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70  
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72

# HW 10, Math 320, Spring 2017

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## Contents

0.1	Section 5.1 problem 10 (page 299) . . . . .	2
0.2	Section 5.1 problem 19 . . . . .	3
0.3	Section 5.1 problem 24 . . . . .	3
0.4	Section 5.1 problem 26 . . . . .	4
0.5	Section 5.1 problem 27 . . . . .	4
0.6	Section 5.1 problem 31 . . . . .	5
0.7	Section 5.1 problem 32 . . . . .	5
0.7.1	Part (a) . . . . .	5
0.7.2	Part (b) . . . . .	6
0.7.3	Part (c) . . . . .	6
0.8	Section 5.1 problem 34 . . . . .	6
0.9	Section 5.1 problem 42 . . . . .	6
0.10	Section 5.1 problem 48 . . . . .	7

## 0.1 Section 5.1 problem 10 (page 299)

### problem

Verify that  $y_1, y_2$  are solutions of the differential equation. Then find a particular solution of the form  $y = c_1 y_1 + c_2 y_2$  that satisfies the initial conditions.  $y'' - 10y' + 25y = 0$  with  $y_1 = e^{5x}, y_2 = xe^{5x}$  and  $y(0) = 3, y'(0) = 13$

### solution

To verify that  $y_1$  or  $y_2$  is solution to the ODE, we plug it into the ODE and see if it gives zero, which is what the RHS is. Since  $y_1' = 5e^{5x}, y_1'' = 25e^{5x}$ , then substituting this into the ODE gives

$$\begin{aligned} y_1'' - 10y_1' + 25y_1 &= 0 \\ 25e^{5x} - 10(5e^{5x}) + 25(e^{5x}) &= 0 \\ 25e^{5x} - 50e^{5x} + 25e^{5x} &= 0 \\ 0 &= 0 \end{aligned}$$

Hence verified. Now we do the same for  $y_2$ . Since  $y_2' = e^{5x} + 5xe^{5x}, y_2'' = 5e^{5x} + 5e^{5x} + 25xe^{5x}$ , then substituting this into the ODE gives

$$\begin{aligned} y_2'' - 10y_2' + 25y_2 &= 0 \\ (5e^{5x} + 5e^{5x} + 25xe^{5x}) - 10(e^{5x} + 5xe^{5x}) + 25(xe^{5x}) &= 0 \\ 5e^{5x} + 5e^{5x} + 25xe^{5x} - 10e^{5x} - 50xe^{5x} + 25xe^{5x} &= 0 \\ 25xe^{5x} - 50xe^{5x} + 25xe^{5x} &= 0 \\ 0 &= 0 \end{aligned}$$

Hence verified. Therefore the general solution is

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

Where the constants are found from initial conditions. Using the first initial condition gives

$$\begin{aligned} y(0) &= 3 \\ c_1 y_1(0) + c_2 y_2(0) &= 3 \\ c_1 (e^{5x})_{x=0} + c_2 (xe^{5x})_{x=0} &= 3 \\ c_1 &= 3 \end{aligned}$$

Hence the solution becomes

$$\begin{aligned} y(x) &= 3y_1(x) + c_2 y_2(x) \\ y' &= 3y_1'(x) + c_2 y_2'(x) \\ &= 3(5e^{5x}) + c_2(e^{5x} + 5xe^{5x}) \end{aligned}$$

Applying the second boundary conditions gives

$$\begin{aligned} y'(0) &= 13 \\ 3(5e^{5x})_{x=0} + c_2(e^{5x} + 5xe^{5x})_{x=0} &= 13 \\ 3(5) + c_2 &= 13 \\ c_2 &= 13 - 15 \\ &= -2 \end{aligned}$$

Therefore the particular solution is

$$\begin{aligned}
y(x) &= c_1 y_1(x) + c_2 y_2(x) \\
&= 3y_1(x) - 2y_2(x) \\
&= 3e^{5x} - 2xe^{5x} \\
&= e^{5x}(3 - 2x)
\end{aligned}$$

## 0.2 Section 5.1 problem 19

problem Show that  $y_1 = 1, y_2 = \sqrt{x}$  are solutions to  $yy'' + (y')^2 = 0$  but that their sum  $y = y_1 + y_2$  is not a solution

solution To show that  $y_1$  and  $y_2$  are solution to the ODE, we plug them into the ODE and see if the result is the same as the RHS. Since  $y_1 = 1$  then  $y_1' = 0, y_1'' = 0$ . Then ODE becomes

$$\begin{aligned}
y_1 y_1'' + (y_1')^2 &= 0 \\
1(0) + 0 &= 0 \\
0 &= 0
\end{aligned}$$

Hence verified. For  $y_2$ , we have  $y_2' = \frac{1}{2\sqrt{x}}, y_2'' = -\frac{1}{4x^{\frac{3}{2}}}$ . Hence the ODE becomes

$$\begin{aligned}
y_2 y_2'' + (y_2')^2 &= 0 \\
\frac{1}{2\sqrt{x}} \left( -\frac{1}{4x^{\frac{3}{2}}} \right) + \left( \frac{1}{2\sqrt{x}} \right)^2 &= 0 \\
\left( -\frac{1}{4x} \right) + \left( \frac{1}{4x} \right) &= 0 \\
0 &= 0
\end{aligned}$$

Hence verified. Now we plugin the sum into the ODE.

$$\begin{aligned}
(y_1 + y_2)(y_1 + y_2)'' + ((y_1 + y_2)')^2 &= 0 \\
(y_1 + y_2)(y_1'' + y_2'') + (y_1' + y_2')^2 &= 0 \\
(y_1 y_1'' + y_1 y_2'') + (y_2 y_1'' + y_2 y_2'') + (y_1')^2 + (y_2')^2 + 2y_1' y_2' &= 0 \\
y_1 y_1'' + y_1 y_2'' + y_2 y_1'' + y_2 y_2'' + (y_1')^2 + (y_2')^2 + 2y_1' y_2' &= 0
\end{aligned}$$

But we found that  $y_1 y_1'' + (y_1')^2 = 0$  and  $y_2 y_2'' + (y_2')^2 = 0$  from earlier. Using these into the LHS of the above simplifies it to

$$y_1 y_2'' + y_2 y_1'' + 2y_1' y_2' = 0$$

But  $y_2'' = -\frac{1}{4x^{\frac{3}{2}}}, y_1'' = 0, y_1' = 0, y_1 = 1$ , then the above becomes

$$-\frac{1}{4} \frac{1}{x^{\frac{3}{2}}} = 0$$

We see that the LHS is not zero. Hence  $y_1 + y_2$  is not a solution to the ODE.

## 0.3 Section 5.1 problem 24

problem Determine whether the pairs of functions are linearly independent or not on the real line.

$$f(x) = \sin^2 x, g(x) = 1 - \cos 2x$$

solution The two functions are L.I. if  $c_1 f(x) + c_2 g(x) = 0$  for each  $x$ , only when  $c_1 = c_2 = 0$ . Or stated differently, two functions are L.D. if there exist  $c_1, c_2$  not all zero, such that  $c_1 f(x) + c_2 g(x) = 0$  for

each  $x$ . To show this, we set up the Wronskian  $W$  and see if it is zero or not. If  $W = 0$  then this mean that the functions are L.D.

$$\begin{aligned} W &= \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = \\ &= \begin{vmatrix} \sin^2 x & 1 - \cos 2x \\ 2 \sin x \cos x & 2 \sin 2x \end{vmatrix} \\ &= 2 \sin^2 x \sin 2x - (1 - \cos 2x)(2 \sin x \cos x) \\ &= 2 \sin^2 x \sin 2x - 2 \sin x \cos x + 2 \cos 2x \sin x \cos x \end{aligned}$$

The RHS of the above simplifies to 0.

$$W = 0$$

Therefore, the functions are linearly dependent.

## 0.4 Section 5.1 problem 26

problem Determine whether the pairs of functions are linearly independent or not on the real line.

$$f(x) = 2 \cos x + 3 \sin x, g(x) = 3 \cos x - 2 \sin x$$

solution To show this, we set up the Wronskian  $W$  and see if it is zero or not. If  $W = 0$  then this mean that the functions are L.D.

$$\begin{aligned} W &= \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = \\ &= \begin{vmatrix} 2 \cos x + 3 \sin x & 3 \cos x - 2 \sin x \\ -2 \sin x + 3 \cos x & -3 \sin x - 2 \cos x \end{vmatrix} \\ &= (2 \cos x + 3 \sin x)(-3 \sin x - 2 \cos x) - (3 \cos x - 2 \sin x)(-2 \sin x + 3 \cos x) \\ &= -13 \cos^2 x - 13 \sin^2 x \\ &= -13 (\cos^2 x + \sin^2 x) \\ &= -13 \end{aligned}$$

Since  $W \neq 0$  then the functions are Linearly independent.

## 0.5 Section 5.1 problem 27

problem Let  $y_p$  be a particular solution of the nonhomogeneous equation  $y'' + py' + qy = f(x)$  and let  $y_h$  be the homogenous solution. Show that  $y = y_h + y_p$  is a solution of the given ODE.

solution since  $y_h$  satisfies the homogenous ODE then we can write

$$y_h'' + py_h' + qy_h = 0 \tag{1}$$

And since  $y_p$  satisfies the nonhomogeneous ODE then we can write

$$y_p'' + py_p' + qy_p = f(x) \tag{2}$$

Adding (1)+(2) gives

$$(y_p'' + y_h'') + p(y_p' + y_h') + q(y_p + y_h) = f(x)$$

But due to linearity of differentiation, then the above can be written as

$$(y_p + y_h)'' + p(y_p + y_p)' + q(y_p + y_h) = f(x)$$

Let  $Y = y_p + y_h$  then

$$Y'' + pY' + qY = f(x)$$

Therefore we showed that  $Y = y_p + y_h$  satisfies the original ODE, hence it is a solution. QED

## 0.6 Section 5.1 problem 31

problem Show that  $y_1 = \sin x^2$  and  $y_2 = \cos x^2$  are L.I. functions, but their Wronskian vanishes at  $x = 0$ . Why does this imply that there is no differential equation of the form  $y'' + p(x)y' + q(x)y = 0$  with both  $p, q$  continuous everywhere, having both  $y_1, y_2$  as solutions?

solution

$$\begin{aligned} W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \\ &= \begin{vmatrix} \sin x^2 & \cos x^2 \\ (2x) \cos x^2 & -(2x) \sin x^2 \end{vmatrix} \\ &= -2x \sin x^2 \cos x^2 - 2x \cos^2 x^2 \sin x^2 \\ &= -2x \left( (\sin x^2)^2 + (\cos x^2)^2 \right) \\ &= -2x \end{aligned}$$

The Wronskian is zero at  $x = 0$  but not zero at other points. It is only when  $W = 0$  everywhere, we say that  $y_1, y_2$  are L.D. We can have L.I. functions, but also have  $W(x_0) = 0$  at some  $x_0$  as in this problem. What this means, is that  $x = 0$  can not be in the domain of the solution for  $y_1, y_2$  to be solutions to the ODE. Hence, since the domain of the solution is everywhere, this means  $x = 0$  is part of the domain, then we conclude that  $y_1, y_2$  can not be both solutions, since they are L.I. at  $x = 0$ .

## 0.7 Section 5.1 problem 32

problem Let  $y_1, y_2$  be two solutions of  $A(x)y'' + B(x)y' + C(x)y = 0$  on open interval  $I$  where  $A, B, C$  are continuous and  $A(x)$  is never zero. (a) Let  $W = W(y_1, y_2)$ . Show that  $A(x) \frac{dW}{dx} = y_1(Ay_2'') - y_2(Ay_1'')$  then substitute for  $Ay_2''$  and  $Ay_1''$  from the original ODE to show that  $A(x) \frac{dW}{dx} = -B(x)W(x)$  (b) Solve this first order ODE equation to deduce Abel's formula  $W(x) = k \exp\left(-\int \frac{B(x)}{A(x)} dx\right)$  where  $k$  is constant. (c) Why does Abel's formula imply that the Wronskian  $W(y_1, y_2)$  is either zero everywhere or non-zero everywhere (as stated in theorem 3)?

solution

### 0.7.1 Part (a)

By definition

$$W(x) = y_1 y_2' - y_2 y_1'$$

Hence

$$\begin{aligned} \frac{dW}{dx} &= y_1' y_2' + y_1 y_2'' - y_2' y_1' - y_2 y_1'' \\ &= y_1 y_2'' - y_2 y_1'' \end{aligned}$$

Therefore

$$\begin{aligned} A(x) \frac{dW}{dx} &= A(x) (y_1 y_2'' - y_2 y_1'') \\ &= y_1 (A(x) y_2'') - y_2 (A(x) y_1'') \end{aligned} \tag{1}$$

But from original ODE,  $A(x)y_1'' + B(x)y_1' + C(x)y_1 = 0$ , therefore

$$A(x)y_1'' = -B(x)y_1' - C(x)y_1 \quad (2)$$

And also from original ODE,  $A(x)y_2'' + B(x)y_2' + C(x)y_2 = 0$ , therefore

$$A(x)y_2'' = -B(x)y_2' - C(x)y_2 \quad (3)$$

Substituting (2,3) into (1) gives

$$\begin{aligned} A(x) \frac{dW}{dx} &= y_1(-B(x)y_2' - C(x)y_2) - y_2(-B(x)y_1' - C(x)y_1) \\ &= -B(x)y_1y_2' - C(x)y_1y_2 + B(x)y_2y_1' + C(x)y_2y_1 \\ &= -B(x)y_1y_2' + B(x)y_2y_1' \\ &= -B(x)(y_1y_2' - y_2y_1') \\ &= -B(x)W(x) \end{aligned} \quad (4)$$

QED.

### 0.7.2 Part (b)

Solving (4).

$$\begin{aligned} A(x) \frac{dW}{dx} + B(x)W(x) &= 0 \\ \frac{dW}{dx} + \frac{B(x)}{A(x)}W(x) &= 0 \end{aligned}$$

Integrating factor is  $\mu = e^{\int \frac{B(x)}{A(x)} dx}$ , hence the above becomes

$$\frac{d}{dx}(\mu W(x)) = 0$$

Integrating gives

$$\begin{aligned} \mu W(x) &= k \\ W(x) &= ke^{-\int \frac{B(x)}{A(x)} dx} \end{aligned}$$

### 0.7.3 Part (c)

Since an exponential function is never zero (for bounded  $\frac{B(x)}{A(x)}$ ), then  $W(x) = ke^{(\cdot)}$  can only be zero if  $k = 0$ . This makes  $W = 0$  everywhere when  $k = 0$ . But if  $k \neq 0$ , then  $W \neq 0$  everywhere. So  $W$  can only be zero everywhere, or not zero everywhere.

## 0.8 Section 5.1 problem 34

problem Apply theorem 5 and 6 to find general solutions of the differential equation  $y'' + 2y' - 15y = 0$

solution The characteristic equation is  $r^2 + 2r - 15 = 0$ , and the roots are

$$\begin{aligned} r_1 &= 3 \\ r_2 &= -5 \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y(x) &= c_1e^{r_1x} + c_2e^{r_2x} \\ &= c_1e^{3x} + c_2e^{-5x} \end{aligned}$$

## 0.9 Section 5.1 problem 42

problem Apply theorem 5 and 6 to find general solutions of the differential equation  $35y'' - y' - 12y = 0$

solution The characteristic equation is  $35r^2 - r - 12 = 0$ , and the roots are

$$r_1 = \frac{3}{5}$$

$$r_2 = -\frac{4}{7}$$

Therefore the solution is

$$\begin{aligned} y(x) &= c_1 e^{r_1 x} + c_2 e^{r_2 x} \\ &= c_1 e^{\frac{3}{5}x} + c_2 e^{-\frac{4}{7}x} \end{aligned}$$

## 0.10 Section 5.1 problem 48

problem Problem gives a general solution  $y(x)$  of a homogeneous second order ODE  $ay'' + by' + cy = 0$  with constant coefficients. Find such an equation  $y(x) = e^x (c_1 e^{x\sqrt{2}} + c_2 e^{-x\sqrt{2}})$

solution We compare the above solution to the general form of the solution given by

$$\begin{aligned} y &= c_1 e^{r_1 x} + c_2 e^{r_2 x} \\ &= c_1 e^{x(1+\sqrt{2})} + c_2 e^{x(1-\sqrt{2})} \end{aligned}$$

We see that

$$r_1 = 1 + \sqrt{2}$$

$$r_2 = 1 - \sqrt{2}$$

This implies that the characteristic equation is

$$\begin{aligned} (r - r_1)(r - r_2) &= 0 \\ (r - (1 + \sqrt{2}))(r - (1 - \sqrt{2})) &= 0 \\ r^2 - 2r - 1 &= 0 \end{aligned}$$

Therefore the ODE is

$$y'' - 2y' - y = 0$$

Where  $a = 1, b = -2, c = -1$ .