NE 548 The Wave Equation

Governs the (small) displacement of an elastic string at position x, time t;
derived using Newton's Law for small amplitude displacements

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

lets first think about the infinite domain $-\infty < x < \infty$

$u = u(x,t)$

same as $\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}\right) = 0$

check: $\frac{\partial^2 u}{\partial t^2} - c\frac{\partial^2 u}{\partial t \partial x} + c\frac{\partial^2 u}{\partial x \partial t} - c^2\frac{\partial^2 u}{\partial x^2} = 0$ ✓

also same as $\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right) = 0$

Let $w = \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}$; $v = \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}$

$\Rightarrow$ 2 1st order wave equations

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0 \quad ; \quad \frac{\partial v}{\partial t} - c\frac{\partial v}{\partial x} = 0$$

$w = w(x,t) \qquad v = v(x,t)$

So first we need to understand these 1^{st}-order wave equations, and then return to the 2^{nd}-order case.

Let $x(t)$ be a moving observer

[In fluids, let $x = x(t)$ "follow a fluid particle"]

Then $w = w(x(t), t)$ satisfies

$$\frac{d}{dt} w = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial t} \quad \text{by chain rule}$$

If the observer moves at constant speed $\frac{dx(t)}{dt} = c$

then the 1D wave equation

$$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0 \quad \text{is equivalent to}$$

$$\left.\begin{array}{l} \frac{d}{dt} w = 0 \quad \text{along} \quad \frac{dx}{dt} = c \\ \\ \text{where } w = w(x(t), t), \quad x = x(t) \end{array}\right\}$$

Thus we have converted the PDE into

2 ODEs by introducing the moving observer

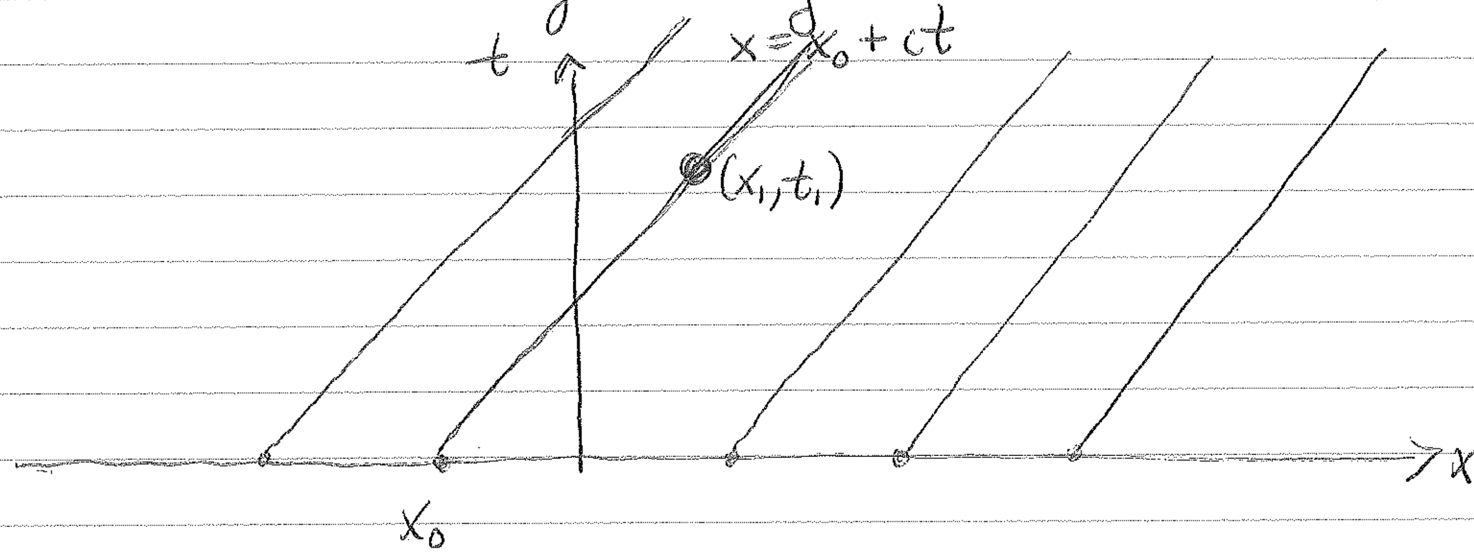
[by following particles]

We are using the "method of characteristics" and the curves given by $\frac{dx(t)}{dt} = c$ are called the "characteristic curves".

In this case $x = x_0 + ct$ are lines with slope c originating at x_0.

$\frac{dw}{dt} = 0$ along $x = x_0 + ct$ means that

w does not change along curves $x = x_0 + ct$:



In the original problem we will be given an initial condition, say $w(x,0) = P(x)$

so we need to solve

$$\frac{dw(x(t),t)}{dt} = 0 \quad \text{along} \quad \frac{dx(t)}{dt} = 0 \quad \text{with}$$

$$w(x(0),0) = P(x(0))$$

Since w does not change in time along characteristics $\Longrightarrow$

$w(x,t) = P(x_0)$ on $x = x_0 + ct$, same as

$w(x,t) = P(x_0)$ on $x_0 = x - ct$ $\Longrightarrow$

$$\boxed{w(x,t) = P(x-ct)}$$

This is a long-winded way of saying

$w(x,t) = P(x-ct)$ satisfies

$\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0$ which we already knew!

Check: Let $\xi = x - ct$, $w(x,t) = P(\xi)$

$$\frac{\partial w}{\partial t} = -c P'(\xi) \; ; \; c\frac{\partial w}{\partial x} = c P'(\xi) \quad \checkmark$$

$\boxed{\text{Example 1}}$ $$\frac{\partial w}{\partial t} + 2\frac{\partial w}{\partial x} = 0$$

$$w(x,0) = P(x) = \begin{cases} 0 & x < 0 \\ 4x & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

w is not changing in time along $x = x_0 + 2t$

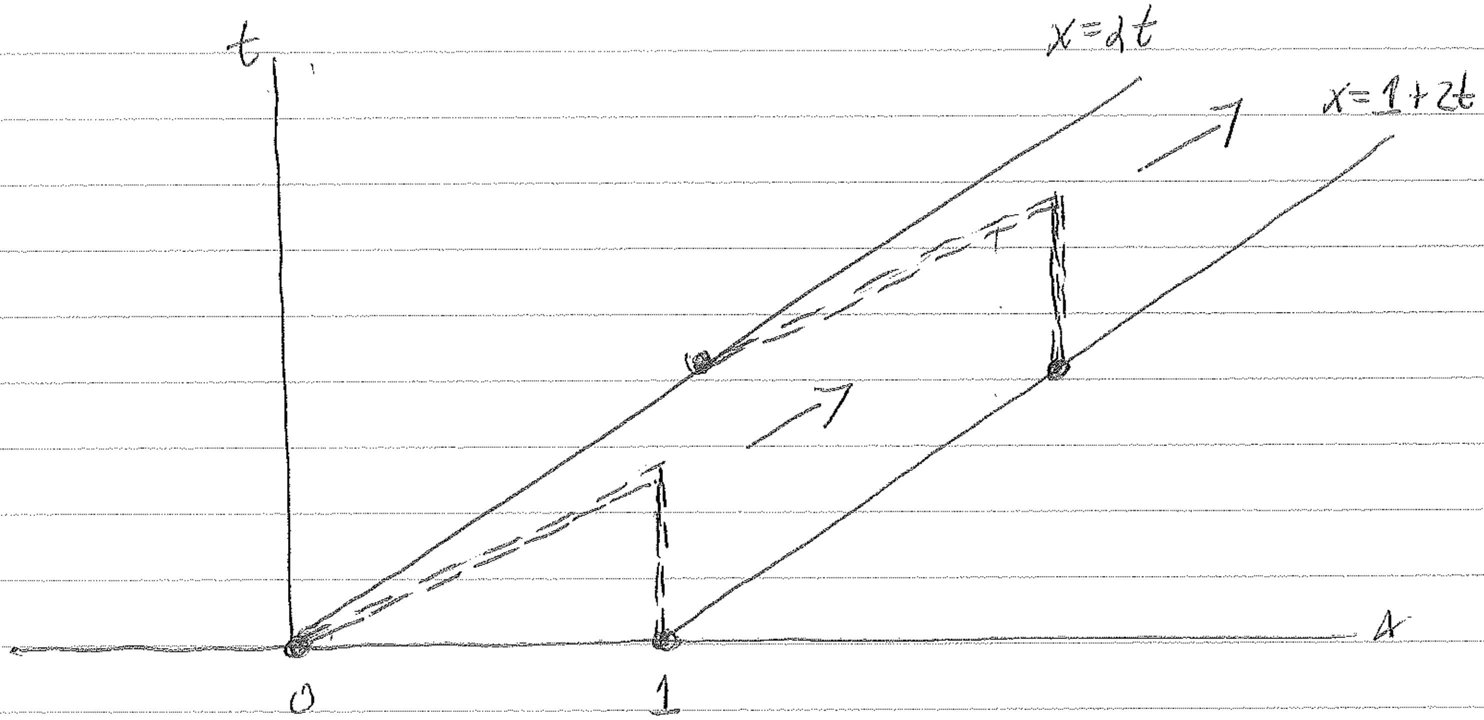
$w(x,t) = P(x_0)$ on $x_0 = x - 2t$

$$= \begin{cases} 0 & x_0 < 0 \\ 4x_0 & 0 \le x_0 \le 1 \\ 0 & x_0 > 1 \end{cases}$$

$$= \begin{cases} 0 & x-2t < 0 \\ 4(x-2t) & 0 \le x-2t \le 0 \\ 0 & x-2t > 1 \end{cases}$$

or also written

$$w(x,t) = \begin{cases} 0 & x < 2t \\ 4(x-2t) & 0 \le x-2t \le 1 \\ 0 & x > 1+2t \end{cases}$$



imagine anothe axis at of the page

a wave of fixed shape moving to the right.

max amplitude of the wave is 4

Simple Application : Transport with Decay

Transport of a radioactively decaying solute in a uniform flow with wave speed c

$$\frac{\partial u(x,t)}{\partial t} + c\,\frac{\partial u(x,t)}{\partial x} = -a\,u(x,t) \qquad u(x,0) = F(x)$$

$-\infty < x < \infty$ [$F(x)$ given for all values of the argument]

$a > 0$ is the decay rate

$u(x,t)$ is concentration [amount per unit volume] of the radioactive solute]

Translate to ODEs by letting

$x = x(t)$; $u = u(x(t), t)$

$x_0 = x(0)$; $u(0) = u(x(0), 0) = F(x_0)$

Then $\dfrac{du(t)}{dt} = -a\,u(t)$ along $\dfrac{dx(t)}{dt} = c$

with solution

$u(t) = A \exp[-at]$ along $x(t) = ct + x_0$

The initial condition $u(0) = F(x_0) \Rightarrow$

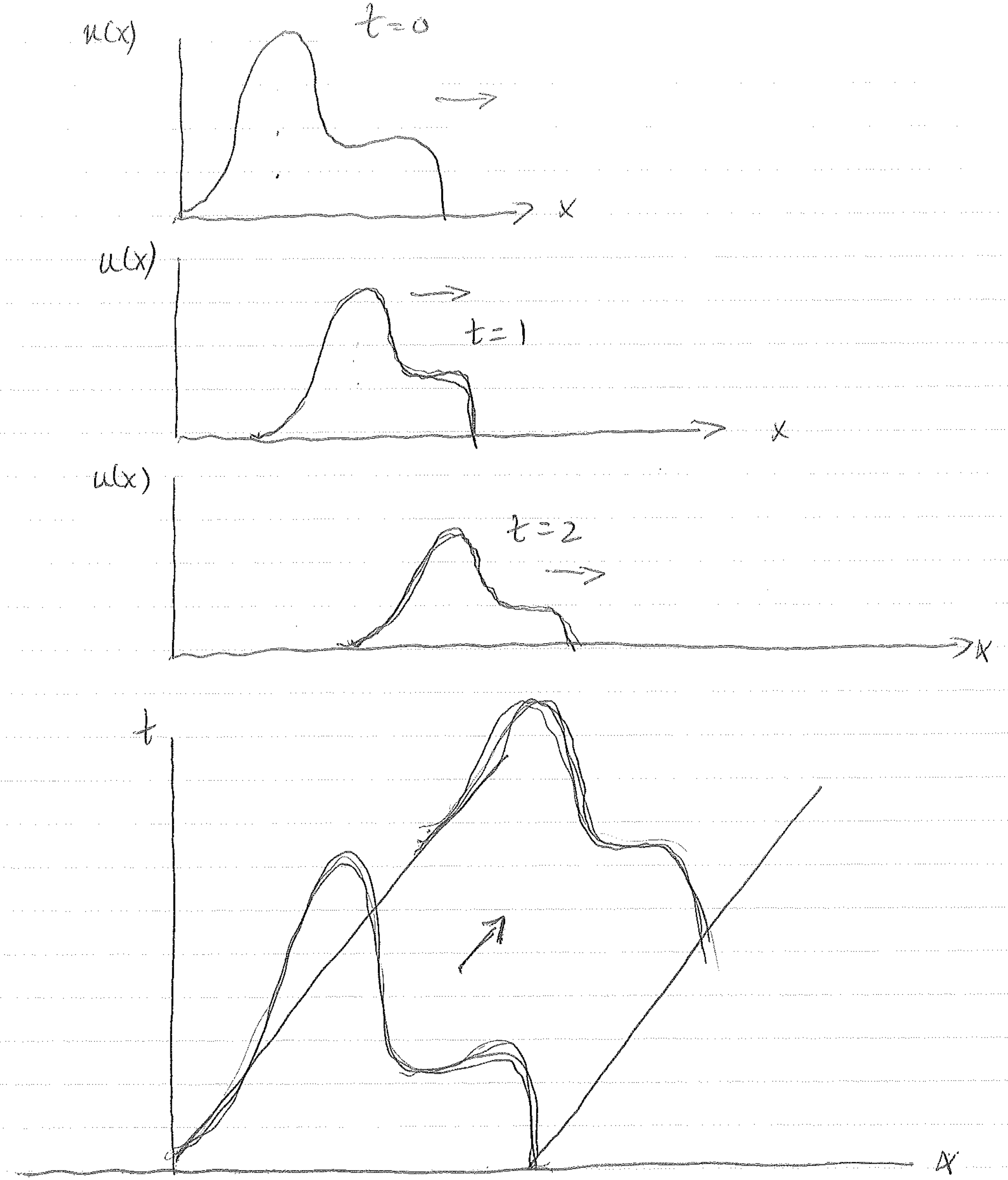
$A = F(x_0)$ a different constant for each value of x_0. on $-\infty < x_0 < \infty$

Finally $u(t) = F(x_0)\exp(-at)$ along $x(t) = ct + x_0$

Translate back to the PDE:

$$u(x,t) = \underbrace{F(x-ct)}_{\text{translation}}\,\underbrace{\exp(-at)}_{\text{decay}}$$

Pictures



$\boxed{\text{Example 3}}$ Linear, non-constant coefficient, 1st-order wave eqn.

$$\frac{\partial w}{\partial t} + 3t^2 \frac{\partial w}{\partial x} = 2tw \qquad w(x,0) = P(x)$$

$$\frac{d}{dt} w(x(t),t) = 2tw \quad \text{along} \quad \frac{dx(t)}{dt} = 3t^2$$

$$\text{or} \quad x(t) = t^3 + x_0$$

$$\Rightarrow \quad \frac{dw}{w} = 2t\,dt \quad \text{along} \quad x(t) = t^3 + x_0$$

$$\Rightarrow \quad \ln|w| = t^2 + C$$

$$w = C^* \exp(t^2) \quad \text{along} \quad x(t) = t^3 + x_0$$

The initial conditions : $w(x(0),0) = P(x(0))$

$$w(x_0, 0) = P(x_0)$$

So $w(x_0,0) = C^* \exp(0) = P(x_0) \Rightarrow C^* = P(x_0)$

$$\Rightarrow \quad w(x,t) = P(x_0)\exp(t^2) \quad \text{or}$$

$$\boxed{w(x,t) = P(x - t^3)\exp(t^2)}$$