Comparing Exact to WKB solution for ODE in lecture 3/2/2017

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This note shows how to obtain exact solution for the ODE given in lecture 3/2/2017, EP 548, and to compare it to the WKB solution for different modes. This shows that the WKB becomes very close to the exact solution for higher modes.

Obtain the exact solution, in terms of BesselJ functions

In[16]:=
```
ClearAll[n, c, y, m, lam];
lam[n_] := (9 n^2 / 49 Pi^4); (*eigenvalues from WKB solution*)
c = Sqrt[6 / 7 Pi^3]; (*normalization value found for WKB*)

y[n_, x_] := c 1/(Pi + x) Sin[n (x^3 + 3 x^2 Pi + 3 Pi^2 x) / (7 Pi^2)];
(*WKB solution found*)
```

Find exact solution

In[18]:=
```
ode = y''[x] + lam (x + Pi)^4 y[x] == 0;
(solExact = y[x] /. First@DSolve[{ode, y[0] == 0}, y[x], x]) // TraditionalForm
```

Out[19]//TraditionalForm=

$$\frac{1}{\sqrt[6]{6}\,\sqrt[24]{\text{lam}}\,J_{\frac{1}{6}}\left(\frac{\sqrt{\text{lam}}\,\pi^3}{3}\right)} c_1\,\Gamma\left(\frac{5}{6}\right)\sqrt[8]{\text{lam}\,(x+\pi)^4}\left(J_{\frac{1}{6}}\left(\frac{\sqrt{\text{lam}}\,\pi^3}{3}\right)J_{-\frac{1}{6}}\left(\frac{\left(\text{lam}\,x^4+4\,\text{lam}\,\pi\,x^3+6\,\text{lam}\,\pi^2\,x^2+4\,\text{lam}\,\pi^3\,x+\text{lam}\,\pi^4\right)^{3/4}}{3\sqrt[4]{\text{lam}}}\right)- J_{-\frac{1}{6}}\left(\frac{\sqrt{\text{lam}}\,\pi^3}{3}\right)J_{\frac{1}{6}}\left(\frac{\left(\text{lam}\,x^4+4\,\text{lam}\,\pi\,x^3+6\,\text{lam}\,\pi^2\,x^2+4\,\text{lam}\,\pi^3\,x+\text{lam}\,\pi^4\right)^{3/4}}{3\sqrt[4]{\text{lam}}}\right)\right)$$

Make function which normalizes the exact solution eigenfunctions and plot each mode eigenfunction with the WKB on the same plot

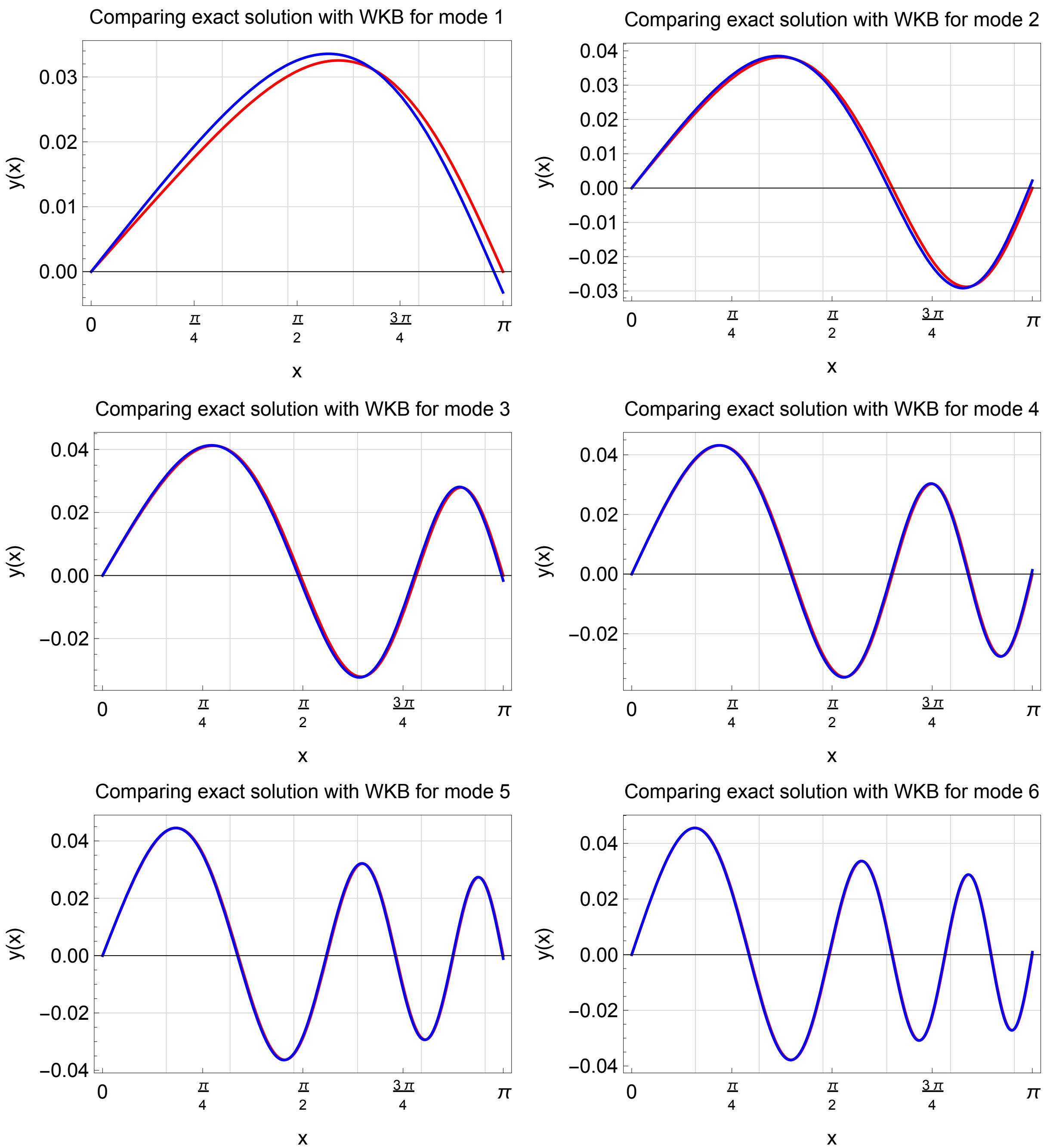
In[65]:=
```
compare[modeNumber_] :=
  Module[{solExact1, int, cFromExact, eigenvalueFromHandSolution, flip},
   eigenvalueFromHandSolution = lam[modeNumber];
   solExact1 = solExact /. lam → eigenvalueFromHandSolution;
   int = Integrate[solExact1^2 * (x + Pi) ^4, {x, 0, Pi}];
   cFromExact = First@NSolve[int == 1, C[1]];
   solExact1 = solExact1 /. cFromExact;
   If[modeNumber > 5, flip = -1, flip = 1];
   Plot[ {y[modeNumber, x], flip * solExact1}, {x, 0, Pi},
    PlotStyle → {Red, Blue}, Frame -> True, FrameLabel → {{"y(x)", None},
      {"x", Row[{"Comparing exact solution with WKB for mode ", modeNumber}]}},
    GridLines → Automatic, GridLinesStyle → LightGray, BaseStyle → 12, ImageSize → 310,
    FrameTicks → {{Automatic, None}, {{0, Pi / 4, Pi / 2, 3 / 4 Pi, Pi}, None}}
   ]
  ];
```

Generate 4 plots, for mode 1, up to 6

These plots show that after mode 5 or 6, the two eigenfunctions are almost exact

In[66]:=
```
plots = Table[compare[n], {n, 6}];
Grid[Partition[plots, 2]]
```

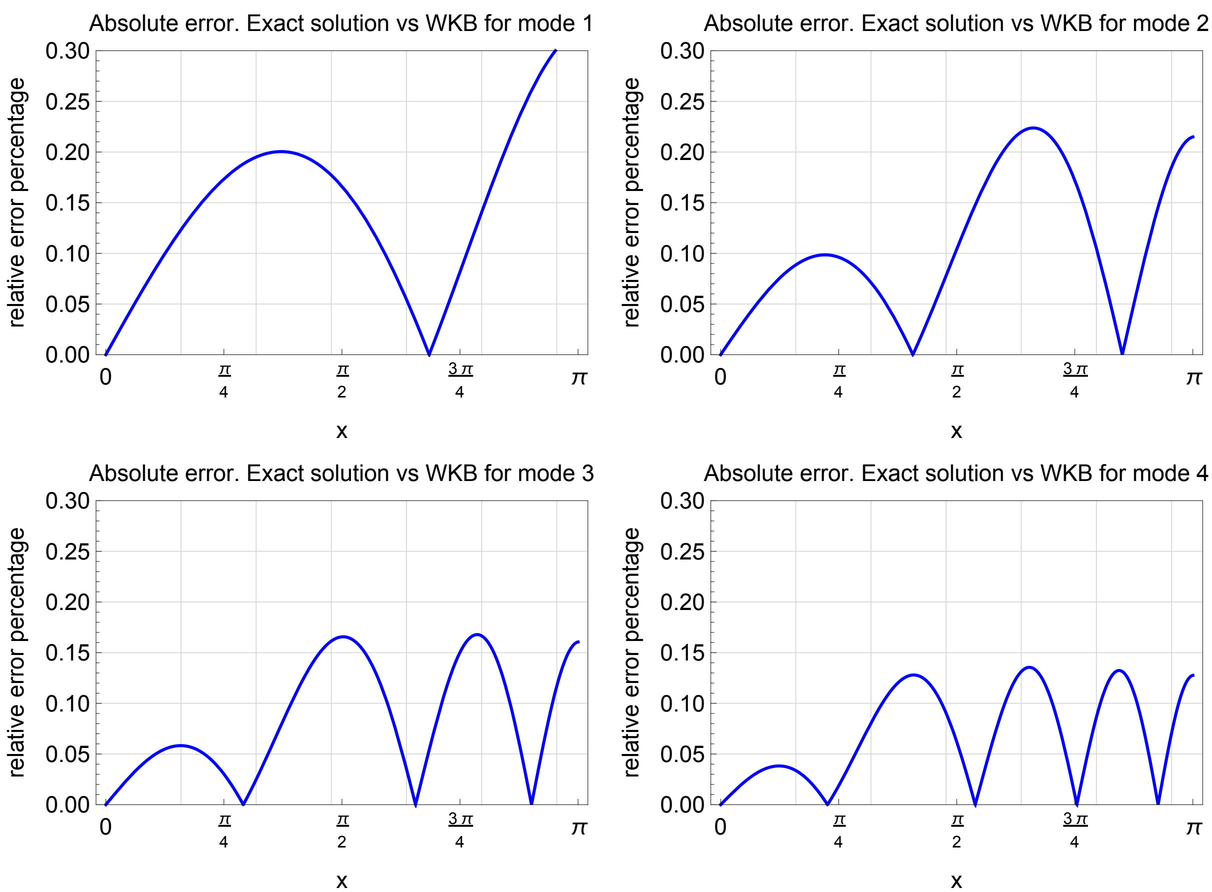
Out[67]=



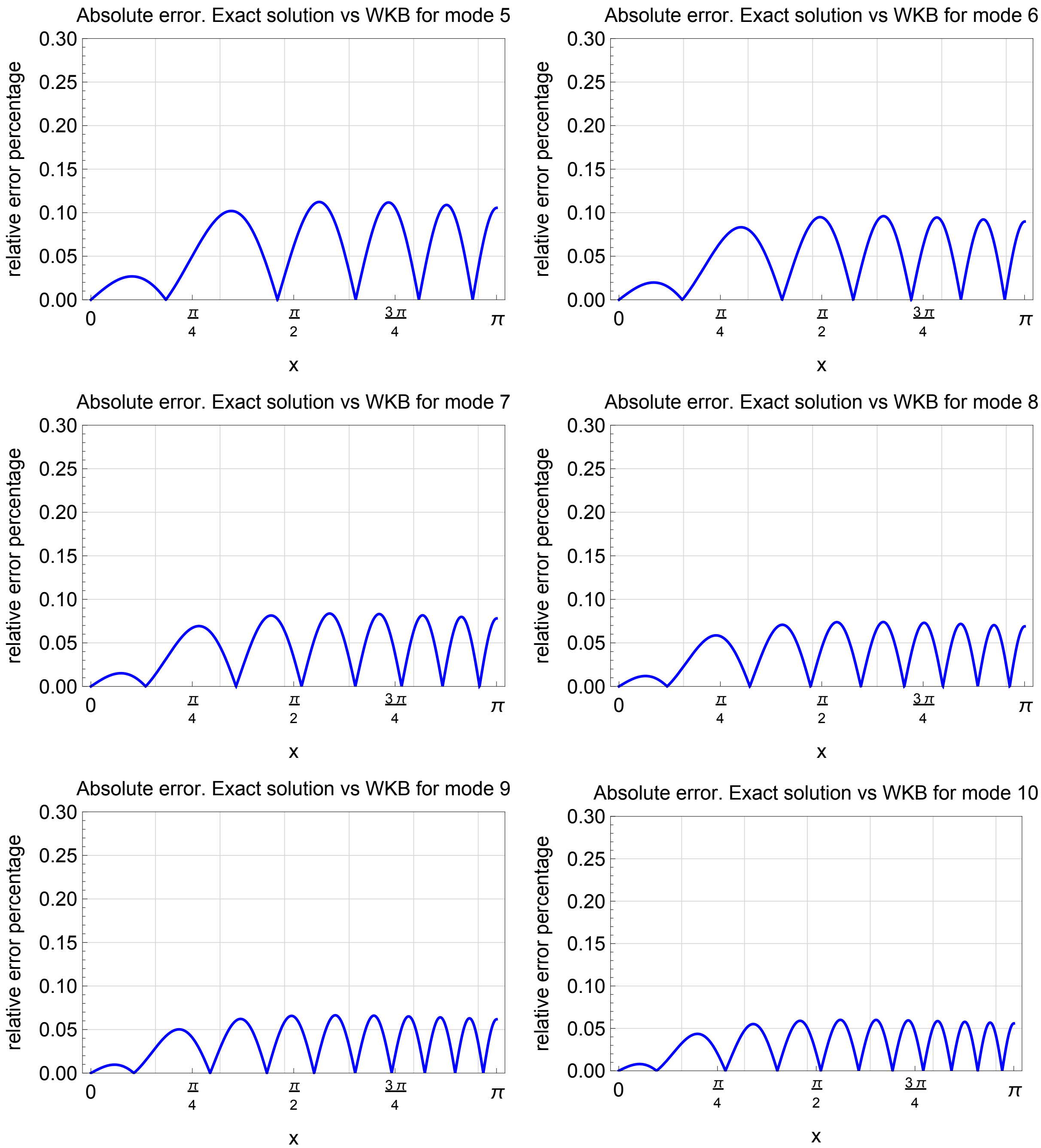
Generate the above again, but now using relative error between the exact and WKB for each mode, to make it more clear

In[68]:=
```
compareError[modeNumber_] :=
  Module[{solExact1, eigenvalueFromHandSolution, int, cFromExact, flip},
   eigenvalueFromHandSolution = lam[modeNumber];
   solExact1 = solExact /. lam → eigenvalueFromHandSolution;
   int = Integrate[solExact1^2 * (x + Pi)^4, {x, 0, Pi}];
   cFromExact = First@NSolve[int == 1, C[1]];
   solExact1 = solExact1 /. cFromExact;
   If[modeNumber > 5, flip = -1, flip = 1];
   Plot[ 100 * Abs[(flip * solExact1 - y[modeNumber, x])], {x, 0, Pi}, PlotStyle →
     {Red, Blue}, Frame -> True, FrameLabel → {{"relative error percentage", None},
      {"x", Row[{"Absolute error. Exact solution vs WKB for mode ", modeNumber}]}},
    GridLines → Automatic, GridLinesStyle → LightGray, BaseStyle → 12, ImageSize → 310,
    FrameTicks → {{Automatic, None}, {{0, Pi/4, Pi/2, 3/4 Pi, Pi}, None}},
    PlotRange → {Automatic, {0, 0.3}}
   ]
  ];
```

In[69]:=
```
plots = Table[compareError[n], {n, 10}]; (*let do 10 modes*)
Grid[Partition[plots, 2]]
```



Out[69]=


Absolute error. Exact solution vs WKB for mode 5
Absolute error. Exact solution vs WKB for mode 6
Absolute error. Exact solution vs WKB for mode 7
Absolute error. Exact solution vs WKB for mode 8
Absolute error. Exact solution vs WKB for mode 9
Absolute error. Exact solution vs WKB for mode 10
relative error percentage
0.00 0.05 0.10 0.15 0.20 0.25 0.30
0 π/4 π/2 3π/4 π
x