Here is an outline for the solution to BO 9.19.

The boundary value problem is

$$\epsilon y'' + \sin(x)y' + 2\sin(x)\cos(x)y = 0, \quad y(0) = \pi, \quad y(\pi) = 0$$

using $\sin(2x) = 2\sin(x)\cos(x)$. Our heuristic argument based on the 1D advection-diffusion equation suggests a boundary layer on the left at $x = 0$. An outer expansion $y_{\text{out}}(x) \sim y_o(x) + \epsilon y_1(x) + O(\epsilon^2)$ gives

$$O(1): \quad y_o' \sim -2\cos(x)y_o$$

with solution

$$y_o(x) \sim A\exp[-2\sin(x)]$$

which cannot satisfy $y(\pi) = 0$ unless $A = 0$ (see below).

For the inner solution near $x = 0$ let $\xi = x/\epsilon^p$. The expansion $y_{\text{in}}(\xi) \sim y_o(\xi) + \epsilon y_1(\xi) + O(\epsilon^2)$ leads to $p = 1/2$ and

$$O(1): \quad y_o''(\xi) \sim -\xi y_o'(\xi), \quad y(\xi = 0) = \pi$$

with solution

$$y_o(\xi) \sim \pi + C\int_0^\xi \exp(-t^2/2)dt$$

$$y_o(\xi) \sim \pi + \sqrt{2}C\int_0^{\xi/\sqrt{2}} \exp(-s^2)ds = \pi + \sqrt{2}C\frac{\sqrt{\pi}}{2}\text{erf}(\xi/\sqrt{2}).$$

On the other side near $x = \pi$, let $\eta = (\pi - x)/\epsilon^p$. The expansion $y_{\text{in}}(\eta) \sim y_o(\eta) + \epsilon y_1(\eta) + O(\epsilon^2)$ leads to $p = 1/2$ and

$$O(1): \quad y_o''(\eta) \sim \eta y_o'(\eta), \quad y(\eta = 0) = 0$$

using $\sin(\pi - \epsilon^{1/2}\eta) = \sin(\epsilon^{1/2}\eta) \sim \epsilon^{1/2}\eta$ for $\epsilon^{1/2}\eta \to 0$. Then

$$y_o(\eta) \sim D\int_0^\eta \exp(t^2/2)dt$$

Now matching

$$\lim_{\xi\to\infty} y_{\text{in}}(\xi) = \lim_{x\to 0} y_{\text{out}}(x)$$

$$\lim_{\eta\to\infty} y_{\text{in}}(\eta) = \lim_{x\to\pi} y_{\text{out}}(x)$$

gives $A = D = 0$ and $C = -\sqrt{2\pi}$. Thus we obtain

$$y(x) \sim \pi - \pi\ \text{erf}(x/\sqrt{2\epsilon}).$$