HW 6, NE 548, Spring 2017

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Contents

0.1 Problem 1

1. Use the Method of Images to solve

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} + Q(x,t), \quad x > 0$$

(a)

$$u(0,t) = 0, \qquad u(x,0) = 0.$$

(b)

$$u(0,t) = 1, \qquad u(x,0) = 0.$$

(c)

$$u_x(0,t) = A(t), \qquad u(x,0) = f(x).$$

Note, I will use k in place of ν since easier to type.

0.1.1 Part (a)

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} + Q(x,t) \\ x &> 0 \\ u(0,t) &= 0 \\ u(x,0) &= 0 \end{aligned}$$

Multiplying both sides by $G(x,t;x_0,t_0)$ and integrating over the domain gives (where in the following G is used instead of $G(x,t;x_0,t_0)$ for simplicity).

$$\int_{x=0}^{\infty}\int_{t=0}^{\infty} Gu_t \, dtdx = \int_{x=0}^{\infty}\int_{t=0}^{\infty} ku_{xx}G \, dtdx + \int_{x=0}^{\infty}\int_{t=0}^{\infty} QG \, dtdx \tag{1}$$

For the integral on the LHS, we apply integration by parts once to move the time derivative from u to G

$$\int_{x=0}^{\infty}\int_{t=0}^{\infty} Gu_t \, dtdx = \int_{x=0}^{\infty} [uG]_{t=0}^{\infty} dx - \int_{x=0}^{\infty}\int_{t=0}^{\infty} G_t u \, dtdx$$

And the first integral in the RHS of (1) gives, after doing integration by parts two times

$$\begin{aligned} \int_{x=0}^{\infty}\int_{t=0}^{\infty} ku_{xx}G \, dtdx &= \int_{t=0}^{\infty} [u_xG]_{x=0}^{\infty} dt - \int_{x=0}^{\infty}\int_{t=0}^{\infty} ku_xG_x \, dtdx \\ &= \int_{t=0}^{\infty} [u_xG]_{x=0}^{\infty} dt - \left(\int_{t=0}^{\infty} [uG_x]_{x=0}^{\infty} dt - \int_{x=0}^{\infty}\int_{t=0}^{\infty} kuG_{xx} \, dtdx\right) \\ &= \int_{t=0}^{\infty} \left([u_xG]_{x=0}^{\infty} - [uG_x]_{x=0}^{\infty}\right) dt + \int_{x=0}^{\infty}\int_{t=0}^{\infty} kuG_{xx} \, dtdx \\ &= \int_{t=0}^{\infty} [u_xG - uG_x]_{x=0}^{\infty} dt + \int_{x=0}^{\infty}\int_{t=0}^{\infty} kuG_{xx} \, dtdx \\ &= -\int_{t=0}^{\infty} [uG_x - u_xG]_{x=0}^{\infty} dt + \int_{x=0}^{\infty}\int_{t=0}^{\infty} kuG_{xx} \, dtdx \end{aligned}$$

Hence (1) becomes

$$\int_{x=0}^{\infty} [uG]_{t=0}^{\infty} dx - \int_{x=0}^{\infty}\int_{t=0}^{\infty} G_t u \, dtdx = \int_{t=0}^{\infty} [u_xG - uG_x]_{x=0}^{\infty} dt + \int_{x=0}^{\infty}\int_{t=0}^{\infty} kuG_{xx} \, dtdx + \int_{x=0}^{\infty}\int_{t=0}^{\infty} GQ \, dtdx$$

Or

$$\int_{x=0}^{\infty}\int_{t=0}^{\infty} -G_t u - kuG_{xx} \, dtdx = -\int_{x=0}^{\infty} [uG]_{t=0}^{\infty} dx - \int_{t=0}^{\infty} [uG_x - u_xG]_{x=0}^{\infty} dt + \int_{x=0}^{\infty}\int_{t=0}^{\infty} GQ \, dtdx \tag{2}$$

We now want to choose $G(x,t;x_0,t_0)$ such that

$$
\begin{aligned}
-G_t u - kuG_{xx} &= \delta(x-x_0)\,\delta(t-t_0)\\
-G_t u &= kuG_{xx} + \delta(x-x_0)\,\delta(t-t_0) \qquad (3)
\end{aligned}
$$

This way, the LHS of (2) becomes $u(x_0,t_0)$. Hence (2) now becomes

$$u(x_0,t_0) = -\int_{x=0}^{\infty} [uG]_{t=0}^{\infty}\,dx - \int_{t=0}^{t_0} [uG_x - u_xG]_{x=0}^{\infty}\,dt + \int_{x=0}^{\infty}\int_{t=0}^{t_0} GQ\ dtdx \tag{4}$$

We now need to find the Green function which satisfies (3). But (3) is equivalent to solution of problem of

$$
\begin{aligned}
-G_t u &= kuG_{xx}\\
G(x,0) &= \delta(x-x_0)\,\delta(t-t_0)\\
-\infty &< x < \infty\\
G(x,t;x_0,t_0) &= 0 \qquad t > t_0\\
G(\pm\infty,t;x_0,t_0) &= 0\\
G(x,t_0;x_0,t_0) &= \delta(x-x_0)
\end{aligned}
$$

But the above problem has a known fundamental solution which we found, but for the forward heat PDE instead of the reverse heat PDE. The fundamental solution to the forward heat PDE is

$$G(x,t) = \frac{1}{\sqrt{4\pi k(t-t_0)}}\exp\left(\frac{-(x-x_0)^2}{4k(t-t_0)}\right) \qquad 0 \le t_0 \le t$$

Hence for the reverse heat PDE the above becomes

$$G(x,t) = \frac{1}{\sqrt{4\pi k(t_0-t)}}\exp\left(\frac{-(x-x_0)^2}{4k(t_0-t)}\right) \qquad 0 \le t \le t_0 \tag{5}$$

The above is the infinite space Green function and what we will use in (4). Now we go back to (4) and simplify the boundary conditions term. Starting with the term $\int_{x=0}^{\infty}[uG]_{t=0}^{\infty}\,dx$. Since $G(x,\infty;x_0,t_0) = 0$ then upper limit is zero. At $t = 0$ we are given that $u(x,0) = 0$, hence this whole term is zero. So now (4) simplifies to

$$u(x_0,t_0) = -\int_{t=0}^{\infty} [uG_x - u_xG]_{x=0}^{\infty}\,dt + \int_{x=0}^{\infty}\int_{t=0}^{\infty} GQ\ dtdx \tag{6}$$

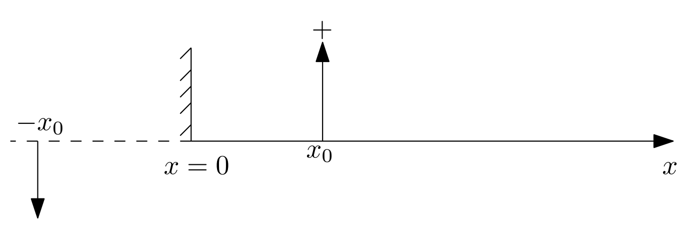
We are told that $u(0,t) = 0$. Hence

$$
\begin{aligned}
[uG_x - u_xG]_{x=0}^{\infty} &= (u(\infty,t)\,G_x(\infty,t) - u_x(\infty,t)\,G(\infty,t)) - (u(0,t)\,G_x(0,t) - u_x(0,t)\,G(0,t))\\
&= (u(\infty,t)\,G_x(\infty,t) - u_x(\infty,t)\,G(\infty,t)) + u_x(0,t)\,G(0,t)
\end{aligned}
$$

We also know that $G(\pm\infty,t;x_0,t_0) = 0$, Hence $G(\infty) = 0$ and also $G_x(\infty) = 0$, hence the above simplifies

$$[uG_x - u_xG]_{x=0}^{\infty} = u_x(0,t)\,G(0,t)$$

To make $G(0,t) = 0$ we place an image impulse at $-x_0$ with negative value to the impulse at x_0. This will make G at $x = 0$ zero.



Therefore the Green function to use is, from (5) becomes

$$G(x,t) = \frac{1}{\sqrt{4\pi k(t_0-t)}}\left(\exp\left(\frac{-(x-x_0)^2}{4k(t_0-t)}\right) - \exp\left(\frac{-(x+x_0)^2}{4k(t_0-t)}\right)\right) \qquad 0 \le t \le t_0$$

Therefore the solution, from (4) becomes

$$u(x_0,t_0) = \int_{x=0}^{\infty}\int_{t=0}^{t_0} \frac{1}{\sqrt{4\pi k(t_0-t)}}\left(\exp\left(\frac{-(x-x_0)^2}{4k(t_0-t)}\right) - \exp\left(\frac{-(x+x_0)^2}{4k(t_0-t)}\right)\right)Q(x,t)\ dtdx \tag{7}$$

Switching the order of x_0, t_0 with x, t gives

$$u(x,t) = \int_{x_0=0}^{\infty}\int_{t_0=0}^{t} \frac{1}{\sqrt{4\pi k(t-t_0)}}\left(\exp\left(\frac{-(x_0-x)^2}{4k(t-t_0)}\right) - \exp\left(\frac{-(x_0+x)^2}{4k(t-t_0)}\right)\right)Q(x_0,t_0)\ dt_0dx_0 \tag{8}$$

Notice, for the terms $(x_0 - x)^2, (x_0 + x)^2$, since they are squared, the order does not matter, so we might as well write the above as

$$u(x,t) = \int_{x_0=0}^{\infty}\int_{t_0=0}^{t} \frac{1}{\sqrt{4\pi k(t-t_0)}}\left(\exp\left(\frac{-(x-x_0)^2}{4k(t-t_0)}\right) - \exp\left(\frac{-(x+x_0)^2}{4k(t-t_0)}\right)\right) Q(x_0,t_0)\ dt_0 dx_0 \tag{8}$$

0.1.2 Part (b)

$$\begin{aligned}\frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} + Q(x,t)\\ x &> 0\\ u(0,t) &= 1\\ u(x,0) &= 0\end{aligned}$$

Everything follows the same as in part (a) up to the point where boundary condition terms need to be evaluated.

$$u(x_0,t_0) = -\int_{x=0}^{\infty}[uG]_{t=0}^{\infty}\,dx - \int_{t=0}^{t_0}[uG_x - u_xG]_{x=0}^{\infty}\,dt + \int_{x=0}^{\infty}\int_{t=0}^{t_0} GQ\ dtdx \tag{4}$$

Where

$$G(x,t) = \frac{1}{\sqrt{4\pi k(t_0-t)}}\exp\left(\frac{-(x-x_0)^2}{4k(t_0-t)}\right) \qquad 0 \le t \le t_0 \tag{5}$$

Starting with the term $\int_{x=0}^{\infty}[uG]_{t=0}^{\infty}\,dx$. Since $G(x,\infty;x_0,t_0) = 0$ then upper limit is zero. At $t=0$ we are given that $u(x,0) = 0$, hence this whole term is zero. So now (4) simplifies to

$$u(x_0,t_0) = -\int_{t=0}^{\infty}[uG_x - u_xG]_{x=0}^{\infty}\,dt + \int_{x=0}^{\infty}\int_{t=0}^{\infty} GQ\ dtdx \tag{6}$$

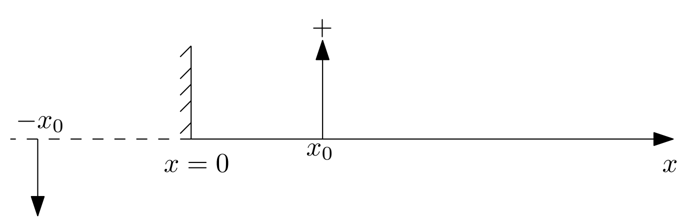
Now

$$[uG_x - u_xG]_{x=0}^{\infty} = (u(\infty,t)G_x(\infty,t) - u_x(\infty,t)G(\infty,t)) - (u(0,t)G_x(0,t) - u_x(0,t)G(0,t))$$

We are told that $u(0,t) = 1$, we also know that $G(\pm\infty,t;x_0,t_0) = 0$, Hence $G(\infty,t) = 0$ and also $G_x(\infty,t) = 0$, hence the above simplifies

$$[uG_x - u_xG]_{x=0}^{\infty} = -G_x(0,t) + u_x(0,t)G(0,t)$$

To make $G(0) = 0$ we place an image impulse at $-x_0$ with negative value to the impulse at x_0. This is the same as part(a)



$$G(x,t) = \frac{1}{\sqrt{4\pi k(t_0-t)}}\left(\exp\left(\frac{-(x-x_0)^2}{4k(t_0-t)}\right) - \exp\left(\frac{-(x+x_0)^2}{4k(t_0-t)}\right)\right) \qquad 0 \le t \le t_0$$

Now the boundary terms reduces to just

$$[uG_x - u_xG]_{x=0}^{\infty} = -G_x(0,t)$$

We need now to evaluate $G_x(0,t)$, the only remaining term. Since we know what $G(x,t)$ from the above, then

$$\frac{\partial}{\partial x}G(x,t) = \frac{1}{\sqrt{4\pi k(t_0-t)}}\left(\frac{-(x-x_0)}{2k(t_0-t)}\exp\left(\frac{-(x-x_0)^2}{4k(t_0-t)}\right) + \frac{2(x+x_0)}{4k(t_0-t)}\exp\left(\frac{-(x+x_0)^2}{4k(t_0-t)}\right)\right)$$

And at $x=0$ the above simplifies to

$$\begin{aligned}G_x(0,t) &= \frac{1}{\sqrt{4\pi k(t_0-t)}}\left(\frac{x_0}{2k(t_0-t)}\exp\left(\frac{-x_0^2}{4k(t_0-t)}\right) + \frac{x_0}{2k(t_0-t)}\exp\left(\frac{-x_0^2}{4k(t_0-t)}\right)\right)\\ &= \frac{1}{\sqrt{4\pi k(t_0-t)}}\left(\frac{x_0}{k(t_0-t)}\exp\left(\frac{-x_0^2}{4k(t_0-t)}\right)\right)\end{aligned}$$

Hence the complete solution becomes from (4)

$$\begin{aligned} u(x_0,t_0) &= -\int_{x=0}^{\infty} [uG]_{t=0}^{\infty} dx - \int_{t=0}^{t_0} [uG_x - u_x G]_{x=0}^{\infty} dt + \int_{x=0}^{\infty}\int_{t=0}^{t_0} GQ \, dtdx \\ &= -\int_{t=0}^{t_0} -G_x(0,t)\, dt + \int_{x=0}^{\infty}\int_{t=0}^{t_0} GQ \, dtdx \end{aligned}$$

Substituting G and G_x in the above gives

$$\begin{aligned} u(x_0,t_0) = \int_{t=0}^{t_0} \frac{1}{\sqrt{4\pi k(t_0-t)}} \left(\frac{x_0}{k(t_0-t)} \exp\left(\frac{-x_0^2}{4k(t_0-t)} \right) \right) dt \\ + \int_{x=0}^{\infty}\int_{t=0}^{t_0} \frac{1}{\sqrt{4\pi k(t_0-t)}} \left(\exp\left(\frac{-(x-x_0)^2}{4k(t_0-t)} \right) - \exp\left(\frac{-(x+x_0)^2}{4k(t_0-t)} \right) \right) Q(x,t) \, dtdx \end{aligned}$$

Switching the order of x_0, t_0 with x, t

$$\begin{aligned} u(x,t) = \int_{t_0=0}^{t} \frac{1}{\sqrt{4\pi k(t-t_0)}} \left(\frac{x}{k(t-t_0)} \exp\left(\frac{-x^2}{4k(t-t_0)} \right) \right) dt_0 \\ + \int_{x_0=0}^{\infty}\int_{t_0=0}^{t} \frac{1}{\sqrt{4\pi k(t-t_0)}} \left(\exp\left(\frac{-(x_0-x)^2}{4k(t-t_0)} \right) - \exp\left(\frac{-(x_0+x)^2}{4k(t-t_0)} \right) \right) Q(x_0,t_0) \, dt_0 dx_0 \end{aligned} \tag{7}$$

But

$$\int_{t_0=0}^{t} \frac{1}{\sqrt{4\pi k(t-t_0)}} \left(\frac{x}{k(t-t_0)} \exp\left(\frac{-x^2}{4k(t-t_0)} \right) \right) dt_0 = \operatorname{erfc}\left(\frac{x}{2\sqrt{t}} \right)$$

Hence (7) becomes

$$u(x,t) = \operatorname{erfc}\left(\frac{x}{2\sqrt{t}} \right) + \int_{x_0=0}^{\infty}\int_{t_0=0}^{t} \frac{1}{\sqrt{4\pi k(t-t_0)}} \left(\exp\left(\frac{-(x_0-x)^2}{4k(t-t_0)} \right) - \exp\left(\frac{-(x_0+x)^2}{4k(t-t_0)} \right) \right) Q(x_0,t_0) \, dt_0 dx_0$$

The only difference between this solution and part(a) solution is the extra term $\operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right)$ due to having non-zero boundary conditions in this case.

0.1.3 part(c)

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} + Q(x,t) \\ x &> 0 \\ \frac{\partial u(0,t)}{\partial x} &= A(t) \\ u(x,0) &= f(x) \end{aligned}$$

Everything follows the same as in part (a) up to the point where boundary condition terms need to be evaluated.

$$u(x_0,t_0) = -\int_{x=0}^{\infty} [uG]_{t=0}^{\infty} dx - \int_{t=0}^{t_0} [uG_x - u_x G]_{x=0}^{\infty} dt + \int_{x=0}^{\infty}\int_{t=0}^{t_0} GQ \, dtdx \tag{4}$$

Starting with the term $\int_{x=0}^{\infty} [uG]_{t=0}^{\infty} dx$. Since $G(x,\infty;x_0,t_0) = 0$ then upper limit is zero. At $t = 0$ we are given that $u(x,0) = f(x)$, hence

$$\begin{aligned} \int_{x=0}^{\infty} [uG]_{t=0}^{\infty} dx &= \int_{x=0}^{\infty} -u(x,0)\, G(x,0)\, dx \\ &= \int_{x=0}^{\infty} -f(x)\, G(x,0)\, dx \end{aligned}$$

Looking at the second term in RHS of (4)

$$[uG_x - u_x G]_{x=0}^{\infty} = (u(\infty,t)\, G_x(\infty,t) - u_x(\infty,t)\, G(\infty,t)) - (u(0,t)\, G_x(0,t) - u_x(0,t)\, G(0,t))$$

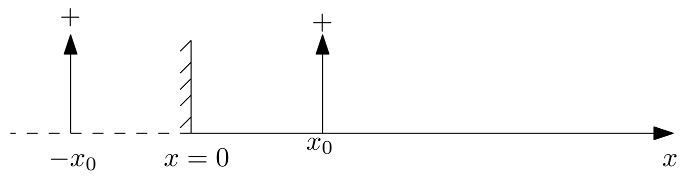
We are told that $u_x(0,t) = A(t)$, we also know that $G(\pm\infty, t; x_0, t_0) = 0$, Hence $G(\infty,t) = 0$ and also $G_x(\infty,t) = 0$. The above simplifies

$$[uG_x - u_x G]_{x=0}^{\infty} = -(u(0,t)\, G_x(0,t) - A(t)\, G(0,t)) \tag{5}$$

We see now from the above, that to get rid of the $u(0,t)\, G_x(0,t)$ term (since we do not know what $u(0,t)$ is), then we now need

$$G_x(0,t) = 0$$

This means we need an image at $-x_0$ which is of same sign as at $+x_0$ as shown in this diagram



Which means the Green function we need to use is the sum of the Green function solutions for the infinite domain problem

$$G(x,t) = \frac{1}{\sqrt{4\pi k(t_0-t)}}\left(\exp\left(\frac{-(x-x_0)^2}{4k(t_0-t)}\right) + \exp\left(\frac{-(x+x_0)^2}{4k(t_0-t)}\right)\right) \qquad 0 \le t \le t_0 \tag{6}$$

The above makes $G_x(0,t) = 0$ and now equation (5) reduces to

$$\begin{aligned}
[uG_x - u_x G]_{x=0}^{\infty} &= A(t)\,G(0,t)\\
&= A(t)\left(\frac{1}{\sqrt{4\pi k(t_0-t)}}\left(\exp\left(\frac{-(-x_0)^2}{4k(t_0-t)}\right) + \exp\left(\frac{-(x_0)^2}{4k(t_0-t)}\right)\right)\right)\\
&= \frac{A(t)}{\sqrt{4\pi k(t_0-t)}}\left(\exp\left(\frac{-x_0^2}{4k(t_0-t)}\right) + \exp\left(\frac{-x_0^2}{4k(t_0-t)}\right)\right)\\
&= \frac{A(t)}{\sqrt{\pi k(t_0-t)}}\exp\left(\frac{-x_0^2}{4k(t_0-t)}\right)
\end{aligned}$$

We now know all the terms needed to evaluate the solution. From (4)

$$u(x_0,t_0) = -\int_{x=0}^{\infty} -f(x)\,G(x,0)\,dx - \int_{t=0}^{t_0} A(t)\,G(0,t)\,dt + \int_{x=0}^{\infty}\int_{t=0}^{t_0} GQ\,dtdx \tag{7}$$

Using the Green function we found in (6), then (7) becomes

$$\begin{aligned}
u(x_0,t_0) = \int_{x=0}^{\infty} f(x)\,\frac{1}{\sqrt{4\pi k t_0}}\left(\exp\left(\frac{-(x-x_0)^2}{4kt_0}\right) + \exp\left(\frac{-(x+x_0)^2}{4kt_0}\right)\right)dx\\
- \int_{t=0}^{t_0} \frac{A(t)}{\sqrt{\pi k(t_0-t)}}\exp\left(\frac{-x_0^2}{4k(t_0-t)}\right)dt\\
+ \int_{x=0}^{\infty}\int_{t=0}^{t_0} \frac{1}{\sqrt{4\pi k(t_0-t)}}\left(\exp\left(\frac{-(x-x_0)^2}{4k(t_0-t)}\right) + \exp\left(\frac{-(x+x_0)^2}{4k(t_0-t)}\right)\right)Q\,dtdx
\end{aligned}$$

Changing the roles of x, t and x_0, t_0 the above becomes

$$\begin{aligned}
u(x,t) = \frac{1}{\sqrt{4\pi k t}}\int_{x_0=0}^{\infty} f(x_0)\left(\exp\left(\frac{-(x_0-x)^2}{4kt}\right) + \exp\left(\frac{-(x_0+x)^2}{4kt}\right)\right)dx_0\\
- \int_{t_0=0}^{t} \frac{A(t_0)}{\sqrt{\pi k(t-t_0)}}\exp\left(\frac{-x^2}{4k(t-t_0)}\right)dt_0\\
+ \int_{x_0=0}^{\infty}\int_{t_0=0}^{t} \frac{1}{\sqrt{4\pi k(t-t_0)}}\left(\exp\left(\frac{-(x_0-x)^2}{4k(t-t_0)}\right) + \exp\left(\frac{-(x_0+x)^2}{4k(t-t_0)}\right)\right)Q(x_0,t_0)\,dt_0dx_0
\end{aligned}$$

Summary

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}}\Delta_1 - \Delta_2 + \Delta_3$$

Where

$$\begin{aligned}
\Delta_1 &= \int_{x_0=0}^{\infty} f(x_0)\left(\exp\left(\frac{-(x_0-x)^2}{4kt}\right) + \exp\left(\frac{-(x_0+x)^2}{4kt}\right)\right)dx_0\\
\Delta_2 &= \int_{t_0=0}^{t} \frac{A(t_0)}{\sqrt{\pi k(t-t_0)}}\exp\left(\frac{-x^2}{4k(t-t_0)}\right)dt_0\\
\Delta_3 &= \int_{x_0=0}^{\infty}\int_{t_0=0}^{t} \frac{1}{\sqrt{4\pi k(t-t_0)}}\left(\exp\left(\frac{-(x_0-x)^2}{4k(t-t_0)}\right) + \exp\left(\frac{-(x_0+x)^2}{4k(t-t_0)}\right)\right)Q(x_0,t_0)\,dt_0dx_0
\end{aligned}$$

Where Δ_1 comes from the initial conditions and Δ_2 comes from the boundary conditions and Δ_3 comes from for forcing function. It is also important to note that Δ_1 is valid for only $t > 0$ and not for $t = 0$.

0.2 Problem 2

2. (a) Solve by the Method of Characteristics:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad x \geq 0, \; t \geq 0$$

$$u(x,0) = f(x), \qquad \frac{\partial u(x,0)}{\partial t} = g(x), \qquad \frac{\partial u(0,t)}{\partial x} = h(t)$$

(b) For the special case $h(t) = 0$, explain how you could use a symmetry argument to help construct the solution.

(c) Sketch the solution if $g(x) = 0$, $h(t) = 0$ and $f(x) = 1$ for $4 \leq x \leq 5$ and $f(x) = 0$ otherwise.

0.2.1 Part (a)

The general solution we will use as starting point is

$$u(x,t) = F(x-ct) + G(x+ct)$$

Where $F(x-ct)$ is the right moving wave and $G(x+ct)$ is the left moving wave. Applying $u(x,0) = f(x)$ gives

$$f(x) = F(x) + G(x) \tag{1}$$

And

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= \frac{dF}{d(x-ct)} \frac{\partial (x-ct)}{\partial t} + \frac{dG}{d(x+ct)} \frac{\partial (x+ct)}{\partial t} \\ &= -cF' + cG' \end{aligned}$$

Hence from second initial conditions we obtain

$$g(x) = -cF' + cG' \tag{2}$$

Equation (1) and (2) are for valid only for positive argument, which means for $x \geq ct$. $G(x+ct)$ has positive argument always since $x \geq 0$ and $t \geq 0$, but $F(x-ct)$ can have negative argument when $x < ct$. For $x < ct$, we will use the boundary conditions to define $F(x-ct)$. Therefore for $x \geq ct$ we solve (1,2) for G, F and find

$$F(x-ct) = \frac{1}{2} f(x-ct) - \frac{1}{2c} \int_0^{x-ct} g(s)\, ds \tag{2A}$$

$$G(x+ct) = \frac{1}{2} f(x+ct) + \frac{1}{2c} \int_0^{x+ct} g(s)\, ds \tag{2B}$$

This results in

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\, ds \qquad x \geq ct \tag{3}$$

The solution (3) is only valid for arguments that are positive. This is not a problem for $G(x+ct)$ since its argument is always positive. But for $F(x-ct)$ its argument can become negative when $0 < x < ct$. So we need to obtain a new solution for $F(x-ct)$ for the case when $x < ct$. First we find $\frac{\partial u(x,t)}{\partial x}$

$$\begin{aligned} \frac{\partial u(x,t)}{\partial x} &= \frac{dF}{d(x-ct)} \frac{\partial (x-ct)}{\partial x} + \frac{dG}{d(x+ct)} \frac{\partial (x+ct)}{\partial x} \\ &= \frac{dF(x-ct)}{d(x-ct)} + \frac{dG(x+ct)}{d(x+ct)} \end{aligned}$$

Hence at $x = 0$

$$h(t) = \frac{dF(-ct)}{d(-ct)} + \frac{dG(ct)}{d(ct)}$$

$$\frac{dF(-ct)}{d(-ct)} = h(t) - \frac{dG(ct)}{d(ct)} \tag{4}$$

Let $z = -ct$, therefore (4) becomes, where $t = \frac{-z}{c}$ also

$$\frac{dF(z)}{dz} = h\left(-\frac{z}{c}\right) - \frac{dG(-z)}{d(-z)} \qquad z < 0$$

$$\frac{dF(z)}{dz} = h\left(-\frac{z}{c}\right) + \frac{dG(-z)}{dz} \qquad z < 0$$

To find $F(z)$, we integrate the above which gives

$$\int_0^z \frac{dF(s)}{ds} ds = \int_0^z h\left(-\frac{s}{c}\right) ds + \int_0^z \frac{dG(-s)}{ds} ds$$
$$F(z) - F(0) = \int_0^z h\left(-\frac{s}{c}\right) ds + G(-z) - G(0)$$

Ignoring the constants of integration $F(0), G(0)$ gives

$$F(z) = \int_0^z h\left(-\frac{s}{c}\right) ds + G(-z) \tag{4A}$$

Replacing $z = x - ct$ in the above gives

$$F(x-ct) = \int_0^{x-ct} h\left(-\frac{s}{c}\right) ds + G(ct-x)$$

Let $r = -\frac{s}{c}$ then when $s = 0$, $r = 0$ and when $s = x - ct$ then $r = -\frac{x-ct}{c} = \frac{ct-x}{c}$. And $\frac{dr}{ds} = -\frac{1}{c}$. Using these the integral in the above becomes

$$F(x-ct) = \int_0^{\frac{ct-x}{c}} h(r)(-c\,dr) + G(ct-x)$$
$$= -c\int_0^{t-\frac{x}{c}} h(r)\,dr + G(ct-x) \tag{5}$$

The above is $F(\cdot)$ when its argument are negative. But in the above $G(ct-x)$ is the same as we found above in (2b), which just replace it argument in 2B which was $x+ct$ with $ct-x$ and obtain

$$G(ct-x) = \frac{1}{2}f(ct-x) + \frac{1}{2c}\int_0^{ct-x} g(s)\,ds \qquad x < ct$$

Therefore (5) becomes

$$F(x-ct) = -c\int_0^{t-\frac{x}{c}} h(r)\,dr + \frac{1}{2}f(ct-x) + \frac{1}{2c}\int_0^{ct-x} g(s)\,ds \qquad x < ct \tag{7}$$

Hence for $x < ct$

$$u(x,t) = F(x-ct) + G(x+ct) \tag{8}$$

But in the above $G(x+ct)$ do not change, and we use the same solution for G for $x \geq ct$ which is in (2B), given again below

$$G(x+ct) = \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(s)\,ds \tag{9}$$

Hence, plugging (7,9) into (8) gives

$$u(x,t) = -c\int_0^{\frac{ct-x}{c}} h(s)\,ds + \frac{1}{2}f(ct-x) + \frac{1}{2c}\int_0^{ct-x} g(s)\,ds + \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(s)\,ds \qquad x < ct$$
$$= -c\int_0^{\frac{ct-x}{c}} h(s)\,ds + \frac{f(ct-x)+f(x+ct)}{2} + \frac{1}{2c}\left(\int_0^{ct-x} g(s)\,ds + \int_0^{x+ct} g(s)\,ds\right)$$

The above for $x < ct$. Therefore the full solution is

$$u(x,t) = \begin{cases} \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds & x \geq ct \\ -c\int_0^{t-\frac{x}{c}} h(s)\,ds + \frac{f(ct-x)+f(x+ct)}{2} + \frac{1}{2c}\left(\int_0^{ct-x} g(s)\,ds + \int_0^{x+ct} g(s)\,ds\right) & x < ct \end{cases} \tag{10}$$

0.2.2 Part(b)

From (10) above, for $h(t) = 0$ the solution becomes

$$u(x,t) = \begin{cases} \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds & x \geq ct \\ \frac{f(ct-x)+f(x+ct)}{2} + \frac{1}{2c}\left(\int_0^{ct-x} g(s)\,ds + \int_0^{x+ct} g(s)\,ds\right) & x < ct \end{cases} \tag{1}$$

The idea of symmetry is to obtain the same solution (1) above but by starting from d'Albert solution (which is valid only for positive arguments)

$$u(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds \qquad x > ct \tag{1A}$$

But by using f_{ext}, g_{ext} in the above instead of f, g, where the d'Albert solution becomes valid for $x < ct$ when using f_{ext}, g_{ext}

$$u(x,t) = \frac{f_{ext}(x+ct)+f_{ext}(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g_{ext}(s)\,ds \qquad x < ct \tag{2}$$

Then using (1A) and (2) we show it is the same as (1). We really need to show that (2) leads to the second part of (1), since (1A) is the same as first part of (1).

The main issue is how to determine f_{ext}, g_{ext} and determine if they should be even or odd extension of f, g. From boundary conditions, in part (a) equation (4A) we found that

$$F(z) = \int_0^z h\left(-\frac{s}{c}\right) ds + G(-z)$$

But now $h(t) = 0$, hence

$$F(z) = G(-z) \tag{3}$$

Now, looking at the first part of the solution in (1). we see that for positive argument the solution has $f(x - ct)$ for $x > ct$ and it has $f(ct - x)$ when $x < ct$. So this leads us to pick f_{ext} being even as follows. Let

$$z = x - ct$$

Then we see that

$$\begin{aligned} f(x - ct) &= f(-(x - ct)) \\ f(z) &= f(-z) \end{aligned}$$

Therefore we need f_{ext} to be an <u>even function</u>.

$$f_{ext}(z) = \begin{cases} f(z) & z > 0 \\ f(-z) & z < 0 \end{cases}$$

But since $F(z) = G(-z)$ from (3), then g_{ext} is also <u>even function</u>, which means

$$g_{ext}(z) = \begin{cases} g(z) & z > 0 \\ g(-z) & z < 0 \end{cases}$$

Now that we found f_{ext}, g_{ext} extensions, we go back to (2). For negative argument

$$u(x,t) = \frac{f_{ext}(x + ct) + f_{ext}(x - ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g_{ext}(s)\, ds \qquad x < ct$$

Since f_{ext}, g_{ext} are even, then using $g_{ext}(z) = g(-z)$ since now $z < 0$ and using $f_{ext}(z) = f(-z)$ since now $z < 0$ the above becomes

$$u(x,t) = \frac{f(-(x + ct)) + f(-(x - ct))}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g_{ext}(s)\, ds \qquad x < ct$$

But $f(-(x + ct)) = f(x + ct)$ since even and $f(-(x - ct)) = f(ct - x)$, hence the above becomes

$$u(x,t) == \frac{f(x + ct) + f(ct - x)}{2} + \frac{1}{2c}\left(\int_{x-ct}^{0} g(-s)\, ds + \int_0^{x+ct} g(s)\, ds\right) \qquad x < ct$$

Let $r = -s$, then $\frac{dr}{ds} = -1$. When $s = x - ct$, $r = ct - x$ and when $s = 0, r = 0$. Then the first integral on the RHS above becomes

$$\begin{aligned} u(x,t) &= \frac{f(x + ct) + f(ct - x)}{2} + \frac{1}{2c}\left(\int_{ct-x}^{0} g(r)(-dr) + \int_0^{x+ct} g(s)\, ds\right) \\ &= \frac{f(x + ct) + f(ct - x)}{2} + \frac{1}{2c}\left(\int_0^{ct-x} g(r)\, dr + \int_0^{x+ct} g(s)\, ds\right) \end{aligned}$$

Relabel r back to s, then

$$u(x,t) = \frac{f(x + ct) + f(ct - x)}{2} + \frac{1}{2c}\left(\int_0^{ct-x} g(s)\, ds + \int_0^{x+ct} g(s)\, ds\right) \qquad x < ct \tag{5}$$

Looking at (5) we see that this is the same solution in (1) for the case of $x < ct$. Hence (1A) and (5) put together give

$$u(x,t) = \begin{cases} \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds & x \geq ct \\ \frac{f(ct-x)+f(x+ct)}{2} + \frac{1}{2c}\left(\int_0^{ct-x} g(s)\, ds + \int_0^{x+ct} g(s)\, ds\right) & x < ct \end{cases} \tag{1}$$

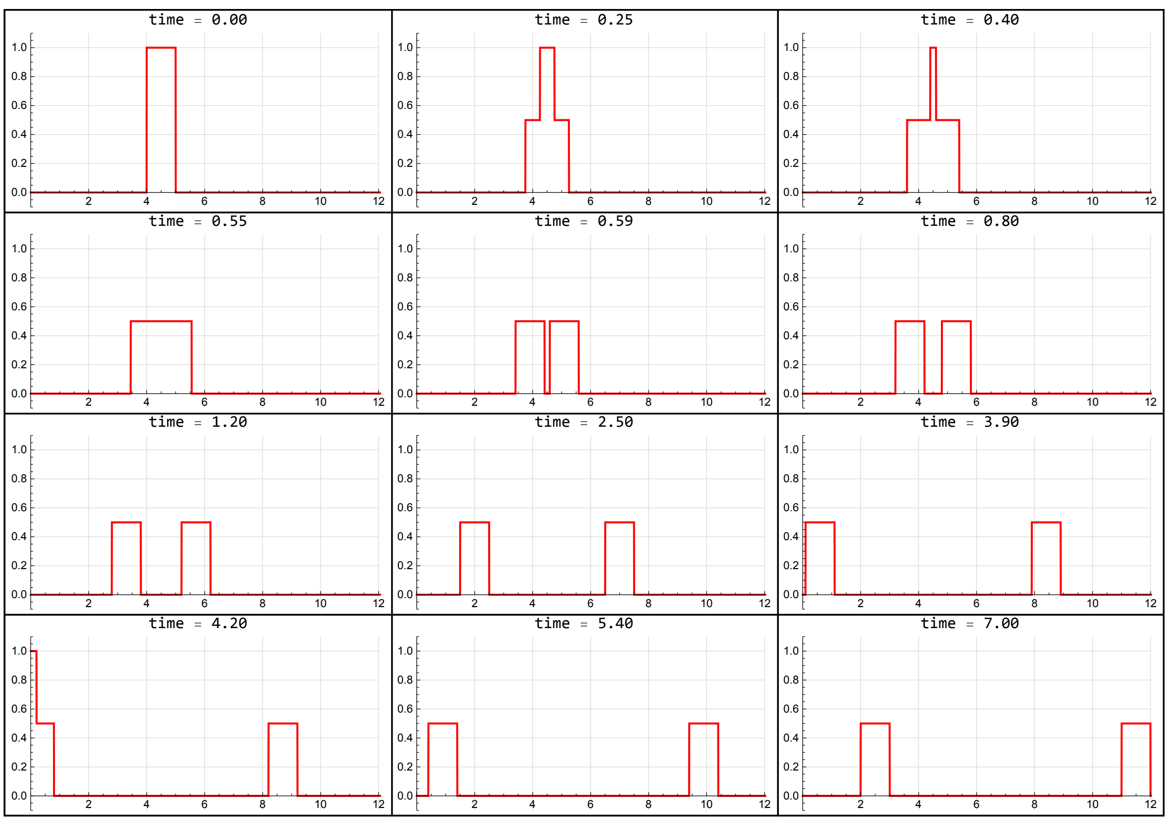
Which is same solution obtain in part(a).

0.2.3 Part(c)

For $g(x) = 0, h(t) = 0$ and $f(x)$ as given, the solution in equation (10) in part(a) becomes

$$u(x,t) = \begin{cases} \frac{f(x+ct)+f(x-ct)}{2} & x \geq ct \\ \frac{f(ct-x)+f(x+ct)}{2} & x < ct \end{cases} \tag{10}$$

A small program we written to make few sketches important time instantces. The left moving wave $G(x + ct)$ hits the boundary at $x = 0$ but do not reflect now as the case with Dirichlet boundary conditions, but instead it remains upright and turns around as shown.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time = 0.55
time = 0.59
time = 0.80
time = 1.20
time = 2.50
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time = 5.40
time = 7.00