Exam 1, NE 548, Spring 2017

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Contents

0.1 problem 3.26 (page 139)

Problem Perform local analysis solution to $(x-1)y'' - xy' + y = 0$ at $x = 1$. Use the result of this analysis to prove that a Taylor series expansion of any solution about $x = 0$ has an infinite radius of convergence. Find the exact solution by summing the series.

solution

Writing the ODE in standard form

$$y''(x) + a(x)y'(x) + b(x)y(x) = 0 \tag{1}$$

$$y'' - \frac{x}{(x-1)}y' + \frac{1}{(x-1)}y = 0 \tag{2}$$

Where $a(x) = \frac{-x}{(x-1)}, b(x) = \frac{1}{(x-1)}$. The above shows that $x = 1$ is singular point for both $a(x)$ and $b(x)$. The next step is to classify the type of the singular point. Is it regular singular point or irregular singular point?

$$\begin{aligned}\lim_{x\to 1}(x-1)a(x) &= \lim_{x\to 1}(x-1)\frac{-x}{(x-1)}\\ &= -1\end{aligned}$$

And

$$\begin{aligned}\lim_{x\to 1}(x-1)^2 b(x) &= \lim_{x\to 1}(x-1)^2\frac{1}{(x-1)}\\ &= 0\end{aligned}$$

Because the limit exist, then $x = 1$ is a regular singular point. Therefore solution is assumed to be a Frobenius power series given by

$$y(x) = \sum_{n=0}^{\infty} a_n (x-1)^{n+r}$$

Substituting this in the original ODE $(x-1)y'' - xy' + y = 0$ gives

$$\begin{aligned}y'(x) &= \sum_{n=0}^{\infty}(n+r)a_n(x-1)^{n+r-1}\\ y''(x) &= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n(x-1)^{n+r-2}\end{aligned}$$

In order to move the $(x-1)$ inside the summation, the original ODE $(x-1)y'' - xy' + y = 0$ is first rewritten as

$$(x-1)y'' - (x-1)y' - y' + y = 0 \tag{3}$$

Substituting the Frobenius series into the above gives

$$\begin{aligned}(x-1)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n(x-1)^{n+r-2}\\ -(x-1)\sum_{n=0}^{\infty}(n+r)a_n(x-1)^{n+r-1}\\ -\sum_{n=0}^{\infty}(n+r)a_n(x-1)^{n+r-1}\\ +\sum_{n=0}^{\infty}a_n(x-1)^{n+r} = 0\end{aligned}$$

Or

$$\begin{aligned}\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n(x-1)^{n+r-1}\\ -\sum_{n=0}^{\infty}(n+r)a_n(x-1)^{n+r}\\ -\sum_{n=0}^{\infty}(n+r)a_n(x-1)^{n+r-1}\\ +\sum_{n=0}^{\infty}a_n(x-1)^{n+r} = 0\end{aligned}$$

Adjusting all powers of $(x-1)$ to be the same by rewriting exponents and summation indices gives

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n(x-1)^{n+r-1} - \sum_{n=1}^{\infty}(n+r-1)a_{n-1}(x-1)^{n+r-1} - \sum_{n=0}^{\infty}(n+r)a_n(x-1)^{n+r-1} + \sum_{n=1}^{\infty}a_{n-1}(x-1)^{n+r-1} = 0$$

Collecting terms with same powers in $(x-1)$ simplifies the above to

$$\sum_{n=0}^{\infty}((n+r)(n+r-1)-(n+r))a_n(x-1)^{n+r-1} - \sum_{n=1}^{\infty}(n+r-2)a_{n-1}(x-1)^{n+r-1} = 0 \tag{4}$$

Setting $n=0$ gives the indicial equation

$$((n+r)(n+r-1)-(n+r))a_0 = 0$$
$$((r)(r-1)-r)a_0 = 0$$

Since $a_0 \neq 0$ then the indicial equation is

$$(r)(r-1)-r = 0$$
$$r^2-2r = 0$$
$$r(r-2) = 0$$

The roots of the indicial equation are therefore

$$r_1 = 2$$
$$r_2 = 0$$

Each one of these roots generates a solution to the ODE. The next step is to find the solution $y_1(x)$ associated with $r=2$. (The largest root is used first). Using $r=2$ in equation (4) gives

$$\sum_{n=0}^{\infty}((n+2)(n+1)-(n+2))a_n(x-1)^{n+1} - \sum_{n=1}^{\infty}na_{n-1}(x-1)^{n+1} = 0$$
$$\sum_{n=0}^{\infty}n(n+2)a_n(x-1)^{n+1} - \sum_{n=1}^{\infty}na_{n-1}(x-1)^{n+1} = 0 \tag{5}$$

At $n \geq 1$, the recursive relation is found and used to generate the coefficients of the Frobenius power series

$$n(n+2)a_n - na_{n-1} = 0$$
$$a_n = \frac{n}{n(n+2)}a_{n-1}$$

Few terms are now generated to see the pattern of the series and to determine the closed form. For $n=1$

$$a_1 = \frac{1}{3}a_0$$

For $n=2$

$$a_2 = \frac{2}{2(2+2)}a_1 = \frac{2}{8}\frac{1}{3}a_0 = \frac{1}{12}a_0$$

For $n=3$

$$a_3 = \frac{3}{3(3+2)}a_2 = \frac{3}{15}\frac{1}{12}a_0 = \frac{1}{60}a_0$$

For $n=4$

$$a_4 = \frac{4}{4(4+2)}a_3 = \frac{1}{6}\frac{1}{60}a_0 = \frac{1}{360}a_0$$

And so on. From the above, the first solution becomes

$$
\begin{aligned}
y_1(x) &= \sum_{n=0}^{\infty} a_n (x-1)^{n+2} \\
&= a_0 (x-1)^2 + a_1 (x-1)^3 + a_2 (x-1)^4 + a_3 (x-1)^4 + a_4 (x-1)^5 + \cdots \\
&= (x-1)^2 \left(a_0 + a_1 (x-1) + a_2 (x-1)^2 + a_3 (x-1)^3 + a_4 (x-1)^4 + \cdots\right) \\
&= (x-1)^2 \left(a_0 + \frac{1}{3} a_0 (x-1) + \frac{1}{12} a_0 (x-1)^2 + \frac{1}{60} a_0 (x-1)^3 + \frac{1}{360} a_0 (x-1)^4 + \cdots\right) \\
&= a_0 (x-1)^2 \left(1 + \frac{1}{3}(x-1) + \frac{1}{12}(x-1)^2 + \frac{1}{60}(x-1)^3 + \frac{1}{360}(x-1)^4 + \cdots\right) \qquad (6)
\end{aligned}
$$

To find closed form solution to $y_1(x)$, Taylor series expansion of e^x around $x = 1$ is found first

$$
\begin{aligned}
e^x &\approx e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{3!}(x-1)^3 + \frac{e}{4!}(x-1)^4 + \frac{e}{5!}(x-1)^5 + \cdots \\
&\approx e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3 + \frac{e}{24}(x-1)^4 + \frac{e}{120}(x-1)^5 + \cdots \\
&\approx e\left(1 + (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{24}(x-1)^4 + \frac{1}{120}(x-1)^5 + \cdots\right)
\end{aligned}
$$

Multiplying the above by 2 gives

$$
2e^x \approx e\left(2 + 2(x-1) + (x-1)^2 + \frac{1}{3}(x-1)^3 + \frac{1}{12}(x-1)^4 + \frac{1}{60}(x-1)^5 + \cdots\right)
$$

Factoring $(x-1)^2$ from the RHS results in

$$
2e^x \approx e\left(2 + 2(x-1) + (x-1)^2 \left(1 + \frac{1}{3}(x-1) + \frac{1}{12}(x-1)^2 + \frac{1}{60}(x-1)^3 + \cdots\right)\right) \qquad (6A)
$$

Comparing the above result with the solution $y_1(x)$ in (6), shows that the (6A) can be written in terms of $y_1(x)$ as

$$
2e^x = e\left(2 + 2(x-1) + (x-1)^2 \left(\frac{y_1(x)}{a_0 (x-1)^2}\right)\right)
$$

Therefore

$$
\begin{aligned}
2e^x &= e\left(2 + 2(x-1) + \frac{y_1(x)}{a_0}\right) \\
2e^{x-1} &= 2 + 2(x-1) + \frac{y_1(x)}{a_0} \\
2e^{x-1} - 2 - 2(x-1) &= \frac{y_1(x)}{a_0}
\end{aligned}
$$

Solving for $y_1(x)$

$$
\begin{aligned}
y_1(x) &= a_0 \left(2e^{x-1} - 2 - 2(x-1)\right) \\
&= a_0 \left(2e^{x-1} - 2 - 2x + 2\right) \\
&= a_0 \left(2e^{x-1} - 2x\right) \\
&= \frac{2a_0}{e} e^x - 2a_0 x
\end{aligned}
$$

Let $\frac{2a_0}{e} = C_1$ and $-2a_0 = C_2$, then the above solution can be written as

$$
y_1(x) = C_1 e^x + C_2 x
$$

Now that $y_1(x)$ is found, which is the solution associated with $r = 2$, the next step is to find the second solution $y_2(x)$ associated with $r = 0$. Since $r_2 - r_1 = 2$ is an integer, the solution can be either case $II(b)(i)$ or case $II(b)(ii)$ as given in the text book at page 72.

From equation (3.3.9) at page 72 of the text, using $N = 2$ since $N = r_2 - r_1$ and where $p(x) = -x$ and $q(x) = 1$ in this problem by comparing our ODE with the standard ODE in (3.3.2) at page 70 given by

$$
y'' + \frac{p(x)}{(x - x_0)} y' + \frac{q(x)}{(x - x_0)^2} y = 0
$$

Expanding $p(x), q(x)$ in Taylor series

$$p(x) = \sum_{n=0}^{\infty} p_n (x-1)^n$$
$$q(x) = \sum_{n=0}^{\infty} q_n (x-1)^n$$

Since $p(x) = -x$ in our ODE, then $p_0 = -1$ and $p_1 = -1$ and all other terms are zero. For $q(x)$, which is just 1 in our ODE, then $q_0 = 1$ and all other terms are zero. Hence

$$\begin{aligned} p_0 &= -1 \\ p_1 &= -1 \\ q_0 &= 1 \\ N &= 2 \\ r &= 0 \end{aligned}$$

The above values are now used to evaluate RHS of 3.3.9 in order to find which case it is. (book uses α for r)

$$0a_N = -\sum_{k=0}^{N-1} \left[(r+k) p_{N-k} + q_{N-k} \right] a_k \tag{3.3.9}$$

Since $N = 2$ the above becomes

$$0a_2 = -\sum_{k=0}^{1} \left[(r+k) p_{2-k} + q_{2-k} \right] a_k$$

Using $r = 0$, since this is the second root, gives

$$\begin{aligned} 0a_2 &= -\sum_{k=0}^{1} \left(kp_{2-k} + q_{2-k} \right) a_k \\ &= -\left(\left(0p_{2-0} + q_{2-0} \right) a_0 + \left(p_{2-1} + q_{2-1} \right) a_1 \right) \\ &= -\left(\left(0p_2 + q_2 \right) a_0 + \left(p_1 + q_1 \right) a_1 \right) \\ &= -\left(0 + q_2 \right) a_0 - \left(p_1 + q_1 \right) a_1 \end{aligned}$$

Since $q_2 = 0, p_1 = -1, q_1 = 1$, therefore

$$\begin{aligned} 0a_2 &= -(0+0) a_0 - (-1+1) a_1 \\ &= 0 \end{aligned}$$

The above shows that this is case $II(b)(ii)$, because the right side of 3.3.9 is zero. This means the second solution $y_2(x)$ is also a Fronbenius series. If the above was not zero, the method of reduction of order would be used to find second solution.

Assuming $y_2(x) = \sum b_n (x-1)^{n+r}$, and since $r = 0$, therefore

$$y_2(x) = \sum_{n=0}^{\infty} b_n (x-1)^n$$

Following the same method used to find the first solution, this series is now used in the ODE to determine b_n.

$$y_2'(x) = \sum_{n=0}^{\infty} nb_n (x-1)^{n-1} = \sum_{n=1}^{\infty} nb_n (x-1)^{n-1} = \sum_{n=0}^{\infty} (n+1) b_{n+1} (x-1)^n$$
$$y_2''(x) = \sum_{n=0}^{\infty} n(n+1) b_{n+1} (x-1)^{n-1} = \sum_{n=1}^{\infty} n(n+1) b_{n+1} (x-1)^{n-1} = \sum_{n=0}^{\infty} (n+1)(n+2) b_{n+2} (x-1)^n$$

The ODE $(x-1)\,y'' - (x-1)\,y' - y' + y = 0$ now becomes

$$
\begin{aligned}
(x-1)\sum_{n=0}^{\infty}(n+1)\,(n+2)\,b_{n+2}\,(x-1)^{n} \\
-(x-1)\sum_{n=0}^{\infty}(n+1)\,b_{n+1}\,(x-1)^{n} \\
-\sum_{n=0}^{\infty}(n+1)\,b_{n+1}\,(x-1)^{n} \\
+\sum_{n=0}^{\infty}b_{n}\,(x-1)^{n} = 0
\end{aligned}
$$

Or

$$
\begin{aligned}
\sum_{n=0}^{\infty}(n+1)\,(n+2)\,b_{n+2}\,(x-1)^{n+1} \\
-\sum_{n=0}^{\infty}(n+1)\,b_{n+1}\,(x-1)^{n+1} \\
-\sum_{n=0}^{\infty}(n+1)\,b_{n+1}\,(x-1)^{n} \\
+\sum_{n=0}^{\infty}b_{n}\,(x-1)^{n} = 0
\end{aligned}
$$

Hence

$$
\sum_{n=1}^{\infty}(n)\,(n+1)\,b_{n+1}\,(x-1)^{n} - \sum_{n=1}^{\infty}nb_{n}\,(x-1)^{n} - \sum_{n=0}^{\infty}(n+1)\,b_{n+1}\,(x-1)^{n} + \sum_{n=0}^{\infty}b_{n}\,(x-1)^{n} = 0
$$

$n = 0$ gives

$$
\begin{aligned}
-(n+1)\,b_{n+1} + b_{n} &= 0 \\
-b_{1} + b_{0} &= 0 \\
b_{1} &= b_{0}
\end{aligned}
$$

$n \geq 1$ generates the recursive relation to find all remaining b_n coefficients

$$
\begin{aligned}
(n)\,(n+1)\,b_{n+1} - nb_{n} - (n+1)\,b_{n+1} + b_{n} &= 0 \\
(n)\,(n+1)\,b_{n+1} - (n+1)\,b_{n+1} &= nb_{n} - b_{n} \\
b_{n+1}\,((n)\,(n+1) - (n+1)) &= b_{n}\,(n-1) \\
b_{n+1} &= b_{n}\frac{(n-1)}{(n)\,(n+1) - (n+1)}
\end{aligned}
$$

Therefore the recursive relation is

$$
b_{n+1} = \frac{b_{n}}{n+1}
$$

Few terms are generated to see the pattern and to find the closed form solution for $y_2\,(x)$.
For $n = 1$

$$
b_{2} = b_{1}\frac{1}{2} = \frac{1}{2}b_{0}
$$

For $n = 2$

$$
b_{3} = \frac{b_{2}}{3} = \frac{1}{3}\frac{1}{2}b_{0} = \frac{1}{6}b_{0}
$$

For $n = 3$

$$
b_{4} = \frac{b_{3}}{3+1} = \frac{1}{4}\frac{1}{6}b_{0} = \frac{1}{24}b_{0}
$$

For $n = 4$

$$
b_{5} = \frac{b_{4}}{4+1} = \frac{1}{5}\frac{1}{24}b_{0} = \frac{1}{120}b_{0}
$$

And so on. Therefore, the second solution is

$$\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} b_n (x-1)^n \\
&= b_0 + b_1(x-1) + b_2(x-1)^2 + \cdots \\
&= b_0 + b_0(x-1) + \frac{1}{2}b_0(x-1)^2 + \frac{1}{6}b_0(x-1)^3 + \frac{1}{24}b_0(x-1)^4 + \frac{1}{120}b_0(x-1)^5 + \cdots \\
&= b_0\left(1 + (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{24}(x-1)^4 + \frac{1}{120}(x-1)^5 + \cdots\right) \qquad (7A)
\end{aligned}$$

The Taylor series for e^x around $x = 1$ is

$$\begin{aligned}
e^x &\approx e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3 + \frac{e}{24}(x-1)^4 + \frac{e}{120}(x-1)^5 + \cdots \\
&\approx e\left(1 + (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{24}(x-1)^4 + \frac{1}{120}(x-1)^5 + \cdots\right) \qquad (7B)
\end{aligned}$$

Comparing (7A) with (7B) shows that the second solution closed form is

$$y_2(x) = b_0 \frac{e^x}{e}$$

Let $\frac{b_0}{e}$ be some constant, say C_3, the second solution above becomes

$$y_2(x) = C_3 e^x$$

Both solutions $y_1(x), y_2(x)$ have now been found. The final solution is

$$\begin{aligned}
y(x) &= y_1(x) + y_2(x) \\
&= \overbrace{C_1 e^x + C_2 x}^{y_1(x)} + \overbrace{C_3 e^x}^{y_2(x)} \\
&= C_4 e^x + C_2 x
\end{aligned}$$

Hence, the exact solution is

$$y(x) = Ae^x + Bx \qquad (7)$$

Where A, B are constants to be found from initial conditions if given. Above solution is now verified by substituting it back to original ODE

$$\begin{aligned}
y' &= Ae^x + B \\
y'' &= Ae^x
\end{aligned}$$

Substituting these into $(x-1)y'' - xy' + y = 0$ gives

$$\begin{aligned}
(x-1)Ae^x - x(Ae^x + B) + Ae^x + Bx &= 0 \\
xAe^x - Ae^x - xAe^x - xB + Ae^x + Bx &= 0 \\
-Ae^x - xB + Ae^x + Bx &= 0 \\
0 &= 0
\end{aligned}$$

To answer the final part of the question, the above solution (7) is analytic around $x = 0$ with infinite radius of convergence since $\exp(\cdot)$ is analytic everywhere. Writing the solution as

$$y(x) = \left(A \sum_{n=0}^{\infty} \frac{x^n}{n!}\right) + Bx$$

The function x have infinite radius of convergence, since it is its own series. And the exponential function has infinite radius of convergence as known, verified by using standard ratio test

$$A \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = A \lim_{n\to\infty} \left|\frac{x^{n+1} n!}{(n+1)! x^n}\right| = A \lim_{n\to\infty} \left|\frac{x n!}{(n+1)!}\right| = A \lim_{n\to\infty} \left|\frac{x}{n+1}\right| = 0$$

For any x. Since the ratio is less than 1, then the solution $y(x)$ expanded around $x = 0$ has an infinite radius of convergence.

0.2 problem 9.8 (page 480)

Problem Use boundary layer to find uniform approximation with error of order $O\left(\varepsilon^2\right)$ for the problem $\varepsilon y'' + y' + y = 0$ with $y(0) = e, y(1) = 1$. Compare your solution to exact solution. Plot the solution for some values of ε.

solution

$$\varepsilon y'' + y' + y = 0 \tag{1}$$

Since $a(x) = 1 > 0$, then a boundary layer is expected at the left side, near $x = 0$. Matching will fail if this was not the case. Starting with the outer solution near $x = 1$. Let

$$y^{out}(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n(x)$$

Substituting this into (1) gives

$$\varepsilon\left(y_0'' + \varepsilon y_1'' + \varepsilon^2 y_2'' + \cdots\right) + \left(y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \cdots\right) + \left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots\right) = 0 \tag{2}$$

Collecting powers of $O\left(\varepsilon^0\right)$ results in the ODE

$$\begin{aligned} y_0' &\sim -y_0 \\ \frac{dy_0}{y_0} &\sim -dx \\ \ln|y_0| &\sim -x + C_1 \\ y_0^{out}(x) &\sim C_1 e^{-x} + O(\varepsilon) \end{aligned} \tag{3}$$

C_1 is found from boundary conditions $y(1) = 1$. Equation (3) gives

$$\begin{aligned} 1 &= C_1 e^{-1} \\ C_1 &= e \end{aligned}$$

Hence solution (3) becomes

$$y_0^{out}(x) \sim e^{1-x}$$

$y_1^{out}(x)$ is now found. Using (2) and collecting terms of $O\left(\varepsilon^1\right)$ gives the ODE

$$y_1' + y_1 \sim -y_0'' \tag{4}$$

But

$$\begin{aligned} y_0'(x) &= -e^{1-x} \\ y_0''(x) &= e^{1-x} \end{aligned}$$

Using the above in the RHS of (4) gives

$$y_1' + y_1 \sim -e^{1-x}$$

The integrating factor is e^x, hence the above becomes

$$\begin{aligned} \frac{d}{dx}\left(y_1 e^x\right) &\sim -e^x e^{1-x} \\ \frac{d}{dx}\left(y_1 e^x\right) &\sim -e \end{aligned}$$

Integrating both sides gives

$$\begin{aligned} y_1 e^x &\sim -ex + C_2 \\ y_1^{out}(x) &\sim -xe^{1-x} + C_2 e^{-x} \end{aligned} \tag{5}$$

Applying boundary conditions $y(1) = 0$ to the above gives

$$\begin{aligned} 0 &= -1 + C_2 e^{-1} \\ C_2 &= e \end{aligned}$$

Hence the solution in (5) becomes

$$\begin{aligned} y_1^{out}(x) &\sim -xe^{1-x} + e^{1-x} \\ &\sim (1-x)e^{1-x} \end{aligned}$$

Therefore the outer solution is

$$\begin{aligned} y^{out}(x) &= y_0 + \varepsilon y_1 \\ &= e^{1-x} + \varepsilon (1-x) e^{1-x} \end{aligned} \tag{6}$$

Now the boundary layer (inner) solution $y^{in}(x)$ near $x = 0$ is found. Let $\xi = \frac{x}{\varepsilon^p}$ be the inner variable. The original ODE is expressed using this new variable, and p is found. Since $\frac{dy}{dx} = \frac{dy}{d\xi}\frac{d\xi}{dx}$ then $\frac{dy}{dx} = \frac{dy}{d\xi}\varepsilon^{-p}$. The differential operator is $\frac{d}{dx} \equiv \varepsilon^{-p}\frac{d}{d\xi}$ therefore

$$\begin{aligned} \frac{d^2}{dx^2} &= \frac{d}{dx}\frac{d}{dx} \\ &= \left(\varepsilon^{-p}\frac{d}{d\xi}\right)\left(\varepsilon^{-p}\frac{d}{d\xi}\right) \\ &= \varepsilon^{-2p}\frac{d^2}{d\xi^2} \end{aligned}$$

Hence $\frac{d^2y}{dx^2} = \varepsilon^{-2p}\frac{d^2y}{d\xi^2}$ and $\varepsilon y'' + y' + y = 0$ becomes

$$\begin{aligned} \varepsilon\left(\varepsilon^{-2p}\frac{d^2y}{d\xi^2}\right) + \varepsilon^{-p}\frac{dy}{d\xi} + y &= 0 \\ \varepsilon^{1-2p}y'' + \varepsilon^{-p}y' + y &= 0 \end{aligned} \tag{7A}$$

The largest terms are $\left\{\varepsilon^{1-2p}, \varepsilon^{-p}\right\}$, balance gives $1 - 2p = -p$ or

$$\boxed{p = 1}$$

The ODE (7A) becomes

$$\varepsilon^{-1}y'' + \varepsilon^{-1}y' + y = 0 \tag{7}$$

Assuming that solution is

$$y_{in}(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$$

Substituting the above into (7) gives

$$\varepsilon^{-1}\left(y_0'' + \varepsilon y_1'' + \cdots\right) + \varepsilon^{-1}\left(y_0' + \varepsilon y_1' + \cdots\right) + \left(y_0 + \varepsilon y_1 + \cdots\right) = 0 \tag{8}$$

Collecting terms with $O\left(\varepsilon^{-1}\right)$ gives the first order ODE to solve

$$y_0'' \sim -y_0'$$

Let $z = y_0'$, the above becomes

$$\begin{aligned} z' &\sim -z \\ \frac{dz}{z} &\sim -d\xi \\ \ln|z| &\sim -\xi + C_4 \\ z &\sim C_4 e^{-\xi} \end{aligned}$$

Hence

$$y_0' \sim C_4 e^{-\xi}$$

Integrating

$$\begin{aligned} y_0^{in}(\xi) &\sim C_4 \int e^{-\xi} d\xi + C_5 \\ &\sim -C_4 e^{-\xi} + C_5 \end{aligned} \tag{9}$$

Applying boundary conditions $y(0) = e$ gives

$$\begin{aligned} e &= -C_4 + C_5 \\ C_5 &= e + C_4 \end{aligned}$$

Equation (9) becomes

$$\begin{aligned} y_0^{in}(\xi) &\sim -C_4 e^{-\xi} + e + C_4 \\ &\sim C_4\left(1 - e^{-\xi}\right) + e \end{aligned} \tag{10}$$

The next leading order $y_1^{in}(\xi)$ is found from (8) by collecting terms in $O\left(\varepsilon^0\right)$, which results in the ODE

$$y_1'' + y_1' \sim -y_0$$

Since $y_0^{in}(\xi) \sim C_4\left(1-e^{-\xi}\right)+e$, therefore $y_0' \sim C_4 e^{-\xi}$ and the above becomes

$$y_1'' + y_1' \sim -C_4 e^{-\xi}$$

The homogenous solution is found first, then method of undetermined coefficients is used to find particular solution. The homogenous ODE is

$$y_{1,h}'' \sim y_{1,h}'$$

This was solved above for y_0^{in}, and the solution is

$$y_{1,h} \sim -C_5 e^{-\xi} + C_6$$

To find the particular solution, let $y_{1,p} \sim A\xi e^{-\xi}$, where ξ was added since $e^{-\xi}$ shows up in the homogenous solution. Hence

$$\begin{aligned} y_{1,p}' &\sim Ae^{-\xi} - A\xi e^{-\xi} \\ y_{1,p}'' &\sim -Ae^{-\xi} - \left(Ae^{-\xi} - A\xi e^{-\xi}\right) \\ &\sim -2Ae^{-\xi} + A\xi e^{-\xi} \end{aligned}$$

Substituting these in the ODE $y_{1,p}'' + y_{1,p}' \sim -C_4 e^{-\xi}$ results in

$$\begin{aligned} -2Ae^{-\xi} + A\xi e^{-\xi} + Ae^{-\xi} - A\xi e^{-\xi} &\sim -C_4 e^{-\xi} \\ -A &= -C_4 \\ A &= C_4 \end{aligned}$$

Therefore the particular solution is

$$y_{1,p} \sim C_4 \xi e^{-\xi}$$

And therefore the complete solution is

$$\begin{aligned} y_1^{in}(\xi) &\sim y_{1,h} + y_{1,p} \\ &\sim -C_5 e^{-\xi} + C_6 + C_4 \xi e^{-\xi} \end{aligned}$$

Applying boundary conditions $y(0)=0$ to the above gives

$$\begin{aligned} 0 &= -C_5 + C_6 \\ C_6 &= C_5 \end{aligned}$$

Hence the solution becomes

$$\begin{aligned} y_1^{in}(\xi) &\sim -C_5 e^{-\xi} + C_5 + C_4 \xi e^{-\xi} \\ &\sim C_5\left(1-e^{-\xi}\right) + C_4 \xi e^{-\xi} \end{aligned} \tag{11}$$

The complete inner solution now becomes

$$\begin{aligned} y^{in}(\xi) &\sim y_0^{in} + \varepsilon y_1^{in} \\ &\sim C_4\left(1-e^{-\xi}\right) + e + \varepsilon\left(C_5\left(1-e^{-\xi}\right) + C_4 \xi e^{-\xi}\right) \end{aligned} \tag{12}$$

There are two constants that need to be determined in the above from matching with the outer solution.

$$\begin{aligned} \lim_{\xi\to\infty} y^{in}(\xi) &\sim \lim_{x\to 0} y^{out}(x) \\ \lim_{\xi\to\infty} C_4\left(1-e^{-\xi}\right) + e + \varepsilon\left(C_5\left(1-e^{-\xi}\right) + C_4\xi e^{-\xi}\right) &\sim \lim_{x\to 0} e^{1-x} + \varepsilon(1-x)e^{1-x} \\ C_4 + e + \varepsilon C_5 &\sim e + \varepsilon e \end{aligned}$$

The above shows that

$$\begin{aligned} C_5 &= e \\ C_4 + e &= e \\ C_4 &= 0 \end{aligned}$$

This gives the boundary layer solution $y^{in}(\xi)$ as

$$\begin{aligned} y^{in}(\xi) &\sim e + \varepsilon e\left(1-e^{-\xi}\right) \\ &\sim e\left(1+\varepsilon\left(1-e^{-\xi}\right)\right) \end{aligned}$$

In terms of x, since $\xi = \frac{x}{\varepsilon}$, the above can be written as

$$y^{in}(x) \sim e\left(1+\varepsilon\left(1-e^{-\frac{x}{\varepsilon}}\right)\right)$$

The uniform solution is therefore

$$\begin{aligned}
y_{\text{uniform}}(x) &\sim y^{in}(x) + y^{out}(x) - y_{match} \\
&\sim \overbrace{e\left(1+\varepsilon\left(1-e^{-\frac{x}{\varepsilon}}\right)\right)}^{y^{in}} + \overbrace{e^{1-x} + \varepsilon(1-x)e^{1-x}}^{y^{out}} - (e+\varepsilon e) \\
&\sim e + e\varepsilon\left(1-e^{-\frac{x}{\varepsilon}}\right) + e^{1-x} + (\varepsilon - \varepsilon x)e^{1-x} - (e+\varepsilon e) \\
&\sim e + e\varepsilon - \varepsilon e^{1-\frac{x}{\varepsilon}} + e^{1-x} + \varepsilon e^{1-x} - \varepsilon x e^{1-x} - e - \varepsilon e \\
&\sim -\varepsilon e^{1-\frac{x}{\varepsilon}} + e^{1-x} + \varepsilon e^{1-x} - \varepsilon x e^{1-x} \\
&\sim e^{1-x}\left(-\varepsilon e^{-\frac{1}{\varepsilon}} + 1 + \varepsilon - \varepsilon x\right)
\end{aligned}$$

Or

$$y_{\text{uniform}}(x) \sim e^{1-x}\left(1+\varepsilon\left(1-x-e^{-\frac{1}{\varepsilon}}\right)\right)$$

With error $O\left(\varepsilon^2\right)$.

The above solution is now compared to the exact solution of $\varepsilon y'' + y' + y = 0$ with $y(0) = e, y(1) = 1$. Since this is a homogenous second order ODE with constant coefficient, it is easily solved using characteristic equation.

$$\varepsilon\lambda^2 + \lambda + 1 = 0$$

The roots are

$$\begin{aligned}
\lambda &= \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a} \\
&= \frac{-1}{2\varepsilon} \pm \frac{\sqrt{1-4\varepsilon}}{2\varepsilon}
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y(x) &= Ae^{\lambda_1 x} + Be^{\lambda_2 x} \\
&= Ae^{\left(\frac{-1}{2\varepsilon}+\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}\right)x} + Be^{\left(\frac{-1}{2\varepsilon}-\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}\right)x} \\
&= Ae^{\frac{-x}{2\varepsilon}}e^{\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}x} + Be^{\frac{-x}{2\varepsilon}}e^{\frac{-\sqrt{1-4\varepsilon}}{2\varepsilon}x} \\
&= e^{\frac{-x}{2\varepsilon}}\left(Ae^{\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}x} + Be^{\frac{-\sqrt{1-4\varepsilon}}{2\varepsilon}x}\right)
\end{aligned}$$

Applying first boundary conditions $y(0) = e$ to the above gives

$$\begin{aligned}
e &= A + B \\
B &= e - A
\end{aligned}$$

Hence the solution becomes

$$\begin{aligned}
y(x) &= e^{\frac{-x}{2\varepsilon}}\left(Ae^{\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}x} + (e-A)e^{\frac{-\sqrt{1-4\varepsilon}}{2\varepsilon}x}\right) \\
&= e^{\frac{-x}{2\varepsilon}}\left(Ae^{\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}x} + e^{1-\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}x} - Ae^{\frac{-\sqrt{1-4\varepsilon}}{2\varepsilon}x}\right) \\
&= e^{\frac{-x}{2\varepsilon}}\left(A\left(e^{\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}x} - e^{\frac{-\sqrt{1-4\varepsilon}}{2\varepsilon}x}\right) + e^{1-\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}x}\right) \qquad (13)
\end{aligned}$$

Applying second boundary conditions $y(1) = 1$ gives

$$1 = e^{\frac{-1}{2\varepsilon}} \left(A \left(e^{\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}} - e^{\frac{-\sqrt{1-4\varepsilon}}{2\varepsilon}} \right) + e^{1-\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}} \right)$$

$$e^{\frac{1}{2\varepsilon}} = A \left(e^{\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}} - e^{\frac{-\sqrt{1-4\varepsilon}}{2\varepsilon}} \right) + e^{1-\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}}$$

$$A = \frac{e^{\frac{1}{2\varepsilon}} - e^{1-\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}}}{e^{\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}} - e^{\frac{-\sqrt{1-4\varepsilon}}{2\varepsilon}}} \tag{14}$$

Substituting this into (13) results in

$$y^{\text{exact}}(x) = e^{\frac{-x}{2\varepsilon}} \left(A \left(e^{\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}x} - e^{\frac{-\sqrt{1-4\varepsilon}}{2\varepsilon}x} \right) + e^{1-\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}x} \right)$$

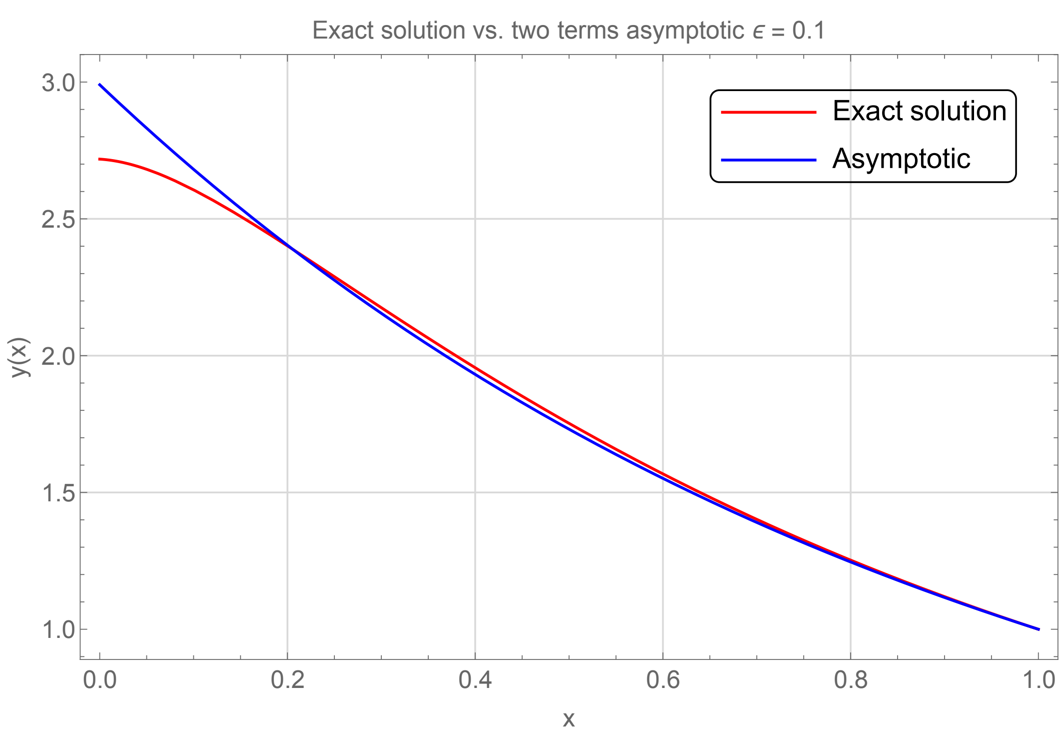
Where A is given in (14). hence

$$y^{\text{exact}}(x) = e^{\frac{-x}{2\varepsilon}} \left(\left(\frac{e^{\frac{1}{2\varepsilon}} - e^{1-\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}}}{e^{\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}} - e^{\frac{-\sqrt{1-4\varepsilon}}{2\varepsilon}}} \right) \left(e^{\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}x} - e^{\frac{-\sqrt{1-4\varepsilon}}{2\varepsilon}x} \right) + e^{1-\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}x} \right)$$

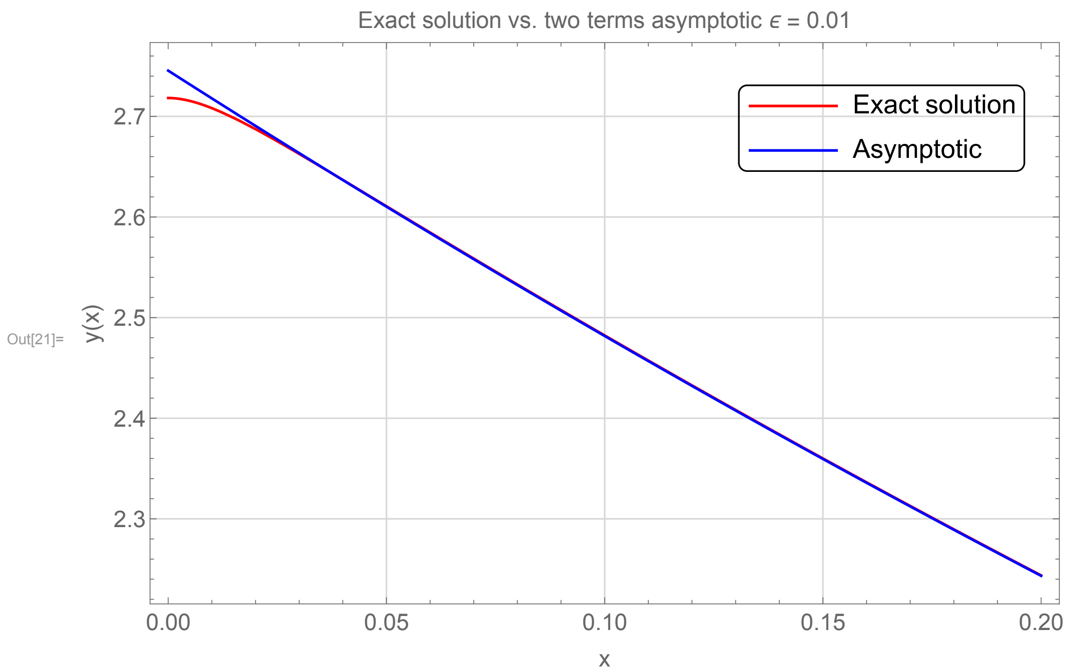
In summary

exact solution	asymptotic solution
$e^{\frac{-x}{2\varepsilon}} \left(\left(\frac{e^{\frac{1}{2\varepsilon}} - e^{1-\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}}}{e^{\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}} - e^{\frac{-\sqrt{1-4\varepsilon}}{2\varepsilon}}} \right) \left(e^{\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}x} - e^{\frac{-\sqrt{1-4\varepsilon}}{2\varepsilon}x} \right) + e^{1-\frac{\sqrt{1-4\varepsilon}}{2\varepsilon}x} \right)$	$e^{1-x} + \varepsilon e^{1-x} \left(1 - x - e^{-\frac{1}{\varepsilon}} \right) + O\left(\varepsilon^2\right)$

The following plot compares the exact solution with the asymptotic solution for $\varepsilon = 0.1$



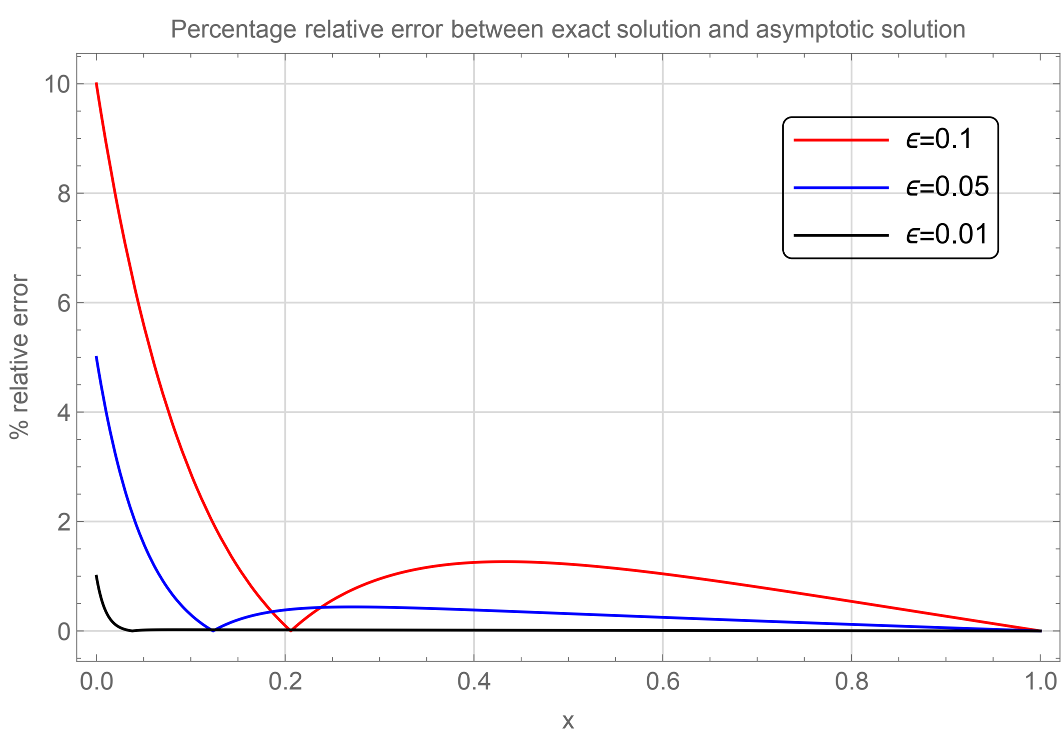
The following plot compares the exact solution with the asymptotic solution for $\varepsilon = 0.01$. The difference was too small to notice in this case, the plot below is zoomed to be near $x = 0$



At $\varepsilon = 0.001$, the difference between the exact and the asymptotic solution was not noticeable. Therefore, to better compare the solutions, the following plot shows the relative percentage error given by

$$100\left|\frac{y^{\text{exact}} - y^{\text{uniform}}}{y^{\text{exact}}}\right|\%$$

For different ε.



Some observations: The above plot shows more clearly how the difference between the exact solution and the asymptotic solution became smaller as ε became smaller. The plot also shows that the boundary layer near $x = 0$ is becoming more narrow are ε becomes smaller as expected. It also shows that the relative error is smaller in the outer region than in the boundary layer region. For example, for $\varepsilon = 0.05$, the largest percentage error in the outer region was less than 1%, while in the boundary layer, very near $x = 0$, the error grows to about 5%. Another observation is that at the matching location, the relative error goes down to zero. One also notices that the matching location drifts towards $x = 0$ as ε becomes smaller because the boundary layer is becoming more narrow. The following table summarizes these observations.

ε	% error near $x = 0$	apparent width of boundary layer
0.1	10	0.2
0.05	5	0.12
0.01	1	0.02

0.3 problem 3

Problem (a) Find physical optics approximation to the eigenvalue and eigenfunctions of the Sturm-Liouville problem are $\lambda \to \infty$

$$
\begin{aligned}
-y'' &= \lambda (\sin(x)+1)^2 y \\
y(0) &= 0 \\
y(\pi) &= 0
\end{aligned}
$$

(b) What is the integral relation necessary to make the eigenfunctions orthonormal? For some reasonable choice of scaling coefficient (give the value), plot the eigenfunctions for $n = 5, n = 20$.

(c) Estimate how large λ should be for the relative error of less than 0.1%

solution

0.3.1 Part a

Writing the ODE as

$$y'' + \lambda (\sin(x)+1)^2 y = 0$$

Let[1]

$$\boxed{\lambda = \frac{1}{\varepsilon^2}}$$

Then the given ODE becomes

$$\varepsilon^2 y''(x) + (\sin(x)+1)^2 y(x) = 0 \tag{1}$$

Physical optics approximation is obtained when $\lambda \to \infty$ which implies $\varepsilon \to 0^+$. Since the ODE is linear and the highest derivative is now multiplied by a very small parameter ε, WKB can therefore be used to solve it. WKB starts by assuming that the solution has the form

$$y(x) \sim \exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right) \qquad \delta \to 0$$

Therefore, taking derivatives and substituting back in the ODE results in

$$
\begin{aligned}
y'(x) &\sim \exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right)\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n'(x)\right) \\
y''(x) &\sim \exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right)\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n'(x)\right)^2 + \exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right)\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n''(x)\right)
\end{aligned}
$$

Substituting these into (1) and canceling the exponential terms gives

$$
\begin{aligned}
\varepsilon^2\left(\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n'(x)\right)^2 + \frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n''(x)\right) &\sim -(\sin(x)+1)^2 \\
\frac{\varepsilon^2}{\delta^2}\left(S_0' + \delta S_1' + \cdots\right)\left(S_0' + \delta S_1' + \cdots\right) + \frac{\varepsilon^2}{\delta}\left(S_0'' + \delta S_1'' + \cdots\right) &\sim -(\sin(x)+1)^2 \\
\frac{\varepsilon^2}{\delta^2}\left(\left(S_0'\right)^2 + \delta\left(2S_1'S_0'\right) + \cdots\right) + \frac{\varepsilon^2}{\delta}\left(S_0'' + \delta S_1'' + \cdots\right) &\sim -(\sin(x)+1)^2 \\
\left(\frac{\varepsilon^2}{\delta^2}\left(S_0'\right)^2 + \frac{2\varepsilon^2}{\delta}S_1'S_0' + \cdots\right) + \left(\frac{\varepsilon^2}{\delta}S_0'' + \varepsilon^2 S_1'' + \cdots\right) &\sim -(\sin(x)+1)^2 \qquad (2)
\end{aligned}
$$

The largest term in the left side is $\frac{\varepsilon^2}{\delta^2}\left(S_0'\right)^2$. By dominant balance, this term has the same order of magnitude as the right side $-(\sin(x)+1)^2$. This implies that δ^2 is proportional to ε^2. For simplicity (following the book) δ can be taken as equal to ε

$$\delta = \varepsilon$$

Using the above in equation (2) results in

$$\left(\left(S_0'\right)^2 + 2\varepsilon S_1'S_0' + \cdots\right) + \left(\varepsilon S_0'' + \varepsilon^2 S_1'' + \cdots\right) \sim -(\sin(x)+1)^2$$

[1] $\lambda = \frac{1}{\varepsilon}$ could also be used. But the book uses ε^2.

Balance of $O(1)$ gives

$$\left(S_0'\right)^2 \sim -(\sin(x)+1)^2 \tag{3}$$

Balance of $O(\varepsilon)$ gives

$$2S_1'S_0' \sim -S_0'' \tag{4}$$

Equation (3) is solved first in order to find $S_0(x)$.

$$S_0' \sim \pm i(\sin(x)+1)$$

Hence

$$\begin{aligned} S_0(x) &\sim \pm i\int_0^x (\sin(t)+1)\,dt + C^{\pm} \\ &\sim \pm i\,(t-\cos(t))_0^x + C^{\pm} \\ &\sim \pm i\,(1+x-\cos(x)) + C^{\pm} \end{aligned} \tag{5}$$

$S_1(x)$ is now found from (4) and using $S_0'' = \pm i\cos(x)$ gives

$$\begin{aligned} S_1' &\sim -\frac{1}{2}\frac{S_0''}{S_0'} \\ &\sim -\frac{1}{2}\frac{\pm i\cos(x)}{\pm i(\sin(x)+1)} \\ &\sim -\frac{1}{2}\frac{\cos(x)}{(\sin(x)+1)} \end{aligned}$$

Hence the solution is

$$S_1(x) \sim -\frac{1}{2}\ln(1+\sin(x)) \tag{6}$$

Having found $S_0(x)$ and $S_1(x)$, the leading behavior is now obtained from

$$\begin{aligned} y(x) &\sim \exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right) \\ &\sim \exp\left(\frac{1}{\varepsilon}(S_0(x)+\varepsilon S_1(x))+\cdots\right) \\ &\sim \exp\left(\frac{1}{\varepsilon}S_0(x)+S_1(x)+\cdots\right) \end{aligned}$$

The leading behavior is only the first two terms (called physical optics approximation in WKB), therefore

$$\begin{aligned} y(x) &\sim \exp\left(\frac{1}{\varepsilon}S_0(x)+S_1(x)\right) \\ &\sim \exp\left(\pm\frac{i}{\varepsilon}(1+x-\cos(x))+C^{\pm}-\frac{1}{2}\ln(1+\sin(x))\right) \\ &\sim \frac{1}{\sqrt{1+\sin x}}\exp\left(\pm\frac{i}{\varepsilon}(1+x-\cos(x))+C^{\pm}\right) \end{aligned}$$

Which can be written as

$$y(x) \sim \frac{C}{\sqrt{1+\sin x}}\exp\left(\frac{i}{\varepsilon}(1+x-\cos(x))\right) - \frac{C}{\sqrt{1+\sin x}}\exp\left(\frac{-i}{\varepsilon}(1+x-\cos(x))\right)$$

In terms of sin and cos the above becomes (using the standard Euler relation simplifications)

$$y(x) \sim \frac{A}{\sqrt{1+\sin x}}\cos\left(\frac{1}{\varepsilon}(1+x-\cos(x))\right) + \frac{B}{\sqrt{1+\sin x}}\sin\left(\frac{1}{\varepsilon}(1+x-\cos(x))\right)$$

Where A, B are the new constants. But $\lambda = \frac{1}{\varepsilon^2}$, and the above becomes

$$y(x) \sim \frac{A}{\sqrt{1+\sin x}}\cos\left(\sqrt{\lambda}(1+x-\cos(x))\right) + \frac{B}{\sqrt{1+\sin x}}\sin\left(\sqrt{\lambda}(1+x-\cos(x))\right) \tag{7}$$

Boundary conditions are now applied to determine A, B.

$$\begin{aligned} y(0) &= 0 \\ y(\pi) &= 0 \end{aligned}$$

First B.C. applied to (7) gives (where now $\sim$ is replaced by $=$ for notation simplicity)

$$
\begin{aligned}
0 &= A\cos\left(\sqrt{\lambda}\left(1-\cos\left(0\right)\right)\right) + B\sin\left(\sqrt{\lambda}\left(1-\cos\left(0\right)\right)\right)\\
0 &= A\cos\left(0\right) + B\sin\left(0\right)\\
0 &= A
\end{aligned}
$$

Hence solution (7) becomes

$$
y\left(x\right) \sim \frac{B}{\sqrt{1+\sin x}}\sin\left(\sqrt{\lambda}\left(1+x-\cos\left(x\right)\right)\right)
$$

Applying the second B.C. $y\left(\pi\right)=0$ to the above results in

$$
\begin{aligned}
0 &= \frac{B}{\sqrt{1+\sin\pi}}\sin\left(\sqrt{\lambda}\left(1+\pi-\cos\left(\pi\right)\right)\right)\\
0 &= B\sin\left(\sqrt{\lambda}\left(1+\pi+1\right)\right)\\
&= B\sin\left(\left(2+\pi\right)\sqrt{\lambda}\right)
\end{aligned}
$$

Hence, non-trivial solution implies that

$$
\begin{aligned}
\left(2+\pi\right)\sqrt{\lambda_n} &= n\pi \qquad n=1,2,3,\cdots\\
\sqrt{\lambda_n} &= \frac{n\pi}{2+\pi}
\end{aligned}
$$

The eigenvalues are

$$
\lambda_n = \frac{n^2\pi^2}{\left(2+\pi\right)^2} \qquad n=1,2,3,\cdots
$$

Hence $\lambda_n \approx n^2$ for large n. The eigenfunctions are

$$
y_n\left(x\right) \sim \frac{B_n}{\sqrt{1+\sin x}}\sin\left(\sqrt{\lambda_n}\left(1+x-\cos\left(x\right)\right)\right) \qquad n=1,2,3,\cdots
$$

The solution is therefore a linear combination of the eigenfunctions

$$
\begin{aligned}
y\left(x\right) &\sim \sum_{n=1}^{\infty} y_n\left(x\right)\\
&\sim \sum_{n=1}^{\infty}\frac{B_n}{\sqrt{1+\sin x}}\sin\left(\sqrt{\lambda_n}\left(1+x-\cos\left(x\right)\right)\right) \qquad \text{(7A)}
\end{aligned}
$$

This solution becomes more accurate for large λ or large n.

0.3.2 Part b

For normalization, the requirement is that

$$
\int_0^{\pi} y_n^2\left(x\right)\overbrace{\left(\sin\left(x\right)+1\right)^2}^{\text{weight}}dx = 1
$$

Substituting the eigenfunction $y_n\left(x\right)$ solution obtained in first part in the above results in

$$
\int_0^{\pi}\left(\frac{B_n}{\sqrt{1+\sin x}}\sin\left(\sqrt{\lambda_n}\left(1+x-\cos\left(x\right)\right)\right)\right)^2\left(\sin\left(x\right)+1\right)^2dx \sim 1
$$

The above is now solved for constant B_n. The constant B_n will the same for each n for normalization. Therefore any n can be used for the purpose of finding the scaling constant. Selecting $n=1$ in the above gives

$$
\begin{aligned}
\int_0^{\pi}\left(\frac{B}{\sqrt{1+\sin x}}\sin\left(\frac{\pi}{2+\pi}\left(1+x-\cos\left(x\right)\right)\right)\right)^2\left(\sin\left(x\right)+1\right)^2dx &\sim 1\\
B^2\int_0^{\pi}\frac{1}{1+\sin x}\sin^2\left(\frac{\pi}{2+\pi}\left(1+x-\cos\left(x\right)\right)\right)\left(\sin\left(x\right)+1\right)^2dx &\sim 1\\
B^2\int_0^{\pi}\sin^2\left(\frac{\pi}{2+\pi}\left(1+x-\cos\left(x\right)\right)\right)\left(\sin\left(x\right)+1\right)dx &\sim 1 \qquad \text{(8)}
\end{aligned}
$$

Letting $u=\frac{\pi}{2+\pi}\left(1+x-\cos\left(x\right)\right)$, then

$$
\frac{du}{dx} = \frac{\pi}{2+\pi}\left(1+\sin\left(x\right)\right)
$$

When $x = 0$, then $u = \frac{\pi}{2+\pi}(1 + 0 - \cos(0)) = 0$ and when $x = \pi$ then $u = \frac{\pi}{2+\pi}(1 + \pi - \cos(\pi)) = \frac{\pi}{2+\pi}(2+\pi) = \pi$, hence (8) becomes

$$B^2 \int_0^\pi \sin^2(u) \frac{2+\pi}{\pi} \frac{du}{dx} dx = 1$$
$$\frac{2+\pi}{\pi} B^2 \int_0^\pi \sin^2(u)\, du = 1$$

But $\sin^2(u) = \frac{1}{2} - \frac{1}{2}\cos 2u$, therefore the above becomes

$$\frac{2+\pi}{\pi} B^2 \int_0^\pi \left(\frac{1}{2} - \frac{1}{2}\cos 2u\right) du = 1$$
$$\frac{1}{2}\frac{2+\pi}{\pi} B^2 \left(u - \frac{\sin 2u}{2}\right)_0^\pi = 1$$
$$\frac{2+\pi}{2\pi} B^2 \left(\left(\pi - \frac{\sin 2\pi}{2}\right) - \left(0 - \frac{\sin 0}{2}\right)\right) = 1$$
$$\frac{2+\pi}{2\pi} B^2 \pi = 1$$
$$B^2 = \frac{2}{2+\pi}$$

Therefore

$$B = \sqrt{\frac{2}{\pi+2}} = 0.62369$$

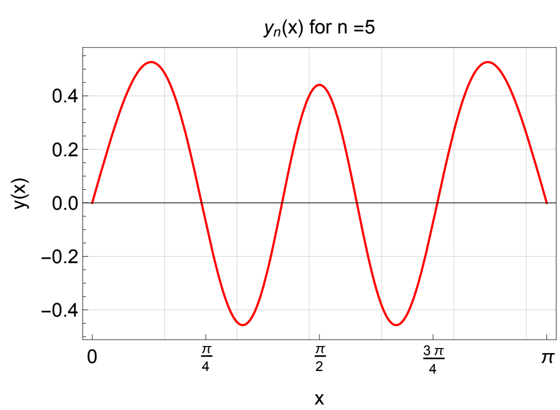
Using the above for each B_n in the solution obtained for the eigenfunctions in (7A), and pulling this scaling constant out of the sum results in

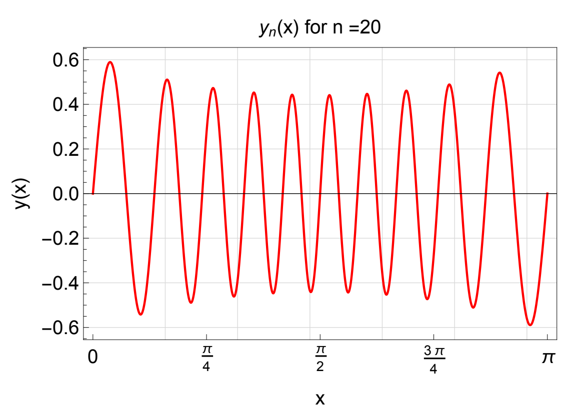
$$y^{\text{normalized}} \sim \sqrt{\frac{2}{\pi+2}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{1+\sin x}} \sin\left(\sqrt{\lambda_n}\,(1 + x - \cos(x))\right) \tag{9}$$

Where

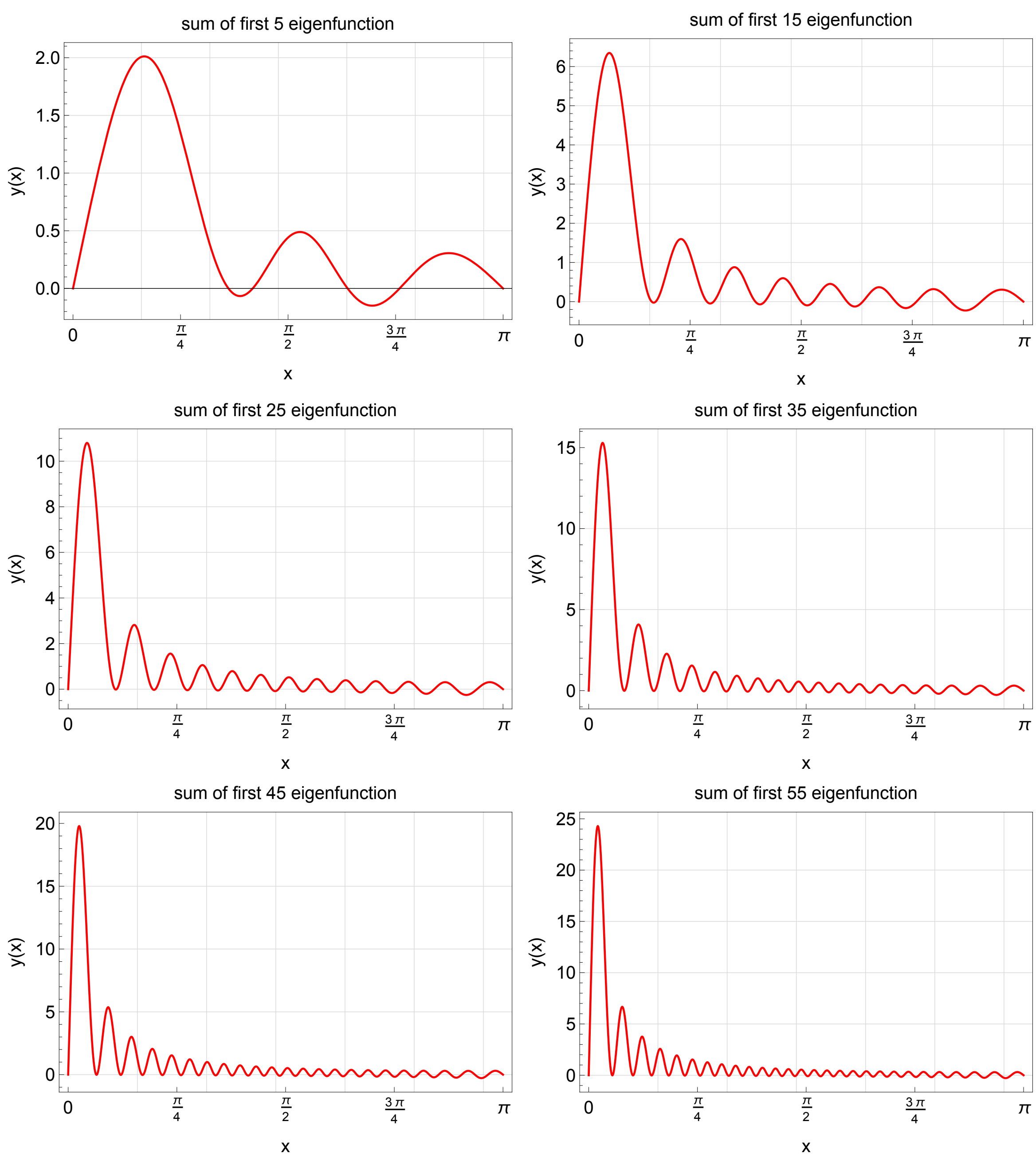
$$\sqrt{\lambda_n} = \frac{n\pi}{2+\pi} \qquad n = 1, 2, 3, \cdots$$

The following are plots for the normalized $y_n(x)$ for n values it asks to show.





The following shows the $y(x)$ as more eigenfunctions are added up to 55.


sum of first 5 eigenfunction
sum of first 15 eigenfunction
sum of first 25 eigenfunction
sum of first 35 eigenfunction
sum of first 45 eigenfunction
sum of first 55 eigenfunction
y(x)
x

0.3.3 Part c

Since approximate solution is

$$y(x) \sim \exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right) \qquad \delta \to 0$$
$$\sim \exp\left(\frac{1}{\delta}S_0(x) + S_1(x) + \delta S_2(x) + \cdots\right) \tag{1}$$

And the physical optics approximation includes the first two terms in the series above, then the relative error between physical optics and exact solution is given by $\delta S_2(x)$. But $\delta = \varepsilon$. Hence (1) becomes

$$y(x) \sim \exp\left(\frac{1}{\varepsilon}S_0(x) + S_1(x) + \varepsilon S_2(x) + \cdots\right)$$

Hence the relative error must be such that

$$|\varepsilon S_2(x)|_{\max} \leq 0.001 \tag{1A}$$

Now $S_2(x)$ is found. From (2) in part(a)

$$\frac{\varepsilon^2}{\delta^2}\left(S_0' + \delta S_1' + \delta^2 S_2' + \cdots\right)\left(S_0' + \delta S_1' + \delta^2 S_2' + \cdots\right) + \frac{\varepsilon^2}{\delta}\left(S_0'' + \delta S_1'' + \delta^2 S_2'' + \cdots\right) \sim -(\sin(x)+1)^2$$
$$\frac{\varepsilon^2}{\delta^2}\left(\left(S_0'\right)^2 + \delta\left(2S_1'S_0'\right) + \delta^2\left(2S_0'S_2' + \left(S_1'\right)^2\right) + \cdots\right) + \frac{\varepsilon^2}{\delta}\left(S_0'' + \delta S_1'' + \delta^2 S_2'' + \cdots\right) \sim -(\sin(x)+1)^2$$
$$\left(\left(S_0'\right)^2 + \varepsilon\left(2S_1'S_0'\right) + \varepsilon^2\left(2S_0'S_2' + \left(S_1'\right)^2\right) + \cdots\right) + \left(\varepsilon S_0'' + \varepsilon^2 S_1'' + \varepsilon^3 S_2'' + \cdots\right) \sim -(\sin(x)+1)^2$$

A balance on $O\left(\varepsilon^2\right)$ gives the ODE to solve to find S_2

$$2S_0'S_2' \sim -\left(S_1'\right)^2 - S_1'' \tag{2}$$

But

$$S_0' \sim \pm i(1+\sin(x))$$
$$\left(S_1'\right)^2 \sim \left(-\frac{1}{2}\frac{\cos(x)}{(\sin(x)+1)}\right)^2$$
$$\sim \frac{1}{4}\frac{\cos^2(x)}{(1+\sin(x))^2}$$
$$S_1'' \sim -\frac{1}{2}\frac{d}{dx}\left(\frac{\cos(x)}{(1+\sin(x))}\right)$$
$$\sim \frac{1}{2}\left(\frac{1}{1+\sin(x)}\right)$$

Hence (2) becomes

$$2S_0'S_2' \sim -\left(S_1'\right)^2 - S_1''$$
$$S_2' \sim -\frac{\left(\left(S_1'\right)^2 + S_1''\right)}{2S_0'}$$
$$\sim -\frac{\left(\frac{1}{4}\frac{\cos^2(x)}{(1+\sin(x))^2} + \frac{1}{2}\left(\frac{1}{1+\sin(x)}\right)\right)}{\pm 2i(1+\sin(x))}$$
$$\sim \pm\frac{i\left(\frac{1}{4}\frac{\cos^2(x)}{(\sin(x)+1)^2} + \frac{1}{2}\left(\frac{1}{1+\sin(x)}\right)\right)}{2(\sin(x)+1)}$$
$$\sim \pm\frac{i\frac{1}{4}\left(\frac{\cos^2(x)+2(1+\sin(x))}{(\sin(x)+1)^2}\right)}{2(\sin(x)+1)}$$
$$\sim \pm\frac{i}{8}\frac{\cos^2(x) + 2(1+\sin(x))}{(1+\sin(x))^3}$$

Therefore

$$\begin{aligned} S_2(x) &\sim \pm\frac{i}{8}\int_0^x \frac{\cos^2(t)+2(1+\sin(t))}{(1+\sin(t))^3}dt \\ &\sim \pm\frac{i}{8}\left(\int_0^x \frac{\cos^2(t)}{(1+\sin(t))^3}dt + 2\int_0^x \frac{1}{(1+\sin(t))^2}dt\right) \end{aligned} \tag{3}$$

To do $\int_0^x \frac{\cos^2(t)}{(1+\sin(t))^3}dt$, I used a lookup integration rule from tables which says $\int \cos^p(t)(a+\sin t)^m\,dt = \frac{1}{(a)(m)}\cos^{p+1}(t)(a+\sin t)^m$, therefore using this rule the integral becomes, where now $m = -3, p = 2, a = 1$,

$$\begin{aligned} \int_0^x \frac{\cos^2 t}{(1+\sin t)^3}dt &= \frac{1}{-3}\left(\frac{\cos^3 t}{(1+\sin t)^3}\right)_0^x \\ &= \frac{1}{-3}\left(\frac{\cos^3 x}{(1+\sin x)^3} - 1\right) \\ &= \frac{1}{3}\left(1 - \frac{\cos^3 x}{(1+\sin x)^3}\right) \end{aligned}$$

And for $\int \frac{1}{(1+\sin(x))^2}dx$, half angle substitution can be used. I do not know what other substitution to use. Using CAS for little help on this, I get

$$\begin{aligned} \int_0^x \frac{1}{(1+\sin t)^2}dt &= \left(-\frac{\cos t}{3(1+\sin t)^2} - \frac{1}{3}\frac{\cos t}{1+\sin t}\right)_0^x \\ &= \left(-\frac{\cos x}{3(1+\sin x)^2} - \frac{1}{3}\frac{\cos x}{1+\sin x}\right) - \left(-\frac{1}{3} - \frac{1}{3}\right) \\ &= \frac{2}{3} - \frac{\cos x}{3(1+\sin x)^2} - \frac{1}{3}\frac{\cos x}{1+\sin x} \end{aligned}$$

Hence from (3)

$$\begin{aligned} S_2(x) &\sim \pm\frac{i}{8}\left(\frac{1}{3}\left(1 - \frac{\cos^3(x)}{(1+\sin(x))^3}\right) + 2\left(\frac{2}{3} - \frac{\cos x}{3(1+\sin x)^2} - \frac{1}{3}\frac{\cos x}{1+\sin x}\right)\right) \\ &\sim \pm\frac{i}{8}\left(\frac{1}{3} - \frac{1}{3}\frac{\cos^3(x)}{(1+\sin(x))^3} + \frac{4}{3} - \frac{2\cos x}{3(1+\sin x)^2} - \frac{2}{3}\frac{\cos x}{1+\sin x}\right) \\ &\sim \pm\frac{i}{24}\left(1 - \frac{\cos^3(x)}{(1+\sin(x))^3} + 4 - \frac{2\cos x}{(1+\sin x)^2} - \frac{2\cos x}{1+\sin x}\right) \end{aligned}$$

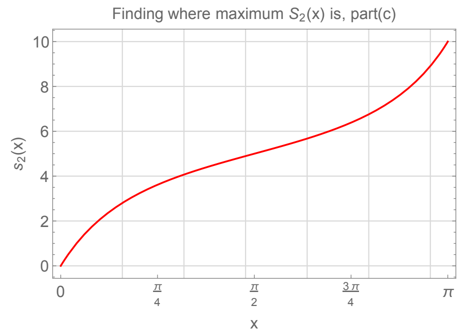
Therefore, from (1A)

$$\begin{aligned} |\varepsilon S_2(x)|_{\max} &\leq 0.001 \\ \left|\varepsilon\frac{i}{24}\left(1 - \frac{\cos^3(x)}{(1+\sin(x))^3} + 4 - \frac{2\cos x}{(1+\sin x)^2} - \frac{2\cos x}{1+\sin x}\right)\right|_{\max} &\leq 0.001 \\ \frac{1}{24}\left|\varepsilon\left(1 - \frac{\cos^3(x)}{(1+\sin(x))^3} + 4 - \frac{2\cos x}{(1+\sin x)^2} - \frac{2\cos x}{1+\sin x}\right)\right|_{\max} &\leq 0.001 \\ \left|\varepsilon\left(1 - \frac{\cos^3(x)}{(1+\sin(x))^3} + 4 - \frac{2\cos x}{(1+\sin x)^2} - \frac{2\cos x}{1+\sin x}\right)\right|_{\max} &\leq 0.024 \end{aligned} \tag{2}$$

The maximum value of $\left(1 - \frac{\cos^3(x)}{(1+\sin(x))^3} + 4 - \frac{2\cos x}{(1+\sin x)^2} - \frac{2\cos x}{1+\sin x}\right)$ between $x = 0$ and $x = \pi$ is now found and used to find ε. A plot of the above shows the maximum is maximum at the end, at $x = \pi$ (Taking the derivative and setting it to zero to determine where the maximum is can also be used).

```
In[324]:= myResult = (1 - Cos[x]^3/(1 + Sin[x])^3 + 4 - 2 Cos[x]/(1 + Sin[x])^2 - 2 Cos[x]/(1 + Sin[x]));
Plot[myResult, {x, 0, Pi}, PlotRange → All, Frame → True, GridLines → Automatic, GridLinesStyle → LightGray,
 FrameLabel → {{"s2(x)", None}, {"x", "Finding where maximum S2(x) is, part(c)"}}, PlotStyle → Red, BaseStyle → 14,
 FrameTicks → {Automatic, {{0, Pi/4, Pi/2, 3/4 Pi, Pi}, None}}, ImageSize → 400]
```

Out[325]=



Therefore, at $x=\pi$

$$\left(1-\frac{\cos^3 x}{(1+\sin x)^3}+4-\frac{2\cos x}{(1+\sin x)^2}-\frac{2\cos x}{1+\sin x}\right)_{x=\pi}=\left(1-\frac{\cos^3(\pi)}{(1+\sin\pi)^3}+4-\frac{2\cos\pi}{(1+\sin\pi)^2}-\frac{2\cos\pi}{1+\sin\pi}\right)$$
$$=10$$

Hence (2) becomes

$$10\varepsilon \leq 0.024$$
$$\varepsilon \leq 0.0024$$

But since $\lambda=\frac{1}{\varepsilon^2}$ the above becomes

$$\frac{1}{\sqrt{\lambda}} \leq 0.0024$$
$$\sqrt{\lambda} \geq \frac{1}{0.0024}$$
$$\sqrt{\lambda} \geq 416.67$$

Hence

$$\boxed{\lambda \geq 17351.1}$$

To find which mode this corresponds to, since $\lambda_n=\frac{n^2\pi^2}{(2+\pi)^2}$, then need to solve for n

$$17351.1=\frac{n^2\pi^2}{(2+\pi)^2}$$
$$n^2\pi^2=(17351.1)(2+\pi)^2$$
$$n=\sqrt{\frac{(17351.1)(2+\pi)^2}{\pi^2}}$$
$$=215.58$$

Hence the next largest integer is used

$$\boxed{n=216}$$

To have relative error less than 0.1% compared to exact solution. Therefore using the result obtained in (9) in part (b) the normalized solution needed is

$$y^{\text{normalized}} \sim \sqrt{\frac{2}{\pi+2}}\sum_{n=1}^{216}\frac{1}{\sqrt{1+\sin x}}\sin\left(\frac{n\pi}{2+\pi}(1+x-\cos(x))\right)$$

The following is a plot of the above solution adding all the first 216 modes for illustration.

```
In[42]:= ClearAll[x, n, lam]
mySol[x_, max_] := Sqrt[2/(Pi + 2)] Sum[1/Sqrt[1 + Sin[x]] Sin[n Pi/(2 + Pi) (1 + x - Cos[x])], {n, 1, max}];

In[46]:= p[n_] := Plot[mySol[x, n], {x, 0, Pi}, PlotRange → All, Frame → True,
  FrameLabel → {{"y(x)", None}, {"x", Row[{"y_n(x) for n =", n}]}}, BaseStyle → 14, GridLines → Automatic,
  GridLinesStyle → LightGray, ImageSize → 600, PlotStyle → Red,
  FrameTicks → {{Automatic, None}, {{0, Pi / 4, Pi / 2, 3 / 4 Pi, Pi}, None}}, PlotRange → All]

In[47]:= p[216]
```

Out[47]=