HW 4, NE 548, Spring 2017

Nasser M. Abbasi

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0.1 problem 10.5 (page 540)

problem Use WKB to obtain solution to

$$\varepsilon y'' + a(x) y' + b(x) y = 0 \tag{1}$$

with $a(x) > 0, y(0) = A, y(1) = B$ correct to order ε.

solution

Assuming

$$y(x) \sim \exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right) \qquad \delta \to 0$$

Therefore

$$y'(x) \sim \frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n'(x)\exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right)$$

$$y''(x) \sim \frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n''(x)\exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right) + \left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n'(x)\right)^2\exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right)$$

Substituting the above into (1) and simplifying gives (writing = instead of $\sim$ for simplicity for now)

$$\varepsilon\left[\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n''(x) + \left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n'(x)\right)^2\right] + \frac{a}{\delta}\sum_{n=0}^{\infty}\delta^n S_n'(x) + b = 0$$

$$\frac{\varepsilon}{\delta}\sum_{n=0}^{\infty}\delta^n S_n''(x) + \frac{\varepsilon}{\delta^2}\left(\sum_{n=0}^{\infty}\delta^n S_n'(x)\sum_{n=0}^{\infty}\delta^n S_n'(x)\right) + \frac{a}{\delta}\sum_{n=0}^{\infty}\delta^n S_n'(x) + b = 0$$

Expanding gives

$$\frac{\varepsilon}{\delta}\left(S_0'' + \delta S_1'' + \delta^2 S_2'' + \cdots\right) + \frac{\varepsilon}{\delta^2}\left(\left(S_0' + \delta S_1' + \delta^2 S_2' + \cdots\right)\left(S_0' + \delta S_1' + \delta^2 S_2' + \cdots\right)\right) + \frac{a}{\delta}\left(S_0' + \delta S_1' + \delta^2 S_2' + \cdots\right) + b = 0$$

Simplifying

$$\left(\frac{\varepsilon}{\delta}S_0'' + \varepsilon S_1'' + \varepsilon\delta S_2'' + \cdots\right) + \left(\frac{\varepsilon}{\delta^2}\left(S_0'\right)^2 + \frac{2\varepsilon}{\delta}\left(S_0'S_1'\right) + \varepsilon\left(2S_0'S_2' + \left(S_1'\right)^2\right) + \cdots\right) + \left(\frac{a}{\delta}S_0' + aS_1' + a\delta S_2' + \cdots\right) + b = 0 \tag{1A}$$

The largest terms in the left are $\frac{\varepsilon}{\delta^2}\left(S_0'\right)^2$ and $\frac{a}{\delta}S_0'$. By dominant balance these must be equal in magnitude. Hence $\frac{\varepsilon}{\delta^2} = O\left(\frac{1}{\delta}\right)$ or $\frac{\varepsilon}{\delta} = O(1)$. Therefore δ is proportional to ε and for simplicity ε is taken as equal to δ, hence (1A) becomes

$$\left(S_0'' + \varepsilon S_1'' + \varepsilon^2 S_2'' + \cdots\right) + \left(\varepsilon^{-1}\left(S_0'\right)^2 + 2S_0'S_1' + \varepsilon\left(2S_0'S_2' + \left(S_1'\right)^2\right) + \cdots\right) + \left(a\varepsilon^{-1}S_0' + aS_1' + a\varepsilon S_2' + \cdots\right) + b = 0$$

Terms of $O\left(\varepsilon^{-1}\right)$ give

$$\left(S_0'\right)^2 + aS_0' = 0 \tag{2}$$

And terms of $O(1)$ give

$$S_0'' + 2S_0'S_1' + aS_1' + b = 0 \tag{3}$$

And terms of $O(\varepsilon)$ give

$$2S_0'S_2' + aS_2' + \left(S_1'\right)^2 + S_1'' = 0$$

$$S_2' = -\frac{\left(S_1'\right)^2 + S_1''}{\left(a + 2S_0'\right)} \tag{4}$$

Starting with (2)

$$S_0'\left(S_0' + a\right) = 0$$

There are two cases to consider.

case 1 $S_0' = 0$. This means that $S_0(x) = c_0$. A constant. Using this result in (3) gives an ODE to solve for $S_1(x)$

$$\begin{aligned} aS_1' + b &= 0 \\ S_1' &= -\frac{b(x)}{a(x)} \\ S_1 &\sim -\int_0^x \frac{b(t)}{a(t)} dt + c_1 \end{aligned}$$

Using this result in (4) gives an ODE to solve for $S_2(x)$

$$\begin{aligned} S_2' &= -\frac{\left(-\frac{b(x)}{a(x)}\right)^2 + \left(-\frac{b(x)}{a(x)}\right)'}{a(x)} \\ &= -\frac{\frac{b^2(x)}{a^2(x)} - \left(\frac{b'(x)}{a(x)} - \frac{b(x)a'(x)}{a^2(x)}\right)}{a(x)} \\ &= -\frac{\frac{b^2(x)}{a^2(x)} - \frac{a(x)b'(x)}{a^2(x)} + \frac{a'(x)b(x)}{a^2(x)}}{a(x)} \\ &= \frac{a(x)b'(x) - b^2(x) - a'(x)b(x)}{a^3(x)} \end{aligned}$$

Therefore

$$S_2 = \int_0^x \frac{a(t)b'(t) - b^2(t) - a'(t)b(t)}{a^3(t)} dt + c_2$$

For case one, the solution becomes

$$\begin{aligned}
y_1(x) &\sim \exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right) \qquad \delta \to 0 \\
&\sim \exp\left(\frac{1}{\varepsilon}\left(S_0(x) + \varepsilon S_1(x) + \varepsilon^2 S_2(x)\right)\right) \qquad \varepsilon \to 0^+ \\
&\sim \exp\left(\frac{1}{\varepsilon}S_0(x) + S_1(x) + \varepsilon S_2(x)\right) \\
&\sim \exp\left(\frac{1}{\varepsilon}c_0 - \int_0^x \frac{b(t)}{a(t)}dt + c_1 + \varepsilon\int_0^x \frac{a(t)b'(t) - b^2(t) - b(t)}{a^3(t)}dt + c_2\right) \\
&\sim C_1 \exp\left(-\int_0^x \frac{b(t)}{a(t)}dt + \varepsilon\int_0^x \frac{a(t)b'(t) - b^2(t) - b(t)}{a^3(t)}dt\right) \qquad (5)
\end{aligned}$$

Where C_1 is a constant that combines all $e^{\frac{1}{\varepsilon}c_0+c_1+c_2}$ constants into one. Equation (5) gives the first WKB solution of order $O(\varepsilon)$ for case one. Case 2 now is considered.

<u>case 2</u> In this case $S_0' = -a$, therefore

$$S_0 = -\int_0^x a(t)\,dt + c_0$$

Equation (3) now gives

$$\begin{aligned}
S_0'' + 2S_0'S_1' + aS_1' + b &= 0 \\
-a' - 2aS_1' + aS_1' + b &= 0 \\
-aS_1' &= a' - b \\
S_1' &= \frac{b - a'}{a} \\
S_1' &= \frac{b}{a} - \frac{a'}{a}
\end{aligned}$$

Integrating the above results in

$$S_1 = \int_0^x \frac{b(t)}{a(t)}dt - \ln(a) + c_1$$

$S_2(x)$ is now found from (4)

$$\begin{aligned}
S_2' &= -\frac{\left(S_1'\right)^2 + S_1''}{\left(a + 2S_0'\right)} \\
&= -\frac{\left(\frac{b-a'}{a}\right)^2 + \left(\frac{b-a'}{a}\right)'}{\left(a + 2(-a)\right)} \\
&= -\frac{\frac{b^2+(a')^2-2ba'}{a^2} + \frac{b'-a''}{a} - \frac{a'b-(a')^2}{a^2}}{-a} \\
&= \frac{b^2 + (a')^2 - 2ba' + ab' - aa'' - a'b - (a')^2}{a^3} \\
&= \frac{b^2 - 2ba' + ab' - aa'' - a'b}{a^3}
\end{aligned}$$

Hence

$$S_2=\int_0^x \frac{b^2(t)-2b(t)a'(t)+a(t)b'(t)-a(t)a''(t)-a'(t)b(t)}{a^3(t)}dt+c_2$$

Therefore for this case the solution becomes

$$\begin{aligned} y_2(x) &\sim \exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right) \qquad \delta\to 0\\ &\sim \exp\left(\frac{1}{\varepsilon}\left(S_0(x)+\varepsilon S_1(x)+\varepsilon^2 S_2(x)\right)\right) \qquad \varepsilon\to 0^+\\ &\sim \exp\left(\frac{1}{\varepsilon}S_0(x)+S_1(x)+\varepsilon S_2(x)\right)\end{aligned}$$

Or

$$y_2(x)\sim \exp\left(\frac{-1}{\varepsilon}\int_0^x a(t)\,dt+c_0\right)\exp\left(\int_0^x\frac{b(t)}{a(t)}dt-\ln(a)+c_1\right)$$
$$\exp\left(\varepsilon\int_0^x \frac{b^2(t)-2b(t)a'(t)+a(t)b'(t)-a(t)a''(t)-a'(t)b(t)}{a^3(t)}dt+c_2\right)$$

Which simplifies to

$$y_2(x)\sim\frac{C_2}{a}\exp\left(\frac{-1}{\varepsilon}\int_0^x a(t)\,dt+\int_0^x\frac{b(t)}{a(t)}dt+\varepsilon\int_0^x\frac{b^2(t)-2b(t)a'(t)+a(t)b'(t)-a(t)a''(t)-a'(t)b(t)}{a^3(t)}dt\right) \tag{6}$$

Where C_2 is new constant that combines c_0, c_1, c_2 constants. The general solution is linear combinations of y_1, y_2

$$y(x)\sim Ay_1(x)+By_2(x)$$

Or

$$y(x)\sim C_1\exp\left(-\int_0^x\frac{b(t)}{a(t)}dt+\varepsilon\int_0^x\frac{a(t)b'(t)-b^2(t)-a'(t)b(t)}{a^3(t)}dt\right)$$
$$+\frac{C_2}{a(x)}\exp\left(\frac{-1}{\varepsilon}\int_0^x a(t)\,dt+\int_0^x\frac{b(t)}{a(t)}dt+\varepsilon\int_0^x\frac{b^2(t)-2b(t)a'(t)+a(t)b'(t)-a(t)a''(t)-a'(t)b(t)}{a^3(t)}dt\right)$$

Now boundary conditions are applied to find C_1, C_2. Using $y(0)=A$ in the above gives

$$A=C_1+\frac{C_2}{a(0)} \tag{7}$$

And using $y(1)=B$ gives

$$B=C_1\exp\left(-\int_0^1\frac{b(t)}{a(t)}dt+\varepsilon\int_0^1\frac{a(t)b'(t)-b^2(t)-a'(t)b(t)}{a^3(t)}dt\right)$$
$$+\frac{C_2}{a(1)}\exp\left(\frac{-1}{\varepsilon}\int_0^1 a(t)\,dt+\int_0^1\frac{b(t)}{a(t)}dt+\varepsilon\int_0^1\frac{b^2(t)-2b(t)a'(t)+a(t)b'(t)-a(t)a''(t)-a'(t)b(t)}{a^3(t)}dt\right)$$

Neglecting exponentially small terms involving $e^{\frac{-1}{\varepsilon}}$ the above becomes

$$\begin{aligned} B=C_1\exp\left(-\int_0^1\frac{b(t)}{a(t)}dt\right)\exp\left(\varepsilon\int_0^1\frac{a(t)b'(t)-b^2(t)-a'(t)b(t)}{a^3(t)}dt\right)\\ +\frac{C_2}{a(1)}\exp\left(\int_0^1\frac{b(t)}{a(t)}dt\right)\exp\left(\varepsilon\int_0^1\frac{b^2(t)-2b(t)a'(t)+a(t)b'(t)-a(t)a''(t)-a'(t)b(t)}{a^3(t)}dt\right)\end{aligned} \tag{8}$$

To simplify the rest of the solution which finds C_1, C_2, let

$$z_1 = \int_0^1 \frac{b(t)}{a(t)} dt$$

$$z_2 = \int_0^1 \frac{a(t)b'(t) - b^2(t) - a'(t)b(t)}{a^3(t)} dt$$

$$z_3 = \int_0^1 \frac{b^2(t) - 2b(t)a'(t) + a(t)b'(t) - a(t)a''(t) - a'(t)b(t)}{a^3(t)} dt$$

Hence (8) becomes

$$B = C_1 e^{-z_1} e^{\varepsilon z_2} + \frac{C_2}{a(1)} e^{z_1} e^{\varepsilon z_3} \tag{8A}$$

From (7) $C_2 = a(0)(A - C_1)$. Substituting this in (8A) gives

$$\begin{aligned}
B &= C_1 e^{-z_1} e^{\varepsilon z_2} + \frac{a(0)(A - C_1)}{a(1)} e^{z_1} e^{\varepsilon z_3} \\
&= C_1 e^{-z_1} e^{\varepsilon z_2} + \frac{a(0)}{a(1)} A e^{z_1} e^{\varepsilon z_3} - \frac{a(0)}{a(1)} C_1 e^{z_1} e^{\varepsilon z_3} \\
B &= C_1 \left(e^{-z_1} e^{\varepsilon z_2} - \frac{a(0)}{a(1)} e^{z_1} e^{\varepsilon z_3} \right) + A \frac{a(0)}{a(1)} e^{z_1} e^{\varepsilon z_3} \\
C_1 &= \frac{B - A \frac{a(0)}{a(1)} e^{z_1} e^{\varepsilon z_3}}{e^{-z_1} e^{\varepsilon z_2} - \frac{a(0)}{a(1)} e^{z_1} e^{\varepsilon z_3}} \\
&= \frac{a(1) B - A a(0) e^{z_1} e^{\varepsilon z_3}}{a(1) e^{-z_1} e^{\varepsilon z_2} - a(0) e^{z_1} e^{\varepsilon z_3}} \qquad (9)
\end{aligned}$$

Using (7), now C_2 is found

$$\begin{aligned}
A &= C_1 + \frac{C_2}{a(0)} \\
A &= \frac{a(1) B - A a(0) e^{z_1} e^{\varepsilon z_3}}{a(1) e^{-z_1} e^{\varepsilon z_2} - a(0) e^{z_1} e^{\varepsilon z_3}} + \frac{C_2}{a(0)} \\
C_2 &= a(0) \left(A - \frac{a(1) B - A a(0) e^{z_1} e^{\varepsilon z_3}}{a(1) e^{-z_1} e^{\varepsilon z_2} - a(0) e^{z_1} e^{\varepsilon z_3}} \right) \qquad (10)
\end{aligned}$$

The constants C_1, C_2, are now found, hence the solution is now complete.

Summary of solution

$$\begin{aligned}
y(x) \sim{}& C_1 \exp\left(-\int_0^x \frac{b(t)}{a(t)} dt + \varepsilon \int_0^x \frac{a(t)b'(t) - b^2(t) - a'(t)b(t)}{a^3(t)} dt \right) \\
&+ \frac{C_2}{a(x)} \exp\left(\frac{-1}{\varepsilon} \int_0^x a(t)\,dt + \int_0^x \frac{b(t)}{a(t)} dt + \varepsilon \int_0^x \frac{b^2(t) - 2b(t)a'(t) + a(t)b'(t) - a(t)a''(t) - a'(t)b(t)}{a^3(t)} dt \right)
\end{aligned}$$

Where

$$\begin{aligned}
C_1 &= \frac{a(1) B - A a(0) e^{z_1} e^{\varepsilon z_3}}{a(1) e^{-z_1} e^{\varepsilon z_2} - a(0) e^{z_1} e^{\varepsilon z_3}} \\
C_2 &= a(0) \left(A - \frac{a(1) B - A a(0) e^{z_1} e^{\varepsilon z_3}}{a(1) e^{-z_1} e^{\varepsilon z_2} - a(0) e^{z_1} e^{\varepsilon z_3}} \right)
\end{aligned}$$

And

$$\begin{aligned} z_1 &= \int_0^1 \frac{b(t)}{a(t)} dt \\ z_2 &= \int_0^1 \frac{a(t)\,b'(t) - b^2(t) - a'(t)\,b(t)}{a^3(t)} dt \\ z_3 &= \int_0^1 \frac{b^2(t) - 2b(t)\,a'(t) + a(t)\,b'(t) - a(t)\,a''(t) - a'(t)\,b(t)}{a^3(t)} dt \end{aligned}$$

0.2 problem 10.6

problem Use second order WKB to derive formula which is more accurate than (10.1.31) for the n^{th} eigenvalue of the Sturm-Liouville problem in 10.1.27. Let $Q(x) = (x+\pi)^4$ and compare your formula with value of E_n in table 10.1

solution

Problem 10.1.27 is

$$y'' + EQ(x)\,y = 0$$

With $Q(x) = (x+\pi)^4$ and boundary conditions $y(0) = 0, y(\pi) = 0$. Letting

$$E = \frac{1}{\varepsilon}$$

Then the ODE becomes

$$\varepsilon y''(x) + (x+\pi)^4\, y(x) = 0 \tag{1}$$

Physical optics approximation is obtained when $\lambda \to \infty$ or $\varepsilon \to 0^+$. Since the ODE is linear, and the highest derivative is now multiplied by a very small parameter ε, WKB can be used to solve it. Assuming the solution is

$$y(x) \sim \exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n S_n(x)\right) \qquad \delta \to 0$$

Then

$$\begin{aligned} y'(x) &\sim \exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n S_n(x)\right)\left(\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n S_n'(x)\right) \\ y''(x) &\sim \exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n S_n(x)\right)\left(\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n S_n'(x)\right)^2 + \exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n S_n(x)\right)\left(\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n S_n''(x)\right) \end{aligned}$$

Substituting these into (1) and canceling the exponential terms gives

$$\begin{aligned} \varepsilon\left(\left(\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n S_n'(x)\right)^2 + \frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n S_n''(x)\right) &\sim -(x+\pi)^4 \\ \frac{\varepsilon}{\delta^2}\left(S_0' + \delta S_1' + \delta^2 S_2' + \cdots\right)\left(S_0' + \delta S_1' + \delta^2 S_2' + \cdots\right) + \frac{\varepsilon}{\delta}\left(S_0'' + \delta S_1'' + \delta^2 S_2'' + \cdots\right) &\sim -(x+\pi)^4 \\ \frac{\varepsilon}{\delta^2}\left(\left(S_0'\right)^2 + \delta\left(2S_1'S_0'\right) + \delta^2\left(2S_0'S_2' + \left(S_0'\right)^2\right) + \cdots\right) + \frac{\varepsilon}{\delta}\left(S_0'' + \delta S_1'' + \delta^2 S_2'' + \cdots\right) &\sim -(x+\pi)^4 \\ \left(\frac{\varepsilon}{\delta^2}\left(S_0'\right)^2 + \frac{\varepsilon}{\delta}\left(2S_1'S_0'\right) + \varepsilon\left(2S_0'S_2' + \left(S_0'\right)^2\right) + \cdots\right) + \left(\frac{\varepsilon}{\delta}S_0'' + \varepsilon^2 S_1'' + \cdots\right) &\sim -(x+\pi)^4 \end{aligned} \tag{2}$$

The largest term in the left side is $\frac{\varepsilon}{\delta^2}\left(S_0'\right)^2$. By dominant balance, this term has the same order of magnitude as right side $-(x+\pi)^4$. Hence δ^2 is proportional to ε and for simplicity, δ can be taken equal to $\sqrt{\varepsilon}$ or

$$\delta = \sqrt{\varepsilon}$$

Equation (2) becomes

$$\left(\left(S_0'\right)^2 + \sqrt{\varepsilon}\left(2S_1'S_0'\right) + \varepsilon\left(2S_0'S_2' + \left(S_0'\right)^2\right) + \cdots\right) + \left(\sqrt{\varepsilon}S_0'' + \varepsilon S_1'' + \cdots\right) \sim -(x+\pi)^4$$

Balance of $O(1)$ gives

$$\left(S_0'\right)^2 \sim -(x+\pi)^4 \tag{3}$$

And Balance of $O\left(\sqrt{\varepsilon}\right)$ gives

$$2S_1'S_0' \sim -S_0'' \tag{4}$$

Balance of $O(\varepsilon)$ gives

$$2S_0'S_2' + \left(S_1'\right)^2 \sim -S_1'' \tag{5}$$

Equation (3) is solved first in order to find $S_0(x)$. Therefore

$$S_0' \sim \pm i(x+\pi)^2$$

Hence

$$\begin{aligned} S_0(x) &\sim \pm i\int_0^x (t+\pi)^2\,dt + C^{\pm} \\ &\sim \pm i\left(\frac{t^3}{3} + \pi t^2 + \pi^2 t\right)_0^x + C^{\pm} \\ &\sim \pm i\left(\frac{x^3}{3} + \pi x^2 + \pi^2 x\right) + C^{\pm} \end{aligned} \tag{6}$$

$S_1(x)$ is now found from (4), and since $S_0'' = \pm 2i(x+\pi)$ therefore

$$\begin{aligned} S_1' &\sim -\frac{1}{2}\frac{S_0''}{S_0'} \\ &\sim -\frac{1}{2}\frac{\pm 2i(x+\pi)}{\pm i(x+\pi)^2} \\ &\sim -\frac{1}{x+\pi} \end{aligned}$$

Hence

$$\begin{aligned} S_1(x) &\sim -\int_0^x \frac{1}{t+\pi}dt \\ &\sim -\ln\left(\frac{\pi+x}{\pi}\right) \end{aligned} \tag{7}$$

$S_2(x)$ is now solved from from (5)

$$\begin{aligned} 2S_0'S_2' + \left(S_1'\right)^2 &\sim -S_1'' \\ S_2' &\sim \frac{-S_1'' - \left(S_1'\right)^2}{2S_0'} \end{aligned}$$

Since $S_1' \sim -\frac{1}{x+\pi}$, then $S_1'' \sim \frac{1}{(x+\pi)^2}$ and the above becomes

$$\begin{aligned} S_2' &\sim \frac{-\frac{1}{(x+\pi)^2} - \left(-\frac{1}{x+\pi}\right)^2}{2\left(\pm i\left(x+\pi\right)^2\right)} \\ &\sim \frac{-\frac{1}{(x+\pi)^2} - \frac{1}{(x+\pi)^2}}{\pm 2i\left(x+\pi\right)^2} \\ &\sim \pm i\frac{1}{\left(x+\pi\right)^4} \end{aligned}$$

Hence

$$\begin{aligned} S_2 &\sim \pm i\left(\int_0^x \frac{1}{\left(t+\pi\right)^4}dt\right) + k^{\pm} \\ &\sim \pm\frac{i}{3}\left(\frac{1}{\pi^3} - \frac{1}{\left(\pi+x\right)^3}\right) + k^{\pm} \end{aligned}$$

Therefore the solution becomes

$$\begin{aligned} y\left(x\right) &\sim \exp\left(\frac{1}{\sqrt{\varepsilon}}\left(S_0\left(x\right) + \sqrt{\varepsilon}S_1\left(x\right) + \varepsilon S_2\left(x\right)\right)\right) \\ &\sim \exp\left(\frac{1}{\sqrt{\varepsilon}}S_0\left(x\right) + S_1\left(x\right) + \sqrt{\varepsilon}S_2\left(x\right)\right) \end{aligned}$$

But $E = \frac{1}{\varepsilon}$, hence $\sqrt{\varepsilon} = \frac{1}{\sqrt{E}}$, and the above becomes

$$y\left(x\right) \sim \exp\left(\sqrt{E}S_0\left(x\right) + S_1\left(x\right) + \frac{1}{\sqrt{E}}S_2\left(x\right)\right)$$

Therefore

$$y\left(x\right) \sim \exp\left(\pm i\sqrt{E}\left(\frac{x^3}{3} + \pi x^2 + \pi^2 x\right) + C^{\pm} - \ln\left(\frac{\pi+x}{\pi}\right) \pm i\frac{1}{\sqrt{E}}\frac{1}{3}\left(\frac{1}{\pi^3} - \frac{1}{\left(\pi+x\right)^3}\right) + k^{\pm}\right)$$

Which can be written as

$$\begin{aligned} y\left(x\right) \sim \left(\frac{\pi+x}{\pi}\right)^{-1} C\exp\left(i\left(\sqrt{E}\left(\frac{x^3}{3} + \pi x^2 + \pi^2 x\right) + \frac{1}{\sqrt{E}}\frac{1}{3}\left(\frac{1}{\pi^3} - \frac{1}{\left(\pi+x\right)^3}\right)\right)\right) \\ - C\left(\frac{\pi+x}{\pi}\right)^{-1}\exp\left(-i\left(\sqrt{E}\left(\frac{x^3}{3} + \pi x^2 + \pi^2 x\right) + \frac{1}{\sqrt{E}}\frac{1}{3}\left(\frac{1}{\pi^3} - \frac{1}{\left(\pi+x\right)^3}\right)\right)\right) \end{aligned}$$

Where all constants combined into $\pm C$. In terms of $\sin/\cos$ the above becomes

$$\begin{aligned} y\left(x\right) \sim& \frac{\pi A}{\pi+x}\cos\left(\sqrt{E}\left(\frac{x^3}{3} + \pi x^2 + \pi^2 x\right) + \frac{1}{\sqrt{E}}\frac{1}{3}\left(\frac{1}{\pi^3} - \frac{1}{\left(\pi+x\right)^3}\right)\right) \\ &+ \frac{\pi B}{\pi+x}\sin\left(\sqrt{E}\left(\frac{x^3}{3} + \pi x^2 + \pi^2 x\right) + \frac{1}{\sqrt{E}}\frac{1}{3}\left(\frac{1}{\pi^3} - \frac{1}{\left(\pi+x\right)^3}\right)\right) \end{aligned} \tag{8}$$

Boundary conditions $y\left(0\right) = 0$ gives

$$\begin{aligned} 0 &\sim A\cos\left(0 + \frac{1}{\sqrt{E}}\left(\frac{1}{\pi^3} - \frac{1}{\pi^3}\right)\right) + B\sin\left(0 + \frac{1}{\sqrt{E}}\frac{1}{3}\left(\frac{1}{\pi^3} - \frac{1}{\pi^3}\right)\right) \\ &\sim A \end{aligned}$$

Hence solution in (8) reduces to

$$y(x) \sim \frac{\pi B}{\pi + x} \sin\left(\sqrt{E}\left(\frac{x^3}{3} + \pi x^2 + \pi^2 x\right) + \frac{1}{\sqrt{E}}\frac{1}{3}\left(\frac{1}{\pi^3} - \frac{1}{(\pi + x)^3}\right)\right)$$

Applying B.C. $y(\pi) = 0$ gives

$$\begin{aligned} 0 &\sim \frac{\pi B}{\pi + \pi} \sin\left(\sqrt{E}\left(\frac{\pi^3}{3} + \pi\pi^2 + \pi^2\pi\right) + \frac{1}{\sqrt{E}}\frac{1}{3}\left(\frac{1}{\pi^3} - \frac{1}{(\pi + \pi)^3}\right)\right) \\ &\sim \frac{B}{2} \sin\left(\sqrt{E}\left(\frac{7}{3}\pi^3\right) + \frac{1}{\sqrt{E}}\left(\frac{7}{24\pi^3}\right)\right) \end{aligned}$$

For non trivial solution, therefore

$$\sqrt{E_n}\left(\frac{7}{3}\pi^3\right) + \frac{1}{\sqrt{E_n}}\left(\frac{7}{24\pi^3}\right) = n\pi \qquad n = 1, 2, 3, \cdots$$

Solving for $\sqrt{E_n}$. Let $\sqrt{E_n} = x$, then the above becomes

$$\begin{aligned} x^2\left(\frac{7}{3}\pi^3\right) + \left(\frac{7}{24\pi^6}\right) &= xn\pi \\ x^2 - \frac{3}{7\pi^2}xn + \frac{1}{8\pi^6} &= 0 \end{aligned}$$

Solving using quadratic formula and taking the positive root, since $E_n > 0$ gives

$$\begin{aligned} x &= \frac{1}{28\pi^3}\left(\sqrt{2}\sqrt{18\pi^2 n^2 - 49} + 6\pi n\right) \qquad n = 1, 2, 3, \cdots \\ \sqrt{E_n} &= \frac{1}{28\pi^3}\left(\sqrt{2}\sqrt{18\pi^2 n^2 - 49} + 6\pi n\right) \\ E_n &= \left(\sqrt{2}\sqrt{18\pi^2 n^2 - 49} + 6\pi n\right)^2 \end{aligned}$$

Table 10.1 is now reproduced to compare the above more accurate E_n. The following table shows the actual E_n values obtained this more accurate method. Values computed from above formula are in column 3.

In[207]:=
```
sol = x /. Last@Solve[x^2 - 3/(7 Pi^2) x n + 1/(8 Pi^6) == 0, x]
```

Out[207]=

$$\frac{3\,n\,\pi + \frac{\sqrt{-49+18\,n^2\,\pi^2}}{\sqrt{2}}}{14\,\pi^3}$$

In[244]:=
```
lam[n_] := (Evaluate@sol)^2
nPoints = {1, 2, 3, 4, 5, 10, 20, 40};
book = {0.00188559, 0.00754235, 0.0169703, 0.0301694, 0.0471397, 0.188559, 0.754235, 3.01694};
exact = {0.00174401, 0.00734865, 0.0167524, 0.0299383, 0.0469006, 0.188395, 0.753977, 3.01668};
hw = Table[N@(lam[n]), {n, nPoints}];
data = Table[{nPoints[[i]], book[[i]], hw[[i]], exact[[i]]}, {i, 1, Length[nPoints]}];
data = Join[{{"n", "En using S0+S1 (book)", "En using S0+S1+S2 (HW)", "Exact"}}, data];
Style[Grid[data, Frame → All], 18]
```

Out[251]=

n	E_n using S0+S1 (book)	E_n using S0+S1+S2 (HW)	Exact
1	0.0018856	0.0016151	0.001744
2	0.0075424	0.00728	0.0073487
3	0.01697	0.016709	0.016752
4	0.030169	0.029909	0.029938
5	0.04714	0.046879	0.046901
10	0.18856	0.1883	0.1884
20	0.75424	0.75398	0.75398
40	3.0169	3.0167	3.0167

The following table shows the relative error in place of the actual values of E_n to better compare how more accurate the result obtained in this solution is compared to the book result

In[252]:=
```
data = Table[{nPoints[[i]],
    100 * Abs@(exact[[i]] - book[[i]]) / exact[[i]],
    100 * Abs@(exact[[i]] - hw[[i]]) / exact[[i]]}, {i, 1, Length[nPoints]}];
data = Join[{{"n", "En using S0+S1 (book) Rel error", "En using S0+S1+S2 (HW) Rel error"}}, data];
Style[Grid[data, Frame → All], 18]
```

Out[254]=

n	E_n using S0+S1 (book) Rel error	E_n using S0+S1+S2 (HW) Rel error
1	8.1181	7.3927
2	2.6359	0.9343
3	1.3007	0.2576
4	0.77192	0.098497
5	0.5098	0.045387
10	0.087051	0.051101
20	0.034219	0.00021643
40	0.0086187	0.000055619

The above shows clearly that adding one more term in the WKB series resulted in more accurate eigenvalue estimate.