Manual Verification of 3.4.28 in Bender and Orszag textbook

Nasser M. Abbasi

December 30, 2019

Current rules I am using in simplifications are

1. $S_0' \ggg S_1'$
2. $S_0'S_1' \ggg \left(S_1'\right)^2$
3. $S_0 \ggg S_1$
4. $(S_0)^n \ggg S_0^{(n)}$ (the first is power, the second is derivative order).

Verify the above are valid for $x \to 0^+$ and well for $x \to \infty$. What can we say about (S') compared to $(S')^2$?

0.1 problem (a), page 88

$$y'' = \frac{1}{x^5} y$$

Irregular singular point at $x \to 0^+$. Let $y = e^{S_0(x)}$ and the above becomes

$$\begin{aligned} y(x) &= e^{S_0(x)} \\ y'(x) &= S_0' e^S \\ y'' &= S_0'' e^{S_0} + \left(S_0'\right)^2 e^{S_0} \\ &= \left(S_0'' + \left(S_0'\right)^2\right) e^{S_0} \end{aligned}$$

Substituting back into $\frac{d^2}{dx^2} y = x^{-5} y$ gives

$$\left(S_0'' + \left(S_0'\right)^2\right) e^S = x^{-5} e^{S_0(x)}$$

$$S_0'' + \left(S_0'\right)^2 = x^{-5}$$

Before solving for S_0, we can do one more simplification. Using the approximation that $\left(S_0'\right)^2 \ggg S_0''$ for $x \to x_0$, the above becomes

$$\left(S_0'\right)^2 \sim x^{-5}$$

Now we are ready to solve for S_0

$$\begin{aligned} S_0' &\sim \omega x^{-\frac{5}{2}} \\ S_0 &\sim \omega \int x^{-\frac{5}{2}} dx \\ &\sim \omega \frac{x^{-\frac{3}{2}}}{-\frac{3}{2}} \\ &\sim -\frac{2}{3}\omega x^{-\frac{3}{2}} \end{aligned}$$

To find leading behavior, let

$$S(x) = S_0(x) + S_1(x)$$

Then $y(x) = e^{S_0(x)+S_1(x)}$ and hence now

$$\begin{aligned} y'(x) &= (S_0(x) + S_1(x))' e^{S_0+S_1} \\ y''(x) &= \left((S_0 + S_1)'\right)^2 e^{S_0+S_1} + (S_0 + S_1)'' e^{S_0+S_1} \end{aligned}$$

Using the above, the ODE $\frac{d^2}{dx^2}y = x^{-5}y$ now becomes

$$\begin{aligned} \left((S_0 + S_1)'\right)^2 e^{S_0+S_1} + (S_0 + S_1)'' e^{S_0+S_1} &\sim x^{-5}e^{S_0+S_1} \\ \left((S_0 + S_1)'\right)^2 + (S_0 + S_1)'' &\sim x^{-5} \\ \left(S_0' + S_1'\right)^2 + S_0'' + S_1'' &\sim x^{-5} \\ \left(S_0'\right)^2 + \left(S_1'\right)^2 + 2S_0'S_1' + S_0'' + S_1'' &\sim x^{-5} \end{aligned}$$

But $S_0' = \omega x^{-\frac{5}{2}}$, found before, hence $\left(S_0'\right)^2 = x^{-5}$ and the above simplifies to

$$\left(S_1'\right)^2 + 2S_0'S_1' + S_0'' + S_1'' = 0$$

Using approximation $S_0'S_1' \ggg \left(S_1'\right)^2$ the above simplifies to

$$2S_0'S_1' + S_0'' + S_1'' = 0$$

Finally, using approximation $S_0'' \ggg S_1''$, the above becomes

$$\begin{aligned} 2S_0'S_1' + S_0'' &= 0 \\ S_1' &\sim -\frac{S_0''}{2S_0'} \\ S_1 &\sim -\frac{1}{2}\ln S_0' + c \\ S_1' &\sim -\frac{1}{2}\ln x^{-\frac{5}{2}} + c \\ S_1' &\sim \frac{5}{4}\ln x + c \end{aligned}$$

Hence, the leading behavior is

$$\begin{aligned} y(x) &= e^{S_0(x)+S_1(x)} \\ &= \exp\left(-\frac{2}{3}\omega x^{-\frac{3}{2}} + \frac{5}{4}\ln x + c\right) \\ &= cx^{\frac{5}{4}}\exp\left(-\omega\frac{2}{3}x^{-\frac{3}{2}}\right) \end{aligned} \tag{1}$$

To verify, using the formula 3.4.28, which is

$$y(x) \sim cQ^{\frac{1-n}{2n}}\exp\left(\omega\int^x Q(t)^{\frac{1}{n}}\,dt\right)$$

In this case, $n = 2$, since the ODE $y'' = x^{-5}y$ is second order. Here we have $Q(x) = x^{-5}$, therefore, plug-in into the above gives

$$\begin{aligned} y(x) &\sim c\left(x^{-5}\right)^{\frac{1-2}{4}}\exp\left(\omega\int^x \left(t^{-5}\right)^{\frac{1}{2}}\,dt\right) \\ &\sim c\left(x^{-5}\right)^{\frac{-1}{4}}\exp\left(\omega\int^x t^{-\frac{5}{2}}dt\right) \\ &\sim cx^{\frac{5}{4}}\exp\left(\omega\left(\frac{x^{-\frac{3}{2}}}{-\frac{3}{2}}\right)\right) \\ &\sim cx^{\frac{5}{4}}\exp\left(-\omega\frac{2}{3}x^{-\frac{3}{2}}\right) \end{aligned} \tag{2}$$

Comparing (1) and (2), we see they are the same.

0.2 problem (b), page 88

$$y''' = xy$$

Irregular singular point at $x \to +\infty$. Let $y = e^{S_0(x)}$ and the above becomes

$$\begin{aligned}
y(x) &= e^{S_0(x)} \\
y'(x) &= S_0' e^{S_0} \\
y'' &= S_0'' e^{S_0} + \left(S_0'\right)^2 e^{S_0} \\
&= \left(S_0'' + \left(S_0'\right)^2\right) e^{S_0} \\
y''' &= \left(S_0'' + \left(S_0'\right)^2\right)' e^{S_0} + \left(S_0'' + \left(S_0'\right)^2\right) S_0' e^{S_0} \\
&= \left(S_0''' + 2S_0'S_0''\right) e^{S_0} + \left(S_0'S_0'' + \left(S_0'\right)^3\right) e^{S_0} \\
&= \left(S_0''' + 3S_0'S_0'' + \left(S_0'\right)^3\right) e^{S_0}
\end{aligned}$$

Substituting back into $y''' = xy$ gives

$$\begin{aligned}
\left(S_0''' + 3S_0'S_0'' + \left(S_0'\right)^3\right) e^{S_0} &= x e^{S_0(x)} \\
S_0''' + 3S_0'S_0'' + \left(S_0'\right)^3 &= x
\end{aligned}$$

Before solving for S_0, we can do one more simplification. Using the approximation that $\left(S_0'\right)^3 \ggg S_0'''$ for $x \to x_0$, the above becomes

$$3S_0'S_0'' + \left(S_0'\right)^3 \sim x$$

In addition, since $S_0' \ggg S_0''$ then we can use the approximation $\left(S_0'\right)^3 \ggg S_0'S_0''$ and the above becomes

$$\begin{aligned}
\left(S_0'\right)^3 &\sim x \\
S_0' &\sim \omega x^{\frac{1}{3}} \\
S_0 &\sim \omega \int x^{\frac{1}{3}} dx \\
S_0 &\sim \omega \frac{3}{4} x^{\frac{4}{3}}
\end{aligned}$$

To find leading behavior, let

$$S(x) = S_0(x) + S_1(x)$$

Then $y(x) = e^{S_0(x)+S_1(x)}$ and hence now

$$\begin{aligned}
y'(x) &= \left(S_0(x) + S_1(x)\right)' e^{S_0+S_1} \\
y''(x) &= \left(\left(S_0 + S_1\right)'\right)^2 e^{S_0+S_1} + \left(S_0 + S_1\right)'' e^{S_0+S_1} \\
&= \left(S_0' + S_1'\right)^2 e^{S_0+S_1} + \left(S_0'' + S_1''\right) e^{S_0+S_1} \\
&= \left(\left(S_0'\right)^2 + \left(S_1'\right)^2 + 2S_0'S_1'\right) e^{S_0+S_1} + \left(S_0'' + S_1''\right) e^{S_0+S_1} \\
&= \left(\left(S_0'\right)^2 + \left(S_1'\right)^2 + 2S_0'S_1' + S_0'' + S_1''\right) e^{S_0+S_1}
\end{aligned}$$

We can take the third derivative

$$y'''(x) \sim \left(\left(S_0'\right)^2 + \left(S_1'\right)^2 + 2S_0'S_1' + S_0'' + S_1''\right)' e^{S_0+S_1}$$
$$+ \left(\left(S_0'\right)^2 + \left(S_1'\right)^2 + 2S_0'S_1' + S_0'' + S_1''\right)(S_0+S_1)' e^{S_0+S_1}$$

$$\sim \left(2S_0'S_0'' + 2S_1'S_1'' + 2S_0''S_1' + 2S_0'S_1'' + S_0''' + S_1'''\right) e^{S_0+S_1}$$
$$+ \left(\left(S_0'\right)^2 + \left(S_1'\right)^2 + 2S_0'S_1' + S_0'' + S_1''\right)\left(S_0' + S_1'\right) e^{S_0+S}$$

$$\sim \left(2S_0'S_0'' + 2S_1'S_1'' + 2S_0''S_1' + 2S_0'S_1'' + S_0''' + S_1'''\right) e^{S_0+S_1}$$
$$+ \left(\left(S_0'\right)^3 + S_0'\left(S_1'\right)^2 + 2\left(S_0'\right)^2 S_1' + S_0'S_0'' + S_0'S_1''\right) e^{S_0+S_1}$$
$$+ \left(S_1'\left(S_0'\right)^2 + \left(S_1'\right)^3 + 2S_0'\left(S_1'\right)^2 + S_1'S_0'' + S_1'S_1''\right) e^{S_0+S_1}$$

$$\sim \left(2S_0'S_0'' + 2S_1'S_1'' + 2S_0''S_1' + 2S_0'S_1'' + S_0''' + S_1'''\right) e^{S_0+S_1}$$
$$+ \left[\left(S_0'\right)^3 + 3S_0'\left(S_1'\right)^2 + 3\left(S_0'\right)^2 S_1' + S_0'S_0'' + S_0'S_1'' + \left(S_1'\right)^3 + S_1'S_0'' + S_1'S_1''\right] e^{S_0+S_1}$$
$$\sim \left(3S_0'S_0'' + 3S_1'S_1'' + 3S_0''S_1' + 3S_0'S_1'' + S_0''' + S_1''' + \left(S_0'\right)^3 + 3S_0'\left(S_1'\right)^2 + 3\left(S_0'\right)^2 S_1' + \left(S_1'\right)^3\right) e^{S_0+S_1}$$
$$\sim \left(\left(S_0'\right)^3 + \left(S_1'\right)^3 + 3S_0'\left(S_1'\right)^2 + 3\left(S_0'\right)^2 S_1' + 3S_0'S_0'' + 3S_1'S_1'' + 3S_0''S_1' + 3S_0'S_1'' + S_0''' + S_1'''\right) e^{S_0+S_1}$$

Lets go ahead and plug-in this into the ODE

$$\left(S_0'\right)^3 + \left(S_1'\right)^3 + 3S_0'\left(S_1'\right)^2 + 3\left(S_0'\right)^2 S_1' + 3S_0'S_0'' + 3S_1'S_1'' + 3S_0''S_1' + 3S_0'S_1'' + S_0''' + S_1''' \sim x$$

Now we do some simplification. $\left(S_0'\right)^3 \ggg S_0'''$ and $\left(S_1'\right)^3 \ggg S_1'''$, hence above becomes

$$\left(S_0'\right)^3 + \left(S_1'\right)^3 + 3S_0'\left(S_1'\right)^2 + 3\left(S_0'\right)^2 S_1' + 3S_0'S_0'' + 3S_1'S_1'' + 3S_0''S_1' + 3S_0'S_1'' \sim x$$

Also, since $S_0'' \ggg S_1''$ then $3S_0'S_0'' \ggg 3S_0'S_1''$

$$\left(S_0'\right)^3 + \left(S_1'\right)^3 + 3S_0'\left(S_1'\right)^2 + 3\left(S_0'\right)^2 S_1' + 3S_0'S_0'' + 3S_1'S_1'' + 3S_0''S_1' \sim x$$

Also, since $\left(S_0'\right)^2 \ggg S_0''$ then $3\left(S_0'\right)^2 S_1' \ggg 3S_0''S_1'$

$$\left(S_0'\right)^3 + \left(S_1'\right)^3 + 3S_0'\left(S_1'\right)^2 + 3\left(S_0'\right)^2 S_1' + 3S_0'S_0'' + 3S_1'S_1'' \sim x$$

Also since $S_1' \ggg S_1''$ then $3\left(S_0'\right)^2 S_1' \ggg 3S_1'S_1''$

$$\left(S_0'\right)^3 + \left(S_1'\right)^3 + 3S_0'\left(S_1'\right)^2 + 3\left(S_0'\right)^2 S_1' + 3S_0'S_0'' \sim x$$

But $S_0' \sim x^{\frac{1}{3}}$ hence$\left(S_0'\right)^3 \sim x$ and the above simplies to

$$\left(S_1'\right)^3 + 3S_0'\left(S_1'\right)^2 + 3\left(S_0'\right)^2 S_1' + 3S_0'S_0'' = 0$$

Using $3S_0'\left(S_1'\right)^2 \ggg \left(S_1'\right)^3$ since $S_0' \ggg S_1'$ then

$$3S_0'\left(S_1'\right)^2 + 3\left(S_0'\right)^2 S_1' + 3S_0'S_0'' = 0$$

Using $3\left(S_0'\right)^2 S_1' \ggg S_0'\left(S_1'\right)^2$ since $\left(S_0'\right)^2 \ggg S_0'$ then

$$3\left(S_0'\right)^2 S_1' + 3S_0'S_0'' = 0$$

No more simplification. We are ready to solve for S_1.

$$\begin{aligned} S_1' &\sim \frac{-S_0'S_0''}{\left(S_0'\right)^2} \\ &\sim \frac{-S_0''}{S_0'} \end{aligned}$$

Hence

$$\begin{aligned} S_1 &\sim -\int \frac{S_0''}{S_0'} dx \\ &\sim -\ln S_0' + c \end{aligned}$$

Since $S_0' \sim x^{\frac{1}{3}}$then the above becomes

$$\begin{aligned} S_1 &\sim -\ln x^{\frac{1}{3}} + c \\ S_1 &\sim -\frac{1}{3}\ln x + c \end{aligned}$$

Hence, the leading behavior is

$$\begin{aligned} y(x) &= e^{S_0(x)+S_1(x)} \\ &= \exp\left(\omega \frac{3}{4} x^{\frac{4}{3}} - \frac{1}{3}\ln x + c\right) \\ &= cx^{\frac{-1}{3}} \exp\left(\omega \frac{3}{4} x^{\frac{4}{3}}\right) \end{aligned} \tag{1}$$

To verify, using the formula 3.4.28, which is

$$y(x) \sim cQ^{\frac{1-n}{2n}} \exp\left(\omega \int^x Q(t)^{\frac{1}{n}}\, dt\right)$$

In this case, $n = 3$, since the ODE $y''' = xy$ is third order. Here we have $Q(x) = x$, therefore, plug-in into the above gives

$$
\begin{aligned}
y(x) &\sim c(x)^{\frac{1-3}{6}} \exp\left(\omega \int^{x} (t)^{\frac{1}{3}}\, dt\right) \\
&\sim c x^{\frac{-1}{3}} \exp\left(\omega \frac{4}{3} x^{\frac{4}{3}}\right)
\end{aligned}
$$

Comparing (1) and (2), we see they are the same.