Here is an outline for the solution to BO 9.6.

Write the equation as

$$y' = (1 + \epsilon x^{-2})y^2 - 2y + 1, \quad y(1) = 1$$

and we want to solve this on $0 \le x \le 1$. For the outer solution, the expansion $y_{\text{out}}(x) \sim y_o(x) + \epsilon y_1(x) + O(\epsilon^2)$ gives

$$O(1): \quad y_o' \sim (y_o - 1)^2, \quad y_o(1) = 1$$

$$O(\epsilon): \quad y_1' \sim 2y_o y_1 + y_o^2 x^{-2} - 2y_1, \quad y_1(1) = 0$$

with solutions

$$y_o(x) \sim 1, \quad y_1(x) \sim 1 - x^{-1}$$

valid away from zero. So $y_{\text{out}}(x) \sim 1 + \epsilon(1 - x^{-1}) + O(\epsilon^2)$.

For the inner solution let $\xi = x/\epsilon^p$. The expansion $y_{\text{out}}(\xi) \sim y_o(\xi) + \epsilon y_1(\xi) + O(\epsilon^2)$ leads to $\xi \propto \epsilon$ (so we take $\xi = \epsilon$ for convenience). The biggest terms are:

$$O(\epsilon^{-1}): \quad y_o' \sim y_o^2 \xi^{-2}$$

with solution

$$y_o(\xi) \sim \xi(1 - A\xi)^{-1}.$$

Now matching

$$\lim_{\xi \to \infty} y_o(\xi) + O(\epsilon) = \lim_{x \to 0} y_o(x) + O(\epsilon)$$

$$\lim_{\xi \to \infty} \xi(1 - A\xi)^{-1} + O(\epsilon) = \lim_{x \to 0} 1 + O(\epsilon)$$

$$\lim_{\xi \to \infty} -A^{-1} + O(\xi^{-1}) + O(\epsilon) = \lim_{x \to 0} 1 + O(\epsilon)$$

gives $A = -1$ with matching region $\epsilon \ll x \ll 1$.

The next-order problem is

$$O(1): \quad y_1' \sim y_o^2 + 2y_o y_1 \xi^{-2} - 2y_o + 1.$$

Use $y_o(\xi)$ from above and the integrating factor method to find

$$y_1(\xi) \sim -\xi(1 + \xi)^{-2} + C\xi^2(1 + \xi)^{-2}.$$

Now matching

$$\lim_{\xi\to\infty} y_o(\xi) + \epsilon y_1(\xi) + O(\epsilon^2) = \lim_{x\to 0} y_o(x) + \epsilon y_1(x) + O(\epsilon^2)$$

$$\lim_{\xi\to\infty} \xi(1+\xi)^{-1} + \epsilon\Bigg(C\xi^2(1+\xi)^{-2} - \xi(1+\xi)^{-2}\Bigg) + O(\epsilon^2) = \lim_{x\to 0} 1 + \epsilon\Bigg(1 - x^{-1}\Bigg) + O(\epsilon^2)$$

$$\lim_{\xi\to\infty} \frac{1}{1+\xi^{-1}} + \epsilon\Bigg(C\xi^2(1+\xi)^{-2} - \xi(1+\xi)^{-2}\Bigg) + O(\epsilon^2) = \lim_{x\to 0} 1 + \epsilon - \frac{\epsilon}{x} + O(\epsilon^2)$$

$$\lim_{\xi\to\infty} 1 - \xi^{-1} + O(\xi^{-2}) + \epsilon C + O(\epsilon\xi^{-1}) + O(\epsilon^2) = \lim_{x\to 0} 1 + \epsilon - \frac{\epsilon}{x} + O(\epsilon^2)$$

and so we choose $C = 1$.

Question for you: what is the matching region? (I think $\epsilon^{1/2} \ll x \ll 1$). Is this correct?

Now we form the uniform approximation $y \sim y_{\text{in}} + y_{\text{out}} - y_{\text{match}} + O(\epsilon^2)$:

$$y \sim 1 + \epsilon(1 - x^{-1}) + \xi(1+\xi)^{-1} + \epsilon\Bigg(\xi^2(1+\xi)^{-2} - \xi(1+\xi)^{-2}\Bigg) - \Bigg(1 + \epsilon - \frac{\epsilon}{x}\Bigg) + O(\epsilon^2)$$

with $\xi = x/\epsilon$ and simplify. Notice that there is no singularity at $x = 0$ the solution is uniformly valid on $0 \leq x \leq 1$.

More questions for you: Does the solution above satisfy the initial condition $y(1) = 1$? If not, can you add an $O(\epsilon^2)$ correction so that $y(1) = 1$? Will you still have a solution uniformly valid to $O(\epsilon^2)$?

I encourage you to make plots so that you can visualize the solution!