HW 6, Math 320, Spring 2017

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Contents

0.1 Section 3.4 problem 8 (page 186) 2
0.2 Section 3.4 problem 15 3
0.3 Section 3.4 problem 20 4
0.4 Section 3.4 problem 27 5
0.5 Section 3.4 problem 29 7
0.6 Section 3.4 problem 30 8
0.7 Section 3.4 problem 32 9
  0.7.1 Part (a) 9
  0.7.2 Part (b) 10
0.8 Section 3.5 problem 13 (page 199) 11
0.9 Section 3.5 problem 19 12
0.10 Section 3.5 problem 24 13
0.11 Section 3.5 problem 30 14
0.12 Section 3.5 problem 32 15
0.13 Section 3.5 problem 34 16
0.14 Section 3.5 problem 35 17

0.1 Section 3.4 problem 8 (page 186)

Problem Calculate AB and BA if defined.

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & -5 & 4 \end{pmatrix}, B = \begin{pmatrix} 3 & 0 \\ -1 & 4 \\ 6 & 5 \end{pmatrix}$$

solution A is 2×3 and B is 3×2, Since inner dimensions agree, then AB is defined and given by 2×2 matrix

$$\begin{aligned} C &= AB \\ &= \begin{pmatrix} 1 & 0 & 3 \\ 2 & -5 & 4 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ -1 & 4 \\ 6 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 21 & 15 \\ 35 & 0 \end{pmatrix} \end{aligned}$$

Now B is 3×2 and A is 2×3, hence inner dimensions agree, and BA is 3×3

$$\begin{aligned} C &= BA \\ &= \begin{pmatrix} 3 & 0 \\ -1 & 4 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 2 & -5 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 0 & 9 \\ 7 & -20 & 13 \\ 16 & -25 & 38 \end{pmatrix} \end{aligned}$$

0.2 Section 3.4 problem 15

<u>Problem</u> ABC matrices are given, verify by computation, that $A(BC) = (AB)C$

$$A = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, B = \begin{pmatrix} 1 & -1 & 2 \end{pmatrix}, C = \begin{pmatrix} 2 & 0 \\ 0 & 3 \\ 1 & 4 \end{pmatrix}$$

<u>solution</u> A is $2 \times 1, B$ is 1×3 and C is 3×2.

$$BC = \begin{pmatrix} 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 5 \end{pmatrix}$$

Hence

$$\begin{aligned} A(BC) &= \begin{pmatrix} 3 \\ 2 \end{pmatrix} \begin{pmatrix} 4 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 12 & 15 \\ 8 & 10 \end{pmatrix} \end{aligned} \tag{1}$$

Now we will do $(AB)C$ and see if we get same result as above

$$AB = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -3 & 6 \\ 2 & -2 & 4 \end{pmatrix}$$

Hence

$$\begin{aligned} (AB)C &= \begin{pmatrix} 3 & -3 & 6 \\ 2 & -2 & 4 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \\ 1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 12 & 15 \\ 8 & 10 \end{pmatrix} \end{aligned} \tag{2}$$

Comparing (1) and (2), we see they are the same. QED.

0.3 Section 3.4 problem 20

Problem Write the system as $Ax=0$ and the find the solution in vector form

$$\begin{aligned} x_1 - 3x_2 + 7x_5 &= 0 \\ x_3 - 2x_5 &= 0 \\ x_4 - 10x_5 &= 0 \end{aligned}$$

Solution

$$\overbrace{\begin{pmatrix} 1 & -3 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -10 \end{pmatrix}}^{A} \overbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}}^{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

To find solution, we need to do Gaussian elimination to obtain Echelon form. But A is already in Echelon form. Hence we start with back substitution phase. From last equation

$$x_4 - 10x_5 = 0$$

Let $x_5 = t$, hence

$$x_4 = 10t$$

From second equation

$$\begin{aligned} x_3 - 2x_5 &= 0 \\ x_3 &= 2t \end{aligned}$$

From first equation

$$\begin{aligned} x_1 - 3x_2 + 7x_5 &= 0 \\ x_1 - 3x_2 &= -7t \end{aligned}$$

Let $x_2 = s$ then

$$x_1 = 3s - 7t$$

Hence solution is

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} &= \begin{pmatrix} 3s-7t \\ s \\ 2t \\ 10t \\ t \end{pmatrix} \\ &= s\begin{pmatrix} 3 & 1 & 0 & 0 & 0 \end{pmatrix} + t\begin{pmatrix} -7 & 0 & 2 & 10 & 1 \end{pmatrix} \end{aligned}$$

0.4 Section 3.4 problem 27

Problem A diagonal matrix is square matrix of form

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & a_{nn} \end{pmatrix}$$

in which every element off the diagonal is zero. Show that the product AB of two $n \times n$ diagonal matrices is again a diagonal matrix. State concise rule for quickly computing AB. Is it clear that $AB = BA$? Explain.

Solution

We want to perform (using 3×3 for illustration) the following.

$$C = AB = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{pmatrix}$$

Let use the matrix multiplication method, where we multiply A by each column of B at a time, to produce one column of the result C. This means the first column of C is

$$c_1 = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} \\ 0 \\ 0 \end{pmatrix}$$

And the second column of C is

$$c_2 = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} 0 \\ b_{22} \\ 0 \end{pmatrix}$$

And third column of C is

$$c_3 = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ b_{33} \end{pmatrix}$$

And so on for larger matrices. Using the above view, shows that c_1 will come out to be (using rules of matrix times vector now)

$$c_1 = \begin{pmatrix} a_{11}b_{11} \\ 0 \\ 0 \end{pmatrix}$$

And c_2 and c_3 will come out to be

$$c_2 = \begin{pmatrix} 0 \\ a_{22}b_{22} \\ 0 \end{pmatrix}$$

$$c_3 = \begin{pmatrix} 0 \\ 0 \\ a_{33}b_{33} \end{pmatrix}$$

And so one for larger matrices. Now we uses these columns to make up C and obtain

$$C = \begin{pmatrix} a_{11}b_{11} & 0 & 0 \\ 0 & a_{22}b_{22} & 0 \\ 0 & 0 & a_{33}b_{33} \end{pmatrix}$$

We see that C is diagonal matrix as well. If we reverse the order of multiplications, BA and follow the same process as above, we will obtain

$$C = \begin{pmatrix} b_{11}a_{11} & 0 & 0 \\ 0 & b_{22}a_{22} & 0 \\ 0 & 0 & b_{33}a_{33} \end{pmatrix}$$

We see if the same Matrix, since number $a_{ii}b_{ii}$ is same as $b_{ii}a_{ii}$. A quick rule to make C is this: Start with C which is all zeros, then multiply each corresponding diagonal elements in A and B and move the result in the diagonal of resulting matrix C. So basically, we just need to multiply diagonal elements.

$$c_{ii} = \begin{cases} a_{ii}b_{ii} & i = 1, 2, 3 \cdots n \\ 0 & \text{otherwise} \end{cases}$$

0.5 Section 3.4 problem 29

Problem If $A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ then show that $A^2=(a+d)A-(ad-bc)I_2$ where I_2 is the 2×2 identity matrix. Thus every 2×2 matrix A satisfies the equation $A^2-(trace\ A)A+(\det A)I=0$ where $\det(A)=ad-bc$ and trace is sum of diagonal elements.

solution

First we find A^2 using matrix-matrix multiplication

$$\begin{aligned} A^2 &= \begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} a & b\\ c & d\end{pmatrix}\\ &= \begin{pmatrix} a^2+bc & ab+bd\\ ac+cd & d^2+bc\end{pmatrix} \end{aligned} \tag{1}$$

Now $trace(A)=a+d$. Hence

$$(trace\ A)A=(a+d)\begin{pmatrix} a & b\\ c & d\end{pmatrix}$$

This is scalar times matrix. Hence

$$\begin{aligned} (trace\ A)A &= \begin{pmatrix} (a+d)a & (a+d)b\\ (a+d)c & (a+d)d\end{pmatrix}\\ &= \begin{pmatrix} a^2+ad & ab+db\\ ac+dc & ad+d^2\end{pmatrix} \end{aligned}$$

And $\det(A)I_2$ is

$$\det(A)I_2=(ad-bc)\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}$$

This is scalar times matrix. Hence

$$\det(A)I_2=\begin{pmatrix} ad-bc & 0\\ 0 & ad-bc\end{pmatrix}$$

From the above, we see that

$$\begin{aligned} (trace\ A)A-\det(A)I_2 &= \begin{pmatrix} a^2+ad & ab+db\\ ac+dc & ad+d^2\end{pmatrix}-\begin{pmatrix} ad-bc & 0\\ 0 & ad-bc\end{pmatrix}\\ &= \begin{pmatrix} \left(a^2+ad\right)-(ad-bc) & ab+db\\ ac+dc & \left(ad+d^2\right)-(ad-bc)\end{pmatrix}\\ &= \begin{pmatrix} a^2+bc & ab+db\\ ac+dc & d^2+bc\end{pmatrix} \end{aligned} \tag{2}$$

If we compare (1) and (2), we see they are the same. Hence we showed that

$$A^2=(trace\ A)A-\det(A)I_2$$

0.6 Section 3.4 problem 30

<u>Problem</u> The formula $A^2 = (trace\ A)\, A - \det(A)\, I_2$ can be used to compute A^2 without explicit matrix multiplication. It follows that $A^3 = (trace\ A)\, A^2 - \det(A)\, A$ and $A^4 = (trace\ A)\, A^3 - \det(A)\, A^2$ and so on. Use this method to determine A^2, A^3, A^4, A^5 given that $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

<u>solution</u>

$$\begin{aligned} trace\ A &= 2+2 = 4 \\ \det A &= 4-1 = 3 \end{aligned}$$

Hence

$$\begin{aligned} A^2 &= (trace\ A)\, A - \det(A)\, I_2 \\ &= 4\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} \end{aligned}$$

And

$$\begin{aligned} A^3 &= (trace\ A)\, A^2 - \det(A)\, A \\ &= 4\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} - 3\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 14 & 13 \\ 13 & 14 \end{pmatrix} \end{aligned}$$

And

$$\begin{aligned} A^4 &= (trace\ A)\, A^3 - \det(A)\, A^2 \\ &= 4\begin{pmatrix} 14 & 13 \\ 13 & 14 \end{pmatrix} - 3\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 41 & 40 \\ 40 & 41 \end{pmatrix} \end{aligned}$$

And

$$\begin{aligned} A^5 &= (trace\ A)\, A^4 - \det(A)\, A^3 \\ &= 4\begin{pmatrix} 41 & 40 \\ 40 & 41 \end{pmatrix} - 3\begin{pmatrix} 14 & 13 \\ 13 & 14 \end{pmatrix} \\ &= \begin{pmatrix} 122 & 121 \\ 121 & 122 \end{pmatrix} \end{aligned}$$

0.7 Section 3.4 problem 32

Problem (a) Suppose that $A=\begin{pmatrix}2 & -1\\ -4 & 3\end{pmatrix}, B=\begin{pmatrix}1 & 5\\ 3 & 7\end{pmatrix}$. Show that $(A+B)^2 \neq A^2+2AB+B^2$. (b) Suppose that A,B are square matrices such that $AB=BA$. Show that $(A+B)^2=A^2+2AB+B^2$

solution

0.7.1 Part (a)

First we find the LHS

$$\begin{aligned}(A+B)^2 &= \left[\begin{pmatrix}2 & -1\\ -4 & 3\end{pmatrix}+\begin{pmatrix}1 & 5\\ 3 & 7\end{pmatrix}\right]^2\\ &= \begin{pmatrix}3 & 4\\ -1 & 10\end{pmatrix}^2\\ &= \begin{pmatrix}3 & 4\\ -1 & 10\end{pmatrix}\begin{pmatrix}3 & 4\\ -1 & 10\end{pmatrix}\\ &= \begin{pmatrix}5 & 52\\ -13 & 96\end{pmatrix} \qquad (1)\end{aligned}$$

Now

$$\begin{aligned}A^2 &= \begin{pmatrix}2 & -1\\ -4 & 3\end{pmatrix}\begin{pmatrix}2 & -1\\ -4 & 3\end{pmatrix}\\ &= \begin{pmatrix}8 & -5\\ -20 & 13\end{pmatrix}\end{aligned}$$

And

$$\begin{aligned}B^2 &= \begin{pmatrix}1 & 5\\ 3 & 7\end{pmatrix}\begin{pmatrix}1 & 5\\ 3 & 7\end{pmatrix}\\ &= \begin{pmatrix}16 & 40\\ 24 & 64\end{pmatrix}\end{aligned}$$

And

$$\begin{aligned}AB &= \begin{pmatrix}2 & -1\\ -4 & 3\end{pmatrix}\begin{pmatrix}1 & 5\\ 3 & 7\end{pmatrix}\\ &= \begin{pmatrix}-1 & 3\\ 5 & 1\end{pmatrix}\end{aligned}$$

Hence

$$2AB = 2\begin{pmatrix}-1 & 3\\ 5 & 1\end{pmatrix} = \begin{pmatrix}-2 & 6\\ 10 & 2\end{pmatrix}$$

Therefore, the RHS $A^2 + 2AB + B^2$ is

$$A^2 + 2AB + B^2 = \begin{pmatrix} 8 & -5 \\ -20 & 13 \end{pmatrix} + \begin{pmatrix} -2 & 6 \\ 10 & 2 \end{pmatrix} + \begin{pmatrix} 16 & 40 \\ 24 & 64 \end{pmatrix}$$

$$= \begin{pmatrix} 22 & 41 \\ 14 & 79 \end{pmatrix} \tag{2}$$

Comparing (1) and (2) we see that are not the same. Hence we showed that, in this example, $(A + B)^2 \neq A^2 + 2AB + B^2$

0.7.2 Part (b)

Now, we assume that $AB = BA$. But since $(A + B)^2 = A^2 + B^2 + AB + BA$ and we are told that $AB = BA$, then

$$\begin{aligned}(A + B)^2 &= A^2 + B^2 + AB + AB \\ &= A^2 + B^2 + 2AB\end{aligned}$$

So only in the case when $AB = BA$ is $(A + B)^2 = A^2 + B^2 + 2AB$. In Part (a), $AB = \begin{pmatrix} -1 & 3 \\ 5 & 1 \end{pmatrix}$, But $BA = \begin{pmatrix} -18 & 14 \\ -22 & 18 \end{pmatrix}$, so in part (a), $AB \neq BA$ and that is why equality failed.

0.8 Section 3.5 problem 13 (page 199)

Problem Find A^{-1} for $\begin{pmatrix} 2 & 7 & 3 \\ 1 & 3 & 2 \\ 3 & 7 & 9 \end{pmatrix}$

solution

We set up AI_3 and perform row operations on A and I at same time, to convert A to I_3. Then A^{-1} will be the on the right side

$$\begin{pmatrix} 2 & 7 & 3 \\ 1 & 3 & 2 \\ 3 & 7 & 9 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_2=R_1]{R_1=R_2} \begin{pmatrix} 1 & 3 & 2 \\ 2 & 7 & 3 \\ 3 & 7 & 9 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_3=R_3-3R_1]{R_2=R_2-2R_1}$$

$$\begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & -2 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & -3 & 1 \end{pmatrix} \xrightarrow{R_3=R_3+2R_2} \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 2 & -7 & 1 \end{pmatrix} \xrightarrow{R_1=R_1-3R_2}$$

$$\begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -3 & 7 & 0 \\ 1 & -2 & 0 \\ 2 & -7 & 1 \end{pmatrix} \xrightarrow{R_2=R_2+R_3} \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -3 & 7 & 0 \\ 3 & -9 & 1 \\ 2 & -7 & 1 \end{pmatrix} \xrightarrow{R_1=R_1-5R_3}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -13 & 42 & -5 \\ 3 & -9 & 1 \\ 2 & -7 & 1 \end{pmatrix}$$

Since the left side is I_3 we stop. Hence

$$A^{-1} = \begin{pmatrix} -13 & 42 & -5 \\ 3 & -9 & 1 \\ 2 & -7 & 1 \end{pmatrix}$$

0.9 Section 3.5 problem 19

<u>Problem</u> Find A^{-1} for $\begin{pmatrix} 1 & 4 & 3 \\ 1 & 4 & 5 \\ 2 & 5 & 1 \end{pmatrix}$

<u>solution</u>

We set up AI_3 and perform row operations on A and I at same time, to convert A to I_3. Then A^{-1} will be the on the right side

$$\begin{pmatrix} 1 & 4 & 3 \\ 1 & 4 & 5 \\ 2 & 5 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \underset{R_3=R_3-2R_1}{\overset{R_2=R_2-R_1}{\longrightarrow}} \begin{pmatrix} 1 & 4 & 3 \\ 0 & 0 & 2 \\ 0 & -3 & -5 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \underset{R_3=R_2}{\overset{R_3=R_2}{\longrightarrow}}$$

$$\begin{pmatrix} 1 & 4 & 3 \\ 0 & -3 & -5 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -2 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \underset{R_3=\frac{R_3}{2}}{\overset{R_2=\frac{R_2}{-3}}{\longrightarrow}} \begin{pmatrix} 1 & 4 & 3 \\ 0 & 1 & \frac{5}{3} \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 0 & \frac{-1}{3} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \overset{R_1=R_1-4R_2}{\longrightarrow}$$

$$\begin{pmatrix} 1 & 0 & -\frac{11}{3} \\ 0 & 1 & \frac{5}{3} \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -\frac{5}{3} & 0 & \frac{4}{3} \\ \frac{2}{3} & 0 & \frac{-1}{3} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \overset{R_2=R_2-\frac{5}{3}R_3}{\longrightarrow} \begin{pmatrix} 1 & 0 & -\frac{11}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -\frac{5}{3} & 0 & \frac{4}{3} \\ \frac{3}{2} & \frac{-5}{6} & \frac{-1}{3} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \overset{R_1=R_1+\frac{11}{3}R_3}{\longrightarrow}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -\frac{7}{2} & \frac{11}{6} & \frac{4}{3} \\ \frac{3}{2} & \frac{-5}{6} & \frac{-1}{3} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

Since the left side is I_3 we stop. Hence

$$A^{-1} = \begin{pmatrix} -\frac{7}{2} & \frac{11}{6} & \frac{4}{3} \\ \frac{3}{2} & \frac{-5}{6} & \frac{-1}{3} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

0.10 Section 3.5 problem 24

Problem Use method of example 8 to find matrix X such that $AX = B$

$$A = \begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix}, B = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 5 & -3 \end{pmatrix}$$

solution

$$AX = B$$

Pre multiply both sides by A^{-1}

$$\begin{aligned} A^{-1}AX &= A^{-1}B \\ I_3 X &= A^{-1}B \\ X &= A^{-1}B \end{aligned} \tag{1}$$

But

$$\begin{aligned} A^{-1} &= \frac{1}{\det(A)} \begin{pmatrix} 7 & -6 \\ -8 & 7 \end{pmatrix} \\ &= \frac{1}{(7 \times 7) - (6 \times 8)} \begin{pmatrix} 7 & -6 \\ -8 & 7 \end{pmatrix} \\ &= \begin{pmatrix} 7 & -6 \\ -8 & 7 \end{pmatrix} \end{aligned}$$

Hence (1) becomes

$$\begin{aligned} X &= \begin{pmatrix} 7 & -6 \\ -8 & 7 \end{pmatrix} \begin{pmatrix} 2 & 0 & 4 \\ 0 & 5 & -3 \end{pmatrix} \\ &= \begin{pmatrix} 14 & -30 & 46 \\ -16 & 35 & -53 \end{pmatrix} \end{aligned}$$

0.11 Section 3.5 problem 30

Problem Suppose that A, B, C are invertible matrices of same size, show that product ABC is invertible and that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

solution

$$\begin{aligned}
(ABC)\left(C^{-1}B^{-1}A^{-1}\right) &= (AB)\left(CC^{-1}\right)\left(B^{-1}A^{-1}\right)\\
&= (AB)\,I\left(B^{-1}A^{-1}\right)\\
&= (AB)\left(B^{-1}A^{-1}\right)\\
&= A\left(BB^{-1}\right)A^{-1}\\
&= AIA^{-1}\\
&= AA^{-1}\\
&I
\end{aligned}$$

And

$$\begin{aligned}
\left(C^{-1}B^{-1}A^{-1}\right)(ABC) &= C^{-1}B^{-1}\left(A^{-1}A\right)BC\\
&= C^{-1}B^{-1}IBC\\
&= C^{-1}B^{-1}BC\\
&= C^{-1}\left(B^{-1}B\right)C\\
&= C^{-1}(I)\,C\\
&= C^{-1}C\\
&= I
\end{aligned}$$

Thus we get I when we multiply ABC on either side by $C^{-1}B^{-1}A^{-1}$. Because the inverse of ABC is unique, this proves that ABC is invertible and that its inverse is $C^{-1}B^{-1}A^{-1}$. QED

0.12 Section 3.5 problem 32

Problem Show that if A is invertible matrix and $AB = AC$ then $B = C$. Thus invertible matrices can be canceled.

solution

Pre multiplying both sides of $AB = AC$ by A^{-1} (which we can do, since we are told A is invertible, then

$$\begin{aligned} A^{-1}AB &= A^{-1}AC \\ \left(A^{-1}A\right)B &= \left(A^{-1}A\right)C \\ IB &= IC \\ B &= C \end{aligned}$$

QED

0.13 Section 3.5 problem 34

Problem Show that a diagonal matrix is invertible iff each diagonal element is non-zero. In this case, state concisely how the inverse matrix is obtained.

solution

An $n \times n$ Matrix A is invertible, if there are elementary row operations which converts A to the identity matrix I_n. Since for a diagonal matrix, we just need to divide each row by its diagonal element in order to make the diagonal element 1 (if it was not already so), then we see immediately, that any diagonal matrix can be converted to I_n this way, unless the diagonal element happened to be zero. Since we can not divide by zero. There are no other operations to make the diagonal element, which is zero, become one. Since all entries above and below the diagonal element (i.e. all elements on the same column as the current zero diagonal element) are zero also by definition. So we are stuck with the zero on the diagonal, and unable to make it 1 using row operations.

Another way to proof this is the following. Since the determinant of diagonal matrix is obtained by just multiplying all the diagonal elements with each others, then if one element is zero, then the whole product is zero, and this means $\det(A) = 0$. But a matrix whose determinant is zero is singular and do not have an inverse. QED.

To obtain the inverse matrix for diagonal matrix with non-zero elements, we simply invert each element on the diagonal. For example

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \frac{1}{a_{11}} & 0 & 0 \\ 0 & \frac{1}{a_{22}} & 0 \\ 0 & 0 & \frac{1}{a_{33}} \end{pmatrix}$$

0.14 Section 3.5 problem 35

Problem Let A be $n \times n$ matrix with either row or column consisting of all zeros. Show that A is not invertible.

solution

An $n \times n$ that has at least one row all zeros, or at least one column all zero, is singular. Meaning its determinant is zero. This is from properties of determinants. Therefore, the matrix is not invertible.

Another proof: A matrix with row all zero, can not have a pivot of 1. Hence it is not possible to transform A to I_n using elementary row operations. Since it is square matrix, if the column is all zeros, then by transposing it, we end up with row which is all zero. Which is the same.