1,5)$$

solution

Let $a\bar{u} + b\bar{v} + c\bar{w} = \bar{t}$, hence

$$a\begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} + b\begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} + c\begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \\ 7 \end{pmatrix}$$
$$\begin{pmatrix} 2 & 4 & 1 \\ 5 & 1 & 1 \\ 3 & -1 & 5 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \\ 7 \end{pmatrix}$$

Applying Gaussian elimination. $R_2 = R_2 - \frac{5}{2}R_1$ gives

$$\begin{pmatrix} 2 & 4 & 1 \\ 0 & -9 & -\frac{3}{2} \\ 3 & -1 & 5 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 7 \\ -\frac{21}{2} \\ 7 \end{pmatrix}$$

$R_3 = R_3 - \frac{3}{2}R_1$ gives

$$\begin{pmatrix} 2 & 4 & 1 \\ 0 & -9 & -\frac{3}{2} \\ 0 & -7 & \frac{7}{2} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 7 \\ -\frac{21}{2} \\ -\frac{7}{2} \end{pmatrix}$$

$R_3 = R_3 - \frac{7}{9}R_2$ gives

$$\begin{pmatrix} 2 & 4 & 1 \\ 0 & -9 & -\frac{3}{2} \\ 0 & 0 & \frac{14}{3} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 7 \\ -\frac{21}{2} \\ \frac{14}{3} \end{pmatrix}$$

Hence

$$\frac{14}{3}c = \frac{14}{3}$$
$$c = 1$$

And from second row

$$
\begin{aligned}
-9b - \frac{3}{2}c &= -\frac{21}{2}\\
-9b - \frac{3}{2} &= -\frac{21}{2}\\
b &= 1
\end{aligned}
$$

And from first row

$$
\begin{aligned}
2a + 4b + c &= 7\\
2a + 4 + 1 &= 7\\
a &= 1
\end{aligned}
$$

Hence

$$\bar{t} = \bar{u} + \bar{v} + \bar{w}$$

0.6 Section 4.1 problem 31

problem Show that the given set V is closed under addition and under multiplication by scalars and is therefore subspace of $\mathbb{R}^3$. V is the set of all (x,y,z) such that $2x = 3y$

solution

What the above says, that given any vector in this space, such as $\bar{v}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ then $y_1 = \frac{2}{3}x_1$. Hence any vector in this space can be written as $\bar{v}_1 = \begin{pmatrix} x_1 \\ \frac{2}{3}x_1 \\ z_1 \end{pmatrix}$. Given two vectors $\bar{v}_1, \bar{v}_2$ in this space, the sum, should also be in this space. let $\bar{u} = \bar{v}_1 + \bar{v}_2$, therefore

$$
\begin{aligned}
\bar{u} &= \begin{pmatrix} x_1 \\ \frac{2}{3}x_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ \frac{2}{3}x_2 \\ z_2 \end{pmatrix}\\
&= \begin{pmatrix} x_1 + x_2 \\ \frac{2}{3}x_1 + \frac{2}{3}x_2 \\ z_1 + z_2 \end{pmatrix}\\
&= \begin{pmatrix} x_1 + x_2 \\ \frac{2}{3}(x_1 + x_2) \\ z_1 + z_2 \end{pmatrix}
\end{aligned}
$$

Hence $\bar{u}$ is also in this space, since its y coordinate is also $\frac{2}{3}$ of its x coordinate. Now check is made for multiplication by scaler. Let $\bar{u} = c\bar{v}$, hence

$$
\begin{aligned}
\bar{u} &= c\begin{pmatrix} x_1 \\ \frac{2}{3}x_1 \\ z_1 \end{pmatrix}\\
&= \begin{pmatrix} cx_1 \\ \frac{2}{3}cx_1 \\ z_1 \end{pmatrix}
\end{aligned}
$$

Hence $\bar{u}$ is also in this space, since its y coordinate is also $\frac{2}{3}$ of its x coordinate. Therefore set V is closed under addition and under multiplication by scalars.

0.7 Section 4.1 problem 35

problem Show that the given set V is not a subspace of $\mathbb{R}^3$. V is the set of all (x,y,z) such that $z \geq 0$.

solution

The above set V is the upper half of the 3D space. (all vectors in the positive z part of 3D). But for this to be subspace, it must be closed under scalar multiplication. Let $\bar{u}$ be a vector in the set V.

Multiplying this vector by $c = -1$, will result in this vector having negative z component, and it will therefore leave the set V. Therefore the set V is not closed under scalar multiplication. Hence V is not a subspace of $\mathbb{R}^3$.

0.8 Section 4.1 problem 40

problem Suppose that $\bar{u},\bar{v},\bar{w}$ are vectors in $\mathbb{R}^3$ such that $\bar{u},\bar{v}$ are L.I. but $\bar{u},\bar{v},\bar{w}$ are L.D. Show that there exist scalars a,b such that $\bar{w} = a\bar{u} + b\bar{v}$

solution

Let $\bar{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \bar{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \bar{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$. Consider the sum $a\bar{u} + b\bar{v}$. Since $\bar{u},\bar{v}$ are L.I. then $a\bar{u} + b\bar{v}$ will produce non-zero vector unless a,b are both zero. Let this vector be $\bar{w}$. Hence $a\bar{u} + b\bar{v} = \bar{w}$. By definition, $\bar{w}$ is linear combination of $\bar{u},\bar{v}$, hence the three vectors $\bar{u},\bar{v},\bar{w}$ are L.D.

A geometrical proof is as follows. Since $\bar{u},\bar{v}$ are L.I. then they span a plane in 3D. This means $\bar{u},\bar{v}$ are basis vector for this 2D plane inside $\mathbb{R}^3$. Now since $\bar{u},\bar{v},\bar{w}$ are L.D. then the vector $\bar{w}$ must also be in the same plane that $\bar{u},\bar{v}$ are its basis. Hence the vector $\bar{w}$ can be expressed in terms of $\bar{u},\bar{v}$. Therefore there exist a,b such that $a\bar{u} + b\bar{v} = \bar{w}$.

0.9 Section 4.2 problem 2 (page 244)

problem A subset W of some n space $\mathbb{R}^n$ is defined by means of a given condition imposed on typical vector $(x_1, x_2, \cdots, x_n)$. Apply theorem 1 to determine whether or not W is subspace of $\mathbb{R}^n$. W is set of all vectors in $\mathbb{R}^3$ such that $x_1 = 5x_2$

solution

From theorem 1, for the subset W to be subspace, it has to at least satisfy being closed under addition of vectors and under scalar multiplication. Let any vector in this space be $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and since $x_1 = 5x_2$ therefore $\bar{x} = \begin{pmatrix} 5x_2 \\ x_2 \\ x_3 \end{pmatrix}$ Hence adding any two such vectors in this space gives

$$\begin{aligned} \bar{x} + \bar{y} &= \begin{pmatrix} 5x_2 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 5y_2 \\ y_2 \\ y_3 \end{pmatrix} \\ &= \begin{pmatrix} 5x_2 + 5y_2 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix} \\ &= \begin{pmatrix} 5\left(x_2 + y_2\right) \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix} \end{aligned}$$

Therefore the sum is also in this set, since its first coordinate is also 5 times its second coordinate. Now scalar multiplication is checked for being closed. Let

$$\begin{aligned} c\bar{x} &= c\begin{pmatrix} 5x_2 \\ x_2 \\ x_3 \end{pmatrix} \\ &= \begin{pmatrix} 5\left(cx_2\right) \\ \left(cx_2\right) \\ cx_3 \end{pmatrix} \end{aligned}$$

Therefore multiplication by scalar is also in this set, since the first coordinate is also 5 times its second coordinate. Therefore W is subspace of $\mathbb{R}^3$

0.10 Section 4.2 problem 8

problem A subset W of some space $\mathbb{R}^n$ is defined by means of a given condition imposed on typical vector $(x_1, x_2, \cdots, x_n)$. Apply theorem 1 to determine whether or not W is subspace of $\mathbb{R}^n$. W is set of all vectors in $\mathbb{R}^2$ such that $x_1^2 + x_2^2 = 0$.

solution

From theorem 1, for the subset W to be subspace, it has to satisfy being closed under addition of vectors and under scalar multiplication. Let any vector in this space be $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \bar{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ be two such vectors such that where $x_1^2 + x_2^2 = 0, y_1^2 + y_2^2 = 0$. Hence adding any two such vectors in this space gives

$$\begin{aligned} \bar{x} + \bar{y} &= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ &= \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} \end{aligned}$$

Checking if $\left(x_1 + y_1\right)^2 + \left(x_2 + y_2\right)^2 = 0$ or not. Expanding

$$\left(x_1 + y_1\right)^2 + \left(x_2 + y_2\right)^2 = \left(x_1^2 + y_1^2 + 2x_1y_1\right) + \left(x_2^2 + y_2^2 + 2x_2y_2\right) \tag{1}$$

But $x_1^2 = -x_2^2$ and $y_1^2 = -y_2^2$ by definition. Substituting this into (1) gives

$$\begin{aligned} \left(x_1 + y_1\right)^2 + \left(x_2 + y_2\right)^2 &= \left(-x_2^2 - y_2^2 + 2x_1y_1\right) + \left(x_2^2 + y_2^2 + 2x_2y_2\right) \\ &= 2\left(x_1y_1 + x_2y_2\right) \end{aligned}$$

Now $x_1 = ix_2$ and $y_1 = \pm iy_2$. Hence the above becomes

$$\begin{aligned} \left(x_1 + y_1\right)^2 + \left(x_2 + y_2\right)^2 &= 2\left((\pm ix_2)\left(\pm iy_2\right) + x_2y_2\right) \\ &= 2\left((ix_2)\left(iy_2\right) + x_2y_2\right) \\ &= 2\left(-x_2y_2 + x_2y_2\right) \\ &= 0 \end{aligned}$$

Therefore closed under multiplication. Checking now if closed under scalar multiplication.

$$\begin{aligned} c\bar{x} &= c\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} cx_1 \\ cx_2 \end{pmatrix} \end{aligned}$$

Hence

$$\begin{aligned} (cx_1)^2 + (cx_2)^2 &= c^2x_1^2 + c^2x_2^2 \\ &= c^2\left(x_1^2 + x_2^2\right) \end{aligned}$$

But $x_1^2 + x_2^2 = 0$. Therefore $(cx_1)^2 + (cx_2)^2 = 0$ and it is closed under scalar multiplication as well. Therefore W is subspace of $\mathbb{R}^2$.

0.11 Section 4.2 problem 11

problem A subset W of some n space $\mathbb{R}^n$ is defined by means of a given condition imposed on typical vector $(x_1, x_2, \cdots, x_n)$. Apply theorem 1 to determine whether or not W is subspace of $\mathbb{R}^n$. W is set of all vectors in $\mathbb{R}^4$ such that $x_1 + x_2 = x_3 + x_4$.

solution

From theorem 1, for the subset W to be subspace, it has to satisfy being closed under addition of vectors and under scalar multiplication. Let any vector in this space be $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \bar{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$ be two such vectors such that $x_1 + x_2 = x_3 + x_4$ and $y_1 + y_2 = y_3 + y_4$. Adding any two such vectors in this space gives

$$\bar{x}+\bar{y}=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}+\begin{pmatrix}y_1\\y_2\\y_3\\y_4\end{pmatrix}$$
$$=\begin{pmatrix}x_1+y_1\\x_2+y_2\\x_3+y_3\\x_4+y_4\end{pmatrix}$$

Checking now if $\left(x_1+y_1\right)+\left(x_2+y_2\right)=\left(x_3+y_3\right)+\left(x_4+y_4\right)$ or not.

$$\left(x_1+y_1\right)+\left(x_2+y_2\right)=\left(x_1+x_2\right)+\left(y_1+y_2\right)$$

But $x_1+x_2=x_3+x_4$ and $y_1+y_2=y_3+y_4$, therefore the above becomes

$$\left(x_1+y_1\right)+\left(x_2+y_2\right)=\left(x_3+x_4\right)+\left(y_3+y_4\right)$$

Hence closed under addition. Checking now if closed under scalar multiplication.

$$c\bar{x}=c\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}$$
$$=\begin{pmatrix}cx_1\\cx_2\\cx_3\\cx_4\end{pmatrix}$$

Hence

$$\left(cx_1\right)+\left(cx_2\right)=c\left(x_1+x_2\right)$$

But $x_1+x_2=x_3+x_4$ hence

$$\begin{aligned}\left(cx_1\right)+\left(cx_2\right)&=c\left(x_3+x_4\right)\\&=\left(cx_3\right)+\left(cx_4\right)\end{aligned}$$

And therefore it is closed under scalar multiplication as well. Hence W is subspace of $\mathbb{R}^4$

0.12 Section 4.2 problem 16

problem Apply method of example 5 to find two solution vectors u,v such that the solution space is the set of all linear combinations of the form $su+tv$

$$\begin{aligned}x_1-4x_2-3x_3-7x_4&=0\\2x_1-x_2+x_3+7x_4&=0\\x_1+2x_2+3x_3+11x_4&=0\end{aligned}$$

solution

$$\begin{pmatrix}1&-4&-3&-7\\2&-1&1&7\\1&2&3&11\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Let $R_2=R_2-2R_1$, hence

$$\begin{pmatrix}1&-4&-3&-7\\0&7&7&21\\1&2&3&11\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Let $R_3 = R_3 - R_1$, hence

$$\begin{pmatrix} 1 & -4 & -3 & -7 \\ 0 & 7 & 7 & 21 \\ 0 & 6 & 6 & 18 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Let $R_3 = R_3 - \frac{6}{7} R_2$, hence

$$\begin{pmatrix} 1 & -4 & -3 & -7 \\ 0 & 7 & 7 & 21 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Last row gives $0x_3 + 0x_4 = 0$. Therefore $x_4 = t, x_3 = s$ are the free parameters. From second row

$$\begin{aligned} 7x_2 + 7x_3 + 21x_4 &= 0 \\ 7x_2 &= -7s - 21t \\ x_2 &= -s - 3t \end{aligned}$$

From first equation

$$\begin{aligned} x_1 - 4x_2 - 3x_3 - 7x_4 &= 0 \\ x_1 &= 4x_2 + 3x_3 + 7x_4 \\ &= 4(-s - 3t) + 3s + 7t \\ &= -s - 5t \end{aligned}$$

Hence solution vector is

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -s - 5t \\ -s - 3t \\ s \\ t \end{pmatrix} &= \begin{pmatrix} -s \\ -s \\ s \\ 0 \end{pmatrix} + \begin{pmatrix} -5t \\ -3t \\ 0 \\ t \end{pmatrix} \\ &= s \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -5 \\ -3 \\ 0 \\ 1 \end{pmatrix} \\ &= s\bar{u} + t\bar{v} \end{aligned}$$

0.13 Section 4.2 problem 22

problem Reduce the given system to echelon form to find a single solution vector u such that the solution space is the set of all scalar multiples of u

$$\begin{aligned} x_1 + 3x_2 + 3x_3 + 3x_4 &= 0 \\ 2x_1 + 7x_2 + 5x_3 - x_4 &= 0 \\ 2x_1 + 7x_2 + 4x_3 - 4x_4 &= 0 \end{aligned}$$

solution

$$\begin{pmatrix} 1 & 3 & 3 & 3 \\ 2 & 7 & 5 & -1 \\ 2 & 7 & 4 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_2 = R_2 - 2R_1$ gives

$$\begin{pmatrix} 1 & 3 & 3 & 3 \\ 0 & 1 & -1 & -7 \\ 2 & 7 & 4 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_2 = R_3 - 2R_1$ gives

$$\begin{pmatrix} 1 & 3 & 3 & 3 \\ 0 & 1 & -1 & -7 \\ 0 & 1 & -2 & -10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$R_3 = R_3 - R_2$ gives

$$\begin{pmatrix} 1 & 3 & 3 & 3 \\ 0 & 1 & -1 & -7 \\ 0 & 0 & -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Last row gives $-x_3 - 3x_4 = 0$. Therefore let $x_4 = t$ be the free parameter. Hence $x_3 = -3t$. From second equation

$$\begin{aligned} x_2 - x_3 - 7x_4 &= 0 \\ x_2 &= x_3 + 7x_4 \\ &= -3t + 7t \\ &= 4t \end{aligned}$$

And from first equation

$$\begin{aligned} x_1 + 3x_2 + 3x_3 + 3x_4 &= 0 \\ x_1 &= -3x_2 - 3x_3 - 3x_4 \\ &= -3(4t) - 3(-3t) - 3t \\ &= -6t \end{aligned}$$

Hence solution vector is

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} &= \begin{pmatrix} -6t \\ 4t \\ -3t \\ t \end{pmatrix} \\ &= t \begin{pmatrix} -6 \\ 4 \\ -3 \\ 1 \end{pmatrix} \\ &= t\bar{u} \end{aligned}$$

0.14 Section 4.2 problem 26

problem Prove: If $\bar{u}$ is a fixed vector in vector space V, then the set of W of all scalar multiples $c\bar{u}$ of $\bar{u}$ is subspace of V

solution

Let

$$W = \{c\bar{u}\}$$

Closure under addition gives

$$\begin{aligned} c_1\bar{u} + c_2\bar{u} &= (c_1 + c_2)\bar{u} \\ &= b\bar{u} \end{aligned}$$

Where $b = c_1 + c_2$. Hence closed under additions. Closure under multiplication by scalar b gives

$$\begin{aligned} bc\bar{u} &= (bc)\bar{u} \\ &= C_1\bar{u} \end{aligned}$$

Where $C_1 = bc$ a new constant. Hence closed under multiplication by scalar b. Therefore W is subspace of V

0.15 Section 4.2 problem 27

problem Let $\bar{u}$ and $\bar{v}$ be fixed vectors in vector space V. Show that the set W of all linear combinations $a\bar{u}+b\bar{v}$ is subspace of V

solution

The set W is $W=\{a\bar{u}+b\bar{v}\}$ which is linear combinations of $\bar{u}$ and $\bar{v}$ where a,b are arbitrary scalar. Closure under addition gives

$$(a_1\bar{u}+b_1\bar{v})+(a_2\bar{u}+b_2\bar{v})=(a_1+a_2)\,\bar{u}+(b_1+b_2)\,\bar{v}$$

But a_1+a_2 and b_1+b_2 are arbitrary scalars, say C_1,C_2 respectively. Hence the above becomes $C_1\bar{u}+C_2\bar{v}$ and this is in W. Hence W is closed under addition. Closure under multiplication by scalar c gives

$$c\,(a\bar{u}+b\bar{v})=ca\bar{u}+cb\bar{v}$$

But ca and cb are arbitrary scalars, say C_1,C_2 respectively. Hence the above becomes $C_1\bar{u}+C_2\bar{v}$ and this is in W. Therefore W is subspace of V.

0.16 Section 4.2 problem 28

problem Suppose A is $n\times n$ matrix and k is constant scalar. Show that the set of all vectors $\bar{x}$ such that $A\bar{x}=k\bar{x}$ is subspace of $\mathbb{R}^n$

solution

Let $W=\{\bar{x}\}$ where $A\bar{x}=k\bar{x}$. To determine if closed under addition, we consider the vector $\bar{x}_1+\bar{x}_2$. This vector should also satisfy $A\,(\bar{x}_1+\bar{x}_2)=k\,(\bar{x}_1+\bar{x}_2)$ for it to be closed. Let us check if this is the case or not.

$$\begin{aligned}A\,(\bar{x}_1+\bar{x}_2)&=A\bar{x}_1+A\bar{x}_2\\&=k\bar{x}_1+k\bar{x}_2\\&=k\,(\bar{x}_1+\bar{x}_2)\end{aligned}$$

Hence it is closed under addition. We will now check closure under scalar multiplication.

$$\begin{aligned}A\,(c\bar{x}_1)&=cA\bar{x}_1\\&=ck\bar{x}\\&=k\,(c\bar{x})\end{aligned}$$

Hence closed under scalar multiplication. Therefore W is subspace of V.

0.17 Section 4.3 problem 6 (page 252)

problem Determine whether the given vectors are L.I. or L.D. Do this by inspection without solving linear system of equations

$$\begin{aligned}\bar{v}_1&=(1,0,0)\\\bar{v}_2&=(1,1,0)\\\bar{v}_3&=(1,1,1)\end{aligned}$$

solution

The equation $c_1\bar{v}_1+c_2\bar{v}_2+c_3\bar{v}_3=\bar{0}$ gives

$$\begin{aligned}c_1\,(1,0,0)+c_2\,(1,1,0)+c_3\,(1,1,1)&=(0,0,0)\\(c_1+c_2+c_3,c_2+c_3,c_3)&=(0,0,0)\end{aligned}$$

Hence $c_3=0$ and $c_2=0$ and $c_1=0$ is the only solution. Therefore definition of linear independence (page 248), the vectors are linearly independent.

0.18 Section 4.3 problem 7

problem Determine whether the given vectors are L.I. or L.D. Do this by inspection without solving linear system of equations

$$v_1 = (2,1,0,0)$$
$$v_2 = (3,0,1,0)$$
$$v_3 = (4,0,0,1)$$

solution

The equation $c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 = \bar{0}$ gives

$$c_1(2,1,0,0) + c_2(3,0,1,0) + c_3(4,0,0,1) = (0,0,0,0)$$
$$(2c_1 + 3c_2 + 4c_3, c_1, c_2, c_3) = (0,0,0,0)$$

Therefore, we see by inspection (comparing terms) that $c_3 = 0, c_2 = 0, c_1 = 0$. Therefore definition of linear independence (page 248), the vectors are linearly independent.

0.19 Section 4.3 problem 15

problem Express the indicated vector w as linear combination of the given vectors v_i if this is possible. If not, show it is impossible

$$\bar{w} = (4,5,6)$$
$$\bar{v}_1 = (2,-1,4)$$
$$\bar{v}_2 = (3,0,1)$$
$$\bar{v}_3 = (1,2,-1)$$

solution

The equation $c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 = \bar{w}$ gives (in matrix form)

$$\begin{pmatrix} 2 & 3 & 1 \\ -1 & 0 & 2 \\ 4 & 1 & -1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

We now solve for c_1, c_2, c_3. Let $R_2 = R_2 + \frac{1}{2}R_1$ therefore

$$\begin{pmatrix} 2 & 3 & 1 \\ 0 & \frac{3}{2} & \frac{5}{2} \\ 4 & 1 & -1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ 6 \end{pmatrix}$$

$R_3 = R_3 - 2R_1$ gives

$$\begin{pmatrix} 2 & 3 & 1 \\ 0 & \frac{3}{2} & \frac{5}{2} \\ 0 & -5 & -3 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ -2 \end{pmatrix}$$

$R_3 = R_3 - \frac{10}{3}R_2$ gives

$$\begin{pmatrix} 2 & -1 & 4 \\ 0 & \frac{3}{2} & \frac{5}{2} \\ 0 & 0 & \frac{16}{3} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ \frac{64}{3} \end{pmatrix}$$

Therefore, since there are no zero pivots at end of forward Gaussian elimination, the solution is unique and not zero. (by backward substitution,

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$$

Hence

$$\bar{w} = c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3$$
$$= 3\bar{v}_1 - 2\bar{v}_2 + 4\bar{v}_3$$

0.20 Section 4.3 problem 20

problem Three vectors v_1, v_2, v_3 are given. If they are L.I., show this. Otherwise, find a nontrivial linear combination of them that is equal to the zero vector.

$$\begin{aligned}\bar{v}_1 &= (1,1,-1,1)\\ \bar{v}_2 &= (2,1,1,1)\\ \bar{v}_3 &= (3,1,4,1)\end{aligned}$$

solution

Here the space is $\mathbb{R}^4$, but only 3 vectors are given. Therefore theorem 3 at page 252 is used. This theorem says that, if we set the A matrix, with its columns as the given vectors above, then the vectors are L.I. iff there is a 3×3 submatrix inside A which has nonzero determinant. To show this, Gaussian eliminating is used.

$$A = \begin{pmatrix}1 & 2 & 3\\ 1 & 1 & 1\\ -1 & 1 & 4\\ 1 & 1 & 1\end{pmatrix} \xrightarrow{R_2=R_2-R_1} \begin{pmatrix}1 & 2 & 3\\ 0 & -1 & -2\\ -1 & 1 & 4\\ 1 & 1 & 1\end{pmatrix} \xrightarrow{R_3=R_3+R_1} \begin{pmatrix}1 & 2 & 3\\ 0 & -1 & -2\\ 0 & 3 & 7\\ 1 & 1 & 1\end{pmatrix} \xrightarrow{R_4=R_4-R_1}$$

$$\begin{pmatrix}1 & 2 & 3\\ 0 & -1 & -2\\ 0 & 3 & 7\\ 0 & -1 & -2\end{pmatrix} \xrightarrow{R_3=R_3+3R_2} \begin{pmatrix}1 & 2 & 3\\ 0 & -1 & -2\\ 0 & 0 & 1\\ 0 & -1 & -2\end{pmatrix} \xrightarrow{R_4=R_4-R_2} \begin{pmatrix}1 & 2 & 3\\ 0 & -1 & -2\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}$$

The above shows that there is a submatrix of size 3×3 which has nonzero determinant. It is the matrix of the first 3 rows

$$\begin{pmatrix}1 & 2 & 3\\ 0 & -1 & -2\\ 0 & 0 & 1\end{pmatrix}$$

This has nonzero determinant. Since it is diagonal, its determinant is the product of diagonal elements. Since no diagonal element is zero, the determinant is not zero. This implies vectors are linearly independent.

0.21 Section 4.3 problem 24

problem The vectors $\bar{v}_i$ are known to be L.I., apply the definition of L.I. to show that the vectors u_i are also L.I.

$$\begin{aligned}\bar{u}_1 &= \bar{v}_1 + \bar{v}_2\\ \bar{u}_2 &= 2\bar{v}_1 + 3\bar{v}_2\end{aligned}$$

solution

We will examine

$$a\bar{u}_1 + b\bar{u}_2 = \bar{0}$$

To see if this is satisfied only for $a = 0, b = 0$.

$$\begin{aligned}a\bar{u}_1 + b\bar{u}_2 &= \bar{0}\\ a(\bar{v}_1 + \bar{v}_2) + b(2\bar{v}_1 + 3\bar{v}_2) &= \bar{0}\\ \bar{v}_1(a + 2b) + \bar{v}_2(a + 3b) &= \bar{0}\end{aligned}$$

But since we are told that $\bar{v}_1, \bar{v}_2$ are L.I., then this implies that $a + 2b = 0$ and $a + 3b = 0$. These two equations we solve now for a, b. These two equations show that $2b = 3b$, which means $b = 0$. Hence $a = 0$ as well. Therefore only solution for $a\bar{u}_1 + b\bar{u}_2 = \bar{0}$ is that $a = b = 0$. This is the same as saying $\bar{u}_1, \bar{u}_2$ are linearly independent.

QED