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This note solves

$$\begin{aligned}\varepsilon y''(x) + (1+x)\,y'(x) + y(x) &= 0\\ y(0) &= 1\\ y(1) &= 1\end{aligned}$$

Where ε is small parameter, using boundary layer theory.

0.0.1 Solution

Since $(1+x) > 0$ in the domain, we expect boundary layer to be on the left. Let $y_{out}(x)$ be the solution in the outer region. Starting with $y(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n$ and substituting back into the ODE gives

$$\varepsilon\left(y_0'' + \varepsilon y_1'' + \cdots\right) + (1+x)\left(y_0' + \varepsilon y_1' + \cdots\right) + \left(y_0 + \varepsilon y_1 + \cdots\right) = 0$$

$O(1)$ terms

Collecting all terms with zero powers of ε

$$(1+x)\,y_0' + y_0 = 0$$

The above is solved using the right side conditions, since this is where the outer region is located. Solving the above using $y_0(1) = 1$ gives

$$y_0^{out}(x) = \frac{2}{1+x}$$

Now we need to find $y_{in}(x)$. To do this, we convert the ODE using transformation $\xi = \frac{x}{\varepsilon}$. Hence $\frac{dy}{dx} = \frac{dy}{d\xi}\frac{d\xi}{dx} = \frac{dy}{d\xi}\frac{1}{\varepsilon}$. Hence the operator $\frac{d}{dx} \equiv \frac{1}{\varepsilon}\frac{d}{d\xi}$. This means the operator $\frac{d^2}{dx^2} \equiv \left(\frac{1}{\varepsilon}\frac{d}{d\xi}\right)\left(\frac{1}{\varepsilon}\frac{d}{d\xi}\right) = \frac{1}{\varepsilon^2}\frac{d^2}{d\xi^2}$. The ODE becomes

$$\begin{aligned}\varepsilon\frac{1}{\varepsilon^2}\frac{d^2 y(\xi)}{d\xi^2} + (1+\xi\varepsilon)\frac{1}{\varepsilon}\frac{dy(\xi)}{d\xi} + y(\xi) &= 0\\ \frac{1}{\varepsilon}y'' + \left(\frac{1}{\varepsilon}+\xi\right)y' + y &= 0\end{aligned}$$

Plugging $y(\xi) = \sum_{n=0}^{\infty} \varepsilon^n y(\xi)_n$ into the above gives

$$\frac{1}{\varepsilon}\left(y_0'' + \varepsilon y_1'' + \cdots\right) + \left(\frac{1}{\varepsilon} + \xi\right)\left(y_0' + \varepsilon y_1' + \cdots\right) + \left(y_0 + \varepsilon y_1 + \cdots\right) = 0$$

$$\frac{1}{\varepsilon}\left(y_0'' + \varepsilon y_1'' + \cdots\right) + \frac{1}{\varepsilon}\left(y_0' + \varepsilon y_1' + \cdots\right) + \xi\left(y_0' + \varepsilon y_1' + \cdots\right) + \left(y_0 + \varepsilon y_1 + \cdots\right) = 0$$

Collecting all terms with smallest power of ε , which is ε^{-1} in this case, gives

$$\frac{1}{\varepsilon}y_0'' + \frac{1}{\varepsilon}y_0' = 0$$

$$y_0'' + y_0' = 0$$

Let $z = y_0'$, the above becomes

$$z' + z = 0$$

$$d\left(e^{\xi}z\right) = 0$$

$$e^{\xi}z = c$$

$$z = ce^{-\xi}$$

Hence $y_0'(\xi) = ce^{-\xi}$. Integrating

$$y_0^{in}(\xi) = -ce^{-\xi} + c_1 \tag{1A}$$

Since c is arbitrary constant, the negative sign can be removed, giving

$$y_0^{in}(\xi) = ce^{-\xi} + c_1 \tag{1A}$$

This is the lowest order solution for the inner $y^{in}(\xi)$. We have two boundary conditions, but we can only use the left side one, where $y_{in}(\xi)$ lives. Hence using $y_0(\xi = 0) = 1$, the above becomes

$$1 = c + c_1$$

$$c_1 = 1 - c$$

The solution (1A) becomes

$$y_0(\xi) = ce^{-\xi} + (1 - c)$$

$$= 1 + c\left(e^{-\xi} - 1\right)$$

Let $c = A_0$ to match the book notation.

$$y_0(\xi) = 1 + A_0\left(e^{-\xi} - 1\right)$$

To find A_0, we match $y_0^{in}(\xi)$ with $y_0^{out}(x)$

$$\lim_{\xi\to\infty} 1 + A_0\left(e^{-\xi} - 1\right) = \lim_{x\to 0^+} \frac{2}{1+x}$$

$$1 - A_0 = 2$$

$$A_0 = -1$$

Hence

$$y_0^{in}(\xi) = 2 - e^{-\xi}$$

$O(\varepsilon)$ terms

We now repeat the process to find $y_1^{in}(\xi)$ and $y_1^{out}(x)$. Starting with $y^{out}(x)$

$$\varepsilon\left(y_0''+\varepsilon y_1''+\cdots\right)+(1+x)\left(y_0'+\varepsilon y_1'+\cdots\right)+\left(y_0+\varepsilon y_1+\cdots\right)=0$$

Collecting all terms with ε^1 now

$$\begin{aligned}\varepsilon y_0''+(1+x)\,\varepsilon y_1'+\varepsilon y_1&=0\\ y_0''+(1+x)\,y_1'+y_1&=0\end{aligned}$$

But we know $y_0=\frac{2}{1+x}$, from above. Hence $y_0''=\frac{4}{(1+x)^3}$ and the above becomes

$$\begin{aligned}(1+x)\,y_1'+y_1&=-\frac{4}{(1+x)^3}\\ y_1'+\frac{y_1}{1+x}&=-\frac{4}{(1+x)^4}\end{aligned}$$

Integrating factor $\mu=e^{\int\frac{1}{1+x}dx}=e^{\ln(1+x)}=1+x$ and the above becomes

$$\begin{aligned}\frac{d}{dx}\left(\mu y_1\right)&=-\mu\frac{4}{(1+x)^4}\\ \frac{d}{dx}\left((1+x)\,y_1\right)&=-\frac{4}{(1+x)^3}\end{aligned}$$

Integrating

$$\begin{aligned}(1+x)\,y_1&=-\int\frac{4}{(1+x)^3}dx+c\\ &=\frac{2}{(1+x)^2}+c\end{aligned}$$

Hence

$$y_1(x)=\frac{2}{(1+x)^3}+\frac{c}{1+x}$$

Applying $y(1)=0$ (notice the boundary condition now becomes $y(1)=0$ and not $y(1)=1$, since we have already used $y(1)=1$ to find leading order). From now on, all boundary conditions will be $y(1)=0$.

$$\begin{aligned}0&=\frac{2}{(1+1)^3}+\frac{c}{1+1}\\ c&=-\frac{1}{2}\end{aligned}$$

Hence

$$y_1(x)=\frac{2}{(1+x)^3}-\frac{1}{2}\frac{1}{(1+x)}$$

Now we need to find $y_1^{in}(\xi)$. To do this, starting from

$$\frac{1}{\varepsilon}\left(y_0''+\varepsilon y_1''+\cdots\right)+\left(\frac{1}{\varepsilon}+\xi\right)\left(y_0'+\varepsilon y_1'+\cdots\right)+\left(y_0+\varepsilon y_1+\cdots\right)=0$$

$$\frac{1}{\varepsilon}\left(y_0''+\varepsilon y_1''+\cdots\right)+\frac{1}{\varepsilon}\left(y_0'+\varepsilon y_1'+\cdots\right)+\xi\left(y_0'+\varepsilon y_1'+\cdots\right)+\left(y_0+\varepsilon y_1+\cdots\right)=0$$

But now collecting all terms with $O(1)$ order, (last time, we collected terms with $O\left(\varepsilon^{-1}\right)$).

$$y_1''+y_1'+\xi y_0'+y_0=0$$

$$y_1''+y_1'=-\xi y_0'-y_0 \tag{1}$$

But we found y_0^{in} earlier which was

$$y_0^{in}(\xi)=1+A_0\left(e^{-\xi}-1\right)$$

Hence $y_0'=-A_0e^{-\xi}$ and the ODE (1) becomes

$$y_1''+y_1'=\xi A_0e^{-\xi}-\left(1+A_0\left(e^{-\xi}-1\right)\right)$$

We need to solve this with boundary conditions $y_1(0)=0$. (again, notice change in B.C. as was mentioned above). The solution is

$$\begin{aligned}y_1(\xi)&=-\xi+A_0\left(\xi-\frac{1}{2}\xi^2e^{-\xi}\right)+A_1\left(1-e^{-\xi}\right)\\&=-\xi+A_0\left(\xi-\frac{1}{2}\xi^2e^{-\xi}\right)-A_1\left(e^{-\xi}-1\right)\end{aligned}$$

Since A_1 is arbitrary constant, and to match the book, we can call $A_2=-A_1$ and then rename A_2 back to A_1 and obtain

$$y_1(\xi)=-\xi+A_0\left(\xi-\frac{1}{2}\xi^2e^{-\xi}\right)+A_1\left(e^{-\xi}-1\right)$$

This is to be able to follow the book. Therefore, this is what we have so far

$$\begin{aligned}y_{out}&=y_0^{out}+\varepsilon y_1^{out}\\&=\frac{2}{1+x}+\varepsilon\left(\frac{2}{(1+x)^3}-\frac{1}{2(1+x)}\right)\end{aligned}$$

And

$$\begin{aligned}y^{in}(\xi)&=y_0^{in}+\varepsilon y_1^{in}\\&=\left(1+A_0\left(e^{-\xi}-1\right)\right)+\varepsilon\left(-\xi+A_0\left(\xi-\frac{1}{2}\xi^2e^{-\xi}\right)+A_1\left(e^{-\xi}-1\right)\right)\\&=A_0e^{-\xi}-\xi\varepsilon-\varepsilon A_1-A_0+\varepsilon A_1e^{-\xi}+\xi\varepsilon A_0-\frac{1}{2}\xi^2\varepsilon A_0e^{-\xi}+1\end{aligned}$$

To find A_0, A_1, we match y_{in} with y_{out}, therefore

$$\lim_{\xi\to\infty}y_{in}=\lim_{x\to0}y_{out}$$

Or

$$\lim_{\xi\to\infty}\left(A_0e^{-\xi}-\xi\varepsilon-\varepsilon A_1-A_0+\varepsilon A_1e^{-\xi}+\xi\varepsilon A_0-\frac{1}{2}\xi^2\varepsilon A_0e^{-\xi}+1\right)=\lim_{x\to 0}\frac{2}{1+x}+\varepsilon\left(\frac{2}{(1+x)^3}-\frac{1}{2(1+x)}\right)$$

Which simplifies to

$$-\xi\varepsilon-\varepsilon A_1-A_0+\xi\varepsilon A_0+1=\lim_{x\to 0}\frac{2}{1+x}+\varepsilon\left(\frac{2}{(1+x)^3}-\frac{1}{2(1+x)}\right)$$

It is easier now to convert the LHS to use x instead of ξ so we can compare. Since $\xi=\frac{x}{\varepsilon}$, then the above becomes

$$-x-\varepsilon A_1-A_0+xA_0+1=\lim_{x\to 0}\frac{2}{1+x}+\varepsilon\left(\frac{2}{(1+x)^3}-\frac{1}{2(1+x)}\right)$$

Using Taylor series on the RHS

$$\begin{aligned}1-A_0-x+A_0x+A_1\varepsilon&=\lim_{x\to 0}2\left(1-x+x^2+\cdots\right)\\&\quad+2\varepsilon\left(1-x+x^2+\cdots\right)\left(1-x+x^2+\cdots\right)\left(1-x+x^2+\cdots\right)-\frac{\varepsilon}{2}\left(1-x+x^2+\cdots\right)\end{aligned}$$

Since we have terms on the the LHS of only $O(1),O(x),O(\varepsilon)$, then we need to keep at least terms with $O(1),O(x),O(\varepsilon)$ on the RHS and drop terms with $O\left(x^2\right),O(\varepsilon x),O\left(\varepsilon^2\right)$ to be able to do the matching. So in the above, RHS simplifies to

$$\begin{aligned}-x-\varepsilon A_1-A_0+xA_0+1&=2(1-x)+2\varepsilon-\frac{\varepsilon}{2}\\-x-\varepsilon A_1-A_0+xA_0+1&=2-2x+2\varepsilon-\frac{\varepsilon}{2}\\-\varepsilon A_1-A_0+x(A_0-1)+1&=2-2x+\frac{3}{2}\varepsilon\end{aligned}$$

Comparing, we see that

$$\begin{aligned}A_0-1&=-2\\A_0&=-1\end{aligned}$$

We notice this is the same A_0 we found for the lowest order. This is how it should always come out. If we get different value, it means we made mistake. We could also match $-A_0+1=2$ which gives $A_0=-1$ as well. Finally

$$\begin{aligned}-\varepsilon A_1&=\frac{3}{2}\varepsilon\\A_1&=-\frac{3}{2}\end{aligned}$$

So we have used matching to find all the constants for y_{in}. Here is the final solution so far

$$y_{out}(x) = \overbrace{\frac{2}{1+x}}^{y_0} + \varepsilon \overbrace{\left(\frac{2}{(1+x)^3} - \frac{1}{2(1+x)}\right)}^{y_1}$$

$$\begin{aligned} y_{in}(\xi) &= \overbrace{1 + A_0\left(e^{-\xi} - 1\right)}^{y_0} + \varepsilon \overbrace{\left(-\xi + A_0\left(\xi - \frac{1}{2}\xi^2 e^{-\xi}\right) + A_1\left(e^{-\xi} - 1\right)\right)}^{y_1} \\ &= 1 - \left(e^{-\xi} - 1\right) + \varepsilon\left(-\xi - \left(\xi - \frac{1}{2}\xi^2 e^{-\xi}\right) - \frac{3}{2}\left(e^{-\xi} - 1\right)\right) \\ &= \frac{3}{2}\varepsilon - e^{-\xi} - 2\xi\varepsilon - \frac{3}{2}\varepsilon e^{-\xi} + \frac{1}{2}\xi^2\varepsilon e^{-\xi} + 2 \end{aligned}$$

In terms of x, since Since $\xi = \frac{x}{\varepsilon}$ the above becomes

$$\begin{aligned} y_{in}(x) &= \frac{3}{2}\varepsilon - e^{-\frac{x}{\varepsilon}} - 2x - \frac{3}{2}\varepsilon e^{-\frac{x}{\varepsilon}} + \frac{1}{2}\frac{x^2}{\varepsilon}e^{-\frac{x}{\varepsilon}} + 2 \\ &= 2 - 2x + \frac{3}{2}\varepsilon + e^{-\frac{x}{\varepsilon}}\left(\frac{1}{2}\frac{x^2}{\varepsilon} - \frac{3}{2}\varepsilon - 1\right) \end{aligned}$$

Hence

$$y_{uniform} = y_{in} + y_{out} - y_{match}$$

Where

$$\begin{aligned} y_{match} &= \lim_{\xi\to\infty} y_{in} \\ &= 2 - 2x + \frac{3}{2}\varepsilon \end{aligned}$$

Hence

$$\begin{aligned} y_{uniform} &= 2 - 2x + \frac{3}{2}\varepsilon + e^{-\frac{x}{\varepsilon}}\left(\frac{1}{2}\frac{x^2}{\varepsilon} - \frac{3}{2}\varepsilon - 1\right) + \frac{2}{1+x} + \varepsilon\left(\frac{2}{(1+x)^3} - \frac{1}{2(1+x)}\right) - \left(2 - 2x + \frac{3}{2}\varepsilon\right) \\ &= e^{-\frac{x}{\varepsilon}}\left(\frac{1}{2}\frac{x^2}{\varepsilon} - \frac{3}{2}\varepsilon - 1\right) + \frac{2}{1+x} + \varepsilon\left(\frac{2}{(1+x)^3} - \frac{1}{2(1+x)}\right) \\ &= \left(\frac{2}{1+x} - e^{-\frac{x}{\varepsilon}} + \frac{1}{2}\frac{x^2}{\varepsilon}e^{-\frac{x}{\varepsilon}}\right) + \varepsilon\left(\frac{2}{(1+x)^3} - \frac{1}{2(1+x)} - \frac{3}{2}e^{-\frac{x}{\varepsilon}}\right) \end{aligned}$$

Which is the same as

$$\begin{aligned} y_{uniform} &= \left(\frac{2}{1+x} - e^{-\xi} + \frac{1}{2}\frac{x^2}{\varepsilon}e^{-\xi}\right) + \varepsilon\left(\frac{2}{(1+x)^3} - \frac{1}{2(1+x)} - \frac{3}{2}e^{-\xi}\right) \\ &= \left(\frac{2}{1+x} - e^{-\xi}\right) + \varepsilon\left(\frac{2}{(1+x)^3} - \frac{1}{2(1+x)} - \frac{3}{2}e^{-\xi} + \frac{1}{2}\xi^2 e^{-\xi}\right) \\ &= \left(\frac{2}{1+x} - e^{-\xi}\right) + \varepsilon\left(\frac{2}{(1+x)^3} - \frac{1}{2(1+x)} + \left(\frac{1}{2}\xi^2 - \frac{3}{2}\right)e^{-\xi}\right) \qquad (1) \end{aligned}$$

Comparing (1) above, with book result in first line of 9.3.16, page 433, we see the same result.

0.0.2 References

1. Advanced Mathematica methods, Bender and Orszag. Chapter 9.

2. Lecture notes. Feb 16, 2017. By Professor Smith. University of Wisconsin. NE 548