HW 11, Math 320, Spring 2017

Nasser M. Abbasi (Discussion section 383, 8:50 AM - 9:40 AM Monday)

December 30, 2019

Contents

0.1 Section 5.2 problem 12 (page 311)

Problem: Use the Wronskian to prove that the given functions are linearly independent on the given interval. $f(x)=x, g(x)=\cos(\ln x), h(x)=\sin(\ln x)$ for $x>0$

solution The Wronskian is

$$\begin{aligned} W(x) &= \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix} \\ &= \begin{vmatrix} x & \cos(\ln x) & \sin(\ln x) \\ 1 & -\sin(\ln x)\frac{1}{x} & \cos(\ln x)\frac{1}{x} \\ 0 & -\cos(\ln x)\frac{1}{x^2}+\sin(\ln)\frac{1}{x^2} & -\sin(\ln x)\frac{1}{x^2}-\cos(\ln x)\frac{1}{x^2} \end{vmatrix} \end{aligned}$$

Expanding along the last row

$$\begin{aligned} W(x) &= W_{32}(-1)^{3+2}A_{32}+W_{33}(-1)^{3+3}A_{33} \\ &= -\left(-\cos(\ln x)\frac{1}{x^2}+\sin(\ln)\frac{1}{x^2}\right)\begin{vmatrix} x & \sin(\ln x) \\ 1 & \cos(\ln x)\frac{1}{x} \end{vmatrix} + \left(-\sin(\ln x)\frac{1}{x^2}-\cos(\ln x)\frac{1}{x^2}\right)\begin{vmatrix} x & \cos(\ln x) \\ 1 & -\sin(\ln x)\frac{1}{x} \end{vmatrix} \\ &= \left(\cos(\ln x)\frac{1}{x^2}-\sin(\ln)\frac{1}{x^2}\right)(\cos(\ln x)-\sin(\ln x)) + \left(\sin(\ln x)\frac{1}{x^2}+\cos(\ln x)\frac{1}{x^2}\right)(\sin(\ln x)+\cos(\ln x)) \end{aligned}$$

Let $\sin(\ln)\frac{1}{x^2}=A, \cos(\ln x)\frac{1}{x^2}=B$, $\cos(\ln x)=a, \sin(\ln x)=b$ then the above is

$$\begin{aligned} W(x) &= (B-A)(a-b)+(A+B)(b+a) \\ &= 2Ab+2Ba \end{aligned}$$

Transforming back

$$\begin{aligned} W(x) &= 2\sin(\ln)\frac{1}{x^2}\sin(\ln x)+2\cos(\ln x)\frac{1}{x^2}\cos(\ln x) \\ &= 2\sin^2(\ln)\frac{1}{x^2}+2\cos^2(\ln x)\frac{1}{x^2} \\ &= \frac{2}{x^2} \end{aligned}$$

Hence, for $x>0$ the Wronskian is not zero. Therefore the functions are L.I.

0.2 Section 5.2 problem 16

Problem: A third order ODE is given, and three L.I. solutions are given. Find a particular solution satisfying the given initial conditions $y'''-5y''+8y'-4y=0$ and $y(0)=1, y'(0)=4, y''(0)=0$ and $y_1=e^x, y_2=e^{2x}, y_3=xe^{2x}$

solution The general solution is

$$\begin{aligned} y &= c_1y_1+c_2y_2+c_3y_3 \\ &= c_1e^x+c_2e^{2x}+c_3xe^{2x} \end{aligned}$$

Hence

$$y'=c_1e^x+2c_2e^{2x}+c_3\left(e^{2x}+2xe^{2x}\right)$$

And

$$y''=c_1e^x+4c_2e^{2x}+c_3\left(2e^{2x}+2e^{2x}+4xe^{2x}\right)$$

From first initial condition we obtain

$$1=c_1+c_2 \tag{1}$$

From second initial condition we obtain

$$4=c_1+2c_2+c_3 \tag{2}$$

And from the third initial condition

$$0=c_1+4c_2+4c_3 \tag{3}$$

We have three equations (1,2,3) to solve for c_1, c_2, c_3.

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 4 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}$$

Augmented matrix is

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 4 \\ 1 & 4 & 4 & 0 \end{pmatrix} \overset{R_2=R_2-R_1}{\underset{R_3=R_3-R_1}{\longmapsto}} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 3 & 4 & -1 \end{pmatrix} \overset{R_3=R_3-3R_2}{\longmapsto} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & -10 \end{pmatrix}$$

We see that $|A| = 1$, since reduced matrix is upper diagonal matrix. Hence the solution is unique. From last row we obtain $c_3 = -10$ and from second row $c_2 + c_3 = 3$ or $c_2 = 3 + 10 = 13$ and from first row $c_1 + c_2 = 1$ or $c_1 = 1 - 13 = -12$, hence the particular solution is

$$\begin{aligned} y(x) &= c_1 e^x + c_2 e^{2x} + c_3 x e^{2x} \\ &= -12e^x + 13e^{2x} - 10xe^{2x} \end{aligned}$$

0.3 Section 5.2 problem 24

Problem: A nonhomogeneous ODE, homogeneous solution y_h and particular solution y_p are given. Find solution that satisfy the initial conditions. $y'' - 2y' + 2y = 2x$ with $y(0) = 4, y'(0) = 8$ and $y_h = c_1 e^x \cos x + c_2 e^x \sin x$ and $y_p = x + 1$

solution The general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 e^x \cos x + c_2 e^x \sin x + x + 1 \end{aligned}$$

Therefore

$$y' = c_1 \left(e^x \cos x - e^x \sin x\right) + c_2 \left(e^x \sin x + e^x \cos x\right) + 1$$

First initial conditions gives

$$\begin{aligned} 4 &= c_1 + 1 \\ c_1 &= 3 \end{aligned}$$

Second initial conditions gives

$$8 = c_1 + c_2 + 1$$

Hence $c_2 = 7 - c_1 = 4$. Therefore the general solution becomes

$$\begin{aligned} y &= 3e^x \cos x + 4e^x \sin x + x + 1 \\ &= e^x (3\cos x + 4\sin x) + x + 1 \end{aligned}$$

0.4 Section 5.2 problem 28

Problem: Show that $1, x, x^2, \cdots, x^n$ are L.I.

solution Using the Wronskian

$$W(x) = \begin{vmatrix} 1 & x & x^2 & x^3 & \cdots & x^n \\ 0 & 1 & 2x & 3x^2 & \cdots & nx^{n-1} \\ 0 & 0 & 2 & 6x & \cdots & n(n-1)x^{n-2} \\ \vdots & \vdots & 0 & 6 & \cdots & n(n-1)(n-2)x^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & & n! \end{vmatrix}$$

Therefore, the resulting Wronskian is an upper diagonal. The determinant of an upper diagonal matrix is the product of the diagonal. We see that there can be no zero element on the diagonal. Hence the determinant is never zero. Therefore $1, x, x^2, \cdots, x^n$ are L.I.

0.5 Section 5.2 problem 30

Problem: Verify that $y_1 = x$ and $y_2 = x^2$ are L.I. solutions on the entire line of the equation $x^2 y'' - 2xy + 2y = 0$ but that $W\left(x, x^2\right)$ vanishes at $x = 0$. Why does these observations not contradicts part (b) of theorem 3?

solution To verify that y_1, y_2 are solution of the ODE, we plugin each into the ODE and see if they satisfy it. For y_1, we obtain

$$x^2 y_1'' - 2xy_1 + 2y_1 = 0$$

But $y_1' = 1, y_1'' = 0$, therefore the above becomes

$$\begin{aligned} -2x + 2x &= 0 \\ 0 &= 0 \end{aligned}$$

Verified. For y_2, where $y_2' = 2x, y_2'' = 2$, we obtain

$$\begin{aligned} x^2 y_2'' - 2xy_2 + 2y_2 &= 0 \\ 2x^2 - 4x^2 + 2x^2 &= 0 \\ 0 &= 0 \end{aligned}$$

Hence verified. Now we need to show that y_1, y_2 are L.I.

$$c_1 y_1 + c_2 y_2 = 0$$

We now solve for c_1, c_2

$$c_1 x + c_2 x^2 = 0$$

Comparing coefficients on the LHS and RHS, we see that $c_1 = 0, c_2 = 0$. Hence this shows that y_1, y_2 are L.I. We now find the Wronskian

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$$

Hence $W(x) = 0$ at $x = 0$. This does not contradicts part (b) of theorem 3, because when we write the ODE in the standard form

$$y_1'' - \frac{2}{x} y_1 + \frac{2}{x^2} y_1 = 0$$

We see that $p_1(x) = -\frac{2}{x}, p_2(x) = \frac{2}{x^2}$. These functions are not continuous at $x = 0$ (there is singularity at $x = 0$). But theorem 3 applies only to the interval where $p_i(x)$ are continuous. Hence does not apply in this case. If $W(x)$ was zero at location other than $x = 0$, only then this will be a contradiction.

0.6 Section 5.3 problem 9 (page 323)

Problem: Find the general solution of the ODE $y'' + 8y' + 25y = 0$

solution This is constant coefficient, linear, second order ODE. The characteristic equation is $r^2 + 8r + 25 = 0$. The roots (using quadratic formula) are

$$\begin{aligned} r_1 &= -4 + 3i \\ r_2 &= -4 - 3i \end{aligned}$$

Hence the general solution is

$$\begin{aligned} y &= c_1 e^{r_1 x} + c_2 e^{r_2 x} \\ &= c_1 e^{(-4+3i)x} + c_2 e^{(-4-3i)x} \\ &= e^{-4x} (c_1 \cos 3x + c_2 \sin 3x) \end{aligned}$$

0.7 Section 5.3 problem 16

Problem: Find the general solution of the ODE $y^{(4)} + 18y'' + 81y = 0$

solution This is constant coefficient, linear, second order ODE. The characteristic equation is $r^4 + 18r^2 + 81 = 0$. Let $r^2 = z$, hence $z^2 + 18z + 81 = 0$. This can be factored to $(z+9)^2 = 0$. Hence the roots are -9 repeated. Therefore $r^2 = -9$ or $r = \pm 3i$. Therefore, the 4 roots are

$$\{3i, -3i, 3i, -3i\}$$

Hence the solution is

$$y = c_1 e^{3ix} + c_2 e^{-3ix} + c_3 x e^{3ix} + c_4 x e^{-3ix}$$

Or

$$\begin{aligned} y &= c_1 \cos 3x + c_2 \sin 3x + x\left(c_3 e^{3ix} + c_4 e^{-3ix}\right) \\ &= c_1 \cos 3x + c_2 \sin 3x + x(c_3 \cos 3x + c_4 \sin 3x) \end{aligned}$$

Or

$$y = (c_1 + xc_3) \cos 3x + c_2 \sin 3x (c_2 + xc_4)$$

0.8 Section 5.3 problem 23

Problem: Solve the initial value problem $y'' - 6y' + 25y = 0, y(0) = 3, y'(0) = 1$

solution This is constant coefficient, linear, second order ODE. The characteristic equation is $r^2 - 6r + 25 = 0$. Using quadratic formula, the roots are

$$\begin{aligned} r_1 &= 3 + 4i \\ r_2 &= 3 - 4i \end{aligned}$$

Hence the general solution is

$$y(x) = e^{3x}(c_1 \cos 4x + c_2 \sin 4x)$$

Hence

$$y'(x) = 3e^{3x}(c_1 \cos 4x + c_2 \sin 4x) + e^{3x}(-c_1 4 \sin 4x + c_2 4 \cos 4x)$$

Applying first initial conditions gives

$$3 = c_1$$

Applying second initial conditions gives

$$\begin{aligned} 1 &= 3(3) + 4c_2 \\ c_2 &= \frac{1-9}{4} = -2 \end{aligned}$$

Hence the solution is

$$y(x) = e^{3x}(3 \cos 4x - 2 \sin 4x)$$

0.9 Section 5.3 problem 26

Problem: Solve the initial value problem $y^{(3)} + 10y'' + 25y' = 0, y(0) = 3, y'(0) = 4, y''(0) = 5$

solution This is constant coefficient, linear, second order ODE. The characteristic equation is $r^3 + 10r^2 + 25r = 0$ or $r\left(r^2 + 10r + 25\right) = 0$, or $r(r+5)^2 = 0$. Hence the roots are $\{0, -5, -5\}$. There are repeated root. Hence the solution is

$$\begin{aligned} y(x) &= c_1 e^{r_1 x} + c_2 e^{r_2 x} + c_3 x e^{r_2 x} \\ &= c_1 + c_2 e^{-5x} + c_3 x e^{-5x} \end{aligned}$$

Hence

$$y' = -5c_2 e^{-5x} + c_3\left(e^{-5x} - 5xe^{-5x}\right)$$

And

$$y'' = 25c_2 e^{-5x} + c_3\left(-5e^{-5x} - 5\left(e^{-5x} - 5xe^{-5x}\right)\right)$$

Applying first IC gives

$$3 = c_1 + c_2$$

Applying second IC gives

$$4 = -5c_2 + c_3$$

Applying third IC gives

$$\begin{aligned} 5 &= 25c_2 + c_3(-5 - 5) \\ &= 25c_2 - 10c_3 \end{aligned}$$

We have three equations to solve for c_1, c_2, c_3.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -5 & 1 \\ 0 & 25 & -10 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

Therefore $R_3 = R_3 + 5R_2$ gives

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -5 & 1 \\ 0 & 0 & -5 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 25 \end{pmatrix}$$

Therefore, from third row, $-5c_3 = 25$ or

$$c_3 = -5$$

From second row

$$-5c_2 + c_3 = 4$$
$$-5c_2 = 4 + 5 = 9$$
$$c_2 = -\frac{9}{5}$$

From first row

$$c_1 + c_2 = 3$$
$$c_1 = 3 + \frac{9}{5}$$
$$= \frac{24}{5}$$

Hence the general solution is

$$\begin{aligned} y &= c_1 + c_2 e^{-5x} + c_3 x e^{-5x} \\ &= \frac{24}{5} - \frac{9}{5} e^{-5x} - 5 x e^{-5x} \\ &= \frac{1}{5}\left(24 - 9e^{-5x} - 25 x e^{-5x}\right) \end{aligned}$$

0.10 Section 5.3 problem 35

Problem: One solution of the differential equation is given, find the second solution $6y^{(4)} + 5y^{(3)} + 25y'' + 20y' + 4y = 0$, and $y_1 = \cos 2x$

solution The characteristic equation is $6r^4 + 5r^3 + 25r^2 + 20r + 4 = 0$. Since $\cos 2x$ is solution, then this implies the roots for this solution must be $r = \pm 2i$, since this is what will give $\cos 2x$ solution. Therefore, there must be factor $\left(r^2 + 4\right)$. Doing long division

$$\frac{6r^4 + 5r^3 + 25r^2 + 20r + 4}{\left(r^2 + 4\right)} = 6r^2 + 5r + 1$$

Hence characteristic equation is

$$\left(r^2 + 4\right)\left(6r^2 + 5r + 1\right)$$
$$\left(r^2 + 4\right)(2r + 1)(3r + 1)$$

Hence the roots are $r_1 = 2i, r_2 = -2i, r_3 = \frac{-1}{2}, r_4 = \frac{-1}{3}$. Therefore the solution is

$$\begin{aligned} y &= c_1 e^{2ix} + c_2 e^{-2ix} + c_3 e^{-\frac{1}{2}x} + c_4 e^{\frac{-1}{3}x} \\ &= c_1 \cos 2x + c_2 \sin 2x + c_3 e^{-\frac{1}{2}x} + c_4 e^{\frac{-1}{3}x} \end{aligned}$$

0.11 Section 5.3 problem 38

Problem: Solve $y^{(3)} - 5y'' + 100y' - 500y = 0$ with $y(0) = 0, y'(0) = 10, y''(0) = 250$ given that $y_1(x) = e^{5x}$ is one particular solution of the differential equation.

solution The characteristic equation is $r^3 - 5r^2 + 100r - 500 = 0$. Since $y_1(x) = e^{5x}$ is one solution, then $(r - 5)$ is one of the roots. Hence by long division

$$\frac{r^3 - 5r^2 + 100r - 500}{r - 5} = r^2 + 100$$

Therefore the factoring of the characteristic equation is

$$(r - 5)\left(r^2 + 100\right) = 0$$

Therefore the roots are $r_1 = 5, r_2 = 10i, r_3 = -10i$ and therefore the solution is

$$y = c_1 e^{5x} + c_2 \cos 10x + c_3 \sin 10x$$

Hence

$$\begin{aligned} y' &= 5c_1 e^{5x} - 10c_2 \sin 10x + 10c_3 \cos 10x \\ y'' &= 25c_1 e^{5x} - 100c_2 \cos 10x - 100c_3 \sin 10x \end{aligned}$$

Applying first IC gives

$$0 = c_1 + c_2$$

Second IC gives

$$10 = 5c_1 + 10c_3$$

Third IC gives

$$250 = 25c_1 - 100c_2$$

We have three equations to solve for c_1, c_2, c_3.

$$\begin{pmatrix} 1 & 1 & 0 \\ 5 & 0 & 10 \\ 25 & -100 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 10 \\ 250 \end{pmatrix}$$

$R_2 = R_2 - 5R_1$ and $R_3 = R_3 - 25R_1$ gives

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -5 & 10 \\ 0 & -125 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 10 \\ 250 \end{pmatrix}$$

$R_3 = R_3 - 25R_2$ gives

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -5 & 10 \\ 0 & 0 & -250 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 10 \\ 0 \end{pmatrix}$$

Hence from last row $c_3 = 0$. From second row $-5c_2 = 10$ or $c_2 = -2$ and from first row $c_1 + c_2 = 0$ or $c_1 = 2$. Hence the solution is

$$\begin{aligned} y &= c_1 e^{5x} + c_2 \cos 10x + c_3 \sin 10x \\ &= 2e^{5x} - 2\cos 10x \end{aligned}$$

0.12 Section 5.3 problem 40

Problem: Find linear homogeneous constant coefficient equation with the given general solution $y(x) = Ae^{2x} + B\cos 2x + C\sin 2x$

solution From the solution, we see that the roots are $r_1 = 2, r_2 = 2i, r_3 = -2i$. Hence the characteristic equation is

$$(r-2)\left(r^2+4\right) = 0$$
$$r^3 - 2r^2 + 4r - 8 = 0$$

Therefore the original ODE is

$$y'''(x) - 2y'' + 4y' - 8y = 0$$

0.13 Section 5.3 problem 49

Problem: Solve $y^{(4)} - y^{(3)} - y'' - y' - 2y = 0$ with $y(0) = 0, y'(0) = 0, y''(0) = 0, y^{(3)} = 30$

solution The characteristic equation is

$$r^4 - r^3 - r^2 - r - 2 = 0$$

By inspection, we see that $r = -1$ is a root. Hence by long division, we have

$$\frac{r^4 - r^3 - r^2 - r - 2}{(r+1)} = r^3 - 2r^2 + r - 2$$

Therefore characteristic equation is

$$(r+1)\left(r^3 - 2r^2 + r - 2\right) = 0$$

By inspection, one of the roots of $r^3 - 2r^2 + r - 2 = 0$ is 2, hence by long division

$$\frac{r^3 - 2r^2 + r - 2}{r-2} = r^2 + 1$$

Therefore characteristic equation becomes

$$(r+1)(r-2)\left(r^2+1\right) = 0$$

Hence roots are $r_1 = -1, r_2 = 2, r_3 = -i, r_4 = i$ and therefore the solution is

$$y = c_1 e^{-x} + c_2 e^{2x} + c_3 \cos x + c_4 \sin x$$

Initial conditions are now applied to find the constants.

$$\begin{aligned} y' &= -c_1 e^{-x} + 2c_2 e^{2x} - c_3 \sin x + c_4 \cos x \\ y'' &= c_1 e^{-x} + 4c_2 e^{2x} - c_3 \cos x - c_4 \sin x \\ y''' &= -c_1 e^{-x} + 8c_2 e^{2x} + c_3 \sin x - c_4 \cos x \end{aligned}$$

From $y(0) = 0$ we obtain

$$0 = c_1 + c_2 + c_3$$

From $y'(0) = 0$

$$0 = -c_1 + 2c_2 + c_4$$

From $y''(0) = 0$

$$0 = c_1 + 4c_2 - c_3$$

And from $y'''(0) = 30$

$$30 = -c_1 + 8c_2 - c_4$$

The 4 equations are solved for c_i

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ 1 & 4 & -1 & 0 \\ -1 & 8 & 0 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 30 \end{pmatrix}$$

$R_2 = R_2 + R_1, R_3 = R_3 - R_1, R_4 = R_4 + R_1$ gives

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 3 & -2 & 0 \\ 0 & 9 & 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 30 \end{pmatrix}$$

$R_3 = R_3 - R_2, R_4 = R_4 - 3R_2$ gives

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & -2 & -4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 30 \end{pmatrix}$$

$R_4 = R_4 - \frac{2}{3}R_3$ gives

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 0 & -\frac{10}{3} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 30 \end{pmatrix}$$

Hence from last row $-\frac{10}{3}c_4 = 30$,then

$$c_4 = -9$$

From 3rd row

$$\begin{aligned} -3c_3 - c_4 &= 0 \\ c_3 &= 3 \end{aligned}$$

From second row

$$\begin{aligned} 3c_2 + c_3 + c_4 &= 0 \\ 3c_2 + 3 - 9 &= 0 \\ c_2 &= 2 \end{aligned}$$

From first row

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_1 + 2 + 3 &= 0 \\ c_1 &= -5 \end{aligned}$$

Hence solution is

$$\begin{aligned} y &= c_1 e^{-x} + c_2 e^{2x} + c_3 \cos x + c_4 \sin x \\ &= -5e^{-x} + 2e^{2x} + 3\cos x - 9\sin x \end{aligned}$$