NEEP 548: Engineering Analysis II 2/27/2011

Problem 9.4-b: Bender & Orszag

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1 Problem Statement

Find the boundary layer solution for

$$\epsilon y'' + (x^2+1)y' - x^3 y = 0 \tag{1}$$

2 Solution

We can expect a boundary layer at $x = 0$.

2.1 Outer Solution

$$(x^2+1)y_o' - x^3 y_o = 0$$
$$y_o' - \left(\frac{x^3}{x^2+1}\right) y_o = 0$$

This is in a form that we can find an integrating factor for,

$$\mu = \exp\left(\int \frac{\mathrm{x}^3}{\mathrm{x}^2+1}\mathrm{dx}\right) = \exp\left[-\frac{1}{2}(\ln[\mathrm{x}^2+1] - \mathrm{x}^2)\right] = (x^2+1)^{-1/2}e^{-x^2/2}$$
$$\implies [y_o(x^2+1)^{-1/2}e^{-x^2/2}]' = 0$$

$$y_o = C_1(x^2+1)^{-1/2}e^{x^2/2}$$

B.C.: $y(1) = 1 \implies 1 = C_1(2)^{-1/2}e^{1/2} \implies C_1 = \sqrt{2}e^{-1/2}$. Therefore, the outer solution is:

$$\boxed{y_o = \sqrt{2}e^{-1/2}(x^2+1)^{-1/2}e^{x^2/2}} \tag{2}$$

2.2 Inner Solution

Define $\xi = x/\epsilon$ such that when $x \sim \epsilon, \qquad \xi \sim 1$

$$\frac{\partial y}{\partial x} = \frac{dy}{d\xi}\frac{d\xi}{dx}$$
$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{\epsilon^2}\frac{d^2 y}{d\xi^2}$$

Plug this into the D. E. to get,

$$\frac{1}{\epsilon}\frac{d^2y}{d\xi^2} + \epsilon\xi^2\frac{dy}{d\xi} + \frac{1}{\epsilon}\frac{dy}{d\xi} - \xi^2\epsilon^2 y = 0 \tag{3}$$

The let the inner solution take the form,

$$y_{in} = y_o(\xi) + \epsilon y_1(\xi) + \epsilon^2 y_2(\xi) + ...$$

Balance terms, order by order,

$$O\left(\frac{1}{\epsilon}\right): \qquad y_o'' + y_o' = 0 \qquad\qquad y(0) = 1$$
$$O(1): \qquad y_1'' + y_1' = 0 \qquad\qquad y(0) = 0$$

To the lowest order,

$$y_{in}'' + y_{in}' = 0 \qquad \text{let} \qquad z = y_{in}'$$

$$\implies \ln z = -\xi + C_2$$
$$\ln y_{in}' = -\xi + C_2$$
$$y_{in}' = C_3 e^{-\xi}$$
$$y_{in} = C_3 e^{\xi} + C_4$$

We can solve for one of the constants by imposing the boundary conditions, $y_{in}(0) = 1 \implies 1 = -C_3 + C_4 \implies C_4 = 1 + C_3$

$$\boxed{y_{in} = 1 + C\left(1 - e^{-\xi}\right)} \tag{4}$$

The other constant we get by matching the 'outer' and 'inner' solutions:

$$\begin{aligned}
\lim_{x\to 0} y_o &= \lim_{\xi\to\infty} y_{in} \\
\lim_{x\to 0} \sqrt{2}e^{-1/2}(x^2+1)^{-1/2}e^{x^2/2} &= \lim_{\xi\to\infty}[1+C(1-e^{-\xi})] \\
\sqrt{2}e^{-1/2} &= 1+C \\
\implies C &= \sqrt{2}e^{-1/2}-1
\end{aligned}$$

Note: $y_{match} = \sqrt{2}e^{-1/2}$

$$y_{tot} = y_{in} + y_{out} - y_{match} \tag{5}$$

$$\boxed{y_{tot} = \sqrt{2}e^{-1/2}(x^2+1)^{-1/2}e^{x^2/2} + e^{-x/\epsilon}(1-\sqrt{2}e^{-1/2})} \tag{6}$$