HW 12, Math 320, Spring 2017

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Contents

0.1 Section 5.1 problem 52 (page 299)

Problem Make the substitution $v = \ln x$ to find general solution for $x > 0$ of the Euler equation $x^2y'' + xy' - y = 0$

solution Let $v = \ln x$. Hence $\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx} = \frac{dy}{dv}\frac{1}{x}$ and

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dv}\frac{1}{x}\right) \\
&= \frac{d^2y}{dv^2}\frac{dv}{dx}\frac{1}{x} + \frac{dy}{dv}\frac{d}{dx}\left(\frac{1}{x}\right) \\
&= \frac{d^2y}{dv^2}\frac{1}{x^2} - \frac{dy}{dv}\frac{1}{x^2}
\end{aligned}$$

Hence the ODE becomes

$$\begin{aligned}
x^2y'' + xy' - y &= 0 \\
x^2\left(\frac{d^2y}{dv^2}\frac{1}{x^2} - \frac{dy}{dv}\frac{1}{x^2}\right) + x\left(\frac{dy}{dv}\frac{1}{x}\right) - y(v) &= 0 \\
\frac{d^2y}{dv^2} - \frac{dy}{dv} + \frac{dy}{dv} - y(v) &= 0 \\
\frac{d^2y}{dv^2} - y(v) &= 0
\end{aligned}$$

This can now be solved using characteristic equation. $r^2 - 1 = 0$ or $r^2 = 1$ or $r = \pm 1$. Hence the solution is

$$y(v) = c_1e^v + c_2e^{-v}$$

But $v = \ln x$, hence

$$\begin{aligned}
y(x) &= c_1e^{\ln x} + c_2e^{-\ln x} \\
&= c_1x + c_1\frac{1}{x}
\end{aligned}$$

The above is the solution.

But an easier method is the following. Let $y = x^r$. Hence $y' = rx^{r-1}, y'' = r(r-1)x^{r-2}$. Substituting this into the ODE gives

$$\begin{aligned}
r(r-1)x^r + rx^r - x^r &= 0 \\
x^r(r(r-1) + r - 1) &= 0
\end{aligned}$$

Since $x^r \neq 0$, we simplify the above and obtain the characteristic equation

$$\begin{aligned}
r(r-1) + r - 1 &= 0 \\
r^2 - 1 &= 0 \\
r^2 &= 1 \\
r &= \pm 1
\end{aligned}$$

Hence

$$\begin{aligned}
y(x) &= c_1x^{r_1} + c_2x^{r_2} \\
&= c_1x + c_2x^{-1}
\end{aligned}$$

For $x > 0$.

0.2 Section 5.1 problem 54

Problem Make the substitution $v = \ln x$ to find general solution for $x > 0$ of the Euler equation $4x^2y'' + 8xy' - 3y = 0$

solution Let $v = \ln x$. Hence $\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx} = \frac{dy}{dv}\frac{1}{x}$ and

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dv}\frac{1}{x}\right) \\
&= \frac{d^2y}{dv^2}\frac{dv}{dx}\frac{1}{x} + \frac{dy}{dv}\frac{d}{dx}\left(\frac{1}{x}\right) \\
&= \frac{d^2y}{dv^2}\frac{1}{x^2} - \frac{dy}{dv}\frac{1}{x^2}
\end{aligned}$$

Hence the ODE becomes

$$
\begin{aligned}
x^2y'' + xy' - y &= 0\\
4x^2\left(\frac{d^2y}{dv^2}\frac{1}{x^2} - \frac{dy}{dv}\frac{1}{x^2}\right) + 8x\left(\frac{dy}{dv}\frac{1}{x}\right) - 3y(v) &= 0\\
4\frac{d^2y}{dv^2} - 4\frac{dy}{dv} + 8\frac{dy}{dv} - 3y(v) &= 0\\
4\frac{d^2y}{dv^2} + 4\frac{dy}{dv} - 3y(v) &= 0
\end{aligned}
$$

This can now be solved using characteristic equation. $4r^2 + 4r - 3 = 0$, whose roots are $r_1 = \frac{-3}{2}, r_2 = \frac{1}{2}$ Hence the solution is

$$
y(v) = c_1 e^{\frac{-3}{2}v} + c_2 e^{\frac{1}{2}v}
$$

But $v = \ln x$, hence

$$
\begin{aligned}
y(x) &= c_1 e^{\frac{-3}{2}\ln x} + c_2 e^{\frac{1}{2}\ln x}\\
&= c_1 x^{\frac{-3}{2}} + c_1 x^{\frac{1}{2}}
\end{aligned}
$$

0.3 Section 5.2 problem 40 (page 311)

Problem Use reduction of order to find second L.I. solution y_2. $x^2y'' - x(x+2)y' + (x+2)y = 0$ with $y_1 = x$ and $x > 0$

solution Let $y = vy_1$, hence

$$
\begin{aligned}
y' &= v'y_1 + vy_1'\\
y'' &= v''y_1 + v'y_1' + v'y_1' + vy_1''\\
&= v''y_1 + 2v'y_1' + vy_1''
\end{aligned}
$$

Therefore the original ODE becomes

$$
\begin{aligned}
x^2y'' - x(x+2)y' + (x+2)y &= 0\\
x^2\left(v''y_1 + 2v'y_1' + vy_1''\right) - x(x+2)\left(v'y_1 + vy_1'\right) + (x+2)\left(vy_1\right) &= 0\\
v''\left(x^2y_1\right) + v'\left(2x^2y_1' - x(x+2)y_1\right) + v\overbrace{\left(x^2y_1'' - x(x+2)y_1' + (x+2)y_1\right)}^{0} &= 0
\end{aligned}
$$

Hence

$$
v''\left(x^2y_1\right) + v'\left(2x^2y_1' - x(x+2)y_1\right) = 0
$$

But $y_1 = x$, hence the above becomes

$$
\begin{aligned}
x^3v'' + v'\left(2x^2 - x(x+2)x\right) &= 0\\
x^3v'' - x^3v' &= 0
\end{aligned}
$$

Since we are told $x > 0$ when we can divide by x^3 and obtain

$$
v'' - v' = 0
$$

To solve the above, let

$$
z = v'
$$

Therefore $z' - z = 0$ or $\frac{d}{dx}(ze^x) = 0$ or $ze^x = c_1$ or $z = c_1e^{-x}$. Therefore the above becomes

$$
v' = c_1e^{-x}
$$

Integrating

$$
v = c_2 - c_1e^{-x}
$$

Since $y = vy_1$ therefore

$$
y = y_1(c_2 - c_1e^{-x})
$$

But $y_1 = x$, hence the complete solution is

$$
y = c_2x - c_1xe^{-x}
$$

Therefore, we see now that the two basis solutions are

$$
\begin{aligned}
y_1 &= x\\
y_2 &= xe^x
\end{aligned}
$$

These can be shown to be L.I. using the Wronskian as follows

$$\begin{aligned} W(x) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} x & xe^x \\ 1 & e^x + xe^x \end{vmatrix} \\ &= xe^x + x^2e^x - xe^x \\ &= x^2e^x \end{aligned}$$

Which is not zero since we are told $x > 0$. Hence indeed the second basis solution y_2 found is L.I. to y_1.

0.4 Section 5.5 problem 9 (page 351)

Problem Find the particular solution for $y'' + 2y' - 3y = 1 + xe^x$

solution First we find the homogenous solution. This will tell us if e^x is one of the basis solutions of not, so we know what to guess. The characteristic equation is

$$\begin{aligned} r^2 + 2r - 3 &= 0 \\ (r-1)(r+3) &= 0 \end{aligned}$$

Hence $y_1 = e^x, y_2 = e^{-3x}$. e^x is a solution to the homogeneous ODE. The guess is therefore

$$\begin{aligned} y_p &= A + (B + Cx)\, xe^x \\ &= A + \left(Bx + Cx^2\right) e^x \qquad (1) \end{aligned}$$

Hence

$$\begin{aligned} y_p' &= (B + 2Cx)\, e^x + \left(Bx + Cx^2\right) e^x \\ &= e^x \left(B + 2Cx + Bx + Cx^2\right) \\ y_p'' &= (2C + B + 2Cx)\, e^x + e^x \left(B + 2Cx + Bx + Cx^2\right) \\ &= e^x \left(2C + B + 2Cx + B + 2Cx + Bx + Cx^2\right) \\ &= e^x \left(2C + 2B + 4Cx + Bx + Cx^2\right) \end{aligned}$$

Plugging into the ODE

$$\begin{aligned} e^x \left(2C + 2B + 4Cx + Bx + Cx^2\right) + 2e^x \left(B + 2Cx + Bx + Cx^2\right) - 3\left(A + \left(Bx + Cx^2\right) e^x\right) &= 1 + xe^x \\ e^x (2C + 2B + 2B) + xe^x (4C + B + 4C + 2B - 3B) + x^2e^x (C + 2C - 3C) - 3A &= 1 + xe^x \\ e^x (2C + 4B) + xe^x (8C) - 3A &= 1 + xe^x \end{aligned}$$

Hence $-3A = 1$ or $A = -\frac{1}{3}$ and

$$8C = 1$$

Or

$$C = \frac{1}{8}$$

And

$$2C + 4B = 0$$

Or

$$B = -\frac{1}{16}$$

Hence particular solution becomes, from (1)

$$\begin{aligned} y_p &= -\frac{1}{3} + \left(-\frac{1}{16}x + \frac{1}{8}x^2\right) e^x \\ &= -\frac{1}{3} + \frac{1}{16}\left(2x^2 - x\right) e^x \end{aligned}$$

0.5 Section 5.5 problem 10

Problem Find the particular solution for $y'' + 9y = 2\cos 3x + 3\sin 3x$

solution First we find the homogenous solution. The characteristic equation is

$$\begin{aligned} r^2+9&=0\\ r^2&=-9\\ r&=\pm 3i \end{aligned}$$

Hence $y_1=e^{3ix}, y_2=e^{-3ix}$ or $y_h=c_1\cos 3x+c_2\sin 3x$. We see that $\cos 3x$ and $\sin 3x$ are already in the homogeneous solution. Therefore the guess is

$$y_p=Ax\cos 3x+Bx\sin 3x$$

Hence

$$\begin{aligned} y_p'&=A\cos 3x-3Ax\sin 3x+B\sin 3x+3Bx\cos 3x\\ y_p''&=-3A\sin 3x-3A\sin 3x-9Ax\cos 3x+3B\cos 3x+3B\cos 3x-9Bx\sin 3x \end{aligned}$$

Substitution into the ODE gives

$$(-3A\sin 3x-3A\sin 3x-9Ax\cos 3x+3B\cos 3x+3B\cos 3x-9Bx\sin 3x)\\ +9\left(Ax\cos 3x+Bx\sin 3x\right)=2\cos 3x+3\sin 3x$$

Or

$$\begin{aligned} -6A\sin 3x-9Ax\cos 3x+6B\cos 3x-9Bx\sin 3x+9Ax\cos 3x+9Bx\sin 3x&=2\cos 3x+3\sin 3x\\ \sin 3x\left(-6A\right)+\cos 3x\left(6B\right)+x\sin 3x\left(-9B+9B\right)+x\cos 3x\left(-9A+9A\right)&=2\cos 3x+3\sin 3x\\ -6A\sin 3x+6B\cos 3x&=2\cos 3x+3\sin 3x \end{aligned}$$

Hence $-6A=3$ or $A=\frac{-1}{2}$ and $6B=2$ or $B=\frac{1}{3}$, therefore the particular solution is

$$\begin{aligned} y_p&=\frac{-1}{2}x\cos 3x+\frac{1}{3}x\sin 3x\\ &=\frac{1}{6}\left(2x\sin 3x-3x\cos 3x\right) \end{aligned}$$

0.6 Section 5.5 problem 16

Problem Find the particular solution for $y''+9y=2x^2e^{3x}+5$

solution From the above problem, we found $y_h=c_1\cos 3x+c_2\sin 3x$. Therefore there are no basis solutions in the RHS which are in the homogenous solution. The guess for the constant term is A. The guess for $2x^2e^{3x}$ is $\left(B_0+B_1x+B_2x^2\right)e^{3x}$, hence

$$\begin{aligned} y_p&=A+\left(B_0+B_1x+B_2x^2\right)e^{3x}\\ y_p'&=\left(B_1+2B_2x\right)e^{3x}+3\left(B_0+B_1x+B_2x^2\right)e^{3x}\\ y_p''&=2B_2e^{3x}+3\left(B_1+2B_2x\right)e^{3x}+3\left(B_1+2B_2x\right)e^{3x}+9\left(B_0+B_1x+B_2x^2\right)e^{3x} \end{aligned}$$

Simplifying

$$\begin{aligned} y_p''&=e^{3x}\left(2B_2+3B_1+3B_1+9B_0\right)+xe^{x}\left(6B_2+6B_2+9B_1\right)+x^2e^{3x}\left(9B_2\right)\\ &=e^{3x}\left(2B_2+6B_1+9B_0\right)+xe^{x}\left(12B_2+9B_1\right)+x^2e^{3x}\left(9B_2\right) \end{aligned}$$

Substitution into the ODE gives

$$\begin{aligned} e^{3x}\left(2B_2+6B_1+9B_0\right)+xe^{x}\left(12B_2+9B_1\right)+x^2e^{3x}\left(9B_2\right)+9\left(A+\left(B_0+B_1x+B_2x^2\right)e^{3x}\right)&=2x^2e^{3x}+5\\ e^{3x}\left(2B_2+6B_1+9B_0\right)+xe^{x}\left(12B_2+9B_1\right)+x^2e^{3x}\left(9B_2\right)+9A+\left(9B_0+9B_1x+9B_2x^2\right)e^{3x}&=2x^2e^{3x}+5\\ e^{3x}\left(2B_2+6B_1+18B_0\right)+xe^{x}\left(12B_2+18B_1\right)+x^2e^{3x}\left(18B_2\right)+9A&=2x^2e^{3x}+5 \end{aligned}$$

Comparing coefficients gives

$$\begin{aligned} 9A&=5\\ 2B_2+6B_1+18B_0&=0\\ 12B_2+18B_1&=0\\ 19B_2&=2 \end{aligned}$$

From last equation $B_2=\frac{1}{9}$. Hence from third equation $18B_1=-\frac{12}{9}$, or $B_1=-\frac{2}{27}$. And from second equation

$$
\begin{aligned}
2B_2 + 6B_1 + 18B_0 &= 0 \\
2\left(\frac{1}{9}\right) + 6\left(-\frac{2}{27}\right) + 18B_0 &= 0 \\
B_0 &= \frac{1}{81}
\end{aligned}
$$

And $A = \frac{5}{9}$. Therefore

$$
\begin{aligned}
y_p &= A + \left(B_0 + B_1 x + B_2 x^2\right) e^{3x} \\
&= \frac{5}{9} + \left(\frac{1}{81} - \frac{2}{27}x + \frac{1}{9}x^2\right) e^{3x} \\
&= \frac{5}{9} + \left(\frac{1}{81} - \frac{6}{81}x + \frac{9}{81}x^2\right) e^{3x} \\
&= \frac{45}{81} + \left(\frac{1}{81} - \frac{6}{81}x + \frac{9}{81}x^2\right) e^{3x} \\
&= \frac{1}{81}\left(45 + e^{3x} - 6xe^{x} + 9x^2 e^{3x}\right)
\end{aligned}
$$

0.7 Section 5.5 problem 25

Problem Setup the form for the particular solution but do not determine the values of the coefficients. $y'' + 3y' + 2y = xe^{-x} - xe^{-2x}$

solution First we find the homogenous solution. The characteristic equation is

$$
\begin{aligned}
r^2 + 3r + 2 &= 0 \\
(r+1)(r+2) &= 0
\end{aligned}
$$

Hence $y_1 = e^{-x}, y_2 = e^{-2x}$. We see that the basis solutions are part of the RHS. Therefore the guess solution is

$$
y_p = x(A_1 + A_2 x) e^{-x} + x(A_3 + A_4 x) e^{-2x}
$$

0.8 Section 5.5 problem 26

Problem Setup the form for the particular solution but do not determine the values of the coefficients. $y'' - 6y' + 13y = xe^{3x} \sin 2x$

solution First we find the homogenous solution. The characteristic equation is

$$
r^2 - 6r + 13 = 0
$$

The roots are $3 \pm 2i$. Hence the homogenous solution is $y_h = c_1 e^{3x} \cos 2x + c_2 e^{3x} \sin 2x$. We see that $e^{3x} \sin 2x$ is already in the homogenous solution. Hence the guess is

$$
\begin{aligned}
y_p &= \overbrace{(A_1 + A_2 x) x}^{x \text{ guess}} \overbrace{(A_3 \sin 2x + A_4 \cos 2x) e^{3x}}^{\sin 2x e^{3x} \text{ guess}} \\
&= \left(A_1 x + A_2 x^2\right) e^{3x} \cos 2x + \left(A_3 x + A_4 x^2\right) e^{3x} \sin 2x
\end{aligned}
$$

0.9 Section 5.5 problem 37

Problem Solve the initial value problem $y''' - 2y'' + y' = 1 + xe^x$ with $y(0) = 0, y'(0) = 0, y''(0) = 1$

solution First we find the homogenous solution. The characteristic equation is

$$
\begin{aligned}
r^3 - 2r^2 + r &= 0 \\
r\left(r^2 - 2r + 1\right) &= 0
\end{aligned}
$$

For $r^2 - 2r + 1 = 0$, it factors into $(r-1)(r-1)$, hence roots are $r_1 = 0, r_2 = 1, r_3 = 1$. Since double roots, the homogenous solution is

$$
y_h = c_1 + c_2 e^x + c_3 x e^x
$$

We notice that both e^x and xe^x is in the RHS. Therefore we need to multiply by x^2. The guess is therefore

$$
\begin{aligned}
y_p &= Ax + x^2 (B + Cx) e^x \\
&= Ax + \left(Bx^2 + Cx^3\right) e^x
\end{aligned}
$$

Therefore

$$\begin{aligned} y_p' &= A + \left(2Bx + 3Cx^2\right) e^x + \left(Bx^2 + Cx^3\right) e^x \\ y_p'' &= (2B + 6Cx) e^x + \left(2Bx + 3Cx^2\right) e^x + \left(2Bx + 3Cx^2\right) e^x + \left(Bx^2 + Cx^3\right) e^x \end{aligned}$$

Simplifying gives

$$\begin{aligned} y_p' &= A + xe^x (2B) + x^2 e^x (3C + B) + x^3 e^x (C) \\ y_p'' &= e^x (2B) + xe^x (6C + 4B) + x^2 e^x (6C + B) + x^3 e^x (C) \\ y_p''' &= e^x (2B) + e^x (6C + 4B) + xe^x (6C + 4B) + 2xe^x (6C + B) + x^2 e^x (6C + B) + 3x^2 e^x (C) + x^3 e^x (C) \\ &= e^x (6B + 6C) + xe^x (6C + 4B + 12C + 2B) + x^2 e^x (6C + B + 3C) + Cx^3 e^x \\ &= e^x (6B + 6C) + xe^x (18C + 6B) + x^2 e^x (9C + B) + Cx^3 e^x \end{aligned}$$

Substitution into the ODE gives

$$y_p''' - 2y_p'' + y_p' = 1 + xe^x$$

Hence

$$\begin{aligned} &e^x (6B + 6C) + xe^x (18C + 6B) + x^2 e^x (9C + B) + Cx^3 e^x \\ &\quad - 2\left(e^x (2B) + xe^x (6C + 4B) + x^2 e^x (6C + B) + x^3 e^x (C)\right) + \\ &\quad A + xe^x (2B) + x^2 e^x (3C + B) + x^3 e^x (C) = 1 + xe^x \end{aligned}$$

Or

$$\begin{aligned} &e^x (6B + 6C) + xe^x (18C + 6B) + x^2 e^x (9C + B) + Cx^3 e^x \\ &\quad - e^x (4B) - xe^x (12C + 8B) - x^2 e^x (12C + 2B) - x^3 e^x (2C) + \\ &\quad A + xe^x (2B) + x^2 e^x (3C + B) + x^3 e^x (C) = 1 + xe^x \end{aligned}$$

Or

$$\begin{aligned} &e^x (6B + 6C - 4B) + xe^x (18C + 6B - 12C - 8B + 2B) + \\ &\quad x^2 e^x (9C + B - 12C - 2B + 3C + B) + x^3 e^x (C - 2C + C) + A = 1 + xe^x \end{aligned}$$

Or

$$e^x (2B + 6C) + xe^x (6C) + A = 1 + xe^x$$

Hence

$$\begin{aligned} 6C &= 1 \\ 2B + 6C &= 0 \\ A &= 1 \end{aligned}$$

Therefore, $C = \frac{1}{6}, B = -\frac{1}{2}$, and the particular solution is

$$\begin{aligned} y_p &= Ax + x^2 (B + Cx) e^x \\ &= x + \left(-\frac{1}{2}x^2 + \frac{1}{6}x^3\right) e^x \end{aligned}$$

Hence the complete solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 + c_2 e^x + c_3 xe^x + x + \left(-\frac{1}{2}x^2 + \frac{1}{6}x^3\right) e^x \end{aligned}$$

Applying initial conditions. $y(0) = 0$ gives

$$0 = c_1 + c_2 \tag{1}$$

And

$$y' = c_2 e^x + c_3 e^x + c_3 xe^x + 1 + \left(-x + \frac{1}{2}x^2\right) e^x + \left(-\frac{1}{2}x^2 + \frac{1}{6}x^3\right) e^x$$

Applying second initial conditions $y'(0) = 0$ gives

$$0 = c_2 + c_3 + 1 \tag{2}$$

And

$$y'' = c_2 e^x + c_3 e^x + c_3 e^x + c_3 xe^x + (-1 + x) e^x + \left(-x + \frac{1}{2}x^2\right) e^x + \left(-x + \frac{1}{2}x^2\right) e^x + \left(-\frac{1}{2}x^2 + \frac{1}{6}x^3\right) e^x$$

Applying initial conditions $y''(0) = 1$ gives

$$\begin{aligned}1 &= c_2 + 2c_3 - 1\\2 &= c_2 + 2c_3\end{aligned}$$

The solution is $c_1 = 4, c_2 = -4, c_3 = 3$, hence the general solution is

$$\begin{aligned}y &= c_1 + c_2 e^x + c_3 x e^x + x + \left(-\frac{1}{2}x^2 + \frac{1}{6}x^3\right) e^x\\&= 4 - 4e^x + 3xe^x + x - \frac{1}{2}x^2 e^x + \frac{1}{6}x^3 e^x\end{aligned}$$

0.10 Section 5.5 problem 49

Problem Use method of variation of parameters to find particular solution $y'' - 4y' + 4y = 2e^{2x}$

solution We need to first find the homogenous solution. The characteristic equation is

$$\begin{aligned}r^2 - 4r + 4 &= 0\\(r-2)(r-2) &= 0\end{aligned}$$

Hence $r_1 = 2$, double root. Therefore

$$\begin{aligned}y_1(x) &= e^{2x}\\y_2(x) &= xe^{2x}\end{aligned}$$

Let

$$y_p = u_1 y_1 + u_2 y_2$$

Where

$$\begin{aligned}u_1 &= -\int \frac{y_2(x) f(x)}{W(x)} dx\\u_2 &= \int \frac{y_1(x) f(x)}{W(x)} dx\end{aligned}$$

Where $f(x) = 2e^{2x}$ and

$$\begin{aligned}W(x) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix}\\&= e^{2x}\left(e^{2x} + 2xe^{2x}\right) - 2xe^{4x}\\&= e^{4x} + 2xe^{4x} - 2xe^{4x}\\&= e^{4x}\end{aligned}$$

Hence

$$u_1 = -\int \frac{xe^{2x}\left(2e^{2x}\right)}{e^{4x}} dx = -\int 2x dx = -x^2$$

And

$$u_2 = \int \frac{e^{2x}\left(2e^{2x}\right)}{e^{4x}} dx = 2x$$

Therefore

$$\begin{aligned}y_p &= u_1 y_1 + u_2 y_2\\&= -x^2 e^{2x} + 2x^2 e^{2x}\\&= x^2 e^{2x}\end{aligned}$$

0.11 Section 5.5 problem 50

Problem Use method of variation of parameters to find particular solution $y'' - 4y = \sinh 2x$

solution We need to first find the homogenous solution. The characteristic equation is

$$\begin{aligned}r^2 - 4 &= 0\\r &= \pm 2\end{aligned}$$

Therefore

$$\begin{aligned}y_1(x) &= e^{2x}\\y_2(x) &= e^{-2x}\end{aligned}$$

Let

$$y_p = u_1 y_1 + u_2 y_2$$

Where

$$u_1 = -\int \frac{y_2(x) f(x)}{W(x)} dx$$
$$u_2 = \int \frac{y_1(x) f(x)}{W(x)} dx$$

Where $f(x) = \sinh 2x = \frac{e^{2x} - e^{-2x}}{2}$ and

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix}$$
$$= -2 - 2 = -4$$

Hence

$$u_1 = -\int \frac{e^{-2x}\left(\frac{e^{2x}-e^{-2x}}{2}\right)}{-4} dx$$
$$= \frac{1}{4}\int e^{-2x}\left(\frac{e^{2x}-e^{-2x}}{2}\right) dx$$
$$= \frac{1}{8}\int \left(1 - e^{-4x}\right) dx$$
$$= \frac{1}{8}\left(x + \frac{e^{-4x}}{4}\right)$$

And

$$u_2 = \int \frac{e^{2x}\left(\frac{e^{2x}-e^{-2x}}{2}\right)}{-4} dx$$
$$= -\frac{1}{8}\int e^{2x}\left(e^{2x} - e^{-2x}\right) dx$$
$$= -\frac{1}{8}\int \left(e^{4x} - 1\right) dx$$
$$= -\frac{1}{8}\left(\frac{e^{4x}}{4} - x\right)$$

Therefore

$$y_p = u_1 y_1 + u_2 y_2$$
$$= \frac{1}{8}\left(x + \frac{e^{-4x}}{4}\right)e^{2x} - \frac{1}{8}\left(\frac{e^{4x}}{4} - x\right)e^{-2x}$$
$$= \left(\frac{1}{8}xe^{2x} + \frac{e^{-2x}}{32}\right) - \frac{1}{8}\left(\frac{e^{2x}}{32} - \frac{xe^{-2x}}{8}\right)$$
$$= \frac{1}{8}xe^{2x} + \frac{e^{-2x}}{32} - \frac{e^{2x}}{32} + \frac{xe^{-2x}}{8}$$
$$= \frac{1}{4}x\left(\frac{e^{2x}+e^{-2x}}{2}\right) + \frac{1}{16}\left(\frac{e^{-2x}-e^{2x}}{2}\right)$$
$$= \frac{1}{4}x\left(\frac{e^{2x}+e^{-2x}}{2}\right) - \frac{1}{16}\left(\frac{e^{2x}-e^{-2x}}{2}\right)$$
$$= \frac{1}{4}x\cosh 2x - \frac{1}{16}\sinh 2x$$
$$= \frac{1}{16}\left(4x\cosh 2x - \sinh 2x\right)$$

0.12 Section 5.5 problem 53

Problem Use method of variation of parameters to find particular solution $y'' + 9y = 2\sec 3x$

solution We need to first find the homogenous solution. The characteristic equation is

$$r^2 + 9 = 0$$
$$r = \pm 3i$$

Therefore

$$y_1(x) = \sin 3x$$
$$y_2(x) = \cos 3x$$

Let

$$y_p = u_1 y_1 + u_2 y_2$$

Where

$$\begin{aligned} u_1 &= -\int \frac{y_2(x) f(x)}{W(x)} dx \\ u_2 &= \int \frac{y_1(x) f(x)}{W(x)} dx \end{aligned}$$

Where $f(x) = 2\sec 3x = \frac{2}{\cos 3x}$ and

$$\begin{aligned} W(x) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin 3x & \cos 3x \\ 3\cos 3x & -3\sin 3x \end{vmatrix} \\ &= -3\sin^2 3x - 3\cos^2 x \\ &= -3 \end{aligned}$$

Hence

$$\begin{aligned} u_1 &= -\int \frac{\cos 3x \left(\frac{2}{\cos 3x}\right)}{-3} dx \\ &= \frac{1}{3}\int 2dx \\ &= \frac{2}{3}x \end{aligned}$$

And

$$\begin{aligned} u_2 &= \int \frac{\sin 3x \left(\frac{2}{\cos 3x}\right)}{-3} dx \\ &= \frac{-2}{3}\int \tan 3x dx \\ &= \frac{-2}{3}\left(\frac{1}{6}\ln \frac{1}{\cos^2(3x)}\right) \end{aligned}$$

Therefore

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 \\ &= \frac{2}{3}x(\sin 3x) + \frac{-2}{3}\left(\frac{1}{6}\ln \frac{1}{\cos^2(3x)}\right)\cos 3x \\ &= \frac{2}{3}x(\sin 3x) - \frac{1}{9}\cos(3x)\ln\left(\frac{1}{\cos^2(3x)}\right) \\ &= \frac{2}{3}x(\sin 3x) + \frac{1}{9}\cos(3x)\ln\left(\cos^2(3x)\right) \\ &= \frac{2}{3}x(\sin 3x) + \frac{2}{9}\cos(3x)\ln|\cos(3x)| \end{aligned}$$

0.13 Section 5.5 problem 61

Problem Find a particular solution to the Euler ODE $x^2y'' + xy' + y = \ln x$ with homogenous solution $y_h = c_1\cos(\ln x) + c_2\sin(\ln x)$

solution We see that

$$\begin{aligned} y_1 &= \cos(\ln x) \\ y_2 &= \sin(\ln x) \end{aligned}$$

Using variation of parameters on the ODE

$$y'' + \frac{1}{x}y' + \frac{1}{x^2}y = \frac{\ln x}{x^2}$$

Where now we use $f(x) = \frac{\ln x}{x^2}$. Let

$$y_p = u_1 y_1 + u_2 y_2$$

Where

$$\begin{aligned} u_1 &= -\int \frac{y_2(x) f(x)}{W(x)} dx \\ u_2 &= \int \frac{y_1(x) f(x)}{W(x)} dx \end{aligned}$$

And

$$
\begin{aligned}
W(x) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos(\ln x) & \sin(\ln x) \\ -\frac{1}{x}\sin(\ln x) & \frac{1}{x}\cos(\ln x) \end{vmatrix} \\
&= \frac{1}{x}\cos^2(\ln x) + \frac{1}{x}\sin^2(\ln x) \\
&= \frac{1}{x}
\end{aligned}
$$

Hence

$$
\begin{aligned}
u_1 &= -\int \frac{\sin(\ln x)\left(\frac{\ln x}{x^2}\right)}{\frac{1}{x}} dx \\
&= -\int \frac{\ln x \sin(\ln x)}{x} dx \\
&= \ln(x)\cos(\ln x) - \sin(\ln x)
\end{aligned}
$$

And

$$
\begin{aligned}
u_2 &= \int \frac{\cos(\ln x)\left(\frac{\ln x}{x^2}\right)}{\frac{1}{x}} dx \\
&= \int \frac{\cos(\ln x)(\ln x)}{x} dx \\
&= \ln(x)\sin(\ln x) + \cos(\ln x)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
y_p &= u_1 y_1 + u_2 y_2 \\
&= (\ln(x)\cos(\ln x) - \sin(\ln x))\cos(\ln x) + (\ln(x)\sin(\ln x) + \cos(\ln x))\sin(\ln x) \\
&= \ln(x)\cos^2(\ln x) - \sin(\ln x)\cos(\ln x) + \ln(x)\sin^2(\ln x) + \sin(\ln)\cos(\ln x) \\
&= \ln(x)\cos^2(\ln x) + \ln(x)\sin^2(\ln x) \\
&= \ln x
\end{aligned}
$$

0.14 Section 5.5 problem 62

Problem Find a particular solution to the Euler ODE $\left(x^2-1\right)y'' - 2xy' + 2y = x^2 - 1$ with homogenous solution $y_h = c_1 x + c_2\left(1+x^2\right)$

solution We see that

$$
\begin{aligned}
y_1 &= x \\
y_2 &= 1 + x^2
\end{aligned}
$$

Using variation of parameters on the ODE

$$
y'' - 2\frac{x}{\left(x^2-1\right)}y' + \frac{2}{\left(x^2-1\right)}y = 1
$$

Where now we use $f(x) = 1$. Let

$$
y_p = u_1 y_1 + u_2 y_2
$$

Where

$$
\begin{aligned}
u_1 &= -\int \frac{y_2(x) f(x)}{W(x)} dx \\
u_2 &= \int \frac{y_1(x) f(x)}{W(x)} dx
\end{aligned}
$$

And

$$
\begin{aligned}
W(x) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & 1+x^2 \\ 1 & 2x \end{vmatrix} \\
&= 2x^2 - \left(1+x^2\right) \\
&= x^2 - 1
\end{aligned}
$$

Hence

$$
\begin{aligned}
u_1 &= -\int \frac{\left(1+x^2\right)(1)}{x^2-1} dx \\
&= -x - \ln(x-1) + \ln(x+1)
\end{aligned}
$$

And

$$\begin{aligned} u_2 &= \int \frac{x}{x^2-1}dx \\ &= \frac{1}{2}\ln(x-1) + \frac{1}{2}\ln(x+1) \end{aligned}$$

Therefore

$$\begin{aligned} y_p &= u_1y_1 + u_2y_2 \\ &= (-x - \ln(x-1) + \ln(x+1))\,x + \left(\frac{1}{2}\ln(x-1) + \frac{1}{2}\ln(x+1)\right)\left(1+x^2\right) \\ &= -x^2 + x\ln\left|\frac{x+1}{x-1}\right| + \frac{1}{2}\left(1+x^2\right)\ln|(x-1)(x+1)| \\ &= -x^2 + x\ln\left|\frac{x+1}{x-1}\right| + \frac{1}{2}\left(1+x^2\right)\ln\left|x^2-1\right| \end{aligned}$$