HW 2, Math 320, Spring 2017

Nasser M. Abbasi (Discussion section 33864 (341-DIS), 8:50 AM - 9:40 AM Monday)

December 30, 2019

Contents

0.1 Section 1.5 problem 18 (page 56)

Problem Find general solution for $xy' = 2y + x^3 \cos x$

Solution It is a good idea to first check if solution exist and if it is unique. Writing the ODE as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{2y + x^3 \cos x}{x} \end{aligned}$$

We see that $f(x,y)$ is continuous for all $x \neq 0$ and for all y. And

$$\frac{\partial f(x,y)}{\partial y} = \frac{2}{x}$$

This is continuous for all $x \neq 0$. Therefore solution exist and unique in some interval which do not include $x = 0$. Now we will solve the ODE.

$$xy' = 2y + x^3 \cos x$$

Dividing by $x \neq 0$ and rearranging gives

$$y' - \frac{2}{x} y = x^2 \cos x$$

We see that the integrating factor $\mu = e^{\int -\frac{2}{x} dx} = e^{-2 \ln x} = \frac{1}{x^2}$. Hence the above ODE can now be written as exact differential by multiplying both side with μ

$$\begin{aligned} \frac{d}{dx}(\mu y) &= \mu \left(x^2 \cos x\right) \\ \frac{d}{dx}\left(\frac{1}{x^2} y\right) &= \frac{1}{x^2}\left(x^2 \cos x\right) \\ \frac{d}{dx}\left(\frac{1}{x^2} y\right) &= \cos x \end{aligned}$$

Integrating both sides gives

$$\begin{aligned} \frac{1}{x^2} y &= \sin x + c \\ y &= x^2 (\sin x + c); \qquad x \neq 0 \end{aligned}$$

0.2 Section 1.5 problem 22

Problem Find solution for $y' = 2xy + 3x^2 e^{x^2}; y(0) = 5$

Solution It is a good idea to first check if solution exist and if it is unique. Writing the ODE as

$$\begin{aligned} y' &= f\left(x,y\right) \\ &= 2xy + 3x^2 e^{x^2} \end{aligned}$$

We see that $f\left(x,y\right)$ is continuous for all x and for all y. And

$$\frac{\partial f\left(x,y\right)}{\partial y} = 2x$$

This is continuous for all x. Therefore solution exist and unique in some interval. Now we will solve the ODE.

$$y' - 2xy = 3x^2 e^{x^2}$$

We see that the integrating factor $\mu = e^{\int -2x dx} = e^{-x^2}$. Hence the above ODE can now be written as exact differential by multiplying both side with μ

$$\begin{aligned} \frac{d}{dx}\left(\mu y\right) &= \mu\left(3x^2 e^{x^2}\right) \\ \frac{d}{dx}\left(e^{-x^2} y\right) &= e^{-x^2}\left(3x^2 e^{x^2}\right) \\ \frac{d}{dx}\left(e^{-x^2} y\right) &= 3x^2 \end{aligned}$$

Integrating both sides

$$e^{-x^2} y = x^3 + c$$

Hence

$$y = e^{x^2}\left(x^3 + c\right)$$

Now initial conditions $y(0) = 5$ are applied to find c. This gives

$$5 = c$$

Hence the complete solution (or the particular solution for this initial conditions) is

$$y = e^{x^2}\left(x^3 + 5\right)$$

0.3 Section 1.5 problem 25

Problem Find solution for $\left(x^2+1\right)y'+3x^3y=6xe^{\frac{-3}{2}x^2}; y(0)=1$

Solution It is a good idea to first check if solution exist and if it is unique. Writing the ODE as

$$\begin{aligned} y' &= f\left(x,y\right) \\ &= \frac{6xe^{\frac{-3}{2}x^2}-3x^3y}{x^2+1} \end{aligned}$$

We see that $f\left(x,y\right)$ is continuous for all x except when $x^2=-1$ or $x=\pm i$. But this is not on real line hence it will not affect us. It is also continuous for all y.

$$\frac{\partial f\left(x,y\right)}{\partial y}=\frac{3x^3}{x^2+1}$$

Again, this is continuous for all x except when $x^2=-1$ or $x=\pm i$. But this is not on real line hence it will not affect us.

Therefore solution exists and unique for all x and y. Now we will solve the ODE.

$$y'+\frac{3x^3}{x^2+1}y=\frac{6xe^{\frac{-3}{2}x^2}}{x^2+1}$$

Integration factor is $\mu=e^{\int\frac{3x^3}{x^2+1}dx}$. To evaluate the integral:

$$\begin{aligned} \int\frac{3x^3}{x^2+1}dx &= 3\int\frac{x^3}{x^2+1}dx \\ &= 3\int x-\frac{x}{x^2+1}dx \\ &= \frac{3}{2}x^2-3\int\frac{x}{x^2+1}dx \end{aligned}$$

Since $\frac{d}{dx}\ln\left(x^2+1\right)=\frac{2x}{x^2+1}$ then by comparing this to the second integral, we see that $\int\frac{x}{x^2+1}=\frac{1}{2}\ln\left(x^2+1\right)$, hence

$$\int\frac{3x^3}{x^2+1}dx=\frac{3}{2}x^2-\frac{3}{2}\ln\left(x^2+1\right)$$

Therefore

$$
\begin{aligned}
\mu &= e^{\int \frac{3x^3}{x^2+1} dx} \\
&= \exp\left(\frac{3}{2}x^2 - \frac{3}{2}\ln\left(x^2+1\right)\right) \\
&= \exp\left(\frac{3}{2}x^2\right)\exp\left(-\frac{3}{2}\ln\left(x^2+1\right)\right) \\
&= \exp\left(\frac{3}{2}x^2\right)\exp\left(\ln\left(x^2+1\right)^{\frac{-3}{2}}\right) \\
&= \frac{e^{\frac{3}{2}x^2}}{\left(x^2+1\right)^{\frac{3}{2}}}
\end{aligned}
$$

Multiplying both sides of the ODE with this integration factor gives

$$
\begin{aligned}
\frac{d}{dx}\left(\mu y\right) &= \frac{6xe^{\frac{-3}{2}x^2}}{x^2+1}\mu \\
\frac{d}{dx}\left(\frac{e^{\frac{3}{2}x^2}}{\left(x^2+1\right)^{\frac{3}{2}}} y\right) &= \frac{6xe^{\frac{-3}{2}x^2}}{x^2+1}\frac{e^{\frac{3}{2}x^2}}{\left(x^2+1\right)^{\frac{3}{2}}} \\
\frac{d}{dx}\left(\frac{e^{\frac{3}{2}x^2}}{\left(x^2+1\right)^{\frac{3}{2}}} y\right) &= \frac{6x}{\left(x^2+1\right)^{\frac{5}{2}}}
\end{aligned}
$$

Integrating both sides

$$
\frac{e^{\frac{3}{2}x^2}}{\left(x^2+1\right)^{\frac{3}{2}}} y = \int \frac{6x}{\left(x^2+1\right)^{\frac{5}{2}}} dx + c \tag{1}
$$

To evaluate $\int \frac{6x}{\left(x^2+1\right)^{\frac{5}{2}}} dx$, let $u = x^2+1$ hence $du = 2xdx$, therefore the integral becomes

$$
\int \frac{6x}{\left(x^2+1\right)^{\frac{5}{2}}} dx = \int \frac{6x}{u^{\frac{5}{2}}}\frac{du}{2x} = 3\int \frac{1}{u^{\frac{5}{2}}} du = 3\int u^{-\frac{5}{2}} du = 3\left(\frac{u^{-\frac{3}{2}}}{-\frac{3}{2}}\right) = -2u^{-\frac{3}{2}}
$$

Hence

$$
\int \frac{6x}{\left(x^2+1\right)^{\frac{5}{2}}} dx = -2\left(x^2+1\right)^{-\frac{3}{2}} = \frac{-2}{\left(x^2+1\right)^{\frac{3}{2}}}
$$

Hence (1) becomes

$$\frac{e^{\frac{3}{2}x^2}}{\left(x^2+1\right)^{\frac{3}{2}}}y = \frac{-2}{\left(x^2+1\right)^{\frac{3}{2}}} + c$$

$$y = \frac{-2}{\left(x^2+1\right)^{\frac{3}{2}}}\frac{\left(x^2+1\right)^{\frac{3}{2}}}{e^{\frac{3}{2}x^2}} + c\frac{\left(x^2+1\right)^{\frac{3}{2}}}{e^{\frac{3}{2}x^2}}$$

$$y = -2e^{\frac{-3}{2}x^2} + c\left(x^2+1\right)^{\frac{3}{2}} e^{\frac{-3}{2}x^2}$$

Applying $y(0) = 1$ gives

$$1 = -2 + c$$

$$c = 3$$

Hence the particular solution is

$$y = -2e^{\frac{-3}{2}x^2} + 3\left(x^2+1\right)^{\frac{3}{2}} e^{\frac{-3}{2}x^2}$$

$$= e^{\frac{-3}{2}x^2}\left(3\left(x^2+1\right)^{\frac{3}{2}} - 2\right)$$

0.4 Section 1.5 problem 27

Problem Solve the differential equation by regarding y as the independent variable rather than x

$$\left(x\left(y\right)+ye^{y}\right)\frac{dy}{dx\left(y\right)}=1$$

Solution

$$\begin{aligned}\frac{dy}{dx\left(y\right)}&=\frac{1}{x\left(y\right)+ye^{y}}\\\frac{dx\left(y\right)}{dy}&=x\left(y\right)+ye^{y}\\\frac{dx\left(y\right)}{dx}-x\left(y\right)&=ye^{y}\end{aligned}$$

For $x\left(y\right)\neq ye^{y}$. Hence

$$\begin{aligned}\frac{dx\left(y\right)}{dy}&=x\left(y\right)+ye^{y}\\\frac{dx\left(y\right)}{dx}-x\left(y\right)&=ye^{y}\end{aligned}$$

Integrating factor is $\mu=e^{-\int dy}=e^{-y}$. Multiplying both sides with μ gives

$$\begin{aligned}\frac{d}{dy}\left(\mu x\right)&=\mu ye^{y}\\\frac{d}{dy}(e^{-y}x)&=y\end{aligned}$$

Integrating both sides

$$e^{-y}x\left(y\right)=\frac{y^{2}}{2}+c$$

Therefore

$$x\left(y\right)=\left(\frac{y^{2}}{2}+c\right)e^{y}$$

0.5 Section 1.5 problem 31

Problem (a) show that $y_c(x) = Ce^{-\int P(x)dx}$ is a general solution of $\frac{dy}{dx} + P(x)y = 0$. (b) Show that $y_p(x) = e^{-\int P(x)dx} \int \left(Q(x)e^{\int P(x)dx}\right) dx$ is a particular solution of $\frac{dy}{dx} + P(x)y = Q(x)$. (c) Suppose that $y_c(x)$ is any general solution of $\frac{dy}{dx} + P(x)y = 0$ and that $y_p(x)$ is any particular solution of $\frac{dy}{dx} + P(x)y = Q(x)$. Show that $y(x) = y_c(x) + y_p(x)$ is a general solution of $\frac{dy}{dx} + P(x)y = Q(x)$

Solution

0.5.1 Part (a)

Given

$$\frac{dy}{dx} + P(x)y = 0$$

Then

$$\frac{dy}{y} = -P(x)dx$$

Integrating both sides

$$\begin{aligned} \ln|y| &= -\int P(x)dx + C \\ y(x) &= Ce^{-\int P(x)dx} \end{aligned}$$

QED. We can also solve this by substituting $y(x) = Ce^{-\int P(x)dx}$ into $\frac{dy}{dx} + P(x)y = 0$ which gives

$$\Delta = \frac{d}{dx}\left(Ce^{-\int P(x)dx}\right) + P(x)Ce^{-\int P(x)dx} \tag{1}$$

But $\frac{d}{dx}e^{g(x)} = g'(x)e^{g(x)}$, hence

$$\begin{aligned} \frac{d}{dx}\left(Ce^{-\int P(x)dx}\right) &= C\frac{d}{dx}\left(-\int P(x)dx\right)e^{-\int P(x)dx} \\ &= -CP(x)e^{-\int P(x)dx} \end{aligned}$$

Therefore (1) becomes

$$\begin{aligned} \Delta &= -CP(x)e^{-\int P(x)dx} + P(x)Ce^{-\int P(x)dx} \\ &= 0 \end{aligned}$$

Hence the solution $y(x) = Ce^{-\int P(x)dx}$ satisfies the ODE. Therefore it is solution.

0.5.2 Part(b)

Given $\frac{dy}{dx}+P(x)y=Q(x)$, the integrating factor is $\mu=e^{\int P(x)dx}$. Multiplying this by both sides of the ODE gives

$$
\begin{aligned}
e^{\int P(x)dx}\frac{dy}{dx}+e^{\int P(x)dx}P(x)y &= e^{\int P(x)dx}Q(x)\\
\frac{d}{dx}\left(e^{\int P(x)dx}y(x)\right) &= e^{\int P(x)dx}Q(x)
\end{aligned}
$$

Integrating both sides

$$
\begin{aligned}
e^{\int P(x)dx}y(x) &= \int e^{\int P(x)dx}Q(x)\,dx+C\\
y(x) &= e^{-\int P(x)dx}\left(\int e^{\int P(x)dx}Q(x)\,dx\right)+Ce^{-\int P(x)dx}
\end{aligned}
$$

For particular $C=0$, we obtain

$$
y_p(x)=e^{-\int P(x)dx}\left(\int e^{\int P(x)dx}Q(x)\,dx\right)
$$

Which is what we asked to show.

0.5.3 Part(c)

Let

$$
\begin{aligned}
y(x) &= y_c(x)+y_p(x)\\
&= Ce^{-\int P(x)dx}+e^{-\int P(x)dx}\left(\int e^{\int P(x)dx}Q(x)\,dx\right)
\end{aligned}
$$

We need now to substitute this in $\frac{dy}{dx}+P(x)y=Q(x)$ and see if it satisfies it. First we find $\frac{dy}{dx}$

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\left[Ce^{-\int P(x)dx}\right]+\frac{d}{dx}\left[e^{-\int P(x)dx}\left(\int e^{\int P(x)dx}Q(x)\,dx\right)\right]\\
&= Ce^{-\int P(x)dx}(-P(x))+\frac{d}{dx}e^{-\int P(x)dx}\left(\int e^{\int P(x)dx}Q(x)\,dx\right)+e^{-\int P(x)dx}\frac{d}{dx}\int e^{\int P(x)dx}Q(x)\,dx\\
&= -CP(x)e^{-\int P(x)dx}+e^{-\int P(x)dx}(-P(x))\left(\int e^{\int P(x)dx}Q(x)\,dx\right)+e^{-\int P(x)dx}\left(e^{\int P(x)dx}Q(x)\right)\\
&= -CP(x)e^{-\int P(x)dx}-P(x)e^{-\int P(x)dx}\left(\int e^{\int P(x)dx}Q(x)\,dx\right)+e^{-\int P(x)dx}\left(e^{\int P(x)dx}Q(x)\right)\\
&= -CP(x)e^{-\int P(x)dx}-P(x)e^{-\int P(x)dx}\left(\int e^{\int P(x)dx}Q(x)\,dx\right)+Q(x)\\
&= -P(x)e^{-\int P(x)dx}\left[\int e^{\int P(x)dx}Q(x)\,dx+C\right]+Q(x)
\end{aligned}
$$

Substituting the above into the left hand side of the given $\frac{dy}{dx}+P(x)y=Q(x)$

$$
\begin{aligned}
LHS &= \frac{dy}{dx}+P(x)y\\
&= -P(x)e^{-\int P(x)dx}\left[\int e^{\int P(x)dx}Q(x)\,dx+C\right]+Q(x)+P(x)\left[Ce^{-\int P(x)dx}+e^{-\int P(x)dx}\left(\int e^{\int P(x)dx}Q(x)\,dx\right)\right]\\
&= \overbrace{-P(x)e^{-\int P(x)dx}\left[\int e^{\int P(x)dx}Q(x)\,dx+C\right]}+Q(x)+\overbrace{P(x)e^{-\int P(x)dx}\left[C+\left(\int e^{\int P(x)dx}Q(x)\,dx\right)\right]}
\end{aligned}
$$

We see that the first term in the RHS above and the third term cancel each others. Hence

$$LHS = Q(x)$$

Which is the right side of the ODE. Hence the solution $y(x)=y_c(x)+y_p(x)$ satisfies the ODE.

QED.

0.6 Section 1.5 problem 37

Problem A 400 gal tank initially contains 100 gal of brine containing 50 lb of salt. Brine containing 1 lb of salt per gallon enters the tank at rate 5 gal/s and the well mixed brine in the tank flows out at rate of 3 gal/s. How much salt will the tank contain when it is full of brine?

Solution

To reduce confusion, let x be the substance which causes the concentration in the Brine. Let $Q(t)$ be the mass (normally called the amount, but saying mass is more clear than saying amount) of x at time t . Hence $Q(0) = 50$ lb. The goal is to find an ODE that describes how $Q(t)$ changes in time. That is, how the mass of x in the tank changes in time. Using

$$\frac{dQ}{dt} = R_{in} - R_{out}$$

Where R_{in} rate of mass of salt entering the tank per second. And R_{out} is rate of mass of salt leaving the tank per second. But

$$R_{in} = 5 \text{ lb/sec}$$

And

$$R_{out} = \frac{Q(t)}{V(t)}\frac{[\text{lb}]}{[\text{gal}]} \times 3\frac{[\text{gal}]}{[\text{second}]} = \frac{3}{V(t)}Q(t)$$

Where $V(t)$ is *current* volume of brine in tank at time t. Hence the ODE is

$$\frac{dQ}{dt} = 5 - \frac{3}{V(t)}Q(t)$$

$$\frac{dQ}{dt} + \frac{3}{V(t)}Q(t) = 5 \tag{1}$$

But we can find $V(t)$. Since initially $V(0) = 100$ gal, and in one second 5 gal enters, and 3 gal exists, then

$$V(t) = 100 + 2t$$

Hence (1) becomes

$$\boxed{\frac{dQ}{dt} + \frac{3}{100+2t}Q(t) = 5}$$

Integrating factor is

$$\mu = e^{\int \frac{3}{100+2t}dt} = e^{3\int \frac{1}{100+2t}dt} = e^{\frac{3}{2}\ln(100+2t)} = (100+2t)^{\frac{3}{2}}$$

Hence (1) becomes

$$\frac{d}{dt}\left(\mu Q\right) = 5\mu$$

Integrating both sides

$$\mu Q = 5\int \mu dt + c$$
$$(100+2t)^{\frac{3}{2}} Q = 5\int (100+2t)^{\frac{3}{2}}\, dt + c$$
$$(100+2t)^{\frac{3}{2}} Q = (100+2t)^{\frac{5}{2}} + c$$

Hence

$$Q(t) = (100+2t) + c\,(100+2t)^{\frac{-3}{2}}$$

But at $t = 0$, $Q(0) = 50$, hence

$$50 = 100 + c\,(100)^{\frac{-3}{2}}$$
$$c = -50\,000$$

Hence the solution is

$$Q(t) = (100+2t) - 50\,000\,(100+2t)^{\frac{-3}{2}} \tag{2}$$

This gives us the mass of salt at time t. What we need now to find out is the time it will take to fill the tank say t_{end}, and use that time to find $Q(t_{end})$ from above. Since initially the tank had 300 gallons remains to be filled, and the flow in is at rate of 5 gal/sec and flow out is at 3 gal/sec, then in one second, the tank will fill up with 2 gallons. Hence it will take

$$t = \frac{300}{2} = 150 \text{ sec}$$

To fill the tank. Using this value of t in (2) gives

$$\begin{aligned} Q(150) &= (100 + 2(150)) - 50\,000\,(100 + 2(150))^{\frac{-3}{2}} \\ &= \frac{1575}{4} \\ &= 393.75 \text{ lb} \end{aligned}$$

0.7 Section 1.5 problem 44

Problem: Figure 1.5.8 shows a slope field and typical solution curves for $y' = x + y$. (a) show that every curve approaches the straight line $y = -x - 1$ as $x \to -\infty$. (b) for each of the five values $y_1 = -10, -5, 0, 5, 10$, determined the initial value y_0 (accurate to 4 decimal points) such that $y(5) = y_1$ for the solution satisfying the initial condition $y(-5) = y_0$

Solution:

0.7.1 Part(a)

$$\begin{aligned} y' &= x + y \\ y' - y &= x \end{aligned}$$

Integrating factor is $\mu = e^{-\int dx} = e^{-x}$. Multiplying the above with μ results in

$$\begin{aligned} \frac{d}{dx}\left(\mu y\right) &= \mu x \\ \frac{d}{dx}\left(e^{-x} y\right) &= e^{-x} x \end{aligned}$$

Integrating both sides

$$e^{-x} y = \int x e^{-x} dx + c$$

But $\int x e^{-x} dx = e^{-x}\left(-1 - x\right)$ using integration by parts. Hence the above becomes

$$\begin{aligned} e^{-x} y &= e^{-x}\left(-1 - x\right) + c \\ y &= \left(-1 - x\right) + c e^{x} \end{aligned} \tag{1}$$

But

$$\lim_{x \to -\infty} e^{x} = 0$$

Hence solution becomes (at large negative x)

$$y = -1 - x$$

Therefore, solution curves approach line $-1 - x$.

0.7.2 Part(b)

The solution is $y = \left(-1 - x\right) + c e^{x}$ from part (a). Using $y(-5) = y_0$, then

$$\begin{aligned} y_0 &= \left(-1 + 5\right) + c e^{-5} \\ y_0 &= 4 + c e^{-5} \\ c &= \left(y_0 - 4\right) e^{5} \end{aligned}$$

Hence solution is

$$\begin{aligned} y &= \left(-1 - x\right) + \left(y_0 - 4\right) e^{5} e^{x} \\ &= \left(-1 - x\right) + \left(y_0 - 4\right) e^{x+5} \end{aligned} \tag{2}$$

Now we need to find y_0 such as $\underline{y(5) = -10}$. From (2)

$$\begin{aligned}-10 &= (-1-5) + \left(y_0 - 4\right) e^{10}\\ y_0 &= (-10+6)\, e^{-10} + 4\\ &= 3.99982\end{aligned}$$

For $\underline{y(5) = -5}$, from (2)

$$\begin{aligned}-5 &= (-1-5) + \left(y_0 - 4\right) e^{10}\\ y_0 &= (-5+6)\, e^{-10} + 4\\ &= 4.00005\end{aligned}$$

For $\underline{y(5) = 0}$ from (2)

$$\begin{aligned}0 &= (-1-5) + \left(y_0 - 4\right) e^{10}\\ y_0 &= 6e^{-10} + 4\\ &= 4.00027\end{aligned}$$

For $\underline{y(5) = 5}$ from (2)

$$\begin{aligned}5 &= (-1-5) + \left(y_0 - 4\right) e^{10}\\ y_0 &= (5+6)\, e^{-10} + 4\\ &= 4.00050\end{aligned}$$

For $\underline{y(5) = 10}$ from (2)

$$\begin{aligned}10 &= (-1-5) + \left(y_0 - 4\right) e^{10}\\ y_0 &= (10+6)\, e^{-10} + 4\\ &= 4.00073\end{aligned}$$

0.8 Section 2.1 problem 3

Problem: Solve $\frac{dx}{dt}=1-x^2; x(0)=3$ and sketch solution

Solution:

$$\begin{aligned}\frac{dx}{dt} &= 1-x^2\\ \frac{dx}{1-x^2} &= dt \end{aligned}\tag{1}$$

For $1-x^2\neq 0$ or for $x\neq\pm1$. But

$$\int\frac{dx}{1-x^2}=\int\frac{dx}{(1+x)(1-x)}$$

Where $\frac{1}{(1+x)(1-x)}=\frac{A}{(1+x)}+\frac{B}{(1-x)}$. But $A=\left(\frac{1}{(1-x)}\right)_{x=-1}=\frac{1}{2}$ and $B=\left(\frac{1}{(1+x)}\right)_{x=1}=\frac{1}{2}$, hence

$$\begin{aligned}\int\frac{dx}{(1+x)(1-x)} &= \frac{1}{2}\int\frac{dx}{(1+x)}+\frac{1}{2}\int\frac{dx}{(1-x)}\\ &= \frac{1}{2}\ln|(1+x)|-\frac{1}{2}\ln|(1-x)|\end{aligned}$$

Therefore (1) becomes

$$\begin{aligned}\frac{1}{2}\ln|(1+x)|-\frac{1}{2}\ln|(1-x)| &= \int dt\\ \ln\left|\frac{(1+x)}{(1-x)}\right| &= \int 2dt\\ \ln\left|\frac{(1+x)}{(1-x)}\right| &= 2t+c\\ \frac{(1+x)}{(1-x)} &= ce^{2t}\\ (1+x) &= (1-x)\,ce^{2t}\\ 1+x &= ce^{2t}-xce^{2t}\\ x+xce^{2t} &= ce^{2t}-1\\ x &= \frac{ce^{2t}-1}{1+ce^{2t}}\end{aligned}$$

Now we use initial conditions $x(0)=3$ to find c

$$\begin{aligned}3 &= \frac{c-1}{1+c}\\ c &= -2\end{aligned}$$

Hence solution is

$$\begin{aligned}x &= \frac{-2e^{2t}-1}{1-2e^{2t}}\\ &= \frac{1+2e^{2t}}{2e^{2t}-1}\end{aligned}$$

Here is a plot of the above solution and two other solutions starting from different initial conditions

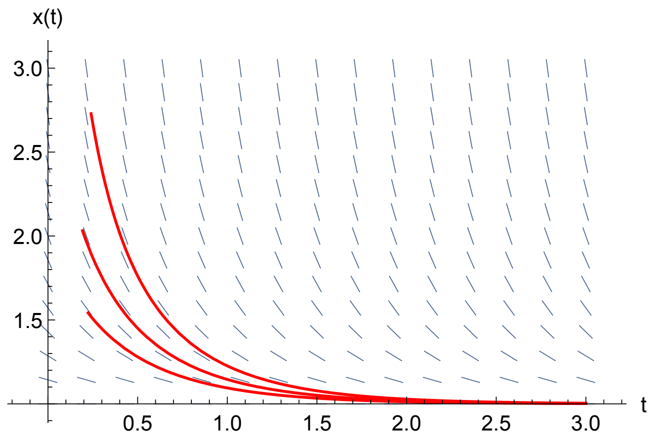


Figure 1: Problem 2.1, 3

0.9 Section 2.1 problem 13

Problem: Consider a breed of rabbits whose birth and death rates β, δ are each proportional to the rabbit population $P = P(t)$ with $\beta > \delta$. (a) Show that $P(t) = \frac{P(0)}{1-kP(0)t}$, where k constant. Note that $P(t) \to \infty$ as $t \to \frac{1}{(kP(0))}$. This is the doomsday. (b) Suppose that $P(0) = 6$ and that there are nine rabbits after ten months. When does doomsday occur?

0.9.1 Part(a)

For doomsday, per book page 86, we use the model that birth rate occur at rate $\beta \propto P^2(t)$ per unit time per population, but in this problem, since death rate is not constant, but also proportional to the rabbit population, then we also make $\delta \propto P^2(t)$ where $\beta > \delta$. Hence we write

$$\frac{dP(t)}{dt} = kP^2(t)$$

Where k is the combined constant of proportionality. This is separable.

$$\begin{aligned} \frac{dP(t)}{P^2(t)} &= kdt \\ \int \frac{dP(t)}{P^2(t)} &= \int kdt \\ -\frac{1}{P} &= kt + c \\ P(t) &= \frac{1}{c - kt} \qquad (1) \end{aligned}$$

Using initial conditions, $t = 0, P(0)$ we find c

$$\begin{aligned} P(0) &= \frac{1}{c} \\ c &= \frac{1}{P(0)} \end{aligned}$$

Hence (1) becomes

$$\begin{aligned} P(t) &= \frac{1}{\frac{1}{P(0)} - kt} \\ &= \frac{P(0)}{1 - P(0)kt} \qquad (2) \end{aligned}$$

0.9.2 Part (b)

Applying initial conditions to (2) in part (a)

$$\begin{aligned} P(t) &= \frac{P(0)}{1 - kP(0)\,t} \\ 9 &= \frac{6}{1 - k(6)(10)} \\ k &= \frac{1}{180} \end{aligned}$$

Hence solution becomes

$$P(t) = \frac{6}{1 - \frac{6}{180}t}$$

When $t = \frac{180}{6} = 30$ months, then $P(t) \to \infty$. Hence <u>30 months is doomsday</u>.

0.10 Section 2.1 problem 15

Problem Consider population $P(t)$ satisfying logistic equation $\frac{dP}{dt} = aP - bP^2$ where $B = aP$ is the time rate at which birth occur and $D = bP^2$ is the rate at which death occur. If the initial population is $P(0)$ and $B(0), D(0)$ are the rates per month at $t = 0$, show that the limiting population is $M = \frac{B(0)P(0)}{D(0)}$

Solution

For the limiting model, per book page 82 (limiting population and carrying capacity), we can use

$$\begin{aligned} \frac{dP}{dt} &= aP - bP^2 \\ &= a\left(1 - \frac{b}{a}P\right)P \\ &= a\left(1 - \frac{P}{M}\right)P \end{aligned}$$

note: In class lecture, the above is written as $\frac{dP}{dt} = r\left(1 - \frac{P}{k}\right)p$, where $r = a$ and $k = M$) But book uses different notations. M is the limiting capacity (or also called equilibrium population). Hence from the above, we see that

$$M = \frac{a}{b} \tag{1}$$

But a, which is the growth rate per time per population is

$$a = \frac{B_0}{P_0}$$

And $D(0) = bP^2(0)$, hence

$$b = \frac{D_0}{P_0^2}$$

Therefore (1) becomes

$$\begin{aligned} M &= \frac{\frac{B_0}{P_0}}{\frac{D_0}{P_0^2}} \\ &= \frac{B_0}{D_0}P_0 \end{aligned}$$

QED.

0.11 Section 2.1 problem 17

Problem Consider rabbit population $P(t)$ satisfying the logistic equation as in problem 15. If the initial population is 240 rabbits and there are 9 births per month and 12 death per month occurring at time $t = 0$, how many months does it take for $P(t)$ to reach 105% of the limiting population M?

Solution The logistic equation, from problem 15 is

$$\frac{dP}{dt} = aP - bP^2$$

From problem 15: Where

$$B = aP$$

Is the time rate at which birth occur and

$$D = bP^2$$

Is the rate at which death occur and $P(t)$ is current size of population. Per problem 15, we know that the limiting population is

$$M = \frac{B(0)P(0)}{D(0)} = \frac{B(0)P(0)}{D(0)}$$

But we are given here, that $P(0) = 240, B(0) = 9$ per month and $D(0) = 12$ per month. This means

$$M = \frac{9(240)}{12} = 180$$

The above is the limiting population size. We now need to solve (1) in order to answer the question

$$\frac{dP}{dt} = aP - bP^2$$

This is separable

$$
\begin{aligned}
\frac{dP}{aP-bP^2} &= dt\\
\int\frac{dP}{aP-bP^2} &= t+c\\
\int\frac{dP}{P(a-bP)} &= t+c\\
\int\frac{1}{aP}-\frac{b}{a(bP-a)}dP &= t+c\\
\frac{1}{a}\ln|P|-\frac{1}{a}\ln|bP-a| &= t+c\\
\frac{1}{a}\ln\left|\left(\frac{bP}{bP-a}\right)\right| &= t+c\\
\frac{1}{a}\ln\left|\left(\frac{bP}{bP-a}\right)\right| &= t+c\\
\ln\left|\left(\frac{bP}{bP-a}\right)\right| &= at+ac\\
\frac{bP}{bP-a} &= c_1e^{at}
\end{aligned}
$$

Where the sign is determined by constant c_1. Hence the above becomes

$$
\begin{aligned}
bP &= c_1e^{at}(bP-a)\\
&= c_1e^{at}bP-c_1ae^{at}\\
bP-c_1e^{at}bP &= -c_1ae^{at}\\
P\left(b-c_1e^{at}b\right) &= -c_1ae^{at}\\
P(t) &= \frac{-c_1ae^{at}}{b-c_1e^{at}b}\\
&= \frac{c_1ae^{at}}{c_1e^{at}b-b}\\
P(t) &= \frac{a}{b-\frac{b}{c_1}e^{-at}}
\end{aligned}
$$

We now need to find c_1 from initial conditions. At $t=0$, $P(0)=240$, hence since $B=aP$ then

$$
\begin{aligned}
a(0) &= \frac{B(0)}{P(0)}=\frac{9}{240}\\
&= \frac{3}{80}
\end{aligned}
$$

And since $D=bP^2$ then

$$
\begin{aligned}
b(0) &= \frac{D(0)}{p(0)^2}=\frac{12}{240^2}\\
&= \frac{1}{4800}
\end{aligned}
$$

Therefore, at $t = 0$, the above solution becomes

$$P(0) = \frac{c_1 a(0) e^{at}}{c_1 e^{at} b(0) - b(0)}$$

$$240 = \frac{c_1 a(0)}{c_1 b(0) - b(0)} = \frac{c_1 \frac{3}{80}}{\frac{1}{4800}(c_1 - 1)}$$

$$240\left(\frac{1}{4800}(c_1 - 1)\right) = c_1 \frac{3}{80}$$

$$\frac{1}{20}c_1 - \frac{1}{20} = c_1 \frac{3}{80}$$

$$\frac{1}{20}c_1 - c_1 \frac{3}{80} = \frac{1}{20}$$

$$c_1\left(\frac{1}{20} - \frac{3}{80}\right) = \frac{1}{20}$$

$$c_1\left(\frac{1}{80}\right) = \frac{1}{20}$$

$$c_1 = 4$$

Hence solution is

$$P(t) = \frac{4ae^{at}}{4e^{at}b - b}$$

$$= \frac{4\left(\frac{3}{80}\right)e^{\frac{3}{80}t}}{4e^{\left(\frac{3}{80}\right)t}\left(\frac{1}{4800}\right) - \frac{1}{4800}}$$

We now solve for t when $P(t) = 105\%$ of M

$$\frac{105}{100}(180) = \frac{4\left(\frac{3}{80}\right)e^{\frac{3}{80}t}(4800)}{4e^{\left(\frac{3}{80}\right)t} - 1}$$

$$189\left(4e^{\left(\frac{3}{80}\right)t} - 1\right) = 720e^{\frac{3}{80}t}$$

$$756e^{\left(\frac{3}{80}\right)t} - 189 = 720e^{\frac{3}{80}t}$$

$$756e^{\left(\frac{3}{80}\right)t} - 720e^{\frac{3}{80}t} = 189$$

$$e^{\frac{3}{80}t} = \frac{189}{36}$$

$$\frac{3}{80}t = \ln\frac{189}{36}$$

$$t = \frac{80}{3}\ln\frac{189}{36}$$

$$= 44.219 \text{ months}$$

0.12 Section 2.1 problem 30

Problem A tumor may be regarded as population of multiplying cells. The birth rate of cells in a tumor decreases exponentially with time so that $\beta(t) = \beta_0 e^{-\alpha t}$ where α, β_0 are positive constants. Hence $\frac{dP}{dt} = \beta_0 e^{-\alpha t} P$ with $P(0) = P_0$. Solve the initial value problem for $P(t) = P_0 e^{\left(\frac{\beta_0}{\alpha}\left(1-e^{-\alpha t}\right)\right)}$. Observe that $P(t)$ approaches finite limiting population $P_0 e^{\left(\frac{\beta_0}{\alpha}\right)}$ as $t \to \infty$.

Solution

$$\frac{dP}{dt} = \beta_0 e^{-\alpha t} P$$

This is separable.

$$\frac{dP}{P} = \beta_0 e^{-\alpha t} dt$$

Integrating

$$\begin{aligned} \ln|P| &= \beta_0 \int e^{-\alpha t} dt \\ &= \beta_0 \frac{e^{-\alpha t}}{-\alpha} + C \end{aligned}$$

Hence

$$P(t) = C e^{-\beta_0 \frac{e^{-\alpha t}}{\alpha}} \tag{1}$$

Applying initial condition on the above gives

$$\begin{aligned} P(0) = P_0 &= C e^{-\beta_0 \frac{1}{\alpha}} \\ C &= P_0 e^{\beta_0 \frac{1}{\alpha}} \end{aligned}$$

Therefore the solution (1) becomes

$$\begin{aligned} P(t) &= P_0 e^{\beta_0 \frac{1}{\alpha}} e^{-\beta_0 \frac{e^{-\alpha t}}{\alpha}} \\ &= P_0 e^{-\beta_0 \frac{e^{-\alpha t}}{\alpha} + \frac{\beta_0}{\alpha}} \\ &= P_0 e^{\frac{\beta_0}{\alpha}\left(1-e^{-\alpha t}\right)} \end{aligned}$$

As $t \to \infty$ then $e^{-\alpha t} \to 0$ since $\alpha > 0$, hence the above becomes

$$P(\infty) = M = P_0 e^{\frac{\beta_0}{\alpha}}$$

The above is the limiting population.

0.13 Section 2.1 problem 31

<u>Problem</u> For tumor in problem 30, suppose that at $t = 0$, there are $P_0 = 10^6$ cells and that $P(t)$ is then increasing at rate 3×10^5 cells per month. After 6 months the tumor has doubled (in size and number of cells). Solve numerially for α and then find the limiting population of tumor.

<u>Solution</u> From problem (30) we found

$$\begin{aligned} P(t) &= P_0 e^{\frac{\beta_0}{\alpha}\left(1-e^{-\alpha t}\right)} \\ &= 10^6 e^{\frac{\beta_0}{\alpha}\left(1-e^{-\alpha t}\right)} \end{aligned}$$

Then, at $t = 0$, we are told $\left(\frac{dP(t)}{dt}\right)_{t=0} = 3\times10^5$ (cells per month). Hence, since $\frac{dP}{dt} = \beta_0 e^{-\alpha t} P$ then at $t = 0$

$$\begin{aligned} 3\times10^5 &= \beta_0 P_0 \\ &= \beta_0 10^6 \end{aligned}$$

Therefore

$$\beta_0 = \frac{3\times10^5}{10^6} = 0.3$$

We also told that after 6 months, the number of cells has doubled. This means, using $t = 6$ (with units of month) that

$$\begin{aligned} P(6) &= 2P_0 \\ 10^6 e^{\frac{\beta_0}{\alpha}\left(1-e^{-6\alpha}\right)} &= 2\times10^6 \end{aligned}$$

But $\beta_0 = 0.3$, hence the above becomes

$$\begin{aligned} e^{\frac{3}{10\alpha}\left(1-e^{-6\alpha}\right)} &= 2 \\ \frac{3}{10\alpha}\left(1-e^{-6\alpha}\right) &= \ln 2 \\ 10\alpha\ln 2 &= 3 - 3e^{-6\alpha} \\ 10\alpha\ln 2 + 3e^{-6\alpha} &= 3 \end{aligned}$$

Using a computer, the solutions are $\alpha = 0$ or <u>$\alpha = 0.3915$</u>

Now the limiting population is found. From $P(t) = P_0 e^{\frac{\beta_0}{\alpha}\left(1-e^{-\alpha t}\right)}$, for large t and since $\alpha > 0$ this becomes

$$\begin{aligned} \lim_{t\to\infty} P(t) &= P_0 e^{\frac{\beta_0}{\alpha}} \\ &= 10^6 e^{\frac{0.3}{0.3915}} \\ &= 2.1518\times10^6 \end{aligned}$$

The above is limit of number of cells for large t.

0.14 Section 2.2 problem 7

Problem Solve for $f(x) = 0$ to find critical points. Then analyze the sign of $f(x)$ to determine if each critical point is stable or not and construct the phase diagram for the differential equation. Next solve the ODE. Finally plot the slope field and verify visually the stability of each critital point.

$$\frac{dx}{dt} = f(x) = (x-2)^2$$

Solution The critial points are x values (dependent variable values) where $f(x) = 0$. Hence

$$\begin{aligned}(x-2)^2 &= 0\\ x &= 2\end{aligned}$$

Since $f(x)$ is always positive, this means if x started at something just below $x = 2$, say $x = 1.5$, then eventually x will reach $x = 2$ and stay there. But if x is started at something just about $x = 2$, say $x = 2.5$, then x will keep increasing away from $x = 2$. This means $x = 2$ is semi stable critial since if we start below it, we reach it, but not if we start about it. Hence the phase diagram is

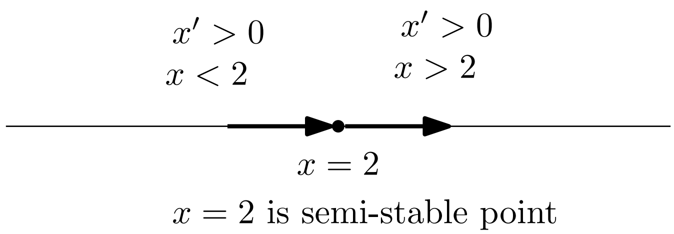


Figure 2: Phase diagram, 2.2 problem 7

Now the ODE is solved $\frac{dx}{dt} = (x-2)^2$. This is non-linear seperable

$$\begin{aligned}\frac{dx}{(x-2)^2} &= dt \qquad x \neq 2\\ \int \frac{dx}{(x-2)^2} &= \int dt\end{aligned}$$

Let $x - 2 = u \rightarrow \frac{du}{dx} = 1$, therefore $\int \frac{dx}{(x-2)^2} = \int \frac{du}{u^2} = -\frac{1}{u} = -\frac{1}{x-2}$ and the above becomes

$$\begin{aligned}-\frac{1}{x-2} &= t + c\\ x &= 2 - \frac{1}{t+c}\end{aligned}$$

Let $x(0) = x_0$, therefore

$$\begin{aligned}x_0 &= 2 - \frac{1}{c}\\ c &= \frac{1}{2 - x_0}\end{aligned}$$

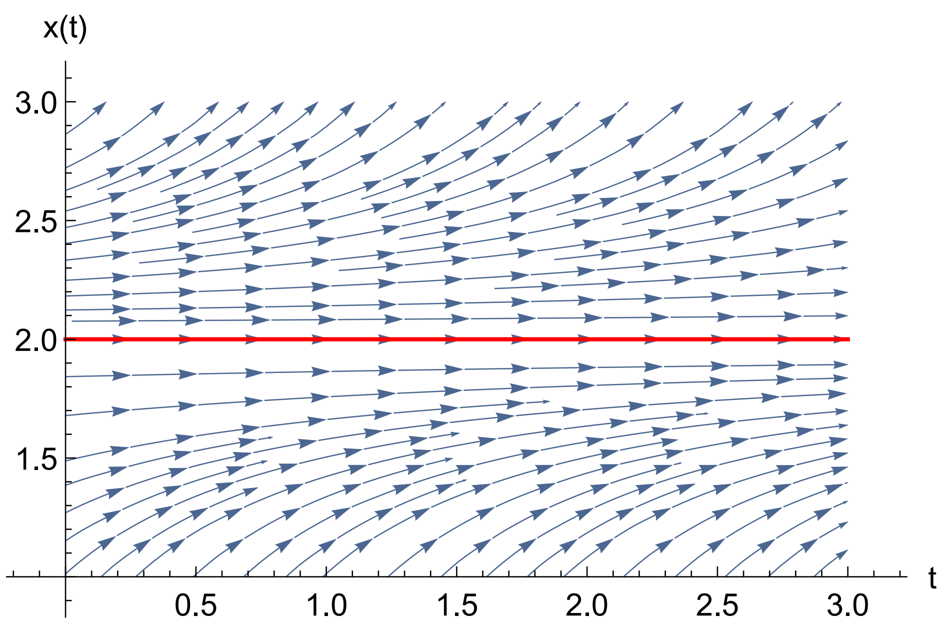
And the solution becomes

$$\begin{aligned}
x &= 2 - \frac{1}{t + \frac{1}{2-x_0}} \\
&= 2 - \frac{2 - x_0}{t(2 - x_0) + 1} \\
&= \frac{2(t(2 - x_0) + 1) - 2 + x_0}{t(2 - x_0) + 1} \\
&= \frac{2t(2 - x_0) + x_0}{t(2 - x_0) + 1} \\
&= \frac{4t - 2tx_0 + x_0}{2t - x_0 t + 1}
\end{aligned}$$

Hence

$$\boxed{x(t) = \frac{(2t-1)x_0 - 4t}{tx_0 - 2t - 1}}$$

Here is slope field plot



From the above plot, we see the solution lines are moving away from $x = 2$ when they start from $x > 2$ but move towards $x = 2$ when starting from $x < 2$.

0.15 Section 2.2 problem 10

Problem Solve for $f(x) = 0$ to find critical points. Then analyze the sign of $f(x)$ to determine if each critical point is stable or not and construct the phase diagram for the differential equation. Next solve the ODE. Finally plot the slope field and verify visually the stability of each critital point.

$$\frac{dx}{dt} = f(x) = 7x - x^2 - 10$$

Solution

The critial points are x values (dependent variable values) where $f(x) = 0$. Hence

$$\begin{aligned} 7x - x^2 - 10 &= 0 \\ x_1 &= 2 \\ x_2 &= 5 \end{aligned}$$

The phase diagram is

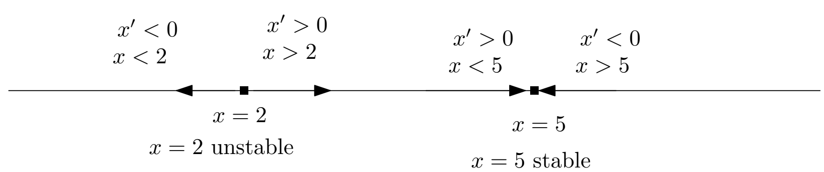


Figure 3: Phase diagram, 2.2 problem 7

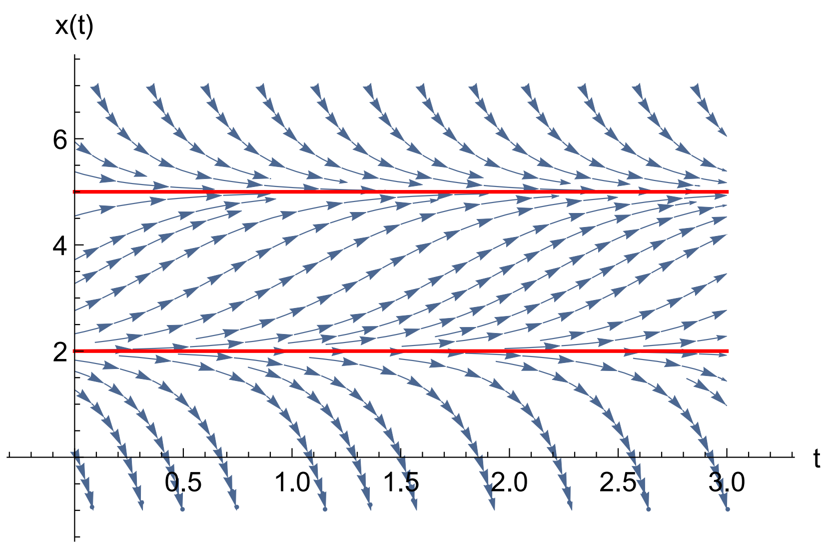
Now the ODE is solved $\frac{dx}{dt} = 7x - x^2 - 10$. This is non-linear seperable

$$\begin{aligned} \frac{dx}{7x - x^2 - 10} &= dt \qquad x \neq 2, x \neq 5 \\ \frac{-dx}{x^2 - 7x + 10} &= dt \\ -\int \frac{dx}{x^2 - 7x + 10} &= \int dt \end{aligned}$$

But $\frac{1}{(x-2)(x-5)} = \frac{A}{(x-2)} + \frac{B}{(x-5)}$, hence $A = \left(\frac{1}{(x-5)}\right)_{x=2} = \left(\frac{1}{-3}\right)$ and $B = \left(\frac{1}{(x-2)}\right)_{x=5} = \frac{1}{3}$ and the above becomes

$$
\begin{aligned}
-\int\left(\frac{1}{-3(x-2)}+\frac{1}{3(x-5)}\right) &= \int dt\\
\int\frac{1}{3(x-2)}-\int\frac{1}{3(x-5)} &= \int dt\\
\frac{1}{3}\int\frac{dx}{(x-2)}-\frac{1}{3}\int\frac{dx}{(x-5)} &= \int dt\\
\frac{1}{3}\ln|x-2|-\frac{1}{3}\ln|x-5| &= \int dt\\
\ln|x-2|-\ln|x-5| &= \int 3dt\\
\ln\left|\frac{x-2}{x-5}\right| &= 3t+c\\
\frac{x-2}{x-5} &= ce^{3t}\\
x-2 &= xce^{3t}-5ce^{3t}\\
x-xce^{3t} &= 2-5ce^{3t}\\
x &= \frac{2-5ce^{3t}}{1-ce^{3t}}
\end{aligned}
$$

Here is slope field plot



From the above plot, we see the solution lines are moving away from $x = 2$ indicating it is unstable and move towards $x = 5$ indicating it is stable.

0.16 Section 2.2 problem 23

Problem Suppose that logistic equation $\frac{dx}{dt} = kx(M-x)$ models a population $x(t)$ of fish in lake that after t months during which no fishing occurs. Now suppose that because of fishing, fish are removed from lake at rate of hx fish per month, with $h > 0$. Thus fish are harvested at a rate propertional to existing fish population, rather than at constant rate of example 4. (a) if $0 < h < kM$, show that population is still logistic. What is the new limiting population. (b) if $h \geq kM.$, show that $x(t) \to 0$ at $t \to \infty$ so that lake is eventually fished out.

Solution

Part (a)

Since fish is removed at rate of hx fish per month, then

$$\begin{aligned}\frac{dx}{dt} &= kx(M-x) - hx\\ &= kx\left((M-x) - \frac{h}{k}\right)\\ &= kx\left(M - \frac{h}{k} - x\right)\\ &= kx\left(\left(M - \frac{h}{k}\right) - x\right)\end{aligned}$$

But $M - \frac{h}{k} > 0$ since $0 < h < kM$, therefore, if we let $\left(M - \frac{h}{k}\right) = \lambda$, then $\frac{dx}{dt} = kx(\lambda - x)$ is still logistic just as $\frac{dx}{dt} = kx(M-x)$ since $\lambda > 0$. $\lambda = M - \frac{h}{k}$ is the new limiting population.

Part (b)

In this case

$$\begin{aligned}\frac{dx}{dt} &= kx\left(\left(M - \frac{h}{k}\right) - x\right)\\ &= kx(\lambda - x)\end{aligned}$$

Now $\lambda < 0$. Solving this ode

$$\begin{aligned}\frac{dx}{x(\lambda - x)} &= k\\ \frac{1}{\lambda x} - \frac{1}{\lambda}\frac{1}{(\lambda - x)} &= k\end{aligned}$$

Integrating

$$\begin{aligned}
\frac{1}{\lambda}\ln|x| - \frac{1}{\lambda}\ln|(\lambda - x)| &= \int k dt \\
\ln\left|\frac{x}{\lambda - x}\right| &= \int \lambda k dt \\
\ln\left|\frac{x}{\lambda - x}\right| &= \lambda k t + c \\
\frac{x}{\lambda - x} &= C e^{\lambda k t} \\
x + x C e^{\lambda k t} &= \lambda C e^{\lambda k t} \\
x(t) &= \frac{\lambda C e^{\lambda k t}}{1 + C e^{\lambda k t}}
\end{aligned}$$

Now, since $\lambda < 0$, then as $t \to \infty$ then $x(t) \to \frac{0}{1} = 0$. Hence the population of fish will die out. (no need to find C first, as the whole term goes to zero). This is what we are asked to show.