HW 4, Math 320, Spring 2017

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0.1 Section 3.1 problem 11 (page 155)

Problem Use method of elimination to determine if linear system is consistent or not. For each consistent system, find the solution if it is unique. Otherwise, describe the infinite solution set in terms of an arbitrary parameter t as in examples 5 and 7.

$$\begin{aligned} 2x + 7y + 3z &= 11 \\ x + 3y + 2z &= 2 \\ 3x + 7y + 9z &= -12 \end{aligned}$$

Solution

We set up the augmented matrix and do forward elimination. The row operations are given on top of each arrow. For example $R_2 = -\frac{1}{2}R_1 + R_2$ mean that row 2 is replaced by $-\frac{1}{2}$ of the first row added to the second row.

$$\begin{pmatrix} 2 & 7 & 3 & 11 \\ 1 & 3 & 2 & 2 \\ 3 & 7 & 9 & -12 \end{pmatrix} \underset{R_3=R_3-\frac{3}{2}R_1}{\overset{R_2=R_2-\frac{1}{2}R_1}{\Longrightarrow}} \begin{pmatrix} 2 & 7 & 3 & 11 \\ 0 & -0.5 & 0.5 & -3.5 \\ 0 & -3.5 & 4.5 & -28.5 \end{pmatrix} \overset{R_2=-7R_2}{\Longrightarrow}$$

$$\begin{pmatrix} 2 & 7 & 3 & 11 \\ 0 & 3.5 & -3.5 & 24.5 \\ 0 & -3.5 & 4.5 & -28.5 \end{pmatrix} \overset{R_3=R_3+R_2}{\Longrightarrow} \begin{pmatrix} 2 & 7 & 3 & 11 \\ 0 & 3.5 & -3.5 & 24.5 \\ 0 & 0 & 1 & -4 \end{pmatrix}$$

Hence the system of equation now is (from the last matrix above)

$$\begin{aligned} 2x + 7y + 3z &= 11 \\ 3.5y - 3.5z &= 24.5 \\ z &= -4 \end{aligned}$$

Since at the last row, we did not get $0 =$ some number, then the system is consistent. This means the system has either a unique solution, or has infinite number of solutions. But since we did not get $0z = 0$, then the system has a unique solution. Now we will find the unique solution by backward substitution. From last equation, we obtain

$$z = -4$$

From the second equation

$$\begin{aligned} 3.5y - 3.5(-4) &= 24.5 \\ y &= 3 \end{aligned}$$

And from the first equation

$$\begin{aligned} 2x + 7(3) + 3(-4) &= 11 \\ x &= 1 \end{aligned}$$

Hence the solution is

$$\begin{aligned} x &= 1 \\ y &= 3 \\ z &= -4 \end{aligned}$$

0.2 Section 3.1 problem 16

Problem Use method of elimination to determine if linear system is consistent or not. For each consistent system, find the solution if it is unique. Otherwise, describe the infinite solution set in terms of an arbitrary parameter t as in examples 5 and 7.

$$\begin{aligned} x - 3y + 2z &= 6 \\ x + 4y - z &= 4 \\ 5x + 6y + z &= 20 \end{aligned}$$

Solution

We set up the augmented matrix and do forward elimination

$$\begin{pmatrix} 1 & -3 & 2 & 6 \\ 1 & 4 & -1 & 4 \\ 5 & 6 & 1 & 20 \end{pmatrix} \xRightarrow{R_2=R_2-R_1} \begin{pmatrix} 1 & -3 & 2 & 6 \\ 0 & 7 & -3 & -2 \\ 5 & 6 & 1 & 20 \end{pmatrix} \xRightarrow{R_3=-5R_1+R_3}$$

$$\begin{pmatrix} 1 & -3 & 2 & 6 \\ 0 & 7 & -3 & -2 \\ 0 & 21 & -9 & -10 \end{pmatrix} \xRightarrow{R_3=-3R_2+R_3} \begin{pmatrix} 1 & -3 & 2 & 6 \\ 0 & 7 & -3 & -2 \\ 0 & 0 & 0 & -4 \end{pmatrix}$$

Hence the last equation of last matrix above, we see that $0z = -4$. Since this result implies $0 = -4$, which is not possible, then there is no solution. The system is inconsistent. There are no solutions.

0.3 Section 3.1 problem 21

Problem Use method of elimination to determine if linear system is consistent or not. For each consistent system, find the solution if it is unique. Otherwise, describe the infinite solution set in terms of an arbitrary parameter t as in examples 5 and 7.

$$\begin{aligned} x + y - z &= 5 \\ 3x + y + 3z &= 11 \\ 4x + y + 5z &= 14 \end{aligned}$$

solution

We set up the augmented matrix and do forward elimination

$$\begin{pmatrix} 1 & 1 & -1 & 5 \\ 3 & 1 & 3 & 11 \\ 4 & 1 & 5 & 14 \end{pmatrix} \overset{R_2=-3R_1+R_2}{\Longrightarrow} \begin{pmatrix} 1 & 1 & -1 & 5 \\ 0 & -2 & 6 & -4 \\ 4 & 1 & 5 & 14 \end{pmatrix} \overset{R_3=-4R_1+R_3}{\Longrightarrow}$$

$$\begin{pmatrix} 1 & 1 & -1 & 5 \\ 0 & -2 & 6 & -4 \\ 0 & -3 & 9 & -6 \end{pmatrix} \overset{R_3=-1.5R_2+R_3}{\Longrightarrow} \begin{pmatrix} 1 & 1 & -1 & 5 \\ 0 & -2 & 6 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since the final equation has $0z = 0$, this means there are infinite number of solutions. Since any z will satisfy this. The system is therefore consistent. Let $z = t$, hence from the second equation we obtain

$$\begin{aligned} -2y + 6z &= -4 \\ -2y + 6t &= -4 \\ y &= \frac{-4 - 6t}{-2} = 2 + 3t \end{aligned}$$

First equation gives

$$\begin{aligned} x + y - z &= 5 \\ x &= 5 - y + z \\ &= 5 - (2 + 3t) + t \\ &= 3 - 2t \end{aligned}$$

Hence solution is

$$\begin{aligned} x &= 3 - 2t \\ y &= 2 + 3t \\ z &= t \end{aligned}$$

0.4 Section 3.1 problem 31

Problem A system has the form

$$a_1 x + b_1 y = 0$$
$$a_2 x + b_2 y = 0$$

Explain by geometric reasoning why such a system has either a unique solution or infinitely many solutions. In the former case, what is the unique solution?

solution These two equations represent two lines in 2D space. These can be written in standard form as

$$y = -\frac{a_1}{b_1} x$$
$$y = -\frac{a_2}{b_2} x$$

We see now, when we compare each equation above to the equation of a line of the form

$$y = mx + c$$

Where m is the slope, and c is the intercept with the y axis, we see that both lines have zero intercept.

This means both lines pass through the origin, but with possibly different slope. Therefore, since both lines pass though one point, then there is either a unique solution, which is the origin in this case, when $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$, or the other case is the infinite number of solutions when the slope is the same, i.e. $\frac{a_1}{b_1} = \frac{a_2}{b_2}$, which means both lines are on top of each others. (same line).

0.5 Section 3.1 problem 33

Problem The linear system

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$
$$a_3x + b_3y = c_3$$

of three equations in two unknowns, represents three lines L_1, L_2, L_3 in xy plane. Figure 3.1.5 shows six possible configurations of these three lines. In each case describe the solution set of the system.

solution

case a No solution. Since there is not one point where the three lines meet at.

case b Unique solution. Since there is a single point where the three lines intersect at.

case c No solution. Since there is not one point where the three lines meet at.

case d No solutions. All lines are parallel. There is not one point where the three lines meet at

case e Unique solution. There is one single point where the three lines intersect. Even though lines L_1, L_2 are on top of each others.

case f Infinite number of solutions. The three lines are on top of each others.

0.6 Section 3.1 problem 34

Problem Consider the linear system

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

of three equations in three unknowns to represent three planes P_1, P_2, P_3 in xyz plane. Describe the solution in each of the following cases. (a) Three planes are parallel and distinct. (b) The three planes coincide. $P_1 = P_2 = P_3$. (c) P_1 and P_2 coincide and are parallel to P_3. (d) P_1, P_2 intersect in a line L that is parallel to P_3. (e) P_1, P_2 intersect in line L that lies in P_3. (f) P_1, P_2 intersect in a line L that intersect P_3 in a single point.

solution

case a No solution exist. Since three planes do not intersect.

case b There are infinite number of solutions. Since intersection is line.

case c No solution Since P_1, P_2 are parallel to P_3

case d No solution. This is similar to case c.

case e Infinite number of solution, since the intersection between all three planes is a line.

case f Unique solution. Since a single point is found on the three planes.

0.7 Section 3.2 problem 11

Problem Use elementary row operations to transform each augmented coefficient matrix to echelon form then solve the system by back substitution

$$
\begin{aligned}
2x_1 + 8x_2 + 3x_3 &= 2 \\
x_1 + 3x_2 + 2x_3 &= 5 \\
2x_1 + 7x_2 + 4x_3 &= 8
\end{aligned}
$$

solution

We set up the augmented matrix and do forward elimination

$$
\begin{pmatrix} 2 & 8 & 3 & 2 \\ 1 & 3 & 2 & 5 \\ 2 & 7 & 4 & 8 \end{pmatrix} \overset{R_2=-\frac{1}{2}R_1+R_2}{\Longrightarrow} \begin{pmatrix} 2 & 8 & 3 & 2 \\ 0 & -1 & \frac{1}{2} & 4 \\ 2 & 7 & 4 & 8 \end{pmatrix} \overset{R_3=-R_1+R_3}{\Longrightarrow}
$$

$$
\begin{pmatrix} 2 & 8 & 3 & 2 \\ 0 & -1 & \frac{1}{2} & 4 \\ 0 & -1 & 1 & 6 \end{pmatrix} \overset{R_3=-R_2+R_3}{\Longrightarrow} \begin{pmatrix} 2 & 8 & 3 & 2 \\ 0 & -1 & \frac{1}{2} & 4 \\ 0 & 0 & \frac{1}{2} & 2 \end{pmatrix}
$$

The above final matrix is now in echelon form. Since the final equation says that $\frac{1}{2}x_3 = 2$, therefore the system is consistent. Doing backward substitution gives

$$
x_3 = 4
$$

From second equation

$$
\begin{aligned}
-x_2 + \frac{1}{2}x_3 &= 4 \\
-x_2 + \frac{1}{2}(4) &= 4 \\
x_2 &= -2
\end{aligned}
$$

And from first equation

$$
\begin{aligned}
2x_1 + 8x_2 + 3x_3 &= 2 \\
2x_1 + 8(-2) + 3(4) &= 2 \\
x_1 &= 3
\end{aligned}
$$

Hence solution is

$$
\begin{aligned}
x_1 &= 3 \\
x_2 &= -2 \\
x_3 &= 4
\end{aligned}
$$

0.8 Section 3.2 problem 18

Problem Use elementary row operations to transform each augmented coefficient matrix to echelon form then solve the system by back substitution

$$\begin{aligned} 3x_1 - 6x_2 + x_3 + 13x_4 &= 15 \\ 3x_1 - 6x_2 + 3x_3 + 21x_4 &= 21 \\ 2x_1 - 4x_2 + 5x_3 + 26x_4 &= 23 \end{aligned}$$

solution

We set up the augmented matrix and do forward elimination

$$\begin{pmatrix} 3 & -6 & 1 & 13 & 15 \\ 3 & -6 & 3 & 21 & 21 \\ 2 & -4 & 5 & 26 & 23 \end{pmatrix} \overset{R_2=-R_1+R_2}{\Longrightarrow} \begin{pmatrix} 3 & -6 & 1 & 13 & 15 \\ 0 & 0 & 2 & 8 & 6 \\ 2 & -4 & 5 & 26 & 23 \end{pmatrix} \overset{R_3=-2R_1+3R_3}{\Longrightarrow}$$

$$\begin{pmatrix} 3 & -6 & 1 & 13 & 15 \\ 0 & 0 & 2 & 8 & 6 \\ 0 & 0 & 13 & 52 & 39 \end{pmatrix} \overset{R_3=-13R_2+2R_3}{\Longrightarrow} \begin{pmatrix} 3 & -6 & 1 & 13 & 15 \\ 0 & 0 & 2 & 8 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The above final matrix is now in echelon form. Since last row gives $0x_i = 0$, then there are infinite number of solutions, as any x will satisfy this. System is therefore consistent.

Let

$$x_4 = t$$

Hence from the second row, we obtain

$$\begin{aligned} 2x_3 + 8x_4 &= 6 \\ 2x_3 + 8t &= 6 \\ x_3 &= \frac{6-8t}{2} = 3 - 4t \end{aligned}$$

And from the first equation

$$\begin{aligned} 3x_1 - 6x_2 + x_3 + 13x_4 = 15 \\ 3x_1 - 6x_2 &= 15 - x_3 - 13x_4 \\ 3x_1 - 6x_2 &= 15 - (3-4t) - 13t \end{aligned}$$

Let $x_2 = s$ then

$$\begin{aligned} 3x_1 - 6s &= 12 - 9t \\ x_1 &= \frac{12 - 9t + 6s}{3} \\ &= 4 - 3t + 2s \end{aligned}$$

Hence the final solution is

$$\begin{aligned} x_1 &= 4 - 3t + 2s \\ x_2 &= s \\ x_3 &= 3 - 4t \\ x_4 &= t \end{aligned}$$

0.9 Section 3.2 problem 24

problem Determine for what value of k each system has (a) unique solution (b) no solution (c) infinite solutions

$$\begin{aligned} 3x + 2y &= 0 \\ 6x + ky &= 0 \end{aligned}$$

solution

We set up the augmented matrix and do forward elimination

$$\begin{pmatrix} 3 & 2 & 0 \\ 6 & k & 0 \end{pmatrix} \overset{R_2=-2R_1+R_2}{\Longrightarrow} \begin{pmatrix} 3 & 2 & 0 \\ 0 & -4+k & 0 \end{pmatrix}$$

Hence, the last equation says that

$$(-4+k)\,y = 0$$

case a A unique solution exist if $k \neq 4$, since in this case y must be zero. Giving the unique solution $\{y \to 0, x \to 0\}$

case b There is no value of k which causes no solution to exist. Since the RHS is zero in the last equation.

case c If $k = 4$, then we have $0y = 0$. Then any value of y will satisfy this. Hence infinite number of solutions.

0.10 Section 3.2 problem 27

problem Determine for what value of k each system has (a) unique solution (b) no solution (c) infinite solutions

$$\begin{aligned} x+2y+z&=3\\ 2x-y-3z&=5\\ 4x+3y-z&=k \end{aligned}$$

solution

We set up the augmented matrix and do forward elimination

$$\begin{pmatrix} 1 & 2 & 1 & 3\\ 2 & -1 & -3 & 5\\ 4 & 3 & -1 & k \end{pmatrix} \xRightarrow{R_2=-2R_1+R_2} \begin{pmatrix} 1 & 2 & 1 & 3\\ 0 & -5 & -5 & -1\\ 4 & 3 & -1 & k \end{pmatrix}$$

$$\xRightarrow{R_3=-4R_1+R_3} \begin{pmatrix} 1 & 2 & 1 & 3\\ 0 & -5 & -5 & -1\\ 0 & -5 & -5 & k-12 \end{pmatrix} \xRightarrow{R_3=-R_2+R_3} \begin{pmatrix} 1 & 2 & 1 & 3\\ 0 & -5 & -5 & -1\\ 0 & 0 & 0 & k-11 \end{pmatrix}$$

Hence, the last equation says that

$$(0)\,z=k-11$$

case a No k exist which gives unique solution. For if $k=11$, then we have $(0)\,z=0$ and this gives infinite solutions. And if $k\neq 11$, then we have $(0)\,z=$ number. Which says there are no solution.

case b If $k\neq 11$, then we have $(0)\,z=$ number. Which says there are no solution.

case c if $k=11$, then we have $(0)\,z=0$ and this gives infinite solutions

0.11 Section 3.2 problem 28

Problem Under what conditions on the constants a,b,c does the systems

$$\begin{aligned} 2x-y+3z &= a\\ x+2y+z &= b\\ 7x+4y+9z &= c \end{aligned}$$

Have unique solution, no solution, infinite number of solutions?

solution

We set up the augmented matrix and do forward elimination

$$\begin{pmatrix} 2 & -1 & 3 & a\\ 1 & 2 & 1 & b\\ 7 & 4 & 9 & c \end{pmatrix} \overset{R_2=2R_2}{\Longrightarrow} \begin{pmatrix} 2 & -1 & 3 & a\\ 2 & 4 & 2 & 2b\\ 7 & 4 & 9 & c \end{pmatrix} \overset{R_2=R_2-R_1}{\Longrightarrow} \begin{pmatrix} 2 & -1 & 3 & a\\ 0 & 5 & -1 & 2b-a\\ 7 & 4 & 9 & c \end{pmatrix}$$

$$\underset{R_3=2R_3}{\overset{R_1=7R_1}{\Longrightarrow}} \begin{pmatrix} 14 & -7 & 21 & 7a\\ 0 & 5 & -1 & 2b-a\\ 14 & 8 & 18 & 2c \end{pmatrix} \overset{R_3=R_3-R_1}{\Longrightarrow} \begin{pmatrix} 14 & -7 & 21 & 7a\\ 0 & 5 & -1 & 2b-a\\ 0 & 15 & -3 & 2c-7a \end{pmatrix}$$

$$\overset{R_3=R_3-3R_2}{\Longrightarrow} \begin{pmatrix} 14 & -7 & 21 & 7a\\ 0 & 5 & -1 & 2b-a\\ 0 & 0 & 0 & 2c-4a-6b \end{pmatrix}$$

Hence, the last equation says

$$\begin{aligned} (0)\,z &= 2c-4a-6b\\ (0)\,z &= c-2a-3b\\ 0\,(z) &= c-(2a+3b) \end{aligned}$$

If the RHS is zero, then we have infinite number of solutions, since then we end up with $(0)\,z=0$, which means any z will satisfy this equation. But if the RHS is not zero, then we end up with $(0)\,z=$ some number. Which is not possible. Therefore we conclude that

If $c=(2a+3b)$ then infinite number of solutions.

If $c\neq(2a+3b)$ then no solution.

It is not possible to obtain a unique solution.

0.12 Problem 3

Write the following as $Ax = b$ and determine for what values of the parameter k the system has (i) unique solution (ii) no solution, (iii) infinite solutions. (a)

$$\begin{aligned} x_1 + 3x_2 &= 8 \\ -x_1 + 2x_2 - x_3 &= 4 \\ 3x_1 + x_2 + 10x_3 &= k \end{aligned}$$

(b)

$$\begin{aligned} -x_2 + 0.5x_3 &= 0 \\ 4x_1 + 2x_2 + 3x_3 &= 2 \\ 2x_1 + 3x_2 + 0.5x_3 &= k \end{aligned}$$

0.12.1 Part (a)

We write it first as $Ax = b$

$$\overbrace{\begin{pmatrix} 1 & 0 & 3 \\ -1 & 2 & -1 \\ 3 & 1 & 10 \end{pmatrix}}^{A} \overbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}^{b} = \overbrace{\begin{pmatrix} 8 \\ 4 \\ k \end{pmatrix}}^{b}$$

We next set up the augmented matrix and do forward elimination

$$\begin{pmatrix} 1 & 0 & 3 & 8 \\ -1 & 2 & -1 & 4 \\ 3 & 1 & 10 & k \end{pmatrix} \overset{R_2=R_1+R_2}{\Longrightarrow} \begin{pmatrix} 1 & 0 & 3 & 8 \\ 0 & 2 & 2 & 12 \\ 3 & 1 & 10 & k \end{pmatrix}$$

$$\overset{R_3=3R_1-R_3}{\Longrightarrow} \begin{pmatrix} 1 & 0 & 3 & 8 \\ 0 & 2 & 2 & 12 \\ 0 & -1 & -1 & 24-k \end{pmatrix} \overset{R_3=R_2+2R_3}{\Longrightarrow} \begin{pmatrix} 1 & 0 & 3 & 8 \\ 0 & 2 & 2 & 12 \\ 0 & 0 & 0 & 60-2k \end{pmatrix}$$

Therefore, from last equation we see that

$$(0)\, x_3 = 30 - k$$

case (i) It is not possible to have unique solution.

case (iI) If $(30 - k) \neq 0$ then there is no solution, since then we have $0x_3$ = some number, which is not possible. Hence for $k \neq 30$, there is no solution.

case (iii) If $(30 - k) = 0$ or $k = 30$, then there are infinite number of solutions.

0.12.2 Part (b)

We write it first as $Ax = b$

$$\overbrace{\begin{pmatrix} 0 & -1 & 0.5 \\ 4 & 2 & 3 \\ 2 & 3 & 0.5 \end{pmatrix}}^{A} \overbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}^{b} = \overbrace{\begin{pmatrix} 0 \\ 2 \\ k \end{pmatrix}}^{b}$$

We next set up the augmented matrix and do forward elimination

$$\begin{pmatrix} 0 & -1 & 0.5 & 0 \\ 4 & 2 & 3 & 2 \\ 2 & 3 & 0.5 & k \end{pmatrix} \overset{\text{swap}(R_2,R_1)}{\Longrightarrow} \begin{pmatrix} 4 & 2 & 3 & 2 \\ 0 & -1 & 0.5 & 0 \\ 2 & 3 & 0.5 & k \end{pmatrix} \overset{R_3=R_1-2R_3}{\Longrightarrow} \begin{pmatrix} 4 & 2 & 3 & 2 \\ 0 & -1 & 0.5 & 0 \\ 0 & -4 & 2 & 2-2k \end{pmatrix}$$

$$\overset{R_3=-4R_2+R_3}{\Longrightarrow} \begin{pmatrix} 4 & 2 & 3 & 2 \\ 0 & -1 & 0.5 & 0 \\ 0 & 0 & 0 & 2-2k \end{pmatrix}$$

Therefore, from last equation we see that

$$(0)\, x_3 = 1 - k$$

case (i) It is not possible to have unique solution.

case (iI) If $(1-k) \neq 0$ then there is no solution, since then we have $0x_3$ = some number, which is not possible. Hence for $k \neq 1$, there is no solution.

case (iii) If $(1-k) = 0$ or $k = 1$, then there are infinite number of solutions.