Exam 2, NE 548, Spring 2017

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0.1 problem 3

3. Here we study the competing effects of nonlinearity and diffusion in the context of Burger's equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \tag{3a}$$

which is the simplest model equation for diffusive waves in fluid dynamics. It can be solved exactly using the Cole-Hopf transformation

$$u = -2\nu \frac{\phi_x}{\phi} \tag{3b}$$

as follows (with 2 steps to achieve the transformation (3b)).

(a) Let $u = \psi_x$ (where the subscript denotes partial differentiation) and integrate once with respect to x.

(b) Let $\psi = -2\nu \ln(\phi)$ to get the diffusion equation for ϕ.

(c) Solve for ϕ with $\phi(x,0) = \Phi(x)$, $-\infty < x < \infty$. In your integral expression for ϕ, use dummy variable η to facilitate the remaining parts below.

(d) Show that

$$\Phi(x) = \exp\left[\frac{-1}{2\nu}\int_{x_o}^{x} F(\alpha)d\alpha\right]$$

where $u(x,0) = F(x)$, with x_o arbitrary which we will take to be positive for convenience below ($x_o > 0$).

(e) Write your expression for $\phi(x,t)$ in terms of

$$f(\eta,x,t) = \int_{x_o}^{\eta} F(\alpha)d\alpha + \frac{(x-\eta)^2}{2t}.$$

(f) Find $\phi_x(x,t)$ and then use equation (3b) to find $u(x,t)$.

0.1.1 Part (a)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \tag{1}$$

Let

$$\begin{aligned} u &= -2\nu \frac{\phi_x}{x} \\ &= \frac{\partial}{\partial x}\left(-2\nu \ln \phi\right) \end{aligned} \tag{1A}$$

0.1.2 Part(b)

Let

$$\psi = -2\nu \ln \phi \tag{2}$$

Hence (1A) becomes

$$u = \frac{\partial}{\partial x}\psi$$

We now substitute the above back into (1) noting first that

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\frac{\partial \psi}{\partial x}$$

Interchanging the order gives

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial}{\partial x}\frac{\partial \psi}{\partial t}\\ &= \frac{\partial}{\partial x}\psi_t\end{aligned}$$

And

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}\frac{\partial \psi}{\partial x}\\ &= \psi_{xx}\end{aligned}$$

And

$$\frac{\partial^2 u}{\partial x^2} = \psi_{xxx}$$

Hence the original PDE (1) now can be written in term of ψ as the new dependent variable as

$$\frac{\partial}{\partial x}\psi_t + \psi_x\left(\psi_{xx}\right) = \nu\psi_{xxx} \tag{3}$$

But

$$\psi_x\left(\psi_{xx}\right) = \frac{1}{2}\frac{\partial}{\partial x}\left(\psi_x^2\right)$$

Using the above in (3), then (3) becomes

$$\begin{aligned}\frac{\partial}{\partial x}\psi_t + \frac{1}{2}\frac{\partial}{\partial x}\left(\psi_x^2\right) &= \nu\psi_{xxx}\\ \frac{\partial}{\partial x}\psi_t + \frac{1}{2}\frac{\partial}{\partial x}\left(\psi_x^2\right) - \nu\frac{\partial}{\partial x}\left(\psi_{xx}\right) &= 0\\ \frac{\partial}{\partial x}\left(\psi_t + \frac{1}{2}\psi_x^2 - \nu\psi_{xx}\right) &= 0\end{aligned}$$

Therefore

$$\psi_t + \frac{1}{2}\psi_x^2 - \nu\psi_{xx} = 0 \tag{4}$$

But from (2) $\psi = -2\nu\ln\phi$, then using this in (4), we now rewrite (4) in terms of ϕ

$$\begin{aligned}\frac{\partial}{\partial t}\left(-2\nu\ln\phi\right) + \frac{1}{2}\left(\frac{\partial}{\partial x}\left(-2\nu\ln\phi\right)\right)^2 - \nu\frac{\partial^2}{\partial x^2}\left(-2\nu\ln\phi\right) &= 0\\ \left(-2\nu\frac{\phi_t}{\phi}\right) + \frac{1}{2}\left(-2\nu\frac{\phi_x}{\phi}\right)^2 - \nu\frac{\partial}{\partial x}\left(-2\nu\frac{\phi_x}{\phi}\right) &= 0\\ -2\nu\frac{\phi_t}{\phi} + 2\nu^2\left(\frac{\phi_x}{\phi}\right)^2 + 2\nu^2\frac{\partial}{\partial x}\left(\frac{\phi_x}{\phi}\right) &= 0\end{aligned}$$

But $\frac{\partial}{\partial x}\left(\frac{\phi_x}{\phi}\right) = \frac{\phi_{xx}}{\phi} - \frac{\phi_x^2}{\phi^2}$, hence the above becomes

$$\begin{aligned}-2\nu\frac{\phi_t}{\phi} + 2\nu^2\left(\frac{\phi_x}{\phi}\right)^2 + 2\nu^2\left(\frac{\phi_{xx}}{\phi} - \frac{\phi_x^2}{\phi^2}\right) &= 0\\ -2\nu\frac{\phi_t}{\phi} + 2\nu^2\left(\frac{\phi_x}{\phi}\right)^2 + 2\nu^2\frac{\phi_{xx}}{\phi} - 2\nu^2\frac{\phi_x^2}{\phi^2} &= 0\\ -2\nu\frac{\phi_t}{\phi} + 2\nu^2\frac{\phi_{xx}}{\phi} &= 0\\ -\frac{\phi_t}{\phi} + \nu\frac{\phi_{xx}}{\phi} &= 0\end{aligned}$$

Since $\phi \neq 0$ identically, then the above simplifies to the heat PDE

$$\begin{aligned}\phi_t &= \nu\phi_{xx} \qquad (5)\\ \phi\left(x,0\right) &= \Phi\left(x\right)\\ -\infty < x &< \infty\end{aligned}$$

0.1.3 Part (c)

Now we solve (5) for $\phi\left(x,t\right)$ and then convert the solution back to $u\left(x,t\right)$ using the Cole-Hopf transformation. This infinite domain heat PDE has known solution (as $\phi\left(\pm\infty,t\right)$ is bounded which is

$$\phi(x,t) = \int_{-\infty}^{\infty} \Phi(\eta) \frac{1}{\sqrt{4\pi \upsilon t}} \exp\left(\frac{-(x-\eta)^2}{4\upsilon t}\right) d\eta \tag{6}$$

0.1.4 Part(d)

Now

$$\Phi(x) = \phi(x,0) \tag{7}$$

But since $u(x,t) = \frac{\partial}{\partial x}(-2\nu \ln \phi)$, then integrating

$$\begin{aligned}
\int_{x_0}^{x} u(\alpha,t)\,d\alpha &= -2\nu \ln \phi \\
\ln \phi &= \frac{-1}{2\nu} \int_{x_0}^{x} u(\alpha,t)\,d\alpha \\
\phi(x,t) &= \exp\left(\frac{-1}{2\nu} \int_{x_0}^{x} u(\alpha,t)\,d\alpha\right)
\end{aligned}$$

Hence at $t = 0$ the above becomes

$$\begin{aligned}
\phi(x,0) &= \exp\left(\frac{-1}{2\nu} \int_{x_0}^{x} u(\alpha,0)\,d\alpha\right) \\
&= \exp\left(\frac{-1}{2\nu} \int_{x_0}^{x} F(\alpha)\,d\alpha\right)
\end{aligned}$$

Where $F(x) = u(x,0)$. Hence from the above, comparing it to (6) we see that

$$\Phi(x) = \exp\left(\frac{-1}{2\nu} \int_{x_0}^{x} F(\alpha)\,d\alpha\right) \tag{8}$$

0.1.5 Part(e)

From (6), we found $\phi(x,t) = \int_{-\infty}^{\infty} \Phi(\eta) \frac{1}{\sqrt{4\pi \upsilon t}} \exp\left(\frac{-(x-\eta)^2}{4\upsilon t}\right) d\eta$. Plugging (8) into this expression gives

$$\begin{aligned}
\phi(x,t) &= \frac{1}{\sqrt{4\pi \upsilon t}} \int_{-\infty}^{\infty} \exp\left(\frac{-1}{2\nu} \int_{x_0}^{\eta} F(\alpha)\,d\alpha\right) \exp\left(\frac{-(x-\eta)^2}{4\upsilon t}\right) d\eta \\
&= \frac{1}{\sqrt{4\pi \upsilon t}} \int_{-\infty}^{\infty} \exp\left(\frac{-1}{2\nu} \int_{x_0}^{\eta} F(\alpha)\,d\alpha - \frac{(x-\eta)^2}{4\upsilon t}\right) d\eta \\
&= \frac{1}{\sqrt{4\pi \upsilon t}} \int_{-\infty}^{\infty} \exp\left(\frac{-1}{2\nu} \left[\int_{x_0}^{\eta} F(\alpha)\,d\alpha + \frac{(x-\eta)^2}{2t}\right]\right) d\eta
\end{aligned} \tag{9}$$

Let

$$\int_{x_0}^{\eta} F(\alpha)\,d\alpha + \frac{(x-\eta)^2}{2t} = f(\eta, x, t)$$

Hence (9) becomes

$$\phi(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi \upsilon t}} e^{\frac{-f(\eta,x,t)}{2\nu}} d\eta \tag{10}$$

0.1.6 Part(f)

From (10)

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi \upsilon t}} \frac{\partial}{\partial x}\left(e^{\frac{-f(\eta,x,t)}{2\nu}}\right) d\eta \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi \upsilon t}} \left(\frac{\partial}{\partial x} f(\eta,x,t)\, e^{\frac{-f(\eta,x,t)}{2\nu}}\right) d\eta \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi \upsilon t}} \left(\frac{\partial}{\partial x}\left[\int_{x_0}^{\eta} F(\alpha)\,d\alpha + \frac{(x-\eta)^2}{2t}\right] e^{\frac{-f(\eta,x,t)}{2\nu}}\right) d\eta
\end{aligned}$$

Using Leibniz integral rule the above simplifies to

$$\frac{\partial \phi}{\partial x} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi \upsilon t}} \left(\frac{(x-\eta)}{t} e^{\frac{-f(\eta,x,t)}{2\nu}} \right) d\eta \tag{11}$$

But

$$u = -2\nu \frac{\phi_x}{\phi}$$

Hence, using (10) and (11) in the above gives

$$u = -2\nu \frac{\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi \upsilon t}} \left(\frac{(x-\eta)}{t} e^{\frac{-f(\eta,x,t)}{2\nu}} \right) d\eta}{\int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi \upsilon t}} e^{\frac{-f(\eta,x,t)}{2\nu}} d\eta}$$

Hence the solution is

$$u(x,t) = -2\nu \frac{\int_{-\infty}^{\infty} \frac{(x-\eta)}{t} e^{\frac{-f(\eta,x,t)}{2\nu}} d\eta}{\int_{-\infty}^{\infty} e^{\frac{-f(\eta,x,t)}{2\nu}} d\eta}$$

Where

$$f(\eta, x, t) = \int_{x_0}^{\eta} F(\alpha)\, d\alpha + \frac{(x-\eta)^2}{2t}$$

0.2 problem 4

4. (a) Use the Method of Images to solve

$$\frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial x^2}+Q(x,t),\quad 0\le x\le L,\quad t\ge 0$$

$$u(x,0)=f(x),\quad \frac{\partial u}{\partial x}(0,t)=0,\quad u(L,t)=A$$

(b) For $A=0$, compare your expression for the solution in (a) to the eigenfunction solution.

0.2.1 Part(a)

$$\begin{aligned}\frac{\partial u}{\partial t}&=v\frac{\partial^2 u}{\partial x^2}+Q(x,t) \qquad (1)\\ 0&\le x\le L\\ t&\ge 0\end{aligned}$$

Initial conditions are

$$u(x,0)=f(x)$$

Boundary conditions are

$$\begin{aligned}\frac{\partial u(0,t)}{\partial x}&=0\\ u(L,t)&=A\end{aligned}$$

Multiplying both sides of (1) by $G(x,t;x_0,t_0)$ and integrating over the domain gives (where in the following G is used instead of $G(x,t;x_0,t_0)$ for simplicity).

$$\int_{x=0}^{L}\int_{t=0}^{\infty}Gu_t\,dtdx=\int_{x=0}^{L}\int_{t=0}^{\infty}vu_{xx}G\,dtdx+\int_{x=0}^{L}\int_{t=0}^{\infty}QG\,dtdx \qquad (1)$$

For the integral on the LHS, we apply integration by parts once to move the time derivative from u to G

$$\int_{x=0}^{L}\int_{t=0}^{\infty}Gu_t\,dtdx=\int_{x=0}^{L}[uG]_{t=0}^{\infty}dx-\int_{x=0}^{L}\int_{t=0}^{\infty}G_tu\,dtdx \qquad (1A)$$

And the first integral in the RHS of (1) gives, after doing integration by parts two times on it

$$\begin{aligned}\int_{x=0}^{L}\int_{t=0}^{\infty}ku_{xx}G\,dtdx&=\int_{t=0}^{\infty}[u_xG]_{x=0}^{L}dt-\int_{x=0}^{L}\int_{t=0}^{\infty}vu_xG_x\,dtdx\\ &=\int_{t=0}^{L}[u_xG]_{x=0}^{L}dt-\left(\int_{t=0}^{\infty}[uG_x]_{x=0}^{L}dt-\int_{x=0}^{L}\int_{t=0}^{\infty}vuG_{xx}\,dtdx\right)\\ &=\int_{t=0}^{\infty}\left([u_xG]_{x=0}^{L}-[uG_x]_{x=0}^{L}\right)dt+\int_{x=0}^{L}\int_{t=0}^{\infty}vuG_{xx}\,dtdx\\ &=\int_{t=0}^{\infty}[u_xG-uG_x]_{x=0}^{L}dt+\int_{x=0}^{L}\int_{t=0}^{\infty}vuG_{xx}\,dtdx\\ &=-\int_{t=0}^{\infty}[uG_x-u_xG]_{x=0}^{L}dt+\int_{x=0}^{L}\int_{t=0}^{\infty}vuG_{xx}\,dtdx \qquad (1B)\end{aligned}$$

Substituting (1A) and (1B) back into (1) results in

$$\int_{x=0}^{L}[uG]_{t=0}^{\infty}dx-\int_{x=0}^{L}\int_{t=0}^{\infty}G_tu\,dtdx=\int_{t=0}^{\infty}[u_xG-uG_x]_{x=0}^{L}dt+\int_{x=0}^{L}\int_{t=0}^{\infty}vuG_{xx}\,dtdx+\int_{x=0}^{L}\int_{t=0}^{\infty}GQ\,dtdx$$

Or

$$\int_{x=0}^{L}\int_{t=0}^{\infty}-G_tu-vuG_{xx}\,dtdx=-\int_{x=0}^{L}[uG]_{t=0}^{\infty}dx-\int_{t=0}^{\infty}[uG_x-u_xG]_{x=0}^{L}dt+\int_{x=0}^{L}\int_{t=0}^{\infty}GQ\,dtdx \qquad (2)$$

We now want to choose $G(x,t;x_0,t_0)$ such that

$$\begin{aligned}-G_tu-vuG_{xx}&=\delta(x-x_0)\,\delta(t-t_0)\\ -G_tu&=vuG_{xx}+\delta(x-x_0)\,\delta(t-t_0) \qquad (3)\end{aligned}$$

This way, the LHS of (2) becomes just $u(x_0,t_0)$. Hence (2) now (after the above choice of G) reduces to

$$u(x_0,t_0) = -\int_{x=0}^{L}[uG]_{t=0}^{\infty}dx - \int_{t=0}^{t_0}[uG_x - u_xG]_{x=0}^{L}dt + \int_{x=0}^{L}\int_{t=0}^{t_0}GQ\,dtdx \tag{4}$$

We now need to find the Green function which satisfies (3). But (3) is equivalent to solution of problem

$$\begin{aligned} -G_t u &= \nu u G_{xx} \\ G(x,0) &= \delta(x-x_0)\,\delta(t-t_0) \\ -\infty &< x < \infty \\ G(x,t;x_0,t_0) &= 0 \qquad t > t_0 \\ G(\pm\infty,t;x_0,t_0) &= 0 \\ G(x,t_0;x_0,t_0) &= \delta(x-x_0) \end{aligned}$$

The above problem has a known fundamental solution which we found before, but for the forward heat PDE instead of the reverse heat PDE as it is now. The fundamental solution to the forward heat PDE is

$$G(x,t) = \frac{1}{\sqrt{4\pi\nu(t-t_0)}}\exp\left(\frac{-(x-x_0)^2}{4\nu(t-t_0)}\right) \qquad 0 \le t_0 \le t$$

Therefore, for the reverse heat PDE the above becomes

$$G(x,t) = \frac{1}{\sqrt{4\pi\nu(t_0-t)}}\exp\left(\frac{-(x-x_0)^2}{4\nu(t_0-t)}\right) \qquad 0 \le t \le t_0 \tag{5}$$

We now go back to (4) and try to evaluate all terms in the RHS. Starting with the first term $\int_{x=0}^{L}[uG]_{t=0}^{\infty}dx$. Since $G(x,\infty;x_0,t_0) = 0$ then the upper limit is zero. But at lower limit $t = 0$ we are given that $u(x,0) = f(x)$, hence this term becomes

$$\begin{aligned} \int_{x=0}^{L}[uG]_{t=0}^{\infty}dx &= \int_{x=0}^{L} -u(x,0)\,G(x,0)\,dx \\ &= \int_{x=0}^{L} -f(x)\,G(x,0)\,dx \end{aligned}$$

Now looking at the second term in RHS of (4), we expand it and find

$$[uG_x - u_xG]_{x=0}^{L} = (u(L,t)\,G_x(L,t) - u_x(L,t)\,G(L,t)) - (u(0,t)\,G_x(0,t) - u_x(0,t)\,G(0,t))$$

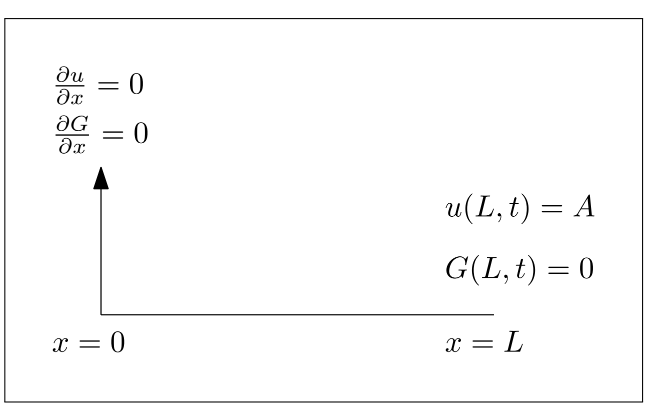
We are told that $u_x(0,t) = 0$ and $u(L,t) = A$, then the above becomes

$$[uG_x - u_xG]_{x=0}^{L} = AG_x(L,t) - u_x(L,t)\,G(L,t) - u(0,t)\,G_x(0,t) \tag{5A}$$

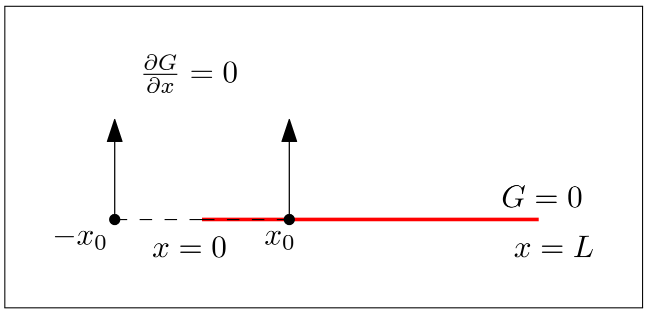
There are still two terms above we do not know. We do not know $u_x(L,t)$ and we also do not know $u(0,t)$. If we can configure, using method of images, such that $G(L,t) = 0$ and $G_x(0,t) = 0$ then we can get rid of these two terms and end up only with $[uG_x - u_xG]_{x=0}^{\infty} = AG_x(L,t)$ which we can evaluate once we know what $G(x,t)$ is.

This means we need to put images on both sides of the boundaries such that to force $G(L,t) = 0$ and also $G_x(0,t) = 0$.

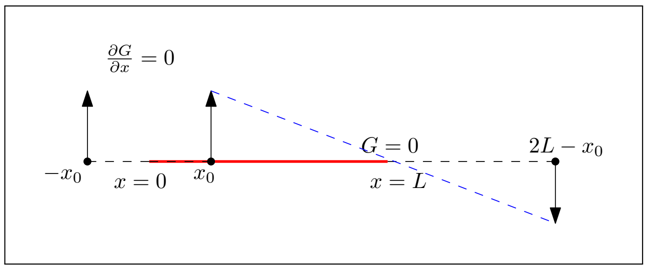
We see that this result agrees what we always did, which is, If the prescribed boundary conditions on u are such that $u = A$, then we want $G = 0$ there. And if it is $\frac{\partial u}{\partial x} = A$, then we want $\frac{\partial G}{\partial x} = 0$ there. And this is what we conclude here also from the above. In other words, the boundary conditions on Green functions are always the homogeneous version of the boundary conditions given on u.



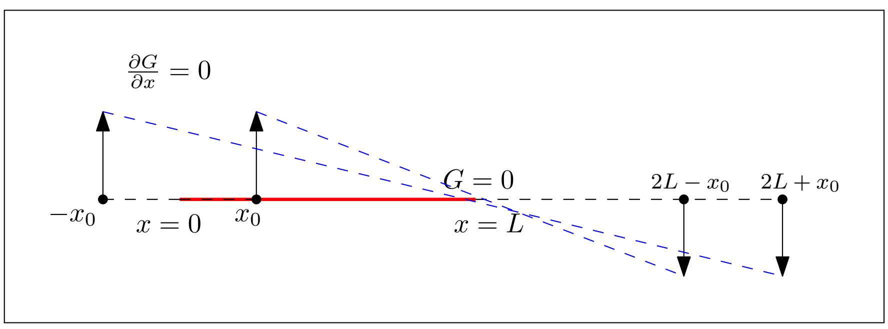
To force $G_x(0,t) = 0$, we need to put same sign images on both sides of $x = 0$. So we end up with this



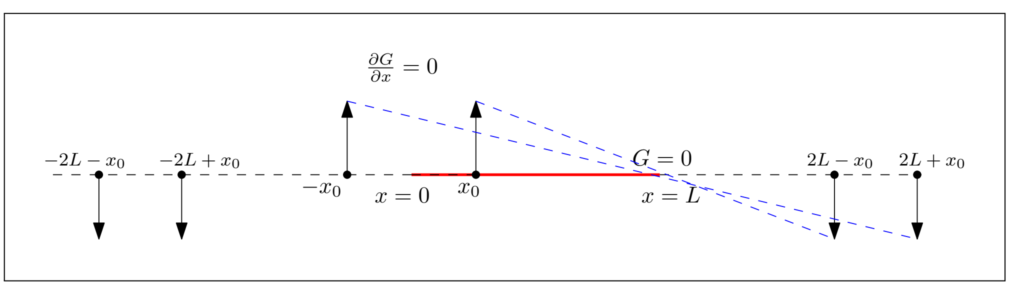
The above makes $G_x(0,t) = 0$ at $x = 0$. Now we want to make $G = 0$ at $x = L$. Then we update the above and put a negative image at $x = 2L - x_0$ to the right of $x = L$ as follows



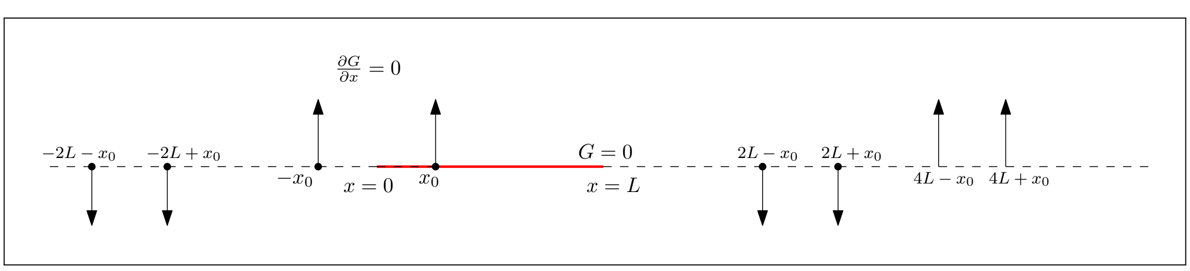
But now we see that the image at $x = -x_0$ has affected condition of $G = 0$ at $x = L$ and will make it not zero as we wanted. So to counter effect this, we have to add another negative image at distance $x = 2L + x_0$ to cancel the effect of the image at $x = -x_0$. We end up with this setup



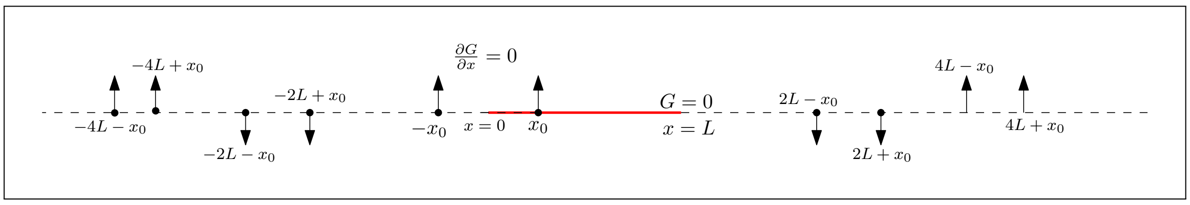
But now we see that the two negative images we added to the right will no longer make $G_x(0,t) = 0$, so we need to counter effect this by adding two negative images to the left side to keep $G_x(0,t) = 0$. So we end up with



But now we see that by putting these two images on the left, we no longer have $G = 0$ at $x = L$. So to counter effect this, we have to put copies of these 2 images on the right again but with positive sign, as follows



But now these two images on the right, no longer keep $G_x(0,t) = 0$, so we have to put same sign images to the left, as follows



And so on. This continues for infinite number of images. Therefore we see from the above, for the positive images, we have the following sum

$$\begin{aligned}\sqrt{4\pi v\left(t_0-t\right)}G\left(x,t;x_0,t_0\right) &= \exp\left(\frac{-\left(x-x_0\right)^2}{4v\left(t_0-t\right)}\right)+\exp\left(\frac{-\left(x+x_0\right)^2}{4v\left(t_0-t\right)}\right)+\exp\left(\frac{-\left(x-\left(4L-x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)\\ &+\exp\left(\frac{-\left(x-\left(-4L+x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)+\exp\left(\frac{-\left(x-\left(4L+x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)\\ &+\exp\left(\frac{-\left(x-\left(-4L-x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)+\cdots\end{aligned}$$

Or

$$G\left(x,t;x_0,t_0\right)=\frac{1}{\sqrt{4\pi v\left(t_0-t\right)}}\left(\sum_{n=-\infty}^{\infty}\exp\left(\frac{-\left(x-\left(4nL-x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)+\exp\left(\frac{-\left(x-\left(-4nL+x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)\right) \tag{6}$$

The above takes care of the positive images. For negative images, we have this sum of images

$$\begin{aligned}\sqrt{4\pi v\left(t_0-t\right)}G\left(x,t;x_0,t_0\right) &= \exp\left(\frac{-\left(x-\left(2L-x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)+\exp\left(\frac{-\left(x-\left(-2L+x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)\\ &+\exp\left(\frac{-\left(x-\left(2L+x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)+\exp\left(\frac{-\left(x-\left(-2L-x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)\\ &+\exp\left(\frac{-\left(x-\left(6L-x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)+\exp\left(\frac{-\left(x-\left(-6L+x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)\\ &+\exp\left(\frac{-\left(x-\left(6L+x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)+\exp\left(\frac{-\left(x-\left(-6L-x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)\cdots\end{aligned}$$

Or

$$\sqrt{4\pi v\left(t_0-t\right)}G\left(x,t;x_0,t_0\right)=-\left(\sum_{n=-\infty}^{\infty}\exp\left(\frac{-\left(x-\left(\left(4n-2\right)L+x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)+\exp\left(\frac{-\left(x-\left(\left(4n-2\right)L-x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)\right) \tag{7}$$

Hence the Green function to use is (6)+(7) which gives

$$\begin{aligned}G\left(x,t;x_0,t_0\right) &= \frac{1}{\sqrt{4\pi v\left(t_0-t\right)}}\left(\sum_{n=-\infty}^{\infty}\exp\left(\frac{-\left(x-\left(4nL-x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)+\exp\left(\frac{-\left(x-\left(-4nL+x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)\right)\\ &-\frac{1}{\sqrt{4\pi v\left(t_0-t\right)}}\left(\sum_{n=-\infty}^{\infty}\exp\left(\frac{-\left(x-\left(\left(4n-2\right)L+x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)+\exp\left(\frac{-\left(x-\left(\left(4n-2\right)L-x_0\right)\right)^2}{4v\left(t_0-t\right)}\right)\right)\end{aligned} \tag{7A}$$

Using the above Green function, we go back to 5A and finally are able to simplify it

$$\left[uG_x-u_xG\right]_{x=0}^{L}=AG_x\left(L,t\right)-u_x\left(L,t\right)G\left(L,t\right)-u\left(0,t\right)G_x\left(0,t\right)$$

The above becomes now (with the images in place as above)

$$\left[uG_x-u_xG\right]_{x=0}^{L}=A\frac{\partial G\left(L,t;x_0,t_0\right)}{\partial x} \tag{8}$$

Since now we know what $G\left(x,t;x_0,t_0\right)$, from (7), we can evaluate its derivative w.r.t. x. (broken up, so it fits on one page)

$$
\begin{aligned}
\frac{\partial G(x,t;x_0,t_0)}{\partial x} &= \frac{1}{\sqrt{4\pi v(t_0-t)}}\sum_{n=-\infty}^{\infty}\frac{-(x-(4nL-x_0))}{2v(t_0-t)}\exp\left(\frac{-(x-(4nL-x_0))^2}{4v(t_0-t)}\right)\\
&+\frac{1}{\sqrt{4\pi v(t_0-t)}}\sum_{n=-\infty}^{\infty}\frac{-(x-(-4nL+x_0))}{2v(t_0-t)}\exp\left(\frac{-(x-(-4nL+x_0))^2}{4v(t_0-t)}\right)\\
&-\frac{1}{\sqrt{4\pi v(t_0-t)}}\sum_{n=-\infty}^{\infty}\frac{-(x-((4n-2)L+x_0))}{2v(t_0-t)}\exp\left(\frac{-(x-((4n-2)L+x_0))^2}{4v(t_0-t)}\right)\\
&-\frac{1}{\sqrt{4\pi v(t_0-t)}}\sum_{n=-\infty}^{\infty}\frac{-(x-((4n-2)L-x_0))}{2v(t_0-t)}\exp\left(\frac{-(x-((4n-2)L-x_0))^2}{4v(t_0-t)}\right)
\end{aligned}
$$

At $x=L$, the above derivative becomes

$$
\begin{aligned}
\frac{\partial G(L,t;x_0,t_0)}{\partial x} &= \frac{1}{\sqrt{4\pi v(t_0-t)}}\sum_{n=-\infty}^{\infty}\frac{-(L-(4nL-x_0))}{2v(t_0-t)}\exp\left(\frac{-(L-(4nL-x_0))^2}{4v(t_0-t)}\right)\\
&+\frac{1}{\sqrt{4\pi v(t_0-t)}}\sum_{n=-\infty}^{\infty}\frac{-(L-(-4nL+x_0))}{2v(t_0-t)}\exp\left(\frac{-(L-(-4nL+x_0))^2}{4v(t_0-t)}\right)\\
&-\frac{1}{\sqrt{4\pi v(t_0-t)}}\sum_{n=-\infty}^{\infty}\frac{-(L-((4n-2)L+x_0))}{2v(t_0-t)}\exp\left(\frac{-(L-((4n-2)L+x_0))^2}{4v(t_0-t)}\right)\\
&-\frac{1}{\sqrt{4\pi v(t_0-t)}}\sum_{n=-\infty}^{\infty}\frac{-(L-((4n-2)L-x_0))}{2v(t_0-t)}\exp\left(\frac{-(L-((4n-2)L-x_0))^2}{4v(t_0-t)}\right) \qquad (9)
\end{aligned}
$$

From (4), we now collect all terms into the solution

$$
u(x_0,t_0) = -\int_{x=0}^{L}[uG]_{t=0}^{\infty}dx - \int_{t=0}^{t_0}[uG_x-u_xG]_{x=0}^{L}dt + \int_{x=0}^{L}\int_{t=0}^{t_0}GQ\;dtdx \qquad (4)
$$

We found $\int_{x=0}^{L}[uG]_{t=0}^{\infty}dx = \int_{x=0}^{L}-f(x)G(x,0)dx$ and now we know what G is. Hence we can find $G(x,0;x_0,t_0)$. It is, from (7A)

$$
\begin{aligned}
G(x,0;x_0,t_0) &= \frac{1}{\sqrt{4\pi v t_0}}\left(\sum_{n=-\infty}^{\infty}\exp\left(\frac{-(x-(4nL-x_0))^2}{4vt_0}\right)+\exp\left(\frac{-(x-(-4nL+x_0))^2}{4vt_0}\right)\right)\\
&-\frac{1}{\sqrt{4\pi v t_0}}\left(\sum_{n=-\infty}^{\infty}\exp\left(\frac{-(x-((4n-2)L+x_0))^2}{4vt_0}\right)+\exp\left(\frac{-(x-((4n-2)L-x_0))^2}{4vt_0}\right)\right) \qquad (10)
\end{aligned}
$$

And we now know what $[uG_x-u_xG]_{x=0}^{L}$ is. It is $A\frac{\partial G(L,t;x_0,t_0)}{\partial x}$. Hence (4) becomes

$$
u(x_0,t_0) = \int_{x=0}^{L}f(x)G(x,0;x_0,t_0)dx - \int_{t=0}^{t_0}A\frac{\partial G(L,t;x_0,t_0)}{\partial x}dt + \int_{x=0}^{L}\int_{t=0}^{t_0}G(x,t;x_0,t_0)Q(x,t)\;dtdx
$$

Changing the roles of x_0,t_0

$$
u(x,t) = \int_{x_0=0}^{L}f(x_0)G(x_0,t_0;x,0)dx_0 - \int_{t=0}^{t}A\frac{\partial G(x_0,t_0;L,t)}{\partial x_0}dt_0 + \int_{x_0=0}^{L}\int_{t_0=0}^{t}G(x_0,t_0;x,t)Q(x_0,t_0)\;dt_0dx_0 \qquad (11)
$$

This completes the solution.

Summary

The solution is

$$
u(x,t) = \int_{x_0=0}^{L}f(x_0)G(x_0,t_0;x,0)dx_0 - \int_{t=0}^{t}A\frac{\partial G(x_0,t_0;L,t)}{\partial x_0}dt_0 + \int_{x_0=0}^{L}\int_{t_0=0}^{t}G(x_0,t_0;x,t)Q(x_0,t_0)\;dt_0dx_0
$$

Where $G(x_0,t_0;x,0)$ is given in (10) (after changing roles of parameters):

$$
\begin{aligned}
G(x_0,t_0;x,0) &= \frac{1}{\sqrt{4\pi v t}}\left(\sum_{n=-\infty}^{\infty}\exp\left(\frac{-(x_0-(4nL-x))^2}{4vt}\right)+\exp\left(\frac{-(x_0-(-4nL+x))^2}{4vt}\right)\right)\\
&-\frac{1}{\sqrt{4\pi v t}}\left(\sum_{n=-\infty}^{\infty}\exp\left(\frac{-(x_0-((4n-2)L+x))^2}{4vt}\right)+\exp\left(\frac{-(x_0-((4n-2)L-x))^2}{4vt}\right)\right) \qquad (10A)
\end{aligned}
$$

and $\frac{\partial G(x_0,t_0;L,t)}{\partial x_0}$ is given in (9) (after also changing roles of parameters):

$$\begin{aligned}\frac{\partial G(x_0,t_0;L,t)}{\partial x_0} &= \frac{1}{\sqrt{4\pi v(t-t_0)}}\sum_{n=-\infty}^{\infty}\frac{-(L-(4nL-x))}{2v(t-t_0)}\exp\left(\frac{-(L-(4nL-x))^2}{4v(t-t_0)}\right)\\ &+\frac{1}{\sqrt{4\pi v(t-t_0)}}\sum_{n=-\infty}^{\infty}\frac{-(L-(-4nL+x))}{2v(t-t_0)}\exp\left(\frac{-(L-(-4nL+x))^2}{4v(t-t_0)}\right)\\ &-\frac{1}{\sqrt{4\pi v(t-t_0)}}\sum_{n=-\infty}^{\infty}\frac{-(L-((4n-2)L+x))}{2v(t-t_0)}\exp\left(\frac{-(L-((4n-2)L+x))^2}{4v(t-t_0)}\right)\\ &-\frac{1}{\sqrt{4\pi v(t-t_0)}}\sum_{n=-\infty}^{\infty}\frac{-(L-((4n-2)L-x))}{2v(t-t_0)}\exp\left(\frac{-(L-((4n-2)L-x))^2}{4v(t-t_0)}\right)\end{aligned}\tag{9A}$$

and $G(x_0,t_0;x,t)$ is given in (7A), but with roles changed as well to become

$$\begin{aligned}G(x_0,t_0;x,t) = &\frac{1}{\sqrt{4\pi v(t-t_0)}}\left(\sum_{n=-\infty}^{\infty}\exp\left(\frac{-(x_0-(4nL-x))^2}{4v(t-t_0)}\right)+\exp\left(\frac{-(x_0-(-4nL+x))^2}{4v(t-t_0)}\right)\right)\\ &-\frac{1}{\sqrt{4\pi v(t-t_0)}}\left(\sum_{n=-\infty}^{\infty}\exp\left(\frac{-(x_0-((4n-2)L+x))^2}{4v(t-t_0)}\right)+\exp\left(\frac{-(x_0-((4n-2)L-x))^2}{4v(t-t_0)}\right)\right)\end{aligned}\tag{7AA}$$

0.2.2 Part(b)

When $A=0$, the solution in part (a) becomes

$$u(x,t)=\int_{x_0=0}^{L} f(x_0)\,G(x_0,t_0;x,0)\,dx_0+\int_{x_0=0}^{L}\int_{t_0=0}^{t} G(x_0,t_0;x,t)\,Q\,dt_0dx_0$$

Where $G(x_0,t_0;x,0)$ is given in (10A) in part (a), and $G(x_0,t_0;x,t)$ is given in (7AA) in part (a). Now we find the eigenfunction solution for this problem order to compare it with the above green function images solution. Since $A=0$ then the PDE now becomes

$$\begin{aligned}\frac{\partial u}{\partial t}&=v\frac{\partial^2 u}{\partial x^2}+Q(x,t)\\ 0&\leq x\leq L\\ t&\geq 0\end{aligned}\tag{1}$$

Initial conditions

$$u(x,0)=f(x)$$

Boundary conditions

$$\begin{aligned}\frac{\partial u(0,t)}{\partial x}&=0\\ u(L,t)&=0\end{aligned}$$

Since boundary conditions has now become homogenous (thanks for $A=0$), we can use separation of variables to find the eigenfunctions, and then use eigenfunction expansion. Let the solution be

$$\begin{aligned}u_n(x,t)&=a_n(t)\,\phi_n(x)\\ u(x,t)&=\sum_{n=1}^{\infty}a_n(t)\,\phi_n(x)\end{aligned}\tag{1A}$$

Where $\phi_n(t)$ are eigenfunctions for the associated homogeneous PDE $\frac{\partial u}{\partial t}=v\frac{\partial^2 u}{\partial x^2}$ which can be found from separation of variables. To find $\phi_n(x)$, we start by separation of variables. Let $u(x,t)=X(x)T(t)$ and we plug this solution back to the PDE to obtain

$$\begin{aligned}XT'&=vX''T\\ \frac{1}{v}\frac{T'}{T}&=\frac{X''}{X}=-\lambda\end{aligned}$$

Hence the spatial ODE is $\frac{X''}{X}=-\lambda$ or $X''+\lambda X=0$ with boundary conditions

$$\begin{aligned}X'(0)&=0\\ X(L)&=0\end{aligned}$$

case $\underline{\lambda=0}$ The solution is $X=c_1+c_2x$. Hence $X'=c_2$. Therefore $c_2=0$. Hence $X=c_1=0$. Trivial solution. So $\underline{\lambda=0 \text{ is not possible}}$.

case $\underline{\lambda>0}$ The solution is $X=c_1\cos\left(\sqrt{\lambda}x\right)+c_2\sin\left(\sqrt{\lambda}x\right)$. and $X'=-\sqrt{\lambda}c_1\sin\lambda x+c_2\sqrt{\lambda}\cos\lambda x$. From first B.C. at $x=0$ we find $0=c_2\sqrt{\lambda}$, hence $c_2=0$ and the solution becomes $X=c_1\cos\left(\sqrt{\lambda}x\right)$. At

$x=L$, we have $0=c_1\cos\left(\sqrt{\lambda}L\right)$ which leads to $\sqrt{\lambda}L=n\frac{\pi}{2}$ for $n=1,3,5,\cdots$ or

$$\sqrt{\lambda_n}=\left(\frac{2n-1}{2}\right)\frac{\pi}{L}\qquad n=1,2,3,\cdots$$
$$\lambda_n=\left(\frac{2n-1}{2}\frac{\pi}{L}\right)^2\qquad n=1,1,2,3,\cdots$$

Hence the $X_n(x)$ solution is

$$X_n(x)=c_n\cos\left(\sqrt{\lambda_n}x\right)\qquad n=1,2,3,\cdots$$

The time ODE is now solved using the above eigenvalues. (we really do not need to do this part, since $A_n(t)$ will be solved for later, and $A_n(t)$ will contain all the time dependent parts, including those that come from $Q(x,t)$, but for completion, this is done)

$$\begin{aligned}\frac{1}{v}\frac{T'}{T}&=-\lambda_n\\ T'+v\lambda_nT&=0\\ \frac{dT}{T}&=-v\lambda_ndt\\ \ln|T|&=-v\lambda_nt+C\\ T&=C_ne^{v\lambda_nt}\end{aligned}$$

Hence the solution to the homogenous PDE is

$$\begin{aligned}u_n(x,t)&=X_nT_n\\ &=c_n\cos\left(\sqrt{\lambda_n}x\right)e^{v\lambda_nt}\end{aligned}$$

Where constants of integration are merged into one. Therefore

$$\begin{aligned}u(x,t)&=\sum_{n=1}^{\infty}X_nT_n\\ &=\sum_{n=1}^{\infty}c_n\cos\left(\sqrt{\lambda_n}x\right)e^{v\lambda_nt}\end{aligned}\tag{2}$$

From the above we see that

$$\boxed{\phi_n(x)=\cos\left(\sqrt{\lambda_n}x\right)}$$

Using this, we now write the solution to $\frac{\partial u}{\partial t}=v\frac{\partial^2u}{\partial x^2}+Q(x,t)$ using eigenfunction expansion

$$u(x,t)=\sum_{n=1}^{\infty}A_n(t)\phi_n(x)\tag{3}$$

Where now $A_n(t)$ will have all the time dependent terms from $Q(x,t)$ as well from the time solution from the homogenous PDE $e^{v\left(\frac{2n-1}{2}\frac{\pi}{L}\right)^2t}$ part. We will solve for $A_n(t)$ now.

In this below, we will expand $Q(x,t)$ using these eigenfunctions (we can do this, since the eigenfunctions are basis for the whole solution space which the forcing function is in as well). We plug-in (3) back into the PDE, and since boundary conditions are now homogenous, then term by term differentiation is justified. The PDE $\frac{\partial u}{\partial t}=v\frac{\partial^2u}{\partial x^2}+Q(x,t)$ now becomes

$$\frac{\partial}{\partial t}\sum_{n=1}^{\infty}A_n(t)\phi_n(x)=v\frac{\partial^2}{\partial x^2}\sum_{n=1}^{\infty}A_n(t)\phi_n(x)+\sum_{n=1}^{\infty}q_n(t)\phi_n(x)\tag{4}$$

Where $\sum_{n=1}^{\infty}q_n(t)\phi_n(x)$ is the eigenfunction expansion of $Q(x,t)$. To find $q_n(t)$ we apply orthogonality as follows

$$\begin{aligned}Q(x,t)&=\sum_{n=1}^{\infty}q_n(t)\phi_n(x)\\ \int_0^L\phi_m(x)Q(x,t)dx&=\int_0^L\left(\sum_{n=1}^{\infty}q_n(t)\phi_n(x)\right)\phi_m(x)dx\\ \int_0^L\phi_m(x)Q(x,t)dx&=\sum_{n=1}^{\infty}\int_0^Lq_n(t)\phi_n(x)\phi_m(x)dx\\ &=\int_0^Lq_m(t)\phi_m^2(x)dx\\ &=q_m(t)\int_0^L\cos^2\left(\sqrt{\lambda_n}x\right)dx\end{aligned}$$

But

$$\int_0^L \cos^2\left(\sqrt{\lambda_n}x\right)dx = \frac{L}{2}$$

Hence

$$q_n(t) = \frac{2}{L}\int_0^L \phi_n(x)\,Q(x,t)\,dx \tag{5}$$

Now that we found $q_n(t)$, we go back to (4) and simplifies it more

$$\begin{aligned}\sum_{n=1}^{\infty} A_n'(t)\,\phi_n(x) &= v\sum_{n=1}^{\infty} A_n(t)\,\phi_n''(x) + \sum_{n=1}^{\infty} q_n(t)\,\phi_n(x)\\ A_n'(t)\,\phi_n(x) &= vA_n(t)\,\phi_n''(x) + q_n(t)\,\phi_n(x)\end{aligned}$$

But since $\phi_n(x) = \cos\left(\sqrt{\lambda_n}x\right)$ then

$$\phi_n'(x) = -\left(\sqrt{\lambda_n}\right)\sin\left(\sqrt{\lambda_n}x\right)$$

And

$$\begin{aligned}\phi_n''(x) &= -\lambda_n\cos\left(\sqrt{\lambda_n}x\right)\\ &= -\lambda_n\phi_n(x)\end{aligned}$$

Hence the above ODE becomes

$$A_n'\phi_n = -vA_n\lambda_n\phi_n + q_n\phi_n$$

Canceling the eigenfunction $\phi_n(x)$ (since not zero) gives

$$A_n'(t) + vA_n(t)\,\lambda_n = q_n(t) \tag{6}$$

We now solve this for $A_n(t)$. Integrating factor is

$$\begin{aligned}\mu &= \exp\left(\int v\lambda_n dt\right)\\ &= e^{\lambda_n v t}\end{aligned}$$

Hence (6) becomes

$$\begin{aligned}\frac{d}{dt}\left(\mu A_n(t)\right) &= \mu q_n(t)\\ e^{\lambda_n v t}A_n(t) &= \int_0^t e^{\lambda_n v s} q_n(s)\,ds + C\\ A_n(t) &= e^{-\lambda_n v t}\int_0^t e^{\lambda_n v s} q_n(s)\,ds + Ce^{-\lambda_n v t}\end{aligned} \tag{7}$$

Now that we found $A_n(t)$, then the solution (3) becomes

$$u(x,t) = \sum_{n=1}^{\infty}\left(e^{-\left(\frac{2n-1}{2}\frac{\pi}{L}\right)^2 vt}\int_0^t e^{\left(\frac{2n-1}{2}\frac{\pi}{L}\right)^2 vs} q_n(s)\,ds + Ce^{-\left(\frac{2n-1}{2}\frac{\pi}{L}\right)^2 vt}\right)\phi_n(x) \tag{8}$$

At $t = 0$, we are given that $u(x,0) = f(x)$, hence the above becomes

$$f(x) = \sum_{n=1}^{\infty} C\phi_n(x)$$

To find C, we apply orthogonality again, which gives

$$\begin{aligned}\int_0^L f(x)\,\phi_m(x)\,dx &= \sum_{n=1}^{\infty}\int_0^L C\phi_n(x)\,\phi_m(x)\,dx\\ &= C\int_0^L \phi_m^2(x)\,dx\\ \int_0^L f(x)\,\phi_m(x)\,dx &= \frac{L}{2}C\\ C &= \frac{2}{L}\int_0^L f(x)\,\phi_n(x)\,dx\end{aligned}$$

Now that we found C, then the solution in (8) is complete. It is

$$\begin{aligned}u(x,t) &= \sum_{n=1}^{\infty} A_n(t)\,\phi_n(x)\\ &= \sum_{n=1}^{\infty}\left(e^{-\lambda_n v t}\int_0^t e^{\lambda_n v s} q_n(s)\,ds + \frac{2}{L}e^{-\lambda_n v t}\int_0^L f(x)\,\phi_n(x)\,dx\right)\phi_n(x)\end{aligned}$$

Or

$$u(x,t) = \sum_{n=1}^{\infty}\left(\phi_n(x)\,e^{-\lambda_n v t}\int_0^t e^{\lambda_n v s} q_n(s)\,ds\right) + \frac{2}{L}\sum_{n=1}^{\infty}\left(\phi_n(x)\,e^{-\lambda_n v t}\int_0^L f(x)\,\phi_n(x)\,dx\right)$$

Summary

The eigenfunction solution is

$$u(x,t) = \sum_{n=1}^{\infty}\left(\phi_n(x)\,e^{-\lambda_n v t}\int_0^t e^{\lambda_n v s} q_n(s)\,ds\right) + \frac{2}{L}\sum_{n=1}^{\infty}\left(\phi_n(x)\,e^{-\lambda_n v t}\int_0^L f(x)\,\phi_n(x)\,dx\right)$$

Where

$$\phi_n(x) = \cos\left(\sqrt{\lambda_n}x\right) \qquad n = 1,2,3,\cdots$$

$$\lambda_n = \left(\frac{2n-1}{2}\frac{\pi}{L}\right)^2 \qquad n = 1,1,2,3,\cdots$$

And

$$q_n(t) = \frac{2}{L}\int_0^L \phi_n(x)\,Q(x,t)\,dx$$

To compare the eigenfunction expansion solution and the Green function solution, we see the following mapping of the two solutions

$$\overbrace{\int_0^L f(x_0)\,G(x_0,t_0;x,0)\,dx_0}^{\frac{2}{L}\sum_{n=1}^{\infty}\left(\phi_n(x)e^{-\lambda_n v t}\int_0^L f(x)\phi_n(x)dx\right)} + \overbrace{\int_0^L\int_0^t G(x_0,t_0;x,t)\,Q(x_0,t_0)\; dt_0 dx_0}^{\sum_{n=1}^{\infty}\left(\phi_n(x)e^{-\lambda_n v t}\int_0^t e^{\lambda_n v s}q_n(s)ds\right)}$$

Where the top expression is the eigenfunction expansion and the bottom expression is the Green function solution using method of images. Where $G(x_0,t_0;x,t)$ in above contains the infinite sums of the images. So the Green function solution contains integrals and inside these integrals are the infinite sums. While the eigenfunction expansions contains two infinite sums, but inside the sums we see the integrals. So summing over the images seems to be equivalent to the operation of summing over eigenfunctions. These two solutions in the limit should of course give the same result (unless I made a mistake somewhere). In this example, I found method of eigenfunction expansion easier, since getting the images in correct locations and sign was tricky to get right.

0.3 problem 5

5. This problem is a simple model for diffraction of light passing through infinitesimally small slits separated by a distance $2a$.

Solve the diffraction equation

$$\frac{\partial u}{\partial t} = \frac{i\lambda}{4\pi}\frac{\partial^2 u}{\partial x^2} \tag{1}$$

with initial source $u(x,0) = f(x) = \delta(x-a) + \delta(x+a),\ a > 0$.

Show that the solution $u(x,t)$ oscillates wildly, but that the intensity $|u(x,t)|^2$ is well-behaved. The intensity $|u(x,t)|^2$ shows that the diffraction pattern at a distance t consists of a series of alternating bright and dark fringes with period $\lambda t/(2a)$.

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &= \frac{i\lambda}{4\pi}\frac{\partial^2 u(x,t)}{\partial x^2} \\ u(x,0) &= f(x) = \delta(x-a) + \delta(x+a)\end{aligned} \tag{1}$$

I will Use Fourier transform to solve this, since this is for $-\infty < x < \infty$ and the solution $u(x,t)$ is assumed bounded at $\pm\infty$ (or goes to zero there), hence $u(x,t)$ is square integrable and therefore we can assume it has a Fourier transform.

Let $U(\xi,t)$ be the spatial part only Fourier transform of $u(x,t)$. Using the Fourier transform pairs defined as

$$\begin{aligned}U(\xi,t) &= \mathscr{F}(u(x,t)) = \int_{-\infty}^{\infty} u(x,t)\, e^{-i2\pi x\xi} dx \\ u(x,t) &= \mathscr{F}^{-1}(U(\xi,t)) = \int_{-\infty}^{\infty} U(\xi,t)\, e^{i2\pi x\xi} d\xi\end{aligned}$$

Therefore, by Fourier transform properties of derivatives

$$\begin{aligned}\mathscr{F}\left(\frac{\partial u(x,t)}{\partial x}\right) &= (2\pi i\xi)\, U(\xi,t) \\ \mathscr{F}\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right) &= (2\pi i\xi)^2\, U(\xi,t)\end{aligned} \tag{2}$$

And

$$\mathscr{F}\left(\frac{\partial u(x,t)}{\partial t}\right) = \frac{\partial U(\xi,t)}{\partial t} \tag{3}$$

Where in (3), we just need to take time derivative of $U(\xi,t)$ since the transform is applied only to the space part. Now we take the Fourier transform of the given PDE and using (2,3) relations we obtain the PDE but now in Fourier space.

$$\begin{aligned}\frac{\partial U(\xi,t)}{\partial t} &= \left(\frac{i\lambda}{4\pi}\right)(2\pi i\xi)^2\, U(\xi,t) \\ &= -\left(\frac{i\lambda}{4\pi}\right) 4\pi^2\xi^2 U(\xi,t) \\ &= \left(-i\lambda\pi\xi^2\right) U(\xi,t)\end{aligned} \tag{4}$$

Equation (4) can now be easily solved for $U(\xi,t)$ since it is separable.

$$\frac{\partial U(\xi,t)}{U(\xi,t)} = \left(-i\lambda\pi\xi^2\right)\partial t$$

Integrating

$$\begin{aligned}\ln|U(\xi,t)| &= \left(-i\lambda\pi\xi^2\right) t + C \\ U(\xi,t) &= U(\xi,0)\, e^{\left(-i\lambda\pi\xi^2\right)t}\end{aligned}$$

Where $U(\xi,0)$ is the Fourier transform of $u(x,0)$, the initial conditions, which is $f(x)$ and is given in the problem. To go back to spatial domain, we now need to do the inverse Fourier transform. By applying the convolution theorem, we know that multiplication in Fourier domain is convolution in spatial domain, therefore

$$\mathscr{F}^{-1}(U(\xi,t)) = \mathscr{F}^{-1}(U(\xi,0)) \circledast \mathscr{F}^{-1}\left(e^{-i\lambda\pi\xi^2 t}\right) \tag{5}$$

But

$$\mathscr{F}^{-1}\left(U\left(\xi,t\right)\right)=u\left(x,t\right)$$
$$\mathscr{F}^{-1}\left(U\left(\xi,0\right)\right)=f\left(x\right)$$

And

$$\mathscr{F}^{-1}\left(e^{-i\lambda\pi\xi^2 t}\right)=\int_{-\infty}^{\infty}e^{-i\lambda\pi\xi^2 t}e^{2i\pi x\xi}d\xi \tag{5A}$$

Hence (5) becomes

$$u\left(x,t\right)=f\left(x\right)\circledast\mathscr{F}^{-1}\left(e^{-i\lambda\pi\xi^2 t}\right)$$

Here, I used Mathematica to help me with the above integral (5A) as I could not find it in tables so far[1]. Here is the result

```
Find inverse Fourier transform, for problem 5, NE 548

In[13]:= InverseFourierTransform[ Exp[-I lam Pi z^2 t], z, x, FourierParameters -> {1,  -2 * Pi}]

Out[13]= e^((i π x^2)/(lam t)) / (Sqrt[2 π] Sqrt[i lam t])
```

Therefore, from Mathematica, we see that

$$\mathscr{F}^{-1}\left(e^{-i\lambda\pi\xi^2 t}\right)=\frac{e^{\frac{i\pi x^2}{\lambda t}}}{\sqrt{2\pi}\sqrt{i}\sqrt{\lambda t}} \tag{6}$$

But[2]

$$\begin{aligned}\sqrt{i}&=\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\\&=\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\\&=e^{i\frac{\pi}{4}}\end{aligned}$$

Hence (6) becomes

$$\mathscr{F}^{-1}\left(e^{-i\lambda\pi\xi^2 t}\right)=\frac{1}{e^{i\frac{\pi}{4}}\sqrt{2\pi\lambda t}}e^{i\frac{\pi x^2}{\lambda t}} \tag{7}$$

Now we are ready to do the convolution in (5A) since we know everything in the RHS, hence

$$\begin{aligned}u\left(x,t\right)&=f\left(x\right)\circledast\frac{1}{e^{i\frac{\pi}{4}}\sqrt{2\pi\lambda t}}e^{i\frac{\pi x^2}{\lambda t}}\\&=\left(\delta\left(x-a\right)+\delta\left(x+a\right)\right)\circledast\frac{1}{e^{i\frac{\pi}{4}}\sqrt{2\pi\lambda t}}e^{i\frac{\pi x^2}{\lambda t}} \qquad (8)\end{aligned}$$

Applying convolution integral on (8), which says that

$$\begin{aligned}f\left(x\right)&=g_1\left(x\right)\circledast g_2\left(x\right)\\&=\int_{-\infty}^{\infty}g_1\left(z\right)g_2\left(x-z\right)dz\end{aligned}$$

Therefore (8) becomes

$$\begin{aligned}u\left(x,t\right)&=\int_{-\infty}^{\infty}\left(\delta\left(z-a\right)+\delta\left(z+a\right)\right)\frac{1}{e^{i\frac{\pi}{4}}\sqrt{2\pi\lambda t}}e^{i\frac{\pi(x-z)^2}{\lambda t}}dz\\&=\frac{1}{e^{i\frac{\pi}{4}}\sqrt{2\pi\lambda t}}\int_{-\infty}^{\infty}\left(\delta\left(z-a\right)+\delta\left(z+a\right)\right)e^{i\frac{\pi(x-z)^2}{\lambda t}}dz\\&=\frac{1}{e^{i\frac{\pi}{4}}\sqrt{2\pi\lambda t}}\left(\int_{-\infty}^{\infty}\delta\left(z-a\right)e^{i\frac{\pi(x-z)^2}{\lambda t}}dz+\int_{-\infty}^{\infty}\delta\left(z+a\right)e^{i\frac{\pi(x-z)^2}{\lambda t}}dz\right)\end{aligned}$$

But an integral with delta function inside it, is just the integrand evaluated where the delta argument become zero which is at $z=a$ and $z=-a$ in the above. (This is called the sifting property). Hence the above integrals are now easily found and we obtain the solution

$$u\left(x,t\right)=\frac{1}{e^{i\frac{\pi}{4}}\sqrt{2\pi\lambda t}}\left(\exp\left(i\frac{\pi\left(x-a\right)^2}{\lambda t}\right)+\exp\left(i\frac{\pi\left(x+a\right)^2}{\lambda t}\right)\right)$$

[1]Trying to do this integral by hand also, but so far having some difficulty..
[2]Taking the positive root only.

The above is the solution we need. But we can simplify it more by using Euler relation.

$$
\begin{aligned}
u(x,t) &= \frac{1}{e^{i\frac{\pi}{4}}\sqrt{2\pi\lambda t}}\left(\exp\left(i\frac{\pi\left(x^2+a^2-2xa\right)}{\lambda t}\right)+\exp\left(i\frac{\pi\left(x^2+a^2+2ax\right)}{\lambda t}\right)\right)\\
&= \frac{1}{e^{i\frac{\pi}{4}}\sqrt{2\pi\lambda t}}\left(\exp\left(\frac{i\pi\left(x^2+a^2\right)}{\lambda t}\right)\exp\left(\frac{-i2\pi xa}{\lambda t}\right)+\exp\left(\frac{i\pi\left(x^2+a^2\right)}{\lambda t}\right)\exp\left(\frac{i2\pi ax}{\lambda t}\right)\right)
\end{aligned}
$$

Taking $\exp\left(\frac{i\pi(x^2+a^2)}{\lambda t}\right)$ as common factor outside results in

$$
\begin{aligned}
u(x,t) &= \frac{\exp\left(\frac{i\pi(x^2+a^2)}{\lambda t}\right)}{e^{i\frac{\pi}{4}}\sqrt{2\pi\lambda t}}\left(\exp\left(i\frac{2\pi ax}{\lambda t}\right)+\exp\left(-i\frac{2\pi xa}{\lambda t}\right)\right)\\
&= \frac{2\exp\left(\frac{i\pi(x^2+a^2)}{\lambda t}\right)}{e^{i\frac{\pi}{4}}\sqrt{2\pi\lambda t}}\left(\frac{\exp\left(i\frac{2\pi ax}{\lambda t}\right)+\exp\left(-i\frac{2\pi xa}{\lambda t}\right)}{2}\right)\\
&= \frac{2\exp\left(\frac{i\pi(x^2+a^2)}{\lambda t}\right)}{e^{i\frac{\pi}{4}}\sqrt{2\pi\lambda t}}\cos\left(\frac{2\pi ax}{\lambda t}\right)\\
&= \frac{\sqrt{2}\exp\left(\frac{i\pi(x^2+a^2)}{\lambda t}-i\frac{\pi}{4}\right)}{\sqrt{\pi\lambda t}}\cos\left(\frac{2\pi ax}{\lambda t}\right)\\
&= \frac{\sqrt{2}\exp\left(i\left(\frac{\pi(x^2+a^2)}{\lambda t}-\frac{\pi}{4}\right)\right)}{\sqrt{\pi\lambda t}}\cos\left(\frac{2\pi ax}{\lambda t}\right)
\end{aligned}
$$

Hence the final solution is

$$
\boxed{u(x,t)=\sqrt{\frac{2}{\pi\lambda t}}\exp\left(i\frac{\pi(x^2+a^2)}{\lambda t}-\frac{\pi}{4}\right)\cos\left(\frac{2\pi a}{\lambda}\frac{x}{t}\right)} \tag{9}
$$

Hence the real part of the solution is

$$
\Re\left(u(x,t)\right)=\sqrt{\frac{2}{\pi\lambda t}}\cos\left(\frac{\pi\left(x^2+a^2\right)}{\lambda t}-\frac{\pi}{4}\right)\cos\left(\frac{2\pi a}{\lambda}\frac{x}{t}\right)
$$

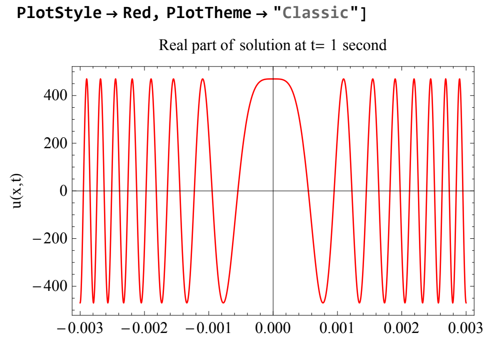
And the imaginary part of the solution is

$$
\Im\left(u(x,t)\right)=\sqrt{\frac{2}{\pi\lambda t}}\sin\left(\frac{\pi\left(x^2+a^2\right)}{\lambda t}-\frac{\pi}{4}\right)\cos\left(\frac{2\pi a}{\lambda}\frac{x}{t}\right)
$$

The $\frac{\pi}{4}$ is just a phase shift. Here is a plot of the Real and Imaginary parts of the solution, using for $\lambda = 600\times10^{-9}meter, a = 1000\times10^{-9}meter$ at $t=1$ second

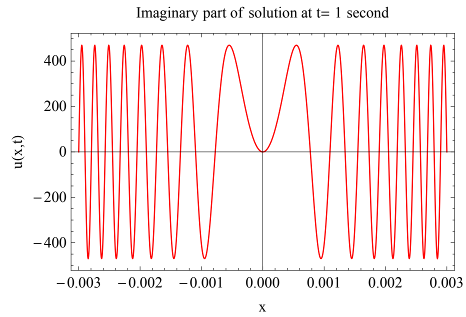
```
In[15]:= a = 1000 * 10^(-9);
         λ = 600 * 10^(-9);

         u[x_, t_] := Sqrt[2/(π λ t)] Exp[I (π (x^2 + a^2))/(λ t) - π/4] Cos[(2 π a)/λ x/t]

In[59]:= Clear[t];
         Plot[Re@u[x, 1], {x, -3000 a, 3000 a}, Frame -> True,
          FrameLabel → {{"u(x,t)", None}, {"x", "Real part of solution at t= 1 second"}}, BaseStyle → 12,
          PlotStyle → Red, PlotTheme → "Classic"]
```

Out[60]=



```
In[61]:= Plot[Im@u[x, 1], {x, -3000 a, 3000 a}, Frame -> True,
          FrameLabel → {{"u(x,t)", None}, {"x", "Imaginary part of solution at t= 1 second"}}, BaseStyle → 12,
          PlotStyle → Red, PlotTheme → "Classic"]
```

Out[61]=

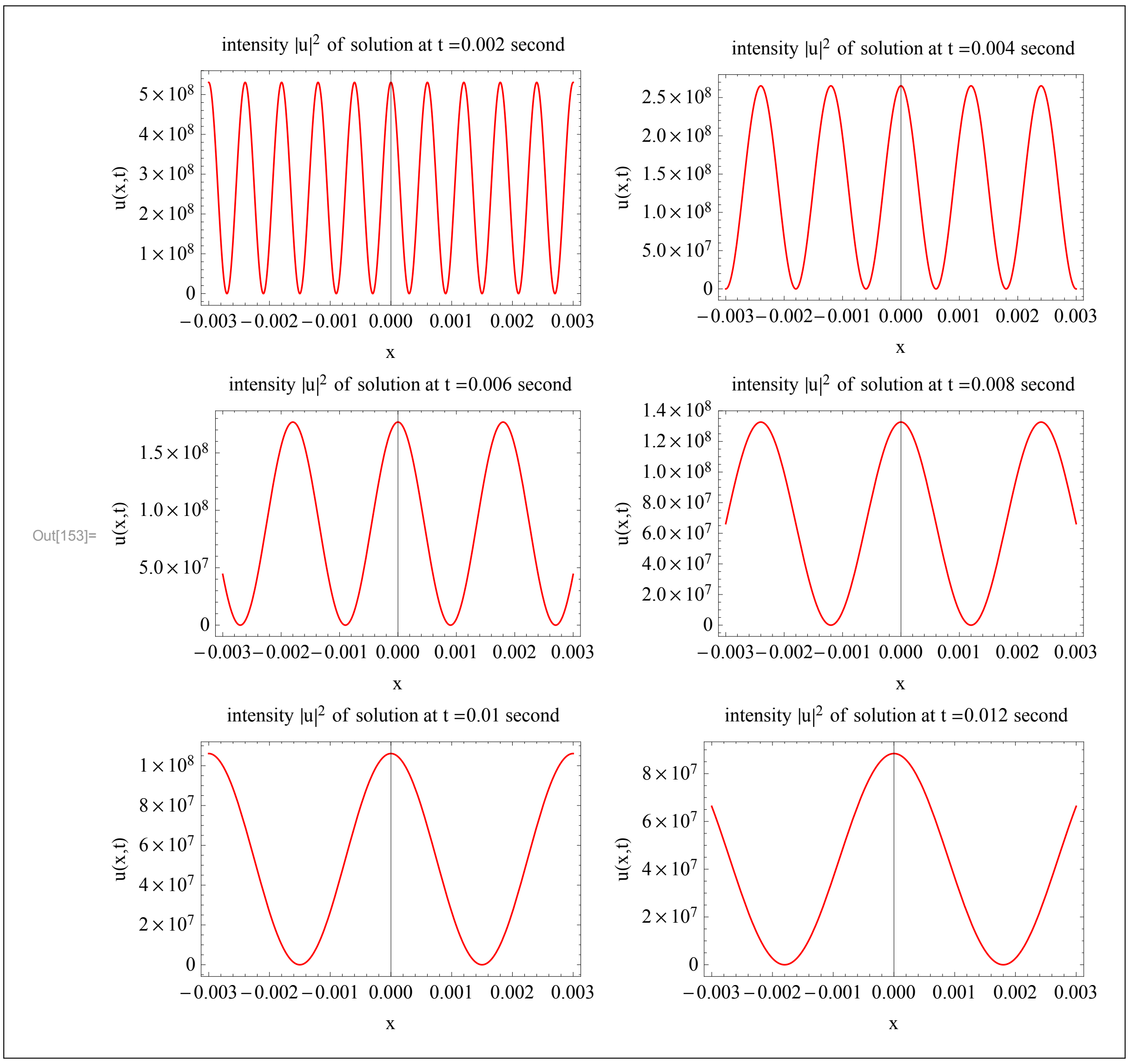


We see the rapid oscillations as distance goes away from the origin. This is due to the x^2 term making the radial frequency value increase quickly with x. We now plot the $|u(x,t)|^2$. Looking at solution in (9), and since complex exponential is ±1, then the amplitude is governed by $\sqrt{\frac{2}{\pi\lambda t}}\cos\left(\frac{2\pi a}{\lambda}\frac{x}{t}\right)$ part of the solution. Hence

$$\boxed{|u(x,t)|^2 = \frac{2}{\pi\lambda t}\cos^2\left(\frac{2\pi a}{\lambda}\frac{x}{t}\right)}$$

These plots show the intensity at different time values. We see from these plots, that the intensity is <u>well behaved</u> in that it does not have the same rapid oscillations seen in the $u(x,t)$ solution plots.

```
In[154]:= a = 1000 * 10^(-9);
          λ = 600 * 10^(-9);
          intensity[x_, t_] := 2/(π λ t) (Cos[(2 π a)/λ x/t])^2
          p =
            Plot[intensity[x, #], {x, -3000 a, 3000 a}, Frame -> True,
               FrameLabel → {{"u(x,t)", None}, {"x", Row[{" intensity |u|^2 of solution at t =", #, " second"}]}},
               BaseStyle → 12, PlotStyle → Red, PlotTheme → "Classic", ImageSize → 300] & /@ Range[0.002, 0.012, 0.002];
```



Now Comparing argument to cosine in above to standard form in order to find the period:

$$\frac{2\pi a}{\lambda}\frac{x}{t} = 2\pi f t$$

Where f is now in hertz, then when $x = t$, we get by comparing terms that

$$\frac{2\pi a}{\lambda}\frac{1}{t} = 2\pi f$$

$$\frac{a}{\lambda}\frac{1}{t} = f$$

But $f = \frac{1}{T}$ where T is the period in seconds. Hence $\frac{a}{\lambda}\frac{1}{t} = \frac{1}{T}$ or

$$\boxed{T = \frac{\lambda t}{a}}$$

So period on intensity is $\frac{\lambda t}{a}$ at $x = t$ (why problem statement is saying period is $\frac{\lambda t}{2a}$?).

0.4 problem 2 (optional)

2. Consider the 1D heat equation in a semi-infinite domain:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}, \quad x \geq 0$$

with boundary conditions: $u(0,t) = \exp(-i\omega t)$ and $u(x,t)$ bounded as $x \to \infty$. In order to construct a real forcing, we need both positive and negative real values of ω. Consider that this forcing has been and will be applied for all time. This "pure boundary value problem" could be an idealization of heating the surface of the earth by the sun (periodic forcing). One could then ask, how far beneath the surface of the earth do the periodic fluctuations of the heat propagate?

(a) Consider solutions of the form $u(x,t;\omega) = \exp(ikx)\exp(-i\omega t)$. Find a single expression for k as a function of (given) ω real, $sgn(\omega)$ and ν real.

Write $u(x,t;\omega)$ as a function of (given) ω real, $sgn(\omega)$ and ν real. To obtain the most general solution by superposition, one would next integrate over all values of ω, $-\infty < \omega < \infty$ (do not do this).

(b) The basic solution can be written as $u(x,t;\omega) = \exp(-i\omega t)\exp(-\sigma x)\exp(i\sigma\, sgn(\omega)\, x)$. Find σ in terms of $|\omega|$ and ν.

(c) Make an estimate for the propagation depth of daily temperature fluctuations.

0.4.1 Part (a)

$$\begin{aligned} \frac{\partial u}{\partial t} &= \nu \frac{\partial^2 u}{\partial x^2} \\ u(0,t) &= e^{-i\omega t} \end{aligned} \tag{1}$$

And $x \geq 0, u(\infty, t)$ bounded. Let

$$u(x,t) = e^{ikx} e^{-i\omega t}$$

Hence

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= -i\omega e^{ikx} e^{-i\omega t} \\ &= -i\omega u(x,t) \end{aligned} \tag{2}$$

And

$$\begin{aligned} \frac{\partial u(x,t)}{\partial x} &= ik e^{ikx} e^{-i\omega t} \\ \frac{\partial^2 u(x,t)}{\partial x^2} &= -k^2 e^{ikx} e^{-i\omega t} \\ &= -k^2 u(x,t) \end{aligned} \tag{3}$$

Substituting (2,3) into (1) gives

$$-i\omega u(x,t) = -\nu k^2 u(x,t)$$

Since $u_\omega(x,t)$ can not be identically zero (trivial solution), then the above simplifies to

$$-i\omega = -\nu k^2$$

Or

$$\boxed{k^2 = \frac{i\omega}{\nu}} \tag{4}$$

Writing

$$\omega = \operatorname{sgn}(\omega)|\omega|$$

Where

$$\operatorname{sgn}(\omega) = \begin{cases} +1 & \omega > 0 \\ 0 & \omega = 0 \\ -1 & \omega < 0 \end{cases}$$

Then (4) becomes

$$k^2 = \frac{i \ \operatorname{sgn}(\omega)|\omega|}{\nu}$$
$$k = \pm\frac{\sqrt{i}\ \sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{\nu}}$$

Since

$$\sqrt{i} = i^{\frac{1}{2}} = \left(e^{i\frac{\pi}{2}}\right)^{\frac{1}{2}} = e^{i\frac{\pi}{4}}$$

Hence k can be written as

$$k = \pm\frac{e^{i\frac{\pi}{4}}\ \sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{\nu}}$$

case A. Let start with the positive root hence

$$k = \frac{e^{i\frac{\pi}{4}}\ \sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{\nu}}$$

Case (A1) $\omega < 0$ then the above becomes

$$\begin{aligned}
k &= \frac{ie^{i\frac{\pi}{4}}\ \sqrt{|\omega|}}{\sqrt{\nu}} = \frac{e^{i\frac{\pi}{2}}e^{i\frac{\pi}{4}}\ \sqrt{|\omega|}}{\sqrt{\nu}} = \frac{e^{i\frac{3\pi}{4}}\ \sqrt{|\omega|}}{\sqrt{\nu}}\\
&= \left(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi\right)\frac{\sqrt{|\omega|}}{\sqrt{\nu}}\\
&= \left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\frac{\sqrt{|\omega|}}{\sqrt{\nu}}\\
&= \left(-\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}} + i\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}}\right)
\end{aligned}$$

And the solution becomes

$$\begin{aligned}
u(x,t) &= \exp(ikx)\exp(-i\omega t)\\
&= \exp\left(i\left(-\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}} + i\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}}\right)x\right)\exp(-i\omega t)\\
&= \exp\left(\left(-\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}}i - \frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}}\right)x\right)\exp(-i\omega t)\\
&= \exp\left(-\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}}x\right)\exp\left(-\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}}ix\right)\exp(-i\omega t)
\end{aligned}$$

We are told that $u(\infty,t)$ is bounded. So for large x we want the above to be bounded. The complex exponential present in the above expression cause no issue for large x since they are oscillatory trig functions. We then just need to worry about $\exp\left(-\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}}x\right)$ for large x. This term will decay for large x since $-\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}}$ is negative (assuming $\nu > 0$ always). Hence positive root hence worked OK when $\omega < 0$. Now we check if it works OK also when $\omega > 0$

case A2 When $\omega > 0$ then k now becomes

$$\begin{aligned}
k &= \frac{e^{i\frac{\pi}{4}}\ \sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{\nu}}\\
&= \frac{e^{i\frac{\pi}{4}}\ \sqrt{\omega}}{\sqrt{\nu}}\\
&= \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\frac{\sqrt{\omega}}{\sqrt{\nu}}\\
&= \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\frac{\sqrt{\omega}}{\sqrt{\nu}}\\
&= \frac{1}{\sqrt{2}}\frac{\sqrt{\omega}}{\sqrt{\nu}} + i\frac{1}{\sqrt{2}}\frac{\sqrt{\omega}}{\sqrt{\nu}}
\end{aligned}$$

And the solution becomes

$$\begin{aligned}
u(x,t) &= \exp(ikx)\exp(-i\omega t)\\
&= \exp\left(i\left(\frac{1}{\sqrt{2}}\frac{\sqrt{\omega}}{\sqrt{\nu}} + i\frac{1}{\sqrt{2}}\frac{\sqrt{\omega}}{\sqrt{\nu}}\right)x\right)\exp(-i\omega t)\\
&= \exp\left(\left(i\frac{1}{\sqrt{2}}\frac{\sqrt{\omega}}{\sqrt{\nu}} - \frac{1}{\sqrt{2}}\frac{\sqrt{\omega}}{\sqrt{\nu}}\right)x\right)\exp(-i\omega t)\\
&= \exp\left(-\frac{1}{\sqrt{2}}\frac{\sqrt{\omega}}{\sqrt{\nu}}x\right)\exp\left(i\frac{1}{\sqrt{2}}\frac{\sqrt{\omega}}{\sqrt{\nu}}x\right)\exp(-i\omega t)
\end{aligned}$$

We are told that $u(\infty,t)$ is bounded. So for large x we want the above to be bounded. The complex exponential present in the above expression cause no issue for large x since they are oscillatory trig functions. We then just need to worry about $\exp\left(-\frac{1}{\sqrt{2}}\frac{\sqrt{\omega}}{\sqrt{\nu}}x\right)$ for large x. This term will decay for large x since $-\frac{1}{\sqrt{2}}\frac{\sqrt{\omega}}{\sqrt{\nu}}$ is negative (assuming $\nu > 0$ always). Hence positive root hence worked OK when $\omega > 0$ as well.

Let check what happens if we use the negative root.

case B. negative root hence

$$k = -\frac{e^{i\frac{\pi}{4}}\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{\nu}}$$

Case (A1) $\omega < 0$ then the above becomes

$$\begin{aligned}
k &= -\frac{ie^{i\frac{\pi}{4}}\sqrt{|\omega|}}{\sqrt{\nu}} = -\frac{e^{i\frac{\pi}{2}}e^{i\frac{\pi}{4}}\sqrt{|\omega|}}{\sqrt{\nu}} = -\frac{e^{i\frac{3\pi}{4}}\sqrt{|\omega|}}{\sqrt{\nu}}\\
&= -\left(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi\right)\frac{\sqrt{|\omega|}}{\sqrt{\nu}}\\
&= -\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\frac{\sqrt{|\omega|}}{\sqrt{\nu}}\\
&= -\left(-\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}} + i\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}}\right)\\
&= \left(\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}} - i\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}}\right)
\end{aligned}$$

And the solution becomes

$$\begin{aligned}
u(x,t) &= \exp(ikx)\exp(-i\omega t)\\
&= \exp\left(i\left(\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}} - i\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}}\right)x\right)\exp(-i\omega t)\\
&= \exp\left(\left(i\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}} + \frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}}\right)x\right)\exp(-i\omega t)\\
&= \exp\left(+\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}}x\right)\exp\left(\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}}ix\right)\exp(-i\omega t)
\end{aligned}$$

We are told that $u(\infty,t)$ is bounded. So for large x we want the above to be bounded. The complex exponential present in the above expression cause no issue for large x since they are oscillatory trig functions. We then just need to worry about $\exp\left(+\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}}x\right)$ for large x. This term will blow up for large x since $+\frac{1}{\sqrt{2}}\frac{\sqrt{|\omega|}}{\sqrt{\nu}}$ is positive (assuming $\nu > 0$ always). Hence we reject the case of negative sign on k. And pick

$$\begin{aligned}
k &= \frac{e^{i\frac{\pi}{4}}\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{\nu}}\\
&= \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\frac{\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{\nu}}
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
u(x,t;\omega) &= \exp(ikx)\exp(-i\omega t)\\
&= \exp\left(i\left(\frac{\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{\nu}}\right)x\right)\exp(-i\omega t)\\
&= \exp\left(i\left(\frac{1}{\sqrt{2\nu}}\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}+i\frac{1}{\sqrt{2\nu}}\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}\right)x\right)\exp(-i\omega t)\\
&= \exp\left(i\frac{1}{\sqrt{2\nu}}\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}x-\frac{1}{\sqrt{2\nu}}\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}\,x\right)\exp(-i\omega t)
\end{aligned}$$

Hence

$$\boxed{u(x,t;\omega)=\exp\left(\frac{-\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{2\nu}}x\right)\exp\left(i\frac{\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{2\nu}}x\right)\exp(-i\omega t)}$$

The general solution $u(x,t)$ is therefore the integral over all ω, hence

$$\begin{aligned}
u(x,t) &= \int_{\omega=-\infty}^{\infty} u(x,t;\omega)\;d\omega\\
&= \int_{-\infty}^{\infty}\exp\left(-\frac{\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{2\nu}}x\right)\exp\left(\frac{i\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{2\nu}}x\right)\exp(-i\omega t)\;d\omega\\
&= \int_{-\infty}^{\infty}\exp\left(-\frac{\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{2\nu}}x\right)\exp\left(i\left[\frac{\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{2\nu}}x-\omega t\right]\right)\;d\omega\\
&= \int_{-\infty}^{\infty}\exp\left(-\frac{\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{2\nu}}x\right)\exp\left(ix\left[\frac{\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{2\nu}}-\omega\frac{t}{x}\right]\right)\;d\omega
\end{aligned}$$

0.4.2 Part (b)

From part (a), we found that

$$\boxed{u(x,t;\omega)=\exp\left(-\frac{\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{2\nu}}x\right)\exp\left(i\frac{\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{2\nu}}x\right)\exp(-i\omega t)} \tag{1}$$

comparing the above to expression given in problem which is

$$u(x,t)=\exp(-\sigma x)\;\exp(i\sigma\operatorname{sgn}(\omega)x)\;\exp(-i\omega t) \tag{2}$$

Therefore, by comparing $\exp(-\sigma x)$ to $\exp\left(-\frac{\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{2\nu}}x\right)$ we see that

$$\boxed{\sigma=\frac{\sqrt{\operatorname{sgn}(\omega)}\sqrt{|\omega|}}{\sqrt{2\nu}}} \tag{3}$$

0.4.3 part (c)

Using the solution found in part (a)

$$u(x,t;\omega)=\exp(-\sigma x)\;\exp(i\sigma\operatorname{sgn}(\omega)x)\;\exp(-i\omega t)$$

To find numerical estimate, assuming $\omega>0$ for now

$$\begin{aligned}
u(x,t;\omega) &= \exp(-\sigma x)\;\exp(i\sigma x)\;\exp(-i\omega t)\\
&= e^{-\sigma x}(\cos\sigma x+i\sin\sigma x)(\cos\omega t-i\sin\omega t)\\
&= e^{-\sigma x}(\cos\sigma x\cos\omega t-i\sin\omega t\cos\sigma x+i\cos\omega t\sin\sigma x+\sin\sigma x\sin\omega t)\\
&= e^{-\sigma x}(\cos(\sigma x)\cos(\omega t)+\sin(\sigma x)\sin(\omega t)+i(\cos(\omega t)\sin(\sigma x)-\sin(\omega t)\cos(\sigma x)))\\
&= e^{-\sigma x}(\cos(t\omega-x\sigma)-i\sin(t\omega-x\sigma))
\end{aligned}$$

Hence will evaluate

$$\begin{aligned}
\operatorname{Re}(u(x,t;\omega)) &= e^{-\sigma x}\operatorname{Re}(\cos(t\omega-x\sigma)-i\sin(t\omega-x\sigma))\\
&= e^{-\sigma x}\cos(t\omega-x\sigma)
\end{aligned}$$

I assume here it is asking for numerical estimate. We only need to determine numerical estimate for σ. For ω, using the period $T=24$ hrs or $T=86400$ seconds, then $\omega=\frac{2\pi}{T}$ is now found. Then we need to determine ν, which is thermal diffusivity for earth crust. There does not seem to be an agreed on value for this and this value also changed with depth inside the earth crust. The value I found that seem mentioned more is 1.2×10^{-6} meter^2 per second. Hence

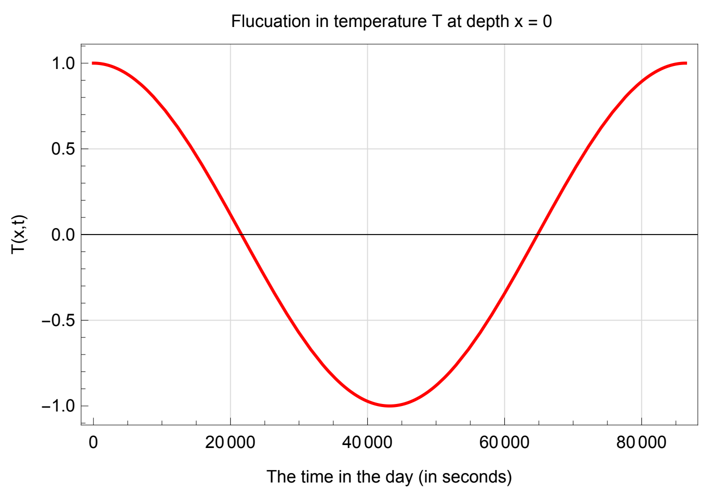
$$\sigma = \frac{\sqrt{\frac{2\pi}{86\,400}}}{\sqrt{2\left(1.2 \times 10^{-6}\right)}}$$
$$= 5.505 \text{ per meter}$$

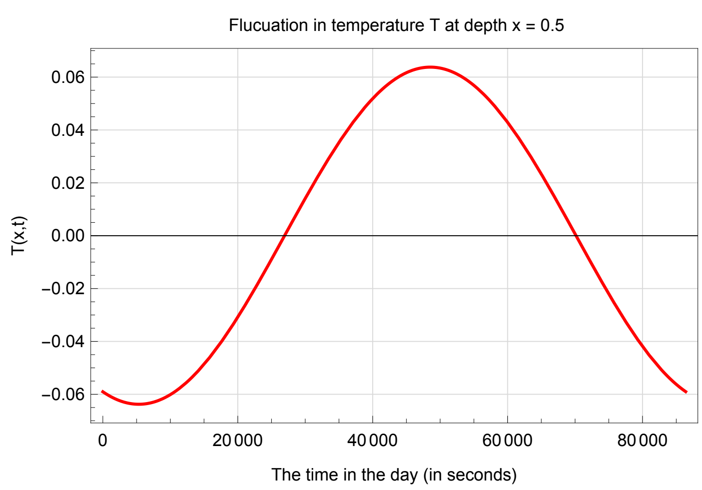
Therefore

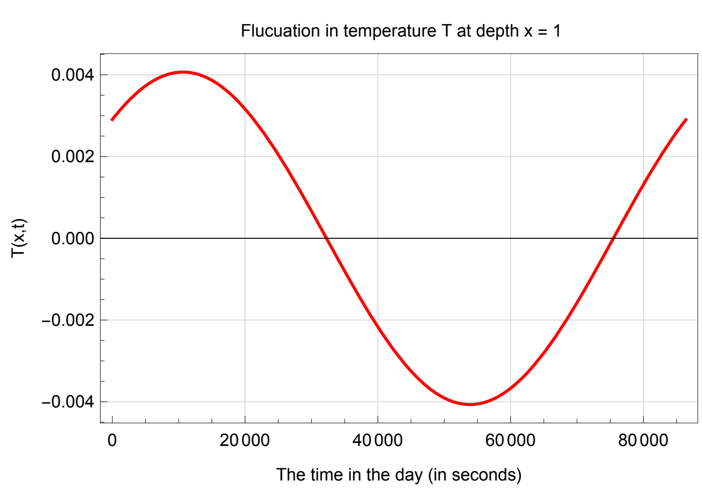
$$\operatorname{Re}\left(u\left(x,t;\omega\right)\right) = e^{-5.505x}\cos\left(t\omega - 5.505x\right)$$

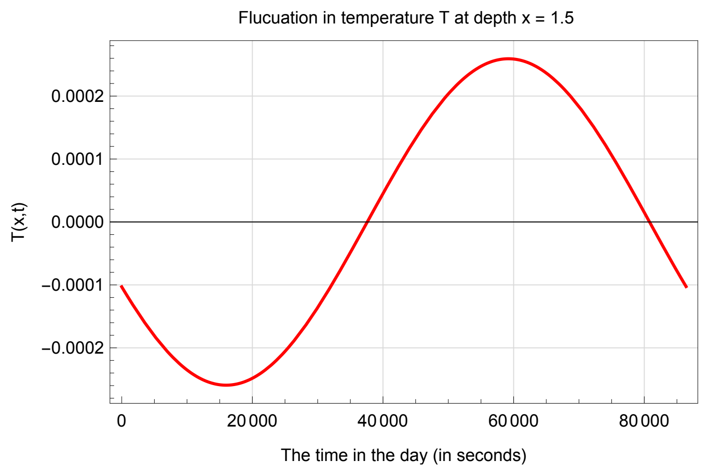
Using $\omega = \frac{2\pi}{86\,400} = 7.29 \times 10^{-5}$ rad/sec. Now we can use the above to estimate fluctuation of heat over 24 hrs period. But we need to fix x for each case. Here $x = 0$ means on the earth surface and x say 10, means at depth 10 meters and so on as I understand that x is starts at 0 at surface or earth and increases as we go lower into the earth crust. Plotting at the above for $x = 0, 0.5, 1, 1.5, 2$ I see that when $x > 2$ then maximum value of $e^{-5.505x}\cos\left(t\omega - 5.505x\right)$ is almost zero. This seems to indicate a range of heat reach is about little more than 2 meters below the surface of earth.

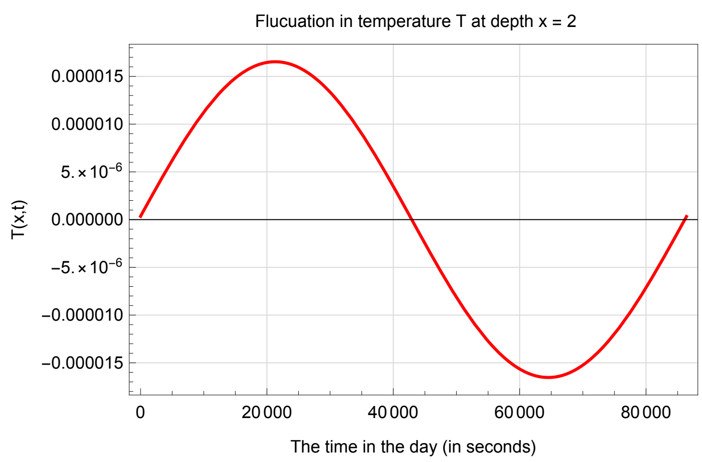
This is a plot of the fluctuation in temperature at different x each in separate plot, then later a plot is given that combines them all.

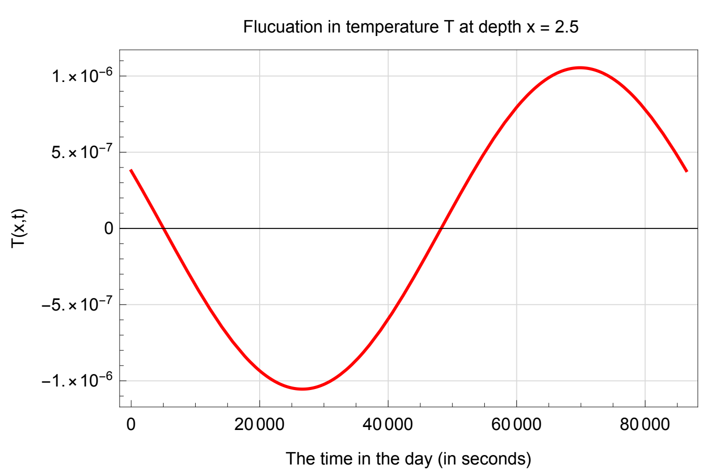




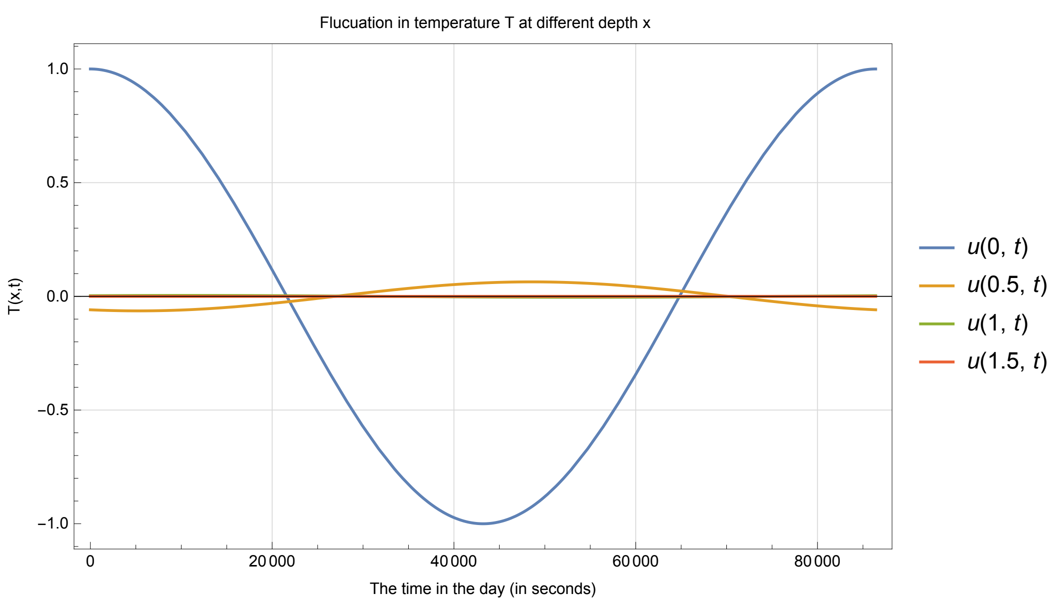








This plot better show the difference per depth, as it combines all the plots into one.


Flucuation in temperature T at different depth x
1.0
0.5
0.0
−0.5
−1.0
T(x,t)
0
20000
40000
60000
80000
The time in the day (in seconds)
u(0, t)
u(0.5, t)
u(1, t)
u(1.5, t)