Example 2

Lets set up the problems for

$$\varepsilon y'' + (x^2+1)y' - x^3 y = 0 \quad y(0) = y(1) = 1$$

Expect a boundary layer near $x = 0$. Why?

$$-(x^2+1)y' \approx \varepsilon y''$$

$$v y' \approx \varepsilon y'' \qquad \begin{array}{c} v < 0 \\ \text{pushing} \\ y \text{ to left} \end{array}, \quad \varepsilon > 0$$

Consider the "outer" problem $x > 0$, $\varepsilon \to 0$

Let $y_{out} = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$

$$\varepsilon(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots)'' + (x^2+1)(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots)'$$
$$-x^3(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots) = 0$$

$$y(1) = 0$$

order by order $\Rightarrow$

ε^0: $(x^2+1)y_0' - x^3 y_0 = 0 \quad y(1) = 1$

ε^1: $(x^2+1)y_1' - x^3 y_1 = -y_0'' \quad y_1(1) = 0$

ε^2: $(x^2+1)y_2' - x^3 y_2 = -y_1'' \quad y_2(1) = 0$ etc.

$$\Rightarrow \quad y_0(x) = \frac{\sqrt{2}}{e^{1/2}} \frac{e^{x^2/2}}{\sqrt{x^2+1}}$$

$$\Rightarrow \quad y_1 = y_0\left[-x + \tfrac{9}{8}\tan^{-1}x - \tfrac{7}{8}\frac{x}{(x^2+1)} + \frac{3}{4}\frac{x}{(1+x^2)^2} + \frac{5}{4} - \frac{9}{32}\pi\right]$$

$$\boxed{y_{ot} = y_0 + \varepsilon y_1 + O(\varepsilon^2)}$$

Now find inner solution let $\xi = \dfrac{x}{\varepsilon}$ $\quad x = O(\varepsilon)$, $\varepsilon \to 0^+$

$$\frac{1}{\varepsilon^2}\varepsilon\frac{d^2y}{d\xi^2} + (\varepsilon^2\xi^2+1)\frac{1}{\varepsilon}\frac{dy}{d\xi} + \varepsilon^3\xi^3 y = 0$$

$$\frac{d^2y}{d\xi^2} + (\varepsilon^2\xi^2+1)\frac{dy}{d\xi} + \varepsilon^4\xi^3 y = 0$$

Let $y_{in}(\xi) = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$

ε^0: $\quad y_0'' + y_0' = 0 \quad y_0(0) = 1$

ε^1: $\quad y_1'' + y_1' = 0 \quad y_1(0) = 0$

ε^2: $\quad y_2'' + \xi^2 y_2' = 0 \quad y_2(0) = 0 \quad$ etc.

$$y_0 = A(1 - e^{-\xi}) + 1$$

$$y_1 = B(1 - e^{-\xi})$$

$$y_{in} = y_0 + \varepsilon y_1 + O(\varepsilon^2)$$

Now match order by order

$$\lim_{\frac{x}{\varepsilon} \to \infty} y_{in} = \lim_{x \to 0} y_{out}$$

ε^0: $\quad \frac{\sqrt{2}}{e^{1/2}} = 1 + A \quad \Rightarrow \quad A = \frac{\sqrt{2}}{e^{1/2}} - 1$